Total variation distance for discretely observed Lévy processes: a Gaussian approximation of the small jumps

Alexandra Carpentier∗
Céline Duval † and Ester Mariucci ‡

Abstract
It is common practice to treat small jumps of Lévy processes as Wiener noise and to approximate its marginals by a Gaussian distribution. However, results that allow to quantify the goodness of this approximation according to a given metric are rare. In this paper, we clarify what happens when the chosen metric is the total variation distance. Such a choice is motivated by its statistical interpretation; if the total variation distance between two statistical models converges to zero, then no test can be constructed to distinguish the two models and they are therefore asymptotically equally informative. We elaborate a fine analysis of a Gaussian approximation for the small jumps of Lévy processes in total variation distance. Non-asymptotic bounds for the total variation distance between \( n \) discrete observations of small jumps of a Lévy process and the corresponding Gaussian distribution are presented and extensively discussed. As a byproduct, new upper bounds for the total variation distance between discrete observations of Lévy processes are provided. The theory is illustrated by concrete examples.

Résumé
Il est habituel d’assimiler les petits sauts d’un processus de Lévy à un mouvement brownien et d’approcher leurs marginales par des distributions gaussiennes. Cependant, les résultats permettant de quantifier cette approximation selon une métrique donnée sont rares. Dans cet article, nous la quantifions pour la distance en variation totale. Un tel choix s’explique par son interprétation statistique: si la distance en variation totale entre deux modèles statistiques tend vers 0, alors il n’existe aucun test permettant de distinguer les deux modèles, qui sont alors asymptotiquement équivalents. Nous contrôlons ici finement la distance en variation totale entre \( n \) incrémentes des petits sauts d’un processus de Lévy et \( n \) variables aléatoires gaussiennes: des bornes non asymptotiques pour la distance en variation totale sont données et discutées. Une conséquence de ces résultats est l’obtention de nouvelles bornes supérieures pour le contrôle en variation totale entre \( n \) incrémentes de deux processus de Lévy. Plusieurs exemples viennent illustrer ces résultats.

Keywords. Lévy processes, Total variation distance, Small jumps, Gaussian approximation, Statistical test.

1 Introduction

Although Lévy processes, or equivalently infinite divisible distributions, are mathematical objects introduced almost a century ago and even though a good knowledge of their basic properties has since long been achieved, they have recently enjoyed renewed interest. This is mainly due to the numerous applications in various fields. To name some examples, Lévy processes or Lévy-type processes (time changed Lévy processes, Lévy driven SDE, etc...) play a central role in mathematical finance, insurance,
telecommunications, biology, neurology, seismology, meteorology and extreme value theory. Examples of applications may be found in the textbooks [3] and [7] whereas the manuscripts [5] and [19] provide a comprehensive presentation of the properties of these processes.

The transition from the purely theoretical study of Lévy processes to the need to understand Lévy driven models used in real life applications has led to new challenges. For instance, the questions of how to simulate the trajectories of Lévy processes and how to make inference (prediction, testing, estimation, etc...) for this class of stochastic processes have become a key issue. The literature concerning these two aspects is already quite large: without any claim of completeness we quote [1], [2], Chapter VI in [3], [4], [6] and Part II in [7]. We specifically focus on statistics and simulation for Lévy processes, because our paper aims to give an exhaustive answer to a recurring question in these areas: When are the small jumps of Lévy processes morally Gaussian?

Before entering into details, we discuss where this question comes from. Thanks to the Lévy-Itô decomposition, the structure of the paths of any Lévy process is well understood and it is well known that any Lévy process $X$ can be decomposed into the sum of three independent Lévy processes: a Brownian motion with drift, a centered martingale $M$ associated with the small jumps of $X$ and a compound Poisson process describing the big jumps of $X$ (see the decomposition (1) in Section 1 below). If the properties of continuously or discretely observed compound Poisson processes and of Gaussian processes are well understood, the same cannot be said for the small jumps $M$. As usual in mathematics, when one faces a complex object a natural reflection is whether the problem can be simplified by replacing the difficult part with an easier but, in a sense, equivalent one. There are various notions of equivalence ranging from the weakest, convergence in law, to the stronger convergence in total variation.

For some time now, many authors have noticed that marginal laws of small jumps of Lévy processes with infinite Lévy measures resemble to Gaussian random variables, see e.g. Figure 1 and 2. This remark has led to propose algorithms of simulation of trajectories of Lévy processes based on a Gaussian approximation of the small jumps, see e.g. [6] or [7], Chapter 6. Regarding estimation procedures, a Gaussian approximation of the small jumps has, to the best of our knowledge, not been exploited yet. A fine control of the total variation distance between these two quantities could open the way of new statistical procedures. The choice of this distance is justified by its statistical interpretation: if the total variation distance between the law of the small jumps and the corresponding Gaussian component converges to zero then no statistical test can be built to distinguish between the two models. In terms of information theory, this means that the two models are asymptotically equally informative.

Investigating the goodness of a Gaussian approximation of the small jumps of a Lévy process in total variation distance makes sense only if one deals with discrete observations. From the continuous
observation of a Lévy process, the problem of separating the continuous part from the jumping part does not arise: the jumps are observed. The measure corresponding to the continuous observation of a continuous Lévy process is orthogonal to the measure corresponding to the continuous observation of a Lévy process with non trivial jump part, see e.g. [12]. However, the situation changes when dealing with discrete observations. The matter of disentangling continuous and discontinuous part of the processes is much more complex. Intuitively, fine techniques are needed to understand whether, between two observations $X_{t_0}$ and $X_{t_1}$, there has been a chaotic continuous behavior, many small jumps, one single bigger jump, or a mixture of these.

A criterion for the weak convergence for marginals of Lévy processes is given by Gnedenko and Kolmogorov [10]:

**Theorem** (Gnedenko, Kolmogorov). Marginals of Lévy processes $X^n = (X^n_t)_{t \geq 0}$ with Lévy triplets $(b_n, \sigma_n^2, \nu_n)$ converge weakly to marginals of a Lévy process $X = (X_t)_{t \geq 0}$ with Lévy triplet $(b, \sigma^2, \nu)$ if and only if

$$b_n \to b \text{ and } \sigma_n^2 \delta_0 + (x^2 \wedge 1)\nu_n(dx) \xrightarrow{w} \sigma^2 \delta_0 + (x^2 \wedge 1)\nu(dx),$$

where $\delta_0$ is the Dirac measure in 0 and $\xrightarrow{w}$ denotes weak convergence of finite measures.

A remarkable fact in the previous statement is the non-separation between the continuous and discontinuous parts of the processes: the law at time $t$ of a pure jumps Lévy process can weakly converge to that of a continuous Lévy process. In particular, if $X$ is a Lévy process with Lévy measure $\nu$ then, for any $\varepsilon > 0$ and $t > 0$, the law of the centered jumps of $X_t$ with magnitude less than $\varepsilon$ converges weakly to a centered Gaussian distribution with variance $t \sigma^2(\varepsilon) := t \int_{|x| < \varepsilon} x^2 \nu(dx)$ as $\varepsilon \to 0$. This result motivates, in [17], the simultaneous estimation the volatility $\sigma$ and the Lévy measure. We aim at understanding this phenomenon, using a notion of closeness stronger than the weak convergence, providing a quantitative translation of the result of Gnedenko and Kolmogorov in total variation distance.

There exist already several results for distances between Lévy processes. Most of them (see for example [9], [12] and [13]) are distances on the Skorohod space, distances between the continuous observation of the processes, and thus out of the scope of this paper. Concerning discretely observed Lévy processes we mention the results in [14] and [15]. Liese [14] proved the following upper bound in total variation distance for marginals of Lévy processes $X^j \sim (b_j, \Sigma_j^2, \nu_j)$, $j = 1, 2$: for any $t > 0$

$$\|\mathcal{L}(X^1_t) - \mathcal{L}(X^2_t)\|_{TV} \leq 2 \sqrt{1 - \left(1 - \frac{H^2(\mathcal{N}(t\bar{b}^j, t\Sigma_j^2), \mathcal{N}(t\bar{b}^j, t\Sigma_j^2))}{2}\right) \exp(-tH^2(\nu_1, \nu_2))}$$

with $\bar{b}^1 = b_1 - \int^1_{-1} x\nu_1(dx)$, $\bar{b}^2 = b_2 - \int^1_{-1} x\nu_2(dx)$ and $H$ denotes the Hellinger distance. This result is the analogous in discrete time of the result of Mémin and Shiryayev [16] for continuously observed Lévy processes. There is a clear separation between the continuous and discontinuous parts of the processes, which is unavoidable on the Skorohod space but that can be relaxed when dealing with the marginals. Clearly, from this kind of upper bounds it is not possible to deduce a Gaussian approximation for the small jumps in total variation: the bound is actually trivial whenever $tH^2(\nu_1, \nu_2) > 1$.

However, such an approximation may hold in total variation as proved in [15], where the convolution structure of Lévy processes with non-zero diffusion coefficients is exploited to transfer results from Wasserstein distances to the total variation distance.

In the present paper we complete the work started in [15], providing a comprehensive answer to the question: Under which asymptotics, and assumptions on the Lévy triplet, does a Gaussian approximation capture the behavior of the small jumps of a discretely observed Lévy process adequately so that the two corresponding statistical models are equally informative? Differently from [15] which deals with marginals, we also establish sharp bounds for the distance between $n$ given increments of the small jumps. Even though from a bound in total variation between marginals one can always deduce
a bound for the $n$ sample using $\|P^\otimes n - Q^\otimes n\|_{TV} \leq \sqrt{2n}\|P - Q\|_{TV}$, this kind of control is in general sub-optimal.

Section 2 gathers the main results of the paper. We first establish in Theorem 1 a non-asymptotic upper bound, which allows to quantify just how “small” the small jumps must be, in terms of the number of observations $n$, their frequency $t$, and some cumulant of the Lévy measure, so that their joint distribution is close in total variation to the corresponding Gaussian distribution with matching mean and variance. Theorem 1 can be sharpened by considering separately large and rare jumps (see Theorem 2). This is crucial when $\varepsilon$ is of much larger order than the standard deviation of the increment. Then, the closest Gaussian distribution to the increments is not the one with matching mean and variance.

These results are optimal in the following sense: whenever our upper bound is large, the total variation distance between the distribution of the Lévy increments and of the corresponding Gaussian random variables can be bounded away from 0 (see Theorems 3 and 4).

To sum up, we provide necessary and sufficient conditions on the cumulant of the Lévy measure, $n$ and $t$, that permits to characterise whether the increments of the process can be well approximated by a Gaussian sequence, in the sense of the total variation distance.

The proof of the lower bound for the total variation is based on the construction of a sharp Gaussian test for Lévy processes. This test combines three ideas, (i) the detection of extreme values that are “too large” for being produced by a Brownian motion, (ii) the detection of asymmetries around the drift in the third moment, and (iii) the detection of too heavy tails in the fourth moment for a Brownian motion. It can be of independent interest as it does not rely on the knowledge of the Lévy triplet of the process and detects optimally the presence of jumps. It uses classical ideas from testing through moments and extreme values [11], and it adapts them to the specific structure of Lévy processes. The closest related work is [18]. We improve on the test proposed there as we go beyond testing based on the fourth moment only, and we tighten the results regarding the presence of rare and large jumps.

The paper is organized as follows. In the remaining of this Section we fix notations. Section 2 is devoted to the analysis of the Gaussian approximation of the small jumps of Lévy processes. More precisely, in Section 2.1 we present upper bounds in total variation distance whereas in Section 2.2 we provide lower bounds proving optimality. Section 2.3 provides a range of conditions under which the assumptions of Theorems 1 and 2 are satisfied and Section 2.4 the resulting rates for stable processes. Most of the proofs are postponed to Section 3. In Appendix A technical results can be found.

Statistical setting and notation

For $X$ a one dimensional Lévy process with Lévy triplet $(b, \Sigma, \nu)$ (we write $X \sim (b, \Sigma, \nu)$), where $b \in \mathbb{R}$, $\Sigma \geq 0$ and $\nu$ is a Borel measure satisfying $\nu(\{0\}) = 0$ and $\int (x^2 \wedge 1)\nu(dx) < \infty$, the Lévy-Itô decomposition gives a decomposition of $X$ as the sum of three independent Lévy processes: a Brownian motion with drift $b$, a centered martingale associated with the small jumps of $X$ and a compound Poisson process associated with the large jumps of $X$. More precisely, for any $\varepsilon > 0$, $X \sim (b, \Sigma, \nu)$ can be decomposed as

$$X_t = b(\varepsilon)t + \Sigma W_t + \lim_{\eta \to 0} \left( \sum_{s \leq t} \Delta X_s 1_{\eta < |\Delta X_s| \leq \varepsilon} - t \int_{\eta < |x| \leq \varepsilon} x\nu(dx) \right) + \sum_{s \leq t} \Delta X_s 1_{|\Delta X_s| > \varepsilon},$$

$$:= b(\varepsilon)t + \Sigma W_t + M_t(\varepsilon) + Z_\varepsilon(\varepsilon), \quad \forall t \geq 0 \tag{1}$$

where $\Delta X_t = X_t - \lim_{s \uparrow t} X_s$ denotes the jump at time $t$ of $X$ and

- the drift is defined as

$$b(\varepsilon) := b + \begin{cases} -\int_{\varepsilon < |x| \leq 1} x\nu(dx) & \text{if } \varepsilon \leq 1, \\
\int_{1 < |x| \leq \varepsilon} x\nu(dx) & \text{if } \varepsilon > 1. \end{cases} \tag{2}$$

4
• $W = (W_t)_{t \geq 0}$ is a standard Brownian motion;
• $M(\varepsilon) = (M_t(\varepsilon))_{t \geq 0}$ is a centered Lévy process (and a martingale) with a Lévy measure $\nu_\varepsilon := \nu 1_{[-\varepsilon,\varepsilon]}$ i.e. it is the Lévy process associated to the jumps of $X$ smaller than $\varepsilon$. More precisely, $M(\varepsilon) \sim (\int_{|x| \leq \varepsilon} x \nu(dx), 0, \nu_\varepsilon)$. Observe that $\int_{|x| \leq \varepsilon} x \nu_\varepsilon(dx) = 0$ for any $\varepsilon \leq 1$. We write $\sigma^2(\varepsilon) = \int x^2 \nu_\varepsilon(dx)$ for the variance at time 1 of $M(\varepsilon)$ and $\mu_k(\varepsilon) = \int_{|x| \leq \varepsilon} x^k \nu_\varepsilon(dx)$ for the $k$-th moment of the Lévy measure $\nu_\varepsilon$.

2.1 Upper bound results

2.1.1 Approximation with the corresponding Gaussian distribution

The total variation distance between two probability measures $P_1$ and $P_2$ defined on the same $\sigma$-field $\mathcal{B}$ is defined as

$$\|P_1 - P_2\|_{TV} := \sup_{B \in \mathcal{B}} |P_1(B) - P_2(B)| = \frac{1}{2} \int \left| \frac{dP_1}{d\mu}(x) - \frac{dP_2}{d\mu}(x) \right| \mu(dx),$$

where $\mu$ is a common dominating measure for $P_1$ and $P_2$. To ease the reading, if $X$ and $Y$ are random variables with densities $f_X$ and $f_Y$ with respect to a same dominating measure, we sometimes write $\|X - Y\|_{TV}$ or $\|f_X - f_Y\|_{TV}$ instead of $\|\mathbb{L}(X) - \mathbb{L}(Y)\|_{TV}$. Finally, we denote by $\mathcal{N}(m, \Sigma^2)$ the law of a Gaussian distribution with mean $m$ and variance $\Sigma^2$. Sometimes, with a slight abuse of notation, we denote with the same symbol $\mathcal{N}(m, \Sigma^2)$ a Gaussian random variable with mean $m$ and variance $\Sigma^2$. The symbol $\#A$ indicates the cardinality of a set $A$. We write for any integer $n$, $\log_+(n) = \log(e \vee n)$.

2 Gaussian approximation for the Lévy process

We investigate under which conditions on $t$, $n$, $\varepsilon$ and the Lévy triplet, a Gaussian approximation of the small jumps (possibly convoluted with the continuous part of the process) is valid. In this section, we compare the increment $X_t(\varepsilon)$ with a Gaussian random variable having the same mean and variance as $X_t(\varepsilon)$. Define

$$\lambda_{\eta, \varepsilon} := \int_{\eta < |x| < \varepsilon} \nu(dx), \quad 0 \leq \eta < \varepsilon, \quad \text{and} \quad \lambda_{0, \varepsilon} = \lim_{\eta \to 0} \lambda_{\eta, \varepsilon},$$

where $\lambda_{0, \varepsilon} = +\infty$ if $\nu$ is an infinite Lévy measure.

**Theorem 1.** For any $\varepsilon > 0$, let $X(\varepsilon) \sim (b, \Sigma^2, \nu_\varepsilon)$ with $b \in \mathbb{R}$, $\Sigma^2 \geq 0$ and $\nu_\varepsilon$ a Lévy measure with support in $[-\varepsilon, \varepsilon]$. For any $t > 0$, set $X_t(\varepsilon) := (X_t(\varepsilon) - tb(\varepsilon))/\sqrt{t(\Sigma^2 + \sigma^2(\varepsilon))}$, and for any $y \in \mathbb{R}$, $\Psi_\varepsilon(y) := \mathbb{E}[e^{iyX_t(\varepsilon)}] = e^{-t\lambda_{0, \varepsilon}}1_{\{|\varepsilon| \leq \varepsilon\}}$. For any $n \geq 1$ such that $\lambda_{0, \varepsilon} \geq 4 \log_+(n)/t$, assume that there exist universal constants $\tilde{c} \in (0, 4^{-1})$, $C > 0$ such that, the following two conditions hold:

$$\int_{|x| \leq \varepsilon} |\Psi^{(k)}_{\varepsilon}(y)|^2 dy \leq C2^k k! n^{-2}, \quad \forall k \in [0, 401 \log_+(n)],$$

$$(\mathcal{H}(\Psi_\varepsilon))$$

$$\varepsilon \leq \tilde{c} \sqrt{t(\Sigma^2 + \sigma^2(\varepsilon))}/\log_+(n).$$

$$(\mathcal{H}_\varepsilon)$$
Then, there exists a constant $C > 0$, depending only on $\bar{c}, C$, such that

$$\|(X_t(\varepsilon))^{\otimes n} - \mathcal{N}(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n}\|_{TV} \leq C \left[ \sqrt{\frac{n\mu_2^2(\varepsilon)}{t^2(\Sigma^2 + \sigma^2(\varepsilon))^4}} + \frac{n\mu_2^2(\varepsilon)}{t(\Sigma^2 + \sigma^2(\varepsilon))^3} + \frac{1}{n} \right].$$

**Remark 1.** The results of Theorem 1 immediately yields to the same bound, using previous notations, for $\|L(X_t(\varepsilon))^{\otimes n} - \mathcal{N}(0, 1))^{\otimes n}\|_{TV}$.

**Examples of implications of Theorem 1.** Theorem 1 is non asymptotic and holds without assuming to work with high or low frequency observations. An explicit relation between intensity and intensity of symmetric jumps of symmetric between the small jumps and the corresponding Gaussian process. To exemplify, consider the small $a_0$ or $1$. Indeed, if it is such that $\varepsilon/\sigma(\varepsilon)$ is very different from a Gaussian process, therefore the total variation distance goes to 1. If $\nu_\varepsilon$ is not symmetric, there exists a universal constant $C$ such that:

$$\|(M_t(\varepsilon))^{\otimes n} - \mathcal{N}(0, t\sigma^2(\varepsilon))^{\otimes n}\|_{TV} \leq C \left( \sqrt{\frac{n\varepsilon^2}{t\sigma^2(\varepsilon)}} + \frac{1}{n} \right).$$

Therefore, a sufficient condition for the total variation distance to be small is given by $\varepsilon/\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, with a rate depending on $n, t$. Unsurprisingly, we observe that the rate of convergence for a Gaussian approximation of the small jumps is faster when the Lévy measure is symmetric.

**Comments on the assumptions** Theorem 1 relies on three distinctive assumptions. First, that the intensity $\lambda_{0, \varepsilon}$ is large enough: if this condition is not satisfied, the asymptotic total variation is either 0 or 1. Indeed, if it is such that $a_n t^{-1} n^{-1} \geq \lambda_{0, \varepsilon}$ where $a_n \to 0$, then, one does never observe any jump with probability going to 1 and the total variation distance goes to 0 as $n \to \infty$. In the noiseless case ($b = 0$ and $\Sigma : 0$) if $a_n t^{-1} n^{-1} \leq \lambda_{0, \varepsilon} \leq A_n \log_+ (n) t^{-1}$ where $a_n \to \infty$ and $A_n \to 0$, the probability of observing at least a time step where one, and only one, jump occurs goes to 1. As well as the probability of having at least one time step where no jump occur goes to 1 as $n \to \infty$. Such a process is very different from a Gaussian process, therefore the total variation distance goes to 1.

The second regularity Assumption ($\mathcal{H}(\Psi_\varepsilon)$) is technical, but does not seem to restrict drastically the class of Lévy processes one can consider. It imposes a condition on the decay of the derivatives of the characteristic function of the rescaled increment, restricted to the event where at least a jump is observed. In Section 2.3 we show that it is satisfied for the class of $\beta$-stable processes (see Proposition 2) which describes well the behavior of many Lévy densities around 0. It also holds as soon as $\Sigma$ is larger than $\sigma(\varepsilon)$ (see Proposition 1). Most compound Poisson processes with Lebesgue continuous jump density and intensity $\lambda_{0, \varepsilon}$ large enough also seem to satisfy it. Examples for which ($\mathcal{H}(\Psi_\varepsilon)$) does not hold are for instance Lévy processes such that $\Sigma = 0$ and $\nu_\varepsilon$ contains a finite number of Dirac masses. However such processes are perfectly detectable from Gaussian observations. The presence of the indicator
function in $\Psi_{\varepsilon}$ permits to include (some) compound Poisson processes in the study. For instance, compound Poisson processes with continuous jump density, for which $\lim_{|y| \to +\infty} \mathbb{E}[\exp(iyX_t(\varepsilon))]$ converges to $e^{-t\lambda_{0,\varepsilon}}$.

The third Assumption ($\mathcal{H}_{\varepsilon}$) of Theorem 1 imposes a constraint on the cumulants of $\nu_{\varepsilon}$. It means that the jumps cannot take values that are too extreme with respect to the standard deviation of the increments. Indeed, if one observes an increment that is much larger than the standard deviation of the increment, it is unlikely that the process is pure noise, and the total variation should not be small. Theorem 2 below allows to get rid of this assumption.

**Relation to other works** To the best of our knowledge, the only work in which non-asymptotic results are found for a Gaussian approximation of the small jumps of Lévy processes is [15] for the case $n = 1$. Proposition 8 in [15] states that, for any $\Sigma > 0$ and $\varepsilon \in (0, 1]$:}

$$\|N(0, t\Sigma^2) * M_t(\varepsilon) - N(0, t(\sigma^2(\varepsilon) + \Sigma^2))\|_{TV} \leq \frac{1}{\sqrt{2\pi t\Sigma^2}} \min\left(2\sqrt{t\sigma^2(\varepsilon)}, \frac{\varepsilon}{2}\right). \tag{6}$$

The assumption $\Sigma > 0$ is an artifact of the proof: a convolutional structure enables to transfer results in Wasserstein distance of order 1 to results in total variation distance, see Proposition 4 in [15].

Applying Theorem 1 with $n = 1$ lead to a trivial bound due to the remainder term $1/n$ in our upper bound. This term is an artifact of the strategy of the proof and can be improved at the expense of additional technicalities. More precisely, this upper bound can be improved in: for any $\kappa > 0$ and under the assumptions of Theorem 1, there exists $C_\kappa > 0$ that depends only on $\kappa, c, C'$ such that

$$CC_\kappa \sqrt{n} \left(\frac{\mu_4(\varepsilon)}{t(\Sigma^2 + \sigma^2(\varepsilon))^2} + \frac{\mu_3(\varepsilon)}{\sqrt{t(\Sigma^2 + \sigma^2(\varepsilon))^{3/2}}}\right) + C \frac{1}{n^\kappa} := CC_\kappa \sqrt{n} r_n + C n^{-\kappa}. \tag{7}$$

This permits to achieve a meaningful bound for the marginals using that for any $n \geq 1$,

$$\|X_t(\varepsilon) - N(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\varepsilon)))\|_{TV} \leq \|(X_t(\varepsilon))^{\circ n} - N(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\varepsilon))^{\circ n})\|_{TV}.$$  

Applying (7) for $n = \lfloor r^{-1/\kappa}_n \rfloor$, we get that there exists $C'_\kappa > 0$ that depends only on $\kappa, c, C'$ such that

$$\|X_t(\varepsilon) - N(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\varepsilon)))\|_{TV} \leq C'_\kappa r_n^{1 - \frac{\kappa}{2}}. \tag{8}$$

Which can be rendered arbitrarily close to $r_n$ for $\kappa$ large enough, at the expense of the increasing constant $C'_\kappa$ in $\kappa$.

This permits to compare our (modified) bound (8) for $n = 1$ and $\Sigma > 0$ and (6) established in [15]. It follows from (3) that (8) gives a tighter rate than (6). However, contrary to (6) our constants are not explicit.

**Total variation distance between Lévy processes** Theorem 1 permits to get an upper bound for the total variation distance between $n$ equidistant observations of the increments of two distinct Lévy processes $X^i \sim (b_i, \Sigma_i^2, \nu_i), i = 1, 2$. Similarly to Theorem 14 obtained in [15] for $n = 1$, relying on the result of Theorem 1 instead of (6) we get: For all $t > 0$, $\varepsilon > 0$ and $n \geq 1$ and under the...
Assumptions of Theorem 1, there exists a positive constant \(C\) such that

\[
\| (X_{kt}^1 - X_{(k-1)t})_{k=1}^n \|_{TV} \leq \frac{\sqrt{nt}}{2\pi \max(\sqrt{\Sigma_1^2 + \sigma_1^2(\varepsilon)}, \sqrt{\Sigma_2^2 + \sigma_2^2(\varepsilon)})} \left| b_1(\varepsilon) - b_2(\varepsilon) \right| + \frac{C}{n} \sum_{i=1}^2 \left( \frac{n \mu_{k,i}(\varepsilon)}{t(\sigma_i^2(\varepsilon) + \Sigma_i^2)^{1/2}} + \frac{n \mu_{k,i}(\varepsilon)}{t(\sigma_i^2(\varepsilon) + \Sigma_i^2)^{1/2}} + \frac{2C}{n} \right) + 1 - \exp \left\{ -nt(\Lambda_1(\varepsilon) - \Lambda_2(\varepsilon)) \right\} + nt(\Lambda_1(\varepsilon) \wedge \Lambda_2(\varepsilon)) \right\} \| \frac{\nu_f}{\Lambda_1(\varepsilon)} - \frac{\nu_f}{\Lambda_2(\varepsilon)} \| \|_{TV},
\]

where \(\Lambda_j(\varepsilon) = \nu_f \mathbb{R}\) if \(\nu_f = \nu_f(\cdot \cap (\mathbb{R} \setminus [-\varepsilon, \varepsilon]))\).

### 2.1.2 Approximation to the closest Gaussian distribution

In Theorem 1, we provide a control of the total variation distance between the increments \(X_t(\varepsilon)\) of a Lévy process \((b, \Sigma^2, \nu_{\varepsilon})\), where \(\nu_{\varepsilon}\) is supported on \([-\varepsilon, \varepsilon]\), and a Gaussian distribution with mean \(b(\varepsilon)t\) and variance \((\Sigma^2 + \sigma^2(\varepsilon))\). However, it may not be the Gaussian distribution closest to \(X_t(\varepsilon)\). If the increment \(X_t(\varepsilon)\) is heavy-tailed and outputs with small probability jumps that are much larger than what a \(\mathcal{N}(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\varepsilon)))\) would typically provide - i.e. if \(\varepsilon\) is much larger than \(\sqrt{t(\Sigma^2 + \sigma^2(\varepsilon))}\) - the best Gaussian approximation of \(X_t(\varepsilon)\) could be different from the “natural approximation” \(\mathcal{N}(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\varepsilon)))\).

In what follows, we define \(\bar{u}\) as the largest real number, smaller than \(\varepsilon\), such that \(\bar{u}\) is of smaller order than \(\sqrt{(\sigma^2(\bar{u}) + \Sigma^2)t}\), up to logarithmic terms. Intuitively, \(\bar{u}\) is the largest quantity smaller than \(\varepsilon\) such that \(X_t(\bar{u})\) is not heavy-tailed compared to what a corresponding Gaussian distribution should be. Therefore \(X_t(\varepsilon) - X_t(\bar{u})\) is “non-Gaussian” and should not be participating to the Gaussian approximation: either it is 0 with high probability, in which case its role is negligible when it comes to the total variation, or it is non-zero for at least one increment implying that the Gaussian approximation cannot hold. For this reason, the Gaussian distribution that is closest to \(X_t(\varepsilon)\) is \(\mathcal{N}(b(\bar{u})t, t(\Sigma^2 + \sigma^2(\bar{u})))\). Finally it is important to remark that the introduction of \(\bar{u}\) permits to remove Assumption (\(\mathcal{H}_4\)).

We now introduce formally the new Gaussian approximation. Set \(u^+ := u^+_{\nu_{\varepsilon}}\) for the largest positive real number such that

\[
\lambda_{u^+, \varepsilon}nt \geq \log_+(n).
\]

Note that such a \(u^+\) exists and is unique whenever \(\nu_{\varepsilon}\) is a Lévy measure such that \(\lambda_{0, \varepsilon} \geq \log_+(n)/nt\), which holds under the assumptions of Theorem 1. Consider the quantity

\[
\bar{u}^*(\varepsilon) = \sup \left\{ u : u \in [u^+, \varepsilon], \frac{\bar{c}}{\sqrt{(\sigma(u)^2 + \Sigma^2)t/\sqrt{n}} + u^+}, \text{ where } \bar{c} \in (0, 4^{-1}) \right\}.
\]

Sometimes we will write \(\bar{u}^*\) instead of \(\bar{u}^*(\varepsilon)\) to lighten the notation.

**Theorem 2.** For any \(\varepsilon > 0\), let \(X(\varepsilon) \sim (b, \Sigma^2, \nu_{\varepsilon})\) with \(b \in \mathbb{R}\), \(\Sigma^2 \geq 0\) and \(\nu_{\varepsilon}\) a Lévy measure with support in \([-\varepsilon, \varepsilon]\). Let \(t > 0\) and \(n \geq 1\) be such that \(\lambda_{0, \varepsilon} \geq 25 \log_+(n)/t\). Denote by \(\bar{X}_t(u) := (X_t(u) - tb(u))/\sqrt{t(\Sigma^2 + \sigma^2(u))}\) and by \(\Psi_{\nu_{\varepsilon}}(y) := E[e^{iy\bar{X}_t(u)}] - e^{-t\lambda_{0, \varepsilon}} \mathbf{1}_{\{\Sigma = 0\}}\), for any \(u > 0\). Furthermore, assume that there exist universal constants \(\bar{c} \in (0, 4^{-1})\) and \(C' > 0\) such that the following two conditions hold:

\[
\int_{\|\nu_{\varepsilon}(\varepsilon)\|}^{+\infty} \left| \Psi_{\nu_{\varepsilon}}^{(k)}(y) \right|^2 dy \leq C'^{-2}k!n^{-2}, \quad \forall k \in [0, 401 \log_+(n)].
\]
Then, if we set \( \tilde{\lambda}^{*} := \lambda_{\tilde{u}^{*}(\tilde{c}), \varepsilon} \), there exists a constant \( C > 0 \) that depends only on \( \tilde{c}, C' \) such that

\[
\min_{B \in \mathbb{R}, S^{2} \geq 0} \left\| (X_{t}(\varepsilon))^{\otimes n} - \mathcal{N}(Bt, tS^{2})^{\otimes n} \right\|_{TV} \leq 1 - e^{-\tilde{\lambda}^{*}nt} + Ce^{-\tilde{\lambda}^{*}nt} \sqrt{\frac{n\mu_{3}(\tilde{u}^{*}(\tilde{c}))}{t^{2}(\Sigma^{2} + \sigma^{2}(\tilde{u}^{*}(\tilde{c})))^{3}}} + \frac{n\mu_{4}(\tilde{u}^{*}(\tilde{c}))}{t(\Sigma^{2} + \sigma^{2}(\tilde{u}^{*}(\tilde{c})))^{3}} + \frac{C'}{n}. \tag{9}
\]

2.2 Lower bound

Theorems 1 and 2 are optimal in the following sense: If the upper bound of Theorem 1 or 2 is not small, then the total variation distance between the random vector associated with the increments of the small jumps - possibly convoluted with Gaussian distributions- and the approximating Gaussian random vector is bounded away from 0 - i.e. is larger than some absolute constant.

To establish such results, we construct an appropriate statistical test that investigates whether the Lévy process is pure noise, and we use the following fact.

**Lemma 1.** Let \( \mathbb{P} \) and \( \mathbb{Q} \) be two probability distributions and \( \Phi \) a test of level \( \alpha_{0} \in (0,1) \) that separates \( \mathbb{P} \) from \( \mathbb{Q} \) from \( n \) i.i.d. observation with power larger than \( 1 - \alpha_{1} \). Then, \( \|\mathbb{P}^{\otimes n} - \mathbb{Q}^{\otimes n}\|_{TV} \geq 1 - \alpha_{0} - \alpha_{1} \).

We will first establish a lower bound corresponding to Theorem 2 as the one for Theorem 1 will be an immediate corollary.

2.2.1 Approximation to the closest Gaussian distribution

Let \( X \sim (b, \Sigma^{2}, \nu) \), with \( b, \Sigma^{2} \) and \( \nu \) possibly unknown and consider the problem of testing whether the process contains jumps of size smaller than \( \varepsilon \) or not, i.e. whether \( \nu_{\varepsilon} = 0 \) or not. Recall that \( \nu_{\varepsilon} = \nu 1_{[-\varepsilon, \varepsilon]} \) and that we defined \( u^{*} \) for the largest positive real number such that \( \lambda_{u^{*}, \varepsilon} nt \geq \log_{+}(n) \). Write now \( u^{*} \) for the largest \( u \in [u^{+}, \varepsilon] \) such that

\[
u^{+} = \sup\{u, u \in [u^{+}, \varepsilon], u \leq \sqrt{t(\Sigma^{2} + \sigma^{2}(u)))\log_{+}(n)^{2}} \} \vee u^{+},
\]

where \( \sup \emptyset = -\infty \). The intuitions behind the quantities \( u^{+} \) and \( u^{*} \) are the following. The quantity \( u^{+} \) is chosen such that, with probability going to 1, there is (i) at least one jump larger than \( u^{+} \) but (ii) not too many of such jumps, i.e. less than \( 2\log_{+}(n) \), and finally (iii) at most one jump larger than \( u^{+} \) per time increment \( t \). Therefore, the discretized process of jumps larger than \( u^{+} \) and smaller than \( \varepsilon \) (in absolute value), does not look Gaussian at all. It is composed of many null entries and a few larger than \( u^{+} \). Now \( u^{+} \) is the largest quantity (larger than \( u^{+} \)) such that \( u^{*} \) is smaller than a number slightly larger than the standard deviation of the increment \( X_{it}(\varepsilon) - X_{(i-1)t}(\varepsilon) \) conditional to the event there are no jumps larger than \( u^{*} \) in \( (i-1)t, it] \). In other words, any increment having a jump larger than \( u^{*} \) is going to be quite visible.

**Theorem 3.** For any \( \varepsilon > 0 \), let \( X(\varepsilon) \sim (b, \Sigma^{2}, \nu_{\varepsilon}) \) with \( b \in \mathbb{R}, \Sigma^{2} \geq 0 \) and \( \nu_{\varepsilon} \) a Lévy measure with support in \([-\varepsilon, \varepsilon]\). For any \( n \geq 1 \) and \( t > 0 \) such that \( \lambda_{\nu, \varepsilon} \geq t^{-1} \), there exists an absolute constant \( C > 0 \) such that the following holds:

\[
\min_{B \in \mathbb{R}, S^{2} \geq 0} \left\| (X_{t}(\varepsilon))^{\otimes n} - \mathcal{N}(Bt, tS^{2})^{\otimes n} \right\|_{TV} \geq \left\{ 1 - C \left[ \frac{t(\Sigma^{2} + \sigma^{2}(u^{*}))^{3}}{n\mu_{3}(u^{*})^{2}} \wedge \frac{t^{2}(\Sigma^{2} + \sigma^{2}(u^{*}))^{4}}{n\mu_{4}(u^{*})^{2}} \right] - \frac{1}{\log n} \right\} \vee \left( 1 - \exp(-\lambda_{\nu^{*}, \varepsilon}nt) - \frac{1}{\log n} \right).
\]

The construction of the test we use to derive Theorem 3 is actually quite involved, we refer to Section 3.3.1 for more details. Here, we illustrate the main ideas. We build an event that occurs with
high probability if \( \nu_\epsilon = 0 \) and with small probability otherwise. This would then allow to bound the total variation distance between the two discretized processes using Lemma 1. The sketch of the proof is the following (all results mentioned below are stated and proved in Section 3 and Appendix A):

- First, we show that \( u^+ \) defined as above satisfies (i)-(ii)-(iii) with probability going to 1 and we bound the deviations of the difference of some of the increments \( X_{2i}(\epsilon) - X_{(2i-1)}(\epsilon) - (X_{(2i-1)})(\epsilon) - X_{2i-1}(\epsilon) \) (Lemma 5).

- Second, we build an estimator of the standard deviation of the increments of the Lévy process \((b, \Sigma^2, \nu_{u^+})\). In order to do so, we use a robust estimator of the mean which drops large increments, and thus the ones larger than \( u^+ \) (Lemma 6).

- From these preliminary steps, we prove that a test comparing the largest entry and the expected standard deviation if \( \nu_\epsilon = 0 \) detects if there is a jump larger than \( u^* \) in the sample (Proposition 3). In the following steps, we focus on tests conditional to the event there is no jump larger than \( u^* \) in the sample - otherwise they are eliminated by the latter test. Two cases remain to be studied.

- If the dominant quantity in Theorem 1 is \( t \mu_4(u^*) \): we first construct a test for detecting if \( t \mu_4(u^*) \) is larger than a constant times \( [t(\Sigma^2 + \sigma^2(u^*))]^3 \), to remove distributions that are too skewed (Proposition 4). Then, we build a test comparing the (estimated) third moment of the increments to the expected behavior if \( \nu_\epsilon = 0 \) (Proposition 5).

- If the dominant quantity in Theorem 1 is \( t \mu_4(u^*) \): we build a test comparing the (estimated) fourth moment of the increments to the expected behavior if \( \nu_\epsilon = 0 \) (Proposition 6).

**Tightness of the lower bound** The bounds on the total variation we establish in Theorems 2 and 3 are tight, up to a \( \log_4(n) \) factor, due to the differences in the definitions\(^2\) of \( \tilde{u}^* \) and \( u^* \), in the following sense. Whenever

\[
\rho_n(u^*) := (\lambda_{u^*,\epsilon} n t) \vee \frac{n \mu_4(u^*)}{t^2(\Sigma^2 + \sigma^2(u^*))^4} \vee \frac{n \mu_6(u^*)}{t(\Sigma^2 + \sigma^2(u^*))^3}
\]

does not converge to 0 with \( n \), the total variation distance does not converge to 0. If \( \rho_n(u^*) \) converges to \(+\infty\) with \( n \), the total variation converges to 1. If \( \rho_n(\tilde{u}^*) \) converges to 0 with \( n \), then the total variation converges to 0 by Theorem 2. Another implication of these bounds is that the Gaussian random variable closest to \( (X_t(\epsilon))_{t \in \mathbb{N}} \) is not necessarily \( \mathcal{N}(b(\epsilon)t, t(\Sigma^2 + \sigma^2(\epsilon)))_{t \in \mathbb{N}} \). When rare and large jumps are present, a tighter Gaussian approximation is provided by \( \mathcal{N}(b(\tilde{u}^*)t, t(\Sigma^2 + \sigma^2(\tilde{u}^*))_{t \in \mathbb{N}}) \).

**A jump detection test** Proof of Theorem 3 is based on the construction of a test of Gaussianity, adapted to Lévy processes, that detects whether the discrete observations are purely Gaussian, or whether they are realizations of a Lévy process with non trivial Lévy measure. More precisely, (see the proof of Theorem 3 for details) we build a uniformly consistent test for the testing problem

\[
H_0 : \nu_\epsilon = 0 \quad \text{against} \quad H_1 : \lambda_{0,\epsilon} = +\infty \quad \text{and} \quad E
\]

where

\[
E = \{ \mu_3(u^*)^2 \geq \frac{Ct(\Sigma^2 + \sigma^2(u^*))^3}{n} \text{ or } \mu_4(u^*)^2 \geq \frac{Ct^2(\Sigma^2 + \sigma^2(u^*))^4}{n} \text{ or a jump larger than } u^* \text{ occurs} \}.
\]

This test is of interest in itself: it does not rely on the knowledge of the Lévy triplet.

\(^1\)Of all increments when \( \nu_\epsilon = 0 \) and of those where a jump larger than \( u^* \) occurs otherwise.

\(^2\)Recall that \( u^* \) is the largest \( u \) larger than \( u^+ \) such that \( u \leq \sqrt{t(\Sigma^2 + \sigma^2(u))}\log_4(n) \), and \( \tilde{u}^* \) is the largest \( u \) larger than \( u^+ \) such that \( u \leq \tilde{c}\sqrt{t(\Sigma^2 + \sigma^2(u))}/\sqrt{\log_4(n)} \) where \( \tilde{c} \) is a constant.
2.2.2 Approximation with the corresponding Gaussian distribution

The following result, is a corollary of Theorem 4.

**Theorem 4.** For any $\varepsilon > 0$, let $X(\varepsilon) \sim (b, \Sigma^2, \nu_\varepsilon)$ with $b \in \mathbb{R}$, $\Sigma^2 \geq 0$ and $\nu_\varepsilon$ a Lévy measure with support in $[-\varepsilon, \varepsilon]$. For any $n \geq 1$ and $t > 0$ such that $\lambda_{0,\varepsilon} \geq t^{-1}$ and $\varepsilon \leq \sqrt{t(S^2 + \sigma^2)\log_t(n)^2}$, there exists an absolute constant $C > 0$ such that the following holds:

$$\| (X_t(\varepsilon))^\otimes n - \mathcal{N}(b(\varepsilon)t, t(S^2 + \sigma^2)) \|_{TV} \geq 1 - C \left[ \frac{t(S^2 + \sigma^2)^3}{n\mu_3(\varepsilon)^2} \wedge \frac{t^2(S^2 + \sigma^2)^4}{n\mu_4(\varepsilon)^2} \right] - \frac{1}{\log n}.$$  

Theorem 4 requires $\varepsilon \leq \sqrt{t(S^2 + \sigma^2(\varepsilon))\log_t(n)^2}$, i.e. that $\varepsilon$ is smaller (up to a multiplicative $\log_t(n)^2$ term) than the standard deviation of the increment. It implies that all moments of order $k$ of the increment can be bounded -up to a constant depending on $k$- by $(\sqrt{t(S^2 + \sigma^2(\varepsilon))\log_t(n)^2})^k$, which is helpful for bounding the deviations of the test statistics. This assumption does not appear in Theorem 3 thanks to the introduction of $u^*$ and considering two different types of tests in the construction of the lower bound: a test for the third and fourth moments and a test for extreme values. This latter test allows to detect -with very high probability- when a jump larger than $u^*$ occurred. The additional assumption $\varepsilon \leq \sqrt{t(S^2 + \sigma^2(\varepsilon))\log_t(n)^2}$ ensures that $u^* = \varepsilon$.

**Improvement of Theorem 4 for mixtures.** An immediate corollary of Theorem 4 (see its proof) is a lower bound on the total variation distance between any two mixture of Gaussian random variables and mixture of Lévy measures concentrated in $[-\varepsilon, \varepsilon]$. More precisely, let $d\Lambda(b, \Sigma^2, \nu_\varepsilon)$ and $d\Lambda'(b, \Sigma^2)$ be two priors on Lévy processes and linear Brownian motions, respectively. Assume that the support of $d\Lambda(b, \Sigma^2, \nu_\varepsilon)$ is included in a set $A$, and that for any $(b, \Sigma^2, \nu_\varepsilon) \in A$, we have $\varepsilon \leq \sqrt{\sigma^2(\varepsilon) + \Sigma^2}$. Then, it holds

$$\| \int (\mathcal{N}(bt, t\Sigma^2) \ast M_{t,\nu}(\varepsilon) ) \otimes n d\Lambda (b, \Sigma^2, \nu) - \int \mathcal{N}(bt, t(S^2 + \sigma^2(\varepsilon))) \otimes n d\Lambda (b, \Sigma^2) \|_{TV} \geq \min_{(b, \Sigma^2, \nu) \in A} \left[ 1 - C \left[ \frac{t(S^2 + \sigma^2(\varepsilon))^3}{n(\mu_3(\varepsilon))^2} \wedge \frac{t^2(S^2 + \sigma^2(\varepsilon))^4}{n(\mu_4(\varepsilon))^2} \right] - \frac{1}{\log n} \right],$$

where $M_{t,\nu}(\varepsilon), \sigma^2(\varepsilon), \mu_3(\varepsilon)$ and $\mu_4(\varepsilon)$ correspond to $M_{t}(\varepsilon), \sigma^2(\varepsilon), \mu_3(\varepsilon)$ and $\mu_4(\varepsilon)$ for the Lévy measure $\nu$. A related result can be achieved for Theorem 3. Note that the corresponding lower bound on the total variation distance is a direct corollary of Theorem 1. The lower bound displayed above is not trivial, it holds because the test that we construct in the proof of Theorem 3 does not depend on the parameters of the Gaussian random variable nor on the Lévy triplet.

2.3 Classes where Assumption $(\mathcal{H}(\Psi))$ is satisfied

Before displaying the results implied by Theorems 1 and 2 on the class of $\beta$-stable processes, we provide two contexts in which Assumptions $(\mathcal{H}(\Psi))$ and $(\mathcal{H}(\nu^*))$ are fulfilled.

**When $\Sigma$ is large enough.** The following proposition, whose proof can be found in Appendix A.4, proves that Assumption $(\mathcal{H}(\Psi))$ is satisfied, whenever $\Sigma$ is large enough - namely, $\sigma(\varepsilon) \lesssim \Sigma$. In this case we can apply directly Theorem 1 and Theorem 2, provided that we apply the previous proposition at $u^*$ instead of $\varepsilon$.

**Proposition 1.** Let $\varepsilon > 0$ and consider a Lévy measure $\nu_\varepsilon$ with support in $[-\varepsilon, \varepsilon]$. Assume that $\lambda_{0,\varepsilon} > 1/t$ and $\varepsilon \leq \sqrt{t(S^2 + \sigma^2(\varepsilon))}$ and for a constant $c_\Sigma > 0$ it holds that $c_\Sigma \Sigma \geq \sigma(\varepsilon)$. Then, there
exists a constant $4^{-1} > \bar{c} > 0$ small enough that depends only on $c_\Sigma$, such that
\[
\int_{\log_+ \{n\}}^{\infty} |\Psi_{\varepsilon}(y)|^2 dy \leq k! n^{-4}, \quad \forall k \in [0, 401 \log_+ \{n\}].
\]

When $\nu_{\varepsilon}$ is polynomially controlled at 0. The following result, whose proof can be found in Appendix A.5, implies that whenever $\nu_{\varepsilon}$ satisfies Assumption (10) below, Assumption $H(\Psi_{\varepsilon})$ is fulfilled. Assumption (10) describes a class of functions that contains any Lévy measure that behaves as the Lévy measure of a stable process in a neighborhood of the origin.

**Proposition 2.** Let $b \in \mathbb{R}$, $\Sigma \geq 0$, $t > 0$, $\varepsilon > 0$, $n \geq 1$ and let $\nu$ be a Lévy measure absolutely continuous with respect to the Lebesgue measure. Suppose that there exists two positive constants $c_+ > c_- > 0$ such that, $\forall x \in [-\varepsilon, \varepsilon] \setminus \{0\}$,
\[
\frac{c_-}{|x|^{\beta+1}} \leq \frac{dv(x)}{dx} \leq \frac{c_+}{|x|^{\beta+1}}, \quad \beta \in (0, 2).
\]
Assume that there exists $c_{\max} \geq 0$ such that $v_{\max}^{\nu} t \geq 1$ and $\log(\Sigma^2 + \sigma^2(\varepsilon))/\log_+ \{n\} \leq c_{\max}$. Then, there exists $4^{-1} > \bar{c} > 0$ small enough depending only on $\beta, c_+, c_-, c_{\max}$ such that if $\varepsilon \leq \bar{c} \sqrt{t(\Sigma^2 + \sigma^2(\varepsilon))}/\log_+ \{n\}$, then it holds that
\[
\int_{\log_+ \{n\}}^{\infty} |\Psi_{\varepsilon}(y)|^2 dy \leq 3k! n^{-4}, \quad \forall k \in [0, 401 \log_+ \{n\}].
\]

**Remark 2.** Whenever there exists a constant $\kappa > 0$ that depends only on $\beta, c_+, c_-$ such that $n^\kappa t \geq 1$, and $(\Sigma^2 + \varepsilon^2 - \beta)n^{-\kappa} \leq 1$, then $c_{\max}$ is an absolute constant and the dependence on $c_{\max}$ in Proposition 2 is not constraining. Moreover, the condition on $\varepsilon$ is the same condition as in Theorems 1 and 2. As $\Sigma^2 + \sigma^2(\varepsilon)$ is of order $(\Sigma^2 + \varepsilon^2 - \beta)$, even in the most constraining case $\Sigma = 0$, $\bar{c}$ can be chosen small enough provided that $\varepsilon^0 \leq \bar{c}_t/\log_+ \{n\}$, for $\bar{c}_t$ chosen small enough (depending on $\bar{c}, \beta, c_+, c_-, c_{\max}$).

### 2.4 Example: Stable processes

We first state the following general result which is a consequence of Proposition 2 and Theorem 1.

**Theorem 5.** Let $b \in \mathbb{R}$, $\Sigma \geq 0$, $t > 0$, $\varepsilon > 0$, $n \geq 1$ and let $\nu$ be a Lévy measure absolutely continuous with respect to the Lebesgue measure such that there exists two positive constants $c_+ > c_- > 0$ for which $\forall x \in [-\varepsilon, \varepsilon] \setminus \{0\}$,
\[
\frac{c_-}{|x|^{\beta+1}} \leq \frac{dv(x)}{dx} \leq \frac{c_+}{|x|^{\beta+1}}, \quad \beta \in (0, 2).
\]
Assume there exists $\kappa > 0$ depending only on $\beta, c_+, c_-$ such that $n^\kappa t \geq 1$.

1. **If $(\Sigma^2 + \varepsilon^2 - \beta)n^{-\kappa} \leq 1$, there exist two constants $C > 0, \bar{c} > 0$ that depend only on $\beta, c_+, c_-, \kappa$ such that**
\[
\min_{B \in \mathbb{R}, S^2 \geq 0} \| (X_t(\varepsilon))^{\otimes n} - \mathcal{N}(Bt, tS^2)^{\otimes n} \|_{TV} \leq 1 - e^{-\lambda_{\bar{c}, t^{\kappa}}} nt
\]
\[
+ Ce^{-\lambda_{\bar{c}, t^{\kappa}}} nt \left( \frac{n^2 \mu_2(\bar{u}^*(\bar{c}))}{2(\Sigma^2 + \sigma^2(\bar{c})))^4} + \frac{n^3 \mu_3(\bar{u}^*(\bar{c}))}{t(\Sigma^2 + \sigma^2(\bar{c}))^3} \right)^{\kappa} + C/n.
\]

2. **If $(\Sigma^2 + \varepsilon^2 - \beta)n^{-\kappa} \leq 1$ and $\varepsilon \leq \bar{c} \sqrt{\Sigma^2 + \sigma^2(\varepsilon)}/\log_+ \{n\}$, it holds for some constant $C > 0$, depending on $\beta, c_+, c_-, \kappa$,
\[
\| (X_t(\varepsilon))^{\otimes n} - \mathcal{N}(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n} \|_{TV} \leq C \sqrt{\frac{n^2 \mu_2(\varepsilon)}{t^2(\Sigma^2 + \sigma^2(\varepsilon))^4}} + \frac{n^3 \mu_3(\varepsilon)}{t(\Sigma^2 + \sigma^2(\varepsilon))^3},
\]
We illustrate the implications of Theorem 5 on the class of infinite stable processes. It is possible to extend the results valid for this example to other types of Lévy processes (e.g., inverse Gaussian processes, tempered stable distributions, etc.) as, around 0, stable measures well approximate many Lévy measures. Let \( \beta \in (0,2) \), \( c_+, c_- \geq 0 \), \( (c_+ = c_- = 1) \) and assume that the Lévy measure \( \nu \) has a density with respect to the Lebesgue measure of the form

\[
\nu(dx) = \frac{c_+}{x^{1+\beta}} \mathbf{1}_{(0,+\infty)}(x) dx + \frac{c_-}{|x|^{1+\beta}} \mathbf{1}_{(-\infty,0)}(x) dx, \quad \forall x \in [-\varepsilon, \varepsilon] \setminus \{0\},
\]

which satisfies Equation (10). Let \( M(\varepsilon) \) be a Lévy process with Lévy triplet \( (-\int_{|x| \leq \varepsilon} x \nu_c(dx), 0, \nu_c) \) where \( \nu_c := \nu|_{|x| \leq \varepsilon} \) and \( b > 0 \), \( \Sigma^2 \geq 0 \), \( t > 0 \), \( \varepsilon > 0 \), \( n \geq 1 \).

In the sequel, we use the symbols \( \approx \), \( \leq \), and \( o(1) \) defined as follows. For \( a, b \in \mathbb{R} \), \( a \approx b \) if there exists \( c > 0 \) depending only on \( \beta, c_+, c_- \) such that \( a = cb \) and \( a \leq b \) if there exists \( c > 0 \) depending only on \( \beta, c_+, c_- \) such that \( a \leq cb \). For a sequence \( (a_n) \) in \( \mathbb{R}^+ \), we have that \( a_n = o(1) \) if \( \lim_{n \to \infty} a_n = 0 \).

We are interested in the question: “Given \( n \) and \( t \), what is the largest (up to a constant) \( \varepsilon^* \geq 0 \) such that it is not possible to distinguish between \( n \) independent realizations of \( \mathcal{N}(b(\varepsilon^*)t, t\Sigma^2) \ast M_t(\varepsilon^*) \) and the closest i.i.d. Gaussian vector?” The answer to this question is provided by Theorem 5. The following two Tables summarize these findings and give the order of magnitude of \( \varepsilon^* \) such that \( i \) if \( \varepsilon/\varepsilon^* = o(1) \), then

\[
\inf_{B \in \mathbb{R}, S \geq 0} \left\| \mathcal{N}(b(\varepsilon^*)t, t\Sigma^2) \ast M_t(\varepsilon^*) \right\|_{TV} \to 0,
\]

and \( ii \) else if \( \varepsilon^*/\varepsilon = o(1) \), then

\[
\inf_{B \in \mathbb{R}, S \geq 0} \left\| \mathcal{N}(b(\varepsilon^*)t, t\Sigma^2) \ast M_t(\varepsilon^*) \right\|_{TV} \to 1.
\]

In all cases we require, additionally to \( \nu \) being the Lévy measure of a \( \beta \)-stable process, that there exists a constant \( \kappa > 0 \) that depends only on \( \beta, c_+, c_- \) such that \( n^\kappa t \geq 1 \), and \( (\Sigma^2 + \varepsilon^{2-\beta})n^{-\kappa} \leq 1 \).

| \( \Sigma^2 \geq (\frac{t}{n})^{\frac{\beta}{\beta-1}} \) | \( \varepsilon^* \approx (\frac{\Sigma^2}{\kappa})^{\frac{1}{\beta-1}} \) | \( \Sigma^2 \geq (\frac{t}{n})^{\frac{\beta}{\beta-1}} \) | \( \varepsilon^* \approx (\frac{\Sigma^2}{\kappa})^{\frac{1}{\beta-1}} \) |
| \( \Sigma^2 \leq (\frac{t}{n})^{\frac{\beta}{\beta-1}} \) | \( \varepsilon^* \approx (\frac{\Sigma^2}{\kappa})^{\frac{1}{\beta-1}} \) | \( \Sigma^2 \leq (\frac{t}{n})^{\frac{\beta}{\beta-1}} \) | \( \varepsilon^* \approx (\frac{1}{n})^{\frac{1}{\beta}} \) |

As expected, for a given value of \( \beta \), the value of \( \varepsilon^* \) is larger when \( \Sigma^2 \) is large, it is also larger for symmetric Lévy densities. Unsurprisingly, in absence of a dominating Gaussian noise, the value of \( \varepsilon^* \) is an increasing function of \( \beta \). In presence of a Gaussian noise the value of \( \varepsilon^* \) decreases with \( \beta \), then only the fourth or third cumulant intervene in the upper bound for the total variation. These cumulants are increasing functions of \( \beta \): the smaller \( \beta \) is, the smaller the upper bound is and the larger \( \varepsilon^* \) gets.

3 Proofs

Several times a compound Poisson approximation for the small jumps of Lévy processes will be used in the proofs, see e.g. the proofs of Theorems 2 and 3. More precisely, for any \( 0 < \eta < \varepsilon \), we will denote by \( M(\eta, \varepsilon) \) the centered compound Poisson process that approximates \( M(\varepsilon) \) as \( \eta \downarrow 0 \), i.e.

\[
M_t(\eta, \varepsilon) := \sum_{s \leq t} \Delta X_s \mathbf{1}_{\eta < |\Delta X_s| \leq \varepsilon} - t \int_{\eta < |x| \leq \varepsilon} x \nu(dx) = \sum_{i=1}^{N_t(\eta, \varepsilon)} Y_i - t \int_{\eta < |x| \leq \varepsilon} x \nu(dx),
\]
where $N(\eta, \varepsilon)$ is a Poisson process with intensity $\lambda_{\eta, \varepsilon} := \int_{|x| \leq \varepsilon} \nu(dx)$ and the $(Y_i)_{i \geq 1}$ are i.i.d. random variables with jump measure

$$\Pr(Y_1 \in B) = \frac{1}{\lambda_{\eta, \varepsilon}} \int_{B \cap \{\eta < |x| \leq \varepsilon\}} \nu(dx), \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

(12)

Then, it is well known (see e.g. [19]) that $M(\eta, \varepsilon)$ converges to $M(\varepsilon)$ almost surely and in $L_2$, as $\eta \to 0$.

3.1 Proof of Theorem 1

3.1.1 Assumptions and notations

We begin by introducing some notations and by reformulating the assumptions of Theorem 1. For a real function $g$ and an interval $I$, we write $g_{I} := g1_{(I)}$. Given a density $\mu$ with respect to a probability measure $\nu$, and a measurable set $A$, we denote by $\Pr(g(A)) = \int_A g(x)\mu(dx)$. Also, we denote by $s^2 := \Sigma^2 + \sigma^2(\varepsilon)$. In what follows, we write $\mu$ for a measure that is the sum of the Lebesgue measure and (countably) many Dirac masses, which dominates the measure associated to $\tilde{X}_t(\varepsilon) = (X_t(\varepsilon) - b(\varepsilon)t)/\sqrt{t(\sigma^2(\varepsilon) + \Sigma^2)}$. Moreover, $f$ will indicate the density, with respect to the measure $\mu$, of the rescaled increment $\tilde{X}_t(\varepsilon)$ and $\varphi$ will be the density, with respect to the measure $\mu$, of a centered Gaussian random variable with unit variance. Whenever we write an integral involving $f$ or $\varphi$ in the sequel, it is with respect to $\mu$ (or the corresponding product measure).

Recall that

$$\Psi(y) := \Psi_\varepsilon(y) = \mathbb{E}[e^{iy\tilde{X}_t(\varepsilon)}] - e^{(-\lambda_0,\varepsilon)nt}1_{\{\varepsilon = 0\}}$$

$$= \exp\left(-\frac{\Sigma^2 y^2}{2} + t\int_{\mathbb{R}} \left(\exp\left(iyu\frac{\sigma}{s\sqrt{t}}\right) - \frac{iuy}{s\sqrt{t}} - 1\right)d\nu(u)\right) - e^{(-\lambda_0,\varepsilon)nt}1_{\{\varepsilon = 0\}}.$$ 

We establish the result under the following assumptions which are implied by the assumptions of Theorem 1. Let $I$ be an integration interval of the form $I := [-c_{sup}\sqrt{\log(n)}, c_{sup}\sqrt{\log(n)}]$, with $c_{sup} \geq 2$ and let us assume here that $n \geq 3$ - but note that the bound on the total variation distance for $n = 3$ is also a bound on the total variation distance for $n = 1$ or $n = 2$.

- Set $K := c_{int}^2\log(n)$, where $c_{int} > 2c_{sup}$. Then, for some constant $c > 1$ it holds that

$$\int_{c\log(n)}^{+\infty} |\Psi^{(k)}|^2 \leq C'k^2k!n^{-2}, \quad \forall \ 0 \leq k \leq K,$$

where $C'$ is a universal constant.

- For some constant $0 < c_p < 1/8$, it holds

$$\Pr(f(I^c)) \leq c_p/n.$$  \hspace{1cm} \text{(H$_0$)}

- For some small enough universal constant $0 < \bar{c} \leq 1$, such that $\bar{c}c \leq \sqrt{\log n}/4$, it holds

$$\varepsilon \leq \bar{c}\sqrt{(\sigma(\varepsilon))^2 + \Sigma^2)/\log(n)} := \tilde{c}s\sqrt{t}/\sqrt{\log(n)} := \tilde{c}_n s\sqrt{t}.$$  \hspace{1cm} \text{(H$_{c\varepsilon}$)}

Note that this assumption is weaker than the Assumption $\mathcal{H}_\varepsilon$ on $\varepsilon$ from Theorem 1. Note also that this assumption permits to simply derive (H$_0$) from the following lemma.

Lemma 2. For $\varepsilon > 0$, $t > 0$ and $n \geq 3$, let $\nu_\varepsilon$ be a (possibly infinite) Lévy measure such that $\lambda_{\varepsilon, \varepsilon} \geq 24\log(n)/t$. Then, whenever $c_{sup} \geq 10$, $\tilde{c}_n \leq 1$ and (H$_{c\varepsilon}$) holds, we get $\Pr(f(I^c)) \leq 3/n^3$. 

14
Lemma 2 implies that under the assumptions of Theorem 1, \((\mathcal{H}_0)\) is satisfied with \(c_p = 3/n^2\).

**Remark 3.** For the proof of Theorem 1, Assumption \((\mathcal{H}_c)\) can be weakened in \(\varepsilon \leq \tilde{c}\sqrt{(\sigma^2 + \Sigma^2)t}\), the extra log is used to establish Lemma 2 related to \((\mathcal{H}_0)\).

- For some constant \(0 < c_m \leq 1/2\), it holds

\[
M := \tilde{c}_m^{-4}\left(\frac{|\mu_3(\varepsilon)|}{\sqrt{t}s^3} + \frac{\mu_4(\varepsilon)}{ts^4}\right) \leq \frac{c_m}{\sqrt{n}}.
\]

**Remark 4.** Assumption \((\mathcal{H}_M)\) will be used in the proof of Theorem 1, it is not limiting as if \((\mathcal{H}_M)\) is not satisfied, the upper bound of Theorem 1 is not small and is therefore irrelevant.

In the sequel, \(C\) stands for a universal constant, whose value may change from line to line.

### 3.1.2 Proof of Theorem 1.

To ease the reading of the proof of Theorem 1, we detail the case where there is a non-zero Gaussian component on \(X(\varepsilon)\) and/or the Lévy measure of \(X(\varepsilon)\) is infinite. The case of compound Poisson processes \((\lambda_{0,\varepsilon} < \infty)\) can be treated similarly, considering separately the sets \(A_n = \{\forall i \leq n, N_i(0, \varepsilon) - N_{i-1}(0, \varepsilon) \geq 1\}\) and its complementary, where \(N(0, \varepsilon)\) is the Poisson process with intensity \(\lambda_{0,\varepsilon}\) associated to the jumps of \(M(\varepsilon)\), see (11). The reason being that on the set \(A_n\) the distribution of the process is absolutely continuous with respect to the Lebesgue measure, and on its complementary it is not. On the set \(A_n\) the techniques employed below can be adapted, and on the complementary set \(A_n^c\), the total variation can be trivially controlled using \(\lambda_{0,\varepsilon} \geq 24\log(n)/t\).

First, by means of a change of variable we get

\[
\|(X_t(\varepsilon))^{\otimes n} - N(\varepsilon t, t(\Sigma^2 + \sigma^2(\varepsilon)))^{\otimes n}\|_{TV} = \|f^{\otimes n} - \varphi^{\otimes n}\|_{TV}.
\]

To bound the total variation distance we consider separately the interval \(\mathcal{I}\) and its complementary (recall that the integrals are with respect to \(\mu^{\otimes n}\)):

\[
\|f^{\otimes n} - \varphi^{\otimes n}\|_{TV} = \|f_\mathcal{I}^{\otimes n} - \varphi_\mathcal{I}^{\otimes n}\|_{TV} + \frac{1}{2} \int_{(\mathcal{I}^c)^c} |f^{\otimes n} - \varphi^{\otimes n}|.
\]

Under \((\mathcal{H}_0)\) and using that \(P_{\varphi^{\otimes n}}(\mathcal{I}^c)^c \leq n(P_{\varphi}(\mathcal{I}^c)) \leq n \exp(-c_{sup}^2 \log(n)/2) \leq 1/n\) for \(c_{sup} \geq 2\), the second term in (13) is bounded by \(\frac{1}{2}(c_p + \frac{1}{n})\). Let us now focus on the first term. Introduce a positive function \(h > 0\) such that

\[
\int h_\mathcal{I} < +\infty, \quad \int \frac{f_\mathcal{I}^2}{h_\mathcal{I}} < +\infty, \quad \int \frac{\varphi_\mathcal{I}^2}{h_\mathcal{I}} < +\infty.
\]

By the Cauchy-Schwarz inequality, we get

\[
\frac{2\|f_\mathcal{I}^{\otimes n} - \varphi_\mathcal{I}^{\otimes n}\|^2}{\|f_\mathcal{I}\|^n} \leq \int \frac{(f_\mathcal{I}^{\otimes n} - \varphi_\mathcal{I}^{\otimes n})^2}{h_\mathcal{I}^{2n}} = \int \frac{(f_\mathcal{I}^{\otimes n})^2}{h_\mathcal{I}^{2n}} + \frac{2f_\mathcal{I}^{\otimes n} \varphi_\mathcal{I}^{\otimes n} + (\varphi_\mathcal{I}^{\otimes n})^2}{h_\mathcal{I}^{2n}}
\]

\[
= \left(\int \frac{(f_\mathcal{I} - \varphi_\mathcal{I})^2}{h_\mathcal{I}} + \frac{2\varphi_\mathcal{I}(f_\mathcal{I} - \varphi_\mathcal{I}) + \varphi_\mathcal{I}^2}{h_\mathcal{I}}\right)^n - 2\left(\int \frac{(f_\mathcal{I} - \varphi_\mathcal{I})\varphi_\mathcal{I} + \varphi_\mathcal{I}^2}{h_\mathcal{I}}\right)^n + \left(\int \frac{\varphi_\mathcal{I}^2}{h_\mathcal{I}}\right)^n.
\]

For \(K = c_{int}^2 \log(n)\), define

\[
h_\mathcal{I}^{-1}(x) = \sqrt{2\pi} 1(x) \sum_{k \leq K} \frac{x^{2k}}{2^k k!}.
\]
and consider the quantities
\[ A^2 := \int \frac{\varphi_T^2}{h_T}, \quad D^2 := \int \frac{(f_T - \varphi_T)^2}{h_T} \quad \text{and} \quad E := \int \frac{\varphi_T}{h_T}(f_T - \varphi_T). \]

It holds:
\[
2\left\| (\int h_T) - \varphi_T^n \right\|_{TV}^2 \leq \left[ D^2 + 2E + A^2 \right] - 2\left[ E + A^2 \right] + A^2n
\]
\[
\leq \sum_{2 \leq k \leq n} \left( \frac{n}{k} \right) \left[ D^2 + 2E \right] A^{2(n-k)} + nD^2A^{2(n-1)} - 2 \sum_{2 \leq k \leq n} \left( \frac{n}{k} \right) E^k A^{2(n-k)}
\]
\[
\leq \sum_{2 \leq k \leq n} \left( \frac{n}{k} \right) 2kD2kA^{2(n-k)} + \sum_{2 \leq k \leq n} \left( \frac{n}{k} \right) 2k[2|E|^k] A^{2(n-k)} + nD^2A^{2(n-1)} + 2 \sum_{2 \leq k \leq n} \left( \frac{n}{k} \right) |E|^k A^{2(n-k)}.
\]

We bound this last term by means of the following Lemma (the proof is postponed in Appendix A.1.2).

**Lemma 3.** Assume \((H_0)\) and suppose that \(c_{int} \geq 2c_{sup} \lor 1\). Then, there exists a universal constant \(c_h > 0\) such that
\[
0 \leq A^2 \leq 1 + c_h/n^2 \quad \text{and} \quad \int h_T \leq 1 + c_h/n^2 \quad \text{and} \quad |E| \leq c_p/n + n^{-c_{sup}/2} + 2c_h/n^2,
\]
where the constant \(c_p\) is defined in \((H_0)\).

Using Lemma 3 and the fact that \(\left( \frac{n}{k} \right) \leq n^k\), from (14) we derive
\[
2\left\| (\int h_T) - \varphi_T^n \right\|_{TV}^2 \leq \exp(c_h) \left[ \sum_{2 \leq k \leq n} 2k(nD^2)^k + \sum_{2 \leq k \leq n} 2k[2|E|^k] + nD^2 + 2 \sum_{2 \leq k \leq n} (n|E|)^k \right]
\]
\[
\leq nD^2 \exp(c_h) \left[ \sum_{2 \leq k \leq n} 2k(nD^2)^{k-1} + 1 \right] + 3 \exp(c_h) \sum_{2 \leq k \leq n} 2k[2|n|E|^k].
\]

Moreover, thanks to Lemma 3, if \(c_{sup} \geq 2\) and \(c_p < \frac{1}{8}\), it holds
\[
\sum_{2 \leq k \leq n} 2k[2|n|E]^k \leq \sum_{2 \leq k \leq n} 4^k(c_p + \frac{2c_h + 1}{n}) \leq \left( 4(c_p + \frac{2c_h + 1}{n}) \right)^2 \frac{1}{1 - 4(c_p + \frac{2c_h + 1}{n})} \leq Cc_p^2,
\]
where \(C\) is a universal constant.

To complete the proof, we are only left to control the order of \(D^2\). Indeed, applying Lemma 3 to bound \(\int h_T\) we derive, for \(c_{sup} \geq 2\) and \(c_p < \frac{1}{8}\),
\[
2\left\| (\int h_T) - \varphi_T^n \right\|_{TV}^2 \leq nD^2 \exp(c_h) \left[ \sum_{2 \leq k \leq n} 2k(nD^2)^{k-1} + 1 \right] + C \exp(c_h)c_p^2.
\]

To control the order of \(nD^2\) in (16), introduce \(G(x) = f(x) - \varphi(x)\) and notice that
\[
D^2 = \int \frac{(f_T - \varphi_T)^2}{h_T} = \int \frac{G^2}{h}.
\]

16
Denote by $P_k(x) = x^k$; the Plancherel formula leads to

\[ D^2 = \sqrt{2\pi} \int \left| \mathbb{1}_J G'(x) \right|^2 \frac{x^{2k}}{2^{k!} k!} dx \leq \sqrt{2\pi} \sum_{k \leq K} \frac{1}{2^{k!} k!} \|P_k G\|_2^2 = \frac{1}{\sqrt{2\pi}} \sum_{k \leq K} \frac{1}{2^{k!} k!} \|\hat{G}^{(k)}\|_2^2, \]  

(17)

using that $\int P_k G = i^k \hat{G}^{(k)}/\sqrt{2\pi}$. Moreover, observe that the Fourier transform of $G$ can be written as

\[ \hat{G}(y) = \exp \left( -\frac{y^2}{2} - i\frac{\mu_3(\varepsilon)}{6\sqrt{t}s^3} y^3 + \sum_{m \geq 4} t \mu_m(\varepsilon) \frac{(yi)^m}{(ts^2)^{m/2} m!} \right) - \exp(-\lambda_0 c t \varepsilon) \mathbb{1}_{\{\Sigma = 0\}} - \exp(-y^2/2) \]

\[ = \exp(-y^2/2) \left[ \exp \left( -i\frac{y^3 \mu_3(\varepsilon)}{6\sqrt{t}s^3} y + \sum_{m \geq 4} t \mu_m(\varepsilon) \frac{(yi)^m}{(ts^2)^{m/2} m!} \right) - 1 \right] - \exp(-\lambda_0 c t \varepsilon) \mathbb{1}_{\{\Sigma = 0\}}, \]

(18)

Assumption $(H_e)$, i.e. $\varepsilon \leq \tilde{c}_n s \sqrt{t}$, implies that $|\mu_m(\varepsilon)| \leq (\tilde{c}_n)^{m-4} \mu_4(\varepsilon) s^{m-4} t^{2m/2}$ for any $m > 4$. Therefore, $\hat{G}(y) = \exp(-y^2/2) \left[ \exp \left( -i\frac{y^3 \mu_3(\varepsilon)}{6\sqrt{t}s^3} y + \sum_{m \geq 4} a_m(\varepsilon) \frac{(yi)^m}{m!} \right) - 1 \right] - \exp(-\lambda_0 c t \varepsilon) \mathbb{1}_{\{\Sigma = 0\}}$.

Then, for any $c > 0$

\[ \int_{c \log(n)}^{-c \log(n)} |\hat{G}(y)|^2 dy \leq \int_{0}^{c \log(n)} \exp(-y^2) \left[ \exp \left( M c \log(n) \right) - 1 \right]^2 dy + 4c \log(n) \exp(-2\lambda_0 c t \varepsilon) \mathbb{1}_{\{\Sigma = 0\}}. \]

Assumption $(H_M)$ ensures that for a small enough universal constant $c_m \geq 0$ we have $M \leq c_m / \sqrt{n}$. In this case, we use a Taylor expansion on $[0, c \log(n)]$ and get if $c_{\tilde{c}_n} \leq 1/2$, and since $\lambda_0 c \geq 24 \log(n)/t$

\[ \int_{-c \log(n)}^{c \log(n)} |\hat{G}(y)|^2 dy \leq C' \int_{0}^{c \log(n)} \exp(-y^2) \left[ M e^{c_{\tilde{c}_n}} \right]^2 dy + 4c/n^2 \leq C'' M^2 + 2/n^2, \]

where $C', C''$ are two universal constants. This together with $(H(\Psi_e))$ permits to bound $\|\hat{G}\|_2^2$ by $M^2$. The $k$th derivatives of $\hat{G}$ are treated similarly, see Lemma 4 below, though the procedure is more cumbersome.

**Lemma 4.** Suppose $(H_e)$ with $\tilde{c}_n \leq 1$, $\tilde{c}_n c \leq 1/4$ and $(H_M)$ with $c_m \leq 1/2$. There exists a constant $C_{c_{\log(n)}}$ that depends on $c_{\log(n)}$ only, such that we have for any $t \in \mathbb{I}$,

\[ |\hat{G}^{(k)}(y)|^2 \leq C_{c_{\log(n)}} k^2 M^2 \sup_{d \leq k-2} \left( 2^{-8(k-d)} \frac{k!}{d!} (k-d)^k \right) \]

\[ \times k^4 (\tilde{c}_n M)^2 e^{2\tilde{c}_n |y|} \|H_{k-1}(y)\phi(y)\|^2 \vee |H_k(y)\hat{G}(y)|^2, \]

where $H_k$ is the Hermite polynomial of degree $k$ and $\phi(y) = e^{-y^2/2}$. Also, there exist two constants $\overline{C}, \varepsilon > 0$ such that

\[ \int_{-c \log(n)}^{c \log(n)} \sup_{d \leq k-2} \left( 2^{-8(k-d)} \frac{k!}{d!} (k-d)^k \right) dy \leq 2k! \left( \overline{C} 2^{k(1-\varepsilon/4)} \vee 1 \right), \quad \forall k \leq K. \]  

(20)
Finally,
\[ \int_{-c \log(n)}^{c \log(n)} \exp(-y^2) e^{2|y|\tilde{c}_n} |H_k(y)|^2 \, dy \leq \frac{4e^{\tilde{c}_n^2}}{\sqrt{2\pi}} k(k!) (1 + \tilde{c}_n^2)^k, \quad \forall k \leq K. \]  
(21)

To complete the proof, we bound the terms appearing in (19), 1 \leq k \leq K, by means of (20) and (21). This leads to a bound on \( D^2 \) using (17), and so to a bound on the total variation thanks to (16) and (13). To that aim, we begin by observing that, for \( |y| \leq c \log(n) \),
\[ |H_k(y)\hat{G}(y)| \leq C |H_k(y)| \exp(-y^2/2) Me^{\tilde{c}_n |y|}, \]
thanks to (18). Equation (21) implies that
\[ \int_{|y| \leq c \log n} \left( k^4(c_n M)^2 e^{2\tilde{c}_n |y|} |H_{k-1}(y)|^2 \exp(-y^2) \right) \forall |H_k(y)\hat{G}(y)|^2 \, dy \leq CM^2 k^5 (k!) (1 + \tilde{c}_n^2)^k, \]
where \( C \) is an absolute constant. Combining (19), (20) and (22) we derive:
\[ \int_{-c \log(n)}^{c \log(n)} |\hat{G}(k)(y)|^2 \, dy \leq C'_{c_{int}} k^5 k! M^2 2^{k - \tilde{c}}, \quad \forall k \leq K, \]
where \( \tilde{c} > 0 \) is a universal constant strictly positive and \( C'_{c_{int}} > 0 \) depends only on \( C_{int} \). Therefore, from (17) joined with Assumption \( (\mathcal{H}(\Psi,)) \), we deduce that
\[ D^2 \leq M^2 C'_{c_{int}} \sqrt{2\pi} \sum_{k \leq K} k^5 2^{k - \tilde{c}} \leq C''_{c_{int}} M^2. \]

In particular, recalling the definitions of \( M \) and \( s \) and using (16), \( (\mathcal{H}_M) \) and Lemma 2 (which ensures that under \( (\mathcal{H}_\varepsilon) \), Assumption \( (\mathcal{H}_0) \) holds with \( c_p = 1/n^2 \)) we finally obtain that
\[ \| f^\otimes n - \varphi^\otimes n \|_{TV} \leq \sqrt{n D^2 \exp(c_h) \sum_{2 \leq k \leq n} 2^k (nD^2)^{k-1} + 1} + C \exp(c_h) c_p^2 + c_p + \frac{3 + 2c}{n} \]
\[ \leq C \sqrt{n} \left( \frac{\mu_3(\varepsilon)}{\sqrt{t(\Sigma^2 + \sigma^2(\varepsilon))^3}} + \frac{\mu_4(\varepsilon)}{t(\Sigma^2 + \sigma^2(\varepsilon))^2} \right) + \frac{4(c + 1)}{n}, \]
where \( C \) depends on \( \tilde{c}_n, c_m, c_{sup} \) and \( c_{int} \). The proof of Theorem 1 is now complete.

### 3.2 Proof of Theorem 2

Consider first the case \( \bar{u}^* = u^+ \). In this case the theorem naturally holds for any \( C \geq 2 \) as \( 1 - e^{-\lambda_{u^+,nt}} = 1 - 1/n \). Assume from now on that \( \bar{u}^* > u^+ \). Note that this implies that \( \bar{u}^* \leq \tilde{c} \sqrt{t(\Sigma^2 + \sigma^2(\bar{u}^+))}/\sqrt{\log_+(n)} \). In this case note that the assumptions of Theorem 1 are satisfied for \( \bar{u}^* \).

Fix \( 0 < u \leq \varepsilon \) and write \( M(\varepsilon) = M(u) + M(u, \varepsilon) \) where \( M(u, \varepsilon) \) is a compound Poisson process with intensity \( \lambda_{u, \varepsilon} \) independent of \( M(u) \), see (11). Decomposing on the values of the Poisson process \( N(u, \varepsilon) \) at time \( nt \), we have
\[ \| (N(b(\varepsilon)t, t\Sigma^2) * M_t(\varepsilon))^{\otimes n} - N(b(\varepsilon)t, t(\Sigma^2 + \sigma^2(\bar{u}^+)))^{\otimes n} \|_{TV} \]
\[ \leq e^{-\lambda u^*, nt} \| (N(b(\varepsilon)t, t\Sigma^2) * M_t(\varepsilon))^{\otimes n} - N(b(\varepsilon)t, (\Sigma^2 + \sigma^2(\bar{u}^+)))^{\otimes n} \|_{TV} + 1 - e^{-\lambda u^*, nt} \]
\[ \leq C e^{-\lambda u^*, nt} \left( \frac{n \mu_2(\bar{u}^+)}{t^2(\Sigma^2 + \sigma^2(\bar{u}^+))^4} + \frac{n \mu_4(\bar{u}^+)}{t(\Sigma^2 + \sigma^2(\bar{u}^+))^2} + \frac{1}{n} \right) + 1 - e^{-\lambda u^*, nt}. \]

18
Indeed, to obtain the last inequality we adapt the result of Theorem 1, as from its proof the result holds regardless the drift and variance of both terms as long as they are equal. Finally, the result follows using
\[
\min_{B \in R, S \geq 0} \| (N(b(\varepsilon) t, t\Sigma^2) * M_t(\varepsilon))^{\otimes n} - N(B t, t\Sigma^2)^{\otimes n} \|_{TV} \leq \\
\| (N(b(\varepsilon) t, t\Sigma^2) * M_t(\varepsilon))^{\otimes n} - N(b(\varepsilon) t, t(\Sigma^2 + \sigma^2(\tilde{\mu}))^{\otimes n} \|_{TV}.
\]

### 3.3 Proof of Theorem 3

#### 3.3.1 Preliminary: Four statistical tests

Let \( X(\varepsilon) \sim (b, \Sigma^2, \nu_\varepsilon), \varepsilon > 0 \). In particular the increments \( X_{it}(\varepsilon) - X_{(i-1)t}(\varepsilon) \) are i.i.d. realizations of
\[
X_t(\varepsilon) = tb(\varepsilon) + \Sigma W_t + M_t(\varepsilon), \quad \text{where } W_t \sim N(0, t).
\]

For any \( n \in N \), set \( \bar{n} = \lfloor n/2 \rfloor \) and define
\[
Z_i(\varepsilon) := \frac{1}{\bar{n}} \sum_{i=1}^{\bar{n}} Z(\varepsilon), \quad \text{and} \quad Z(\varepsilon) = \max\{ Z_i(\varepsilon), 1 \leq i \leq \bar{n} \},
\]
where for any sequence \( a \), the sequence \( a_{(\cdot)} \) is a reordering of \( a \) by increasing order.

For any \( 0 < u \leq \varepsilon \), we write \( X(\varepsilon) \) as \( X(\varepsilon) = \overline{X}(u, \varepsilon) + M(u, \varepsilon) \), where
\[
\overline{X}(u, \varepsilon) = \frac{1}{n} \sum_{i=n/2+1}^{n} \left( X_{it}(\varepsilon) - X_{(i-1)t}(\varepsilon) \right),
\]
is a Lévy process with jumps of size smaller (or equal) than \( u \) and \( M(u, \varepsilon) = M_t(\varepsilon) - M_t(u) \) is a pure jump Lévy process with jumps of size between \( u \) and \( \varepsilon \). In accordance with the notation introduced in (11), we write \( N(u, \varepsilon) \) for the number of jumps larger than \( u \) and smaller than \( \varepsilon \), that is, for any \( t > 0 \), \( N_t(u, \varepsilon) \) is a Poisson random variable with mean \( t \lambda_{u, \varepsilon} \).

Furthermore, in order to present the test needed to prove Theorem 3, we introduce the following notations:
\[
\overline{X}_{t,n}(\varepsilon) := \frac{1}{n} \sum_{i=n/2+1}^{n} \left( X_{it}(\varepsilon) - X_{(i-1)t}(\varepsilon) \right),
\]
\[
\overline{Y}_{n,3}(\varepsilon) := \frac{1}{n/2} \sum_{i=1}^{n/2} \left( \left( X_{it}(\varepsilon) - X_{(i-1)t}(\varepsilon) \right) - \overline{X}_{t,n}(\varepsilon) \right)^2,
\]
\[
\overline{Y}_{n,2}(\varepsilon) := \frac{1}{n/4} \sum_{i=1}^{n/4} Z_i^2(\varepsilon), \quad \overline{Y}_{n,2}(\varepsilon) := \frac{1}{n/4} \sum_{i=n/4+1}^{n/2} Z_i^2(\varepsilon),
\]
\[
\overline{Y}_{n,4}(\varepsilon) := \frac{1}{n/2} \sum_{i=1}^{n/2} Z_i^4(\varepsilon), \quad \overline{Y}_{n,6}(\varepsilon) := \frac{1}{n/2} \sum_{i=1}^{n/2} Z_i^6(\varepsilon),
\]
\[
T_n^{(3)}(\varepsilon) := \frac{1}{1 - (n - \lfloor n/2 \rfloor - 2)} \overline{Y}_{n,3}(\varepsilon), \quad T_n^{(4)}(\varepsilon) := \frac{1}{4} \left( \overline{Y}_{n,4}(\varepsilon) - 3 \overline{Y}_{n,2}(\varepsilon) \overline{Y}_{n,2}(\varepsilon) \right).
\]

By definition, \( \overline{X}_{t,n}(\varepsilon) \) is the empirical version of \( E[X_t(\varepsilon)] \) computed on the second half of the sample only and \( \overline{Y}_{n,3}(\varepsilon) \) (resp. \( \overline{Y}_{n,6}(\varepsilon) \)) is an estimator of \( E[(X_t(\varepsilon) - tb(\varepsilon))^2] \) (resp. of \( 8E[(X_t(\varepsilon) - tb(\varepsilon))^6] \)) computed on the first half of the sample. Moreover, since \( E[(X_t(\varepsilon) - b(\varepsilon)t)^2] = t\mu_3(\varepsilon) \), using Corollary
Lemma 5. There exists a universal sequence $\alpha_n \to 0$ such that $P(\xi_n) \geq 1 - \alpha_n$.

Lemma 6. There exists a universal sequence $\alpha_n \to 0$ such that $P(\xi_n') \geq 1 - \alpha_n$.

Lemma 7. There exist a universal sequence $\alpha_n \to 0$ and a universal constant $C > 0$ such that the following holds.

Whenever $\nu_x = 0$, with probability larger than $1 - \alpha_n$ we have

$$C(\sqrt{t\Sigma^2})^6 \geq \overline{Y}_{n,6}(\varepsilon).$$

In any case, with probability larger than $1 - \alpha_n$ and conditional on $N_{nt}(u^+, \varepsilon) = 0$, it holds

$$\overline{Y}_{n,6}(\varepsilon) \geq \frac{t\mu_6(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^3}{2}.$$

Observe that Lemmas 5, 6 and 7 joined with Equation (28), imply the existence of two absolute sequences $\alpha_n \to 0$ and $\beta_n \to 0$ such that

$$P(\xi_n \cap \xi_n') \geq 1 - \alpha_n,$$

$$P(\xi_n \cap \xi_n' \cap \xi_n'') \geq 1 - \beta_n.$$ (28)

We are now ready to introduce the four tests we use to establish Theorem 3:

$$\Phi_n^{(\text{max})} = 1\{Z_n(\varepsilon) \geq \log(n)^{3/2}S_n(\varepsilon)\}, \quad \Phi_n^{(6)} = 1\{\overline{Y}_{n,6}(\varepsilon) \geq cS_n^6(\varepsilon)\},$$

$$\Phi_n^{(3)} = 1\{|T_n^{(3)}(\varepsilon)| \geq \frac{c}{\sqrt{\alpha}}\sqrt{S_n^2(\varepsilon)}\}, \quad \Phi_n^{(4)} = 1\{|T_n^{(4)}(\varepsilon)| \geq \frac{c}{\sqrt{\alpha}}S_n^4(\varepsilon)\sqrt{\frac{1}{n}}\}.$$ (29)

Their properties are investigated in Propositions 3, 4, 5 and 6 below. Finally recall that for any $\varepsilon > 0$, the null hypothesis that we consider is $H_0 : \nu_x = 0$. 

20
Proposition 3. Under $H_0$, for any $n > e^{4\sqrt{n}}$, it holds that $\xi_n \cap \xi'_n \subset \{\Phi_n^{(\text{max})} = 0\}$. Moreover, for any $n > e^2$, it holds $\xi_n \cap \xi'_n \cap \{N_{nt}(u^*, \varepsilon) \geq 1\} \subset \{N_{nt}(u^*, \varepsilon) \geq 1\} \cap \{\Phi_n^{(\text{max})} = 1\}$.

Proposition 4. There exist $c > 0$ a universal constant and $C_c$ depending only on $c$ such that the following holds, for $n$ large enough. Under $H_0$, it holds that $\xi'_n \cap \xi'_n \subset \{\Phi_n^{(6)} = 0\}$. Moreover if $t\mu_0(u^*) \geq C_c t^3(\Sigma^2 + \sigma^2(u^*))^3$, then $\xi'_n \cap \xi'_n \cap \{N_{nt}(u^*, \varepsilon) = 0\} \subset \{N_{nt}(u^*, \varepsilon) = 0\} \cap \{\Phi_n^{6} = 1\}$.

Proposition 5. Let $\alpha > 2\log(n)^{-1}$. Let $c > 0$ and $c' > 0$ be a large enough absolute constant and let $C_{c,c'} > 0$ be a large enough absolute constant depending only on $c$ and $c'$. Then, the following holds.

Under $H_0$, $\Phi_n^{(3)} = 0$ with probability larger than $1 - \alpha - P(\xi'_n)$. Under the hypothesis $H_0^{(3)}$, $\mu_3(u^*) > \rho_n^{(3)}$ and conditional to the event $N_{nt}(u^*) = 0$, if

$$u^* > u^+, \; t\mu_0(u^*) \leq c't^3(\Sigma^2 + \sigma^2(u^*))^3, \; \rho_n^{(3)} \geq C_{c,c'} \frac{t(\Sigma^2 + \sigma^2(u^*))^3}{\sqrt{n\alpha}},$$

it holds that $\Phi_n^{(3)} = 1$ with probability larger than $1 - \alpha - P(\xi'_n)$.

Proposition 6. Let $\alpha > 2\log(n)^{-1}$. Let $c > 0$ and $c' > 0$ be a large enough absolute constant and let $C_{c,c'} > 0$ be a large enough absolute constant depending only on $c$ and $c'$. Then, the following holds.

Under $H_0$, it holds that $\Phi_n^{(4)} = 0$ with probability larger than $1 - \alpha - P(\xi'_n)$. Under the hypothesis $H_0^{(4)}$, $\mu_4(u^*) > \rho_n^{(4)}$ and conditional to the event $N_{nt}(u^*, \varepsilon) = 0$, if

$$u^* > u^+, \; \rho_n^{(4)} \geq C_{c,c'} \frac{t(\Sigma^2 + \sigma^2(u^*))^2}{\sqrt{n\alpha}},$$

it holds that $\Phi_n^{(4)} = 1$ with probability larger than $1 - \alpha - P(\xi'_n)$.

3.3.2 Proof of Theorem 3

Let $(\bar{b}, \Sigma^2, \nu_\varepsilon)$ be a Lévy triplet where $\nu_\varepsilon$ is a Lévy measure with support in $[-\varepsilon, \varepsilon]$. Assume that we want to test

$$H_0 : \nu_\varepsilon = 0, \quad \text{against} \quad H_1 : (b, \Sigma^2, \nu_{\varepsilon}) = (\bar{b}, \Sigma^2, \nu_{\varepsilon}).$$

We write $\bar{\mu}, \bar{\lambda}$ and $\bar{u}^*$ for all the quantities related to $(\bar{b}, \Sigma^2, \nu_{\varepsilon})$.

We can choose $c^{(3)}, c^{(4)}, c^{(6)} > 0$ large enough universal constants and $C^{(3)}, C^{(4)}, C^{(6)} > 0$ large enough depending only on $c^{(3)}, c^{(4)}, c^{(6)}$, and an absolute sequence $\alpha_n$ that converges to 0 such that Propositions 3, 4, 5 and 6 hold. Set

$$\alpha = \left\{ \left( \frac{C^{(3)}}{\bar{\mu}_3(u^*)} \sqrt{\frac{t(\Sigma^2 + \sigma^2(u^*))^3}{\sqrt{n}}} \right)^2 \wedge \left( \frac{C^{(4)}}{\bar{\mu}_4(u^*)} \frac{t(\Sigma^2 + \sigma^2(u^*))^2}{\sqrt{2n}} \right)^2 \right\} \vee \alpha_n.$$

Write $i = 3$ if $\left( \frac{C^{(3)}}{\bar{\mu}_3(u^*)} \sqrt{\frac{t(\Sigma^2 + \sigma^2(u^*))^3}{\sqrt{n}}} \right)^2 \leq \left( \frac{C^{(4)}}{\bar{\mu}_4(u^*)} \frac{t(\Sigma^2 + \sigma^2(u^*))^2}{\sqrt{2n}} \right)^2$ and $i = 4$ otherwise. In the remaining of the proof $\alpha_n$ denotes a vanishing sequence whose value may change from line to line.

Case 1: $1 - \exp(\bar{\lambda}_{u^*, \epsilon} nt) \geq 1 - \alpha$. In this case, consider the test $\Phi_n = \Phi_n^{(\text{max})}$. If $X \sim (b, \Sigma^2, \nu_{\varepsilon})$ is in $H_0$ (i.e. $\nu_{\varepsilon} = 0$), an application of Proposition 3 and Lemmas 5 and 6 yields $P(\Phi_n = 0) \geq 1 - \alpha_n$.

If, instead, $X$ is such that $(b, \Sigma^2, \nu_{\varepsilon}) = (\bar{b}, \Sigma^2, \nu_{\varepsilon})$, by means of Proposition 3 and Lemmas 5, 6 we get

$$P(\Phi_n = 1) \geq P\left( \{N_{nt}(\bar{u}^*, \varepsilon) \neq 0\} \cap \xi_n \cap \xi'_n \geq 1 - \exp(-\bar{\lambda}_{u^*, \epsilon} nt) - \alpha_n.\right.$$

So by Lemma 1 it follows that the total variation between the observations of $n$ increments of $X$ at the sampling rate $t$ and the closest Gaussian random variable is larger than $1 - \exp(-\bar{\lambda}_{u^*, \epsilon} nt) - \alpha_n$.  

21
Case 2: \( 1 - \exp(-\lambda_u,\varepsilon nt) \leq 1 - \alpha \). In this case consider the test

\[
\Phi_{n,c(i),c(6),\alpha} = \Phi_{n,\max}^{(i)} \vee \Phi_{n,c(i),\alpha}^{(i)} \vee \Phi_{n,c(6)}^{(i)}.
\]

If \( X \) is in \( H_0 \) (i.e. \( \nu = 0 \)), by Propositions 3, 4, 5 and 6 we have that

\[
P(\Phi_{n,c(i),c(6),\alpha} = 0) \geq 1 - \alpha_n.
\]

If \( X \) is such that \((b, \Sigma^2, \nu) = (\tilde{b}, \tilde{\Sigma}^2, \tilde{\nu})\), we distinguish two cases.

- If \( t\mu(u^*) \geq C(6)(t(\Sigma^2 + \sigma^2(u^*)))^3 \): Propositions 3, 4 yield

  \[
P(\Phi_{n,c(i),c(6),\alpha} = 1) \geq P(\{N_{nt}(\tilde{u}^*, \varepsilon) \neq 0\} \cap \xi_n \cap \xi_n') + P(\{N_{nt}(\tilde{u}^*, \varepsilon) = 0\}(1 - \alpha_n).
\]

- If \( t\mu(u^*) < C(6)(t(\Sigma^2 + \sigma^2(u^*)))^3 \): Propositions 3, 5, 6 joined with \( \{u^* > u^+\} \) yield

  \[
P(\Phi_{n,c(i),c(6),\alpha} = 1) \geq P(\{N_{nt}(\tilde{u}^*, \varepsilon) \neq 0\} \cap \xi_n \cap \xi_n') + P(\{N_{nt}(\tilde{u}^*, \varepsilon) = 0\}(1 - \alpha_n).
\]

In both cases we conclude that,

\[
P(\Phi_{n,c(i),c(6),\alpha} = 1) \geq 1 - \alpha \exp\left(-\lambda_u,\varepsilon nt\right) - \alpha_n.
\]

By Lemma 1 we deduce that the total variation distance between the observations of \( n \) increments of \( X \) at the sampling rate \( t \) and the closest Gaussian random variable is larger than \( 1 - 2\alpha - \alpha_n \).

Acknowledgements. The work of A. Carpentier is partially supported by the Deutsche Forschungsgemein- schaft (DFG) Emmy Noether grant MuSyAD (CA 1488/1-1), by the DFG - 314838170, GRK 2297 MathCoRe, by the DFG GRK 2433 DAEDALUS, by the DFG CRC 1294 'Data Assimilation', Project A03, and by the UFA-DFH through the French-German Doktorandenkolleg CDFA 01-18.

The work of E. Mariucci has been partially funded by the Federal Ministry for Education and Research through the Sponsorship provided by the Alexander von Humboldt Foundation, by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 314838170, GRK 2297 MathCoRe, and by Deutsche Forschungsgemeinschaft (DFG) through grant CRC 1294 'Data Assimilation'.

References


A Technical results

A.1 Proofs of the auxiliary Lemmas used in the proof of Theorem 1

A.1.1 Proof of Lemma 2

Consider a compound Poisson approximation of the increment \( \widetilde{M}_t(\varepsilon) := \frac{(M_t(\varepsilon))}{\sqrt{t(\sigma^2(\varepsilon) + \Sigma^2)}} \). Let 0 < \( \eta \) < \( \varepsilon \), and define

\[
\widetilde{M}_t(\eta, \varepsilon) = \frac{\sum_{i=0}^{N_t(\eta, \varepsilon)} Y_i - \int_{t \leq |x| \leq \eta} x \, d\nu}{\sqrt{t \sigma^2(\varepsilon) + \Sigma^2}} = M_t(\eta, \varepsilon) + \frac{N_t(\eta, \varepsilon) \int_{t \leq |x| \leq \eta} x \, d\nu - t \int_{t \leq |x| \leq \eta} x \, d\nu}{\sqrt{t \sigma^2(\varepsilon) + \Sigma^2}}.
\]
where \( \overline{M}_t(\eta, \varepsilon) = \sum_{i=0}^{N_t(\eta, \varepsilon)} (Y_i - \lambda_{\eta, \varepsilon}^{-1} \int_{|x| \leq \varepsilon} x \, d\nu) / \sqrt{ts^2} \), and \( N_t(\eta, \varepsilon) \), \( \lambda_{\eta, \varepsilon} \) and the sequence \( (Y_i)_{i \geq 0} \) are defined as in (11) and (12). Note that for any \( N \), \( \mathbb{E}[\overline{M}_t(\eta, \varepsilon)]N_t(\eta, \varepsilon) = 0 \), and if \( |N_t(\eta, \varepsilon) - t\lambda_{\eta, \varepsilon}/2| \leq t\lambda_{\eta, \varepsilon}/2 \) we have \( \lambda_{\eta, \varepsilon}^{-1} \mathbb{E}Y_i = \int_{|x| \leq \varepsilon} x^2 \, d\nu \leq \sigma^2(\varepsilon) \leq s^2 \) and \( \mathbb{V}[\overline{M}_t(\eta, \varepsilon)]N_t(\lambda, \varepsilon) = N \leq 2 \). Finally, the random variables \( |Y_i| \) are bounded by \( \varepsilon \). For any \( N \) such that \( |N - t\lambda_{\eta, \varepsilon}| \leq t\lambda_{\eta, \varepsilon}/2 \), the Bernstein’s inequality, conditional on \( N_t(\eta, \varepsilon) = N \), leads to

\[
P(|\overline{M}_t(\eta, \varepsilon)| > c_{sup} \sqrt{\log(n)/2} |N_t(\lambda, \varepsilon) = N) \leq 2 \exp\left(- \frac{1}{8} \frac{c_{sup} \log n}{2 + \frac{c_{sup} \log n}{\varepsilon}} \right),
\]

where we used (\( \mathcal{H}_\varepsilon \)). Therefore, for any \( N \) such that \( |N - t\lambda_{\eta, \varepsilon}| \leq t\lambda_{\eta, \varepsilon}/2 \), it holds

\[
P(|\overline{M}_t(\eta, \varepsilon)| > c_{sup} \sqrt{\log(n)/2} |N_t(\eta, \varepsilon) = N) \leq n^{-3},
\]

if \( c_{sup} \geq 10 \). Now by assumption on \( \lambda_{0, \varepsilon} \), there exists \( \overline{\eta} := \eta_t > 0 \) such that for any \( \eta \leq \overline{\eta} \), we have \( t\lambda_{\eta, \varepsilon} \geq 1 \) and \( \frac{24\log(n)}{t} \leq \lambda_{\eta, \varepsilon} \). Moreover, for \( \eta \leq \overline{\eta} \), since \( N_t(\eta, \varepsilon) \) is a Poisson random variable of parameter \( t\lambda_{\eta, \varepsilon} \geq 1 \), we have for any \( 0 \leq x \leq \sqrt{t\lambda_{\eta, \varepsilon}} \)

\[
P(|N_t(\eta, \varepsilon) - t\lambda_{\eta, \varepsilon}| \geq x \sqrt{t\lambda_{\eta, \varepsilon}/2}) \leq \exp(-x^2/8).
\]

This implies that for \( x := \sqrt{24\log(n)} \leq \sqrt{t\lambda_{\eta, \varepsilon}} \),

\[
P(|N_t(\lambda, \varepsilon) - t\lambda_{\eta, \varepsilon}| \geq \sqrt{6t\lambda_{\eta, \varepsilon} \log(n)}) \leq n^{-3}.
\]

Removing the conditioning on \( N_t(\eta, \varepsilon) \) (noting that \( \sqrt{6t\lambda_{\eta, \varepsilon} \log(n)} \leq t\lambda_{\eta, \varepsilon}/2 \)) we get

\[
P\left(|\overline{M}_t(\eta, \varepsilon) + \frac{N_t(\eta, \varepsilon)}{\sqrt{ts^2}} \lambda_{\eta, \varepsilon}^{-1} \int_{|x| \leq \varepsilon} x \, d\nu(x) - \frac{t}{\sqrt{ts^2}} \int_{|x| \leq \varepsilon} x \, d\nu(x) \right| > c_{sup} \sqrt{\log(n)}) \leq 2n^{-3},
\]

using that \( c_{sup} \geq 10 \) and that by the Cauchy-Schwarz inequality \( |\int_{|x| \leq \varepsilon} x \, d\nu(x)| \leq \sqrt{\lambda_{\eta, \varepsilon} \sigma(\varepsilon)} \). Taking the limit as \( \eta \to 0 \) we get

\[
P\left(|\overline{M}_t(\varepsilon)| > c_{sup} \sqrt{\log(n)}) \leq 2n^{-3}.
\]

As \( \tilde{\chi}_t(\varepsilon) = \frac{N(\sigma^2t)}{\sqrt{t\sigma^2t + \Sigma^2}} + \overline{M}_t(\varepsilon) \), by Gaussian concentration we get \( \mathbb{P}(|\tilde{\chi}_t(\varepsilon)| > c_{sup} \sqrt{\log(n)}) \leq 3n^{-3} \). This implies the result whenever \( c_{sup} \geq 10 \).

**A.1.2 Proof of Lemma 3**

In the following, we will use repeatedly that for \( n \geq 1 \), \( n! \geq (n/e)^n \). For any \( x \in \mathcal{I} \), using that \( \sqrt{K} \geq 2c_{sup} \sqrt{\log(n)} \) (since \( c_{int} \geq 2c_{sup} \)), we derive that

\[
|\sqrt{2\pi} \exp(x^2/2) - h_x^{-1}(x)| = \sqrt{2\pi} \mathbb{1}_{(x)} \sum_{k=K+1}^{+\infty} \frac{x^{2k}/k!}{2^{2k}k!} \leq \sqrt{2\pi} \mathbb{1}_{(x)} \sum_{k=K+1}^{+\infty} \frac{(\sqrt{K}/2)^{2k}e^k}{2^{2k}k!} \leq \frac{1}{1 - e^{-1/4 \sqrt{K}}}. 
\]

Therefore, for \( c_{int} \geq 1 \) and \( x \in \mathcal{I} \), we have

\[
|\varphi^{-1}(x) - h^{-1}(x)| = |\sqrt{2\pi} \exp(x^2/2) - h_x^{-1}(x)| \leq \frac{1}{1 - e^{-1/4 \sqrt{K}}}. 
\]
Equation (30) implies that
\[ A = \int \frac{\varphi^2_I}{h_I} = \int \left( \varphi_I + \frac{\varphi^2_I}{h_I^2} - \varphi^{-1}_I \right) \leq 1 + \frac{1}{1 - e/4 n^2}, \]
and
\[ \left| \int h_I - \varphi_I \right| = \left| \int h_I \varphi^{-1} - h^{-1} \right| \leq \int h(x) \varphi(x) \sqrt{2\pi} \exp(x^2/2) - h^{-1}(x) \leq \frac{1}{1 - e/4 n^2}, \]
since, by definition, \( h_I \leq 1/\sqrt{2\pi} \). This, together with \( \Pr_{\varphi}(I^c) \leq n^{-c_{\text{sup}}^2/2} \), leads to the second inequality in (15). Finally, using (30), we get
\[ |E| = \left| \int \varphi_I h^{-1}_I (f_I - \varphi_I) - \int (f_I - \varphi_I) \right| = \left| \int \varphi_I (h^{-1}_I - \varphi^{-1}_I)(f_I - \varphi_I) \right| \leq 2 \frac{1}{1 - e/4 n^2}. \]
By means of \((H_0)\) and using that \( \Pr_{\varphi}(I^c) \leq n^{-c_{\text{sup}}^2/2} \), we derive
\[ \left| \int (f_I - \varphi_I) \right| \leq \left| \Pr_f(I^c) - \Pr_{\varphi}(I^c) \right| \leq n^{-c_{\text{sup}}^2/2} + c_p/n, \]
hence the bound on \(|E|\) is established.

A.1.3 Proof of Lemma 4
Write \( \hat{G} + \exp(-\lambda_0,\epsilon n)1_{\{\Sigma=0\}} = \phi V \), where \( \phi(y) = \exp(-y^2/2) \) and \( V = \exp(g-1), g(y) = -iy^3 \frac{\mu_4(\epsilon)}{6\sqrt{8}^3} + i \frac{\mu_4(\epsilon)}{6} \sum m_{\geq 4} a_m (\frac{m^m}{m!}) \) with \( a_m = \frac{\mu_4(\epsilon)}{(6\sqrt{8}^3)^{m/2}} \). Recall that \( M = \bar{c}_n^{m/2} \frac{\mu_4(\epsilon)}{\sqrt{8}^3} + \mu_4(\epsilon) \). We start with two preliminary Lemmas.

Lemma 8. Suppose that \((H_{\epsilon})\) holds true with \( \bar{c}_n \leq 1 \). Then, for any \( m \geq 1 \), we have
\[ |V(m)| = |(\exp(g) - 1)m| \leq 2m e^{M \bar{c}_n^{m/2}} \max_{1 \leq u \leq m} u^{m-u} (\bar{c}_n M)^u e^{\bar{c}_n |y|}. \]

Proof of Lemma 8. First, note that for any \( j \geq 1 \), we have
\[ g^{(j)}(y) = -i y^3 (\frac{\mu_3(\epsilon)}{6\sqrt{8}^3}) + \frac{\mu_4(\epsilon)}{t s^4} \sum m_{\geq (j+4)} a_m \frac{i^m y^{m-j}}{(m-j)!}, \]
and
\[ |g^{(j)}(y)| \leq \left( \frac{\mu_3(\epsilon)}{\sqrt{8}^3} + \frac{\mu_4(\epsilon)}{t s^4} \right) \sum m_{\geq (j+3)} |a_m| \frac{y^{m-j}}{(m-j)!}, \]
where \( a_3 = 1 \). Using that for any \( m \geq 4 \) it holds that \( a_m \leq \bar{c}_n^{m-4} \) with \( \bar{c}_n \leq 1 \), we derive
\[ |g^{(j)}(y)| \leq \left( \frac{\mu_3(\epsilon)}{\sqrt{8}^3} + \frac{\mu_4(\epsilon)}{t s^4} \right) \sum m_{\geq (j+3)} \bar{c}_n^{m-4} \frac{|y|^{m-j}}{(m-j)!} \leq \bar{c}_n^{m-4} |y| \bar{c}_n. \] (31)
Let us write
\[ R_m = \frac{(\exp(g))^{(m)}}{\exp(g)} \]
and note that \( R_{m+1} = R_m^{(1)} + g^{(1)}(R_m) \). For any \( d \geq 0 \)
\[ |R_{m+1}^{(d)}| = |R_m^{(d+1)} + (g^{(1)}(R_m) \cdot R_m^{(d)}| \leq |R_m^{(d+1)}| + \sum_{j \leq d} C_d^j |g^{(d-j+1)}| |R_m^{(j)}|, \] (32)
by the Leibniz formula. For }m \geq 1\text{, let us consider the following induction assumption:}

\[ H(m) : \forall d \in \mathbb{N}, |R_{m+1}^{(d)}| \leq 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^u e^{\bar{c}_n |y|u} u^{m-u} u^d. \]

\( H(1) \) is true since \( R_1 = g^{(1)} \) and by (31) we obtain \( |g^{(j)}| \leq M \bar{c}_n e^{\bar{c}_n |y|} \) for \( j \geq 1 \). Let us now assume that \( H(m) \) holds for some \( m \geq 1 \). By (32), joined with \( H(m) \) and (31), we get

\[
|R_{m+1}^{(d)}| \leq 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^u e^{\bar{c}_n |y|u} u^{m-u} u^d + \left[ M \bar{c}_n \exp(\bar{c}_n |y|) \right] \sum_{j \leq d} C_j^d \left[ 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^u e^{\bar{c}_n |y|u} u^{m-u} u^j \right].
\]

It follows that

\[
|R_{m+1}^{(d)}| \leq 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^u e^{\bar{c}_n |y|u} u^{m-u} u^d + \left[ 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^{u+1} e^{\bar{c}_n |y|(u+1)} u^{m-u} (1+u)^d \right]
\]

\[
\leq 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^u e^{\bar{c}_n |y|u} u^{m+1-u} u^d + \left[ 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^{u+1} e^{\bar{c}_n |y|(u+1)} (u+1)^m u^d \right],
\]

where we used the binomial formula for the second equation. Finally,

\[
|R_{m+1}^{(d)}| \leq 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^u e^{\bar{c}_n |y|u} u^{m+1-u} u^d + \left[ 2^m \max_{u \in \{1, \ldots, m\}} (\bar{c}_n M)^{u+1} e^{\bar{c}_n |y|(u+1)} (u+1)^m u^d \right]
\]

\[
\leq 2^{m+1} \max_{u \in \{1, \ldots, m+1\}} (\bar{c}_n M)^u e^{\bar{c}_n |y|u} u^{m+1-u}.
\]

Therefore, \( H(m+1) \) holds and the induction hypothesis is thus proven. In particular,

\[
|R_m| \leq 2^m \max_{u \in \{1, \ldots, m\}} u^{m-u} (\bar{c}_n M)^u e^{\bar{c}_n |y|u},
\]

and for any \( m \geq 1 \) we obtain

\[
|V^{(m)}| = |(\exp(g) - 1)^{(m)}| \leq |R_m| \|\exp(g)\| \leq 2^m e^{Me^{\bar{c}_n |y|}} \max_{1 \leq u \leq m} u^{m-u} (\bar{c}_n M)^u e^{\bar{c}_n |y|u}.
\]

\[ \square \]

**Lemma 9.** Suppose that \( (\mathcal{H}_z) \) holds true with \( \bar{c}_n \leq 1 \). Then, for any \( k \)

\[
|\tilde{G}^{(k)}(y)|^2 \leq \left( \frac{k^2 e^{2Me^{\bar{c}_n |y|}}}{d \leq k-1} \right) \left( \frac{k}{d} \right)^2 |\phi^{(d)}(y)|^2 2^{k-d} \max_{1 \leq u \leq k-d} u^{2(k-d-u)} (\bar{c}_n M)^{2u} e^{2\bar{c}_n |y| u} \sqrt{|\phi^{(k)}|^2 V^2}.
\]

**Proof of Lemma 9.** By the binomial formula we bound \( |\tilde{G}^{(k)}| \leq k \sup_{d \leq k} C_k^d |\phi^{(d)}| V^{(k-d)} \). Finally, an application of Lemma 8 yields

\[
|V^{(m)}| = |(\exp(g) - 1)^{(m)}| \leq 2^m e^{Me^{\bar{c}_n |y|}} \max_{1 \leq u \leq m} u^{m-u} (\bar{c}_n M)^u e^{\bar{c}_n |y|u}.
\]

\[ \square \]

26
Proof of Equation (19) in Lemma 4. An application of Lemma 9 yields

\[ |\widehat{G}^{(k)}(y)|^2 \leq \left( k^2 e^{2M_{\nu}y/\nu_n} \sup_{d \leq k-1} \left( \frac{k}{d} \right)^2 |\phi^{(d)}(y)|^2 \sum_{1 \leq u \leq k-d} u^{2(k-d-u)} \left( \frac{c_n M}{u} \right)^{2u} e^{2c_n y/\nu_n} \right) \vee \left( |\phi^{(k)}|^2 V^2 \right). \]

The term \(|\phi^{(k)}|^2 V^2\) leads to the last term in (19) since \(H_m \phi = \phi^{(m)}\) for any \(m\) and \(|V| = |\phi^{-1} \hat{G}|\).

The term corresponding to \(d = k - 1\) leads to the second term in (19), using \(\mathcal{H}_M\) and the fact that for \(|y| \leq c \log(n)\), \(e^{2M_{\nu}y/\nu_n} \leq 2c_n n^{\gamma - \frac{1}{2}} < e\) whenever \(c_n \leq \frac{1}{2}, c_m \leq \frac{1}{2}\).

We control the remaining term using the decomposition \((\bar{c}_n M)^{2u} - ((\bar{c}_n M)^{2u}) (\bar{c}_n M)^{u-1} (\bar{c}_n M)^{2u} = \left( \frac{\bar{c}_n M}{u} \right)^{2u} \left( \frac{\bar{c}_n M}{u} \right)^{u-1} \left( \frac{\bar{c}_n M}{u} \right)^{2u}\).

First notice that by means of \((\mathcal{H}_M)\) together with \(c_m \leq \frac{1}{2}\), we deduce that for any integer \(u \geq 2\) and \(t\) such that \(|y| \leq c \log(n)\), if \(c_n \leq \frac{1}{4}\) it holds

\[ M^{u-1} \exp(2c_n y/\nu_n) \leq 1 \quad \text{and} \quad e^{2M_{\nu}y/\nu_n} \leq 1. \]

Moreover, for any \(u \geq 2,\)

\[ 2^{2(k-d)} \max_{2 \leq u \leq k-d} u^{2(k-d-u)} (\bar{c}_n M)^{u(\gamma - 1/2)} \leq \left( M(k-d)^{k-d} \right) \vee \left[ 2^{2(k-d)} \max_{(k-d)/4 \leq u \leq k-d} e(2(k-d) \log(u) - \frac{1}{4} \log(n)(u-1) M^{(u-1)}) \right]. \]

Since \(k \leq K = c_{\text{int}}^{\gamma} \log(n)\), we know that \(2(k-d) \log(u) - \log(n)(u-1)/4\) is negative whenever \(4c_{\text{int}}^{\gamma} \log(u) \leq u\). Thus, using \((\mathcal{H}_M)\), it follows that for \(k \leq c_{\text{int}}^{\gamma} \log(n)\) there exists a constant \(C_{\text{int}}\), that depends only on \(c_{\text{int}}\), such that

\[ 2^{2(k-d)} \max_{2 \leq u \leq k-d} u^{2(k-d-u)} (\bar{c}_n M)^{(u-1)} \leq C_{\text{int}} 2^{-8(k-d)} (k-d)^{k-d}. \]

This completes the proof of Equation (19).

Proof of Equation (20) in Lemma 4. By means of the Stirling approximation

\[ \sqrt{\frac{\pi k}{2}} \left( \frac{k}{e} \right)^k \leq k! \leq 2 \sqrt{\pi k} \left( \frac{k}{e} \right)^k, \quad \forall k \geq 1, \quad (33) \]

we derive that if \(Z \sim \mathcal{N}(0, \omega^2)\) then,

\[ \mathbb{E}[Z^{2m}] \leq 4(2\omega^2)^m m^m e^m \leq 4(2\omega^2)^m m!, \quad \forall m \geq 1. \quad (34) \]

By Plancherel theorem and (34) applied with \(\omega^2 = 1/2\), we deduce

\[ \int |\phi^{(m)}(y)|^2 dy = \|P_m \phi\|^2 = \frac{1}{2\pi} \int x^{2m} \exp(-x^2) dx \leq \frac{4}{\sqrt{2\pi}} \frac{m^m}{e^m}, \quad \forall m \geq 1. \quad (35) \]

Equation (35) and (33) imply

\[ \int e^{\log(n)} k^2 M^2 \sup_{d \leq k-2} \left[ 2^{-8(k-d)} \left( \frac{k}{d} \right)^2 (k-d)^{k-d} \left| \phi^{(d)}(y) \right|^2 \right] dy \leq k^2 M^2 \sup_{d \leq k-2} \left[ 2^{-8(k-d)} \left( \frac{k}{d} \right)^2 (k-d)^{k-d} 2d^2 e^d \right] \leq k^2 M^2 \sup_{d \leq k-2} \left[ 2^{-8(k-d)} e^{k-d} k \left( \frac{k}{d} \right) \right] \leq 2k^2 k! M^2 \sup_{d \leq k-2} \left[ 2^{-4(k-d)} \left( \frac{k}{d} \right) \right]. \]

27
where we used that $2^{-4}e \leq 1$. Moreover, we observe that by sub-Gaussian concentration of the binomial distribution there exist $\overline{C}, \overline{c} > 0$ universal constant such that $rac{(k)}{2} \leq \overline{C}e^{-|(d-k/2)/\sqrt{\pi})|^2}$. Therefore,

$$
\int_{-\log(n)}^{\log(n)} k^2 M^2 \sup_{d \leq k-2} \left[ 2^{-8(k-d)} \left( \frac{k}{d} \right)^2 (k-d)^{k-d} |\phi^{(d)}(y)|^2 \right] dy 
\leq 2k^2 k! M^2 \left[ \sup_{d \leq k/4} \left[ 2^{-4d} \left( \frac{k}{d} \right) \right] \vee \sup_{k/4+1 \leq d \leq k-2} \left[ 2^{-4d} \left( \frac{k}{d} \right) \right] \right] 
\leq 2k^2 k! M^2 \left[ \sup_{d \leq k/4} \left[ 2^{-4d} \overline{C}^2 k e^{-1/(2\sqrt{\pi})} \right] \vee 1 \right].
$$

Finally, we get

$$
\int_{-\log(n)}^{\log(n)} k^2 M^2 \sup_{d \leq k-2} \left[ 2^{-8(k-d)} \left( \frac{k}{d} \right)^2 (k-d)^{k-d} |\phi^{(d)}(y)|^2 \right] dy 
\leq 2k^2 k! M^2 \left[ \sup_{d \leq k/4} \left[ 2^{-4d} \overline{C}^2 k e^{-1/16} \right] \vee 1 \right] 
\leq 2k^2 k! M^2 \left[ \overline{C}^2 k e^{-1/16} \right] \vee 1,
$$

as desired.

**Proof of Equation (21) in Lemma 4.** First, we begin by observing that

$$
\int_{|y| \leq \log(n)} \exp(-y^2) e^{2|y|c_n} |H_k(y)|^2 dy = 2e^{\overline{c}^2} \int_{0 \leq y \leq \log(n)} \exp(-(y - \overline{c})^2) |H_k(y)|^2 dy 
= 2e^{\overline{c}^2} \int_{-\overline{c} \leq y \leq \log(n) - \overline{c}} \exp(-y^2) |H_k(y + \overline{c})|^2 dy 
\leq 2e^{\overline{c}^2} \int_\mathbb{R} \exp(-y^2) |H_k(y + \overline{c})|^2 dy.
$$

By means of the following property of Hermite polynomials

$$
H_k(y + \overline{c}) = \sum_{u=0}^{k} \binom{k}{u} H_u(y) c_n^{k-u}
$$

joined with the Cauchy Schwarz inequality, we derive

$$
H_k(y + \overline{c})^2 \leq k \sum_{u=0}^{k} \binom{k}{u} \overline{c}^{2k-2u} |H_u(y)|^2.
$$

Finally, using (35) and recalling the definition of Hermite polynomials, we get

$$
\int_\mathbb{R} \exp(-y^2) |H_k(y + \overline{c})|^2 dy \leq 2e^{\overline{c}^2} k \sum_{u=0}^{k} \binom{k}{u} \overline{c}^{2k-2u} \int_\mathbb{R} \exp(-y^2) |H_u(y)|^2 dy 
\leq \frac{4e^{\overline{c}^2}}{\sqrt{2\pi}} k \sum_{u=0}^{k} \binom{k}{u} \overline{c}^{2k-2u} u! = \frac{4e^{\overline{c}^2}}{\sqrt{2\pi}} k! \sum_{u=0}^{k} \frac{k!}{u!(k-u)!} \overline{c}^{2k-2u}
\leq \frac{4e^{\overline{c}^2}}{\sqrt{2\pi}} k!(1 + \overline{c}^2)^k,
$$

which completes the proof.
A.2 Proofs of the Propositions involved in the proof of Theorem 3

Since it will be used several times in the rest of the paper, we write BCI for the Bienaymé-Chebyshev inequality which states that, if $Z$ is a random variable with finite variance, then with probability larger than $1 - \alpha$, it holds

$$E[Z] - \sqrt{V(Z)/\alpha} \leq Z \leq E[Z] + \sqrt{V(Z)/\alpha}.$$ 

Also, to lighten the notation, we will sometimes avoid indexing with $\epsilon$, writing for example $X$ instead of $X(\epsilon)$, or $Z_i$ instead of $Z_i(\epsilon)$ and so on. For the same reason, we will sometimes write $N(u)$ instead of $N(u, \epsilon)$, for $0 < u \leq \epsilon$. Finally, in several occasions we will use that $\sigma(u^+) \leq \sigma(u)$.

A.2.1 Proof of Proposition 3

Under $H_0$: By means of Equations (23) and (24) we have that

$$\max_i Z_i(\omega) \leq 4\sqrt{t\Sigma^2 \log(n)}, \quad \forall \omega \in \xi_n \quad \text{and} \quad S_n(\omega') \geq \frac{\sqrt{t\Sigma^2}}{\pi}, \quad \forall \omega' \in \xi'_n.$$ 

Therefore, for $n > e^{4\sqrt{\pi}}$, on the event $\xi_n \cap \xi'_n$ we have that $Z_{(\bar{m})} < \log(n)^{3/2} S_n$, and thus $\Phi^{(\max)}_n = 0$ as desired.

If a jump larger than $u^*$ occurs: If $u^* = \epsilon$, then Proposition 3 is satisfied as $\lambda_{\epsilon, \epsilon} = 0$, i.e. no jumps larger than $\epsilon$ happen. Assume from now on that $u^* < \epsilon$. By definition of $u^*$, and using that $\sigma(u)$ increases with $u$, we have that $u^* \geq \sqrt{(\Sigma^2 + \sigma^2(u^*))t \log(n)}$. Let us assume that $N_{\text{int}}(u^*) \geq 1$, i.e. from now on we always condition by this event. This assumption, combined with (26), implies that on $\xi_n$ there exists $i$ such that $N_{\text{int}}(u^*) - N_{(i-1)}(u^*) = 1$, and therefore $|M_{\text{int}}(u^*, \epsilon) - M_{(i-1)}(u^*, \epsilon) - (M_{(i-1)}(u^*, \epsilon) - M_{(i-2)}(u^*, \epsilon))| \geq u^*$. In addition, by means of Equation (26), we also know that on $\xi_n$

$$|X_{\text{int}}(u^*) - X_{(i-1)}(u^*) - (X_{(i-1)}(u^*) - X_{(i-2)}(u^*))| \leq 2\sqrt{(\Sigma^2 + \sigma^2(u^*))t \log(n)}.$$ 

Recalling the definition of $u^*$ and taking $n > e^2$ we can conclude that, on $\xi_n$, it holds that $Z_i \geq u^*/2$.

Furthermore, by Equation (27) we know that on $\xi_n$

$$S_n \leq 2\sqrt{2t(\Sigma^2 + \sigma^2(u^+))} \leq 2\sqrt{2t(\Sigma^2 + \sigma^2(u^*))},$$

which allows to conclude that, for $n > e^2$, on $\xi_n \cap \xi'_n$, it holds

$$Z_i \geq \frac{S_n \log(n)^2}{2\sqrt{2}} > S_n \log(n)^{3/2},$$

that is $\Phi^{(\max)}_n = 1$, as desired.

A.2.2 Proof of Proposition 4

Under $H_0$: By means of Equations (24) and (25), for any $\omega' \in \xi'_n$ and $\omega'' \in \xi''_n$, we have

$$S_n(\omega') \geq \frac{\sqrt{t\Sigma^2}}{\pi} \quad \text{and} \quad \Upsilon_{n, 6}(\omega'') \leq C(\sqrt{t\Sigma^2})^6.$$ 

Therefore, on $\xi''_n \cap \xi'_n$, we have $\Upsilon_{n, 6} < C\pi^3 S_n^6$, and thus $\Phi^{(6)}_{n, c} = 0$, as desired.
If \( t\mu_6(u^*) \) is large and no large jump occurs: On the one hand, by Equation (25) we know that, on \( \xi''_n \cap \{N_{nt}(u^*) = 0\} \), it holds

\[
\Upsilon_{n,6} \geq \frac{1}{2} \left[ 2t\mu_6(u^*) + (t(\Sigma^2 + \sigma^2(u^*))^3 \right].
\]

On the other hand, on \( \xi'_n \), by means of Equation (27), we have that \( S_n \leq 2\sqrt{2t(\Sigma^2 + \sigma^2(u^*))} \). Thus, denoting by \( C_c \) an absolute constant depending only on \( c \), whenever \( t\mu_6(u^*) \geq C_c\varepsilon, \sigma^2(u^*) \), it holds that \( \Upsilon_{n,6} > cS_n^6 \), on \( \xi''_n \cap \xi' \cap \{N_{nt}(u^*) = 0\} \), for \( n \) large enough. We therefore conclude that \( \Phi_{n,c}^{(6)} = 1 \), as desired.

A.2.3 Proof of Proposition 5

We begin with some preliminary results proven in Section A.3.

Lemma 10. For \( n \) larger than a universal constant, \( \varepsilon > 0 \) and any \( \log(n)^{-1} < \alpha \leq 1 \), there exist an event \( \xi''_n \) of probability larger than \( 1 - \alpha \) and two universal constants \( c, C > 0 \) such that the following holds:

\[
|E[T_n^{(3)}(\varepsilon)|\xi''_n] - t\mu_3(\varepsilon)| \leq c\frac{\varepsilon^{3/2}(\Sigma^2 + \sigma^2(\varepsilon))^{3/2}}{n\alpha},
\]

and
\[
V(T_n^{(3)}(\varepsilon)|\xi''_n) \leq C \frac{(t\mu_6(\varepsilon) + t(\Sigma^2 + \sigma^2(\varepsilon))^3)}.\]

Corollary 1. For any \( \varepsilon > 0 \) and any \( \log(n)^{-1} < \alpha \leq 1 \), there exists an event \( \xi''_n \) of probability larger than \( 1 - \alpha \) and two universal constants \( c, C > 0 \) such that the following holds:

\[
|E[T_n^{(3)}(\varepsilon)|\xi''_n, N_{nt}(u^*, \varepsilon) = 0] - t\mu_3(u^*)| \leq c\frac{\varepsilon^{3/2}(\Sigma^2 + \sigma^2(u^*))^{3/2}}{n\alpha},
\]

and
\[
V(T_n^{(3)}(\varepsilon)|\xi''_n, N_{nt}(u^*, \varepsilon) = 0) \leq C \frac{(t\mu_6(u^*) + t(\Sigma^2 + \sigma^2(u^*))^3)}.\]

Proof of Proposition 5. For some given \( \alpha \), let \( \xi''_n \) be an event as in Corollary 1. If \( 3 \leq k \leq 6 \), thanks to the hypothesis (29) on \( t\mu_6(u^*) \), there exists an universal constant \( C > 0 \) such that

\[
V(T_n^{(3)}|N_{nt}(u^*) = 0, \xi''_n) \leq C \frac{(t(\Sigma^2 + \sigma^2(u^*))^3}.\]

Therefore, using BCI, we have

\[
P \left( \left| \frac{T_n^{(3)} - t\mu_3(u^*)}{\sqrt{C \frac{t(\Sigma^2 + \sigma^2(u^*))^3}} \right| > c/C + 1 \right) \leq 2\alpha. \tag{36}
\]

Under \( H_0: \) \( \mu_3(u^*) \) and \( \sigma^2(u^*) \) are zero and thus

\[
P \left( \left| \frac{T_n^{(3)}}{\sqrt{C \frac{3^3 \Sigma^6}} \right| > \frac{1}{\sqrt{\alpha}} \right) \leq \alpha.
\]

Therefore, recalling the definition of \( \xi'_n \) (see Equation (24)), we have that for \( c > 0 \) a large enough absolute constant, with probability larger than \( 1 - \alpha - P(\xi' \xi''_n) \), it holds \( |T_n^{(3)}| \leq S_n^6 c/\sqrt{\alpha n} \), which means that the test is accepted with probability larger than \( 1 - 2\alpha \).
Under $H_{1,\rho_0}^{(3)}(\rho)$ and conditional to $N_{n}(u^*) = 0$ : There exists a constant $C_c > 0$, depending only on $c > 0$, such that

$$t|\mu_3(u^*)| \geq t\rho \geq \frac{C_c}{\sqrt{n\alpha}} \sqrt{(t(\Sigma^2 + \sigma^2(u^*)))^2}.$$  

This implies by Equation (36) and for $C_c$ large enough depending only on $c$ that with probability larger than $1 - \alpha$

$$|T_n^{(3)}| \geq \frac{C_c}{2\sqrt{n\alpha}} \sqrt{\frac{T^3(\Sigma^2 + \sigma^2(u^*))^3}{\sqrt{n}}}.$$  

For $C_c$ large enough depending only on $c$, by definition of $\xi_n$ (see (24)) we have that with probability larger than $1 - \alpha - \mathbb{P}(\xi_n^c)$ that $|T_n^{(3)}| \geq \frac{C_c}{\sqrt{n\alpha}}$. The test is rejected with probability larger than $1 - \alpha - \mathbb{P}(\xi_n^c)$.

\[\square\]

A.2.4 Proof of Proposition 6

Proposition 6 can be proved with arguments very similar to those used in the proof of Proposition 5.

Lemma 11. For any $\varepsilon > 0$ it holds $\mathbb{E}[T_n^{(4)}(\varepsilon)] = t\mu_4(\varepsilon)$. For $n$ larger than an absolute constant and for some universal constant $C > 0$, it holds

$$\mathbb{V}(T_n^{(4)}(\varepsilon)) \leq \frac{C}{n} \left(t\mu_8(\varepsilon) + (t(\Sigma^2 + \sigma^2(\varepsilon)))^4\right).$$  

Corollary 2. For any $\varepsilon > 0$, it holds $\mathbb{E}[T_n^{(4)}(\varepsilon)|N_{n1}(u^*, \varepsilon) = 0] = t\mu_4(u^*)$. Moreover, there exists a universal constant $C > 0$ such that

$$\mathbb{V}(T_n^{(4)}(\varepsilon)|N_{n1}(u^*, \varepsilon) = 0) \leq \frac{C}{n} \left(t\mu_8(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^4\right).$$  

\[37\]

Proof of Proposition 6. The proof follows the same scheme as the one in Lemma 5. Here we only observe that Equation (37) implies

$$\mathbb{V}(T_n^{(4)}(\varepsilon)|N_{n1}(u^*, \varepsilon) = 0) \leq \frac{C}{n} \left(t\mu_8(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^4\right) \leq \frac{C}{n} \left((u^*)^4t\mu_4(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^4\right),$$  

since $\mu_8(u^*) \leq (u^*)^4\mu_4(u^*)$. By means of BCI we thus deduce that

$$\mathbb{P}\left(\frac{T_n^{(4)} - t\mu_4(u^*)}{\sqrt{\frac{C}{n} \left((u^*)^4t\mu_4(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^4\right)}} > \frac{1}{\sqrt{\alpha}}|N_{n1}(u^*) = 0\right) \leq \alpha,$$

which, using that $\alpha \geq \log(n^{-1})$ and the definition of $u^*$, implies

$$\mathbb{P}\left(\frac{T_n^{(4)} - \frac{1}{2}t\mu_4(u^*)}{\sqrt{\frac{C}{n} \left((u^*)^4t\mu_4(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^4\right)}} < \frac{1}{\sqrt{\alpha}}|N_{n1}(u^*) = 0\right) \leq \alpha,$$

for some universal constant $C'$ and for $n$ larger than a universal constant.

\[\square\]

A.3 Proofs of the Lemmas involved in the proof of Theorem 3

Hereafter, when there is no ambiguity we drop the dependency $\varepsilon$, writing for example $X$ instead of $X(\varepsilon)$, or $Z_i$ instead of $Z_i(\varepsilon)$ and so on. For the same reason, we sometimes write $N(u)$ instead of $N(u, \varepsilon)$.
A.3.1 Proof of Lemma 1

Let \( \Phi \) be such a test for \( H_0 : P \) against \( H_1 : Q \). The conditions on \( \Phi \) lead to

\[
\|P^\otimes n - Q^\otimes n\|_{TV} \geq \|P_{H_0}(\Phi = 0) - P_{H_1}(\Phi = 0)\| \geq 1 - \alpha_1 - \alpha_0.
\]

A.3.2 Proof of Lemma 5

If \( \nu \neq 0 \): Under \( H_0 \) we know that all \( Z_i \) are i.i.d. realizations of the absolute values of centered Gaussian random variables with variance \( 2t\Sigma^2 \). By Gaussian concentration and using that \( \tilde{n} = \lfloor n/2 \rfloor \leq n \), with probability larger than \( 1 - 1/n \) it holds that \( \max_{i \leq \tilde{n}} Z_i \leq 4\sqrt{t\Sigma^2 \log(n)} \).

If \( \nu \neq 0 \): By BCI, with probability larger than \( 1 - \alpha \), it holds \( |N_n t(u^+, \varepsilon) - t\lambda_{u^+} \varepsilon| \leq \sqrt{t\lambda_{u^+} \varepsilon^2}/\alpha \), i.e. for \( \alpha = 4\log(n)^{-1} \) we have that with probability larger than \( 1 - 4\log(n)^{-1} \)

\[
\log(n)/2 \leq N_n t(u^+, \varepsilon) \leq 2\log(n).
\]

Furthermore, observe that

\[
\mathbb{P}(N_{it}(u^+, \varepsilon) - N_{(i-2)t}(u^+, \varepsilon) \leq 1) = \exp(-2t\lambda_{u^+} \varepsilon) + 2t\lambda_{u^+} \varepsilon \exp(-2t\lambda_{u^+} \varepsilon)
\]

\[
= \exp(-2 \log(n)/n + 2\frac{\log(n)}{n} \exp(-2 \log(n)/n)).
\]

It follows

\[
\mathbb{P}\left(\{\forall i \leq n, N_{it}(u^+, \varepsilon) - N_{(i-2)t}(u^+, \varepsilon) \leq 1\}\right) = \left(\exp(-2 \log(n)/n + 2\frac{\log(n)}{n} \exp(-2 \log(n)/n))\right)^n
\]

\[
= \exp(-2 \log(n))(1 + 2 \log(n)/n)^n \rightarrow 1,
\]

at a rate which does not depend on \( \nu, \varepsilon, b, \Sigma \).

Finally, by BCI, with probability larger than \( 1 - \alpha \), we have

\[
|\bar{X}_{it}(u^*) - \bar{X}_{(i-1)t}(u^*) - b(u^*)| \leq \sqrt{(\Sigma^2 + \sigma^2(u^*))t\alpha^{-1}}.
\]

So, conditional on \( \{1 \leq N_{nt}(u^+, \varepsilon) \leq 2\log(n)\} \), with probability larger than \( 1 - \log(n)^{-1} \), we have that \( \forall i \ s.t. N_{it}(u^+, \varepsilon) - N_{(i-1)t}(u^+, \varepsilon) \neq 0 \)

\[
|\bar{X}_{it}(u^*) - \bar{X}_{(i-1)t}(u^*) - b(u^*)| \leq \sqrt{(\Sigma^2 + \sigma^2(u^*))t \log(n)}.
\]

We conclude observing that, conditional on \( \{1 \leq N_{nt}(u^+, \varepsilon) \leq 2\log(n)\} \), with probability larger than \( 1 - \log(n)^{-1} \), we have \( \forall i \ s.t. N_{it}(u^+, \varepsilon) - N_{(i-2)t}(u^+, \varepsilon) \neq 0 \)

\[
|\bar{X}_{it}(u^*) - \bar{X}_{(i-1)t}(u^*) - (\bar{X}_{(i-1)t}(u^*) - \bar{X}_{(i-2)t}(u^*))| \leq 2\sqrt{(\Sigma^2 + \sigma^2(u^*))t \log(n)}.
\]

A.3.3 Proof of Lemma 6

Preliminary. Denote by \( \tilde{Z}_i = |(X_{it}(u^*) - X_{(i-1)t}(u^*)) - (X_{(i-1)t}(u^*) - X_{(i-2)t}(u^*))| \) and assume that \( \xi_n \) holds. We begin by observing that, for any \( \omega \in \xi_n \), we have:

\[
\frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}-4\log(n)} \tilde{Z}_i(\omega) \leq S_n(\omega) \leq \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} \tilde{Z}_i(\omega).
\]

(38)
To show (38), let \( I := \{ i : \tilde{Z}_i = Z_i \} \), that is the set where no jumps of size larger than \( u^+ \) occur between \((i-2)t\) and \( it\). By means of the positivity of the variables \( \tilde{Z}_i \) and \( Z_i \), we get

\[
\frac{1}{n} \sum_{1 \leq i \leq \tilde{n} - 2 \log(n), i \in I} Z(i) \leq S_n = \frac{1}{n} \sum_{i=1}^{\tilde{n} - 2 \log(n)} Z(i).
\]

Moreover, since \( \#I^c \leq 2 \log(n) \) on \( \xi_n \), we have

\[
\frac{1}{n} \sum_{1 \leq i \leq \tilde{n} - 2 \log(n), i \notin I} Z(i) \leq S_n \leq \frac{1}{n} \sum_{i \in I} Z_i.
\]

Using again that \( \mathbb{P}\{ \#I^c \leq 2 \log(n) \cap \xi_n \} = 1 \), the definition of \( I \) and the fact that \( \tilde{Z}_i, Z_i \) are positive, we obtain (38).

**Control when \( \xi'_n \) is given by (27).** Note that \( \mathbb{E}\tilde{Z}_i^2 = 2t(\Sigma^2 + \sigma^2(u^+)) \), so by Cauchy-Schwartz inequality \( \mathbb{E}\tilde{Z}_i \leq \sqrt{2t(\Sigma^2 + \sigma^2(u^+))} \). It follows by BCI, that with probability larger than \( 1 - 1/n \)

\[
\frac{1}{n} \sum_{i=1}^{\tilde{n}} \tilde{Z}_i \leq 2\sqrt{2t(\Sigma^2 + \sigma^2(u^+))}.
\]

Then, on \( \xi_n \), by Equation (38), with probability larger than \( 1 - 1/n \) it holds \( S_n \leq 2\sqrt{2t(\Sigma^2 + \sigma^2(u^+))} \).

**Control when \( \xi''_n \) is given by (24).** In this case \( \nu_i = 0 \), and thus \( Z_i = \tilde{Z}_i \) are i.i.d. and distributed as the absolute value of a centered Gaussian random variable with variance \( 2t\Sigma^2 \). By Gaussian concentration it then follows that with probability larger than \( 1 - \alpha \)

\[
\max |Z_i| \leq 2\sqrt{2t\Sigma^2 \log(2/\alpha)}.
\]

Using that \( \mathbb{E}||\mathcal{N}(0,1)|| = \sqrt{2/\pi} \) and BCI, we conclude that with probability larger than \( 1 - \alpha \)

\[
\frac{1}{n} \sum_{i=1}^{\tilde{n} - 4 \log(n)} \tilde{Z}_i \geq \frac{2}{\pi} \sqrt{2t\Sigma^2} - \sqrt{\frac{1}{n\alpha}} \sqrt{2t\Sigma^2} - \frac{4 \log(n)}{n} \sqrt{2t\Sigma^2 \log(2/\alpha)}.
\]

By Equation (38), with probability larger than \( 1 - 2^9 \pi n^{-1} \), it holds \( S_n \geq \sqrt{\Sigma^2}/\sqrt{\pi} \).

**A.3.4 Preliminaries for the proofs of Lemmas 7, 10 and 11**

If \( Y \sim \mathcal{N}(m, \Sigma^2) \) its moments can be computed through the recursive formula:

\[
\mathbb{E}[Y^k] = (k - 1)\mathbb{V}(Y)\mathbb{E}[Y^{k - 2}] + \mathbb{E}[Y]\mathbb{E}[Y^{k - 1}], \quad k \in \mathbb{N}.
\]

**Lemma 12.** Let \( Y \sim \mathcal{N}(m, \Sigma^2) \), then it holds

\[
\begin{align*}
\mathbb{E}[Y^3] &= 3\Sigma^2 m + m^3, \quad \mathbb{E}[Y^4] = 3\Sigma^4 + 6m^2\Sigma^2 + m^4, \quad \mathbb{E}[Y^5] = 15m\Sigma^4 + 10m^3\Sigma^2 + m^5, \\
\mathbb{E}[Y^6] &= 15\Sigma^6 + 45m^2\Sigma^4 + 15m^4\Sigma^2 + m^6, \quad \mathbb{E}[Y^7] = 105m^3\Sigma^4 + 105m^3\Sigma^2 + 21m^5\Sigma^2 + m^7, \\
\mathbb{E}[Y^8] &= 105\Sigma^8 + 420m^2\Sigma^6 + 210m^4\Sigma^4 + 28m^6\Sigma^2 + m^8.
\end{align*}
\]

Similarly, using the series expansion of the characteristic function, together with the Lévy-Kintchine formula, we get

\[
\mathbb{E}[M_t(\varepsilon)^k] = \frac{d^k}{du^k} \exp \left( t \int_{|x| \leq \varepsilon} (e^{iux} - 1 - iux)\nu_x(dx) \right) \bigg|_{u=0}, \quad \forall \varepsilon > 0.
\]

(39)
Lemma 13. For $\varepsilon > 0$, set $\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} y^2 \nu(dy)$ and $\mu_k(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} y^k \nu(dy)$, $k \geq 3$. We have

\[
\mathbb{E}[M_1(\varepsilon)] = 0, \quad \mathbb{E}[(M_1(\varepsilon))^2] = t\sigma^2(\varepsilon), \quad \mathbb{E}[(M_1(\varepsilon))^3] = t\mu_3(\varepsilon)
\]

\[
\mathbb{E}[(M_1(\varepsilon))^4] = t\mu_4(\varepsilon) + 3t^2\sigma^4(\varepsilon), \quad \mathbb{E}[(M_1(\varepsilon))^5] = t\mu_5(\varepsilon) + 10t^2\sigma^2(\varepsilon)\mu_3(\varepsilon),
\]

\[
\mathbb{E}[(M_1(\varepsilon))^6] = t\mu_6(\varepsilon) + t^2(10\mu_3(\varepsilon)^2 + 15\sigma^2(\varepsilon)\mu_4(\varepsilon)) + 15t^3\sigma^6(\varepsilon),
\]

\[
\mathbb{E}[(M_1(\varepsilon))^7] = t\mu_7(\varepsilon) + t^2(21\sigma^2(\varepsilon)\mu_5(\varepsilon) + 35\mu_3(\varepsilon)\mu_4(\varepsilon)) + 105t^3\sigma^4(\varepsilon)\mu_3(\varepsilon),
\]

\[
\mathbb{E}[(M_1(\varepsilon))^8] = t\mu_8(\varepsilon) + t^2(35\mu_4(\varepsilon)^2 + 56\mu_3(\varepsilon)\mu_5(\varepsilon) + 28\sigma^2(\varepsilon)\mu_6(\varepsilon))
\]

\[
+ t^3(280\sigma^2(\varepsilon)\mu_4(\varepsilon)^2 + 210\sigma^4(\varepsilon)\mu_4(\varepsilon)) + 105t^4\sigma^8(\varepsilon).
\]

More generally if $\nu_e$ is a Lévy measure such that $\int_{|x| \leq \varepsilon} \nu(dx) > t^{-1}$, then for any $k \geq 2$ even integer it holds that

\[
\mathbb{E}[M_k(\varepsilon)] \leq C_k \left( t\mu_k(\varepsilon) + (t\sigma^2(\varepsilon))^{k/2} \right),
\]

where $\overline{C}_k > 0$ is a constant that depends only on $k$.

Proof of Lemma 13. The explicit first 8 moments are computed using Equation (39). We prove now Lemma 13. For $\eta, \varepsilon$, such that

\[
\int_{|y| \leq \varepsilon} \nu_e(dy) > t^{-1},
\]

we derive

\[
\mathbb{E}[M_1(\eta, \varepsilon)] = 0, \quad \mathbb{E}[(M_1(\eta, \varepsilon))^2] = t\sigma^2(\eta, \varepsilon), \quad \mathbb{E}[(M_1(\eta, \varepsilon))^3] = t\mu_3(\eta, \varepsilon),
\]

\[
\mathbb{E}[(M_1(\eta, \varepsilon))^4] = t\mu_4(\eta, \varepsilon) + 3t^2\sigma^4(\eta, \varepsilon), \quad \mathbb{E}[(M_1(\eta, \varepsilon))^5] = t\mu_5(\eta, \varepsilon) + 10t^2\sigma^2(\eta, \varepsilon)\mu_3(\eta, \varepsilon),
\]

\[
\mathbb{E}[(M_1(\eta, \varepsilon))^6] = t\mu_6(\eta, \varepsilon) + t^2(10\mu_3(\eta, \varepsilon)^2 + 15\sigma^2(\eta, \varepsilon)\mu_4(\eta, \varepsilon)) + 15t^3\sigma^6(\eta, \varepsilon),
\]

\[
\mathbb{E}[(M_1(\eta, \varepsilon))^7] = t\mu_7(\eta, \varepsilon) + t^2(21\sigma^2(\eta, \varepsilon)\mu_5(\eta, \varepsilon) + 35\mu_3(\eta, \varepsilon)\mu_4(\eta, \varepsilon)) + 105t^3\sigma^4(\eta, \varepsilon)\mu_3(\eta, \varepsilon),
\]

\[
\mathbb{E}[(M_1(\eta, \varepsilon))^8] = t\mu_8(\eta, \varepsilon) + t^2(35\mu_4(\eta, \varepsilon)^2 + 56\mu_3(\eta, \varepsilon)\mu_5(\eta, \varepsilon) + 28\sigma^2(\eta, \varepsilon)\mu_6(\eta, \varepsilon))
\]

\[
+ t^3(280\sigma^2(\eta, \varepsilon)\mu_4(\eta, \varepsilon)^2 + 210\sigma^4(\eta, \varepsilon)\mu_4(\eta, \varepsilon)) + 105t^4\sigma^8(\eta, \varepsilon).
\]

We therefore deduce that there exists a constant $\overline{C}_k$, only depending on $k$, such that

\[
\mathbb{E}[M_k(\eta, \varepsilon)] \leq \overline{C}_k \left( t\mu_k(\eta, \varepsilon) + (t\sigma^2(\eta, \varepsilon))^{k/2} \right).
\]
In particular, we conclude that the family of random variables \( (M_t(\eta, \varepsilon))^k \) of 0 < \( \eta \leq \varepsilon \) is uniformly integrable. Therefore, using also that the family of processes \( M(\eta, \varepsilon) \) converges almost surely to \( M_t(\varepsilon) \), as \( \eta \to 0 \), we derive that \( \lim_{\eta \to 0} \mathbb{E}[|M_t(\eta, \varepsilon)|^k] = \mathbb{E}[|M_t(\varepsilon)|^k] \). Making \( \eta \) converge to 0 in (41) gives the desired result.

\[ \text{Corollary 3. For } \varepsilon > 0, \text{ set } \sigma^2(\varepsilon) = \int_{|y| \leq \varepsilon} y^2 \nu(dy) \text{ and } \mu_k(\varepsilon) := \int_{|y| \leq \varepsilon} y^k \nu(dy), \ k \geq 2. \text{ It holds that} \]

\[
\mathbb{E}[X_t(\varepsilon) - b(\varepsilon)t] = 0, \quad \mathbb{E}[(X_t(\varepsilon) - b(\varepsilon)t)^3] = t\mu_3(\varepsilon), \quad \mathbb{E}[X_t(\varepsilon) - X_{t,n}(\varepsilon)] = 0, \\
\mathbb{E}[(X_t(\varepsilon) - X_{t,n}(\varepsilon))^3] = \left(1 - \frac{1}{(n - \lfloor n/2 \rfloor)^2}\right)t\mu_3(\varepsilon), \quad \mathbb{E}[Z_t^2(\varepsilon)] = 2t(\Sigma^2 + \sigma^2(\varepsilon)), \\
\mathbb{E}[Z_t^4(\varepsilon)] = 4t\mu_4(\varepsilon) + 12(t(\Sigma^2 + \sigma^2(\varepsilon)))^2, \quad \mathbb{E}[Z_t^6(\varepsilon)] \geq 2t\mu_6(\varepsilon) + (t(\Sigma^2 + \sigma^2(\varepsilon)))^3.
\]

\[ \text{Proof. The result is obtained combining Lemmas 12 and 13 with the independence between } (X_{2t}(\varepsilon) - X_t(\varepsilon)) \text{ and } X_t(\varepsilon). \]

\[ \text{Corollary 4. For } \varepsilon > 0, \text{ set } \sigma^2(\varepsilon) = \int_{|y| \leq \varepsilon} y^2 \nu(dy) \text{ and } \mu_k(\varepsilon) := \int_{|y| \leq \varepsilon} y^k \nu(dy), \ k \geq 2. \text{ It holds for any } k \geq 2, \ k \text{ even integer that} \]

\[
\mathbb{E}[(X_t(\varepsilon) - b(\varepsilon)t)^k] \leq \overline{C}_k(t\mu_k(\varepsilon) + (t(\Sigma^2 + \sigma^2(\varepsilon)))^{k/2}), \\
\mathbb{E}[Z_t^k(\varepsilon)] \leq \overline{C}_k(t\mu_k(\varepsilon) + (t(\Sigma^2 + \sigma^2(\varepsilon)))^{k/2}),
\]

where \( \overline{C}_k \) is a constant that depends only on \( k \).

\[ \text{Proof of Corollary 4. By means of Lemma 13 we have} \]

\[
\mathbb{E}[(X_t(\varepsilon) - b(\varepsilon)t)^k] = \mathbb{E}(M_t(\varepsilon) + \Sigma W_t)^k \leq 2^{k-1}\left(\mathbb{E}[M_t(\varepsilon)^k] + \Sigma^k\mathbb{E}[W_t^k]\right) \\
\leq C_k(t^{k/2}\Sigma^k + t\mu_k(\varepsilon) + (t\sigma^2(\varepsilon))^{k/2}),
\]

where \( C_k \) is a constant that depends on \( k \) only.

\[ \text{A.3.5 Proof of Lemma 7} \]

If \( \nu_c = 0 \). By Corollary 4, there exist universal constants \( \overline{C}_6 \) and \( \overline{C}_{12} \) such that \( \mathbb{E}[Z_t^6] \leq \overline{C}_6 t^3 \Sigma^6 \) and \( \mathbb{E}[Z_t^{12}] \leq \overline{C}_{12} t^6 \Sigma^{12} \). Using that \( Z_t \) are i.i.d. we get

\[
\mathbb{E}[Y_{n,6}] \leq \overline{C}_6 t^3 \Sigma^6, \quad \mathbb{E}[Y_{n,6}] \leq \overline{C}_{12} t^6 \Sigma^{12}.
\]

Therefore, by means of BCI, with probability larger than \( 1 - \log(n)^{-1} \) conditional to \( N_{nt}(u^*) = 0 \) we have

\[
\overline{Y}_{n,6} \leq \overline{C}_6 t^3 \Sigma^6 + \sqrt{\frac{\overline{C}_{12} \log(n)}{n - \lfloor n/2 \rfloor}}(t\Sigma^2)^6,
\]

which allows to deduce that for \( n \) larger than a universal constant, with probability larger than \( 1 - \log(n)^{-1} \), we have \( \overline{Y}_{n,6} \leq 2\overline{C}_6 t^3 \Sigma^6 \).
If $\nu \neq 0$. By Corollaries 3 and 4, conditional to $N_{\nu}(u^*) = 0$, for any $i$ it holds
\[
\mathbb{E}[Z^i_t|N_{\nu}(u^*) = 0] \geq t\mu_0(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^3
\]
\[
\mathbb{E}[Z^i_{12}|N_{\nu}(u^*) = 0] \leq C_{12}(t\mu_2(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^6),
\]
where $C_{12}$ is the universal constant from Corollary 4. Using that the $Z_i$ are i.i.d. we obtain
\[
\mathbb{E}[\overline{Y}_{n,6}|N_{\nu}(u^*) = 0] \geq t\mu_6(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^6
\]
\[
\mathbb{V}(\overline{Y}_{n,6}|N_{\nu}(u^*) = 0) \leq \frac{C_{12}}{n - \lfloor n/2 \rfloor^2}(t\mu_2(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^6).
\]
It follows by BCI that with probability larger than $1 - \log(n)^{-1}$, conditional to $N_{\nu}(u^*) = 0$, we have
\[
\overline{Y}_{n,6} \geq t\mu_6(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^3 - \sqrt{\frac{C_{12} \log(n)}{(n - \lfloor n/2 \rfloor^2)}}(t\mu_2(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^6).
\]
Since $\mu_{12}(u^*) \leq (u^*)^6\mu_6(u^*)$, with probability larger than $1 - \log(n)^{-1}$ and conditional to $N_{\nu}(u^*) = 0$, we have
\[
\overline{Y}_{n,6} \geq t\mu_6(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^3 - \sqrt{\frac{C_{12} \log(n)}{(n - \lfloor n/2 \rfloor^2)}}((u^*)^6t\mu_6(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^6).
\]
Finally, by means of the definition of $u^*$ and for $n$ larger than a universal constant, with probability larger than $1 - \log(n)^{-1}$ conditional to $N_{\nu}(u^*) = 0$, it holds
\[
\overline{Y}_{n,6} \geq \frac{t\mu_6(u^*) + (t(\Sigma^2 + \sigma^2(u^*)))^3}{2}.
\]

### A.3.6 Proof of Lemma 10

Note that $\mathbb{E}[\overline{X}_{t,n}(\varepsilon)] = b(\varepsilon)t$ and that $\mathbb{V}(\overline{X}_{t,n}(\varepsilon)) = (n - \lfloor n/2 \rfloor)^{-1}t(\Sigma^2 + \sigma^2(\varepsilon))$. Write
\[
\xi_{n}''' = \{|\overline{X}_{t,n}(\varepsilon) - b(\varepsilon)t| \leq r_n\} \text{ where } r_n := r_n(\alpha) = \frac{t(\Sigma^2 + \sigma^2(\varepsilon))}{\alpha(n - \lfloor n/2 \rfloor)^T}.
\]
By BCI we have for any $0 < \alpha \leq 1$
\[
\mathbb{P}(\xi_{n}''') \geq 1 - \alpha. \tag{42}
\]
Conditional on $\xi_{n}'''$, by Corollary 3 and the definition of $r_n$,
\[
|\mathbb{E}[\overline{Y}_{n,6}|\xi_{n}'''] - t\mu_3(\varepsilon)| \leq r_n^3 + 3t(\Sigma^2 + \sigma^2(\varepsilon))r_n \leq 100\frac{(t(\Sigma^2 + \sigma^2(\varepsilon)))^{3/2}}{\sqrt{n\alpha}},
\]
for $\alpha \geq \log(n)^{-1}$.

Now we compute $\mathbb{V}(T_{n}^{(3)})$. Since $\overline{X}_{t,n}$ and $\overline{X}_{t,n}'$ are independent, as they are computed on two independent samples, the elements of the sum are independent of each other conditional on the second half of the sample. Then, conditional on the second half of the sample,
\[
\mathbb{V}(T_{n}^{(3)}|\xi_{n}''') \leq \left(1 - \frac{1}{(n - \lfloor n/2 \rfloor)^2}\right)^2 \frac{1}{(n/2)^2} \sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{E}\left[\left|X_{it}(\varepsilon) - X_{i(\varepsilon) \varepsilon} - \overline{X}_{t,n}(\varepsilon)\right|^3\right]^2 |\xi_{n}'''
\]
\[
\leq \frac{16}{n} \mathbb{E}\left[\left|X_{it}(\varepsilon) - X_{i(\varepsilon) \varepsilon} - t\varepsilon\right|^6 + \left|\overline{X}_{t,n}(\varepsilon) - t\varepsilon\right|^6 |\xi_{n}'''
\]
\[
\leq \frac{16}{n} \left[C_{6}(t\mu_6(\varepsilon) + (t(\Sigma^2 + \sigma^2(\varepsilon)))^3) + r_n^6\right],
\]
where $C_{6}$ is the constant from Corollary 4. Hence, by Equation (42), there exists a universal constant $C$ such that
$$\mathbb{V}(T_{n}^{(3)}; \xi_{n}^{(3)}) \leq \frac{C}{n} \left[ t \mu_{0}(\varepsilon) + [t(\Sigma^{2} + \sigma^{2}(\varepsilon))]^{3} \right].$$

A.3.7 Proof of Lemma 11

The main ingredient of the proof consists in establishing expansions of $\mathbb{V}(T_{n}^{(4)})$. Computations are cumbersome but not difficult, we only give the main tools here but we do not provide all computations. By Corollary 3 and since $\mathbb{V}_{n,2}$ and $\mathbb{V}_{n,2}'$ are independent, we have
$$\mathbb{E}[\mathbb{Y}_{n,2}] = 2t(\Sigma^{2} + \sigma^{2}(\varepsilon)), \quad \mathbb{E}[\mathbb{Y}_{n,2}\mathbb{Y}_{n,2}'] = 4t^{2}(\Sigma^{2} + \sigma^{2}(\varepsilon))^{2},$$
$$\mathbb{E}[\mathbb{Y}_{n,4}] = 12t^{2}(\Sigma^{2} + \sigma^{2}(\varepsilon))^{2} + 4t\mu_{4}(\varepsilon).$$

In particular, $\mathbb{E}[T_{n}^{(4)}] = t\mu_{4}(\varepsilon)$. Next, we have $\mathbb{V}[T_{n}^{(4)}] \leq 9\mathbb{V}[\mathbb{Y}_{n,2}\mathbb{Y}_{n,2}'] + \mathbb{V}[\mathbb{Y}_{n,4}]$. We analyse these two terms separately.

Since the $Z_{i}$ in the sum composing $\mathbb{Y}_{n,4}$ are i.i.d. we have
$$\mathbb{V}(\mathbb{Y}_{n,4}) \leq \frac{1}{[n/2]^{2}} \sum_{i=1}^{[n/2]} \mathbb{E}(Z_{i}^{2}) \leq \frac{1}{[n/2]^{2}} 2^{n} C_{8} \left( t \mu_{8}(\varepsilon) + [t(\Sigma^{2} + \sigma^{2}(\varepsilon))]^{4} \right),$$
$$\leq \frac{C}{n} \left( t \mu_{8}(\varepsilon) + [t(\Sigma^{2} + \sigma^{2}(\varepsilon))]^{4} \right),$$
where $C_{8}$ is the constant from Corollary 4, and where $C$ is a universal constant.

Similarly, as the $Z_{i}$ in the sums composing $\mathbb{Y}_{n,2}$ and $\mathbb{Y}_{n,2}'$ are i.i.d. we have
$$\mathbb{V}(\mathbb{Y}_{n,2}\mathbb{Y}_{n,2}') \leq 4\mathbb{E}[\mathbb{Y}_{n,2}]\mathbb{V}(\mathbb{Y}_{n,2}') \leq \frac{4C_{4}}{[n/4]} \left( t \mu_{4}(\varepsilon) + [t(\Sigma^{2} + \sigma^{2}(\varepsilon))]^{2} \right)$$
$$\times \left[ \frac{C_{4}}{[n/4]} \left( t \mu_{4}(\varepsilon) + [t(\Sigma^{2} + \sigma^{2}(\varepsilon))]^{2} \right) + 4t^{2}(\Sigma^{2} + \sigma^{2}(\varepsilon))^{2} \right]$$
$$\leq \frac{C'}{n} \left( [t(\Sigma^{2} + \sigma^{2}(\varepsilon))]^{4} + \frac{t^{2}\mu_{4}(\varepsilon)^{2}}{n} \right),$$
where $C_{4}$ is the constant from Corollary 4, and where $C'$ is a universal constant. Combining this with the last displayed equation, completes the proof.

A.4 Proof of Proposition 1

Write $X_{t}(\varepsilon) = (X_{t} - t\mu(\varepsilon))/\sqrt{t\sigma^{2}}$ for the rescaled increment, $\hat{M}_{t}(\varepsilon) = M_{t}(\varepsilon)/\sqrt{t} \sigma$ for the pure-jump part of the rescaled increment and $s^{2} = \sigma^{2}(\varepsilon) + \Sigma^{2}$. We first state the following lemma, proven at the end of the proof of this proposition.

Lemma 14. Assume that $t\lambda_{0,0} > 1$ and $\varepsilon \leq \sqrt{t(\Sigma^{2} + \sigma^{2}(\varepsilon))}$. It holds for any $x > 0$ that
$$\mathbb{P}(\lvert \hat{M}_{t}(\varepsilon) \rvert > x) \leq 2 \exp \left( - \frac{x^{2}}{17} \right) + 2 \exp(-x/2).$$

Using Lemma 14, we get $\mathbb{P}(\lvert \hat{M}_{t}(\varepsilon) \rvert \geq x) \leq 2 \exp(-x^{2}/17) + 2 \exp(-x/2).$ This implies that for $\mu$ being the measure corresponding to the distribution of the rescaled jump component $\lvert \hat{M}_{t}(\varepsilon) \rvert$, and for any $k \in \mathbb{N}^{*}$
$$\mathbb{E}[\lvert \hat{M}_{t}(\varepsilon) \rvert^{k}] = k \int_{0}^{\infty} x^{k-1} \mu(dx) \leq 2k \int_{0}^{\infty} x^{k-1} \left( \exp(-x^{2}/17) + \exp(-x/2) \right) dx \leq 4 \times 3^{k+2}k!. \quad (43)$$
The rescaled increment $\tilde{X}_t(\varepsilon)$ has characteristic function given by $\Psi = \Phi_1\Psi_2$, for $\Phi_1(y) = \exp(-\frac{y^2s^2}{2})$ and $\Psi_2$ the characteristic function of $\tilde{M}_t(\varepsilon)$. As $\int \mu(dx) = 1$ and $(e^{iu(y)})^{(d)} = (ui)^d e^{iu}$, we get using Equation (43) for any $d \in \mathbb{N}^*$, $|\Psi_2^{(d)}(y)| \leq 4 \times 3^{d+2}d!$. This implies that

$$|\Psi^{(k)}(y)|^2 = \left| \sum_j \binom{k}{j} \frac{\Phi_1^{(j)}(y)}{\Psi_2^{(k-j)}} \right|^2 \leq 16 \times 3^{2k+4} \max_{j \leq k} |\Phi_1^{(j)}(y)|^2 ((k-j)!)^2. \quad (44)$$

Set $r^2 = \Sigma^2/s^2$ and note that $(c_2^2 + 1)^{-1/2} \leq r \leq 1$. We have by definition of Hermite polynomials

$$\Phi_1^{(j)}(y) = r^j H_j(rt) \exp(-y^2/2).$$

Since $|H_j(u)| \leq j \times j!(|u|^j + 1)$ we have that

$$\int_{c \log(n)}^{+\infty} |\Phi_1^{(j)}(y)|^2 dy \leq 2r^{2j} j^2 \int_{c \log(n)}^{+\infty} (|y|^{2j} + 1) \exp(-y^2 r^2) dy$$

$$\leq j^{3r^2j} \exp(-c^2 \log(n)^2 r^2/2) \int_{c \log(n)}^{+\infty} (|y|^{2j} + 1) \exp(-y^2 r^2/2) dy$$

$$\leq (2j)^{5j} \sqrt{2\pi(c_2^2 + 1)} \exp \left(-\frac{c^2 \log(n)^2}{2(c_2^2 + 1)} \right).$$

By Equation (44) it follows

$$\int_{c \log(n)}^{+\infty} |\Psi^{(k)}(y)|^2 dy \leq 16 \times 3^{2k+4} \max_{j \leq k} ((k-j)!/2)^j \sqrt{2\pi(c_2^2 + 1)} \exp \left(-\frac{c^2 \log(n)^2}{2(c_2^2 + 1)} \right)$$

$$\leq 16 \times 3^{2k+4} e^{5j} k^{2k} \sqrt{2\pi(c_2^2 + 1)} \exp \left(-\frac{c^2 \log(n)^2}{2(c_2^2 + 1)} \right),$$

taking $c$ large enough depending only on $c_2$ completes the proof.

Proof of Lemma 14. For $\varepsilon \geq \eta > 0$, write $v_{\eta,\varepsilon} = \lambda_{\eta,\varepsilon}^{-1} \int_{|x| \leq \varepsilon} x^2 dv$. Using notations and definitions from the proof of Lemma 2, we have for $\varepsilon \geq \eta > 0$ and using $\lambda_{\eta,\varepsilon} v_{\eta,\varepsilon} \leq s^2$ we get for all $x > 0$

$$\mathbb{P}(|\tilde{M}_t(\eta,\varepsilon)| > x | N_t(\eta,\varepsilon) = N) \leq 2 \exp \left( -\frac{1}{2} \frac{x^2}{\lambda_{\eta,\varepsilon} + \frac{x}{\sqrt{ts^2}}} \right). \quad (45)$$

Now by assumption on $\lambda_{\eta,\varepsilon}$, there exists $\eta > 0$ such that for any $\eta \leq \eta$, we have $t\lambda_{\eta,\varepsilon} \geq 1$. Moreover, for $\eta \leq \eta$, since $N_t(\eta,\varepsilon)$ is a Poisson random variable of parameter $t\lambda_{\eta,\varepsilon} \geq 1$, we have using the Chernoff bound,

$$\mathbb{P}(N_t(\eta,\varepsilon) - t\lambda_{\eta,\varepsilon} \geq x) \leq \exp(-x^2/(2e^4 t\lambda_{\eta,\varepsilon})), \quad 0 \leq x \leq e^2 t\lambda_{\eta,\varepsilon}$$

$$\mathbb{P}(N_t(\eta,\varepsilon) - t\lambda_{\eta,\varepsilon} \geq x) \leq \exp(-x/2), \quad x \geq e^2 t\lambda_{\eta,\varepsilon}.$$ Combining Equation (45) and the last displayed equation, one gets for for $\eta \leq \eta$ and $x > 0$

$$\mathbb{P}(|\tilde{M}_t(\eta,\varepsilon)| > x) \leq \mathbb{P} \left\{ \{ |\tilde{M}_t(\eta,\varepsilon)| > x \cap N_t(\eta,\varepsilon) \leq (1 + e^2) t\lambda_{\eta,\varepsilon} + x \} \cup \{ N_t(\eta,\varepsilon) \geq (1 + e^2) t\lambda_{\eta,\varepsilon} + x \} \right\}$$

$$\leq 2 \exp \left( -\frac{1}{2} \frac{t\lambda_{\eta,\varepsilon}(1+e^2)+x}{t\lambda_{\eta,\varepsilon}} + \frac{e^2}{\sqrt{ts^2}} \right) \text{exp}(-x/2) \leq 2 \exp \left( -\frac{x^2}{17} \right) + 2 \exp(-x/2),$$

where we use $\varepsilon/\sqrt{ts^2} \leq 1$. The last equation holds for any $\eta \leq \eta$. Having $\eta \to 0$, we conclude as in the proof of Lemma 2 to transfert the result from $\tilde{M}_t(\varepsilon)$ to $\tilde{M}_t$. $\square$
A.5 Proof of Proposition 2

Let $\beta \in (0, 2)$ and consider a Lévy measure $\nu$ that has a density with respect to the Lebesgue measure such that there exist two positive constants $c_+ > c_- > 0$ with

$$\frac{c_-}{|x|^\beta} \leq \frac{d\nu(x)}{dx} \leq \frac{c_+}{|x|^\beta}, \quad \forall x \in [-\varepsilon, \varepsilon] \setminus \{0\}.$$

Let $X(\varepsilon) \sim (b, \Sigma^2, \nu 1_{|x| \leq \varepsilon})$. The characteristic function of $X_1(\varepsilon)$ is $\Psi = \exp(\tilde{\psi})$, with

$$\tilde{\psi}(y) = iy - ty \Sigma^2/2 + t \int_{-\varepsilon}^\varepsilon (e^{iyu} - 1 - iyu) d\nu(u).$$

The rescaled increment, denoted by $\tilde{X}_t(\varepsilon)$, has characteristic function $\Psi = \exp(\psi)$ with

$$\psi(y) = -y^2 \frac{\Sigma^2}{2(\Sigma^2 + \sigma^2(\varepsilon))} + t \int_{|u| \leq \varepsilon} (e^{iyu/\sqrt{s^2\varepsilon^2}} - 1 - iyu/\sqrt{s^2\varepsilon^2}) d\nu(u),$$

where $\sigma^2(\varepsilon) = \int_{|u| \leq \varepsilon} u^2 d\nu(u) \in [\frac{2c_-}{2-\beta} \varepsilon^{2-\beta} \frac{2c_+}{2-\beta} \varepsilon^{2-\beta}]$. From now on, write $s^2 = \Sigma^2 + \sigma^2(\varepsilon)$. We have

$$\psi(y) = -y^2 \frac{\Sigma^2}{2s^2} + t \int_{|u| \leq \varepsilon} (e^{iyu/\sqrt{s^2\varepsilon^2}} - 1 - iyu/\sqrt{s^2\varepsilon^2}) d\nu(u).$$

Note that the cumulants of $\tilde{X}_t(\varepsilon)$ are such that $\mu_1(\varepsilon) = 0$, $\mu_2(\varepsilon) = 1$ and for all $k \geq 3$

$$|\mu_k(\varepsilon)| \leq \left[\frac{2c_-}{k-\beta} \varepsilon^{k-\beta} \frac{2c_+}{k-\beta} \varepsilon^{k-\beta}\right].$$

In the sequel we show that Assumption ($H(\Psi_{\varepsilon})$) holds, under the assumption

$$s^2 t/\varepsilon^2 \geq \tilde{C} \log(n), \quad (46)$$

where $\tilde{C} = \tilde{c}^{-2} > 0$ is a large enough constant, i.e. $a := \frac{\varepsilon}{s \sqrt{t}} \leq \frac{\tilde{c}}{\sqrt{\log(n)}}$.

A.5.1 Preliminary technical Lemmas

Lemma 15. There exists $c_\beta > 0$ a constant that depends only on $\beta, c_+, c_-$ such that the following holds

$$\text{Re}(\psi(y)) \leq -c_\beta y^2 1_{(y \leq \sqrt{s^2\varepsilon})} - c_\beta \frac{ty\beta}{\sqrt{s^2\varepsilon^2}} 1_{(y \geq \sqrt{s^2\varepsilon})} - \frac{\Sigma^2}{2s^2} y^2 1_{(y \geq \sqrt{s^2\varepsilon})}, \quad y > 0,$$

where Re(y) denotes the real part of y.

Proof of Lemma 15. It holds

$$\psi(y) = -y^2 \frac{\Sigma^2}{2s^2} + t \int_{-\varepsilon}^{\varepsilon} (e^{iyu/\sqrt{s^2\varepsilon^2}} - 1 - iyu/\sqrt{s^2\varepsilon^2}) d\nu(u) = -y^2 \frac{\Sigma^2}{2s^2} + I.$$

We now focus on the study of $I$. Doing the change of variable $v = yu/\sqrt{s^2}$ we get

$$I = t \sqrt{s^2} \int_{-\varepsilon}^{\varepsilon} (e^{iv} - 1 - iv) \frac{d\nu(v/\sqrt{s^2}/y)}{y}.$$
If \( y \varepsilon \leq \sqrt{ts^2} \), for \( 0 \leq v \leq 1 \), there exists an absolute constant \( c > 0 \) such that \( \text{Re}(e^{iv} - 1 - iv) = \cos(v) - 1 \leq -c\varepsilon^2 \). In particular,

\[
\text{Re}(I) \leq 2t\sqrt{ts^2}c_-y^\beta \int_0^{\frac{\sqrt{ts^2}}{v}} \frac{-cv^2}{(v\sqrt{ts^2})^{\beta+1}} dv
\]

\[
= -2tcc_- \frac{y^\beta}{\sqrt{ts^2}} \int_0^{\frac{\sqrt{ts^2}}{v}} v^{1-\beta} dv = -\frac{2cc_-}{2 - \beta} y^2 \frac{\varepsilon^2 - \beta}{s^2}.
\]

Since \( \sigma^2(\varepsilon) \in [2c_- \frac{\varepsilon^2 - \beta}{s^2}, 2c_+ \frac{\varepsilon^2 - \beta}{s^2}] \), whenever \( y \varepsilon \leq \sqrt{ts^2} \) we have

\[
\text{Re}(\psi(y)) \leq -\frac{cc_-}{c_+} \sigma^2(\varepsilon) y^2 - \frac{\Sigma^2}{s^2} y^2 \leq -\frac{cc_-}{c_+} \leq 1 y^2,
\]

using \( s^2 = \sigma^2(\varepsilon) + \Sigma^2 \).

If \( y \varepsilon > \sqrt{ts^2} \), then \( \text{Re}(e^{iv} - 1 - iv) = \cos(v) - 1 \leq 0 \) for any \( v \in \mathbb{R} \). Therefore,

\[
\text{Re}(I) \leq 2tcc_- \frac{y^\beta}{\sqrt{ts^2}} \int_0^1 \frac{\cos(v) - 1}{v^{\beta+1}} dv \leq 2tcc_- \frac{y^\beta}{\sqrt{ts^2}} \int_0^1 \frac{\cos(v) - 1}{v^{\beta+1}} dv
\]

\[
\leq -2tcc_- \frac{y^\beta}{\sqrt{ts^2}} \int_0^1 v^{1-\beta} dv = -\frac{2tcc_-}{2 - \beta} \frac{y^\beta}{\sqrt{ts^2}},
\]

where we used the previous bound on \( \cos(v) \). It follows that whenever \( y \varepsilon \geq \sqrt{ts^2} \)

\[
\text{Re}(\psi(y)) \leq -y^2 \frac{\Sigma^2}{2s^2} - 2t \frac{cc_-}{2 - \beta} \frac{y^\beta}{\sqrt{ts^2}}.
\]

Putting together the cases \( \{y \varepsilon > \sqrt{ts^2}\} \) and \( \{y \varepsilon \leq \sqrt{ts^2}\} \) completes the proof. \( \square \)

**Lemma 16.** There exists \( c_\beta > 0 \) a constant that depends only on \( \beta, c_+, c_- \) such that the following holds for \( y > 0 \)

\[
|\psi^{(1)}(y)| \leq c_\beta y^1_{\{y \varepsilon \leq \sqrt{ts^2}\}} + c_\beta \frac{y^{\beta-1}}{\sqrt{ts^2}} 1_{\{y \varepsilon > \sqrt{ts^2}\}} + \frac{\Sigma^2}{s^2} y^1_{\{y \varepsilon > \sqrt{ts^2}\}}.
\]

**Proof of Lemma 16.** First, it holds

\[
\psi^{(1)}(y) = -\frac{\Sigma^2}{s^2} y + t \int_{-\varepsilon}^{\varepsilon} \frac{iu}{\sqrt{ts^2}} (e^{iu/\sqrt{ts^2}} - 1) dv(u).
\]

The proof is similar to that of Lemma 15 replacing \( e^{iu} - 1 - iv \) with \( iue^{iyv} - iv \) which is of order \( yv^2 \) close to 0. This leads to the bound

\[
|\psi^{(1)}(y)| \leq \frac{\Sigma^2}{s^2} + c_\beta \frac{y^{\beta-1}}{s^2} 1_{\{y \varepsilon \leq \sqrt{ts^2}\}} + c_\beta \frac{y^{\beta-1}}{\sqrt{ts^2}} 1_{\{y \varepsilon > \sqrt{ts^2}\}}.
\]

\( \square \)

**Lemma 17.** For any integer \( d \geq 2 \) we have

\[
|\psi^{(d)}(y)| \leq c_\beta \left( \frac{\varepsilon}{\sqrt{ts^2}} \right)^{d-2} + \frac{\Sigma^2}{s^2} 1_{\{d=2\}} \leq c_\beta \left( \frac{\varepsilon}{\sqrt{ts^2}} \right)^{d-2} := c_\beta a^{d-2},
\]

where \( c_\beta > 0 \) is a constant that depends only on \( \beta, c_+, c_- \).
Proof of Lemma 17. We begin by observing that, for any \( m \geq 2 \), it holds
\[
\psi^{(m)}(y) = \frac{t}{\sqrt{ts^2 + 1}} \int_{-\varepsilon}^\infty e^{iuy/\sqrt{ts^2}} (ui)^m \nu(u) \, du.
\]
The proof directly follows by performing the change of variable \( v = yu/\sqrt{ts^2} \) and using that \( \beta < 2 \). \qed

The quantities \( c_{\text{int}} \) and \( K \) defined as in Section 3.1.1 appear in Lemmas 18 and 19 and in Section A.5.2.

Lemma 18. Assume that there exists \( C \geq c_{\text{int}} \) such that \( \frac{\varepsilon^2}{\varepsilon^2} \geq C^2 \log(n) \). Then, there exists \( C_\beta > 0 \) that depends only on \( \beta, c_+, c_- \) such that the following holds. For all \( m \leq K := c_{\text{int}}^2 \log(n) \),
\[
\left| \frac{(\exp(\psi))^{(m)}}{\exp(\psi)} \right| \leq C_\beta f^m,
\]
where
\[
f(y) = (c_\beta \vee 1)\sqrt{ts^2} \varepsilon^{-1} 1_{\{y \leq \sqrt{ts^2} \leq 1\}} + (c_\beta \vee 1) t \frac{y^{\beta-1}}{\sqrt{ts^2}} \varepsilon 1_{\{y > \sqrt{ts^2} \}} + \varepsilon \sum_{2} \sqrt{s} y 1_{\{y > \sqrt{ts^2} \}},
\]
and \( c_\beta \) is defined in Lemma 16.

Proof of Lemma 18. In accordance with the previous notation, we set \( a := \frac{\varepsilon^2}{\varepsilon^2} \) and we observe that \( K a^2 = c_{\text{int}}^2 \log n \frac{\varepsilon^2}{\varepsilon^2} \leq c_{\text{int}}^2 \frac{\varepsilon^2}{\varepsilon^2} \leq 1 \). The proof is similar to the proof of Lemma 8 considering instead the induction hypothesis \( H(m) : \forall d \in \mathbb{N}, |R_{m(d)}| \leq (4(c_\beta \vee 1))^m f^m (1 + ma)^d \). Assumption \( H(1) \) holds since \( R_1 = \psi^{(1)} \) and Lemma 17 gives \( |\psi^{(d)}| \leq c_\beta d^{\beta-2} \) for \( d \geq 2 \). To show that \( H(m) \) implies \( H(m+1) \) we use Lemma 17 joined with \( 0 \leq f^{-1} \leq a \) and \( ma^2 \leq 1 \). \qed

Lemma 19. Set \( y_{\text{min}} = c \log(n) \), \( a = \frac{\varepsilon^2}{\varepsilon^2} \) and assume that \( 1 \leq c_{\text{int}} \leq c \). For any \( m \leq K \), and any \( y \in [y_{\text{min}} \wedge a^{-1}, a^{-1}] \) there exists a constant \( C_\beta > 0 \), depending only on \( \beta, c_+, c_- \), such that
\[
\left| \frac{(\exp(\psi))^{(m)}}{\exp(\psi)} \right| \leq C_\beta \tilde{f}^m,
\]
where \( \tilde{f}(y) = (c_\beta \vee 1) t \) and \( c_\beta \) is defined as in Lemma 16.

Proof of Lemma 19. The proof is similar to the proof of Lemma 8 considering instead the induction hypothesis, for \( C_\beta \geq 4 \) and \( y \in [y_{\text{min}} \wedge a^{-1}, a^{-1}] \), \( H(m) : \forall d \in \mathbb{N}, |R_{m(d)}^d| \leq C_\beta^m f^m (1 + m)^d \). The result holds since for all \( m \leq K \), we have by assumption that \( (1 + m)^d \leq 1 + K \leq 2y_{\text{min}} \). \qed

A.5.2 Proof Proposition 2

For any integer \( k \geq 1 \) we write
\[
\int_{c \log(n)}^\infty |\Psi^{(k)}(y)|^2 dy = \int_{c \log(n)}^\infty |\Psi^{(k)}(y)|^2 1_{\left(\frac{1}{y}, \infty\right)}(y) dy + \int_{c \log(n)}^\infty |\Psi^{(k)}(y)|^2 1_{\left(0, \frac{1}{y}\right)}(y) dy =: T_1 + T_2.
\]
Control of $T_1$. First, on the interval $[\frac{1}{2}, \infty)$, by Lemma 18 there exists a constant $\bar{c}_\beta$, that depends only on $\beta, c_+, c_-$, such that

$$f(y) \leq \bar{c}_\beta \left( t \frac{y^{\beta - 1}}{\sqrt{ls^{2\beta}}} + \frac{y^{2\Sigma^2}}{s^2} \right).$$

Moreover, by Lemma 15, there exists a constant $\bar{c}_\beta' > 0$, that depends only on $\beta, c_+, c_-$, such that

$$|\exp(\psi(y))| \leq \exp \left( -2\bar{c}_\beta' \left( \frac{y^{2\Sigma^2}}{s^2} \right) \vee \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right) \right), \quad ya > 1,$$

and

$$|\exp(\psi(y))| \leq \exp(-2\bar{c}_\beta'/a^2), \quad ya > 1.$$

Therefore, by Equation (46)

$$|\exp(\psi(y))| \leq n^{-\kappa(C)} \bar{c}_\beta := n^{-\kappa(C)}, \quad ya > 1. \quad (47)$$

Using the previous inequalities, it follows from Lemma 18 that there exists $C_\beta > 0$ depending only on $\beta, c_+, c_-$ such that

$$\int_{(1/a)\vee(c\log(n))}^{+\infty} |\psi^{(m)}(y)|^2 dy \leq \exp(-\bar{c}_\beta'/a^2) \int_{1/a}^{+\infty} C_\beta^{2m} \bar{c}_\beta^{2m} \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} + \frac{y^{2\Sigma^2}}{s^2} \right)^{2m} \exp \left( -\bar{c}_\beta' \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} + \frac{y^{2\Sigma^2}}{s^2} \right) \right) dy$$

$$\leq \exp(-\bar{c}_\beta'/a^2)(C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right)^{2m} \exp \left( -\bar{c}_\beta' \left( \frac{y^{2\Sigma^2}}{s^2} \right) \vee \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right) \right) dy$$

$$+ \exp(-\bar{c}_\beta'/a^2)(C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left( \frac{y^{2\Sigma^2}}{s^2} \right)^{2m} \exp \left( -\bar{c}_\beta' \left( \frac{y^{2\Sigma^2}}{s^2} \right) \vee \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right) \right) dy$$

$$\leq A_1 + A_2.$$

We first consider the term $A_1$. By definition of $A_1$, we immediately get

$$A_1 \leq \exp(-\bar{c}_\beta'/a^2)(C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right)^{2m} \exp \left( -\bar{c}_\beta' \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right) \right) dy.$$

Using $\beta < 2$ and (46) which implies $1/a \geq \sqrt{C}\sqrt{\log(n)} > 1$, we obtain

$$A_1 \leq \exp(-\bar{c}_\beta'/a^2)(C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right)^{2m} \exp \left( -\bar{c}_\beta' \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right) \right) dy$$

$$\leq \exp(-\bar{c}_\beta'/a^2)(C_\beta \bar{c}_\beta)^{2m} \int_{1/a}^{+\infty} \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right)^{2m} \exp \left( -\bar{c}_\beta' \left( \frac{ty^\beta}{\sqrt{ls^{2\beta}}} \right) \right) dy.$$

We then consider the change of variable $v = ty^\beta/\sqrt{ls^{2\beta}}$ and use Equation (47) to get

$$A_1 \leq n^{-\kappa(C)}(C_\beta \bar{c}_\beta)^{2m} \sqrt{\frac{ls^2}{\beta^4}} \int_{1/e^0}^{+\infty} v^{m+1/\beta-1} \exp(-\bar{c}_\beta' v) dv.$$

Finally,

$$A_1 \leq n^{-\kappa(C)} \sqrt{\frac{ls^2}{\beta^4}} (C_\beta \bar{c}_\beta)^{2m} (\bar{c}_\beta')^{-m+1/\beta} \Gamma(m+1/\beta).$$
It follows that, for $C$ large enough and depending only on $c_{\text{int}}, \beta, c_+, c_-, c_{\text{max}}$ and for any $m \leq K = c_{\text{int}}^2 \log(n)$, $n\!^{c_{\text{max}}} \geq 1$ and $\log(s)/\log(n) \leq c_{\text{max}}$, we have $A_1 \leq n^{-4}m!$. Let us now consider the term $A_2$. Note that if $\Sigma = 0$ then $A_2 = 0$; in the sequel we assume that $\Sigma \neq 0$. It holds

$$A_2 \leq \left[ \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \int_{s^2/\Sigma^2 \vee 1/a}^{+\infty} \left( \frac{y\Sigma^2}{s^2} \right)^{2m} \exp \left( -\left( \frac{y^2\Sigma^2}{s^2} \right) \right)dy \right]$$

$$+ \left[ \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \int_{1/a}^{s^2/\Sigma^2 \vee 1/a} \left( \frac{y\Sigma^2}{s^2} \right)^{2m} \exp \left( -\left( \frac{y^2\Sigma^2}{s^2} \right) \right)dy \right] = A_{2,1} + A_{2,2}.$$ 

We first consider the term $A_{2,1}$. Using (34), we have

$$A_{2,1} \leq \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \left( \frac{\Sigma^2}{s^2} \right)^{2m} \int_{s^2/\Sigma^2}^{+\infty} y^{2m} \exp \left( -\frac{y^2\Sigma^2}{s^2} \right)dy$$

$$\leq \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \left( \frac{\Sigma^2}{s^2} \right)^{2m} \int_{0}^{s^2/\Sigma^2} y^{2m} \exp \left( -\frac{y^2\Sigma^2}{s^2} \right)dy$$

$$\leq \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \left( \frac{\Sigma^2}{s^2} \right)^{2m} \exp \left( -\tau'_\beta \frac{s^2}{\Sigma^2} \right) \times 2\sqrt{2\pi} \frac{2m!}{\left( \frac{s^2}{\Sigma^2} \right)^{m+\frac{1}{2}}}$$

$$\leq \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \left( \frac{\Sigma^2}{s^2} \right)^{1/2} \exp \left( -\tau'_\beta \frac{s^2}{\Sigma^2} \right) \times \frac{10}{\sqrt{\tau'_\beta}} \left( \frac{2\sqrt{C_{\beta}^2\varphi}}{\tau'_\beta} \right)^m m!.$$ 

Then, from (47) and for a constant $C$ large enough depending only on $\beta, c_+, c_-, c_{\text{int}}$, it holds for any $m \leq K = c_{\text{int}}^2 \log(n)$ that $A_{2,1} \leq n^{-4}m!$. Now, we consider the second term,

$$A_{2,2} = \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \left( \frac{\Sigma^2}{s^2} \right)^{2m} \int_{1/a}^{s^2/\Sigma^2 \vee 1/a} y^{2m} \exp \left( -\frac{ty^2}{\sqrt{s^2}} \right)dy$$

$$\leq \exp(-\tau'_\beta/a^2)(C_{\beta}^2\varphi)2m \int_{1/a}^{s^2/\Sigma^2 \vee 1/a} \exp \left( -\frac{ty^2}{\sqrt{s^2}} \right)dy.$$ 

We apply the change of variable $v = ty^2/\sqrt{s^2}$

$$A_{2,2} \leq \exp(-\tau'_\beta/a^2) \frac{\sqrt{s^2}}{\beta^2 t^{1/\beta}}(C_{\beta}^2\varphi)2m \left( \frac{s^2}{\Sigma^2} \right)^{1/\beta} \int_{t^{-1/\beta}}^{+\infty} \left( \frac{\sqrt{s^2}}{t} v \right)^{1/\beta - 1} \exp(-\tau'_\beta v)dv$$

$$\leq \exp(-\tau'_\beta/a^2) \frac{1}{\beta^2} (C_{\beta}^2\varphi)2m \frac{\sqrt{s^2}}{t^{1/\beta}}(\tau'_\beta)^{-1/\beta + 1} \Gamma(1/\beta).$$ 

So for any $m \leq K = c_{\text{int}}^2 \log(n)$, $n\!^{c_{\text{max}}} \geq 1$ and $\log(s)/\log(n) \leq c_{\text{max}}$ we have if $C$ is large enough depending only on $c_{\text{int}}, \beta, c_+, c_-, c_{\text{max}}$ (see (47)) that $A_{2,2} \leq n^{-4}m!$. 

Gathering both bounds on $A_{2,1}, A_{2,2}$, for any $m \leq K = c_{\text{int}}^2 \log(n)$ we have, if $C$ is large enough depending only on $c_{\text{int}}, \beta, c_+, c_-, c_{\text{max}}$, that $A_2 = A_{2,1} + A_{2,2} \leq 2n^{-4}m!$. Finally, gathering all terms, we derive that $(\mathcal{H}(\psi(x)))$ holds on the set $[1/a, \infty)$:

$$\int_{(1/a)\vee(c \log(n))}^{+\infty} |\psi^{(m)}(y)|^2 dy \leq A_1 + A_2 \leq 3n^{-4}m!, \quad m \leq K = c_{\text{int}}^2 \log(n).$$
Control of $T_2$. Whenever $1/a \leq y_{\min} = \log(n)$, $T_2 = 0$. In what follows, assume that $1/a > y_{\min}$.

By definition of $\tilde{f}$ in Lemma 19 there exists $c_\beta > 0$, depending only on $\beta, c_+, c_-$, such that $\tilde{f}(y) \leq \tau_\beta y$. Moreover, Lemma 15 implies that there exists $c'_\beta > 0$, depending only on $\beta, c_+, c_-$, such that $|\exp(\psi(y))| \leq \exp(-2c'_\beta y_{\min}^2)$, $y \in [y_{\min}, 1/a]$. Then, by (46)

$$|\exp(\psi(y))| \leq n^{-c'\beta} := n^{-\kappa(c)}, \quad y \in [y_{\min}, 1/a].$$

(48)

This, together with Lemma 19, implies that there exists a constant $C_\beta > 0$ that depends only on $\beta, c_+, c_-$ such that

$$\int_{c\log(n)}^{1/a} |\Psi^{(m)}(y)|^2 dy \leq C_\beta^{2m} \exp(-C'_\beta y_{\min}^2) \int_{c\log(n)}^{1/a} (\tau_\beta y)^{2m} \exp(-\tau_\beta y^2) dy.$$

Using (34) and (48) we get

$$\int_{c\log(n)}^{1/a} |\Psi^{(m)}(y)|^2 dy \leq 2\sqrt{2\pi} \exp(-C'_\beta y_{\min}^2) C_\beta^{2m} (2\tau_\beta)^{2m} 2^m m! (\tau'_\beta)^{-m+1/2}$$

$$\leq 10n^{-\kappa(c)} m! \sqrt{\tau'_\beta} (4C_\beta \tau_\beta / \sqrt{\tau'_\beta})^{2m}.$$

We conclude taking $c \geq 0$ a large enough constant depending only on $\beta, c_+, c_-, c_{\text{int}}$ (see (48))

$$\int_{c\log(n)}^{1/a} |\Psi^{(m)}(y)|^2 dy \leq n^{-4} m!, \quad \forall m \leq K = c_{\text{int}}^2 \log(n).$$

Conclusion. Putting both parts of the integral together we have

$$\int_{c\log(n)}^{+\infty} |\Psi^{(m)}(y)|^2 dy \leq 3n^{-4} m!, \quad \forall m \leq K = c_{\text{int}}^2 \log(n).$$

This concludes the proof of Proposition 2.