Abstract

In this paper, we prove absence of temperature chaos for the two-dimensional discrete Gaussian free field using the convergence of the full extremal process, which has been obtained recently by Biskup and Louidor. This means that the overlap of two points chosen under Gibbs measures at different temperatures has a nontrivial distribution. Whereas this distribution is the same as for the random energy model when the two points are sampled at the same temperature, we point out here that they are different when temperatures are distinct: more precisely, we prove that the mean overlap of two points chosen under Gibbs measures at different temperatures for the DGFF is strictly smaller than the REM’s one. Therefore, although neither of these models exhibits temperature chaos, one could say that the DGFF is more chaotic in temperature than the REM.

1 Introduction

1.1 Spin glasses and chaos phenomenon

The phenomenon of chaos in temperature or disorder is a classical problem in spin glasses, which was discovered by Fisher and Huse [30] for the Edwards-Anderson model, and Bray and Moore [20] for the Sherrington-Kirkpatrick model. It arose from the discovery that, for some models, a small change in the external parameters (such as temperature or disorder) can induce a dramatic change of the overall energy landscape or may modify the location of the ground state and the organization of the pure states for the Gibbs measure. In the last decades, this phenomenon has been studied extensively by physicists for various models, see Rizzo [42] for a recent survey. Several mathematical results have also been obtained by Chen and Panchenko [23], Chen [22], Auffinger and Chen [5], Panchenko [40], Subag [44], Chen and Panchenko [24] and Ben Arous, Subag and Zeitouni [6] in recent years.

This paper concerns the problem of chaos in temperature, which can be described more precisely as follows. The overlap between two configurations is defined as the normalized covariance between energies of these configurations. Temperature chaos means that, if one samples independently two spin configurations from Gibbs measures at different temperatures but with fixed disorder, then their overlap will be almost deterministic. This phenomenon happens when the set of most likely configurations under the Gibbs measure changes radically when the temperature varies only slightly: in spin glass models, the Gibbs measure at a given temperature
is concentrated near some fixed level of energy and on some pure states and both of them can change with temperature.

Some spin glass models display temperature chaos and some others do not. For example, Subag [44] proved recently the absence of chaos in temperature for spherical pure $p$-spin models with $p \geq 3$, while Ben Arous, Subag and Zeitouni [6] proved chaos in temperature for some spherical mixed $p$-spin models. Both results rely on a precise geometrical description of the Gibbs measure and hold at low temperature ($\beta$ large enough), in part of the one-step replica symmetry breaking (1-RSB) regime. For the spherical pure $p$-spin models, the supports of the Gibbs measures are close to each other for different (low enough) temperatures, while, for spherical mixed $p$-spin models, the Gibbs measure concentrates on thin spherical bands which depend on the temperature. This difference explains partially why chaos in temperature occurs for the second class of models but not for the first one.

In order to shed some light on the mysteries of the Parisi theory for mean field spin glasses, Derrida introduced in the 80’s the random energy model (REM) [27], where the Gaussian energy levels are assumed to be independent, and its generalization, the generalized random energy model (GREM) [28], whose correlations are given by a tree structure of finite depth. These tractable models have been extensively studied and allowed, in particular, to investigate the phenomenon of replica symmetry breaking. We refer to Bolthausen [12] and Kistler [34] for connection to spin glass theory. Let us mention here that Kurkova [35] proved the absence of chaos in temperature for the REM; this will be made more precise later.

Natural hierarchical models with an infinite number of levels are the branching Brownian motion (BBM) and the branching random walk (BRW), see e.g. the seminal paper by Derrida and Spohn [29], who introduced directed polymers on trees (a BRW with i.i.d. displacements) as an infinite hierarchical extension of the GREM for spin glasses. Physicists suggested that Gaussian BRW and BBM should belong to a universality class called log-correlated Gaussian fields. We refer to the works by Carpentier et Le Doussal [21], Fyodorov and Bouchaud [31, 32] and Fyodorov, Le Doussal and Rosso [33] for connection with spin glass theory. Such a model is the two-dimensional discrete Gaussian free field that we study in this paper and will be described precisely in the next subsection. It appears that this model has an implicit hierarchical structure similar to BRW. The lecture notes of Biskup [7] give a general and excellent account of recent results about two-dimensional discrete Gaussian free field.

In this paper, we prove absence of temperature chaos for the two-dimensional discrete Gaussian free field using the convergence of the full extremal process recently proved by Biskup and Louidor [9]. We also show that the mean overlap between two points sampled independently according to Gibbs measures at different temperatures is strictly smaller than the REM’s one, which might be surprising since the overlap distribution is the same for both models if the two points are sampled at the same temperature.

1.2 The two-dimensional discrete Gaussian free field

We consider in this paper the two-dimensional discrete Gaussian free field (DGFF) on a lattice approximation of a domain $D \subset \mathbb{R}^2$. More precisely, let $D$ be a bounded open set of $\mathbb{R}^2$ such that its boundary $\partial D$ has only a finite number of connected components, each of which has a positive diameter and is a $C^1$ path (i.e. the range of a $C^1$ function from $[0, 1]$ to $\mathbb{R}^2$). We denote by $\text{dist}_{\infty}$ the $\ell_\infty$-distance on $\mathbb{Z}^2$. Let $(D_N)_{N \geq 1}$ be the sequence of subsets of $\mathbb{Z}^2$ defined by

$$D_N := \left\{ x \in \mathbb{Z}^2 : \text{dist}_{\infty}\left( x, D^c \right) > \frac{1}{N} \right\}.$$  

These assumptions are slightly more restrictive than those in the recent papers by Biskup and Louidor [11, 10, 9], but only to avoid some technical issues due to boundary effects.
The discrete Gaussian free field on $D_N$ is the Gaussian process $(h^N_x)_{x \in D_N}$ with covariance given by the Green function $G_N$ of the simple random walk on $\mathbb{Z}^2$ killed upon exiting $D_N$. See (A.1) for a detailed definition of $G_N$. In the sequel, we will skip the dependence in $N$ for the field $h^N$, denoting simply $(h_x)_{x \in D_N}$. In comparison with spin glass theory, $D_N$ plays the role of the configurations set and $-h_x$ is the energy of configuration $x$. The Gibbs measure at inverse temperature $\beta > 0$ associated with the DGFF is defined by

$$G_{\beta,N} := \frac{1}{Z_{\beta,N}} \sum_{x \in D_N} e^{\beta h_x} \delta_x,$$

where $Z_{\beta,N} := \sum_{x \in D_N} e^{\beta h_x}$ is the partition function. Moreover, the overlap between $x, y \in D_N$ can be defined by

$$q_N(x, y) := \frac{\mathbb{E}[h_x h_y]}{\max_{z \in D_N} \mathbb{E}[h^2_z]} = \frac{G_N(x, y)}{\max_{z \in D_N} G_N(z, z)}.$$

Note that $q_N(x, y) \in [0, 1]$ and behaves roughly like $1 - \frac{\log |x-y|}{\log N}$ at least for points $x$ and $y$ far enough from the boundary $\partial D_N$ (see Lemma A.2). The DGFF displays a one-step replica symmetry breaking in spin glass terminology: asymptotically, the overlap between two points chosen independently according to $G_{\beta,N}$ is concentrated at 0 for $\beta \leq \beta_c := \sqrt{2\pi}$ (see Proposition 2.1), but is supported by 0 and 1 when $\beta > \beta_c$ (see [4] for the DGFF on the square lattice). Our focus in this paper is the behavior of the overlap between points chosen according to Gibbs measures at different temperatures, especially in the supercritical case where more interesting phenomena occur.

Recently, many progresses have been made in the study of the extremes of the DGFF. Concerning the maximum of the DGFF, it has been proved by Bolthausen, Deuschel and Zeitouni [13], Bramson and Zeitouni [19] and Bramson, Ding and Zeitouni [18], in the case of the square lattice, that $\max_{x \in D_N} h_x - m_N$ converges in distribution towards a nontrivial limit, where $m_N$ is an explicit sequence. Biskup and Louidor described the limit in [8] and extended the result to more general domains in [11]. Then, Biskup and Louidor [9] proved the convergence of the extremal process, which describes the field seen from position $m_N$. In particular, they show that extremal points are gathered in clusters of diameter $O(1)$ centered at each local maximum of height $m_N + O(1)$ and these local maxima are at a distance of order $N$ from each other. See Section 2.2 for a precise statement of their result.

Points contributing to the extremal process are also the points supporting supercritical Gibbs measures, hence the convergence of the extremal process leads to a precise asymptotic description of $G_{\beta,N}$ for $\beta > \beta_c$. In spin glass terminology, the pure states under $G_{\beta,N}$ are the clusters centered at local maxima of height $m_N + O(1)$: if two points are in the same pure state, then they have an overlap close to 1 and, if they are in two different pure states, their overlap is close to 0. This explains the one-step replica symmetry breaking behavior of the model. Moreover, the ordered weights of the pure states under $G_{\beta,N}$ follows asymptotically a Poisson–Dirichlet distribution of parameter $\beta_c/\beta$ denoted PD($\beta_c/\beta$), which is the law of $(e^{\beta \xi_k} / \sum_j e^{\beta \xi_j}, k \geq 1)$ where $\downarrow$ stands for the decreasing rearrangement and $(\xi_k)_{k \geq 1}$ are the atoms of a Poisson point process on $\mathbb{R}$ with intensity $e^{-\beta \xi} dh$, see Biskup and Louidor [9, Corollary 2.7] for a precise statement. We also mention that a weaker version of this Poisson–Dirichlet convergence for the overlap distribution has been previously obtained by Arguin and Zindy [3, 4].

Results concerning the DGFF are often compared to their counterpart for the random energy model. The REM is defined here, for each $N \geq 1$, as $(h^\text{REM}_x)_{x \in D_N}$, which are i.i.d. centered Gaussian random variables with variance $\max_{x \in D_N} G_N(x, x)$, in order to be comparable to the DGFF. The overlap between $x, y \in D_N$ is defined by

$$q^\text{REM}_N(x, y) := \frac{\mathbb{E}[h^\text{REM}_x h^\text{REM}_y]}{\max_{z \in D_N} G_N(z, z)} = \mathbb{1}_{\{x=y\}}.$$
and the REM exhibits also a one-step replica symmetry breaking behavior for $\beta > \beta_c$. Moreover, the supercritical Gibbs measure $\mathcal{G}_{\beta,N}^{\text{REM}}$ for $\beta > \beta_c$ is also carried by extremal points, which are uniformly distributed over $D_N$. Hence the pure states are here the singletons formed by extremal points and are at distance of order $N$ of each other. The pure states have also Gibbs weight following asymptotically the Poisson–Dirichlet distribution of parameter $\beta_c/\beta$ and therefore the overlap under $(\mathcal{G}_{\beta,N}^{\text{REM}})^\otimes 2$ has asymptotically the same law as for the DGFF. See Figure 1. Note that results in that direction have also been obtained by Bovier and Kurkova [16, 17] for more elaborate variants of the REM; the GREM and the CREM.

### 1.3 Overlap distribution at two different temperatures

In the previous subsection, we saw that the pure states of the supercritical Gibbs measure of the DGFF are points with height $m_N + O(1)$. In particular, they do not depend on the temperature and, for $\beta, \beta' > \beta_c$, two points chosen independently according to $\mathcal{G}_{\beta,N}$ and $\mathcal{G}_{\beta',N}$ have a positive probability to be in the same pure state, but also to be in different pure states: therefore, their overlap can be either 0 or 1 and there is clearly no temperature chaos for the DGFF. This is made more precise in our first result, stating the convergence of the overlap under $\mathcal{G}_{\beta,N} \otimes \mathcal{G}_{\beta',N}$.

**Theorem 1.1.** Let $\beta, \beta' > 0$.

(i) If $\beta \leq \beta_c$ or $\beta' \leq \beta_c$, then, for all $a \in (0,1)$,

$$
\mathcal{G}_{\beta,N} \otimes \mathcal{G}_{\beta',N}(q_N(u,v) \geq a) \xrightarrow{N \to \infty} 0, \quad \text{in } L^1.
$$

(ii) If $\beta > \beta_c$ and $\beta' > \beta_c$, then, for all $a \in (0,1)$,

$$
\mathcal{G}_{\beta,N} \otimes \mathcal{G}_{\beta',N}(q_N(u,v) \geq a) \xrightarrow{N \to \infty} Q(\beta, \beta'), \quad \text{in distribution,}
$$

with

$$
Q(\beta, \beta') := \frac{\sum_{k \geq 1} e^{\beta(\xi_k + X_{\beta,k})} e^{\beta'(\xi_k + X_{\beta',k})}}{\left(\sum_{k \geq 1} e^{\beta(\xi_k + X_{\beta,k})}\right) \left(\sum_{k \geq 1} e^{\beta'(\xi_k + X_{\beta',k})}\right)}.
$$
where \((X_{\beta,k}, X_{\beta',k})_{k \geq 1}\) are independent copies of a couple of real random variables \((X_{\beta}, X_{\beta'})\) defined in (2.4) and \((\xi_k)_{k \geq 1}\) are the atoms of a Poisson point process with intensity \(e^{-\beta_k h} dh\) independent of \((X_{\beta,k}, X_{\beta',k})_{k \geq 1}\). Moreover, \(\mathbb{E}[e^{\beta_k X_{\beta}}] < \infty\) and \(\mathbb{E}[e^{\beta_k X_{\beta'}}] < \infty\).

In other words, this result proves the convergence of the pushforward of the measure \(G_{\beta,N} \otimes G_{\beta',N}\) on \(D_N^2\), by the function \(q_N\), which is a random measure on \([0,1]\). The limit is either \(\delta_0\) if \(\beta \wedge \beta' \leq \beta_c\), or \((1 - Q(\beta, \beta'))\delta_0 + Q(\beta, \beta')\delta_1\) otherwise. Part (ii) of this theorem is essentially a consequence of the convergence of the full extremal process obtained by Biskup and Louidor [9]: note that the \(e^{\beta(\xi_k + X_{\beta,k})}/(\sum_{j \geq 1} e^{\beta(\xi_j + X_{\beta,j})})\) for \(k \geq 1\) are the asymptotic weights of the clusters under \(G_{\beta,N}\), so \(Q(\beta, \beta')\) is simply the probability of choosing two points in the same cluster, when they are chosen proportionally to their Gibbs weights with inverse temperature \(\beta\) and \(\beta'\) respectively.

For the REM, Kurkova [35] proved the following result: if \(\beta, \beta' > \beta_c\), then, for any \(a \in (0,1)\), we have the following convergence in distribution

\[
\begin{align*}
G_{\beta,N}^{\text{REM}} \otimes G_{\beta',N}^{\text{REM}} (q_N^{\text{REM}}(u,v) \geq a) \xrightarrow[N \to \infty]{(d)} Q_{\text{REM}}(\beta, \beta') := \sum_{k \geq 1} \frac{e^{\beta_k + \beta' k}}{(\sum_{k \geq 1} e^{\beta_k})(\sum_{k \geq 1} e^{\beta' k})},
\end{align*}
\]

where the \((\xi_k)_{k \geq 1}\) are also the atoms of a Poisson point process with intensity \(e^{-\beta_k h} dh\). In the case \(\beta = \beta'\), it is known that \(Q(\beta, \beta)\) and \(Q_{\text{REM}}(\beta, \beta)\) have the same distribution: this follows from the simple fact that

\[
(\xi_k + X_{\beta,k})_{k \geq 1} \xrightarrow{(d)} \left(\xi_k + \beta_c^{-1} \log \mathbb{E}[e^{\beta_k X_{\beta}}]\right)_{k \geq 1},
\]

where the equality in distribution holds with both sides seen as point processes (see for example [14, Proposition 8.7]).

One may ask whether \(Q(\beta, \beta')\) and \(Q_{\text{REM}}(\beta, \beta')\) have the same distribution when \(\beta \neq \beta'\). The answer is negative and our second result shows that, in mean, the overlap is less likely to be close to 1 under \(G_{\beta,N} \otimes G_{\beta',N}\) that under \(G_{\beta,N}^{\text{REM}} \otimes G_{\beta',N}^{\text{REM}}\).

**Theorem 1.2.** For any \(\beta, \beta' > \beta_c\) such that \(\beta \neq \beta'\), we have

\[
\mathbb{E}[Q(\beta, \beta')] < \mathbb{E}[Q_{\text{REM}}(\beta, \beta')].
\]

The reason behind this inequality is the following. For the REM, the weight of a pure state depends only on its height: a likely pure state under \(G_{\beta,N}^{\text{REM}}\) will also be likely under \(G_{\beta',N}^{\text{REM}}\). Rather, the weight of a pure state for the DGFF depends both on the height of the local maximum and on the geometry of the cluster around it, hence some pure states are likely for \(\beta\) close to \(\beta_c\) and unlikely for large \(\beta\), or vice-versa. Therefore, it is more difficult to choose twice the same pure state under the Gibbs measures at two different temperatures for the DGFF than for the REM and one could say that the DGFF is more chaotic in temperature than the REM. This result leads to the picture displayed in Figure 2 for the mean overlap in both models.

**Remark 1.3.** Theorem 1.2 can be seen as the comparison of \(\mathbb{E}[(p, q)] = \mathbb{E}[(\sum_{i \geq 1} p_i q_i)]\), for two different couples \((p, q)\) with marginal laws \(PD(\beta_c/\beta)\) and \(PD(\beta_c/\beta')\). In the case of the REM, the couple is characterized by the relation \(q_i = p^{\beta/\beta'}_i/(\sum_{k \geq 1} p^{\beta/\beta'}_k)\), and one could ask if this particular choice reaches the minimum in the Wasserstein distance between the measures \(PD(\beta/\beta)\) and \(PD(\beta/\beta')\) on the Hilbert space \(\ell^2\) of square-summable sequences. The answer is negative by [25, Theorem 2.3] and the fact that \(\mathbb{P} \otimes \mathbb{P}(\{(\omega, \omega') : \langle p(\omega) - p(\omega'), q(\omega) - q(\omega') \rangle < 0\}) > 0\) in that case.

**Remark 1.4.** In the case where \(\beta < \infty\) and \(\beta' = \infty\), it is easier to see that the overlap distribution for the DGFF should be different from the REM’s one. We can define \(G_{\infty,N}\) as the measure
which gives mass 1 to the point where \( \max_{x \in D_N} h_x \) is reached. Then, assuming that \((\xi_k)_{k \geq 1}\) is ranked in the decreasing order, one can prove in this case that the limiting overlap distribution is given by
\[
Q(\beta, \infty) = \frac{e^{\beta(\xi_1 + X_{\beta,1})}}{\sum_{k \geq 1} e^{\beta(\xi_k + X_{\beta,k})}},
\]
which is asymptotically the probability that the point chosen according to \( G_{\beta,N} \) is in the same cluster as the highest particle. On the other hand, we have \( Q^{REM}(\beta, \infty) = e^{\beta \xi_1} / (\sum_{k \geq 1} e^{\beta \xi_k}) \).

Then, it follows from (1.2) that \( Q(\beta, \infty) \) is stochastically strictly dominated by \( Q^{REM}(\beta, \infty) \), because with positive probability \( \max_{k \geq 1} (\xi_k + X_{\beta,k}) \) is not reached at \( k = 1 \) (this is true as soon as \( X_\beta \) is not a.s. constant, which holds by the proof of Lemma 3.5).

\textit{Remark 1.5.} Similar results could be proved for the branching Brownian motion or the branching random walk. Indeed, using the convergence of the extremal process of the BBM \([1, 2, 15]\) or of the BRW \([37, 38]\), one can deduce Theorem 1.1.(ii) (see in particular the proof of Theorem 4.3 in \([38]\)). For Theorem 1.2, the method used here is quite general and works also for BBM and BRW, aside from Lemma 3.5 that should be adapted.

1.4 Organization of the paper

The paper is organized as follows. Theorem 1.1 is proved in Section 2 and Theorem 1.2 in Section 3. Appendix A contains well-known results concerning the two-dimensional Gaussian free field.

2 Convergence of the overlap distribution

2.1 Proof of Part (i) of Theorem 1.1

The proof of Part (i) of Theorem 1.1 will be a direct consequence of the following result, which relies on classical arguments in spin glass theory.

\textbf{Proposition 2.1.} If \( \beta \leq \beta_c \), then for any \( a \in (0, 1) \),
\[
G_{\beta,N}^{GFF}(q_N(u,v) \geq a) \xrightarrow{N \to \infty} 0, \quad \text{in } L^1.
\]

The key observation is that the free energy contains all information about the mean overlap under the measure \( E[G_{\beta,N}^{GFF}] \). The free energy is defined by \( f_N(\beta) := (\log Z_{\beta,N}) / (\log N^2) \), where we recall \( Z_{\beta,N} \) denotes the partition function. The free energy converges as \( N \to \infty \) and
\[
\lim_{N \to \infty} f_N(\beta) = f(\beta) := \begin{cases} 1 + (\beta / \beta_c)^2, & \text{if } \beta \leq \beta_c, \\ 2\beta / \beta_c, & \text{if } \beta \geq \beta_c, \end{cases} \quad \text{in } L^1. \quad (2.1)
\]
This convergence is a consequence of [26, Theorem 1.3] in the case of the square lattice (see [4] for details of the argument). For general domains, this is a consequence of [10, Theorem 2.1], which sharpens considerably the results of [26].

Proof. Since $\mathbb{E}[f_N(\beta)]$ is a convex function of $\beta$, by a standard result of convexity (see e.g. Proposition I.3.2 in [43]), at each point of differentiability of $f$, the limit of the derivatives equals the derivative of the limit. It follows from (2.1) that $f$ is differentiable at any $\beta > 0$, hence we get

$$f'(\beta) = \lim_{N \to \infty} \frac{d}{d\beta} \mathbb{E}[f_N(\beta)] = \lim_{N \to \infty} \frac{1}{\log N^2} \sum_{x \in D_N} \mathbb{E} \left[ \frac{h_x e^{\beta h_x}}{\sum_{z \in D_N} e^{\beta h_z}} \right].$$

Applying Gaussian integration by part (see Lemma A.1) with respect to the factor $h_x$, it follows that

$$f'(\beta) = \lim_{N \to \infty} \frac{\beta}{\log N^2} \left( \sum_{x \in D_N} \mathbb{E} \left[ h_x^2 \right] + \sum_{x \neq y \in D_N} \mathbb{E}[h_x h_y] \mathbb{E} \left[ \frac{e^{\beta h_x} e^{\beta h_y}}{\sum_{z \in D_N} e^{\beta h_z}} \right] \right).$$

In order to deal with the first sum, we introduce $D_{N,\delta} := \{ x \in D_N : d(x, D_N^c) > N^{1-\delta} \}$ for some small $\delta \in (0, 1)$. It follows from (A.2) and (A.3), that $\max_{x \in D_N} G_N(x, x) = \frac{\beta}{\pi} \log N + O(N)$ and $G_N(x, x) \geq \frac{2(1-\delta)}{\pi} \log N + O(N)$ uniformly in $x \in D_{N,\delta}$ as $N \to \infty$. On the other hand, we have $\mathbb{E} [G_{\beta,N}(D_{N,\delta}^c)] \to 0$ as $N \to \infty$ by Lemma A.3. Combining this and letting $\delta \to 0$, we get

$$f'(\beta) = \lim_{N \to \infty} \frac{\beta}{\frac{\beta}{\pi}} \left( 1 - \mathbb{E} \left[ G_{\beta,N}^2 \right] \right).$$

Moreover, it follows from (2.1) that $f'(\beta) = \beta/\pi$ for any $\beta \leq \beta_c$, and therefore $\mathbb{E} [G_{\beta,N}^2] \to 0$ as $N \to \infty$, which concludes the proof.

Proof of Part (i) of Theorem 1.1. By symmetry, we can assume that $\beta \leq \beta_c$. Moreover, by Lemma A.2.(i), it is enough to prove that $G_{\beta,N} \otimes G_{\beta',N}(\|u-v\| \leq N^{1-a}) \to 0$ as $N \to \infty$ in $L^1(\mathbb{P})$, for any $a \in (0, 1)$. For this, we consider a partition of $D_N$ by considering the intersection of $D_N$ with a partition of $\mathbb{R}^2$ with boxes of side-length $N^{1-a}$: let $B$ denote this partition of $D_N$. Then, we have the following upper bound

$$G_{\beta,N} \otimes G_{\beta',N}(\|u-v\| \leq N^{1-a}) \leq \sum_{B \in B} G_{\beta,N}(B) G_{\beta',N}(\{ y \in D_N : \exists x \in B : \|x-y\| \leq N^{1-a} \})$$

$$\leq \left( \max_{B \in B} G_{\beta,N}(B) \right) \sum_{B \in B} G_{\beta',N}(\{ y \in D_N : \exists x \in B : \|x-y\| \leq N^{1-a} \})$$

$$\leq 9 \left( \max_{B \in B} G_{\beta,N}(B) \right) \sum_{B \in B} G_{\beta',N}(B) = 9 \max_{B \in B} G_{\beta,N}(B),$$

noting that the factor 9, which appears in the second inequality, comes from the fact that if $x \in B$ and $\|x-y\| \leq N^{1-a}$, then $y$ belongs either to $B$ or to one of the 8 boxes neighbor to $B$, see Figure 3. On the other hand, for any $B \in B$, we have

$$\left( \max_{B \in B} G_{\beta,N}(B) \right)^2 = \max_{B \in B} G_{\beta,N}^2(B^2) \leq G_{\beta,N}^2(\|u-v\| \leq \sqrt{2} N^{1-a}) \frac{L^1(\mathbb{P})}{N \to \infty} 0,$$

where the convergence follows from Proposition 2.1 combined with Lemmas A.2.(ii) and A.3. This concludes the proof. \qed
2.2 Proof of Part (ii) of Theorem 1.1

Part (ii) of Theorem 1.1 is essentially a consequence of the convergence of the extremal process of the DGFF proved by Biskup and Louidor [9, Theorem 2.1], that we now state here. The full extremal process is defined as the following random measure on $D \times \mathbb{R} \times \mathbb{R}^{2}$:

$$
\eta_{N,r} := \sum_{x \in D_{N}} \mathbb{1}_{[h_{x} = \max_{y \in \Lambda_{r}(x)} h_{y}]} \delta_{x/N} \otimes \delta_{h_{x} - m_{N}} \otimes \delta_{(h_{x} - h_{x+\zeta})_{\zeta \in \mathbb{Z}^{2}}},
$$

where $\Lambda_{r}(x) := \{ y \in D_{N} : \| x - y \| \leq r \}$ and $m_{N} := \frac{1}{\sqrt{2\pi}} \log N - \frac{3}{2\sqrt{2\pi}} \log \log N$. It encodes the rescaled position of the local maxima, their centered value seen from position $m_{N}$ and the field seen from this local maximum. Then, there exist a random finite Borel measure $Z^{D}$ on $D$ and a probability measure $\nu$ on $(\mathbb{R}^{+})^{\mathbb{Z}^{2}}$ such that, for any sequence $(r_{N})_{N \geq 1}$ of positive real numbers with $r_{N} \to \infty$ and $N/r_{N} \to \infty$,

$$
\eta_{N,r_{N}} \xrightarrow{\text{law}} \eta := \text{PPP}\left(Z^{D}(dz) \otimes e^{-\beta_{c}h} dh \otimes \nu(d\phi)\right),
$$

in the sense that, for any continuous function $f : D \times (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}^{2} \to \mathbb{R}$ with compact support, $\eta_{N,r_{N}}(f) = \int f \, d\eta_{N,r_{N}}$ converges in distribution towards $\eta(f)$, where we set $\mathbb{R} := \mathbb{R} \cup \{-\infty, \infty\}$.

The random measure $Z^{D}$ has been studied in [11] and identified as the critical Liouville quantum gravity obtained by exponentiating the continuous GFF on $D$. Moreover, Biskup and Louidor [9] give an explicit description of the cluster law $\nu$, see Subsection 3.3 for more details.

As a consequence of the convergence of the full extremal process, Biskup and Louidor [9, Theorem 2.6] prove the convergence of the following random measure on $D$:

$$
\rho_{\beta,N} := \sum_{x \in D_{N}} e^{\beta(h_{x} - m_{N})} \delta_{x/N},
$$

for $\beta > \beta_{c}$. Their result is extended to the multi-dimensional case in the following proposition. For this, we introduce another family of random measures $(\rho_{\beta})_{\beta > \beta_{c}}$ defined as follows. Conditionally on $Z^{D}$, let $(\chi_{k})_{k \geq 1}$ be an i.i.d. sequence of random variables with distribution $Z^{D}/Z^{D}(D)$, $(\xi_{k})_{k \geq 1}$ a Poisson point process with intensity $e^{-\beta_{c}h} dh$ and $(\phi^{k})_{k \geq 1}$ be an i.i.d. sequence of random variables with distribution $\nu$ such that $(\chi_{k})_{k \geq 1}$, $(\xi_{k})_{k \geq 1}$ and $(\phi^{k})_{k \geq 1}$ are independent. Then, for $\beta > \beta_{c}$, we set

$$
\rho_{\beta} := Z^{D}(D)^{\beta/\beta_{c}} \sum_{k \geq 1} e^{\beta(\xi_{k} + X_{\beta,k})} \delta_{\chi_{k}}, \quad \text{with} \quad X_{\beta,k} := \frac{1}{\beta} \log \sum_{x \in \mathbb{Z}^{2}} e^{-\beta\phi^{k}_{x}},
$$

where $X_{\beta,k}$ is well-defined by [9, Theorem 2.6]. Note that, in [9, Theorem 2.6], the definition of the limiting measure is simplified by using (1.2) to get rid of the $X_{\beta,k}$’s, but this cannot be done simultaneously for several $\beta$’s.

Figure 3 – The set $D_{N}$ is partitionned using a square grid of span $N^{1-a}$. If $x$ is in a given box and $\| x - y \| \leq N^{1-a}$, then $y$ belongs to one of the hatched boxes.
Proposition 2.2. For any $p \geq 1$ and $\beta_1, \ldots, \beta_p > \beta_c$, we have
\[
\rho_{\beta_1, N} \otimes \cdots \otimes \rho_{\beta_p, N} \xrightarrow{N \to \infty} \rho_{\beta_1} \otimes \cdots \otimes \rho_{\beta_p}, \quad \text{in distribution,}
\]
on the space of Radon measures on $\mathcal{D}^p$ endowed with the vague convergence.

Proof. The argument of Biskup and Louidor [9, Theorem 2.6] works in the multi-dimensional case, by replacing the quantity $\sum_{x \in D_N} e^{h_x - m_N} f(x/N)$ by $\sum_{x \in D_N} \sum_{i=1}^p e^{h_i (x/N)} f_i(x/N)$ for some test functions $f_1, \ldots, f_p$. See the first arXiv version [39] of this paper for more details. $\square$

Before proving Part (ii) of Theorem 1.1, we recall some other known results on supercritical Gibbs measures. Let $\beta > \beta_c$. Firstly, it follows from Equations (2.23) and (6.58) of [9] that the supercritical Gibbs measure is mainly supported by extremal points: for any $\eta > 0$,
\[
\lim_{\ell \to \infty} \limsup_{N \to \infty} \mathbb{P}(\mathcal{G}_{\beta, N}(\{x \in D_N : h_x - m_N \notin [-\ell, \ell]\} > \eta)) = 0. \tag{2.5}
\]
Combining this with [9, Lemma B.11], we get that, for any $\beta, \beta' > \beta_c$,
\[
\lim_{r \to \infty} \limsup_{N \to \infty} \mathbb{P}(\mathcal{G}_{\beta, N} \otimes \mathcal{G}_{\beta', N}(r < \|u - v\| < N/r) > \eta) = 0, \tag{2.6}
\]
which means that two points sampled accordingly to two supercritical Gibbs measures are typically either very close to each other or very far.

Proof of Part (ii) of Theorem 1.1. First note that it follows from Lemmas A.2 and A.3 combined with (2.6) that
\[
\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \mathbb{P}(\mathcal{G}_{\beta, N} \otimes \mathcal{G}_{\beta', N}(q_N(u, v) \geq a) - \mathcal{G}_{\beta, N} \otimes \mathcal{G}_{\beta', N}(\|u - v\| \leq \varepsilon N) > \eta) = 0. \tag{2.7}
\]
For any $\varepsilon > 0$, let $f_{\varepsilon} : D^2 \to \mathbb{R}_+$ be a continuous function such that $1_{\{|z - z'| \leq \varepsilon\}} \leq f_{\varepsilon}(z, z') \leq 1_{\{|z - z'| \leq 2\varepsilon\}}$. Then, we have
\[
\frac{\rho_{\beta, N} \otimes \rho_{\beta', N}(f_{\varepsilon}/2)}{\rho_{\beta, N} \otimes \rho_{\beta', N}(D^2)} \leq \mathcal{G}_{\beta, N} \otimes \mathcal{G}_{\beta', N}(\|u - v\| \leq \varepsilon N) \leq \frac{\rho_{\beta, N} \otimes \rho_{\beta', N}(f_{\varepsilon})}{\rho_{\beta, N} \otimes \rho_{\beta', N}(D^2)}.
\]
Moreover, by Proposition 2.2, we get
\[
\frac{\rho_{\beta, N} \otimes \rho_{\beta', N}(f_{\varepsilon})}{\rho_{\beta, N} \otimes \rho_{\beta', N}(D^2)} \xrightarrow{N \to \infty} \frac{\rho_{\beta} \otimes \rho_{\beta'}(f_{\varepsilon})}{\rho_{\beta} \otimes \rho_{\beta'}(D^2)}, \quad \text{a.s.}
\]
\[
\frac{\rho_{\beta} \otimes \rho_{\beta'}(\{(z, z) : z \in D\})}{\rho_{\beta} \otimes \rho_{\beta'}(D^2)} \xrightarrow{\varepsilon \to 0} \frac{\rho_{\beta} \otimes \rho_{\beta'}(\{(z, z) : z \in D\})}{\rho_{\beta} \otimes \rho_{\beta'}(D^2)},
\]
which is equal to $Q(\beta, \beta')$, because $Z^D$ has no atom by [11, Theorem 2.1]. The fact that $\mathbb{E}[e^{h_{X, N}}] < \infty$ is proved in [9, Theorem 2.6]. $\square$

3 Comparison with the overlap for the REM

3.1 Structure of the proof of Theorem 1.2

In this subsection, we prove Theorem 1.2 up to some postponed lemmas. Therefore, we consider some $\beta, \beta' > \beta_c$ such that $\beta \neq \beta'$. In the case $\beta = \beta'$, we saw in the introduction that the classical argument to get rid of the $X_{\beta, k}$’s in $Q(\beta, \beta)$ is the fact that $(\xi_k + X_{\beta, k})_{k \geq 1}$ has the same
distribution as \((\xi_k + c_\beta)_{k \geq 1}\) where \(c_\beta\) is a constant (see (1.2)). Here, we have instead to work with the joint distribution \((\xi_k + X_{\beta,k}, \xi_k + X_{\beta',k})_{k \geq 1}\) for \(\beta \neq \beta'\). The following lemma shows that, if we apply the previous change of point process to the first coordinate, the random shift in the second coordinate is still independent of \((\xi_k)_{k \geq 1}\), although this can seem counter-intuitive at first sight. This result can be found for example in [41, Lemma 2.1].

**Lemma 3.1.** Let \(c_\beta := \beta_c^{-1} \log \mathbb{E}[e^{\beta X_\beta}]\) and \(Y\) be a random variable such that, for any measurable function \(f : \mathbb{R} \to \mathbb{R}_+\),

\[
\mathbb{E}[f(Y)] = \frac{\mathbb{E}[e^{\beta X_\beta} f(X_{\beta'} - X_\beta)\]}{\mathbb{E}[e^{\beta X_\beta}]},
\]

Then, the point process \((\xi_k + X_{\beta,k}, \xi_k + X_{\beta',k})_{k \geq 1}\) with values in \(\mathbb{R}^2\) has the same distribution as \((\xi_k + c_\beta, \xi_k + c_\beta + Y_k)_{k \geq 1}\), where \((Y_k)_{k \geq 1}\) are i.i.d. copies of \(Y\) independent of \((\xi_k)_{k \geq 1}\).

Now, our aim is to prove that the random perturbations \(Y_k\) on the second coordinate of \((\xi_k + c_\beta, \xi_k + c_\beta + Y_k)_{k \geq 1}\) play a negative role in maximizing \(Q(\beta, \beta')\). This is done in the following lemma, stated in a more general setting and shown in Subsection 3.2.

**Lemma 3.2.** Let \((p_n)_{n \geq 1}\) and \((q_n)_{n \geq 1}\) be nonincreasing deterministic sequences of nonnegative real numbers such that \(\sum_{n \geq 1} p_n = 1\). Let \((A_n)_{n \geq 1}\) be a sequence of i.i.d. positive random variables. We set

\[
\tilde{p}_n := \frac{A_n p_n}{\sum_{k \geq 1} A_k p_k}, \quad \forall n \geq 1.
\]

Then, we have

\[
\mathbb{E} \left[ \sum_{n \geq 1} \tilde{p}_n q_n \right] \leq \sum_{n \geq 1} p_n q_n.
\]

Moreover, if \(A_1\) is not almost surely constant, \((q_n)_{n \geq 1}\) is not constant and, for any \(n \geq 1, p_n > 0\), then the inequality in (3.2) is strict.

**Remark 3.3.** In order to maximize the inner product \(\sum_{n \geq 1} p_n q_n\), the sequences \((p_n)_{n \geq 1}\) and \((q_n)_{n \geq 1}\) have to be ordered in the same way, as in particular in the assumption of the lemma. The random perturbation on \((\tilde{p}_n)_{n \geq 1}\) may disturb this shared ordering and, hence, tends to reduce the inner product. However, note that, without the expectation, the inequality in (3.2) can be wrong with positive probability. Therefore, apart from the case where \(\beta\) or \(\beta'\) is infinite (see Remark 1.4), it is not clear whether \(Q(\beta, \beta')\) is stochastically dominated by \(Q_{\text{REM}}(\beta, \beta')\) or not.

**Remark 3.4.** One could also assume that \(\sum_{n \geq 1} q_n = 1\) and perturb the sequence \((q_n)_{n \geq 1}\), by defining \(\tilde{q}_n := B_n q_n / \sum_{k \geq 1} B_k q_k\) with \((A_n, B_n)_{n \geq 1}\) a sequence of i.i.d. random variables with values in \((0, \infty)^2\). But then, we do not necessarily have \(\mathbb{E}[\sum_{n \geq 1} \tilde{p}_n \tilde{q}_n] \leq \sum_{n \geq 1} p_n q_n\). For this reason, in order to prove Theorem 1.2, we first have to use Lemma 3.1, so that only one of the two sequences is perturbed (in comparison with the REM).

Looking at Lemma 3.2, one can observe that, in order to get a strict inequality in Theorem 1.2, we still need the following lemma, which is proved in Subsection 3.3.

**Lemma 3.5.** The random variable \(Y\) defined by (3.1) is not almost surely constant.

This result is the only step in the proof of Theorem 1.2 which is specific to the DGFF, because it depends on the law \(\nu\) of the decorations in the limit of the full extremal process. Now, with these three lemmas, we can proceed to the proof of Theorem 1.2.
Proof of Theorem 1.2. Applying Lemma 3.1, we get

\[
\mathbb{E}[Q(\beta, \beta')] = \mathbb{E} \left[ \frac{\sum_{n \geq 1} e^{\beta \xi_n} e^{\beta'(\xi_n + Y_n)}}{\left( \sum_{k \geq 1} e^{\beta_k} \right) \left( \sum_{k \geq 1} e^{\beta_k(\xi_k + Y_k)} \right)} \right].
\]

We can assume that the atoms \((\xi_k)_{k \geq 1}\) are ranked in decreasing order. Then, Lemma 3.2 with \(p_n := e^{\beta \xi_n}/(\sum_{k \geq 1} e^{\beta_k})\), \(q_n := e^{\beta' \xi_n}/(\sum_{k \geq 1} e^{\beta_k})\) and \(A_n := e^{\beta' Y_n}\), where \(A_n\) is not almost surely constant by Lemma 3.5, implies that, almost surely,

\[
\mathbb{E} \left[ \frac{\sum_{n \geq 1} e^{\beta \xi_n} e^{\beta'(\xi_n + Y_n)}}{\left( \sum_{k \geq 1} e^{\beta_k} \right) \left( \sum_{k \geq 1} e^{\beta_k(\xi_k + Y_k)} \right)} \right] < \frac{\sum_{n \geq 1} e^{\beta \xi_n} e^{\beta' \xi_n}}{\left( \sum_{k \geq 1} e^{\beta_k} \right) \left( \sum_{k \geq 1} e^{\beta_k(\xi_k + Y_k)} \right)} = Q_{\text{REM}}(\beta, \beta'),
\]

recalling the definition of \(Q_{\text{REM}}(\beta, \beta')\) from (1.1). Taking the expectation proves the result. □

3.2 Random perturbation of a nonincreasing sequence

Proof of Lemma 3.2. If \(\sum_{n \geq 1} A_n p_n = \infty\) a.s., then \(\hat{p}_n = 0\) a.s. for any \(n \geq 1\) and therefore (3.2) holds clearly and the inequality is strict as soon as \((q_n)_{n \geq 1}\) is not constant equal to zero. Otherwise, we have \(\sum_{n \geq 1} A_n p_n < \infty\) a.s. by Kolmogorov’s zero-one law and we work under this assumption until the end of the proof. Using also that \(\sum_{n \geq 1} p_n = 1\) and \((q_n)_{n \geq 1}\) is bounded, it follows that all forthcoming sums converge absolutely almost surely. Note also that if \((q_n)_{n \geq 1}\) is constant, then (3.2) is clear, hence we assume that it is not.

We introduce \(S := \sum_{n \geq 1} (p_n - \hat{p}_n)q_n\) and our aim is to prove that \(\mathbb{E}[S] \geq 0\). Using that \(\sum_{k \geq 1} p_k = 1\), we have

\[
S = \sum_{n \geq 1} \frac{(\sum_{j \geq 1} A_k p_k) p_n - (\sum_{k \geq 1} A_k p_k) A_n p_n}{\sum_{j \geq 1} A_j p_j} q_n = \frac{1}{\sum_{j \geq 1} A_j p_j} \sum_{n,k \geq 1} (A_k - A_n)p_k p_n q_n.
\]

Then, switching the role of \(k\) and \(n\) in one of the sums, we get

\[
\sum_{n,k \geq 1} (A_k - A_n)p_k p_n q_n = \sum_{n,k \geq 1} A_k p_k p_n q_n - \sum_{n,k \geq 1} A_n p_k p_n q_n = \sum_{n,k \geq 1} A_n p_k (q_k - q_n).
\]

Therefore, applying dominated convergence theorem, we have

\[
\mathbb{E}[S] = \sum_{n,k \geq 1} \mathbb{E} \left[ \frac{A_n}{\sum_{j \geq 1} A_j p_j} p_k (q_k - q_n) \right] = \sum_{n \geq 1} y_n p_n x_n,
\]

where we set, for any \(n \geq 1\),

\[
x_n := \sum_{k \geq 1} p_k (q_k - q_n) \quad \text{and} \quad y_n := \mathbb{E} \left[ \frac{A_n}{\sum_{j \geq 1} A_j p_j} \right].
\]

Note that \(x_n = (\sum_{k \geq 1} q_k p_k) - q_n\) is nondecreasing in \(n\), with \(x_1 \leq 0\) and \(\lim_{n \to \infty} x_n > 0\) (because \(\sum_{k \geq 1} p_k = 1\) and \((q_n)_{n \geq 1}\) is nonincreasing and not constant), and therefore there exists \(n_0 \geq 1\) such that \(x_n \leq 0\) for \(n \leq n_0\) and \(x_n > 0\) for \(n > n_0\). Moreover, we have \(\sum_{n \geq 1} p_n x_n = 0\), where the \(n_0\) first terms are nonpositive and the other ones are nonnegative. Therefore, in order to prove that \(\mathbb{E}[S] = \sum_{n \geq 1} y_n p_n x_n\) is nonnegative, it is sufficient to prove that \((y_n)_{n \geq 1}\) is a nondecreasing sequence. Indeed, it follows then that

\[
\sum_{n \geq 1} y_n p_n x_n \geq \sum_{n \geq 1} y_{n_0} p_n x_n = 0,
\]

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by distinguishing the case \( n \leq n_0 \) and the case \( n > n_0 \).

For any \( n > m \geq 1 \), our aim is now to prove that \( y_n \geq y_m \). Setting \( B := \sum_{j \notin \{m,n\}} A_j p_j \), we have

\[
y_n - y_m = E \left[ \frac{A_n - A_m}{A_n p_n + A_m p_m + B} \right].
\]

We distinguish two cases. On the event \( \{A_n \geq A_m\} \), since \( p_n \leq p_m \), we have \( A_n p_n + A_m p_m \leq (A_n + A_m)(p_n + p_m)/2 \) and so

\[
\frac{A_n - A_m}{A_n p_n + A_m p_m + B} \geq \frac{A_n - A_m}{(A_n + A_m)(p_n + p_m)/2 + B}. \tag{3.3}
\]

On the other hand, on the event \( \{A_n < A_m\} \), we have \( A_n p_n + A_m p_m \geq (A_n + A_m)(p_n + p_m)/2 \) and (3.3) holds too. Therefore, we get

\[
y_n - y_m \geq E \left[ \frac{2(A_n - A_m)}{(A_n + A_m)(p_n + p_m) + 2B} \right] = 0, \tag{3.4}
\]

because \( A_n \) and \( A_m \) have the same distribution and play a symmetric role in the last expectation (and \( B \) is independent of \( (A_n, A_m) \)). It proves that \( (y_n)_{n \geq 1} \) is a nondecreasing sequence and therefore that \( E[S] \geq 0 \).

Now, we assume in addition that \( A_1 \) is not almost surely constant and, for each \( n \geq 1 \), \( p_n > 0 \). Our aim is to prove that

\[
E[S] = \sum_{n \geq 1} y_n p_n x_n > 0. \tag{3.5}
\]

First note that, for any \( n > m \geq 1 \), if \( p_n < p_m \), then \( y_n > y_m \). Indeed, on the event \( \{A_n < A_m\} \), we have in this case \( A_n p_n + A_m p_m > (A_n + A_m)(p_n + p_m)/2 \) and, therefore,

\[
\frac{A_n - A_m}{A_n p_n + A_m p_m + B} > \frac{A_n - A_m}{(A_n + A_m)(p_n + p_m)/2 + B}.
\]

But the event \( \{A_n < A_m\} \) has positive probability (because \( A_n \) and \( A_m \) are independent with the same non-constant distribution) and, thus, we get a strict inequality in (3.4). We can now prove (3.5). Recall that \( \sum_{n \geq 1} p_n x_n = 0 \), where the \( n_0 \) first terms are nonpositive and the other ones are positive (because now \( p_n > 0 \)). Since \( p_n \downarrow 0 \) and \( p_{n_0} > 0 \), there exists \( n_1 > n_0 \) such that \( p_{n_1} < p_{n_0} \) and therefore \( y_{n_1} > y_{n_0} \). Then, using that \( p_{n_1} x_{n_1} > 0 \), we get the following strict inequality

\[
\sum_{n \geq 1} y_n p_n x_n > y_{n_0} \sum_{n \geq 1} p_n x_n = 0,
\]

which concludes the proof.

\[\square\]

3.3 Decorations play a nontrivial role

In this subsection we prove Lemma 3.5, which states that the random variable \( Y \), whose distribution depends on the decoration law \( \nu \), is not almost surely constant. The contrary would have been surprising, but, in order to prove it properly, we first need to show a basic property of the law \( \nu \) in Lemma 3.6 below.

Let us start by recalling some results from [9] concerning the decoration law \( \nu \) for the DGFF. Let \( \nu^0 \) denote the law of the DGFF in \( \mathbb{Z}^2 \setminus \{0\} \). Its covariance is given by

\[
\text{Cov}_\nu(\phi_x, \phi_y) = a(x) + a(y) - a(x - y),
\]

where \( a : \mathbb{Z}^2 \to \mathbb{R} \) is the potential kernel of the simple symmetric random walk started from zero (see Appendix A for more details). In this subsection, we denote by \( \mathbb{E}_\nu \) and \( \mathbb{E}_{\nu^0} \) the expectations under which \( \phi = (\phi_x)_{x \in \mathbb{Z}^2} \) has respectively law \( \nu \) and \( \nu^0 \). Let \( C^\infty_c(\mathbb{R}^{\mathbb{Z}^2}) \)
denote the set of continuous bounded functions on $\mathbb{R}^2$ that depend only on a finite number of coordinates in $\mathbb{Z}^2$. Then, Biskup and Louidor [9, Theorem 2.3 and Proposition 5.8] proved the following description for the decoration law: for any $f \in C^{boc}_b(\mathbb{R}^2)$,

$$E_\nu[f(\phi)] = \lim_{r \to \infty} E_\nu,\rho[f(\psi) \mid \forall x \in \Lambda_r(0), \psi_x \geq 0],$$

where $\psi := \phi + \frac{2}{\sqrt{y}} a$, (3.6)

recalling that $\Lambda_r(0) = \{x \in \mathbb{Z}^2 : \|x\| \leq r\}$. The main tool for the proof of Lemma 3.5 is the following lemma, that describes the conditional law of one coordinate of the decoration given the other coordinates.

**Lemma 3.6.** For any $y \in \mathbb{Z}^2 \setminus \{0\}$, under $\nu$, the conditional law of $\phi_y$ given $(\phi_x)_{x \neq y}$ is a normal law with mean $\frac{1}{4} \sum_{x \sim y} \phi_x$ and variance 1 conditioned to be nonnegative.

**Proof.** Let $f \in C^{boc}_b(\mathbb{R}^2)$ that does not depend on the $y$-coordinate. Let $h \in C_0(\mathbb{R})$. Applying the domain Markov property to the DGFF $\phi$ with law $\nu^0$, conditionally on the $\phi_x$ for $x \neq y$, $\phi_y$ has the same law as $Z + \frac{1}{4} \sum_{x \sim y} \phi_x$, where $Z$ is independent of $(\phi_x)_{x \neq y}$ with law $N(0, 1)$. On the other hand, by [36, Proposition 4.4.2], the potential kernel $a$ is discrete harmonic on $\mathbb{Z}^2 \setminus \{0\}$, so we have $a(y) = \frac{1}{4} \sum_{x \sim y} a(x)$. Therefore, we get

$$E_\nu \left[ f(\psi) h(\psi_y) \prod_{x \in \Lambda_r(0)} 1_{\{\psi_x \geq 0\}} \right] = E_\nu \left[ f(\psi) \left( \prod_{x \in \Lambda_r(0) \setminus \{y\}} 1_{\{\psi_x \geq 0\}} \right) \int h \left( z + \frac{1}{4} \sum_{x \sim y} \psi_x \right) 1_{\{z + \frac{1}{4} \sum_{x \sim y} \psi_x \geq 0\}} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz \right].$$

(3.7)

Now, we define a function $H : \mathbb{R} \to \mathbb{R}$ by

$$H(t) := \frac{\int_{\mathbb{R}} h(z + t) \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz}{\int_{\mathbb{R}} 1_{\{z + t \geq 0\}} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz},$$

which is a continuous bounded function. Then, (3.7) is equal to

$$E_\nu \left[ f(\psi) \left( \prod_{x \in \Lambda_r(0) \setminus \{y\}} 1_{\{\psi_x \geq 0\}} \right) H \left( \frac{1}{4} \sum_{x \sim y} \psi_x \right) \int_{\mathbb{R}} 1_{\{z + \frac{1}{4} \sum_{x \sim y} \psi_x \geq 0\}} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz \right] = E_\nu \left[ f(\psi) H \left( \frac{1}{4} \sum_{x \sim y} \psi_x \right) \prod_{x \in \Lambda_r(0)} 1_{\{\psi_x \geq 0\}} \right].$$

(3.8)

using the Markov domain property as before. Applying (3.6) to the left-hand side of (3.7) and to the right-hand side of (3.8), we get $E_\nu[f(\phi)h(\phi_y)] = E_\nu[f(\phi)H(\frac{1}{4} \sum_{x \sim y} \phi_x)]$. Recalling the definition of function $H$ and since this equality holds for any $f \in C^{boc}_b(\mathbb{R}^2)$ not depending on the $y$-coordinate and any $h \in C_0(\mathbb{R})$, it proves the claimed result. $\square$

**Proof of Lemma 3.5.** By contradiction, assume that $Y$ is almost surely constant. Then, $X_{\beta'} - X_{\beta}$ is a.s. constant and, recalling the definition of $X_{\beta}$ in (2.4), there exists some constant $c > 0$ such that, $\nu$-a.s., $\sum_{x \in \mathbb{Z}^2} e^{-\beta \phi_x} = c (\sum_{x \in \mathbb{Z}^2} e^{-\beta' \phi_x})^{\beta/\beta'}$. Fix some $y \in \mathbb{Z}^2 \setminus \{0\}$. We have

$$E_\nu \left[ e^{-\beta \phi_y} + \sum_{x \neq y} e^{-\beta \phi_x} - c \left( e^{-\beta \phi_y} + \sum_{x \neq y} e^{-\beta' \phi_x} \right)^{\beta/\beta'} \right] (\phi_x)_{x \neq y} = 0, \ \nu$'$-a.s.$

But, given $(\phi_x)_{x \neq y}$, $\phi_y$ has a distribution with positive density on $(0, \infty)$ by Lemma 3.6. Recalling that $\sum_{x \neq y} e^{-\beta \phi_x} \geq 1$, it shows that there exist some $s, s' \geq 1$ such that, for almost every $z \in (0, \infty)$, $e^{-\beta z} + s - c(e^{-\beta z} + s')^{\beta/\beta'} = 0$. This cannot be true for $\beta \neq \beta'$ and, therefore, one gets a contradiction. $\square$
A Some useful results

In this appendix, we state some known results used throughout the paper. The first result is the well-known Gaussian integration by part, whose proof can be found in [45, Equation (A.17)].

Lemma A.1. Let \((X, Z_1, \ldots, Z_d)\) be a centered Gaussian random vector. Then, for any \(C^1\) function \(F: \mathbb{R}^d \to \mathbb{R}\) of moderate growth at infinity, we have

\[
\mathbb{E}[XF(Z_1, \ldots, Z_d)] = \sum_{i=1}^{d} \mathbb{E}[XZ_i] \mathbb{E}\left[\frac{\partial F}{\partial z_i}(Z_1, \ldots, Z_d)\right].
\]

Furthermore, Theorem 4.4.4 of [36] gives the asymptotic behavior of function \(a\) started from zero, which is a function \(a: \mathbb{Z}^2 \to \mathbb{R}\) defined by

\[
a(x) := \sum_{n \geq 0} (\mathbb{P}(S_n = 0) - \mathbb{P}(S_n = x))
\]

and can also been written explicitely as a double integral, see [36, Proposition 4.4.3]. Indeed, by [36, Proposition 4.6.2] and since \(D_N\) is finite, the following relation between these functions holds for any \(x, y \in \mathbb{Z}^2\),

\[
G_N(x, y) = \mathbb{E}_x \left[ \sum_{k=0}^{\tau_N-1} 1_{\{S_k = y\}} \right].
\]  \hspace{1cm} (A.1)

The Green function is related to the potential kernel of the simple symmetric random walk started from zero, which is a function \(a: \mathbb{Z}^2 \to \mathbb{R}\) defined by

\[
a(x) := \sum_{n \geq 0} (\mathbb{P}(S_n = 0) - \mathbb{P}(S_n = x))
\]

and can also been written explicitely as a double integral, see [36, Proposition 4.4.3]. Indeed, by [36, Proposition 4.6.2] and since \(D_N\) is finite, the following relation between these functions holds for any \(x, y \in \mathbb{Z}^2\),

\[
G_N(x, y) = \mathbb{E}_x [a(S_{\tau_N} - y)] - a(x - y). \quad \text{(A.2)}
\]

Furthermore, Theorem 4.4.4 of [36] gives the asymptotic behavior of function \(a\) as \(\|x\| \to \infty\), we have

\[
a(x) = \frac{2}{\pi} \log \|x\| + \frac{2\gamma + \log 8}{\pi} + O\left(\|x\|^{-2}\right), \quad \text{(A.3)}
\]

where \(\gamma\) is Euler’s constant. From these tools, we can easily deduce the following result, relating the overlap between two points to their euclidean distance.

Lemma A.2.  

(i) There exists a constant \(c_1 > 0\) such that, for any \(N \geq 1\), \(0 < \alpha < 1\) and any \(x, y \in D_N\),

\[
q_N(x, y) \geq \alpha \quad \Rightarrow \quad \|x - y\| \leq c_1 N^{1-\alpha}.
\]

(ii) There exists a constant \(c_2 > 0\) such that, for any \(N \geq 1\), \(0 < \delta < \alpha < 1\) and any \(x, y \in D_{N,\delta} := \{x \in D_N: d(x, D_N^c) > N^{1-\delta}\}\),

\[
\|x - y\| \leq c_2 N^{1-\alpha} \quad \Rightarrow \quad q_N(x, y) \geq \alpha - \delta.
\]

Proof. Recall that \(q_N(x, y) = G_N(x, y) / \max_{z \in D_N} G_N(z, z)\). Combining (A.2) and (A.3), first note that

\[
\max_{x \in D_N} G_N(x, z) = \frac{2}{\pi} \log N + O_N(1) \quad \text{as} \quad N \to \infty.
\]

Using again (A.2) and (A.3), we get the following upper bound, uniformly in \(x, y \in D_N\),

\[
q_N(x, y) \leq 1 - \frac{\log \|x - y\|}{\log N} + O_N((\log N)^{-1}), \quad \text{(A.4)}
\]

and the following lower bound, uniformly in \(x, y \in D_{N,\delta}\),

\[
q_N(x, y) \geq 1 - \delta - \frac{\log \|x - y\|}{\log N} + O_N((\log N)^{-1}). \quad \text{(A.5)}
\]

Part (i) of the lemma follows from (A.4) and Part (ii) from (A.5).
In the previous lemma, we saw that the overlap is properly related to the distance if the points of interest are not too close from the boundary of $D_N$. The following result shows that, for any inverse temperature $\beta > 0$, the Gibbs measure of $D^c_{N,\delta} := D_N \setminus D_{N,\delta}$ is negligible.

**Lemma A.3.** Let $\delta > 0$ and recall that $D_{N,\delta} = \{ x \in D_N : d(x, D^c_N) > N^{1-\delta} \}$. For any $\beta > 0$, $$G_{\beta, N}(D^c_{N,\delta}) \xrightarrow{N \to \infty} 0, \text{ in } L^1.$$

**Proof.** This result is shown in [4, Lemma 3.1] in the case of the square lattice $V_N = (0, N)^2 \cap \mathbb{Z}^2$, but the proof works also in our more general framework because we also have $\#D_{N,\delta}^c = O(N^{2-\delta})$ under our assumptions. Indeed, we assumed that $\partial D$ has a finite number of connected components, each of which is a $C^1$ path: therefore, there exists $C > 0$ such that, for any $\varepsilon > 0$, $\partial D$ can be covered by at most $C \varepsilon^{-1}$ balls of radius $\varepsilon$. Fix some $N \geq 1$, there exist $M \leq C N^\delta / 2$ and $z_1, \ldots, z_M$ such that $\partial D \subset \bigcup_{i=1}^M \overline{B}(z_i, 2N^{-\delta})$. Then, recalling the definition of $D_N$ and $D_{N,\delta}$, note that

$$\frac{D^c_{N,\delta}}{N} \subset \left\{ z \in D : d(z, \partial D) \leq 2N^{-\delta} \right\} \cap \frac{\mathbb{Z}^2}{N} \subset \bigcup_{i=1}^M \left( B(z_i, 4N^{-\delta}) \cap \frac{\mathbb{Z}^2}{N} \right).$$

It follows that $\#D^c_{N,\delta} \leq M 4(4N^{1-\delta})^2 \leq C' N^{2-\delta}$, which is the announced result.

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**References**


