Hard–edge asymptotics of the Jacobi growth process

Mark Cerenzia and Jeffrey Kuan *

Abstract

We introduce a two parameter \((\alpha, \beta > -1)\) family of interacting particle systems with determinantal correlation kernels expressible in terms of Jacobi polynomials \(\{P_k^{(\alpha,\beta)}\}_{k \geq 0}\). The family includes previously discovered Plancherel measures for the infinite–dimensional orthogonal and symplectic groups. The construction uses certain multivariate BC–type orthogonal polynomials that generalize the characters of these groups.

The local asymptotics near the hard edge where one expects distinguishing behavior yields the multi–time \((\alpha, \beta)\)–dependent discrete Jacobi kernel and the multi–time \(\beta\)–dependent hard–edge Pearcey kernel. The hard–edge Pearcey kernel has previously appeared in the asymptotics of non–intersecting squared Bessel paths at the hard edge.

In their study of the perturbed chiral GUE ensemble, Patrick Desrosiers and Peter J. Forrester \[24\] (2008) introduce a process with a (fixed-time) kernel, depending on a parameter \(\beta\), which can be written in terms of the Bessel function \(J_\beta\) of the first kind. Later, in the asymptotic analysis of an interacting particle system near a hard edge with a reflecting barrier, Borodin–Kuan \[14\] (2009) find a multi–time kernel, which at fixed times corresponds to the \(\beta = -1/2\) case of a \(\beta\)–dependent kernel arising in the work \[35\] of Kuijlaars, Martinez-Finkelshtein, Wielonsky (2010) on non-intersecting squared Bessel paths at the hard edge at 0. The authors conjectured that this kernel is the same as the one from \[24\]. Indeed, Delvaux–Vető \[23\] (2014) extend the results to a multi–time \(\beta\)–dependent kernel called the hard–edge Pearcey kernel, and prove that it generalizes the kernel of \[24\] and the kernel of \[35\] for all \(\beta > -1\). The terminology goes back to the Pearcey kernel, which has previously occurred at the soft edge of certain processes (see e.g. \[1, 3, 5, 13, 17, 18, 27, 28, 36, 40, 42\]).

More recent work \[19\] constructs an interacting particle system similar to that of \[14\], where jumps into the wall are suppressed instead of reflected. The resulting asymptotic kernel ends up being the \(\beta = 1/2\) case of the hard–edge Pearcey kernel. In light of these results, it is natural to ask for a \(\beta\)–dependent interacting particle system with asymptotics converging to the \(\beta\)–dependent hard–edge Pearcey kernel. This paper will construct a two parameter \((\alpha, \beta > -1)\) family of interacting particle systems whose limiting behavior is the hard–edge Pearcey kernel.

The methods in \[14\] and \[19\] use the representation theory of the orthogonal and symplectic groups, respectively. The connection to the parameter \(\beta\) occurs through the characters, which can be written as a determinant of the Jacobi polynomials \(P^{(\alpha,\beta)}\) for \((\alpha = \pm 1/2, \beta = -1/2)\) (in the orthogonal case) and \((\alpha = \pm 1/2, \beta = 1/2)\) (in the symplectic case). The determinantal formulae and branching rules provided in the final section of the work of Okounkov and Olshanski \[38\] on BC–type orthogonal polynomials indicate that it should be possible to extend these methods to general \(\alpha, \beta > -1\), and it is through these Jacobi polynomials that the process is defined. See also \[20, 21\] and \[33\] for interacting particle systems with walls which are related to representation theory.

*Electronic addresses: cerenzia@princeton.edu, kuan@math.columbia.edu
Let us now describe the model in more detail. In orthogonal/symplectic Gelfand-Tsetlin patterns, level \( n \in \mathbb{Z}_{\geq 0} := \{1, 2, 3 \ldots \} \) has \( r_n := [(n + 1)/2] \) particles \( \mathcal{X}_k^n(\gamma) \), \( 1 \leq k \leq r_n \), dynamically evolving according to the time parameter \( \gamma \geq 0 \) from the \textit{densely packed} initial condition \( \mathcal{X}_k^n(0) = r_n - k \). Each particle has two unit rate exponential clocks, one for rightward jumps and one for leftward jumps, all of which are independent. Suppose the right clock rings of \( \mathcal{X}_k^n \). If \( \mathcal{X}_k^n = \mathcal{X}_{k-1}^{n-1} \), then the right jump attempt is suppressed (a “block” occurs). Otherwise, take the largest \( c \geq 1 \) such that

\[
\mathcal{X}_k^n = \mathcal{X}_k^{n+1} + (r_{n+1} - r_n) = \cdots = \mathcal{X}_k^{n+(c-1)} + \sum_{\ell=1}^{c-1}(r_{n+\ell+1} - r_{n+\ell}),
\]

then all \( c \) of these particles jump to the right (a “push” occurs). Similarly, leftward jump attempts by \( \mathcal{X}_k^n \) can either be suppressed by a particle on the previous level \( n - 1 \) in a state where it can push \( \mathcal{X}_k^n \) by a right jump, or the left jump of \( \mathcal{X}_k^n \) can push a particle on the next level \( n + 1 \) whose rightward jumps would be blocked by \( \mathcal{X}_k^n \). Lastly, we need to declare what occurs if \( n \geq 1 \) is odd and the left clock of a “wall–particle” \( \mathcal{X}_r^n \) rings when the particle is at position 0 (i.e., at the wall). If such wall-jumps are reflected (i.e., a right jump is attempted), we call the dynamics for such point patterns \textit{orthogonal Plancherel Growth}, and \textit{symplectic Plancherel Growth} if such wall-jumps are suppressed. These dynamics can be depicted as follows (where the smaller arrows indicate which clock rings, and \( \Rightarrow \) indicates a blocked jump):

\[
\begin{array}{cccc}
| x_3^5 & x_2^5 & x_1^5 | & | x_3^5 & x_2^5 \rightarrow x_1^5 | & | x_3^5 & x_2^5 & x_1^5 | & | x_3^5 & x_2^5 & x_1^5 |\\
| x_2^4 & x_1^4 | & | x_2^4 & x_1^4 | & | x_2^4 \leftrightarrow x_1^4 | & | x_2^4 \leftrightarrow x_1^4 | & | x_2^4 \leftrightarrow x_1^4 |\\
| x_2^n & x_1^n | & | x_2^n & x_1^n | & | x_2^n \leftrightarrow x_1^n | & | x_2^n \leftrightarrow x_1^n | & | x_2^n \leftrightarrow x_1^n | & \cdots (1) \\
| x_1^1 | & | x_1^1 | & | x_1^1 | & | x_1^1 | & | x_1^1 |
\end{array}
\]

Note the blocking and pushing of a particle by straddling particles on the previous level serves to maintain \textit{interlacing conditions}, which in the chosen coordinates read as

\[
\begin{cases}
x_k^{n+1} \leq x_k^n < x_k^{n+1}, & \text{if } n \text{ even} \\
x_k^{n+1} \leq x_k^n \leq x_k^{n+1}, & \text{if } n \text{ odd}
\end{cases}, \quad 1 \leq k \leq r_n.
\]

To make this visualization easier, see Figures 2 and 3 which compares the dynamics in shifted coordinates.

We now view either system of interacting particles as determining a random subset \( \mathcal{X}(\gamma) := \{ \mathcal{X}_k^n(\gamma) \}_{1 \leq k \leq r_n} \) of \( \mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0} \). We write \( \xi_\gamma \) for the distribution on \( 2^{\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}} \) of such point configurations (we refer to this distribution as a \textit{point process}). Borodin–Kuan [14] studied the local asymptotics of such systems and their results are summarized in Figure 3. Their arguments rely on the fundamental result that this point process is determinantal with an explicitly computable kernel; take \( \alpha = \beta = -1/2 \) in Theorem 2.1 for a precise statement. As indicated by this statement, certain orthogonal polynomials underlie the correlation structure, and the dynamics given by driving \textit{simple} random walks follow\[1\] from the three–term recurrence

\[
 xp_n(x) = \frac{1}{2}[p_{n+1}(x) + p_{n-1}(x)], \quad n > 0
\]

\[1\] If one considers the single particle on the first level, its jump rates are given by the coefficients in the three–term recurrence relation.
Figure 1: Three possible initial steps of either orthogonal or symplectic Plancherel growth. The arrows indicate the direction that the particle will attempt to jump next; see Figure 2 for the fourth step. The tiling determined by their positions suggests a stepped-surface interpretation.

Figure 2: The fourth step illustrates the distinctive wall behaviors: reflection (left) and suppression (right). Simulations of the orthogonal and symplectic cases indicate similar average behavior.

satisfied by the relevant first/fourth (orthogonal case) and second/third (symplectic case) Chebyshev polynomials (see e.g. [37] for the definition of the third and fourth Chebyshev polynomials). Indeed, these are the only four Jacobi polynomials $P_{k(\alpha,\beta)}^\kappa$ that satisfy the recurrence (3) (after an appropriate scaling – see (3) below). The cases $\beta = \pm1/2$ of the multi-time versions of the kernels arise from the asymptotics at $-1$ of these polynomials.

Having reviewed these two archetypes, we can quickly introduce the Jacobi growth process. Fix
Figure 3: Summary of Borodin–Kuan [14] results. The hydrodynamic limit capturing average behavior indicates a densely-packed region of inactivity (“frozen”) and a central region of activity (“liquid”), with the remaining area unreached (“empty”). Compare with Figure 4 of [14].

\[ \alpha, \beta > -1. \]

Consider the evolving point configuration \( X(\alpha, \beta) := \{ (X(\alpha, \beta))_k^n \}_{n \geq 1} \) of particles in \( \mathbb{Z}_0 \times \mathbb{Z}_0 \) with the same point pattern preserved by push–block interactions as in the orthogonal and symplectic cases, except we now declare that if a particle is at position \( \lambda \geq 0 \) on level \( n \geq 1 \) and is allowed to jump to position \( \mu \), then it will do so at the rate

\[
w_n(\lambda, \mu) := \begin{cases} 
\frac{\lambda + \beta + 1}{2\lambda + \alpha + \beta + 2} & n \text{ even, } \lambda \geq 0, \mu = \lambda + 1 \\
\frac{\lambda + \alpha + \beta + 1}{2\lambda + \alpha + \beta + 2} & n \text{ odd, } \lambda \geq 1, \mu = \lambda - 1 \\
\frac{\lambda + \alpha + \beta + 1}{2\lambda + \alpha + \beta + 2} & n \text{ even, } \\
\frac{\lambda + \beta}{2\lambda} & n \text{ odd, } \lambda \geq 1, \mu = \lambda - 1 \\
\frac{2(\lambda + 1)}{2\lambda + \alpha + \beta + 1} & n \text{ even, } \lambda \geq 0, \mu = \lambda + 1 \\
\frac{\lambda + \alpha + \beta + 1}{2\lambda + \alpha + \beta + 2} & n \text{ odd, } \lambda \geq 1, \mu = \lambda - 1 \\
\frac{\lambda + \alpha + \beta + 1}{2\lambda + \alpha + \beta + 2} \\
\frac{\lambda + \beta}{2\lambda} & \lambda \geq 1, \mu = \lambda - 1 \\
\frac{2(\alpha + 1)}{(\alpha + \beta + 2)} & n \text{ even, } \\
\frac{\lambda + \alpha + \beta + 1}{2\lambda + \alpha + \beta + 2} & n \text{ odd, } \lambda \geq 1, \mu = \lambda - 1 \\
\frac{\lambda + \beta}{2\lambda} & \lambda \geq 1, \mu = \lambda - 1 \\
\end{cases}, \quad (4)
\]

where \( w_n(0, -1) = 0 \). The initial condition is once again the densely packed initial condition \( X(\alpha, \beta)(0) = r_n - k \). The reader can check that the particle system corresponds to the orthogonal case if \( \alpha = \beta = -1/2 \) and to the symplectic case if \( -\alpha = \beta = 1/2 \); in particular, the leftmost particles on odd levels have rate \( \frac{2(\alpha + 1)}{(\alpha + \beta + 2)} \) right jumps off of the wall at 0. Also note that the jump rates converge to 1/2 as \( \lambda \to \infty \). These rates have a more enlightening and compact form in terms of quantities arising from Jacobi polynomials (see equation (12) below).

It is worth mentioning that there are several other asymptotic regimes that could be pursued. In addition to investigating the edge asymptotic behavior, [14] also identifies the incomplete beta kernel (defined in [39]) as describing the asymptotics in the bulk of the system. Some formal calculations based on the results of Chapter 8 of [41] suggest the same is true for the Jacobi growth process, but we do not pursue the rigorous details here. These formal calculations should also imply Gaussian free field fluctuations in the height function, using the results of [32].

The paper [12] identifies and [19] constructs the limit of orthogonal and symplectic Plancherel growth under diffusive scaling (i.e., the bottom left corner of Figure 3). The fact that the drift of the rates (4) of the driving random walks vanishes under this scaling suggests the Jacobi growth process shares the same diffusion limit. See [6] for recent work on such systems.
1 Jacobi Polynomials

In order to state the main results of this paper, let us first introduce some notation concerning the Jacobi polynomials.

Define the Jacobi Polynomials \( \{P_k^{(\alpha,\beta)}\}_{k \geq 0} \) by the initial condition \( P_0^{(\alpha,\beta)}(x) \equiv 1 \) and by the recurrence

\[
xP_k^{(\alpha,\beta)}(x) = A_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(x) + B_k^{(\alpha,\beta)} P_{k-1}^{(\alpha,\beta)}(x) + C_k^{(\alpha,\beta)} P_{k+1}^{(\alpha,\beta)}(x), \quad k \geq 0, \tag{5}
\]

where \( P_{-1}^{(\alpha,\beta)} \equiv 0 \) and where

\[
A_k^{(\alpha,\beta)} := \frac{(\beta - \alpha)(\alpha + \beta)}{(2k + \alpha + \beta)(2k + \alpha + \beta + 2)}, \quad C_k^{(\alpha,\beta)} := \frac{2(k+1)(1+\alpha+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)}, \quad k \geq 0, \tag{6}
\]

\[
B_k^{(\alpha,\beta)} := \frac{2(k+\alpha)(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}, \quad k \geq 1.
\]

Observe that for any values of \( \alpha \) and \( \beta \),

\[
\lim_{k \to \infty} A_k^{(\alpha,\beta)} = 0, \quad \lim_{k \to \infty} B_k^{(\alpha,\beta)} = \lim_{k \to \infty} C_k^{(\alpha,\beta)} = \frac{1}{2}.
\]

The \( P_k^{(\alpha,\beta)} \) are orthogonal with respect to the weight \( w_{(\alpha,\beta)}(x) := (1-x)\alpha(1+x)\beta 1_{[-1,1]}(x) \), with normalization \( P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)} \), so that the leading coefficient is given by

\[
a_k^{(\alpha,\beta)} := \frac{\Gamma(2k+\alpha+\beta+1)}{2^{2k}k!\Gamma(k+\alpha+\beta+1)} = \prod_{\ell=0}^{k-1} (C_\ell^{(\alpha,\beta)})^{-1}, \tag{7}
\]

and the inner product by

\[
\langle P_k^{(\alpha,\beta)}, P_\ell^{(\alpha,\beta)} \rangle_{(\alpha,\beta)} := \int_\mathbb{R} P_k^{(\alpha,\beta)}(x) P_\ell^{(\alpha,\beta)}(x) w_{(\alpha,\beta)}(x) dx = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)} \delta_{k\ell}. \tag{8}
\]

We will always write \( \tilde{P}_k^{(\alpha,\beta)} := P_k^{(\alpha,\beta)}/\|P_k^{(\alpha,\beta)}\| \) so that \( \langle \tilde{P}_k^{(\alpha,\beta)}, \tilde{P}_\ell^{(\alpha,\beta)} \rangle_{(\alpha,\beta)} = \delta_{k\ell} \). We will frequently make use of the orthogonal decomposition

\[
T(x) = \sum_{k=0}^{\infty} \langle \tilde{P}_k^{(\alpha,\beta)}, T \rangle_{(\alpha,\beta)} \tilde{P}_k^{(\alpha,\beta)}(x) \tag{9}
\]

for \( T \in C^1[-1,1] \). See Szegö [1] for more details on this discussion.

Fix \( \alpha, \beta > -1 \); the parameter \( \beta \) will hereafter be suppressed from notation whenever possible. Write

\[
\alpha_n := \begin{cases} \alpha + 1, & \text{if } n \text{ even} \\ \alpha, & \text{if } n \text{ odd} \end{cases}
\]

The use of this \( \alpha_n \) has occurred before, see (7.4) of [2]. Besides the orthonormal scaling \( \tilde{P}_k^{(\alpha)} \), it will be convenient to define\( \tilde{P}_k^{(\alpha_n)} \)

\[
\tilde{P}_k^{(\alpha_n)} = \tilde{c}_k^{n} P_k^{(\alpha_n)}, \tag{10}
\]

\[
\tilde{P}_k^{(\alpha_n)} = \tilde{c}_k^{n} P_k^{(\alpha_n)}.
\]

\footnote{Note that there is an abuse of notation here, in that the scalings \( \tilde{P}_k^{(\alpha_n)} \) and \( \tilde{P}_k^{(\alpha_n)} \) depend on the parity of \( n \). So for example, \( \tilde{P}_k^{(\alpha+1)} \) should be interpreted as \( \tilde{c}_k^{2} P_k^{(\alpha+1)} \) and \( \tilde{P}_k^{(\alpha)} \) should be interpreted as \( \tilde{c}_k^{1} P_k^{(\alpha)} \).}
where
\[ c_k^\alpha := (2k + \alpha_n + \beta + 1) \cdot \begin{cases} \frac{\Gamma(k+1)\Gamma(\alpha+1)}{2\Gamma(\alpha+k+2)}, & \text{n even} \\ \frac{\Gamma(k+\alpha+\beta+2)}{\Gamma(k+\alpha+k+2)\Gamma(k+\beta+1)}, & \text{n odd} \end{cases}, \quad c_k^\beta := \begin{cases} \frac{\Gamma(k+\alpha+\beta+2)}{\Gamma(k+\alpha+k+2)\Gamma(\alpha+1)}, & \text{n even} \\ \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)}, & \text{n odd} \end{cases}. \tag{9} \]

Define
\[ \phi_n(k) := c_k^\alpha / c_k^\beta, \quad \tilde{\phi}_n(k,m) := \phi_n(k) \cdot \begin{cases} 1_{(k<m)}, & \text{if n even} \\ 1_{(k\leq m)}, & \text{if n odd} \end{cases}. \]

We also adopt the convention \( \phi_n(-1, \cdot) \equiv 1 \). Note that the facts
\[ \phi_n(s) \tilde{P}_s^{(\alpha_n)} = \tilde{P}_s^{(\alpha_n)}, \quad \tilde{P}_k^{(\alpha)}(1) = 1, \tag{10} \]
and that the scalings are related by
\[ \frac{\tilde{P}_k^{(\alpha_n)} \tilde{P}_k^{(\alpha_m)}}{2^{\alpha+\beta+1}\Gamma(\alpha+1)} = \tilde{P}_k^{(\alpha_n)} \tilde{P}_k^{(\alpha_m)}. \tag{11} \]

Finally, with this notation established, we can rewrite the rate [4] compactly as
\[ w_n(\lambda, \mu) = \begin{cases} \frac{c_{\lambda+1}^\alpha B_{\lambda+1}^{(\alpha_n, \beta)}}{c_{\lambda}^\alpha c_{\lambda+1}^\beta C_{\lambda+1}^{(\alpha_n, \beta)}} C_{\lambda+1}^{(\alpha_n, \beta)}, & \lambda \geq 0, \mu = \lambda + 1 \\ \frac{c_{\lambda}^\alpha c_{\lambda-1}^\alpha C_{\lambda}^{(\alpha_n, \beta)}}{c_{\lambda+1}^\beta C_{\lambda+1}^{(\alpha_n, \beta)}} B_{\lambda}^{(\alpha_n, \beta)}, & \lambda \geq 1, \mu = \lambda - 1 \end{cases}. \tag{12} \]

## 2 Statement of main theorems

Below, we write \( \xi^\gamma = \xi^\gamma_{(\alpha, \beta)} \) for the distribution on \( 2^{\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}} \) of the random point configuration \( \mathcal{X}_{(\alpha, \beta)}(\gamma) \in 2^{\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}} \) determined by the Jacobi growth dynamics.

**Theorem 2.1.** Fix \( \alpha, \beta > -1 \) and \( \gamma \geq 0 \). Then the correlation functions \( \{\rho_k^\gamma\}_{k \geq 1} \) of the Jacobi growth process are determinantal: for \( z_1, \ldots, z_k \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0} \),
\[ \rho_k^\gamma(z_1, \ldots, z_k) := \xi^\gamma(\{E \in 2^{\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}} | E \supset \{z_1, \ldots, z_k\}\}) = \det[K_{(\alpha, \beta)}^\gamma(z_i, z_j)]_{i,j=1}^k, \]
where the (nonsymmetric) kernel \( K_{(\alpha, \beta)}^\gamma \) is given by
\[ K_{(\alpha, \beta)}^\gamma((s, n), (t, m)) := \frac{1}{2\pi i} \int \left( \int_{-1}^1 e^{x\gamma} \frac{\tilde{P}_s^{(\alpha_n)}(x)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)} \frac{\tilde{P}_t^{(\alpha_m)}(u)}{(1-u)^{r_n+\alpha_n}(1+x)^{\beta}} du \right) dx \right) du + 1_{(n \geq m)} \int_{-1}^1 e^{x\gamma} \frac{\tilde{P}_s^{(\alpha_n)}(x)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)} \frac{\tilde{P}_t^{(\alpha_m)}(x)(1-x)^{r_n-\alpha_n}(1+x)^{\beta}}{dx}. \tag{13} \]

for \( (s, n), (t, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0} \), where the complex integral is a positively oriented (i.e., counterclockwise) simple loop around \([-1, 1]\).

Near the wall of the particle system, we identify an \((\alpha, \beta)\)-dependent family of Jacobi kernels and a \(\beta\)-dependent family of hard–edge Pearcey kernels. We first focus on the large time limit at a finite distance from the wall. The discrete Jacobi kernel arising here is a generalization of the one in [14], which was only defined there in the case \( \beta = -1/2 \). We remark that this generalization was rigorously discovered independently and concurrently as the work [13] of Borodin-Olshanski, which first appeared online around the same time as the current work.
Theorem 2.2. (Edge Limit: Discrete Jacobi Kernel) Fix $\alpha, \beta > -1$. Assume $\gamma \geq 0$, $(s_i, n_i) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$, $1 \leq i \leq k$, depend on $N$ in such a way that $\gamma/N \to \tau > 0$, $r_{n_i}/N := [(n_i + 1)/2]/N \to \eta > 0$ but the $s_i$ are fixed and finite. Assume the parity of each $n_i$ is constant and that the differences $r_{n_i} - r_{n_j}$ are of constant order. Then as $N \to \infty$,

$$
\rho_{k}^\gamma((s_1, n_1), \ldots, (s_k, n_k)) \to \begin{cases} 
\det[J_{(\alpha, \beta)}((s_i, n_i), (s_j, n_j)|1 - \eta/\tau)_{i,j=1}^k, & \text{if } 1 - \eta/\tau > -1 \\
1, & \text{if } 1 - \eta/\tau \leq -1 
\end{cases}
$$

where the discrete Jacobi kernel $J_{(\alpha, \beta)}$ is given by

$$
J_{(\alpha, \beta)}((s, n), (t, m)|\epsilon) := \int_{-1}^{1} [1_{(n \geq m)} - 1_{[-1, \epsilon]}(x)] \frac{\hat{P}_s^{(\alpha \epsilon)}(x)}{2^{\alpha+1} \Gamma(\alpha+1)} \hat{P}_t^{(\alpha \epsilon)}(x)(1-x)^{r_{n-m} + \alpha \epsilon}(1+x)^{\beta \epsilon} dx.
$$

(14)

Define the particle–hole involution as the map $\Delta$ on $2^{\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}}$ given by $\Delta(\chi) := (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}) \setminus \chi$. The push–forward $\xi_\Delta := \xi^\gamma \circ \Delta^{-1}$ of the point process $\xi^\gamma$ furnishes another point process which also possesses a determinantal correlation structure; see Proposition 3.1 of [13] and equation (27). Our main result shows that the kernel arising in our setting at the critical point is the hard–edge Pearcey kernel from [23]. Note in particular the result is independent of $\alpha > -1$. This is due to the Mehler–Heine formula, which gives asymptotics of the Jacobi polynomials independent of $\alpha$; see equation (26). Below, recall that $J_\beta$ is the Bessel function of the first kind, and $I_\beta$ is the modified Bessel function of the first kind, defined by

$$
J_\beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\beta + 2k}}{k! \Gamma(\beta + k + 1)}, \quad I_\beta(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\beta + 2k}}{k! \Gamma(\beta + k + 1)}.
$$

The hard–edge Pearcey kernel $L^\beta$ of [23] is defined by

$$
(L^\beta(x, y))^{\beta/2} = -1_{(t,s)} \frac{1}{t-s} \exp \left( -\frac{x+y}{t-s} \right) I_\beta \left( \frac{2\sqrt{xy}}{t-s} \right) + \frac{2}{\pi i} \int_{C} dv \int_{0}^{\infty} du \left( \frac{u}{v} \right)^\beta \frac{uv}{v^2 - u^2} e^{u^4/4 + su^2} J_\beta(2\sqrt{yu}) J_\beta(2\sqrt{xv}),
$$

where $C$ consists of two rays, one from $e^{i\pi/4} \infty$ to 0 and one from 0 to $e^{-i\pi/4} \infty$. Note that the $(x/y)^{\beta/2}$ term is a conjugating factor which disappears in the determinant.

Theorem 2.3. (Edge Limit: Hard–edge Pearcey Kernel) Fix $\alpha, \beta > -1$. Let $\rho_{\Delta}^\gamma$ denote the $k$th correlation function of the point process $\xi_\Delta$: for distinct $z_1, \ldots, z_k \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$,

$$
\rho_{\Delta}^\gamma(z_1, \ldots, z_k) := \xi_\Delta^\gamma(\{E \in 2^{\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}} | E \supset \{z_1, \ldots, z_k\}\}).
$$

Assume $\gamma \geq 0$, $(s_i, n_i) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$, $1 \leq i \leq k$, depend on $N$ in such a way that $\gamma/N \to 1/2$, $s_i/N^{1/4} \to \nu_i > 0$, and $(r_{n_i} - N)/\sqrt{N} \to \sigma_i$. Assume the parity of each $n_i$ is constant. Then as $N \to \infty$

$$
N^{k/4} \cdot \rho_{\Delta}^\gamma((s_1, n_1), \ldots, (s_k, n_k)) \to \det[L^\beta(\sigma_i, \nu_i^2; \sigma_j, \nu_j^2) \cdot 2\sqrt{\nu_i \nu_j}]_{i,j=1}^k.
$$
Remark 2.4. The expression $2\sqrt{\nu_i \nu_j}$ arises due to the change of variables $x = \nu_i^2, y = \nu_j^2$ in the spatial dimensions. This change of variables is related to the fact that $L^\beta$ occurred in the context of squared Bessel paths. Indeed, it can be written as

$$2\sqrt{\nu_i \nu_j} = \sqrt{\frac{dx}{dv_i}} \frac{dy}{dv_j}. $$

See Remark 7.1 below for a more in–depth explanation.

Let us review the outline of the remainder of the paper. Section 3 will prove some necessary identities for the Jacobi polynomials. Section 4 constructs probability measures on the interlacing particle configurations, and Section 5 constructs a Markov process preserving these measures. Section 6 computes an explicit formula for the correlation kernel, and Section 7 takes the asymptotics of this kernel. Section 8 contains technical proofs.

Acknowledgements The authors would like to thank Alexei Borodin for valuable discussion. M.C. was supported by NSF grant DMS–0806591 and J.K. was supported by the Minerva Foundation and NSF grant DMS–1502665.

3 Some ancillary results

We now collect some results that are fundamental for computing the correlation kernel in Section 6 and for uncovering the intertwining relationship in Section 5. The first generalizes Lemma 2.4 and 2.5 of [14].

Proposition 3.1. The Jacobi polynomials satisfy the following identities: for any $T \in C^1[-1, 1]$,

1. $\sum_{r=0}^{s} \tilde{P}_{r}^{(\alpha)} = \tilde{P}_{s}^{(\alpha+1)}$, for all $s \geq 0$.

2. $\sum_{r=s+1}^{\infty} \langle \tilde{P}_{r}^{(\alpha)}, T \rangle_{\alpha} = \langle \tilde{P}_{s}^{(\alpha+1)}, T(1) - T \rangle_{\alpha}$

3. $\sum_{r=0}^{s-1} P_{r}^{(\alpha+1)} = \frac{\tilde{P}_{s}^{(\alpha+1)}}{x-1}$, for all $s \geq 1$.

4. $\sum_{r=s}^{\infty} \langle P_{r}^{(\alpha+1)}, T \rangle_{\alpha+1} = \langle \tilde{P}_{s}^{(\alpha)}, T \rangle_{\alpha}$

Proof. See section 8.

For smooth $E \in C^\infty[-1, 1]$, denote the $m$th Taylor Remainder of $E$ about 1 by

$$R_m^E(x) := \begin{cases} E(x), & m \leq 0 \\ E(x) - \sum_{k=0}^{m-1} \frac{E^{(k)}(1)}{k!} (x-1)^k, & m \geq 1 \end{cases}$$

and let

$$\Psi_{n-l}(s) = \Psi_{n-l}^E(s) := \left( \frac{P_{s}^{(\alpha_n)}}{2^{\alpha+n+1} \Gamma(\alpha+1)}, (x-1)^{r-n-l} R_{r-n}^E \right)_{\alpha+1}. \tag{15}$$

The next result generalizes Theorem 4.4 in [14].

Lemma 3.2. For any $E \in C^\infty[-1, 1]$, the functions $\Psi_{n-l}^n(s), n \geq 1, l \in \mathbb{Z}$, satisfy the composition rule

$$\left( \phi_{n-1} * \Psi_{r-n-l}^n \right)(s) := \sum_{r \geq 0} \phi_{n-1}(s, r) \Psi_{r-n-l}^n(r) = \Psi_{r-n-1-l}^{n-1}(s).$$

Proof. See section 8.
4 Consistent series of probability measures on partitions

For \( n \geq 1 \), we write \( r_n := \lfloor (n + 1)/2 \rfloor \). Let \( \mathbb{J}_n \) denote the set of partitions \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_{r_n} \geq 0) \) of nonnegative integers of length at most \( r_n \) (the length of a partition \( \lambda \) is the number of nonzero terms). For \( \lambda \in \mathbb{J}_n, \mu \in \mathbb{J}_{n+1} \), write \( \lambda \prec \mu \) if \( \lambda \) interlaces \( \mu \) in the sense

\[
\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \ldots \geq \mu_{r_n} \geq \lambda_{r_n} \geq \mu_{r_n+1}.
\]

For \( \lambda \in \mathbb{J}_n \), the transformation \( \tilde{\lambda}_i := \lambda_i + r_n - i \) arises naturally and frequently in our formulas. For the construction of Section 6.1 define the collection \( \mathbb{J}_{n,\text{seq}} \) of all finite sequences \( u = (u^1, \ldots, u^n) \) of partitions \( u^i \in \mathbb{J}_i \), \( 1 \leq i \leq n \), of length \( n \). Let \( \mathbb{J}_{n,\text{paths}} \subset \mathbb{J}_{n,\text{seq}} \) denote the subset of Gelfand-Tsetlin patterns, i.e., finite interlaced sequences \( u = (u^1 < \cdots < u^n) \). Let \( \mathbb{J}_{\infty,\text{seq}}, \mathbb{J}_{\infty,\text{paths}} \) denote infinite versions of each of these sets. We adopt the following notational convention:

**Important Notational Convention.** Quantities with indices that overflow (e.g. \( \lambda_{r_n+1} \) for \( \lambda \in \mathbb{J}_n \)) are set to 0 and those with indices that underflow (e.g. \( \lambda_0 \) for \( \lambda \in \mathbb{J}_n \)) are set to \( \infty \).

We now collect two elementary results that will allow us to express our probability measures on partition sequences in determinantal form and to work with the resulting expressions. The proof of the first, which is straightforward and now standard, follows along the same lines as Proposition 3.5 of [13], so we only state the version we need.

**Lemma 4.1.** Indication of interlacement between two partitions \( \lambda \in \mathbb{J}_n, \mu \in \mathbb{J}_{n+1} \) takes the determinantal form

\[
1_{(\lambda \prec \mu)} = \begin{cases} 
\det[\tilde{\lambda}_i \prec \tilde{\mu}_i]_{i,j=1}^{n+1}, & \text{if } n \text{ even} \\
\det[\tilde{\lambda}_i \leq \tilde{\mu}_i]_{i,j=1}^{n+1}, & \text{if } n \text{ odd}
\end{cases}
\]

where \( \tilde{\lambda}_i := \lambda_i + r_n - i \).

For reference, we recite a variant of the Cauchy–Binet formula, also referred to as the Andreiev identity [4].

**Lemma 4.2.** (Lemma 2.1, [14]) For each \( k \geq 0 \), let \( f_k, g_k \) be functions on \( \mathbb{C} \) such that \( \sum_{k=0}^{\infty} f_k(w_i)g_k(z_j) \) converges absolutely for some \( w_i, z_j \in \mathbb{C}, 1 \leq i, j \leq n \). Then

\[
\sum_{k_1 > \ldots > k_n \geq 0} \det[f_{k_i}(w_i)]_{i,j=1}^{n} \det[g_{k_i}(z_j)]_{i,j=1}^{n} = \det \left[ \sum_{k=0}^{\infty} f_k(w_i)g_k(z_j) \right]_{i,j=1}^{n}.
\]

We now introduce the fundamental measures on partitions (equation (18) below) and their cotransitions (see equation (16) below and Section 5.2 of [31] for the origin of this terminology). These cotransitions link the measures, analogous to how transition probabilities link measures on a state space; however, these be properly called transitions, because the measures live on different spaces. For \( \lambda \in \mathbb{J}_n \), let

\[
d_{\lambda}^{(n)}(x_1, \ldots, x_{r_n}) = d_{\lambda}^{(\alpha, \beta;n)}(x_1, \ldots, x_{r_n}) := \frac{\det \left[ \tilde{P}_{\lambda}^{(\alpha)}(x_j) \right]_{i,j=1}^{r_n}}{\det[x_j^{r_n-i}]_{i,j=1}^{r_n}}.
\]

Note that this is well defined at \( x_i = x_j \) because the denominator is a Vandermonde determinant, which divides the numerator. Up to a constant, \( d_{\lambda}^{(n)} \) is a \( BC_{r_n} \) orthogonal polynomial at \( \theta = 1 \), see Proposition 7.1 of [38]. Furthermore, it has been previously shown in Proposition 7.5 of [38] that \( d_{\lambda}^{(n)}(1, \ldots, 1) \) is non-negative for all \( \lambda \in \mathbb{J}_n \). For the next result, recall the definition of \( \phi_n \) in section
Proposition 4.3. The quantities

\[ T_{n-1}^n(\lambda, \mu) := \prod_{k=1}^{r_{n-1}} \phi_{n-1}(\tilde{\mu}_k) \frac{d_{\hat{\lambda}}^{(n)}(1, \ldots, 1)}{d_{\hat{\lambda}}^{(n)}(1, \ldots, 1)} = \det[\phi_{n-1}(\tilde{\mu}_i, \tilde{\lambda}_j)]_{i,j=1}^{r_{n-1}} \frac{d_{\hat{\lambda}}^{(n)}(1, \ldots, 1)}{d_{\hat{\lambda}}^{(n)}(1, \ldots, 1)} \]  

are cotransition probabilities from \( \lambda \in \mathbb{J}_n \) down to \( \mu \in \mathbb{J}_{n-1} \), \( n \geq 2 \). In particular, this furnishes stochastic operators \( T_{n-1}^n \).

Proof. The second equality of (16) follows from Lemma 4.1 and the convention \( \phi_{n-1}(-1, \cdot) \equiv 1 \) for \( n \) odd.

Now we prove the sum-to–unity property. If \( n \) is odd so that \( \alpha_n = \alpha \) and \( r_n = r_{n-1} \), then

\[
\det \left[ \frac{\tilde{P}^{(\alpha_n)}(x_j)}{\det[x_j^{r_{n-1}}]} \right]_{i,j=1}^{r_{n-1}} = \det \left[ \frac{\tilde{P}^{(\alpha_n)}(x_j) - \tilde{P}^{(\alpha)}(x_j)}{x_j - 1} \right]_{i,j=1}^{r_{n-1}} = \sum_{\lambda \in \mathbb{J}_{n-1} \ni \mu < \lambda} \prod_{k=1}^{r_{n-1}} \phi_{n-1}(\tilde{\mu}_k) \frac{\det[\tilde{P}^{(\alpha_n-1)}(x_j)]_{i,j=1}^{r_{n-1}}}{\det[x_j^{r_{n-1}}]} \frac{\det[\tilde{P}^{(\alpha)}(x_j)]_{i,j=1}^{r_{n-1}}}{\det[x_j^{r_{n-1}}]},
\]

where we have used the fact \( \tilde{P}^{(\alpha_n)}(1) = 1 \) and indicated properties of the determinants in the first equality, the identity (31) in the second, and the fact \( \tilde{P}^{(\alpha_n)} = \phi_n(s) \tilde{P}^{(\alpha_n)} \) in the last. If instead \( n \) is even, then \( r_n = r_{n-1} \) and the same identity holds without the evaluation “\( x_{r_n} = 1 \)” by using the first item in Proposition 3.1.

Remark 4.4. The proposition shows that \( d_{\hat{\lambda}}^{(n)}(1, \ldots, 1) \) can be written as a linear combination of \( d_{\mu}^{(n-1)}(1, \ldots, 1) \), a property that is called a branching rule. Iterating the branching shows the quantities \( d_{\hat{\lambda}}^{(n)}(1, \ldots, 1) \) have a combinatorial expression as a sum of weights over all paths of length \( r_n \) ending at \( \lambda \), which was already established as (7.5) and (7.6) in the proof of Proposition 7.5 of Okounkov–Olshanski [38]. The branching weights are built from the coefficients \( B(m, \ell) \) of equation (7.14) in [38], which are given in our notation by

\[
B(m, \ell) = \phi_n(m) \phi_{n-1}(\ell) = \frac{C_m^\ell C_{\ell+1}^{n-1}}{C_m^{n-1} C_{\ell}^{n}}, \quad n \text{ even}.
\]

Let \( \psi \in C^1[-1, 1] \). The cotransition operators \( T_{n-1}^n \) serve to link the series of measures on \( \mathbb{J}_n \) defined by (recalling that \( \Psi \) was defined in (15))

\[
P_n^{\psi}(\lambda) := \left[ \frac{\tilde{P}^{(\alpha_n, \alpha+1)}(x-1)^{r_n-i} \psi}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \right]_{i,j=1}^{r_n} d_{\hat{\lambda}}^{(n)}(1, \ldots, 1) = \det \left[ \psi_{r_n-i}(\tilde{\lambda}_j) \right]_{i,j=1}^{r_n} \frac{d_{\hat{\lambda}}^{(n)}(1, \ldots, 1)}{d_{\hat{\lambda}}^{(n)}(1, \ldots, 1)}.
\]  

(18)

Proposition 4.5. The series of measures \( \{P_n^{\psi}\}_{n=1}^{\infty} \) is consistent in the sense

\[
P_n^{\psi}(\mu) = (P_n^{\psi} T_{n-1}^n)(\mu) := \sum_{\lambda \in \mathbb{J}_n} P_n^{\psi}(\lambda) T_{n-1}^n(\lambda, \mu), \quad n \geq 2,
\]

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and each satisfy
\[ \sum_{\mu \in \mathcal{J}_n} P^n(\mu) \frac{d(n)}{\mu}(x_1, \ldots, x_n) = \psi(x_1) \cdots \psi(x_n), \]
so in particular, have mass \( \psi(1)^n \). Moreover, if \( \psi \equiv 1 \) then \( P^n(\lambda) = \delta_0(\lambda) \).

**Proof.** See section 8. \qed

## 5 Multilevel Markov process

### 5.1 Single level dynamics

The next proposition dictates a natural way to transition between the measures \( P^n \) on a single level \( \mathcal{J}_n \). Recall that \( P^n \) was defined in (18). For any fixed \( \psi \in C^1[-1,1] \), define a \( \mathcal{J}_n \times \mathcal{J}_n \)-matrix \( T^n_\psi \) with entries (recalling the scaling \( 11 \))

\[
T^n_\psi(\mu, \lambda) := \det \left[ \frac{\langle \tilde{P}^{(\alpha)}(\mu), \tilde{P}^{(\alpha)}(\lambda) \rangle}{\alpha n} \right]_{\|=1}^{\|=r_n}, \]

The superscript \( \psi \) in \( T^n_\psi \) will always be included to distinguish it from \( T^{n+1} \). We now seek a class of functions \( \psi \) in \( C^1[-1,1] \) for which the matrix \( T^n_\psi \) will be stochastic for all \( n \geq 1 \). In light of Theorem 5.2 of [38], a natural class of functions to consider is the set \( \mathcal{Y} \) of convex combinations of products of functions of the form

\[ e^{\gamma(x-1)}, \quad 1 + p(x-1), \quad \frac{1}{1 - q(x-1)}. \]

Indeed, if \( T^n_\psi \) is stochastic for all \( n \geq 1 \), then by Proposition 5.1 below \( P^n_\psi \) is a probability measure for all \( n \geq 1 \), so \( \psi \in \mathcal{Y} \). However, the converse is not true: see Remark 5.2 below.

The next result generalizes statements in Section 3.1 of [14].

**Proposition 5.1.** Let \( \psi, \psi_1, \psi_2 \in C^1[-1,1] \). Let \( \psi_m(x) := e^{\gamma(x-1)}. \)

1. \( \sum_{\mu \in \mathcal{J}_n} P^{\psi_1}_n(\mu) T^{\psi_2}_n(\mu, \lambda) = P^{\psi_1 \psi_2}_n(\lambda), \) and similarly \( T^{\psi_1}_n T^{\psi_2}_n = T^{\psi_1 \psi_2}_n. \)
2. \( \sum_{\lambda \in \mathcal{J}_n} T^{\psi}_n(\mu, \lambda) = \psi(1)^{r_n}. \)
3. If \( \psi_m \in C^1[-1,1] \) converge uniformly to \( \psi \), then \( T^{\psi_m}_n(\mu, \lambda) \rightarrow T^{\psi}_n(\mu, \lambda) \) as \( m \rightarrow \infty. \)
4. Set \( \phi(x) = 1 + a(x-1) \) where \( a \) is non-negative. For \( m \) odd, \( T^{\psi}_n \) is stochastic if

\[
a^{-1} \geq \min \left( 2, \frac{4\alpha^3 + \alpha^2(8\beta + 9) + 4\alpha (\beta^2 + 3\beta + 1) + \beta(3\beta + 4)}{\alpha(\alpha + \beta + 1)(\alpha + \beta + 2)}, \right.
\]
\[
\left. 4\alpha^3 + 8(2\alpha^2 + 3\alpha(\beta + 1)) + \alpha^2(8\beta + 9) + (4\alpha + 8)(\beta^2 + 3\beta + 1) + 24(\alpha + \beta + 1) + \beta(3\beta + 4) + 16 \right) \frac{1}{(\alpha + \beta + 2)(\alpha + \beta + 3)(\alpha + \beta + 4)}, \]

In particular, this lower bound is equal to 2 if and only if the two inequalities \((2\alpha + 2\beta + 7)(\alpha^2 - \beta^2) \leq 0 \) and \((2\alpha + 2\beta + 3)(\alpha^2 - \beta^2) \leq 0\) hold.
For \( n \) even, \( T_n^\phi \) is stochastic if
\[
a^{-1} \geq \min \left( 2, \frac{4\alpha^3 + \alpha^2(8\beta + 21) + \alpha(4\beta^2 + 28\beta + 34) + 7\beta^2 + 24\beta + 17}{(\alpha + \beta + 1)(\alpha + \beta + 2)(\alpha + \beta + 3)}, \right.
\[
\left. \frac{4\alpha^3 + \alpha^2(8\beta + 37) + \alpha(4\beta^2 + 52\beta + 114) + 15\beta^2 + 96\beta + 129}{(\alpha + \beta + 3)(\alpha + \beta + 4)(\alpha + \beta + 5)} \right).
\]

In particular, the lower bound is equal to \( 2 \) if and only if the two inequalities \( (2(\alpha + 1) + 3)(\alpha + 1) \geq (2\beta + 3)\beta \) and
\[
2\alpha^3 + \alpha^2(2\beta + 13) + \alpha(-2\beta^2 + 4\beta + 20) + 2\beta + 9 \geq \beta^2(2\beta + 9)
\]
hold.

5. For any fixed \( \alpha, \beta > -1 \), the semigroup \( \{T_n^\psi\}_{n \geq 0} \) operating on the Banach space of absolutely summable functions \( l^1(\mathbb{J}_n) \) on \( \mathbb{J}_n \) is stochastic and satisfies \( \|T_n^\psi - Id\|_{l^1(\mathbb{J}_n)} \to 0 \).

Proof. See section \([8]\). \(\square\)

**Remark 5.2.** In Proposition 3.3 of \([14]\), the values \( \alpha = -1/2 \) and \( \beta = -1/2 \) are considered with the conclusion that \( T_n^\phi \) is stochastic for \( \phi(x) = 1 + a(x - 1) \) when \( a \) is bounded above by \( 1/2 \). Therefore, Proposition 5.1.4 is a generalization of that result.

For \( \phi(x) = 1 + a(x - 1) \) with \( a > 1/2 \) there are values of \( n \) for which \( T_n^\phi \) is not stochastic. For example, consider \( T_n^\phi \) for \( n = 1, 2 \). By \([5]\),
\[
(1 + a(x - 1))P_k^{(\alpha)}(x) = (1 + a(A_k^{(\alpha,\beta)} - 1))P_k^{(\alpha)}(x) + aB_k^{(\alpha,\beta)}P_{k-1}^{(\alpha)}(x) + aC_k^{(\alpha,\beta)}P_{k+1}^{(\alpha)}(x).
\]
Since \( B_k^{(\alpha,\beta)}, C_k^{(\alpha,\beta)} \) are positive and \( 1 > \frac{|\beta - \alpha|}{\alpha + \beta + 2} = |A_0| \geq |A_1| \geq \ldots \) with \( \lim_{k \to \infty} |A_k| = 0 \), then \( T_n^\phi \) will be stochastic if and only if
\[
a \leq \frac{1}{1 - A_0^{(\alpha_n,\beta)}} = \frac{\alpha_n + \beta + 2}{2\alpha_n + 2}.
\]

Due to results of \([8]\), it might be possible to interpret \( T_n^\phi \) as a discrete–time particle system with Bernoulli jumps, but this direction is not pursued here.

**Remark 5.3.** When \( \psi = (1 - q(x - 1))^{-1} \), the matrix \( T_n^\psi \) has been previously considered in \([33, 34]\) with the values \( |\alpha| = |\beta| = 1/2 \) arising from the representation theory of the orthogonal and symplectic groups. In those cases, there was the property that \( T_n^\psi(\lambda, \mu) \) was nonzero only if there existed a \( \nu \) such that \( \nu < \lambda, \mu \), which lead to the corresponding Markov process being interpreted as a discrete–time particle system having geometric jump rates with states \( \nu \) at half–integer times. However, one can check computationally that this property will not hold for general \( \alpha, \beta > -1 \).
5.2 Intertwined multilevel dynamics: discrete steps

We now implement the procedure for constructing multilevel dynamics with certain restrictions introduced in Borodin–Ferrari [10], based on the construction in Diaconis–Fill [25]. These stochastic operators \( T_{n-1}^n \) defined above ensure that the interlacing condition is preserved when we link single level dynamics to a multilevel evolution, and the following intertwining relationship is key.

**Proposition 5.4.** Fix \( \psi \in C^1([-1, 1]) \) with \( \psi(1) = 1 \). For \( n \geq 1 \), the operators \( T_n^\psi \) and \( T_{n+1}^\psi \) satisfy the intertwining relations

\[
\Delta_{n+1} := T_{n+1}^\psi T_n^\psi = T_n^\psi T_{n+1}^\psi.
\]

**Proof.** See section 8. \( \square \)

Because the intertwining relationship holds, the Borodin–Ferrari construction [10] can be applied. For \( u, t \in \mathbb{J}_{n, paths} \) define

\[
L_{k+1}^\psi(u, t) := \frac{T_k^\psi (u_{k+1}, t_{k+1})T_k^\psi (t_{k+1}, t_k)}{\Delta_k^\psi (u_{k+1}, t_k)} 1_{\Delta_k^\psi (u_{k+1}, t_k) \neq 0}.
\]

(19)

In words, it is the probability of the transition \( u_{k+1} \rightarrow t_{k+1} \rightarrow t_k \) by a jump and cotransition conditional on \( u_{k+1} \rightarrow t_k \) occurring by such steps; note this quantity only depends on \( u_{k+1}, t_k, t_{k+1} \).

Define the transition operator

\[
A_n^\psi(u, t) := T_1^\psi(u_1^1, t_1^1) \cdot \prod_{k=1}^{n-1} L_{k+1}^\psi(u, t),
\]

which is stochastic when \( \psi(x) = e^{\gamma(x-1)} \) for \( \gamma \geq 0 \) (by Propositions 4.3 and 5.1.5).

Just as the stochastic (for suitable \( \psi \)) operators \( T_n^\psi \) account for transitions between the probability measures \( P_n^\psi \) on \( \mathbb{J}_n \) (the first part of Proposition 5.1.1), there is analogous statement for \( A_n^\psi \):

**Proposition 5.5.** Let \( m_n(x_n) \) be a probability measure on \( \mathbb{J}_n \). Consider the evolution of the measure \( m_n(x_n)T_{n-1}^n(x_n, x_{n-1}) \cdots T_1^\psi(x_2, x_1) \) on \( \mathbb{J}_{n, paths} \) under \( A_n^\psi \), and denote by \( (x_1(j), \ldots, x_n(j)) \) the result after \( j = 0, 1, 2, \ldots \) steps. Then for any \( k_1 \geq k_2 \geq \ldots \geq k_n \geq 0 \), the joint distribution of

\[
(x_n(1), \ldots, x_n(k_n), x_{n-1}(k_n), x_{n-1}(k_n+1), \ldots, x_{n-1}(k_{n-1}),
\]

\[
x_{n-2}(k_{n-1}), \ldots, x_2(k_2), x_1(k_2), \ldots, x_1(k_1))
\]

coincides with the evolution of \( m_n \) under the transition matrices

\[
\begin{pmatrix}
T_1^\psi, & \cdots, & T_1^\psi \\
T_{k_1}^\psi, & \cdots, & T_{k_1}^\psi \\
T_{k_2}^\psi, & \cdots, & T_{k_2}^\psi
\end{pmatrix},
\]

Proof. See Proposition 2.5 of [10]. \( \square \)

This proposition, along with Proposition 5.1.1, implies that \( A_n^\psi \) supplies an evolution of signed measures on \( \mathbb{J}_{n, paths} \) of the form

\[
P_{n, paths}^\psi(t) := P_n^\psi(t^n) \cdot \prod_{k=1}^{n-1} T_{k}^{t_{k+1}^1, t_k^k}, \quad t \in \mathbb{J}_{n, paths},
\]

(20)

in the sense

\[
(P_{n, paths}^\psi A_n^\psi)(t) = P_{n, paths}^\psi(t), \quad \psi_1, \psi_2 \in C^1[-1, 1], \quad t \in \mathbb{J}_{n, paths}.
\]

(21)
5.3 Description of continuous-time multilevel dynamics

We now explain how to obtain the interacting particle system described in the introduction. Recall that in general, if $Q$ is a matrix with countably many rows and columns such that its rows add up to 0, its off–diagonal entries are non–negative, and its diagonal entries are uniformly bounded, then there is a unique continuous–time Markov chain with $Q$ as its generator (see e.g. Proposition 2.10 of [2]). In words, this Markov chain satisfies

- In state $i$, a jump takes place after exponential waiting time with parameter $-Q_{ii}$.
- The system makes a jump to state $j$ with probability $-Q_{ij}/Q_{ii}$.

With this in mind, we aim for the following, noting that it is similar to Theorem 3.12 of [14].

**Proposition 5.6.** There exists a matrix $Q_n$ with rows and columns indexed by $\mathbb{J}_{n,\text{paths}}$ such that for all $\gamma \geq 0$, and any $\phi \in C^1[-1,1]$ satisfying $\phi(1) \neq 0$,

$$e^{\gamma Q_n} \cdot P_{n,\text{paths}}^\phi = A_{n,\gamma}^\phi \cdot P_{n,\text{paths}}^\phi.$$ 

**Proof.** Let $\mathfrak{B}_n$ be the Banach space defined as the completion of the subspace of $l^1(\mathbb{J}_{n,\text{paths}})$ consisting of measures of the form (20) corresponding to functions $\psi \in C^1[-1,1]$. The stochastic operators $\{A_{n,\gamma}^\psi\}_{\gamma \geq 0}$ form a semigroup on $\mathfrak{B}_n$ by (21). Hence (see, e.g. Theorem 2.6.1 of [29] or Proposition 3.13 of [14]), if suffices to show that

$$\|A_{n,\gamma}^\psi - Id\|_{\mathfrak{B}_n} \xrightarrow{\gamma \to 0} 0.$$

Showing this for $\{A_{n,\gamma}^\psi\}_{\gamma \geq 0}$ follows from the same property of $\{T_{n,\gamma}^\psi\}_{\gamma \geq 0}$ (the fifth part of Proposition 5.1): the form (20) implies

$$\sum_{u \in \mathbb{J}_{n,\text{paths}}} |P_{n,\text{paths}}^\psi(u) - P_{n,\text{paths}}^\psi(u)| = \sum_{u^\gamma \in \mathbb{J}_n} |P_{n,\text{paths}}^\psi(u^\gamma) - P_{n,\text{paths}}^\psi(u^\gamma)| \sum_{k=1}^{n-1} \prod_{t_{k-1} \leq t < t_k} T_{t_k}^k(u^{k-1}, u^k)$$

$$= \sum_{u^\gamma \in \mathbb{J}_n} |P_{n,\text{paths}}^\psi(u^\gamma) - P_{n,\text{paths}}^\psi(u^\gamma)|,$$

so we have

$$\|A_{n,\gamma}^\psi - Id\|_{\mathfrak{B}_n} = \|T_{n,\gamma}^\psi - Id\|_{l^1(\mathbb{J}_n)} \xrightarrow{\gamma \to 0} 0,$$

as needed. 

To describe the particle system, we endeavor to arrive at a description of $\tilde{Q}_n = \frac{d}{d\gamma}\big|_{\gamma=0} A_{n,\gamma}^\psi$. This involves computing

$$\frac{T_k^\psi(u^k, t^k)T_{k-1}^k(t^k, t^{k-1})}{\Delta_{k-1}(u^k, t^{k-1})}$$

(22)

to second order (note in this expression the ratios of the “$d_n(\cdot)$” terms and the factors “$\prod_{\ell=1}^{r_{k-1}} \phi_{k-1}(\tilde{t}_{\ell}^{-1})$” cancel). By construction, we arrive at the same particle pattern and push–block dynamics as in the orthogonal and symplectic cases, but with different jump rates. More precisely, if a particle is at position $\lambda \geq 0$ on level $n \geq 1$ and is allowed to jump to position $\mu$, then it will do so at a rate

$$\frac{\left\langle P_{\lambda,\gamma}^{(\alpha_0)}, P_{\mu,\gamma}^{(\alpha_0)} x \right\rangle_{\alpha_0}}{2^{\alpha+\beta+1} \Gamma(\alpha+1)} = \begin{cases} \frac{\hat{c}_{\lambda+1}^{\alpha_0}}{\hat{c}_{\lambda}^{\alpha_0}} B_{\lambda+1}^{(\alpha_0, \beta)} = \frac{\hat{c}_{\lambda+1}^{\alpha_0}}{\hat{c}_{\lambda}^{\alpha_0}} C_{\lambda+1}^{(\alpha_0, \beta)}, & \lambda \geq 0, \mu = \lambda + 1 \\ \frac{\hat{c}_{\lambda-1}^{\alpha_0}}{\hat{c}_{\lambda}^{\alpha_0}} C_{\lambda-1}^{(\alpha_0, \beta)} = \frac{\hat{c}_{\lambda-1}^{\alpha_0}}{\hat{c}_{\lambda}^{\alpha_0}} B_{\lambda-1}^{(\alpha_0, \beta)}, & \lambda \geq 1, \mu = \lambda - 1 \end{cases}.$$
Let us exemplify the calculation. Assume the system is in a state so that a wall jump is possible at an odd level \( k \geq 1 \), i.e., a state \( t \in \mathbb{J}_{\text{n.paths}} \) that satisfies \( t^k r_k = t^k \rightarrow 0 \), \( t^k r_{k-1} \geq 1 \), so \( t^k r_{k-1} \geq 2 \), and if \( k \geq 3 \), \( t^{k-1} r_{k-1} = t^{k-1} \rightarrow 0 \geq 1 \). Consider a transition to \( u \in \mathbb{J}_{\text{n.paths}} \) that agrees with \( t \in \mathbb{J}_{\text{n.paths}} \) except that \( u^k r_k = u^k \rightarrow 0 \). Noting that \( \psi_{\gamma}(x) = 1 + \gamma(x - 1) + O(\gamma^2) \) for small \( \gamma > 0 \), we use \( \left[35\right] \) to compute, up to second-order in \( \gamma \), (omitting the superscript “\( k \)

det \left[ \frac{1}{2^{\alpha+\beta+1} \Gamma(\alpha+1)} \left( \tilde{P}_{\alpha}^{(\alpha)} \right) \right]_{i,j=1}^{r_k} \begin{align*}
&= \begin{bmatrix}
1 - \gamma + \gamma A_{t_1} & \gamma \frac{c_{t_2}}{c_{t_2}+1} C_{t_2-1} & 0 & 0 \\
\gamma \frac{c_{t_2}}{c_{t_2}+1} C_{t_2-1} & 1 - \gamma + \gamma A_{t_2} & \gamma \frac{c_{t_3}}{c_{t_3}+1} C_{t_3-1} & 0 \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & \vdots & \vdots & \ddots & (1 - \gamma) + \gamma A_{t_{k-1}} & \gamma \frac{c_{t_{k}}}{c_{t_{k}}+1} C_{t_{k}-1} & \gamma \frac{c_{t_{k+1}}}{c_{t_{k+1}}+1} C_{t_{k+1}-1} & \gamma \frac{c_{t_{k+2}}}{c_{t_{k+2}}+1} C_{t_{k+2}-1} \\
\end{bmatrix} \\
&\approx \text{det} \begin{bmatrix}
2(\alpha+1) + O(\gamma^2). \\
\end{bmatrix}
\end{align*}

Similar calculations in the denominator of \( \left[22\right] \) yield an order of \( 1 + O(\gamma) \). Since the terms \( \text{“}d^{(\alpha)}(\cdot)\text{”} \) occurring in the numerator and denominator all cancel, we arrive at

\[
\frac{T_k^{\psi_{\gamma}}(t^k, u^k) T_{k-1}^{\psi_{\gamma}}(u^k, u^{k-1})}{\Delta_k^{\psi_{\gamma}}(u^k, t^k)} = \frac{\gamma \cdot 2(\alpha+1)}{(\alpha+\beta+2)} + O(\gamma^2) \cdot \frac{1 + O(\gamma)}{1 + O(\gamma)} = \left[ \frac{\gamma \cdot 2(\alpha+1)}{(\alpha+\beta+2)} + O(\gamma^2) \right] (1 + O(\gamma)) = \gamma \cdot \frac{2(\alpha+1)}{(\alpha+\beta+2)} + O(\gamma^2).
\]

where the second equality uses the geometric series. The other cases can be handled similarly.

### 6 Determinantal correlations

#### 6.1 Probability measures on partition paths

For \( u \in \mathbb{J}_{\text{n.seq}} \), consider the cylinder sets \( C_u := \{ t \in \mathbb{J}_{\text{\infty.seq}} \mid t^1 = u^1, \ldots, t^n = u^n \} \). We now realize the measures \( P^{\gamma} \) on \( \mathbb{J}_n \) of Proposition [4.5] as embedded in a single probability measure \( P^{\gamma} \) on \( \mathbb{J}_{\infty.seq} \) by the prescription

\[
P^{\gamma}(C_u) := P^{\gamma}_{\text{n.paths}}(u) = \left( P^{\psi_{\gamma}}_{\text{n.}}(u^n) \cdot \prod_{k=1}^{n-1} T_{k+1}^k(u^{k+1}, u^k) \right), \quad u \in \mathbb{J}_{\text{n.seq}}.
\]

Note that \( P^{\gamma} \) supported on \( \mathbb{J}_{\text{n.paths}} \subset \mathbb{J}_{\infty.seq} \), and the consistency relation of Proposition [4.5] guarantee \( P^{\gamma} \) is well-defined. The mapping

\[
\mathbb{J}_{\infty.seq} \ni u \mapsto \{ u^n_k = u^n_k + (r_n - k) \mid n \geq 1, \ 1 \leq k \leq r_n \}.
\]

pushes \( P^{\gamma} \) forward to determine a probability measure \( \xi^{\gamma} \) on \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0} \), which the last section showed is the fixed-time distribution of the random point configuration \( \mathcal{X}^{(\alpha, \beta)}_{\gamma} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0} \) described in the introduction. We will also refer to this point process as the \( (\alpha, \beta)\text{-Plancherel point process} \) (this title does not depend on the choice of coordinates).
6.2 Computation of correlation kernel

For any \( n \geq 1 \), a point configuration \( \mathbb{X}_n := \{ x^m_k \in \mathbb{Z} \mid 1 \leq m \leq n, 1 \leq k \leq r_m \} \) in \( \mathbb{Z}_{\geq 0} \times [n] \) determines the cylinder set \( C_u := \widehat{\pi}^{-1}(\mathbb{X}_n) \) with \( u \in \mathbb{J}_{n,seq} \) satisfying \( \tilde{u}_m^m = x^m_k, 1 \leq m \leq n \). Using Lemma 4.1, our push-forward measure \( \xi^r \) satisfies the main expression (13) for \( K \).

Proof. See section 8.

For any \( n, m \geq 1 \),

\[
\sum_{k=1}^{r_m} \Psi^n_{r_m-k}(s) \Phi^m_{r_m-k}(t) = \int \frac{\widehat{P}^{(\alpha_n)}_s(u)}{E(u)(w-1)^{r_m-k+1}} du
\]

\[
= \int \frac{(x-1)^{r_m-r_n} R^{E}_{r_m-r_n} (x-1)^{r_n-r_m} \widehat{P}^{(\alpha_m)}_t(u)(u-1)^{r_n-r_m}}{x-u} \frac{w}{E(u)(w-1)^{r_m-k+1}} du.
\]

Note how the terms in (25) simplify if \( r_n \geq r_m \), since in this case \( R^{E}_{r_m-r_n} \equiv E \).

Proof. See Section 8.

Now plug the expression (25) into (24). If \( r_n \geq r_m \), the expression (25) simplifies, quickly leading to the main expression (13) for \( K^r \) in Theorem 2.1. (note we have multiplied by the conjugating determinant form)
factor \((-1)^{r_n-r_m}\), which vanishes in the determinant. But if \(r_n < r_m\), we get (13) along with the additional term
\[
\frac{1}{2\pi i} \int_{C} \left( \frac{\mathcal{P}_s^{(\alpha_n)}(x)}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \frac{\mathcal{P}_t^{(\alpha_m)}(u)}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \frac{(u-1)^{r_n-r_m} R_{r_m-r_n}(u)}{E(u)(x-u)} \right) \, du
\]
\[
+ \left( \frac{\mathcal{P}_s^{(\alpha_n)}}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \right) \frac{\mathcal{P}_t^{(\alpha_m)}(x-1)^{r_n-r_m} R_{r_m-r_n}}{E} \, \alpha_n
\]
The residue of the first term at \(u = x\) exactly cancels the second, and since
\[
(u-1)^{r_n-r_m} R_{r_m-r_n}(u) = \sum_{k=r_m-r_n}^{\infty} \frac{E(k)(1)}{k!} (u-1)^{k+r_n-r_m},
\]
there is no residue at \(u = 1\) (note of course \(E = \psi_\gamma\) does not have a pole at 1 either). This completes the proof of Theorem 2.1.

7 Edge asymptotic analysis

7.1 Finite distance from the wall: Jacobi kernel

Note that an essentially identical asymptotic analysis previously appeared in [14]; however, we repeat the details here.

Assume that \(\gamma, r_n, r_m\) depend on \(N\) in such a way that \(\gamma/N \to \tau > 0\) and \(r_n/N, r_m/N \to \eta > 0\), but that \(s, t\) are fixed and finite. Assume the parity of each \(n_i\) is constant and that the differences \(r_{n_i} - r_{m_i}\) are of constant order. Let \(A(z) := \tau z + \eta \log(1 - z)\) and note \(1 - \eta/\tau\) is a zero of \(A'(z)\). Write our kernel as
\[
\frac{1}{2\pi i} \int_{-\infty}^{1} \left( \frac{e^{-N(A(1-\eta/\tau)-A(x))}}{e^{N(A(u)-A(1-\eta/\tau))}} \frac{\mathcal{P}_s^{(\alpha_n)}(x)}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \frac{\mathcal{P}_t^{(\alpha_m)}(u)}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \frac{(1-x)^{\alpha_n}(1+x)^{\beta}}{x-u} \right) \, dx
\]
\[
+ 1_{(n \geq m)} \left( \frac{\mathcal{P}_s^{(\alpha_n)}}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \right) \frac{(x-1)^{r_n-r_m} \mathcal{P}_t^{(\alpha_m)}}{x} \, \alpha_n
\]
Now deform the \(u\)-contour, as in Figure 4, to be a steepest ascent loop remaining in the region \(\Re(A(u)) - A(1-\eta/\tau)) > 0\) and passing through the critical point \(1 - \eta/\tau\). Let us briefly explain why the contour deformation exists. Note that \(\Re(A(z)) = \Re(A(z))\), so the image in Figure 4 must be symmetric about the real axis. Along the real axis, we have \(\Re(A(z)) = \tau z + \eta \log |1 - z|\). Therefore, \(\Re(A(z) - A(1 - \eta/\tau)) < 0\) in a neighborhood of 1, and there is a unique point \(w > 1\) such that \(\Re(A(z) - A(1 - \eta/\tau)) > 0\) for \(z > w\) and \(\Re(A(z) - A(1 - \eta/\tau)) < 0\) for \(1 < z < w\). To analyze the case when \(z < 1\), note that \(\frac{d}{dz}\Re(A(z)) = \tau - \eta(1 - z)^{-1}\), which is positive for \(z < 1 - \eta/\tau\) and negative for \(1 - \eta/\tau < z < 1\), which implies that \(\Re(A(z) - A(1 - \eta/\tau)) < 0\) for all \(z \in (-\infty, 1) \setminus \{1 - \eta/\tau\}\). This establishes the existence of the contour deformation.

After the deformations are made, there is the convergence
\[
\frac{e^{-N(A(1-\eta/\tau)-A(x))}}{e^{N(A(u)-A(1-\eta/\tau))}} \to 0,
\]
so the integrand converges to 0. However, if \( 1 - \eta / \tau > -1 \), the deformations cause the double integral to pick up residues at \( u = x \) given by

\[
-\frac{1}{2^{n+\beta+1}\Gamma(\alpha+1)} \int_{-1}^{1-\eta/\tau} \tilde{P}_s^{(\alpha)}(x)\tilde{P}_t^{(\alpha_m)}(x)(x-1)^{\alpha_n-r_m}(1-x)^{\alpha_n}(1+x)^{\beta} dx.
\]

Adding these residues to the \( 1_{(n \geq m)} \) term, and including the conjugating factor \((-1)^{\alpha_n-r_m}\) shows convergence to the discrete Jacobi kernel.

If \( 1 - \eta / \tau < -1 \), the unit interval already lies in the region \( \mathcal{R}(A(1 - \eta / \tau) - A(x)) > 0 \), so the double integral term tends to zero without picking up residues. Thus, \( \det[K^\gamma] \) converges to a triangular matrix whose diagonal entries are given by the \( 1_{(n \geq m)} \) term, which equals 1. This completes the proof of Theorem 2.2.

7.2 Hard–edge Pearcey kernel

Assume that \( \gamma, s, t \) and \( r_n \) depend on \( N \) in such a way that in the \( N \to \infty \) limit, \( \gamma/N \to 1/2 \), \( s/N^{1/4} \to \nu_1 > 0 \), \( t/N^{1/4} \to \nu_2 > 0 \), and lastly that \( (r_n - N)/\sqrt{N} \to \sigma_1 \), \( (r_m - N)/\sqrt{N} \to \sigma_2 \). Assume also that \( n, m \) have constant parity. The Mehler–Heine formula (Chapter 8 of Szegő [41]) and the symmetry relation \( P_n^{(\alpha, \beta)}(-z) = (-1)^n P_n^{(\beta, \alpha)}(z) \) tells us that for any \( \alpha, \beta > -1 \),

\[
\lim_{N \to \infty} N^{-\beta/4}(-1)^{\lceil\nu_1 N^{1/4}\rceil} P_n^{(\alpha, \beta)} \left( \frac{z}{\sqrt{N}} - 1 \right) = \left( \frac{\sqrt{2z}}{2} \right)^{-\beta} J_\beta(\nu_1 \sqrt{2z}).
\] (26)
Note that the right-hand-side only depends on $\beta$. Recall the particle-hole involution $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}) \setminus (X(\gamma))$ of $X_{(\alpha, \beta)}$ is also determinantal with correlation kernel (here, $\psi_{\gamma}(x) = e^{\gamma x}$)

$$K_{\Delta}((s, n), (t, m)) := \delta_{s,t} \delta_{n,m} - K^\alpha((s, n), (t, m))$$

$$= -\frac{1}{2\pi i} \oint_{-1}^{1} \oint_{-1}^{1} \psi_{\gamma}(x) \frac{\overline{P}_s(\alpha_n)}{2^{\alpha + 1} \Gamma(\alpha + 1)} \hat{P}_t(\alpha_m) \frac{(x - 1)^{\alpha_n}(1 + x)^{\beta}}{(u - 1)^{\alpha_m}} dx du$$

$$- 1_{(n>m)} \left( \overline{P}_s(\alpha_n) \right) \frac{\overline{P}_s(\alpha_n)}{2^{\alpha + 1} \Gamma(\alpha + 1)} \left( x - 1 \right)^{\alpha_n} \hat{P}_t(\alpha_m) \right\}_{\alpha_n},$$

where we have used the fact that

$$1_{(n=m)} \delta_{s,t} = 1_{(n=m)} \left( \overline{P}_s(\alpha_n) \right) \frac{\overline{P}_s(\alpha_n)}{2^{\alpha + 1} \Gamma(\alpha + 1)} \left( x - 1 \right)^{\alpha_n} \hat{P}_t(\alpha_m) \right\}_{\alpha_n}. \quad (27)$$

Deforming the $u$-contour in the double integral term of (27) as in Figure 5, we endeavor to find

Figure 5: Steepest descent deformations. The yellow region signifies the region $\Re(A(u) - A(-1)) > 0$ and the white elsewhere.

the contribution at $-1$. Making the substitutions $x' = N^{1/2}(x + 1)$ and $u' = N^{1/2}(u + 1)$, we have, as $N \to \infty$,

$$N^{1+\beta/2} \frac{(1 - x)^{\alpha_n}(1 + x)^{\beta}}{x - u} du dx = N^{1+\beta/2} N^{1/2} (2 - x' N^{-1/2})^{\alpha_n}(x')^{\beta} N^{-\beta/2} \frac{2^{\alpha_n}(x')^{\beta}}{x' - u'} \cdot N^{-1} du' dx' \quad (28)$$

To obtain the asymptotics of the quantity $\tilde{c}_{s,t}$ defined in [9], first recall Stirling’s formula:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left( \frac{z}{e} \right)^z (1 + O(z^{-1})) \text{ as } z \to \infty.
From this formula, it is not hard to see that for fixed \( w \),

\[
\lim_{z \to \infty} z^{-w} \frac{\Gamma(z + w)}{\Gamma(z)} = \lim_{z \to \infty} z^{-w} \sqrt{\frac{z}{z + w}} e^{-w} \left( \frac{z + w}{z} \right)^{z+w} = \lim_{z \to \infty} \sqrt{\frac{1}{1 + w/z}} e^{-w} \left( 1 + \frac{w}{z} \right)^{z-w} (z+w)^w = 1.
\]

Therefore (using \( f(N) \sim g(N) \) to indicate that \( f(N)/g(N) \to 1 \))

\[
\bar{c}_s^n \sim \begin{cases} \nu_1 N^{1/4} \cdot \Gamma(\alpha + 1) (\nu_1 N^{1/4})^{-(\alpha+1)}, & n \text{ even}, \\ 2\nu_1 N^{1/4} \cdot (\nu_1 N^{1/4})^\alpha, & n \text{ odd}, \end{cases}
\]

and

\[
\bar{c}_t^n \sim \begin{cases} (\nu_2 N^{1/4})^{\alpha+1}, & m \text{ even}, \\ \Gamma(\alpha + 1) (\nu_2 N^{1/4})^{-\alpha}, & m \text{ odd}. \end{cases}
\]

By introducing the conjugating factor \( c := (N^{-1/4})^{(2\alpha+1)1_{(n \text{ odd})} - (2\alpha+1)1_{(m \text{ odd})}} \), which drops out when taking the determinant, one sees that

\[
\frac{N^{-1/4}}{\Gamma(\alpha + 1)} c \cdot \bar{c}_s^n \bar{c}_t^m \rightarrow \begin{cases} (\nu_2)^{\alpha+1}(\nu_1)^{-\alpha}, & n \text{ even, } m \text{ even}, \\ \Gamma(\alpha + 1)(\nu_1\nu_2)^{-\alpha}, & n \text{ even, } m \text{ odd}, \\ \Gamma(\alpha + 1)(\nu_1\nu_2)^{\alpha+1}, & n \text{ odd, } m \text{ even}, \\ (\nu_2)^{-\alpha}(\nu_1)^{\alpha+1}, & n \text{ odd, } m \text{ odd}, \end{cases}
\]

which can itself be written as \( \sqrt{\nu_1\nu_2} \cdot c' \), where \( c' \) is the conjugating factor

\[
c' := \Gamma(\alpha + 1)^{1_{(n \text{ even})} - 1_{(m \text{ even})}} \frac{\nu_2}{\nu_1} (-1)^{n\alpha+1/2}.
\]

Putting (28) and (29) together with the Mehler–Heine formula (26), we get

\[
(-1)^{s-t} \frac{c}{c'} \cdot N^{-1/4} N^{(1+\beta)/2} N^{-\beta/2} \cdot \frac{\bar{c}_s^n \bar{c}_t^m}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} P_s^{(\alpha_n)}(x) P_t^{(\alpha_m)}(u) (1 - x)^\alpha (1 + x)^\beta \frac{du}{x - u} dx
\]

\[
\rightarrow \sqrt{\nu_1\nu_2} \frac{x'}{u'} \frac{J_\beta(\nu_1 \sqrt{2x'}) J_\beta(\nu_2 \sqrt{2u'})}{x' - u'} \frac{du'dx'}{x' - u'}.
\]

The remainder of the integrand satisfies

\[
(-1)^{r_n-r_m} 2^{r_m-r_n} e^{\gamma x} (x - 1)^{r_n} e^{\gamma u} (u - 1)^{r_m} = e^{-N(A(-1)-A(x)) + \sigma_1 \sqrt{N(\log(1-x) - \log 2)}} e^{N(A(u)-A(-1)) + \sigma_2 \sqrt{N(\log(1-u) - \log 2)}} (1 + O(N^{-1})) \rightarrow e^{-(x')^2/8 - \sigma_1 x'/2} e^{-(u')^2/8 - \sigma_2 u'/2},
\]

where \( A(z) := z/2 + \log(1-z) \) and the last approximation follows from, respectively, second and first order Taylor expansions about \(-1\). Hence, we have (omitting conjugating factors)

\[
\frac{N^{1/4}}{2\pi i} \int_{-\infty}^{\infty} e^{\gamma x} \frac{\widetilde{P}_s^{(\alpha_n)}(x)}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \widetilde{P}_t^{(\alpha_m)}(u) (x - 1)^{r_n} (1 - x)^\alpha (1 + x)^\beta \frac{du}{u - x}
\]

\[
\rightarrow \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\langle \nu_1 \rangle \langle \nu_2 \rangle + \sigma_2 u' - \sigma_1 x'} \left( \frac{x'}{u'} \right)^{\beta/2} \sqrt{\nu_1\nu_2} J_\beta(\nu_1 \sqrt{2x'}) J_\beta(\nu_2 \sqrt{2u'}) \frac{du'dx'}{x' - u'}.
\]
Turning to the other term of $K_\Delta^\alpha$, note the following two asymptotic relations:

\[ (-2)^{n-r_m} (x-1)^{n-r_m} = \left( 1 - \frac{x'}{2\sqrt{N}} \right)^{(s_1-s_2)\sqrt{N}} \rightarrow e^{-\frac{(s_1-s_2)x'}{2}}. \]

Then the single-integral term satisfies

\[ \frac{e^{-t}N^{1/4}}{2^{a+\beta+1}N^{\alpha+1}} P_s^{(\alpha_n)}(x) P_t^{(\alpha_m)}(x) C(x) \rightarrow \sqrt{\nu_1 \nu_2} J_\beta(\nu_1 \sqrt{2x'}) J_\beta(\nu_2 \sqrt{2x'}) dx'. \]

Then the single-integral term satisfies

\[ \frac{(-2)^{n-r_m} \Gamma(\alpha+1) 1_{(n, \text{ even})} - 1_{(n, \text{ odd})}}{N^{1/4}(2\alpha+1)1_{(n, \text{ odd})} - (2\alpha+1)1_{(n, \text{ even})}} \times \int_0^\infty e^{-\frac{(s_1-s_2)x'}{\nu_1 \nu_2}} \sqrt{\nu_1 \nu_2} J_\beta(\nu_1 \sqrt{2x'}) J_\beta(\nu_2 \sqrt{2x'}) dx' \rightarrow 1_{(\sigma_1 > \sigma_2)} \left( \int_0^\infty e^{-\frac{(s_1-s_2)x'}{\nu_1 \nu_2}} \sqrt{\nu_1 \nu_2} J_\beta(\nu_1 \sqrt{2x'}) J_\beta(\nu_2 \sqrt{2x'}) dx' \right). \]

It remains to show that the expression for the kernel here matches the hard-edge Pearcey kernel $L^\beta$. By the identity (see e.g. equation (25) of [7]),

\[ \int_0^\infty J_\beta(a_1 x) J_\beta(a_2 x) e^{-\gamma^2 x^2} dx = \frac{1}{2\gamma^2} \exp \left( -\frac{a_1^2 + a_2^2}{4\gamma^2} \right) I_\beta \left( \frac{a_1 a_2}{2\gamma^2} \right), \quad \text{Re} \beta > -1, \quad \text{Re} \gamma^2 > 0, \]

with $x' = 2x^2$, $\gamma^2 = (\sigma_1 - \sigma_2), a_1 = 2\nu_1, a_2 = 2\nu_2$, then

\[ 1_{(\sigma_1 > \sigma_2)} \left( \int_0^\infty e^{-\frac{(s_1-s_2)x'}{\nu_1 \nu_2}} \sqrt{\nu_1 \nu_2} J_\beta(\nu_1 \sqrt{2x'}) J_\beta(\nu_2 \sqrt{2x'}) dx' \right) \]

\[ = 1_{(\sigma_1 > \sigma_2)} \sqrt{\nu_1 \nu_2} \frac{2}{\sigma_1 - \sigma_2} \exp \left( -\frac{\nu_1^2 + \nu_2^2}{\sigma_1 - \sigma_2} \right) I_\beta \left( \frac{2\nu_1 \nu_2}{\sigma_1 - \sigma_2} \right). \]

By the substitutions $x' = 2x^2, u' = 2u^2,$

\[ \frac{1}{2\pi i} \int_0^\infty \int_{-i\infty}^{i\infty} e^{(u')^2 - (u')^2 + \frac{2s_2 u' - s_1 x'}{x'}} \left( \frac{x'}{u'} \right)^{\beta/2} \sqrt{\nu_1 \nu_2} J_\beta(\nu_1 \sqrt{2x'}) J_\beta(\nu_2 \sqrt{2u'}) \frac{du'dx'}{x' - u'} \]  

\[ = \sqrt{\nu_1 \nu_2} \frac{4}{\pi i} \int_0^\infty \int_C e^{u^4 - x^4 + \frac{2s_2 u^2 - s_1 x^2}{x^2}} \left( \frac{x}{u} \right)^\beta J_\beta(2\nu_1 x) J_\beta(2\nu_2 u) \frac{du dx}{x^2 - u^2}, \]

where $C$ consists of two rays, one from $e^{i\pi/4} \infty$ to $0$ and one from $0$ to $e^{-i\pi/4} \infty$. The final expression is therefore

\[ L^\beta(\sigma_1, \nu_1^2; \sigma_2, \nu_2^2) \cdot 2\sqrt{\nu_1 \nu_2}, \]

finishing the proof of Theorem 2.3.

**Remark 7.1.** The expression $2\sqrt{\nu_1 \nu_2}$ can be understood with the following heuristics. If $\mathcal{X}$ is a determinantal point process on $\mathbb{R}$ with correlation kernel $K(\cdot, \cdot)$ and correlation functions $\rho_k$, then (see e.g. (2.6) and (2.16) of [30])

\[ \mathbb{E}[\text{# of pairs of distinct points of } \mathcal{X} \text{ in } A \subset \mathbb{R}] = \int_{A \times A} \rho_2(x, y) dx dy \]

\[ = \int_{A \times A} (K(x, x)K(y, y) - K(x, y)K(y, x)) dx dy. \]
If \( \phi : \mathbb{R} \to \mathbb{R} \) is a differentiable bijection, and \( \phi(\mathcal{X}) \) is a determinantal point process with correlation kernel \( \tilde{K}(\cdot, \cdot) \), then

\[
\mathbb{E}[\# \text{ of pairs of distinct points of } \tilde{\mathcal{X}} \text{ in } \phi(A)] = \int_{\phi(A \times A)} \left( \tilde{K}(\tilde{x}, \tilde{x})\tilde{K}(\tilde{y}, \tilde{y}) - \tilde{K}(\tilde{x}, \tilde{y})\tilde{K}(\tilde{y}, \tilde{x}) \right) d\tilde{x} d\tilde{y}
\]

where the last equality is due to the change of variables \( \tilde{x} = \phi(x), \tilde{y} = \phi(y) \). These two integrals must be equal, so \( K \) and \( \tilde{K} \) are related by

\[
K(x, y) = \tilde{K}(\phi(x), \phi(y)) \sqrt{\phi'(x)\phi'(y)}.
\]

Note that this can be written equivalently as

\[
\sqrt{\frac{\phi'(y)}{\phi'(x)}} K(x, y) = \tilde{K}(\phi(x), \phi(y))\phi'(y),
\]

which is how it is expressed in Corollary 2.25 of [23].

8 Technical Proofs

8.1 Proof of Proposition 3.1

For the first identity, note that if \( p(x) \) is any polynomial of degree \( < s \), then

\[
\left\langle \sum_{r=0}^{s} \hat{P}^{(\alpha)}(x)^{2}, p \cdot (1 - x) \right\rangle_{\alpha + 1} = \sum_{r=0}^{s} \left\langle \hat{P}^{(\alpha)}(x), p \cdot (1 - x) \right\rangle_{\alpha} \hat{P}^{(\alpha)}(1) = 2^{\alpha + \beta + 1}\Gamma(\alpha + 1) \cdot p(x)(1 - x)|_{x=1} = 0,
\]

where we have used the facts \( \hat{P}^{(\alpha)}(1) = 1 \) and \( [11] \) along with orthogonal decomposition \( (8) \). Hence, by uniqueness of orthogonal polynomials, \( \sum_{r=0}^{s} P^{(\alpha)}_{r} \) must be proportional to \( P^{(\alpha+1)}_{s} \) and comparing leading coefficients tells us that the proportionality constant is (see equations \((6)\) and \((9)\))

\[
(2s + \alpha + \beta + 1)\frac{\Gamma(s + \alpha + \beta + 1)}{\Gamma(s + \beta + 1)} \cdot \frac{a^{(\alpha,\beta)}_{s}}{a^{(\alpha+1,\beta)}_{s}} = \frac{\Gamma(s + \alpha + \beta + 2)}{\Gamma(s + \beta + 1)},
\]

as required. On the other hand, combining this identity with the Christoffel–Darboux formula gives

\[
\hat{P}^{(\alpha+1)}_{s}(x) = \sum_{r=0}^{s} \hat{P}^{(\alpha)}_{r}(x)\hat{P}^{(\alpha)}_{r}(1) = \frac{2^{\alpha + \beta + 1}\Gamma(\alpha + 1)a^{(\alpha,\beta)}_{s}}{\|P^{(\alpha,\beta)}_{s}\|^{2}a^{(\alpha+1,\beta)}_{s}} \cdot \left( \frac{P^{(\alpha,\beta)}_{s+1}(x)P^{(\alpha,\beta)}_{s}(1) - P^{(\alpha,\beta)}_{s}(x)P^{(\alpha,\beta)}_{s+1}(1)}{x - 1} \right).
\]

Multiply both sides of equation \((30)\) by

\[
\phi_{n}(s) = \frac{c^{n}_{s}}{\tilde{c}^{n}_{s}} = \frac{(2s + \alpha + \beta + 2)\Gamma(s + 1)\Gamma(\alpha + 1)\Gamma(s + \beta + 1)}{2 \cdot \Gamma(s + \alpha + 2)\Gamma(s + \alpha + \beta + 2)}, \text{ n even}
\]
and recall

\[ P_s^{(\alpha, \beta)}(1) = \frac{\Gamma(s + \alpha + 1)}{\Gamma(s + 1) \Gamma(\alpha + 1)}. \]

Then the expression (30) becomes

\[ \hat{P}_{r+1}^{(\alpha + 1)} = \frac{\hat{P}_{r+1}^{(\alpha, \beta)}(x) - \hat{P}_{r}^{(\alpha, \beta)}(x)}{x - 1}, \quad (31) \]

which telescopes when summing over \( r \) to give the third identity

For the second and fourth identities, first note that since \( P_0^{(\alpha)} \equiv 1, \)

\[ \langle \hat{P}_s^{(\alpha+1)}, 1 \rangle = \sum_{r=0}^{s} \langle \hat{P}_r^{(\alpha)}, P_r^{(\alpha)} \rangle_\alpha = (\beta + \alpha + 1) \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + 1)} \sum_{r=0}^{\infty} \langle \hat{P}_r^{(\alpha)}, P_r^{(\alpha)} \rangle_\alpha = 2^{\alpha+\beta+1} \Gamma(\alpha + 1). \]

This fact along with orthogonal decomposition (8) gives

\[ \langle \hat{P}_s^{(\alpha+1)}, T(1) \rangle_\alpha = 2^{\alpha+\beta+1} \Gamma(\alpha + 1) T(1) = 2^{\alpha+\beta+1} \Gamma(\alpha + 1) \sum_{r=0}^{\infty} \langle \hat{P}_r^{(\alpha)}, T \rangle_\alpha \hat{P}_r^{(\alpha)}(1) = \sum_{r=0}^{\infty} \langle \hat{P}_r^{(\alpha)}, T \rangle_\alpha. \]

Then subtract \( \langle \hat{P}_s^{(\alpha+1)}, T \rangle_\alpha = \sum_{r=0}^{s} \langle \hat{P}_r^{(\alpha)}, T \rangle_\alpha \) from both sides to arrive at the second identity. For the last identity, note

\[ \langle \sum_{r=0}^{s-1} \hat{P}_r^{(\alpha+1)}, T \rangle_\alpha^{\alpha+1} \rightarrow \langle 1, T \rangle_\alpha, \]

where the term disappears necessarily from convergence of orthogonal decompositions (8). Subtracting the finite sum \( \sum_{r=0}^{s-1} \langle \hat{P}_s^{(\alpha+1)}, T \rangle_\alpha = \langle 1, \hat{P}_s^{(\alpha)}, T \rangle_\alpha \) from the right side completes the proof.

### 8.2 Proof of Proposition 3.2

Fix \( n \) odd so \( r_{n-1} = r_n - 1 \) and \( \alpha_n = \alpha \). Then using part (3) of Proposition 3.1 with \( T_m(x) := (x - 1)^{-m} R_m^E(x) \) and the fact \( T_m(1) := \lim_{x \to 1} T_m(x) = E^{(m)}(1)/m! \), we get using (10)

\[
\sum_{r \geq 0} \phi_{n-1}(s, r) \Psi_{r-n}^n(l(r)) = \frac{\phi_{n-1}(s)}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \sum_{r=s+1}^{\infty} \langle \hat{P}_r^{(\alpha)}, (x - 1)^{r-n} R_l^E_{l-r_n} \rangle_\alpha \\
= \frac{1}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \langle \phi_{n-1}(s) \hat{P}_s^{(\alpha+1)}, \frac{T_{l-r_n} - T_{l-r_n}(1)}{x - 1} \rangle_\alpha^{\alpha+1} \\
= \frac{1}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \langle \hat{P}_s^{(\alpha+1)}, \frac{R_{l-r_n-1}^E(x)}{(x - 1)^{l-r_n-1}} \rangle_\alpha^{\alpha+1} = \Psi_{r_{n-1}-l}^{n-1}(s),
\]

The case \( n \) even instead involves \( r_n = r_{n-1}, \alpha_n = \alpha + 1 \), and the other part of Proposition 3.1

\[
\sum_{r \geq 0} \phi_{n-1}(s, r) \Psi_{r-n}^n(l(r)) = \frac{1}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \sum_{r=s}^{\infty} \langle \hat{P}_r^{(\alpha+1)}, (x - 1)^{r-n} R_l^E_{l-r_n} \rangle_{\alpha+1} \\
= \frac{1}{2^{\alpha+\beta+1} \Gamma(\alpha + 1)} \langle \phi_{n-1}(s) \hat{P}_s^{(\alpha)}, (x - 1)^{r-n-1} R_l^E_{l-r_n-1} \rangle_\alpha^{\alpha+1} = \Psi_{r_{n-1}-l}^{n-1}(s).\]
8.3 Proof of Proposition 4.5

For either parity of $n$, we deduce the second identity by computing

$$
\sum_{\mu \in \mathbb{F}_n} P_n^{\psi}(\mu) d_{\mu}^{(n)}(x_1, \ldots, x_{r_n}) = \frac{\det \left[ \sum_{k=0}^{\infty} \tilde{P}_k^{(\alpha)}(x-1)^{r_n-i} \tilde{P}_k^{(\alpha)}(x) \right]_{i,j=1}^{r_n}}{\det [x_j^{r_n-i}]_{i,j=1}^{r_n}} = \frac{\det \left[ (x_j-1)^{r_n-i} \psi(x_j) \right]_{i,j=1}^{r_n}}{\det [(x_j-1)^{r_n-i}]_{i,j=1}^{r_n}} = \prod_{j=1}^{r_n} \psi(x_j),
$$

where we have used Lemma 4.2 and orthogonal decomposition (32). Similarly, if $\psi \equiv 1$, then

$$
P_n^{\psi}(0) = \frac{\det \left[ \sum_{k=0}^{\infty} \tilde{P}_k^{(\alpha)}(x-1)^{r_n-i} \tilde{P}_k^{(\alpha)}(x) \right]_{i,j=1}^{r_n}}{\det [x_j^{r_n-i}]_{i,j=1}^{r_n}} = \frac{\det \left[ (x_j-1)^{r_n-i} \right]_{i,j=1}^{r_n}}{\det [(x_j-1)^{r_n-i}]_{i,j=1}^{r_n}} = 1
$$

If instead $\lambda \neq 0$, then $P_n^{\psi}(\lambda) = 0$ since the first expression above will have at least one row of zeros by orthogonality to polynomials of lower degree.

Now assume $n - 1$ odd, so that $\alpha_{n-1} = \alpha$ and $r_n = r_{n-1}$. Then we have for $\mu \in \mathbb{J}_{n-1}$

$$
\sum_{\lambda \in \mathbb{J}_n} P_n^{\psi}(\lambda) T_{n-1}^{\mu}(\lambda, \mu) = d_{\mu}^{(n-1)}(1, \ldots, 1) \sum_{\lambda \in \mathbb{J}_n} \det[\phi_{n-1}(\tilde{\mu}_i, \tilde{\lambda}_j)]_{i,j=1}^{r_n} \det \left[ \Psi_{r_n-1}^{n, \psi}(\tilde{\lambda}_j) \right]_{i,j=1}^{r_n}
$$

$$
= d_{\mu}^{(n-1)}(1, \ldots, 1) \det \left[ (\phi_{n-1} \ast \Psi_{r_n-1}^{n, \psi})(\tilde{\mu}_i) \right]_{i,j=1}^{r_n}
$$

$$
= d_{\mu}^{(n-1)}(1, \ldots, 1) \det \left[ \Psi_{r_n-1}^{n, \psi}(\tilde{\mu}_i) \right]_{i,j=1}^{r_n-1} = P_n^{\psi}(\mu).
$$

where we have used Lemma 4.2 and the composition rule Lemma 3.2 The exact same calculations of (33) hold for the case $n - 1$ even (where instead $\alpha_{n-1} = \alpha + 1$ and $r_n = r_{n-1} - 1$), except to justify its last equality, we need the fact that the $r_n$th row where the conventions $\tilde{\mu}_n \equiv -1$ and $\phi_{n-1}(-1, \cdot) \equiv 1$ has $j$th entry

$$
(\phi_{n-1} \ast \Psi_{r_n-1}^{n, \psi})(s) = \sum_{s \geq 0} \Psi_{r_n-1}^{n, \psi}(s) = \sum_{s \geq 0} \left( \frac{\tilde{P}_s^{(\alpha)}}{2^{\alpha+\beta+1} \Gamma(\alpha+1)}, (x-1)^{r_n-j} \psi \right)_{\alpha+1} = \delta_{r_n,j},
$$

where we have used the fact $\tilde{P}_s^{(\alpha)}(1) = 1$ for $n$ odd and orthogonal decomposition (32) in the last equality. This completes the proof.
8.4 Proof of Proposition 5.1

Both items of the first point follow readily from Lemma 4.2 and orthogonal decomposition (8); for example,

\[
\sum_{\mu \in \mathcal{I}_n} P_\mu^{\psi_1}(\mu) T_\mu^{\psi_2}(\mu, \lambda) = d_\lambda^{(n)}(1, \ldots, 1) \sum_{\mu \in \mathcal{I}_n} \det \left[ \left\langle \bar{P}_\mu^{(\alpha_n)}(x), (x - 1)^{r_n - i} \psi_1 \right\rangle_{\alpha_n} \right]^{r_n}_{i,j=1} \det \left[ \left\langle \hat{P}_\mu^{(\alpha_n)}(x), (x - 1)^{r_n - i} \psi_1 \right\rangle_{\alpha_n} \right]^{r_n}_{i,j=1} = d_\lambda^{(n)}(1, \ldots, 1) \cdot \det \left[ \left\langle \sum_{k=0}^{\infty} \bar{P}_k^{(\alpha_n)}(x), (x - 1)^{r_n - i} \psi_1 \right\rangle_{\alpha_n} \right]^{r_n}_{i,j=1} = d_\lambda^{(n)}(1, \ldots, 1) \cdot \det \left[ \left\langle \hat{P}_k^{(\alpha_n)}(x), (x - 1)^{r_n - i} \psi_1 \right\rangle_{\alpha_n} \right]^{r_n}_{i,j=1} = P_\mu^{\psi_1 \psi_2}(\lambda).
\]

The second point is similar to (32):

\[
\sum_{\lambda \in \mathcal{I}_n} T_\lambda^{\psi}(\mu, \lambda) = \frac{1}{d^{(n)}(1, \ldots, 1)} \sum_{\lambda \in \mathcal{I}_n} \det \left[ \left\langle \bar{P}_\lambda^{(\alpha_n)}(x), \bar{P}_\lambda^{(\alpha_n)}(x) \right\rangle_{\alpha_n} \right]^{r_n}_{i,j=1} \det \left[ \hat{P}_\lambda^{(\alpha_n)}(x) \right]^{r_n}_{i,j=1} = \frac{1}{d^{(n)}(1, \ldots, 1)} \det \left[ \left\langle \sum_{k=0}^{\infty} \bar{P}_k^{(\alpha_n)}(x), \bar{P}_k^{(\alpha_n)}(x) \right\rangle_{\alpha_n} \right]^{r_n}_{i,j=1} \det \left[ \hat{P}_k^{(\alpha_n)}(x) \right]^{r_n}_{i,j=1} = \frac{1}{d^{(n)}(1, \ldots, 1)} \det \left[ \bar{P}_\mu^{(\alpha_n)}(x) \right]^{r_n}_{i,j=1} \det \left[ \hat{P}_k^{(\alpha_n)}(x) \right]^{r_n}_{i,j=1} = \psi(1)^{r_n},
\]

where Lemma 4.2 and orthogonal decomposition (8) were used in the second and third equalities. The third point of the proposition is a straightforward application of the dominated convergence theorem.

For the fourth point, it suffices by the second point to show that the entries are non-negative. Recall that we write \( \bar{P}_k^{(\alpha_n)} = e_k^\alpha P_k^{(\alpha_n)} \) and \( \hat{P}_k^{(\alpha_n)} = \bar{e}_k^\alpha P_k^{(\alpha_n)} \). Then using the three term recurrence (5) and orthogonality relations, we then compute

\[
\frac{(\bar{P}_\mu^{(\alpha_n)}, \bar{P}_\lambda^{(\alpha_n)} \phi)_{\alpha_n}}{2^{\alpha + \beta + 1} \Gamma(\alpha + 1)} = (1 - a) \delta_{\mu, \lambda} + a \frac{e^{\alpha}}{e^{\mu}} [A^{(\alpha_n, \beta)}_{\lambda} \delta_{\mu, \lambda} + B^{(\alpha_n, \beta)}_{\lambda} \delta_{\mu, \lambda - 1} + C^{(\alpha_n, \beta)}_{\lambda} \delta_{\mu, \lambda + 1}].
\]

If \( \bar{\lambda}_i > \bar{\mu}_i + 1 \) for some \( 1 \leq i \leq r_n \), then \( \bar{\lambda}_k > \bar{\mu}_l + 1 \) for \( 1 \leq k \leq i \leq l \leq r_n \), which implies \( T_\mu^{\psi}(\mu, \lambda) = 0 \) (the resulting matrix admits a 2 x 2 block form with an off-diagonal block of 0’s and a diagonal block with a zero vector). The same conclusion of course holds if \( \bar{\mu}_i > \bar{\lambda}_i + 1 \), so assume \( |\bar{\lambda}_i - \bar{\mu}_i| \leq 1 \) for all \( 1 \leq i \leq r_n \). If \( |\bar{\mu}_i - \bar{\mu}_{i+1}| = 1 \) for some \( 1 \leq i < r_n \), then \( |\bar{\mu}_i - \bar{\lambda}_{i+1}| \leq 1 \) implies \( |\bar{\lambda}_i - \bar{\mu}_{i+1}| > 1 \) and similarly \( |\bar{\lambda}_i - \bar{\mu}_{i+1}| \leq 1 \) implies \( |\bar{\lambda}_{i+1} - \bar{\mu}_i| > 1 \). In either case, \( T_\mu^{\psi}(\mu, \lambda) \) breaks into a product of determinants, one of which is the \( i \times i \) upper left corner minor of the matrix defining \( T_\mu^{\psi}(\mu, \lambda) \). Iterating this argument reduces consideration to the case where \( |\bar{\mu}_i - \bar{\mu}_{i+1}| \leq 1 \) for all \( 1 \leq i < r_n \), which means \( \mu_i \) are all equal to some \( p \in \mathbb{Z}_{\geq 0} \). Now the two blocks corresponding
to \( \{ i : \lambda_i = p \pm 1 \} \) are triangular with nonnegative entries (since \( B_k^{(\alpha_n, \beta)} C_k^{(\alpha_n, \beta)} \geq 0 \)) and straddle a tridiagonal block corresponding to \( \{ i : \lambda_i = p = \mu_i \} \). Write \( q^* := \max \{ i : \lambda_i = p = \mu_i \} \), \( q_* := \min \{ i : \lambda_i = p = \mu_i \} \), \( q := q^* - q_* + 1 \), and set \( \mu := \bar{\mu}_q \). The last argument has further reduced consideration to the determinant of the \( q \times q \) tridiagonal matrix

\[
\begin{pmatrix}
1 - a + a A_k^{(\alpha, \beta)}, & a \frac{c_n}{c_{n-1}} C_k^{(\alpha, \beta)} & 0 & 0 \\
 a \frac{c_n}{c_{n-1}} B_k^{(\alpha, \beta)} & 1 - a + a A_{k-1}^{(\alpha, \beta)} & a \frac{c_n}{c_{n-2}} C_{k-1}^{(\alpha, \beta)} & 0 \\
0 & a \frac{c_n}{c_{n-2}} B_{k-1}^{(\alpha, \beta)} & 1 - a + a A_{k-2}^{(\alpha, \beta)} & \ddots \\
\vdots & \vdots & \ddots & \ddots 
\end{pmatrix}
\]

(36)

In general, if a tridiagonal matrix has the property that each diagonal entry of a given row is greater than or equal to the sum of the off-diagonals in that row, then all of its principal minors are non-negative (see page 5 of [26]). In particular, its determinant is non-negative. This leads us to show that the expressions

\[
a \left[ \frac{c_n}{c_0} B_1^{(\alpha, \beta)} - A_0^{(\alpha, \beta)} + 1 \right], \quad a \left[ \frac{c_n}{c_{n+1}} B_{k+1}^{(\alpha, \beta)} + \frac{c_n}{c_{n-1}} C_{k-1}^{(\alpha, \beta)} - A_k^{(\alpha, \beta)} + 1 \right], \quad k \geq 1
\]

are less than or equal to \( 1 \). Since the formula defining \( C_{k-1}^{(\alpha, \beta)} \) is equal to zero when \( k = 0 \), it suffices to focus on the second expression. When \( n \) is odd, we compute

\[
\frac{c_n}{c_{n+1}} B_{k+1}^{(\alpha, \beta)} + \frac{c_n}{c_{n-1}} C_{k-1}^{(\alpha, \beta)} - A_k^{(\alpha, \beta)} =
\frac{k + \alpha + \beta + 1}{2k + \alpha + \beta + 1} \left( 2(k + \alpha + 1) + \frac{2k}{2k + \alpha + 2} \right) - \frac{2k}{2k + \alpha + 1} - \frac{\beta^2 - \alpha^2}{(2k + \alpha + \beta)(2k + \alpha + 1)}.
\]

With some algebra, it can be shown that the desired inequality is equivalent to the condition that \( a^{-1} \) is bounded below by

\[
\frac{4a^3 + a^2(8\beta + 9) + (4a + 8k)(\beta^2 + 3\beta + 1) + \beta(3\beta + 4) + 16k^3 + 24k^2(\alpha + \beta + 1) + 8k(2a^2 + 3\alpha(\beta + 1))}{(\alpha + \beta + 2k)(\alpha + \beta + 2k + 1)(\alpha + \beta + 2k + 2)}.
\]

By taking a derivative with respect to \( k \), it can be seen that this quantity as a function of \( k \) can only have real critical points if \( k \) is a real solution of a cubic polynomial, which can be solved explicitly to find

\[
k = \frac{1}{8} \left( -4\alpha - 4\beta + \sqrt{7 + 4\sqrt{3}} + \frac{1}{\sqrt{7 + 4\sqrt{3}}} - 5 \right),
\]

which is approximately \((-2.17836 - 4\alpha - 4\beta)/8\) and thus less than 1 for \( \alpha_n, \beta > -1 \). Therefore, this quantity is monotonic on \( k \in [1, \infty) \), so it suffices to evaluate it at \( k = \infty, 0, 1 \), which yields the three terms in the lower bound in the fourth statement. Setting the two terms from \( k = 0, 1 \) equal to 2 yields the two inequalities \((2\alpha + 2\beta + 7)(\alpha^2 - \beta^2) \leq 0\) and \((2\alpha + 2\beta + 3)(\alpha^2 - \beta^2) \leq 0\).

\[3\]

Once it is shown that the expressions in [37] are less than or equal to 1, the Gershgorin circle theorem implies that the eigenvalues of the matrix in [36] have non-negative real part. Since any tridiagonal matrix with non-negative off-diagonal entries must have real eigenvalues, this implies that all the eigenvalues of the matrix in [36] are non-negative. Thus, the determinant is non-negative. We thank the anonymous referee for this insightful observation.
Similarly, when $n$ is even,

$$\frac{\bar{c}_k^{n+1} B^{(\alpha, \beta)}_k + \bar{c}_{k-1}^{n} C^{(\alpha, \beta)}_{k-1} - A^{(\alpha, \beta)}_k}{(2k + \alpha + \beta + 3)(2k + \alpha + \beta + 2)} + \frac{2(\alpha + k + 1)(k + \alpha + \beta + 1)}{(2k + \alpha + \beta + 2)(2k + \alpha + \beta + 1)} - \frac{\beta^2 - \alpha^2_n}{(2k + \beta + \alpha_n)(2k + \alpha_n + \beta + 1)}.$$ 

With some algebra, $a^{-1}$ is bounded below by

$$4\alpha^3 + \alpha^2(8\beta + 21) + \alpha(4\beta^2 + 28\beta + 34) + 7\beta^2 + 24\beta + 16k^3 + 24k^2(\alpha + \beta + 2) + 8k(2\alpha^2 + \alpha(3\beta + 7) + \beta^2 + 6\beta + 6) + 17$$

$$(\alpha + \beta + 2k + 1)(\alpha + \beta + 2k + 2)(\alpha + \beta + 2k + 3)$$

Again, this quantity as a function of $k$ can only have real critical points at

$$k = \frac{1}{8} \left( -4\alpha - 4\beta + \sqrt{7 + 4\sqrt{3} + \frac{1}{\sqrt{7 + 4\sqrt{3}}} - 9} \right),$$

which is again less than 1 for $\alpha, \beta > -1$. Plugging in $k = 0, 1$ yields the two quantities in the lower bound, and setting those quantities equal to 2 yields the two inequalities.

To conclude the last statement, note that by the third and first points

$$T^{\psi, \gamma}_n = \lim_{m \to \infty} T^{(1+\gamma(x-1)/m)}_n = \lim_{m \to \infty} T^{1+\gamma(x-1)/m}_n \ldots T^{1+\gamma(x-1)/m}_n.$$  \hspace{1cm} (38)

By the second point and fourth points respectively, the entries of $T^{\psi, \gamma}_n$ have rows which sum to 1 and are positive. Additionally,

$$\|T^{\psi, \gamma}_n - Id\|_{\psi, \gamma} = \sup_{\lambda, \mu \in \mathbb{J}_n} \sum_{\lambda \in \mathbb{J}_n} (\delta_{\lambda \mu} - T^{\psi, \gamma}_n(\lambda, \mu)) = \sup_{\lambda, \mu \in \mathbb{J}_n} (2 - 2T^{\psi, \gamma}_n(\lambda, \lambda)).$$ \hspace{1cm} (39)

Note by (38) that $T^{\psi, \gamma}_n(\lambda, \lambda) \geq \lim_{m \to \infty} T^{(1+\gamma(x-1)/m)}_n(\lambda, \lambda)$. The identity (39) then tells us it is sufficient to show $\inf_{\lambda \in \mathbb{J}_n} |T^{(1+\gamma(x-1)/m)}_n(\lambda, \lambda)| \to 1$ as $\gamma \to 0$. But our argument above tells us that $T^{(1-\gamma)(m-x-\gamma)/m}_n(\lambda, \lambda)$ breaks apart into a product of determinants of $q \times q$ matrices of the form \hspace{1cm} (36). Thus we are interested in

$$\inf_{\mu \geq q} \det \left[ (1 - a) I_q + a^q \cdot \begin{bmatrix} A_\mu & \bar{c}_{n-1}^{\mu} C_{n-1} & 0 & 0 \\ \bar{c}_{n}^{\mu} B_\mu & \bar{c}_{n}^{\mu} C_{n-1} & 0 & 0 \\ 0 & \bar{c}_{n-1}^{\mu} B_{\mu-1} & A_{\mu-1} & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \right].$$

For large enough $m$, this infimum is nonzero and in fact positive by our reasoning above. By continuity, it goes to 1 as $\gamma \to 0$, as required.

### 8.5 Proof of Proposition 5.4

First assume $n$ is even, so that $\alpha_n = \alpha + 1, \alpha_{n+1} = \alpha, r_n = r_{n+1} - 1$, and $\phi_n(s, t) = \phi_n(s)1_{s < t}$. For $\mu \in \mathbb{J}_{n+1}$ and $\lambda \in \mathbb{J}_n$, Lemma \hspace{1cm} (4.1) allows us to write

$$(T^{\psi}_n T^{\psi}_n)(\mu, \lambda) = \frac{d^{(n)}_{\lambda}(1, \ldots, 1)}{d^{(n+1)}_{\mu}(1, \ldots, 1)} \sum_{z \in \mathbb{J}_n} \det [\phi_n(z, \lambda, \mu)]_{i,j=1}^{r_n} \det \left[ \frac{\hat{P}^{(\alpha+1)}_{\bar{z}_j, \lambda} \hat{P}^{(\alpha+1)}_{\lambda, \psi}}{2^{\alpha+\beta+1}(\alpha + 1)} \right]^{r_n}_{i,j=1}.$$
Expand the first determinant along the \( r_{n+1} \)-column, where the convention \( z_{r_{n+1}} \equiv -1 \) applies. For each \( 1 \leq l \leq r_{n+1} \), the \( l \)th resulting summand is

\[
(-1)^{r_{n+1}+l} \sum_{z \in J_n} \det \left[ \phi_n(z_j, \tilde{\mu}_i) \right]_{1 \leq i \leq l \leq r_{n+1}} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \hat{P}_j^{(\alpha+1)}, \hat{P}_k^{(\alpha+1)} \psi \right\rangle_{\alpha+1} \right]_{i,j=1}^{r_n}
\]

\[
= (-1)^{r_{n+1}+l} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \sum_{k=0}^{\tilde{\mu}_i-1} \hat{P}_k^{(\alpha+1)}, \hat{P}_j^{(\alpha+1)} \psi \right\rangle_{\alpha+1} \right]_{1 \leq i \neq l \leq r_{n+1}}^{1 \leq j \leq r_n}
\]

\[
= (-1)^{r_{n+1}+l} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \left( 1 - \frac{\hat{P}_i^{(\alpha)}}{\tilde{\mu}_i} \right) \psi, \hat{P}_j^{(\alpha+1)} \psi \right\rangle_{\alpha} \right]_{1 \leq i \neq l \leq r_{n+1}}^{1 \leq j \leq r_n}
\]

\[
= (-1)^{r_{n+1}+l} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \left( \hat{P}_i^{(\alpha)} \psi \right) (1) - \hat{P}_i^{(\alpha)} \psi, \phi_n(\tilde{\lambda}_j) \hat{P}_j^{(\alpha+1)} \psi \right\rangle_{\alpha} \right]_{1 \leq i \neq l \leq r_{n+1}}^{1 \leq j \leq r_n}
\]

\[
= (-1)^{r_{n+1}+l} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \hat{P}_i^{(\alpha)} \psi, \phi_n(\tilde{\lambda}_j) \sum_{k=0}^{\infty} \hat{P}_k^{(\alpha)} \right\rangle_{\alpha} \right]_{1 \leq i \neq l \leq r_{n+1}}^{1 \leq j \leq r_n}
\]

where the first equality follows from Lemma 4.2 and (10), the second equality follows from Proposition 3.1.3, the third from the definition of the pairing \( \langle \cdot, \cdot \rangle_{\alpha+1} \), the fourth from \( \langle \hat{P}_i^{(\alpha)} \psi \rangle (1) = 1 \), the fifth from Proposition 3.1.2, and the sixth from the definition of \( \phi(\cdot, \cdot) \). Summing this calculation over \( 1 \leq l \leq r_n \) gives

\[
(T_n^{\alpha+1} T_n^{\psi}) (\mu, \lambda) = \sum_{l=1}^{r_{n+1}} (-1)^{r_{n+1}+l} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \hat{P}_i^{(\alpha)} \psi, \phi_n(\tilde{\lambda}_j, k) \hat{P}_k^{(\alpha)} \right\rangle_{\alpha} \right]_{1 \leq i \neq l \leq r_{n+1}}^{1 \leq j \leq r_n}
\]

\[
= \frac{d^{(\alpha)}(1, \ldots, 1)}{d^{(\alpha+1)}(\mu, 1, \ldots, 1)} \sum_{l=1}^{r_{n+1}} (-1)^{r_{n+1}+l} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \hat{P}_i^{(\alpha)} \psi, \phi_n(\tilde{\lambda}_j, k) \hat{P}_k^{(\alpha)} \right\rangle_{\alpha} \right]_{1 \leq i \neq l \leq r_{n+1}}^{1 \leq j \leq r_n}
\]

\[
= \frac{d^{(\alpha)}(1, \ldots, 1)}{d^{(\alpha+1)}(\mu, 1, \ldots, 1)} \sum_{l=1}^{r_{n+1}} \det \left[ \frac{1}{2^{\alpha+1} \Gamma(\alpha+1)} \left\langle \hat{P}_i^{(\alpha)} \psi, \phi_n(\tilde{\lambda}_j, k) \hat{P}_k^{(\alpha)} \right\rangle_{\alpha} \right]_{1 \leq i \neq l \leq r_{n+1}}^{1 \leq j \leq r_n}
\]

\[
= (T_n^{\psi} T_n^{\alpha+1}) (\mu, \lambda),
\]

where the first equality follows from the cofactor expansion of the determinant, the second equality follows by Lemma 4.2.

The much more straightforward “n odd” case involves the other components of Proposition 3.1 and 4.1 and is left to the reader.
8.6 Proof of Proposition 6.1

The proof relies on linear algebraic details that can be found, e.g., from a combination of Theorem 4.2 of [9] and Lemma 3.4 of [11], which in turn rely on the Eynard-Mehta theorem in the manner of [16]. For an exposition of these details in our setting, the appendix of [19] is relevant to this case, as long as one uses the calculation (34) to justify that the $r_n \times r_n$ matrix $M$ with $(i,j)$ entry

$$\left(\phi_{q_{i-1}} \ast \phi_{q_i} \ast \cdots \ast \phi_{q_{n-1}} \ast \Psi_{r_{n-j}}^n\right)(-1), \quad q_k := 2k - 1,$$

is upper triangular. Thus, it suffices to identify the functions used there with the functions in the statement above.

By orthogonal decomposition (8), we have for $1 \leq k, l \leq r_n$

$$\sum_{s \geq 0} \Psi_{r_n-k}^n(s) \Phi_{r_n-l}^n(s) = \frac{1}{2\pi i} \oint \sum_{s \geq 0} \langle \tilde{P}_s^{(\alpha)}, (x - 1)^{r_n-k} E \rangle_{\alpha} \tilde{P}_s^{(\alpha)}(w) \frac{1}{E(w)(w - 1)^{r_n-l+1}} dw = \frac{1}{2\pi i} \oint (w - 1)^{k-l+1} dw = \delta_{kl}.$$

Note also that $\Phi_{r_n-k}^n(t)$ is a polynomial in $t$ of the same degree as $(\phi_{q_{k-1}} \ast \phi_{q_k})_{(1)}(-1, t)$, where $q_k = 2k - 1$. These items confirm that $\{\Phi_{r_n-k}^n(t)\}_{k=1}^m$ is the unique basis of the linear span of $\{(\phi_{q_{k-1}} \ast \phi_{q_k})(-1, t)\}_{k=1}^m$ that is biorthogonal to the $\{\Psi_{r_n-k}(s)\}_{k=1}^m$.

We also need to show $\phi^{(n,m)} = \phi^{(n,m)} \ast \cdots \ast \phi^{(n,m)}$ for $n < m$. First assume $m = n + 1$. If $n$ is odd, then $r_n = r_{n+1}$, $\alpha_n = \alpha$, $\alpha_{n+1} = \alpha + 1$, and $\phi_n(s, t) = \phi_n(s) \cdot 1_{(s \leq t)}$. Using (10), we have

$$\phi_n(s, t) = \frac{1}{2\pi i} \oint \left( \frac{\tilde{P}_s^{(\alpha)}(u)}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \right)_{\alpha} du,$$

where the last equality computes the residue at $u = x$, and where we used Proposition 3.1 and orthogonality. Similarly, if $n$ is even, $\phi_n(s, t) = \phi_n(s) \cdot 1_{(s \leq t)}$, $r_n = r_{n+1} - 1$, and the $\alpha$’s switch, so that

$$\phi_n(s, t) = \frac{1}{2\pi i} \oint \left( \frac{\tilde{P}_s^{(\alpha)}(u)}{2\alpha + \beta + 1 \Gamma(\alpha + 1)} \right)_{\alpha+1} du,$$

where now the last equality computes residues at both $u = x$ and $u = 1$, and uses the fact that $\tilde{P}_t^{(\alpha)}(1) = 1$. The full statement for general $n < m$ then follows by induction.
8.7 Proof of Lemma 6.2

If \( r_m \leq r_n \), then using the geometric sum identity \( \sum_{k=1}^{q} \left( \frac{u-1}{x-1} \right)^k = \left( \frac{u-1}{x-1} \right) \left( 1 - \left( \frac{u-1}{x-1} \right)^q \right) \), we have

\[
\sum_{k=1}^{r_m} \Psi_{r_n-k}^n(s) \Phi_{r_m-k}^m(t) = \frac{1}{2\pi i} \oint \frac{\hat{P}_t^{(\alpha_m)}}{E(u)} \left( \frac{\hat{P}_s^{(\alpha_n)}}{2^{a+\beta+1} \Gamma(\alpha+1)} \left( \frac{x-1}{u-1} \right)^{r_n} E(x) \left( 1 - \left( \frac{u-1}{x-1} \right)^{r_m} \right) \right) \, du.
\]

and taking the residue at \( u = x \) of the second summand in the inner product yields (25). If instead \( r_m > r_n \), write

\[
\sum_{k=1}^{r_m} \Psi_{r_n-k}^n(s) \Phi_{r_m-k}^m(t) = \sum_{k=1}^{r_n} \Psi_{r_n-k}^n(s) \Phi_{r_m-k}^m(t) + \sum_{k=r_n+1}^{r_m} \Psi_{r_n-k}^n(s) \Phi_{r_m-k}^m(t).
\]

The first summand is treated as in (40):

\[
\sum_{k=1}^{r_n} \Psi_{r_n-k}^n(s) \Phi_{r_m-k}^m(t) = \frac{1}{2\pi i} \oint \frac{\hat{P}_t^{(\alpha_m)}}{E(u)} \left( \frac{\hat{P}_s^{(\alpha_n)}}{2^{a+\beta+1} \Gamma(\alpha+1)} \left( \frac{x-1}{u-1} \right)^{r_n} E(x) \left( 1 - \left( \frac{u-1}{x-1} \right)^{r_m} \right) \right) \, du.
\]

For the second summand, we use the geometric sum identity repeatedly to get

\[
\sum_{k=r_n+1}^{r_m} \left( E(x) - E(1) \right) \frac{(u-1)^k}{(x-1)^{k-r_n}} = 1_{(r_n \geq r_n+1)} \sum_{k=r_n+2}^{r_m} \sum_{r=1}^{k-r_n-1} \frac{E(r)(1)}{r!} \frac{(u-1)^k}{(x-1)^{k-r_n-r}}
\]

\[
= \left( E(x) - E(1) \right) \frac{(u-1)^{r_n+1}}{x-u} \left( 1 - \frac{u-1}{x-1} \right)^{r_m-r_n}
\]

\[
- 1_{(r_n \geq r_n+1)} \sum_{r=1}^{r_m-r_n-1} \frac{E(r)(1) u^{r_n+r+1}}{r!} \frac{(u-1)^{r_m-r_n-r}}{x-u} \left( 1 - \frac{u-1}{x-1} \right)^{r_m-r_n-r}.
\]

and then arrive at

\[
\sum_{k=r_n+1}^{r_m} \Psi_{r_n-k}^n(s) \Phi_{r_m-k}^m(t) = -\frac{1}{2\pi i} \oint \frac{\hat{P}_t^{(\alpha_m)}}{2^{a+\beta+1} \Gamma(\alpha+1)} \left( \frac{x-1}{u-1} \right)^{r_n-r_m} \frac{R_{r_m-r_n}^E(u)}{E(u)(x-u)} \, du
\]

\[
+ \frac{1}{2\pi i} \oint \frac{\hat{P}_t^{(\alpha_m)}}{2^{a+\beta+1} \Gamma(\alpha+1)} \left( \frac{x-1}{u-1} \right)^{r_m-r_n} \frac{R_{r_m-r_n}^E(u) - E(u) + E(x)}{E(u)(x-u)} \, du
\]

Taking the residue at \( u = x \) of the first term and combining with the first summand completes the proof.

References


