Disagreement coupling of Gibbs processes with an application to Poisson approximation

GÜNTER LAST* and MORITZ OTTO†

June 1, 2022

Abstract

We discuss a thinning and an embedding procedure to construct finite Gibbs processes with a given Papangelou intensity. Extending the approach in [18, 17] we will use this to couple two finite Gibbs processes with different boundary conditions. As one application we will establish Poisson approximation of point processes derived from certain infinite volume Gibbs processes via dependent thinning. As another application we shall discuss empty space probabilities of certain Gibbs processes.

Keywords: Gibbs process, disagreement coupling, Poisson approximation, Papangelou intensity, Poisson thinning, Poisson embedding, empty space probabilities

AMS MSC 2010: 60G55, 60D05, 60K35

1 Introduction

In the seminal paper [38] the authors introduced disagreement percolation for discrete Markov random fields to control boundary effects and to establish new uniqueness criteria for Gibbs measures on graphs. The idea is to locally couple two fields with different boundary conditions and to control the disagreement with a stochastically dominating percolation process. In [18, 17] the method was developed for Gibbs processes in continuum. The first aim of this paper is to establish disagreement coupling in great generality using modern point process theory. This way we shall also close a gap left in [18, 17]. The second main aim is to combine this coupling with a recent result from [6] (generalizing a classical result from [1]) to obtain Poisson approximation of point processes derived of Gibbsian functionals. We shall also discuss empty space probabilities of Gibbs processes.

Gibbs processes form an important class of point processes. In (mathematical) physics they describe the thermodynamical behavior of interacting particles; see [34]. As a mathematical model they are much more versatile than the completely independent Poisson process (see e.g. [9]). They are also quite popular in spatial statistics; see e.g. [30, 7]. Getting a handle on distributional properties of Gibbs processes is not easy. However,

*guenter.last@kit.edu, Karlsruhe Institute of Technology, Institute of Stochastics, 76131 Karlsruhe, Germany.
†moritz.otto@ovgu.de, Otto von Guericke University Magdeburg, Institute for Mathematical Stochastics (IMST), Universitätsplatz 2, 39106 Magdeburg, Germany.
starting with the seminal paper [35], the last decade has seen some efforts to understand some asymptotic properties of functionals of Gibbs processes in infinite volume, at least if the process is in a certain sense close to a Poisson process. Our paper aims at adding to this development.

Let us shortly summarize the structure and main results of the paper. Section 2 collects some basic facts on Gibbs processes based on the classical point process approach from [31] and [27]. The state space is assumed to be Borel but is otherwise not required to have any topological properties. The main result of Section 4 is the thinning representation from Theorem 4.3 which extends a result in [17] to general Borel spaces. Our proof is based on Lebesgue–Stieltjes calculus and is different from the one given in [17]. In Section 5 we prove with Theorem 5.1 a version of the thinning representation based on embedding into a Poisson process on a suitable product space. This result is quite convenient for coupling purposes and should be compared with the classical Poisson embedding of marked point processes based on stochastic intensities; see [5]. This embedding does not require any boundedness assumptions on the Papangelou intensity.

In Section 6 we formulate and study the disagreement coupling of two (local) Gibbs processes with the same Papangelou intensity but different boundary conditions. To this end we consider a symmetric and measurable relation on the state space equipping each point configuration with a graph structure. At each point of the state space the Papangelou intensity is then assumed to depend only on the cluster connected to this point. Theorem 6.3 generalizes Theorem 3.1 in [17] and shows that the points of disagreement of two recursively defined point processes are connected to the boundary conditions. The main part of the proof (which seems to be missing in [17]) is devoted to checking that the coupled processes have the desired Gibbs distribution. Our main tool here is a basically well-known spatial Markov property of the Poisson process. We work here with a rather general definition of a stopping set given in the Appendix of [24]. We have chosen the terminology “disagreement coupling” as opposed to “disagreement percolation” because this coupling might be potentially useful also beyond a percolation setting. In fact, Theorem 6.3 does not require the absence of percolation in the dominating Poisson process. But indeed, so far all current applications of disagreement coupling require the absence of percolation. The main example are proofs of uniqueness of certain Gibbs distributions; see [18, 38, 17]. A very recent result in this area is [3], a paper that makes crucial use of the general setting in Theorem 6.3. In [2] disagreement coupling was used to establish exponential decorrelation of certain Gibbs processes.

In Section 7 we discuss empty space probabilities of Gibbs processes. In particular we show that a large class of Gibbs processes are Poisson-like as defined in [35]. This property is needed in the proof of Theorem 10.1 but we believe that it is of importance in its own right.

Section 8 contains some (basically well-known) material on Palm distributions of a Gibbs process which is needed later. In Theorem 9.1 we prove a bound on the total variation distance between an appropriately scaled thinning of a marked Gibbs process in $\mathbb{R}^d$ with bounded Papangelou intensity and a Poisson process. Our proof adapts a coupling technique for Poisson approximation from [32, Theorem 3.3] to Gibbs processes. In doing so it exploits the disagreement coupling studied in Section 6 for a Gibbs process and its Palm version. Similarly as in [38, 17] we assume that a random graph defined on the points of a dominating Poisson process does not percolate. In fact we need an
exponentially small cluster size; see (9.13). This condition appears to be weaker than subcriticality of an associated spatial branching type process, as assumed in [35]; see [2] for a short discussion. Our general bound involves expectations with respect to the Gibbs distribution. This cannot be avoided. Still, apart from subcriticality of the dominating percolation model, our total variation bound is more explicit and more general than the one presented in [36]. We demonstrate this in Section 10 by applying Theorem 9.1 to Matérn type I thinnings of Gibbs processes. In this situation the bound from Theorem 9.1 becomes very concrete.

2 Gibbs processes

Let \((\mathbb{X}, \mathcal{X})\) be a Borel space equipped with a \(\sigma\)-finite measure \(\lambda\) and define the set ring \(\mathcal{X}_0 := \{B \in \mathcal{X} : \lambda(B) < \infty\}\). Let \(\mathcal{N} = \mathcal{N}(\mathbb{X})\) be the space of all measures on \(\mathbb{X}\) which are \(\mathbb{N}_0\)-valued on \(\mathcal{X}_0\) (where \(\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}\)) and let \(\mathcal{N}\) denote the smallest \(\sigma\)-field such that \(\mu \mapsto \mu(B)\) is measurable for all \(B \in \mathcal{X}\). A point process is a random element \(\eta\) of \(\mathcal{N}\), defined over some fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The intensity measure of \(\eta\) is the measure \(\mathbb{E}[\eta]\) defined by \(\mathbb{E}[\eta](B) := \mathbb{E}\eta(B), \ B \in \mathcal{X}\). For any \(\sigma\)-finite measure \(\nu\) on \(\mathbb{X}\) we let \(\Pi_{\nu}\) denote the distribution of a Poisson process (see e.g. [25]) on \(\mathbb{X}\) with intensity measure \(\nu\). Of particular interest for us are the distributions \(\Pi_{\alpha \lambda_B}\), where \(\alpha \geq 0\) and \(\lambda_B := \lambda(B \cap \cdot)\) is the restriction of \(\lambda\) to some \(B \in \mathcal{X}_0\). We use the latter notation for any measure on \(\mathbb{X}\). For each \(B \in \mathcal{X}\) we define \(\mathcal{N}_B := \{\mu \in \mathcal{N} : \mu(B^c) = 0\} = \{\mu_B : \mu \in \mathcal{N}\}\). For \(\mu \in \mathcal{N}\) we write \(x \in \mu\) if \(\mu(\{x\}) > 0\). If \(\mu(\{x\}) \in \{0, 1\}\) for all \(x \in \mathbb{X}\) then \(\mu\) is called simple.

Let \(\kappa : \mathbb{X} \times \mathcal{N} \to \mathbb{R}_+\) be measurable. A point process \(\xi\) on \(\mathbb{X}\) is called a Gibbs process with Papangelou intensity (PI) \(\kappa\) if

\[
\mathbb{E}\left[\int f(x, \xi) \, \xi(dx)\right] = \mathbb{E}\left[\int f(x, \xi + \delta_x) \kappa(x, \xi) \, \lambda(dx)\right],
\]

for each measurable \(f : \mathbb{X} \times \mathcal{N} \to \mathbb{R}_+\). The latter are the GNZ equations named after Georgii, Nguyen and Zessin [13, 31].

In the present generality it is not known, whether Gibbs processes exist. Partial answers (under varying assumptions on \(\kappa\)) are given for instance in [34, 26, 10, 12, 20]. A considerably more challenging topic is the distributional uniqueness of a Gibbs process. From a mathematical point of view not much is known in this regard. We refer to [9, 20] for recent surveys.

Equation (2.1) can be generalized. For \(m \in \mathbb{N}\) we define a measurable function \(\kappa_m : \mathbb{X}^m \times \mathcal{N} \to \mathbb{R}_+\) by

\[
\kappa_m(x_1, \ldots, x_m, \mu) := \kappa(x_1, \mu)\kappa(x_2, \mu + \delta_{x_1}) \cdots \kappa(x_m, \mu + \delta_{x_1} + \cdots + \delta_{x_{m-1}}).
\]

Note that \(\kappa_1 = \kappa\). If \(\xi\) is a Gibbs process with PI \(\kappa\) then we have for each \(m \in \mathbb{N}\) and for each measurable \(f : \mathbb{X}^m \times \mathcal{N} \to \mathbb{R}_+\) that

\[
\mathbb{E}\left[\int f(x_1, \ldots, x_m, \xi)^{m}(d(x_1, \ldots, x_m))\right] = \mathbb{E}\left[\int f(x_1, \ldots, x_m, \xi + \delta_{x_1} + \cdots + \delta_{x_m}) \kappa_m(x_1, \ldots, x_m, \xi) \, \lambda^m(d(x_1, \ldots, x_m))\right],
\]

(2.2)
where $\mu^{(m)}$ is the $m$-th factorial measure of $\mu$; see [25]. This follows by induction, using that
\[
\kappa_{m+1}(x_1, \ldots, x_m, \mu) = \kappa_m(x_1, \ldots, x_m, \mu) \kappa(\mu + \delta_{x_1} + \cdots + \delta_{x_m}).
\]
For each $m \in \mathbb{N}$ let $\tilde{\kappa}_m: \mathcal{X}^m \times \mathbb{N} \to \mathbb{R}_+$ be the symmetrization of $\kappa_m$ in the first $m$ arguments. The Hamiltonian $H: \mathbb{N} \times \mathbb{N} \to (-\infty, \infty]$ (based on $\kappa$) is defined by
\[
H(\mu, \psi) := \begin{cases} 
0, & \text{if } \mu(\mathcal{X}) = 0, \\
-\log \tilde{\kappa}_m(x_1, \ldots, x_m, \psi), & \text{if } \mu = \delta_{x_1} + \cdots + \delta_{x_m}, \\
\infty, & \text{if } \mu(\mathcal{X}) = \infty.
\end{cases} \tag{2.3}
\]
For $B \in \mathcal{X}_0$ the partition function $Z_B: \mathbb{N} \to [0, \infty]$ is defined by
\[
Z_B(\psi) := \int e^{-H(\mu, \psi)} \Pi_\lambda (d\mu), \quad \psi \in \mathbb{N}. \tag{2.4}
\]
Since $H(0, \psi) = 0$ for all finite $\psi \in \mathbb{N}$ we have that
\[
Z_B(\psi) \geq e^{-\lambda(B)}, \quad B \in \mathcal{X}_0.
\]
Note that the right-hand side is positive. For $\nu \in \mathbb{N}$ the Gibbs measure $\Pi_{B,\nu}$ on $\mathbb{N}$ is defined by
\[
\Pi_{B,\nu} := Z_B(\nu)^{-1} \int 1\{\mu \in \cdot\} e^{-H(\mu, \nu)} \Pi_\lambda (d\mu) \tag{2.5}
\]
provided that $Z_B(\nu) < \infty$. If $Z_B(\nu) = \infty$ we set $\Pi_{B,\nu} := \Pi_\lambda$. This measure is concentrated on $\mathbb{N}_B$.

It was proved in [27, 31] that if $\xi$ is a Gibbs process with $\Pi \kappa$ then $\kappa_m(\cdot, \xi)$ is almost surely symmetric for each $m \in \mathbb{N}$,
\[
\mathbb{P}(Z_B(\xi_{B^c}) < \infty) = 1, \quad B \in \mathcal{X}_0, \tag{2.6}
\]
and, for each measurable $f: \mathbb{N} \to \mathbb{R}_+$,
\[
\mathbb{E}[f(\xi_B) \mid \xi_{B^c}] = \int f(\mu) \Pi_{B,\xi_{B^c}} (d\mu), \quad B \in \mathcal{X}_0, \tag{2.7}
\]
where relations involving conditional expectations are assumed to hold almost surely. These are the DLR-equations; see [34, 21, 26]. Note that (2.7) implies
\[
\mathbb{P}(\xi(B) = 0 \mid \xi_{B^c}) = e^{-\lambda(B)} Z_B(\xi_{B^c})^{-1}, \quad B \in \mathcal{X}_0. \tag{2.8}
\]

A natural requirement on $\kappa$ is the cocycle formula
\[
\kappa(x, \mu) \kappa(\mu + \delta_x) = \kappa(\mu, \mu + \delta_y), \tag{2.9}
\]
which should hold at least for $\lambda^2 \otimes \Pi_\lambda$-a.e. $(x, y, \mu)$. (Otherwise one cannot hope for the existence of a Gibbs process.) It then follows for each $m \in \mathbb{N}$ that
\[
\kappa_m(x_1, \ldots, x_m, \mu) = \tilde{\kappa}_m(x_1, \ldots, x_m, \mu), \tag{2.10}
\]
for $\lambda^m \otimes \Pi_\lambda$-a.e. $(x_1, \ldots, x_m, \mu)$. Even though not stated in the present generality, the following result was proved in the seminal work [31].
Theorem 2.1. Assume that $\kappa$ satisfies (2.9) for each $\mu \in \mathcal{N}$ and for $\lambda^2$-a.e. $(x, y)$. Assume that $\xi$ is a point process satisfying (2.6) and (2.7). Then $\xi$ is a Gibbs process with PI $\kappa$.

In fact, the proof of Theorem 2.1 shows the following.

Corollary 2.2. Assume that $\lambda(X) < \infty$. Assume also that $\kappa$ satisfies the assumption of Theorem 2.1 and that $Z_X := \int e^{-H(\mu, 0)} \Pi_\lambda(d\mu) < \infty$. Then

$$Z_X^{-1} \int 1\{\mu \in \cdot\} e^{-H(\mu, 0)} \Pi_\lambda(d\mu)$$

(2.11)

is the distribution of a Gibbs process with PI $\kappa$.

For $B \in \mathcal{X}_0$ and $\psi \in \mathcal{N}$ we define $\kappa_{\psi} : X \times \mathcal{N} \to \mathbb{R}_+$ by

$$\kappa_{\psi}(x, \mu) := \kappa(x, \psi + \mu)$$

(2.12)

Let $\kappa_{B, \psi}$ denote the restriction of $\kappa_{\psi}$ to $B \times \mathcal{N}_B$. Corollary 2.2 shows that the conditional distribution $\mathbb{P}(\xi_B \in \cdot | \xi_{B^c})$ is almost surely a Gibbs process with PI $\kappa_{B, \xi_{B^c}}$; see also [2, Lemma 2.5].

In Section 6 we shall need the following property of Gibbs measures. Let $B, C \in \mathcal{X}_0$ with $C \subset B$ and suppose that $\nu \in \mathcal{N}_{X \setminus B}$ satisfies $Z_B(\nu) < \infty$. Then $Z_{B \cup C}(\nu + \mu_C) < \infty$ for $\Pi_{B, \nu}$-a.e. $\mu$ and

$$\Pi_{B, \nu} = \int \int 1\{\mu_C + \mu' \in \cdot\} \Pi_{B \cup C, \nu + \mu_C}(d\mu') \Pi_{B, \nu}(d\mu).$$

(2.13)

This is the DLR-equation for a Gibbs process with PI $\kappa_{B, \nu}$. It can be proved directly, using the definition (2.5) of a Gibbs measure and the fact that a Poisson process on $B$ is the sum of two independent Poisson processes on $C$ and $B \setminus C$.

3 Assumptions and Examples

In this paper a measurable function $\kappa : X \times \mathcal{N} \to \mathbb{R}_+$ will always denote a PI of a Gibbs process. To simplify the presentation we shall always assume (sometime without further mentioning) that $\kappa$ satisfies the cocycle identity (2.9) for all $(x, y, \mu) \in X^2 \times \mathcal{N}$. Very much as in the literature we will make two types of assumptions on $\kappa$, namely stability assumptions and assumptions on the local dependence of $\kappa(x, \mu)$ on $\mu$. We say that $\psi \in \mathcal{N}$ is stable on $W \in \mathcal{X}$ if

$$Z_B(\psi_{B^c}) < \infty, \quad B \in \mathcal{X}_0 \cap W,$$

(Dom1)

where $\mathcal{X}_0 \cap W := \{B \cap W : B \in \mathcal{X}_0\}$. If this holds for $W = X$ then we say that $\psi \in \mathcal{N}$ is stable. We say that $\kappa$ is locally stable if there exists a measurable function $\alpha : X \to (0, \infty)$ satisfying

$$\kappa(x, \mu) \leq \alpha(x), \quad (x, \mu) \in X \times \mathcal{N}(X), \quad \text{and} \quad \int_B \alpha(x) \lambda(dx) < \infty, \quad B \in \mathcal{X}_0.$$  

(Dom2)
Remark 3.1. A sufficient condition for $\psi \in \mathbb{N}$ to be stable on $W \in \mathcal{X}_0$ is as follows. Suppose that $\alpha_{W,\psi}: W \to \mathbb{R}_+$ is measurable and satisfies $\int_W \alpha_{W,\psi}(x) \lambda(dx) < \infty$. If
\[
\sup\{\kappa(x, \mu_W + \psi_W \cdot) : \mu \in \mathbb{N}\} \leq \alpha_{W,\psi}(x), \quad x \in W, \mu \in \mathbb{N}, \tag{3.1}
\]
then the definition of $Z_B$ shows that (Dom1) holds. In particular, if $\kappa$ is locally stable, then all $\psi \in \mathbb{N}$ are stable.

To state our second type of assumptions we assume that $\sim$ is a symmetric relation on $\mathbb{X}$ such that $\{(x, y) : x \sim y\}$ is a measurable subset of $\mathbb{X}^2$. Given $x \in \mathbb{X}$ and $A \subset \mathbb{X}$ we write $x \sim A$ if there exists $z \in A$ such that $x \sim z$. We say that $x, z \in \mathbb{X}$ are connected via $A$ if there exist $n \in \mathbb{N}_0$ and $z_1, \ldots, z_n \in A$ such that $z_i \sim z_{i+1}$ for $i \in \{0, \ldots, n\}$ where $z_0 := x$ and $z_{n+1} := z$. We use this terminology also for counting measures instead of $A$. Define the cluster $C(x, \mu) \in \mathbb{N}$ of $x \in \mathbb{X}$ in $\mu \in \mathbb{N}$ by
\[
C(x, \mu) := \int 1\{y \in \cdot\} 1\{y \neq x \text{ and } x \text{ and } y \text{ are connected via } \mu\} \mu(dy).
\]
We will consider the assumption
\[
\kappa(x, \mu) = \kappa(x, C(x, \mu)), \quad (x, \mu) \in \mathbb{X} \times \mathbb{N}. \tag{Loc1}
\]
For $x \in \mathbb{X}$ we let $N_x := \{y \in \mathbb{X} \setminus \{x\} : y \sim x\}$. A stronger version of (Loc1) is
\[
\kappa(x, \mu) = \kappa(x, \mu_{N_x}), \quad (x, \mu) \in \mathbb{X} \times \mathbb{N}. \tag{Loc2}
\]
Of course, in such a general setting the assumptions (Loc1) and (Loc2) do not put a restriction on $\kappa$. It is the specific choice of $\sim$ together with further assumptions on $\lambda$, which will make them meaningful.

The following examples will illustrate the forgoing assumptions.

Example 3.2. (Gibbs processes with pair potential) Suppose that $U$ is a pair potential on $\mathbb{X}$ that is a measurable and symmetric function $U: \mathbb{X} \times \mathbb{X} \to (\infty, \infty]$. We assume that there exists $C > 0$ such that
\[
\sum_{i<j} U(x_i, x_j) \geq -Cn, \quad x_1, \ldots, x_n \in \mathbb{X}, \quad n \in \mathbb{N}.
\]
This is a classical stability assumption on a pair potential; see [34]. Let $\alpha > 0$ and assume that $\kappa$ is given by
\[
\kappa(x, \mu) = \alpha \exp\left[-\int U(x, y) \mu(dy)\right], \quad x \in \mathbb{X}, \mu \in \mathbb{N},
\]
whenever the integral in the right-hand side exists. Let $W \in \mathcal{X}_0$ and $\psi \in \mathbb{N}$. A simple calculation shows that $Z_W(\psi_W \cdot) < \infty$ if
\[
\int_W \exp\left[-\int U(x, y) \psi(dy)\right] \lambda(dx) < \infty. \tag{3.2}
\]
Assume moreover that $U$ is bounded from below and let $\psi \in \mathbf{N}$ satisfy (3.2). Then it is easy to see that $\psi$ is stable on $W$ and also that $\psi$ is a regular boundary condition for $W$. Define $x \sim y \iff U(x,y) \neq 0$. Then (Loc2) holds. If $U \geq 0$, then (Dom2) holds.

The existence of Gibbs processes with a pair potential was shown under varying assumptions. We refer here to the seminal work [33, 34] (treating the case $X = \mathbb{R}^d$), to [20] (working on a complete separable metric space and assuming $U \geq 0$) and to [12] (working on $\mathbb{R}^d$ but allowing for infinite range pair potentials which may take negative values).

**Example 3.3. (Strauss process)** Suppose that $X = \mathbb{R}^d \times \mathbb{R}^+$ and $\lambda = \lambda_d \otimes \mathbb{Q}$ for some probability measure $\mathbb{Q}$ on $[0, \infty)$. Define $(x,r) \sim (y,s) \iff \|x-y\| \leq r + s$. Consider for $\alpha > 0$ and $\beta \in [0, 1]$ the PI

$$\kappa(x,r,\mu) := \alpha \beta^{\mu(N(x,r))}, \quad x \in \mathbb{R}^d, \ r > 0, \ \mu \in \mathbb{N}.$$ 

A Gibbs process $\xi$ with this PI is called Strauss process. For $\beta = 0$ this is a hard-core process. It is easy to see that $\kappa$ satisfies (Dom2) and (Loc2).

**Example 3.4. (Area interaction process)** As for the Strauss process, let $X := \mathbb{R}^d \times \mathbb{R}^+$ and $\lambda := \lambda_d \otimes \mathbb{Q}$ for some probability measure $\mathbb{Q}$ on $[0, \infty)$. Define $(x,r) \sim (y,s) \iff \|x-y\| \leq r + s$. For $(x,r) \in X$ and $\mu \in \mathbb{N}$ define

$$V(x,r,\mu) := \lambda_d \left( B(x,r) \setminus \bigcup_{(y,s) \in \mu^{-1}(x,r)} B(y,s) \right).$$

Consider for $\alpha > 0$ and $\beta \in [0, 1]$ the PI

$$\kappa(x,r,\mu) := \alpha \beta^{V(x,r,\mu)}, \quad x \in \mathbb{R}^d, \ r > 0, \ \mu \in \mathbb{N}.$$ 

Then (Loc2) holds. Gibbs processes with this PI are called area interaction processes. Their existence is shown in [33] for $\mathbb{Q}$ with finite support and in [8] for dimension $d \leq 2$. If $\mathbb{Q}([r_1, \infty)) = 1$ for some $r_1 > 0$, then (Dom2) is satisfied.

**Example 3.5. (Continuum random cluster model)** Let $X := \mathbb{R}^d \times \mathbb{R}^+$ and $\lambda := \lambda_d \otimes \mathbb{Q}$ for some probability measure $\mathbb{Q}$ on $[0, \infty)$. Define $(x,r) \sim (y,s) \iff \|x-y\| \leq r + s$. Let $\alpha > 0$, $q > 0$ and $k(x,r,\mu)$ denote the number of connected components in $\mu$ connected to $(x,r)$ in $\mu + \delta(x,r)$ and define

$$\kappa(x,r,\mu) = \alpha q^{1-k(x,r,\mu)}, \quad x \in \mathbb{R}^d, \ r > 0, \ \mu \in \mathbb{N}.$$ 

It is clear that (Loc1) holds. If $q \geq 1$ then (Dom2) holds. Moreover it was shown in [11] that in both cases there exists a Gibbs process $\xi$ with PI $\kappa$. If there are $0 < r_1 < r_2 < \infty$ such that $\mathbb{Q}([r_1, r_2]) = 1$, then there exists a constant $c_d$ (depending on the dimension $d$) such that $\kappa(x,r,\mu) \leq q^{1-c_d}$ and (Dom2) holds.

**Example 3.6. (Widom-Rowlinson model)** For $m \geq 2$ let $M := \{1, \ldots, m\}$, $X := \mathbb{R}^d \times \mathbb{R}^+ \times M$ and $\lambda := \lambda_d \otimes \mathbb{Q} \otimes \mathbb{U}$, where $\mathbb{Q}$ is a probability measure on $[0, \infty)$ and $\mathbb{U}$ denotes uniform distribution on $M$. Define

$$(x,r,\ell) \sim (y,s,k) \iff \|x-y\| \leq r + s \text{ and } \ell \neq k, \quad x, y \in \mathbb{R}^d, \ r, s \in \mathbb{R}^+, \ \ell, k \in M.$$
and let
\[
\kappa(x, r, \ell, \mu) = \begin{cases} 
0, & \text{if } \mu(N(x, r, \ell)) > 0, \\
\alpha, & \text{otherwise}. 
\end{cases}
\]

If we think of the elements of $\mathbb{M}$ as colors, the construction rule of this model forbids overlapping of two balls of different colors. It was first introduced in [39] for deterministic radii. For its existence we refer to [14, Remark 4.2]. Clearly, (Dom2) and (Loc2) are satisfied.

## 4 Thinning presentation of finite Gibbs processes

In this section we assume that the measure $\lambda$ is finite and diffuse. We consider a measurable function $\kappa: X \times N \to \mathbb{R}_+$ satisfying the stability assumption (Dom2). We know from [15] (see [2] for the infinite case) that a Gibbs process with PI $\kappa$ is stochastically dominated by a Poisson process with intensity measure $\alpha \lambda$, defined by $\alpha \lambda(d\mu) := \alpha(x) \lambda(d\mu)$. In this section we shall reestablish this result by means of a rather explicit thinning construction, introduced in [17, Proposition 4.1]. (The proof was amended in the latest preprint version of that paper.) We work here in greater generality using fundamental properties of Poisson processes and standard techniques from Lebesgue–Stieltjes calculus.

We need to introduce some notation. By definition of a Borel space there exists an injective measurable mapping $\varphi: X \to \mathbb{R}$ such that $\varphi(X)$ and the inverse mapping $\varphi^{-1}: \varphi(X) \to X$ are measurable. We introduce a total order $\leq$ on $X$ by writing $x \leq y$ if $\varphi(x) \leq \varphi(y)$. We write $x < y$ if $x \leq y$ and $x \neq y$. As usual we can then define \( (x, y) = \{z \in X : x < z \leq y\} \). Other intervals are defined analogously. For instance we write \( (-\infty, y) := \{z \in X : z < y\} \). Let $N^* \equiv N^*(X)$ denote the set of all simple and finite elements of $N$. It is easy to see that the mapping $(x, \mu) \mapsto (\mu(-\infty, x), \mu(-\infty, x])$ is measurable on $X \times N^*$. We abbreviate $\mu_x := \mu(-\infty, x)$.

Assume that $\kappa: X \times N^* \to \mathbb{R}_+$ is a measurable function satisfying the cocycle assumption (2.9) for all $(x, y, \mu) \in X^2 \times N^*$. We define the partition functions by (2.4). Define a function $p: X \times N^* \to [0, 1]$ by

\[
p(x, \psi) := \kappa(x, \psi_x) \frac{Z_{(x, \infty)}(\psi_x + \delta_x)}{Z_{(x, \infty)}(\psi_x)}, \quad (x, \psi) \in X \times N^*,
\] (4.1)

where $\infty/\infty := 0$. Since

\[
Z_{(x, \infty)}(\psi_x + \delta_x) = \int e^{-H(\mu, \psi + \delta_x)} \Pi(\lambda(d\mu)),
\]

\[
Z_{(x, \infty)}(\psi_x) = \int e^{-H(\mu, \psi)} \Pi(\lambda(d\mu)),
\]

the function $p$ is measurable. It turns out that it also satisfies (Loc2).

**Lemma 4.1.** We have that

\[
p(x, \psi) \leq \alpha(x), \quad (x, \psi) \in X \times N^*.
\] (4.2)
Proof: Let \( x \in \mathbb{X} \) and \( \psi \in \mathbb{N}^* \). By the multivariate Mecke equation ([25, Theorem 4.4]),

\[
\kappa(x, \psi_x) Z(x, \infty)(\psi_x + \delta_x) = \kappa(x, \psi_x) \int e^{-H(\mu, \psi_x + \delta_x)} \, d\Pi_{\lambda(x, \infty)}(d\mu)
\]

\[
= \kappa(x, \psi_x) e^{-\lambda(x, \infty)} \left( 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \kappa_m(x_1, \ldots, x_m, \psi_x + \delta_x) \lambda_{(x, \infty)}^m(d(x_1, \ldots, x_m)) \right)
\]

\[
= e^{-\lambda(x, \infty)} \left( \kappa(x, \psi_x) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \kappa_{m+1}(x_1, \ldots, x_m, \psi_x) \lambda_{(x, \infty)}^m(d(x_1, \ldots, x_m)) \right).
\]

Since

\[
\kappa_{m+1}(x_1, \ldots, x_m, \psi_x) = \kappa(x, \psi_x + \delta_{x_1} + \cdots + \delta_{x_m}) \kappa_m(x_1, \ldots, x_m, \psi_x)
\]

and \( \kappa(x, \cdot) \leq \alpha(x) \) by (Loc2), the above is bounded by

\[
\alpha(x) e^{-\lambda(x, \infty)} \left( 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\mathbb{X}^m} \kappa_m(x_1, \ldots, x_m, \psi_x) \lambda_{(x, \infty)}^m(d(x_1, \ldots, x_m)) \right)
\]

which equals \( \alpha(x) Z(x, \infty)(\psi_x) \). This proves (4.2). \( \square \)

Let us define a kernel \( K_{\kappa, \alpha} \) from \( \mathbb{N}^* \) to \( \mathbb{N}^* \) by

\[
K_{\kappa, \alpha}(\mu, \cdot) := \sum_{\psi \leq \mu} 1\{\psi \in \cdot\} \prod_{x \in \psi} \alpha(x)^{-1} p(x, \psi) \prod_{x \in \mu - \psi} (1 - \alpha(x)^{-1} p(x, \psi)), \quad \mu \in \mathbb{N}^*, \quad (4.3)
\]

The next lemma shows that \( K_{\kappa, \alpha} \) is a probability kernel.

**Lemma 4.2.** We have that \( K_{\kappa, \alpha}(\mu, \mathbb{N}^*) = 1 \) for all \( \mu \in \mathbb{N}^* \).

**Proof:** We use induction on the number of point of \( \mu \). Obviously, \( K_{\kappa, \alpha}(0, \mathbb{N}^*) = 1 \), where 0 is the zero measure.

Let \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n, y \in \mathbb{X} \) with \( x_1 < \cdots < x_n < y \). Define \( \mu := \delta_{x_1} + \cdots + \delta_{x_n} \).

From the definition of \( K_{\kappa, \alpha} \) we obtain that we have

\[
K_{\kappa, \alpha}(\mu + \delta_y, \mathbb{N}^*) = \sum_{\psi \leq \mu + \delta_y} \prod_{x \in \psi} \alpha(x)^{-1} p(x, \psi) \prod_{x \in \mu - \psi} (1 - \alpha(x)^{-1} p(x, \psi))
\]

\[
= \sum_{\psi \leq \mu} \left( \prod_{x \in \psi + \delta_y} \alpha(x)^{-1} p(x, \psi + \delta_y) \prod_{x \in \mu - \psi} (1 - \alpha(x)^{-1} p(x, \psi + \delta_y))
\]

\[
+ \prod_{x \in \psi} \alpha(x)^{-1} p(x, \psi) \prod_{x \in \mu + \delta_y - \psi} (1 - \alpha(x)^{-1} p(x, \psi)) \right).
\]

Since \( p(x, \mu) = p(x, \mu_x) \) for all \( (x, \mu) \in \mathbb{X} \times \mathbb{N}^* \) and \( x_1 < \cdots < x_n < y \), we obtain that the above equals

\[
\sum_{\psi \leq \mu} \left( \alpha(y)^{-1} p(y, \psi + \delta_y) \prod_{x \in \psi} \alpha(x)^{-1} p(x, \psi) \prod_{x \in \mu - \psi} (1 - \alpha(x)^{-1} p(x, \psi))
\]

\[
+ (1 - \alpha(y)^{-1} p(y, \psi)) \prod_{x \in \psi} \alpha(x)^{-1} p(x, \psi) \prod_{x \in \mu - \psi} (1 - \alpha(x)^{-1} p(x, \psi)) \right).
\]
Since \( p(y, \psi + \delta y) = p(y, \psi) \), this equals
\[
\sum_{\psi \leq \mu} \prod_{x \in \psi} \alpha(x)^{-1} p(x, \psi) \prod_{x \in \mu - \psi} (1 - \alpha(x)^{-1} p(x, \psi)) = K_{\kappa, \alpha}(\mu, \mathbb{N}^*).
\]
This finishes the induction. \( \square \)

The following thinning representation is the main result of this section. It extends [17, Proposition 4.1] to general Borel spaces.

**Theorem 4.3.** Assume that \( \lambda \) is finite and diffuse and that (Dom2) holds. Then
\[
Q_{\kappa, \alpha} := \int \int 1\{\psi \in \cdot\} K_{\kappa, \alpha}(\mu, d\psi) \Pi_{\alpha \lambda}(d\mu) \tag{4.4}
\]
is the distribution of a Gibbs process with PI \( \kappa \).

**Proof:** The proof will use Corollary 4.5, to be proved later. Let \( f : \mathbb{N}^* \to \mathbb{R}_+ \) be measurable. Then
\[
\int \int f(\psi) Q_{\kappa, \alpha}(d\psi) = \sum_{n=0}^{\infty} \int \int 1\{\psi(\mathbb{X}) = n\} f(\psi) K_{\kappa, \alpha}(\mu, d\psi) \Pi_{\alpha \lambda}(d\mu).
\]
Since \( \alpha \lambda \) is diffuse, a Poisson process with this distribution is simple ([25, Proposition 6.9]), so that the above equals
\[
\sum_{n=0}^{\infty} \int \int f(\delta x_1 + \cdots + \delta x_n) 1\{x_1 < \cdots < x_n\} \prod_{i=1}^{n} \alpha(x_i)^{-1} p(x_i, \delta x_1 + \cdots + \delta x_n) \prod_{x \in \mu - (\delta x_1 + \cdots + \delta x_n)} (1 - \alpha(x_i)^{-1} p(x, \delta x_1 + \cdots + \delta x_n)) \mu^{(n)}(d(x_1, \ldots, x_n)) \Pi_{\alpha \lambda}(d\mu),
\]
where, for \( n = 0 \), we interpret \( \delta x_1 + \cdots + \delta x_n \) as the zero measure. By the multivariate Mecke equation ([25, Theorem 4.4]), this equals
\[
\sum_{n=0}^{\infty} \int f(\delta x_1 + \cdots + \delta x_n) \prod_{i=1}^{n} p(x_i, \delta x_1 + \cdots + \delta x_n) P(x_1, \ldots, x_n) \lambda_n(d(x_1, \ldots, x_n)),
\]
where \( \lambda_n \) is the restriction of \( \lambda^n \) to \( \{x_1 < \cdots < x_n\} \) and
\[
P(x_1, \ldots, x_n) := \int \prod_{x \in \mu} (1 - \alpha(x)^{-1} p(x, \delta x_1 + \cdots + \delta x_n)) \Pi_{\alpha \lambda}(d\mu).
\]
By [25, Exercise 3.6] we have that
\[
P(x_1, \ldots, x_n) = \exp \left[ - \int p(x, \delta x_1 + \cdots + \delta x_n) \lambda(dx) \right],
\]

so that
\[
\int f(\psi) \Pi_{\kappa, \lambda}(d\psi) = \sum_{n=0}^{\infty} \int f(\delta_{x_1} + \cdots + \delta_{x_n}) \prod_{i=1}^{n} p(x_i, \delta_{x_1} + \cdots + \delta_{x_n}) \\
\times \exp \left( - \int p(x, \delta_{x_1} + \cdots + \delta_{x_n}) \lambda(dx) \right) \lambda_n(d(x_1, \ldots, x_n)).
\]

By assumption (Dom2) and Remark 3.1 all elements of \(N^*\) are stable. Inserting above the result of Corollary 4.5 and the definition (4.1), we obtain that
\[
\int f(\psi) Q_{\kappa, \alpha}(d\psi) = Z_{X}(0)^{-1} e^{-\lambda(X)} \sum_{n=0}^{\infty} \int f(\delta_{x_1} + \cdots + \delta_{x_n}) \kappa_n(x_1, \ldots, x_n, 0) \\
\times 1\{x_1 < \cdots < x_n\} \lambda^n(d(x_1, \ldots, x_n)),
\]
where \(\kappa_0 \equiv 1\). Since \(\lambda\) is diffuse and \(\kappa_n(\cdot, 0)\) is symmetric \(\lambda^n\)-a.e., we obtain that
\[
\int f(\psi) Q_{\kappa, \lambda}(d\psi) = Z_{X}(0)^{-1} \int f(\mu) e^{-H(\mu, 0)} \Pi_{\lambda}(d\mu).
\]

Therefore the assertion follows from Corollary 2.2. 

The following lemma provides useful information on the function \(p\). It does not require assumption (Dom2). Note that \(\psi_{-\infty} = 0\) and \(Z_{\emptyset}(\psi) = 1\) for each \(\psi \in N^*\). We set \((\infty, \infty) := \emptyset\).

**Lemma 4.4.** Suppose that \(\psi \in N^*\) is stable and let \(z, w \in X \cup \{-\infty, \infty\}\) such that \(z < w\). Then
\[
\int 1\{x \in (z, w]\} p(x, \psi) \lambda(dx) = \lambda(z, w) + \log Z_{(z,\infty)}(\psi_{(-\infty,z]}) - \log Z_{(w,\infty)}(\psi_{(-\infty,w)})) \\
+ \int (\log Z_{(z,\infty)}(\psi_x + \delta_x) - \log Z_{(z,\infty)}(\psi_z))(\psi_{(z,w)})(dx).
\]

**Proof:** For \(x \in X\) and \(\psi \in N^*\) we abbreviate \(Z_x(\psi) := Z_{(x,\infty)}(\psi)\). Since \(\lambda\) is diffuse, we have \(\Pi_{\lambda}(\{\mu \in N : \mu(\{x\}) \geq 1\}) = 0\). Therefore we obtain from definition (2.4) that \(Z_x(\psi) = Z_{(x,\infty)}(\psi)\). Since \(\psi\) is stable, this is a finite number. In the following we write \(\lambda(x, \infty) := \lambda((x, \infty))\) and similar for other intervals. By definition,
\[
Z_x(\psi_x) = e^{-\lambda(x, \infty)} + \sum_{n=1}^{\infty} \int e^{-H(\mu(x, \infty), \psi_x)} 1\{\mu(x, \infty) = n\} \Pi_{\lambda}(d\mu) \\
= e^{-\lambda(x, \infty)} + \sum_{n=1}^{\infty} \int \int e^{-H(\mu(y, \infty), \psi_x)} 1\{y \in (x, \infty)\} \\
\times 1\{\mu(x, y) = 0, \mu(y, \infty) = n - 1\} \mu(dy) \Pi_{\lambda}(d\mu),
\]
where we have used that \(\Pi_{\lambda}\) is the distribution of a simple point process. By the Mecke equation this equals
\[
e^{-\lambda(x, \infty)} + \sum_{n=1}^{\infty} \int \int e^{-H(\mu(y, \infty) + \delta_y, \psi_x)} 1\{y \in (x, \infty)\} \\
\times 1\{\mu(x, y) = 0, \mu(y, \infty) = n - 1\} \Pi_{\lambda}(d\mu) \lambda(dy),
\]
where we have used that a Poisson process with a diffuse intensity measure does not have fixed atoms. Using the complete independence of a Poisson process and then the formula for empty space probabilities, we obtain that

\[ Z_x(\psi_x) = e^{-\lambda(x,\infty)} + \int \int 1\{y \in (x, \infty)\} e^{-\lambda(y) \psi_x} \Pi_\lambda(d\mu) \lambda(dy). \]

By the definition of the Hamiltonian we have that

\[ e^{-H(\mu,y,\psi_x)} = \kappa(y, \psi_x) e^{-H(\mu,y,\psi_x+\delta_y)}, \]

so that

\[ Z_x(\psi_x) = e^{-\lambda(x,\infty)} + \lambda e^\lambda \int 1\{y \in (x, \infty)\} e^{-\lambda(y) \kappa(y,\psi_x)} Z_y(\psi_x + \delta_y) \lambda(dy), \quad (4.6) \]

where \( \lambda(y) := \lambda(-\infty,y) \).

We now argue that we can assume that \( \Xi = \mathbb{R} \). Consider the Borel isomorphism \( \varphi : \Xi \to U := \varphi(\Xi)\), interpreted as a mapping from \( \Xi \) to \( \mathbb{R} \). We define a diffuse measure \( \lambda' \) on \( \mathbb{R} \) by \( \lambda' := \lambda \circ \varphi^{-1} \) (the image of \( \lambda \) under \( \varphi \)). Recall that \( \mathcal{N}^*(\mathbb{R}) \) denotes the set of all finite and simple counting measures on \( \mathbb{R} \) and define a measurable mapping \( \kappa' : \mathbb{R} \times \mathcal{N}^*(\mathbb{R}) \to \mathbb{R}_+ \) by \( \kappa'(u,\mu) := 1\{u \in U\} \kappa(\varphi^{-1}(u),\mu_U \circ \varphi) \). (If \( \mu_U = \delta_{s_1} + \cdots + \delta_{s_n} \) then \( \mu_U \circ \varphi = \delta_{\varphi^{-1}(s_1)} + \cdots + \delta_{\varphi^{-1}(s_n)} \).) Then \( \kappa' \) satisfies the cocycle assumption (2.9) and we denote the associated partition functions by \( Z'_B \) for \( B \in \mathcal{B}(\mathbb{R}) \) (the system of Borel subsets of \( \mathbb{R} \)). It is easy to show that

\[ Z'_B(\mu) = Z_{\varphi^{-1}(B)}(\mu_U \circ \varphi), \quad \mu \in \mathcal{N}^*(\mathbb{R}), \ B \in \mathcal{B}(\mathbb{R}). \quad (4.7) \]

Let \( \mu := \psi \circ \varphi^{-1} \) and suppose that (4.5) holds with \((\Xi, \lambda, \kappa, \psi)\) replaced with \((\mathbb{R}, \lambda', \kappa', \mu)\).

This means that for all \( s,t \in \mathbb{R} \cup \{-\infty, \infty\} \)

\[
\int 1\{u \in (s,t]\} p'(u,\mu) \lambda'(du) = \lambda'(s,t] + \log Z'(s,\infty) (\mu(-\infty,s]) - \log Z'(t,\infty) (\mu(-\infty,t])
\]

\[ + \int \left( \log Z'(u,\infty) (\mu_u + \delta_u) - \log Z'(u,\infty) (\mu_u) \right) \mu(s,t] (du), \]

where \( p' \) is defined in terms of \( \kappa' \) as \( p \) in terms of \( \kappa \). Applying this formula with \((s,t) := (\varphi(z), \varphi(w))\) (where \((\varphi(-\infty), \varphi(\infty)) := (-\infty, \infty)\)) and using (4.7), yields (4.5).

From now on we will assume that \( \Xi = \mathbb{R} \). \( a \in \mathbb{R} \cup \{-\infty, \infty\} \) we set \( \psi_a^+ := \psi[a,\infty) \). Let \( a,b \in \mathbb{R} \cup \{-\infty, \infty\} \) such that \( \psi(a,b) = 0 \). Then

\[ \psi_x = \psi_a^+, \quad x \in (a,b]. \]

Since the left-hand side of (4.6) is finite, so is the right-hand side. This shows that the function \( x \mapsto f(x) := Z_x(\psi_x) \) from \( (a,b] \) to \( \mathbb{R} \) is continuous and of totally bounded variation. Now we use Lebesgue-Stieltjes calculus; see e.g. [23, Appendix A4]. By the product rule and \( d_x e^{-\lambda(x,\infty)} = -e^{-\lambda(x,\infty)} \lambda(dx) \) we have that

\[
\begin{align*}
\frac{df(x)}{dx} &= e^{-\lambda(x,\infty)} \lambda(dx) + e^{\lambda(x)} \left[ \int_x^\infty e^{-\lambda(y)} \kappa(y,\psi^+_a Z_y(\psi^+_a+\delta_y) \lambda(dy) \right] \lambda(dx) \\
&\quad - \kappa(x,\psi_a) Z_x(\psi^+_a + \delta_x) \lambda(dx),
\end{align*}
\]

12
where we note that $\kappa(x, \psi_a)Z_a(\psi_a^+ + \delta_x) < \infty$ for $\lambda$-a.e. $x \in [a, b]$. Therefore,

$$
df(x) = e^{-\lambda(x, \infty)}\lambda(dx) + (f(x) - e^{-\lambda(x, \infty)}) \lambda(dx) - \kappa(x, \psi_a)Z_x(\psi_a^+ + \delta_x) \lambda(dx)
$$

that is

$$
dZ_x(\psi_x) = Z_x(\psi_x) \lambda(dx) - \kappa(x, \psi_a)Z_x(\psi_a^+ + \delta_x) \lambda(dx), \quad \text{on } (a, b]. \quad (4.8)
$$

Therefore we obtain from the definition (4.1) that

$$
\int_a^b p(x, \psi) \lambda(dx) = \lambda(a, b) - \int_a^b Z_x(\psi_x)^{-1} dZ_x(\psi_x)
$$

$$
= \lambda(a, b) + \log Z_a(\psi_a^+) - \log Z_b(\psi_a^+), \quad (4.9)
$$

where we have used [23, Corollary A4.11]. Note that

$$
\lim_{z \to -\infty} Z_z(\psi_z^+) = Z_X(0), \quad \lim_{z \to \infty} Z_z(\psi_z^+) = 1.
$$

There exist $n \in \mathbb{N}_0$ and $x_1, \ldots, x_n \in X$ such that $x_1 < \cdots < x_n$ and $\psi = \delta_{x_1} + \cdots + \delta_{x_n}$. If $n = 0$, then the assertion (4.5) follows from (4.9). Hence we can assume that $n \geq 1$. By (4.9) we have for each $i \in \{0, \ldots, n\}$,

$$
\int_{x_i}^{x_{i+1}} p(x, \psi) \lambda(dx) = \lambda(x_i, x_{i+1}) + \log Z_{x_i}(\psi_{(-\infty, x_i]}) - \log Z_{x_{i+1}}(\psi_{(-\infty, x_i]}), \quad (4.10)
$$

where $x_0 := -\infty$ and $x_{n+1} := \infty$.

Let $z, w \in \mathbb{R}$ with $z < w$. Then there exist $j, k \in \{0, \ldots, n\}$ with $j \leq k$ such that $z \in [x_j, x_{j+1})$ and $w \in [x_k, x_{k+1})$. For $i \in \{0, \ldots, n\}$ we set $\psi^i := \delta_{x_1} + \cdots + \delta_{x_i} = \psi_{(-\infty, x_i]}$. Let us first assume that $x_k < w$. From (4.9) and (4.10) we then derive that

$$
\int_z^w p(x, \psi) \lambda(dx) = \int_z^{x_{j+1}} p(x, \psi) \lambda(dx) + \sum_{i=j+1}^{k-1} \int_{x_i}^{x_{i+1}} p(x, \psi) \lambda(dx) + \int_{x_k}^w p(x, \psi) \lambda(dx)
$$

$$
= \lambda(z, w) + \log Z_z(\psi^j) - \log Z_{x_{j+1}}(\psi^j)
$$

$$
+ \sum_{i=j+1}^{k-1} \log Z_{x_i}(\psi^i) - \sum_{i=j}^{k-1} \log Z_{x_{i+1}}(\psi^i) + \log Z_{x_k}(\psi^k) - \log Z_w(\psi^k).
$$

Reordering terms yields,

$$
\int_z^w p(x, \psi) \lambda(dx)
$$

$$
= \lambda(z, w) + \log Z_z(\psi_z^+) - \log Z_{w}(\psi_w) + \sum_{i=j+1}^k \log Z_{x_i}(\psi^i) - \sum_{i=j}^{k-1} \log Z_{x_{i+1}}(\psi^i)
$$

$$
= \lambda(z, w) + \log Z_z(\psi_z^+) - \log Z_{w}(\psi_w) + \sum_{i=j+1}^{k} (\log Z_{x_i}(\psi^i) - \log Z_{x_i}(\psi^{i-1})).
$$
This is equivalent to (4.5). If \( w = x_k \) we have that
\[
\int_z^w p(x, \psi) \lambda(dx) = \lambda(z, w) + \log Z_x(\psi^i) + \sum_{i=j+1}^{k-1} \log Z_{x_i}(\psi^i) - \sum_{i=j}^{k-1} \log Z_{x_{i+1}}(\psi^i)
\]
\[
= \lambda(z, w) + \log Z_x(\psi^i) - \log Z_{x_k}(\psi^{k-1}) + \sum_{i=j+1}^{k-1} \left( \log Z_{x_i}(\psi^i) - \log Z_{x_{i+1}}(\psi^i) \right),
\]
which is again equivalent to (4.5).

The following corollary has been crucial in the proof of Theorem 4.3.

**Corollary 4.5.** Suppose that \( \psi \in \mathbb{N}^* \) is stable. Then
\[
\exp \left[ - \int p(x, \psi) \lambda(dx) \right] = e^{-\lambda(x)} Z_{\mathbb{X}}(0)^{-1} \prod_{y \in \psi} \frac{Z_{(y,\infty)}(\psi_y)}{Z_{(y,\infty)}(\psi_y + \delta_y)}. \tag{4.11}
\]
In particular \( \int p(x, \psi) \lambda(dx) < \infty \).

**Proof:** Taking in Lemma 4.4 \( z = -\infty \) and \( w = \infty \) yields the first assertion. To prove the second, we take \( y \in \psi \). Then \( Z_{(y,\infty)}(\psi_y + \delta_y) = Z_{(y,\infty)}(\psi_{(-\infty,y)}) \). Since \( \psi \) is stable, this is finite. Therefore the right-hand side of (4.11) does not vanish and the result follows.

## 5 Poisson embedding of finite Gibbs processes

As in Section 4 we consider a diffuse finite measure \( \lambda \) on \( \mathbb{X} \) and a measurable function \( \kappa: \mathbb{X} \times \mathbb{N} \to \mathbb{R}_+ \), satisfying the cocycle assumption (2.9) for all \( (x, y, \mu) \in \mathbb{X}^2 \times \mathbb{N} \). In this section we construct a (finite) Gibbs process by a recursively defined embedding into a Poisson process on \( \mathbb{X} \times \mathbb{R}_+ \) with intensity measure \( \lambda \otimes \lambda_1 \), where \( \lambda_1 \) denotes Lebesgue measure on \( \mathbb{R}_+ \). For (marked) point processes on \( \mathbb{R}_+ \) this embedding technique is well-known; see [5]. To the best of our knowledge it has never been used in a spatial setting. As in Theorem 4.3 we use the function \( p \) defined by (4.1). However, we do not need the local stability assumption (Dom2). Even if this assumption holds, we find it more convenient to work with embedding rather than with a probabilistic thinning version of Theorem 4.3.

Let \( \mathbb{N}_f^*(\mathbb{X} \times \mathbb{R}_+) \) denote the space of all simple counting measures \( \psi \) on \( \mathbb{X} \times \mathbb{R}_+ \) such that \( \psi(B) < \infty \) for each measurable \( B \subset \mathbb{X} \times \mathbb{R}_+ \) with \( (\lambda \otimes \lambda_1)(B) < \infty \). (Again we equip this space with the standard \( \sigma \)-field.) Recall that \( \mathbb{N}^* \) denotes the space of simple and finite counting measures on \( \mathbb{X} \) and note that the elements of \( \mathbb{N}_f^*(\mathbb{X} \times \mathbb{R}_+) \) are not assumed to be finite. Using the total ordering on \( \mathbb{X} \) we define for each \( n \in \mathbb{N} \) a mapping \( x_n(\cdot): \mathbb{N}_f^*(\mathbb{X} \times \mathbb{R}_+) \to \mathbb{X} \cup \{-\infty, \infty\} \) as follows. Let \( \psi \in \mathbb{N}_f^*(\mathbb{X} \times \mathbb{R}_+) \). If \( \int p(x, 0) \lambda(dx) = \infty \), then we set \( x_1(\psi) := -\infty \). If \( \int p(x, 0) \lambda(dx) < \infty \) then
\[
\psi(\{(x, t) \in \mathbb{X} \times \mathbb{R}_+: t \leq p(x, 0)\}) < \infty
\]
and we set
\[
x_1(\psi) := \min\{x \in \mathbb{X} : \text{there exists } t \geq 0 \text{ such that } (x, t) \in \psi \text{ and } t \leq p(x, 0)\}, \tag{5.1}
\]
where \( \min \emptyset := \infty \). Inductively we define \( x_n(\psi) \) for all \( n \in \mathbb{N} \). If \( x_n(\psi) \notin X \) then we set \( x_{n+1}(\psi) := x_n(\psi) \). If \( x_n(\psi) \in X \) and

\[
\int 1\{x_n(\psi) < x\} p(x, \delta_{x_1(\psi)} + \cdots + \delta_{x_n(\psi)}) \lambda(dx) = \infty,
\]

then we set \( x_{n+1}(\psi) := -\infty \). Otherwise we define

\[
x_{n+1}(\psi) := \min\{x > x_n(\psi) : \text{there exists } t \geq 0 \text{ s.t. } (x, t) \in \psi \text{ and } t \leq p(x, \delta_{x_1(\psi)} + \cdots + \delta_{x_n(\psi)})\}.
\]

Define the embedding operator \( T : N_0^*(X \times \mathbb{R}_+) \to N^*(X) \) by

\[
T(\psi) := 1\{\tau(\psi) < \infty\} \sum_{n=1}^{\tau(\psi)} \delta_{x_n(\psi)},
\]

where \( \tau(\psi) := \sup\{n \geq 1 : x_n(\psi) \in X\} \). Equation (5.4) below provides an alternative representation of \( T \), which shows that \( T \) is measurable.

**Theorem 5.1.** Assume that \( \lambda \) is finite and diffuse and that \( \Pi_\lambda \)-a.e. \( \psi \in N^* \) are stable. Assume that \( \Phi \) is a Poisson process on \( X \times \mathbb{R}_+ \) with intensity measure \( \lambda \otimes \lambda_1 \). Then \( T(\Phi) \) is a Gibbs process with PI \( \kappa \).

**Proof:** Given \( n \in \mathbb{N}_0, x_1, \ldots, x_n \in X \) and \( i \in \{0, \ldots, n\} \) we define

\[
A_i(x_1, \ldots, x_n) := \{(x, t) \in X \times \mathbb{R}_+ : x_i < x < x_{i+1}, t \leq p(x, \delta_{x_1} + \cdots + \delta_{x_i})\},
\]

where we set \( x_0 := -\infty \) and \( x_{n+1} := \infty \) with the convention that \( -\infty < y < \infty \) for each \( y \in X \). For \( n \in \mathbb{N}_0 \) we set \( B_n(x_1, \ldots, x_n) := \bigcup_{i=0}^n A_i(x_1, \ldots, x_n) \). Let \( H := \{\psi \in N^*(X) : \int p(x, \psi) \lambda(dx) < \infty\} \) and define

\[
C_n := \{(x_1, \ldots, x_n) \in X^n : \delta_{x_1} \in H, \delta_{x_1} + \delta_{x_2} \in H, \ldots, \delta_{x_1} + \cdots + \delta_{x_n} \in H\},
\]

where \( C_0 := 1\{0 \in H\} \). By definition of \( T \) we have for all \( B \in \mathcal{X} \)

\[
T(\psi)(B) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \int 1\{t_1 \leq p(x_1, 0), \ldots, t_n \leq p(x_n, \delta_{x_1} + \cdots + \delta_{x_{n-1}})\} 1\{x_1 < \cdots < x_n\}
\]

\[
\times 1\{(x_1, \ldots, x_{n-1}) \in C_{n-1}, \psi(B_n(x_1, \ldots, x_n)) = 0\} 1\{x_i \in B\} \psi^{(n)}(d(x_1, t_1, \ldots, x_n, t_n)).
\]

This shows that \( T \) is measurable.

We next prove that

\[
\mathbb{P}(x_n(\Phi) = -\infty) = 0, \quad n \in \mathbb{N}.
\]

First we note that \( x_1(\Phi) = -\infty \) iff \( \int p(x, 0) \lambda(dx) = \infty \). Since \( \Pi_\lambda(\{0\}) > 0 \), the empty measure 0 is stable, so that Corollary 4.5 shows that this case cannot occur. For \( n \in \mathbb{N} \) we have that

\[
\mathbb{P}(x_{n+1}(\Phi) = -\infty) \leq \sum_{k=1}^{n} \mathbb{P}(T(\Phi)(X) = k, x_{k+1}(\Phi) = -\infty).
\]
Hence, it suffices to show that \( P(T(\Phi)(X) = n, x_{n+1}(\Phi) = -\infty) = 0 \) for all \( n \in \mathbb{N} \). We have

\[
P(x_{n+1}(\Phi) = -\infty) = \mathbb{E} \left[ \int 1\{t_1 \leq p(x_1, 0), \ldots, t_n \leq p(x_n, \delta_{x_1} + \cdots + \delta_{x_{n-1}})\} 1\{x_1 < \cdots < x_n\}
\times 1\{\Phi(B_{n-1}(x_1, \ldots, x_n)) = 0, (x_1, \ldots, x_n) \notin C_n\} \Phi^n(d(x_1, t_1, \ldots, x_n, t_n)) \right].
\]

By the multivariate Mecke equation this equals

\[
\mathbb{E} \left[ \int p(x_1, 0) \cdots p(x_n, \delta_{x_1} + \cdots + \delta_{x_{n-1}}) 1\{x_1 < \cdots < x_n\}
\times 1\{\Phi(B_{n-1}(x_1, \ldots, x_n)) = 0, (x_1, \ldots, x_n) \notin C_n\} \lambda^n(d(x_1, \ldots, x_n)) \right]
= \int p(x_1, 0) \cdots p(x_n, \delta_{x_1} + \cdots + \delta_{x_{n-1}}) 1\{x_1 < \cdots < x_n\}
\times P(\Phi(B_{n-1}(x_1, \ldots, x_n)) = 0) 1\{(x_1, \ldots, x_n) \notin C_n\} \lambda^n(d(x_1, \ldots, x_n)).
\]

Our assumptions allow to apply Corollary 4.5, so that

\[
\int 1\{x_n < x\} p(x, \delta_{x_1} + \cdots + \delta_{x_{n-1}}) \lambda(dx) < \infty \tag{5.6}
\]

for \( \lambda^n \)-a.e. \((x_1, \ldots, x_n)\). Hence (5.5) follows.

Now we take a measurable \( f: \mathbb{N}^*(X) \to \mathbb{R}_+ \). Taking into account (5.5), we obtain for each \( n \in \mathbb{N}_0 \) similarly as above that

\[
\mathbb{E}[f(T(\Phi))] = \sum_{n=0}^{\infty} \int f(\delta_{x_1} + \cdots + \delta_{x_n}) p(x_1, 0) \cdots p(x_n, \delta_{x_1} + \cdots + \delta_{x_{n-1}}) 1\{x_1 < \cdots < x_n\}
\times P(\Phi(B_n(x_1, \ldots, x_n)) = 0) \lambda^n(d(x_1, \ldots, x_n)),
\]

with an obvious interpretation of the summand for \( n = 0 \). Whenever \( x_1 < \cdots < x_n \) we have that

\[
P(\Phi(B_n(x_1, \ldots, x_n)) = 0) = \prod_{i=0}^{n} P(\Phi(A_i(x_1, \ldots, x_n)) = 0)
= \prod_{i=0}^{n} \exp \left[ - \int 1\{x_i < x < x_{i+1}\} p(x, \delta_{x_1} + \cdots + \delta_{x_i}) \lambda(dx) \right].
\]

Using here Lemma 4.4, we can conclude the assertion as in the proof of Theorem 4.3. \( \square \)

**Remark 5.2.** Suppose that \( \kappa \) satisfies assumption (Dom2). Then we can replace \( \Phi \) by its restriction to \( \{(x, t) \in X \times \mathbb{R}_+: t \leq \alpha(x)\} \).

Later we need the following useful consequence of Theorem 5.1. Here we do not assume that the measure \( \lambda \) is finite. For technical reasons we assume that \( X \) is a complete separable metric space and that each set in \( X_0 \) is bounded.
Lemma 5.3. Let ξ, ξ' be two Gibbs processes on X with Papangelou intensities κ, κ' which both satisfy (Dom2) with the same function α. Then there exists a Poisson process η with intensity measure αλ and point processes ξ, ξ', such that ξ \overset{d}{=} \tilde{\xi}, ξ' \overset{d}{=} \tilde{\xi}' and \tilde{\xi} ≤ η and \tilde{\xi}' ≤ η almost surely.

Proof: Let B ∈ X₀. The restriction ξ_B of ξ to B is a Gibbs process with PI

\[ \kappa^B(x, \mu) := \int \kappa(x, \mu + \psi)\mathbb{P}(\xi_X \in d\psi \mid \xi_B = \mu), \quad (x, \mu) \in B \times N_B. \]

Note that \( \kappa^B \leq \alpha \). A similar assertion applies to ξ' and its PI \( \kappa'^B \). Let Φ be a Poisson process on \( X \times \mathbb{R}_+ \) with intensity measure \( \lambda \otimes \lambda_+ \). Since \( \lambda \) is diffuse we can assume that Φ is simple, that is a random element of \( X \) process on \( \xi \) measure \( \lambda \).

In this section we return to the general setting of Section 2, that is we consider a

6 Disagreement coupling

Let Φ be a Poisson process on \( X \times \mathbb{R}_+ \) with intensity measure \( \alpha \lambda \) and point processes \( \tilde{\xi}, \tilde{\xi}' \), such that \( \tilde{\xi} \overset{d}{=} \xi, \tilde{\xi}' \overset{d}{=} \xi' \) and \( \tilde{\xi} \leq \eta \) and \( \tilde{\xi}' \leq \eta \) almost surely.

Proof: Let B ∈ X₀. The restriction ξ_B of ξ to B is a Gibbs process with PI

\[ \kappa^B(x, \mu) := \int \kappa(x, \mu + \psi)\mathbb{P}(\xi_X \in d\psi \mid \xi_B = \mu), \quad (x, \mu) \in B \times N_B. \]

By Theorem 5.1 we have that \( \chi^B = \xi_B \) and \( \chi'^B = \xi'_B \). By definition of the embedding and Lemma 4.1 we have that \( \chi^B \) and \( \chi'^B \) are both smaller than

\[ \Phi^B := \int 1\{x \in \cdot, x \in B, t \leq \alpha(x)\} \Phi(d(x, t)), \]

which is a Poisson process with intensity measure \( \alpha \lambda \). Now we argue exactly as in the proof of [16, Corollary 3.4]. We take a sequence \( (B_n)_n \) of bounded and closed sets such that \( B_n \uparrow X \). Then \( \chi^{B_n} \overset{d}{=} \xi, \chi'^{B_n} \overset{d}{=} \xi' \) and \( \Phi^{B_n} \overset{d}{=} \eta \), where \( \eta \) is a Poisson process with intensity measure \( \alpha \lambda \) and where we refer to [22, Chapter 4] for the theory of weak convergence of point process distributions. It follows from [21, Theorem 16.3] that the above sequences are all tight. A standard argument shows that \( (\chi^{B_n}, \chi'^{B_n}, \Phi^{B_n}), n \in \mathbb{N} \), is tight and hence converges in distribution to \( (\xi, \xi', \eta) \) along some subsequence. Then \( \xi \overset{d}{=} \tilde{\xi} \) and \( \xi' \overset{d}{=} \tilde{\xi}' \). Moreover, \( \eta \) is a Poisson process with intensity measure \( \alpha \lambda \). We can assume that \( (\tilde{\xi}, \tilde{\xi}', \eta) \) is defined on the original probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Since the set \( \{ (\mu, \mu', \psi) \in N^3 : \mu \leq \psi, \mu' \leq \psi \} \) is closed, it follows from the Portmanteau theorem that

\[ \mathbb{P}(\xi \leq \eta, \xi' \leq \eta) = 1. \]

6 Disagreement coupling

In this section we return to the general setting of Section 2, that is we consider a \( \sigma \)-finite measure \( \lambda \) along with a measurable \( \kappa : X \times N \to \mathbb{R}_+ \) satisfying (2.9) for all \( (x, y, \mu) \in X^2 \times N \). We assume that \( \lambda \) is diffuse. For \( \psi \in N \) and \( B \in X_0 \) we recall the definition (2.12) of \( \kappa_\psi \) and that \( \kappa_{B, \psi} \) is the restriction of \( \kappa_\psi \) to \( B \times N_B \). Using \( \kappa_{B, \psi} \) instead of \( \kappa \) we can define the function \( p_{B, \psi} : W \times N^*(B) \to [0, 1] \) by (4.1) and the function \( T_{B, \psi} : N^*_t(B \times \mathbb{R}_+) \to N^*(B) \) by (5.3). Note that this mapping depends on the chosen ordering on \( B \). Let \( \Phi \) be a Poisson process on \( X \times \mathbb{R}_+ \) with intensity measure \( \lambda \otimes \lambda_1 \). We will use Theorem 5.1 by applying the mapping \( T_{Z, \psi} \) to the restriction \( \Phi_{Z \times \mathbb{R}_+} \) for suitable random sets \( Z \) and (recursively defined) point processes \( \Psi \).

In the following we fix \( W \in X_0 \). We say that \( \psi \in N_{W^\psi} \) is a regular boundary condition (for \( W \)) if

\[ Z_B(\mu_{W \setminus B} + \psi) < \infty, \quad B \subset W, B \in \mathcal{X}, \Pi_\lambda \text{-a.e.} \ \mu. \]
This means that there exists a measurable \( H \subset \mathbb{N} \) such that \( \Pi_\lambda(H) = 1 \) and such that \( \mu_{W \setminus B} + \psi \) is for all \( \mu \in H \) stable on \( W \); cf. (Dom1).

**Remark 6.1.** If the PI \( \kappa \) satisfies (Dom2), then each \( \psi \in \mathcal{N}_{W^c} \) is a regular boundary condition for \( W \in \mathcal{X}_0 \); see Remark 3.1.

**Remark 6.2.** Let \( \xi \) be a Gibbs process with PI \( \kappa \) and let \( W, B \in \mathcal{X}_0 \) with \( B \subset W \). From (2.6) and the DLR-equations we obtain that

\[
\int 1\{Z_B(\mu_{W \setminus B} + \psi) < \infty\} \Pi_{W,\psi}(d\mu) = 1, \quad \mathbb{P}(\xi_{B^c} \in \cdot)-\text{a.e. } \psi,
\]

where \( \Pi_{W,\psi} \) is the conditional distribution of \( \xi_W \) given \( \xi_{W^c} = \psi \). Therefore (6.1) seems to be a reasonable assumption, at least if the exceptional set is allowed to depend on \( B \subset W \). If (Dom1) holds for all \( \psi \in \mathcal{N} \), then each boundary condition is regular for \( W \).

We construct a special coupling of two Gibbs processes on \( W \in \mathcal{X}_0 \) with Papangelou intensities \( \kappa_{W,\psi} \) and \( \kappa_{W,\psi'} \) for regular boundary conditions \( \psi, \psi' \in \mathcal{N}_{W^c} \). This extends the coupling construction in [17] (see also [2, Theorem 1]) to general Borel spaces. While the coupling in [17] is based on the thinning construction from Theorem 4.3, we find it more convenient to work with the more explicit Poisson embedding from Theorem 5.1. In particular we can then apply the spatial Markov property of the underlying Poisson process in a smooth and rigorous way, to establish the Gibbs property of the marginals and hence to add the arguments missing in [17]. In fact this is the main part of the proof.

Even though our coupling does not require the local stability assumption (Dom2), the latter is crucial for our Theorem 9.1.

Let \( W \in \mathcal{X}_0 \) and \( \psi, \psi' \in \mathcal{N}_{W^c} \). Recursively we define a sequence of mappings \( Y_1, Y_2, \ldots \) from \( \Omega \) to the Borel subsets of \( W \), along with point processes \( \chi_1, \chi_2, \ldots \) and \( \chi_1', \chi_2', \ldots \). Given such sequences we define, for \( n \in \mathbb{N} \), \( W_n := Y_1 \cup \cdots \cup Y_n \), \( \xi_n := (\chi_n)_{Y_n} \) and \( \xi_n' := (\chi_n')_{Y_n} \). The recursion starts with \( Y_1 := \{x \in W : x \sim (\psi + \psi')\} \) and

\[
(\chi_1, \chi_1') := (T_{W,\psi}(\Phi_{W \times \mathbb{R}^+}), T_{W,\psi'}(\Phi_{W \times \mathbb{R}^+})).
\]

The latter definition uses the ordering induced by a Borel isomorphism. We will modify the original isomorphism by first assuming that \( \varphi(W) \subset (0, 1] \) and then setting

\[
\varphi_Y := \frac{\varphi_{Y_1}}{2} + \left( \frac{1 + \varphi}{2} \right)_{W \setminus Y_1} \quad \text{in} \quad (6.2)
\]

where the lower set index denotes restriction. Then \( \varphi_Y \) is a Borel isomorphism and each point in \( Y_1 \) is smaller than every point in \( W \setminus Y_1 \). This fact will become relevant later on.

For \( n \in \mathbb{N} \) we define

\[
Y_{n+1} := \begin{cases} 
W \setminus W_n, & \text{if } \chi_n(Y_n) + \chi_n'(Y_n) = 0, \\
\{x \in W \setminus W_n : x \sim (\chi_n + \chi_n')_{Y_n}\}, & \text{if } \chi_n(Y_n) + \chi_n'(Y_n) > 0,
\end{cases}
\]

and

\[
(\chi_{n+1}, \chi_{n+1}') := (T_{W \setminus W_n, \psi + \xi_1 + \cdots + \xi_n}(\Phi_{W \setminus W_n \times \mathbb{R}^+}), T_{W \setminus W_n, \psi' + \xi_1' + \cdots + \xi_n'}(\Phi_{W \setminus W_n \times \mathbb{R}^+})). \quad (6.4)
\]
Here we use the Borel isomorphism

\[ \varphi_{n+1} := \frac{\varphi_{Y_n+1}}{2} + \left( \frac{1 + \varphi}{2} \right)_{W \setminus W_{n+1}}. \]  

(6.5)

Since \( \varphi_{n+1} \) depends on \( Y_{n+1} \) it is a random mapping and in fact a mapping from \( \Omega \times (W \setminus W_n) \) to \((0, 1]\). Note that if \( \chi_n(Y_n) + \chi'_n(Y_n) = 0 \) then \( Y_1 \cup \cdots \cup Y_{n+1} = W \). In that case we have \( \xi_{n+1} = \xi'_{n+1} \) (see the final part of the proof of Theorem 6.3) and \( Y_i = \emptyset \) for \( i \geq n + 2 \).

Let us briefly record some measurability properties. By the measurability property of \( \sim \) it follows that \( Y_1 \) is measurable, while the measurability of \( \chi_1, \chi'_1 \) (as functions on \( \Omega \)) follows from the measurability of \( T_{W,\psi} \) and \( T_{W,\psi'} \). Therefore \( \xi_1, \xi_2 \) are measurable as well. It follows that \( Y_2 \) is graph-measurable, that is \((x, \omega) \mapsto 1\{x \in Y_2(\omega)\}\) is measurable on \( W \times \Omega \). Using the explicit representation (5.4) for \( T_{W \setminus W_1, \kappa_{\psi} + \xi_1 + \ldots + \xi_n} \) we see that \( \chi_2 \) is measurable. Proceeding this way, it follows inductively that each \( Y_n \) (and hence each \( W_n \)) is graph-measurable and that \( \chi_1, \chi_2, \ldots, \chi'_1, \chi'_2, \ldots, \xi_1, \xi_2, \ldots \) and \( \xi'_1, \xi'_2, \ldots \) are measurable and hence point processes.

Define

\[ \xi := \sum_{n=1}^{\infty} \xi_n, \quad \xi' := \sum_{n=1}^{\infty} \xi'_n. \]

For \( \mu, \mu' \in \mathbb{N} \) let \(|\mu - \mu'| \in \mathbb{N}\) denote the total variation measure of \( \mu - \mu' \) and define

\[ |\xi - \xi'| := \sum_{n=1}^{\infty} |\xi_n - \xi'_n|. \]

**Theorem 6.3.** Assume that \( \kappa \) satisfies (Loc1). Suppose that \( W \in X_0 \) and let \( \psi, \psi' \in \mathbb{N}_{W^c} \) be two regular boundary conditions for \( W \). Construct \( \xi \) and \( \xi' \) as above. Then \( \xi \) is a Gibbs process on \( W \) with PI \( \kappa_{W,\psi} \) while \( \xi' \) is a Gibbs process on \( W \) with PI \( \kappa_{W,\psi'} \). Every point in \( |\xi - \xi'| \) is connected via \( \xi + \xi' \) to \( \psi + \psi' \). Moreover, if \( \kappa \) satisfies (Dom2), then the support of \( \xi + \xi' \) is contained in the support of \( \int 1\{x \in \cdot \cap W, t \leq \alpha(x)\} \Phi(d(x,t)) \) which is a Poisson process with intensity measure \( \alpha_{W} \).

**Remark 6.4.** Theorem 6.3 applies for all boundary conditions to the Strauss process, the continuum random cluster model and the Widom-Rowlinson model; see Section 3. Moreover, it applies to all boundary conditions satisfying (3.2) in a Gibbs process with a pair potential. For the area interaction process, Theorem 6.3 applies if \( Q([r_1, \infty)) = 1 \) for some \( r_1 > 0 \).

**Proof of Theorem 6.3.** We need some more notation. For each measurable set \( B \subset W \) and each \( \mu \in \mathbb{N}_{B^c} \) define \( p_{B,\mu} \) by (4.1) with \((X, \kappa)\) replaced by \((B, \kappa_{B,\mu})\). Define

\[ Y_n^* := \{(x, t) \in X \times \mathbb{R}_+ : x \in Y_n, t \leq p_n(x)\}, \]

(6.6)

where

\[ p_n(x) := \max\{p_{W \setminus W_{n-1}, \psi + \xi_1 + \ldots + \xi_{n-1} (x, \chi_n), p_{W \setminus W_{n-1}, \psi' + \xi'_1 + \ldots + \xi'_{n-1} (x, \chi'_n)} \} \]
and \( W_0 \coloneqq \emptyset \). Since \( Y_n \) is graph-measurable, so is \( Y^*_n \). Set \( S_n \coloneqq \bar{Y}_1 \cup \cdots \cup \bar{Y}_n \). A crucial tool for our proof is the spatial Markov property

\[
\mathbb{P}(\Phi_{(W \times \mathbb{R}_+)} \backslash S_n \in \cdot \mid \Phi_{S_n}) = \Pi(\lambda S_1)_{(W \times \mathbb{R}_+)} \backslash S_n), \quad \mathbb{P}\text{-a.s. on } \{\eta(S_n) < \infty\}. \tag{6.7}
\]

This follows from [24, Theorem A.3], once we will have proved that \( S_n \) is a stopping set. The latter means that \( S_n \) is graph-measurable and

\[
S_n(\Phi_{S_n} + \mu_{(W \times \mathbb{R}_+)} \backslash S_n) = S_n(\Phi), \quad \mu \in \mathbb{N}_+^*(W \times \mathbb{R}_+). \tag{6.8}
\]

Here we (slightly) abuse our notation by interpreting \( S_n \) as a mapping on \( \mathbb{N}_+^*(W \times \mathbb{R}_+) \).

To check (6.8), we prove inductively that \( \xi_0^* := (\xi_1, \xi_1', \ldots, \xi_n, \xi_n') \) and \( \bar{Y}_1, \ldots, \bar{Y}_n \) do not change if the points in \( \Phi_{(W \times \mathbb{R}_+)} \backslash S_n \) are replaced by an arbitrary configuration. For \( n = 1 \) this follows from (6.2) and the definitions of \( T_{W, \psi}(\Phi_{W \times \mathbb{R}_+}) \) and \( T_{W, \psi}(\Phi_{W \times \mathbb{R}_+}) \). Suppose that it is true for some \( n \in \mathbb{N} \). In particular, changing the points of \( \Phi \) in \( (W \times \mathbb{R}_+) \backslash S_{n+1} \) does not change \( \xi_n^* \) and hence also not \( Y_{n+1} \). Hence it follows from (6.5) that \( (\xi_{n+1}, \xi_{n+1}') \) does not change either. By definition (6.6) \( Y_{n+1}^* \) does not change.

Since \( \psi \) and \( \psi' \) are regular boundary conditions for \( W \) it follows from Theorem 5.1 that \( \chi \) (resp. \( \chi' \)) is a Gibbs process with PI \( \kappa_{W, \psi} \) (resp. \( \kappa_{W, \psi'} \)). We shall prove by induction that

\[
\chi_{n+1} + \sum_{m=1}^{n} \xi_m \overset{d}{=} \chi_1, \chi'_{n+1} + \sum_{m=1}^{n} \xi'_m \overset{d}{=} \chi'_1, \quad n \in \mathbb{N}_0, \tag{6.9}
\]

and

\[
\mathbb{P}(\Phi(S_{n+1}) < \infty) = 1. \tag{6.10}
\]

The case \( n = 0 \) of (6.9) is trivial. Since \( \psi \) and \( \psi' \) are regular boundary conditions we can recall from the proof of Theorem 5.1 that \( \int p_W(x, \psi) \lambda(dx) + \int p_W(x, \psi') \lambda(dx) < \infty \). We assert that the first coordinate of each point of \( \Phi_{W \times \mathbb{R}_+} \) in \( \bar{Y}_1 \) is either a point of \( \chi_1 \) or \( \chi'_1 \) (or both). To see this let \((x, t) \in \Phi_{Y_1 \times \mathbb{R}_+} \) such that \( p_{W, \psi}(x, \chi_1) \leq t \). Suppose that \( \xi_1 = \delta_{x_1} + \cdots + \delta_{x_n} \), where \( n \in \mathbb{N}_0 \) and \( x_1 < \cdots < x_n \). By definition (5.1) of the smallest point of \( T_{W, \psi} \) we must then have that \( \xi_1 = (\chi_1)_{Y_1} \neq 0 \) and \( x \geq x_1 \). From the recursion (5.2) we can also exclude the case \( x > x_n \). So either \( n = 1 \) and \( x = x_1 \) or \( n \geq 2 \) and there exists \( m \in \{1, \ldots, n-1\} \) such that \( x_m < x \leq x_{m+1} \). In the latter case we have by definition of \( p_{W, \psi} \) (see (4.1)) that \( p_{W, \psi}(x, \chi_1) = p_{W, \psi}(x, \delta_{x_1} + \cdots + \delta_{x_m}) \). Therefore we obtain again from the recursion (5.2) that \( x_{m+1} \leq x \) and hence \( x = x_{m+1} \). This proves our (auxiliary) assertion. Hence we can conclude that

\[
\Phi(S_1) \leq \chi_1(W_1) + \chi'_1(W_1), \quad \mathbb{P}\text{-a.s.} \tag{6.11}
\]

which is finite.

Let \( n \in \mathbb{N} \) and assume that (6.9) and (6.10) hold for \( n - 1 \). Let \( f : \mathbb{N}_W \to \mathbb{R}_+ \) be measurable. We have that

\[
I := \mathbb{E}[f(\xi_1 + \cdots + \xi_n + \chi_{n+1})] = \mathbb{E}\left[\mathbb{E}[f(\xi_1 + \cdots + \xi_n + T_{W \backslash W_n, \kappa_\psi + \xi_1 + \cdots + \xi_n} \Phi_{S_1})] \mid \Phi_{S_n}\right],
\]

20
where $S_0 := \emptyset$ and $S_n^c := (W \times \mathbb{R}_+) \setminus S_n$. We have seen above that
\[
(\omega, \mu) \mapsto (\xi_1(\omega), \ldots, \xi_n(\omega), T_{W \setminus W_n(\omega), \kappa_{\psi + \xi_1(\omega) + \cdots + \xi_n(\omega)}}(\mu))
\]
can be written as a function of $(\Phi_{S_n}(\omega), \mu)$. As at (5.4) it can be shown that this function is in fact measurable. To deal with $I$ we shall use (6.7), Theorem 5.1 and a standard property of conditional expectations; see [21, Theorem 6.4].

In order to apply Theorem 5.1 we need to check that $\xi_1 + \cdots + \xi_n + \psi$ is almost surely a regular boundary condition for $W \setminus W_n$. By assumption (6.1) there exists a measurable $H \subset N_W$ such that $\Pi_\lambda(H) = 1$ and
\[
H \subset \{ \mu \in N_W : Z_B(\mu_{W \setminus B} + \psi) < \infty \text{ for each measurable } B \subset W \}.
\]
By induction hypothesis $\xi_1 + \cdots + \xi_n$ is the restriction of the Gibbs process $\xi_1 + \cdots + \xi_{n-1} + \chi_n$ to $W_n$. Hence we obtain from Lemma 6.5 below that
\[
\mathbb{P}(\xi_1 + \cdots + \xi_n + \mu_{W \setminus W_n} \in H) = 1, \quad \text{\Pi}_\lambda\text{-a.e. } \mu.
\]
And if $\xi_1 + \cdots + \xi_n \in H$ then $\xi_1 + \cdots + \xi_n + \psi$ is indeed a regular boundary condition for $W \setminus W_n$.

Now we are allowed to apply (6.7) and Theorem 5.1 to obtain that
\[
I = \mathbb{E}\left[ \int f(\xi_1 + \cdots + \xi_n + \mu') \Pi_{W \setminus W_n, \psi + \xi_1 + \cdots + \xi_n}(d\mu') \right],
\]
where we recall the definition (2.5) of a Gibbs measure. Assume that $n = 1$. Since $\chi_1$ is Gibbs we obtain that
\[
I = \int \int f(\mu_{Y_1} + \mu') \Pi_{W \setminus Y_1, \psi + \mu_{Y_1}}(d\mu') \Pi_{W, \psi}(d\mu).
\]  
(6.12)

Using (2.13) in (6.12) with $(B, C) = (W, Y_1)$ we obtain that $I = \int f(\mu) \Pi_{W, \psi}(d\mu)$, that is, the first part of (6.9) for $n = 1$. Assume now that $n \geq 2$. Define the event $A_n := \{(\chi_{n-1} + \chi_{n-1}) (Y_{n-1}) = 0\}$. Then we have that $I = I_1 + I_2$, where
\[
I_1 := \mathbb{E}[1_{A_n} \int f(\xi_1 + \cdots + \xi_n + \mu') \Pi_{W \setminus W_n, \psi + \xi_1 + \cdots + \xi_n}(d\mu')],
\]
and $I_2$ is defined in the obvious way. On the event $A_n$ we have that $Y_n = W \setminus W_{n-1}$, $W_n = W$ and $\xi_n = \chi_n$. Therefore,
\[
I_1 = \mathbb{E}[1_{A_n} f(\xi_1 + \cdots + \xi_n)]
= \mathbb{E}\left[ 1_{A_n} \int f(\xi_1 + \cdots + \xi_{n-1} + \mu') \Pi_{W \setminus W_{n-1}, \psi + \xi_1 + \cdots + \xi_{n-1}}(d\mu) \right],
\]
where we have again used (6.7) and Theorem 5.1. On the event $\Omega \setminus A_n$ we have that $\xi_n = (\chi_n)_{Y_n}$.
Therefore we obtain from (6.7) and Theorem 5.1
\[
I_2 = \mathbb{E}\left[ 1_{\Omega \setminus A_n} \int f(\xi_1 + \cdots + \xi_{n-1} + \mu_{Y_n} + \mu') \times \Pi_{W \setminus W_n, \psi + \xi_1 + \cdots + \xi_{n-1} + \mu_{Y_n}}(d\mu') \Pi_{W \setminus W_{n-1}, \psi + \xi_1 + \cdots + \xi_{n-1}}(d\mu) \right].
\]  
(6.13)
Applying (2.13) to the right-hand side of (6.13) with \( B := W \setminus W_{n-1} \) and \( C := W \setminus Y_n \) yields
\[
I_2 = \mathbb{E}\left[ 1_{\Omega \setminus A_n} \int f(\xi_1 + \cdots + \xi_{n-1} + \mu) \Pi_{W \setminus W_{n-1}, \psi+\xi_1+\cdots+\xi_{n-1}}(d\mu) \right]
\]
and hence
\[
I = \mathbb{E}\left[ \int f(\xi_1 + \cdots + \xi_{n-1} + \mu) \Pi_{W \setminus W_{n-1}, \psi+\xi_1+\cdots+\xi_{n-1}}(d\mu) \right]
= \mathbb{E}[f(\xi_1 + \cdots + \xi_{n-1} + \chi)],
\]
where the second equality comes again from Theorem 5.1. This shows the first part of (6.9) for \( n \geq 2 \). Of course the second part follows in the same way.

Since \( \xi_1 + \cdots + \xi_n + \psi \) is almost surely a regular boundary condition for \( W \setminus W_n \) we obtain exactly as at (6.11) that
\[
\Phi(S_{n+1}) \leq \chi_{n+1}(W_{n+1}) + \chi'_{n+1}(W_{n+1}), \quad \mathbb{P}\text{-a.s.} \quad (6.14)
\]
This shows (6.10) and finishes the inductive proof of (6.9) and (6.10). We shall only use (6.9).

For each \( m \in \mathbb{N} \) we have that
\[
\mathbb{P}(\xi(W) > k) = \lim_{n \to \infty} \mathbb{P}((\xi_1 + \cdots + \xi_n)(W) > k) \leq \mathbb{P}(\chi_1 > k).
\]
Therefore \( \mathbb{P}(\xi(W) < \infty) = 1 \), so that almost surely \( \xi_n(W) = 0 \) for all sufficiently large \( n \). Since the same holds for the point processes \( \xi'_n \) we have by definition of the recursion that \( \chi_n = 0 \) for large enough \( n \). Therefore we obtain for each bounded measurable \( f : N_W \to \mathbb{R} \) by bounded convergence
\[
\mathbb{E}[f(\xi)] = \lim_{n \to \infty} \mathbb{E}\left[f\left(\sum_{m=1}^{n} \xi_m\right)\right] = \lim_{n \to \infty} \mathbb{E}\left[f\left(\chi_{n+1} + \sum_{m=1}^{n} \xi_m\right)\right], \quad (6.15)
\]
so that \( \xi \) is Gibbs with PI \( \kappa_\psi \).

Finally we let \( n \in \mathbb{N} \) be the smallest integer such that \( \chi_n(Y_n) + \chi'_n(Y_n) = 0 \). Then \( Y_{n+1} = W \setminus W_n \) and (Loc1) implies that
\[
\kappa_{W \setminus W_n, \psi+\xi_1+\cdots+\xi_n} = \kappa_{W \setminus W_n, \psi'+\xi'_1+\cdots+\xi'_n}.
\]
Hence \( \xi_{n+1} = \xi'_{n+1} \) and \( \xi_i = \xi'_i = 0 \) for \( i \geq n + 2 \). Therefore each point from \( |\xi - \xi'| \) must lie in \( W_n \). By definition all those points are connected via \( \xi_{W_n} + \xi'_{W_n} \) to \( \psi + \psi' \). The final assertion follows from the definition (6.4) and Lemma 4.1.

The following lemma has been used in the preceding proof.

**Lemma 6.5.** Let \( W \in X_0 \) and suppose that \( \xi \) is a Gibbs process on \( W \). Let \( S \) be a graph-measurable mapping from \( \Omega \) into \( X \cap W \). Then \( \int \mathbb{P}(S + \mu_{W \setminus S} \in \cdot) \Pi_\lambda(d\mu) \ll \Pi_{\lambda_W} \).
Proof. Let $f: \mathbb{N} \to \mathbb{R}_+$ be measurable such that $\int f(\mu) \Pi_{\lambda W}(d\mu) = 0$. Since a Poisson process is completely independent and $\lambda(S) \leq \lambda(W) < \infty$ we have

\[
I := \mathbb{E} \left[ \int f(\xi_S + \mu_{W \setminus S}) \Pi_\lambda(d\mu) \right] = \mathbb{E} \left[ e^{\lambda(S)} \int 1\{\mu(S) = 0\} f(\xi_S + \mu_{W \setminus S}) \Pi_\lambda(d\mu) \right] \leq e^{\lambda(W)} \mathbb{E} \left[ \int f(\xi_S + \mu_{W}) \Pi_\lambda(d\mu) \right].
\]

By [19, Theorem 1.1] (applying to general Borel spaces), $\mathbb{P}(\xi_S \in \cdot) \ll \mathbb{P}(\xi \in \cdot)$. Hence we also have $\mathbb{P}(\xi_S \in \cdot) \ll \Pi_{\lambda W}$ and we let $g: \mathbb{N} \to \mathbb{R}_+$ denote the corresponding density. Then

\[
I \leq e^{\lambda(W)} \iint g(\psi_W)f(\psi_W + \mu_W) \Pi_\lambda(d\mu) \Pi_\lambda(d\psi).
\]

Noting that

\[
\iint f(\psi_W + \mu_W) \Pi_\lambda(d\mu) \Pi_\lambda(d\psi) = \int f(\mu_W) \Pi_{2\lambda}(d\mu)
\]

and that $\Pi_{2\lambda W}$ and $\Pi_{\lambda W}$ are equivalent, we obtain that $I = 0$. This concludes the proof. \hfill \Box

Remark 6.6. Consider the assumptions of Theorem 6.3. We assert that the mapping $(\omega, \psi, \psi') \mapsto (\xi(\omega), \xi'(\omega))$ is measurable. Since measurability issues can be a little tricky at times, we give here an explicit argument. Let $\Psi$ be a point process on $X$ and $Z$ a graph-measurable mapping from $\Omega$ into $X_0$. Let $B \in \mathcal{X}$. By (5.4) we have that

\[
T_{Z,\Psi}(\Phi_{Z \times \mathbb{R}_+})(B) = \sum_{n=1}^{\infty} \int 1\{t_1 \leq p(x_1, \Psi), \ldots, t_n \leq p(x_n, \Psi + \delta x_1 + \cdots + \delta x_{n-1})\} \times 1\{x_1 \cdots < n\} 1\{x_1, \ldots, x_n \in Z\} 1\{(x_1, \ldots, x_{n-1}) \in C_{n-1}\} \times 1\{\Phi(B_n(x_1, \ldots, x_n) \cap (Z \times \mathbb{R}_+)) = 0\} \sum_{i=1}^{n} 1\{x_i \in B\} \Phi^{(n)}(d(x_1, t_1, \ldots, x_n, t_n)).
\]

Here the order $<$ is allowed to depend measurably on $\omega \in \Omega$. Writing $R_\omega$ for this order, the measurability means that $(\omega, x, y) \mapsto 1\{x \leq R_\omega y\}$ is measurable. It follows that $T_{Z,\Psi}(\Phi_{Z \times \mathbb{R}_+})(B)$ is a random variable which in turn implies that $T_{Z,\Psi}(\Phi_{Z \times \mathbb{R}_+})$ is a point process. Using this fact together with the recursive construction of $(\xi, \xi')$ shows that $(\xi, \xi')$ is indeed jointly measurable in $\omega \in \Omega$ and the boundary conditions $\psi$ and $\psi'$.

7 Bounds for empty space probabilities

In this section we apply the previous results to obtain upper bounds for empty space probabilities of a Gibbs process. These probabilities are important characteristics of a point process. In fact, they determine the distribution of a simple point process (see e.g. [25]) and have many applications, for instance in stochastic geometry (see e.g. [7]) or in quantifying the clustering of a point process (see [4]). We shall need an upper bound in the proof of Theorem 10.1.
A naive approach to bound the empty space probability for a given boundary condition from above would be to bound the partition function from below and to exploit the relation (2.8). This is often not very promising since the partition function is hardly accessible. Our approach is different. We start with the embedding representation of a (finite) Gibbs process from Section 5 that involves the function $p$ from (4.1). This enables us to use bounds for fractions of partition functions (which are often considerably easier to find than bounds for the partition function itself) to obtain bounds for empty space probabilities.

Let $(\mathbb{X}, \mathcal{X})$ be a Borel space equipped with a $\sigma$-finite and diffuse measure $\lambda$. We consider a Gibbs process $\xi$ on $\mathbb{X}$ with PI $\kappa$ satisfying the cocycle condition (2.9) and assume that (Dom2) and (Loc1) hold.

Let be a Poisson process on $\mathbb{X}$ with intensity measure $\alpha \lambda$ and let $B \in \mathcal{X}_0$. By [15, Theorem 1.1] (and [2, Lemma 2.5]) the process $\xi_B$, conditioned on $\xi_{B^c}$, is stochastically dominated by $\eta$. In particular

$$P(\xi(B) = 0 \mid \xi_{B^c}) \geq e^{-((\alpha \lambda)(B))}, \quad P\text{-a.s.}$$  \hspace{1cm} (7.1)

The next result provides an inequality in the converse direction. For $B \in \mathcal{X}_0$ we set

$$B_\xi := \{x \in B : x \not\sim \xi_{B^c}\} = B \setminus \bigcup_{y \in \xi_{B^c}} N_y.$$  \hspace{1cm} (7.2)

**Theorem 7.1.** Suppose that $\xi$ is a Gibbs process whose PI $\kappa$ satisfies (Dom2) and (Loc1). Let $B \in \mathcal{X}_0$. Then

$$P(\xi(B) = 0 \mid \xi_{B^c}) \leq \exp \left[ - \int 1\{x \in B_\xi\} e^{-((\alpha \lambda)(N_x \cap B))}\kappa(x, 0) \lambda(dx) \right], \quad P\text{-a.s.}$$  \hspace{1cm} (7.3)

**Proof:** We have mentioned at the end of Section 2 that the conditional distribution $P(\xi_B \in \cdot \mid \xi_{B^c})$ is that of a Gibbs process with PI $\kappa_{B, \xi_{B^c}}$. By assumption (Dom2) and Remark 3.1 we can apply Theorem 5.1 (with $\kappa = \kappa_{B, \xi_{B^c}}$ and $\mathbb{X} = B$) to construct this Gibbs distribution. By definition of the embedding operator and the properties of the underlying Poisson process we have that

$$P(\xi(B) = 0 \mid \xi_{B^c}) = \exp \left[ - \int 1\{x \in B\} p_{B, \xi_{B^c}}(x, 0) \lambda(dx) \right], \quad P\text{-a.s.}$$  \hspace{1cm} (7.4)

Recall from (4.1) that

$$p_{B, \xi_{B^c}}(x, 0) = \kappa(x, \xi_{B^c}) \frac{Z_{B_x}(\xi_{B^c} + \delta_x)}{Z_{B_x}(\xi_{B^c})}, \quad x \in B,$$

where $B_x = B \cap (x, \infty)$ and the intervals are defined with respect to the order (induced by the Borel isomorphism) on $\mathbb{X}$. Assume that $x \in B_\xi$. If $\mu \in \mathbb{N}_B$ satisfies $\mu(N_x) = 0$, then assumption (Loc1) shows that

$$H(\mu, \xi_{B^c} + \delta_x) = H(\mu, \xi_{B^c}).$$  \hspace{1cm} (7.5)

Therefore

$$Z_{B_x}(\xi_{B^c} + \delta_x) \geq \int 1\{\mu(N_x) = 0\} e^{-H(\mu, \xi_{B^c})} \Pi_{\lambda_{B_x}}(d\mu),$$
Since a Poisson process is completely independent we obtain that
\[
Z_{B_x}(\xi_{B^c} + \delta_x) \geq e^{\lambda(N_x \cap B_x)} \int e^{-H(\mu_{N_x} \xi_{B^c})} \Pi_{\lambda_{B_x}} (d\mu) = e^{-\lambda(N_x \cap B_x)} Z_{B_x \setminus N_x}(\xi_{B^c}).
\]

Using the definition of the Hamiltonian we further obtain that
\[
Z_{B_x}(\xi_{B^c}) = \int e^{-H(\mu_{N_x} \xi_{B^c})} e^{-H(\mu_{N_x} \xi_{B^c} + \mu_{N_x})} \Pi_{\lambda_{B_x}} (d\mu) \leq \int e^{-H(\mu_{N_x} \xi_{B^c})} \prod_{y \in \mu_{N_x}} \alpha(y) \Pi_{\lambda_{B_x}} (d\mu),
\]
where we have used our assumption (Dom2). By the independence properties of a Poisson process this implies
\[
Z_{B_x}(\xi_{B^c}) \leq \exp \left[ \int 1\{y \in N_x \cap B_x\}(\alpha(y) - 1) \lambda(dy) \right] Z_{B_x \setminus N_x}(\xi_{B^c}).
\]

Therefore we obtain for \(x \in B_\xi\) that
\[
p_{B,\xi}(x,0) \geq \exp \left[ -\lambda(N_x \cap B_x) + \int 1\{y \in N_x \cap B_x\}(1 - \alpha(y)) \lambda(dy) \right] \kappa(x,0)
\geq e^{-\lambda(N_x \cap B_x)} \kappa(x,0).
\]

Inserting this into (7.3) gives the result. □

**Remark 7.2.** If \(N_x = \{x\}\) for each \(x \in \mathbb{X}\), then (Loc1) means that \(\xi\) is a Poisson process with intensity measure \(\kappa(x,0)\lambda(dx)\). Then (7.2) is an identity. In a sense this boundary case is obtained in the limit as \(N_x \downarrow \{x\}\) for each \(x \in \mathbb{X}\).

**Remark 7.3.** It is interesting to note that the preceding theorem holds for any reference measure \(\lambda\). In particular they do not require the clusters \(C(x,\eta)\) (or \(C(x,\xi)\)) to be finite. This is in contrast to the assumptions of Lemma 3.3 in the seminal paper [35], which applies only to a restricted range of parameters.

In the following example we apply Theorem 7.1 to general Gibbs particle processes with deterministically bounded grains.

**Example 7.4.** Let \(\mathbb{X} := C^d\) denote the space of compact and nonempty subsets (particles) of \(\mathbb{R}^d\). We equip \(C^d\) with the Hausdorff metric and the associated Borel \(\sigma\)-field \(\mathcal{B}(C^d)\). For \(K \in C^d\) let \(z(K)\) denote the center of the circumscribed sphere of \(K\). Define
\[
\lambda(\cdot) := \iint 1\{K + x \in \cdot\}\mathcal{Q}(dK)\lambda_d(dx),
\]
where \(\lambda_d\) denotes Lebesgue measure on \(\mathbb{R}^d\) and \(\mathcal{Q}\) is a probability measure on \(C^d\) satisfying
\[
\mathcal{Q}(\{K \in C^d : C \subset B(o,R)\}) = 1
\]

25
for some fixed $R > 0$, where $o$ denotes the origin in $\mathbb{R}^d$. Assume that (Dom2) holds. Assume that (Loc1) holds with respect to the relation $\sim$ defined by $K \sim L$ if $K \cap L \neq \emptyset$. Finally we assume the translation invariance $\kappa(K, \mu) = \kappa(K - x, \theta_x \mu)$ for $K \in \mathcal{C}^d$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{N}$, where $\theta_x \mu$ is defined by $\theta_x \mu(B) := \mu(B + x)$ for measurable sets $B \subset \mathbb{X}$. Let $\mathcal{C}^d_t := \{K \in \mathcal{C}^d \: z(K) \in B(o, t)\}$ for $t \geq 2R$. Then Theorem 7.1 implies that
\[
\mathbb{P}(\xi(\mathcal{C}_t^d) = 0 \mid \xi_{B^c}) \leq e^{-c_0(t-2R)^d}, \quad \mathbb{P}\text{-a.s., } t \geq 2R, \tag{7.5}
\]
where
\[
c_0 := e^{-\alpha \kappa d 2^d R^d} \int \kappa(K, 0) \mathbb{Q}(dK). \tag{7.6}
\]
If $c_0 > 0$ (that is if $\int \kappa(K, 0) \mathbb{Q}(dK) > 0$) then we have
\[
\limsup_{t \to \infty} t^{-d} \log \mathbb{P}(\xi(\mathcal{C}_t^d) = 0 \mid \xi_{B^c}) \leq -c_0 \quad \mathbb{P}\text{-a.s.}, \tag{7.7}
\]

**Remark 7.5.** The authors of [35] called the property (7.7) *Poisson-like* and established it under assumptions similar to those in Corollary 9.7. Our method does not require the absence of percolation or a related subcriticality property.

In the following result we replace the assumption (Loc1) by the stronger assumption (Loc2).

**Theorem 7.6.** Suppose that $\xi$ is a Gibbs process whose PI $\kappa$ satisfies (Dom2) and (Loc2). Let $B \in \mathcal{X}_0$. Then
\[
\mathbb{P}(\xi(B) = 0 \mid \xi_{B^c}) \leq \exp \left[ - \int 1\{x \in B\} e^{-\alpha \lambda(N_x \cap B)} \kappa(x, \xi_{B^c}) \lambda(dx) \right], \quad \mathbb{P}\text{-a.s.} \tag{7.8}
\]

**Proof:** Let $x \in B$ and $\mu \in \mathcal{N}_B$ such that $\mu(N_x) = 0$. By assumption (Loc2) the identity (7.4) remains true, so that the proof of Theorem 7.1 applies. \hfill $\Box$

**Corollary 7.7.** Let the assumptions of Theorem 7.6 be satisfied. Assume moreover that there exists a measurable $\kappa' : \mathbb{X} \times \mathcal{N} \to \mathbb{R}_+$ such that $\kappa \geq \kappa'$ and $\kappa'(x, \cdot)$ is decreasing for each $x \in \mathbb{X}$. Then
\[
\mathbb{P}(\xi(B) = 0) \leq \mathbb{E} \exp \left[ - \int 1\{x \in B\} e^{-\alpha \lambda(N_x \cap B)} \kappa'(x, \eta_{B^c}) \lambda(dx) \right]. \tag{7.9}
\]

**Proof:** The point process $\xi_{B^c}$ is stochastically dominated by $\eta_{B^c}$. Hence the result follows upon replacing in (7.8) $\kappa$ by $\kappa'$ and then taking expectations. \hfill $\Box$

**Example 7.8.** Assume that $\mathbb{X} := \mathbb{R}^d \times \mathbb{R}_+$ equipped with the product measure $\lambda := \lambda_d \otimes \mathbb{Q}$, where $\mathbb{Q}$ is a probability measure on $\mathbb{R}_+$ satisfying
\[
\int s^d \mathbb{Q}(ds) < \infty. \tag{7.10}
\]
Assume that $\xi$ is a Gibbs process on $\mathbb{X}$ with PI $\kappa$ satisfying (Dom2) for some constant $\alpha > 0$. We define a relation $\sim$ on $\mathbb{X}$ by

$$(x, r) \sim (y, s) \iff \|x - y\| \leq r + s$$  \hspace{1cm} (7.11)$$
and assume that (Loc2) holds. Assume that there exist $\beta' > 0$ and $\beta \in [0, 1)$ such that

$$\kappa(x, r, \mu) \geq \beta' \beta^\mu(N(x, r)),$$  \hspace{1cm} (7.12)$$
where $\beta > 0$. Applying Corollary 7.7 and take $B$, $c > 0$, for some $B$. These assumptions are satisfied by the Strauss process discussed in Example 3.3. We find that $\kappa(r + s)^d \mathbb{Q}(ds) \leq cr^d$ for some $c > 0$. Applying Jensen’s inequality to the probability measure $\frac{1}{\kappa_d r} \lambda_{B_t} \otimes \mathbb{Q}$, we find that

$$
\mathbb{P}(\xi(B_t \times \mathbb{R}_+) = 0) \leq \frac{1}{\kappa_d r} \mathbb{E} \int \int 1\{x \in B_t\} e^{-\alpha \lambda(N(x, r)) \beta' \beta^\mu(N(x, r)) \cap (B_t \times \mathbb{R}_+)} dx \mathbb{Q}(dr).
$$  \hspace{1cm} (7.13)$$

We have for all $(x, r) \in \mathbb{R}^d \times \mathbb{R}_+$ that

$$\alpha \lambda(N(x, r)) = \alpha \int \int 1\{\|y - x\| < r + s\} dy \mathbb{Q}(ds) = \alpha \int \kappa_d (r + s)^d \mathbb{Q}(ds) \leq cr^d$$

for some $c > 0$. Applying Jensen’s inequality to the probability measure $\frac{1}{\kappa_d r} \lambda_{B_t} \otimes \mathbb{Q}$, we find that

$$
\mathbb{P}(\xi(B_t \times \mathbb{R}_+) = 0) \leq \frac{1}{\kappa_d r} \mathbb{E} \int \int 1\{x \in B_t\} \exp \left[ -\beta' \kappa_d r^d e^{-cr^d} \beta^\mu(N(x, r)) \cap (B_t \times \mathbb{R}_+) \right] dx \mathbb{Q}(dr),
$$  \hspace{1cm} (7.14)$$

where $Z_{t,x,r}$ has a Poisson distribution with parameter $\alpha \lambda(N(x, r) \cap (B_t^c \times \mathbb{R}_+))$.

We treat (7.14) distinguishing by the value of $\beta^\mu(N(x, r))$. Let $a \in (0, d)$. Then

$$
\frac{1}{\kappa_d r} \mathbb{E} \int \int 1\{t^d \beta^\mu(N(x, r)) \geq t^a\} 1\{x \in B_t\} \exp \left[ -\beta' \kappa_d r^d e^{-cr^d} \beta^\mu(N(x, r)) \cap (B_t \times \mathbb{R}_+) \right] dx \mathbb{Q}(dr)
$$

$$\leq \frac{1}{\kappa_d r} \mathbb{E} \int \int 1\{t^d \beta^\mu(N(x, r)) \geq t^a\} 1\{x \in B_t\} \exp \left[ -\beta' e^{-cr^d} t^a \right] dx \mathbb{Q}(dr)
$$

$$\leq \int \exp \left[ -\beta' e^{-cr^d} t^a \right] \mathbb{Q}(dr).
$$

By Jensen’s inequality this can be bounded by

$$\exp \left[ -\beta' t^a \int e^{-cr^d} \mathbb{Q}(dr) \right].$$

We further have that

$$
\frac{1}{\kappa_d r} \mathbb{E} \int \int 1\{t^d \beta^\mu(N(x, r)) \leq t^a\} 1\{x \in B_t\} \exp \left[ -\beta' \kappa_d r^d e^{-cr^d} \beta^\mu(N(x, r)) \cap (B_t \times \mathbb{R}_+) \right] dx \mathbb{Q}(dr)
$$

$$\leq \frac{1}{\kappa_d r} \mathbb{E} \int \int 1\{x \in B_t\} \mathbb{P}(\beta^\mu(N(x, r)) > t^d - a) dx \mathbb{Q}(dr).$$
By Markov’s inequality
\[ P(\beta^{-d}Z_{t,x,r} > t^{d-a}) \leq t^{a-d}E[e^{vZ_{t,x,r}}(\beta^{-1}-1)]. \]
Hence the above integral is bounded by
\[ \frac{1}{\kappa_d t^d} \int 1\{x \in B_t\} e^{-\alpha d} t^{a-d} dx \mathbb{Q}(dr) = t^{a-d} \int e^{-\alpha d} \mathbb{Q}(dr). \]

Altogether we obtain polynomial decay of \( P(\xi(B_t \times \mathbb{R}_+)) = 0). \)

Remark 7.9. The polynomial decay in Example 7.8 might be suboptimal. But it has been derived under the minimal integrability assumption (7.10). This is in contrast to Example 7.4, where the range of interaction between neighbors has been assumed to be deterministically bounded.

Remark 7.10. Since (Dom2) and (Loc2) hold for the Strauss process, the continuum random cluster model and the Widom-Rowlinson model, Theorem 7.1 and Theorem 7.6 apply to these models. Moreover, they apply to Gibbs processes with a pair potential if (3.2) holds and to the area interaction process if \( \mathbb{Q}([r_1, \infty)) = 1 \) for some \( r_1 > 0). \)

8 Palm measures and thinnings

To prepare our results on Poisson approximation in the next section we define and briefly discuss Palm measures and dependent thinnings of a Gibbs process.

Again we work in the general setting of Section 2 and let \( (X, \mathcal{X}) \) be a Borel space equipped with a \( \sigma \)-finite measure \( \lambda \). Let \( \xi \) and \( \chi \) be two point processes on \( X \) and assume that \( \chi \) has a \( \sigma \)-finite intensity measure \( E[\chi] \). The Palm distributions \( P_{\xi|\chi}^x \) (of \( \xi \) w.r.t. \( \chi \)), \( x \in X \), are a family of probability measures on \( N(X) \) such that \((x,A) \mapsto P_{\xi|\chi}^x (A) \) is a probability kernel and
\[ E \int h(x, \xi) \chi(dx) = \int h(x, \mu) P_{\xi|\chi}^x (d\mu) E[\chi](dx) \] (8.1)
for all measurable \( h: X \times N(X) \to [0, \infty) \). One can interpret \( P_{\xi|\chi}^x (\cdot) \) as conditional distribution given that \( \chi \) has a point at \( x \). We refer to [22] for the existence and an in depth discussion.

Let \( \xi \) be a Gibbs process on \( X \) with PI \( \kappa \) satisfying the cocycle condition (2.9). Assume given a measurable function \( g: X \times N \to \{0, 1\} \) satisfying the hereditary property
\[ \{(y, \mu) \in X \times N : g(x, \mu + \delta_y) = 0\} \subset \{(y, \mu) \in X \times N : g(x, \mu) = 0\}, \quad x \in X. \] (8.2)
Using \( g \) we define a (measurable) thinning operator \( \Gamma: N \to N \) by
\[ \Gamma(\mu)(B) := \int 1\{x \in B\} g(x, \mu - \delta_x) \mu(dx), \quad B \in \mathcal{X}, \mu \in N. \] (8.3)
Lemma 8.1. For $E[\Gamma(\xi)]$-a.e. $x \in X$ the Palm probability measure $\mathbb{P}_{x}^{\xi|\Gamma(\xi)}$ is the distribution of a Gibbs process with PI $\kappa^x$ given by

$$\kappa^x(y, \mu) := \kappa(y, \mu + \delta_x) \frac{g(x, \mu + \delta_y)}{g(x, \mu)}, \quad (y, \mu) \in X \times \mathbb{N},$$

where $0/0 := 0$.

Proof: By the GNZ equation (2.1), the intensity measure of $\Gamma(\xi)$ is given by

$$E[\Gamma(\xi)](B) = E \int 1\{x \in B\} g(x, \xi - \delta_x) \xi(dx) = E \int 1\{x \in B\} g(x, \xi) \kappa(x, \xi) \lambda(dx).$$

This measure is $\sigma$-finite. The remainder of the proof is quite standard; see e.g. [2] for the case $g \equiv 1$. \hfill \Box

9 Poisson approximation of Gibbsian functionals

In this section we let $(Y, \mathcal{Y})$ be a complete separable metric space equipped with the Borel $\sigma$-field $\mathcal{Y}$ and some probability measure $Q$. We consider the product space $X := \mathbb{R}^d \times Y$ equipped with the product measure $\lambda := \lambda_d \otimes Q$, where $\lambda_d$ denotes Lebesgue measure on $\mathbb{R}^d$. For a (signed) measure $\rho$ on $X$ and a Borel set $B \subset \mathbb{R}^d$ we abbreviate $\rho_B := \rho_{B \times Y}$ and $\rho(B) := \rho(B \times Y)$. We consider a Gibbs process $\xi$ on $X$ with PI $\kappa$ satisfying the cocycle condition (2.9) and (Dom2) for some $\alpha > 0$. Let $\sim$ be a measurable symmetric relation on $X$ and assume that (Loc1) holds.

We consider a measurable function $g : X \times N \to \{0, 1\}$ satisfying (8.2). Let $R \subset \mathbb{R}^d$ be a compact set and let $R(x) := R + x$ for all $x \in \mathbb{R}^d$. We assume that for all $(x, r, \mu) \in \mathbb{R}^d \times Y \times N(X),

$$g(x, r, \mu) = g(x, r, \mu_{R(x)}).$$

(9.1)

Let $W \subset \mathbb{R}^d$ be a compact set. We wish to approximate the restriction of the point process $\Gamma(\xi)$ to $W \times Y$ by a Poisson process $\nu$ on $W \times Y$.

To the best of our knowledge, Poisson approximation of (derived) Gibbs processes is so far only discussed in very few articles in the literature. In [36, Theorem 3.A] Stein’s method (through [1, Theorem 2.4]) is used to obtain bounds on the total variation distance between a finite thinned Gibbs process and a Poisson process. However, as [36, Theorem 4.I] and the examples given thereafter show, it is not easy to exploit these bounds. (One needs to bound distances between densities and empty space probabilities.) In Theorem 9.1 and Corollary 9.7 below, we treat a wider class of (scaled) thinned Gibbs processes and give very explicit bounds on their total variation distance to a Poisson process. Our proof exploits [6, Theorem 3.1] and is based on a coupling of the thinned Gibbs process and its Palm version. This technique is applied in [6] and [32] to Poisson approximation of thinned Poisson processes.

Let $\mu$ be a simple counting measure on $X$. We can extend $\bar{\mu} := \mu(\cdot \times Y)$ to a graph $G(\mu)$ as follows. Let $x, y \in \bar{\mu}$ be distinct and let $r, s \in Y$ such that $(x, r), (y, s) \in \mu$. \hfill \Box
We draw an edge between \( x \) and \( y \) if \( (x, r) \sim (y, s) \). For Borel sets \( A, B \subset \mathbb{R}^d \) we write \( A \xrightarrow{\mu} B \) if there exist \( \mu \)-points \( x \in A \) and \( y \in B \) which belong to the same component of \( G(\mu) \).

Recall that the Kantorovich-Rubinstein (KR) distance between (the distributions) of two finite point processes \( \xi' \) and \( \xi'' \) on \( \mathbb{X} \) is defined by

\[
\text{d}_{\text{KR}}(\xi', \xi'') := \sup_{h \in \text{Lip}(\mathbb{X})} |\mathbb{E}h(\xi') - \mathbb{E}h(\xi'')|,
\]

where \( \text{Lip}(\mathbb{X}) \) is the class of all measurable 1-Lipschitz functions with respect to the total variation metric \( d_{\text{TV}} \) between two (finite) measures on \( \mathbb{X} \). Note that convergence in the KR distance implies convergence in distribution and that the KR distance dominates the total variation distance

\[
\text{d}_{\text{TV}}(\xi', \xi'') := \sup_A |\mathbb{P}(\xi' \in A) - \mathbb{P}(\xi'' \in A)|,
\]

where the supremum is taken over all measurable subsets \( A \) of \( \text{N}(\mathbb{X}) \).

In the next theorem we will, in addition to the set \( R \) (introduced at (9.1)), consider another compact set \( S \subset \mathbb{R}^d \) that contains the origin \( o \in \mathbb{R}^d \). We define \( S(x) := S + x, x \in \mathbb{R}^d \) and let \( W + S := \{x + y : x \in W, y \in S\} \) be the (compact) Minkowski sum of \( W \) and \( S \). Note that \( W \subset W + S \) since \( o \in S \).

**Theorem 9.1.** Let \( \xi \) be a Gibbs process on \( \mathbb{R}^d \times \mathbb{Y} \) with a PI \( \kappa \) satisfying (Loc1) and (Dom2) for some \( \alpha > 0 \). Let \( R \subset \mathbb{R}^d, S \subset \mathbb{R}^d \) with \( o \in S \) and \( W \subset \mathbb{R}^d \) be compact sets such that \( R \subset S \). Define \( \Gamma \) by (8.3), where \( g \) is assumed to satisfy (8.2) and (9.1). Let \( \nu \) be a Poisson process with finite intensity measure \( \mathbb{E}[\nu] \). Then

\[
d_{\text{KR}}(\Gamma(\xi)_W, \nu) \leq d_{\text{TV}}(\mathbb{E}[\Gamma(\xi)_W], \mathbb{E}[\nu]) + T_1 + T_2 + T_3,
\]

where

\[
T_1 := 2 \iint 1\{x, y \in W\} \mathbb{E}[g(x, r, \xi)\kappa(x, r, \xi)] \mathbb{E}[g(y, s, \xi)\kappa(y, s, \xi)]
\times 1\{S(x) \cap S(y) \neq \emptyset\} \mathbb{Q}^2(d(r, s)) \mathbb{d}(x, y),
\]

\[
T_2 := 2 \iint 1\{x, y \in W\} \mathbb{E}[g(x, r, \xi + \delta_{(y,s)})\kappa(x, r, \xi + \delta_{(y,s)})g(y, s, \xi + \delta_{(x,r)})\kappa(y, s, \xi)]
\times 1\{S(x) \cap S(y) \neq \emptyset\} \mathbb{Q}^2(d(r, s)) \mathbb{d}(x, y),
\]

\[
T_3 := 2\alpha \iint 1\{x, y \in W\} 1\{S(x) \cap S(y) = \emptyset\} \mathbb{P}(R(y) \xrightarrow{\eta} (W + S)^c \cup R(x))
\times \mathbb{E}[g(x, r, \xi)\kappa(x, r, \xi)] \mathbb{Q}(dr) \mathbb{d}(x, y),
\]

where \( \eta \) is a Poisson process on \( \mathbb{R}^d \times \mathbb{Y} \) with intensity measure \( \alpha \lambda_d \otimes \mathbb{Q} \).

**Remark 9.2.** One should compare Theorem 9.1 with [6, Theorem 4.1] that considers Poisson process approximation of functionals of a Poisson process. The terms \( T_1 \) and \( T_2 \) on the right-hand side of (9.2) are analogous to the terms \( E_2 \) and \( E_3 \) in [6, Theorem 4.1]. The term \( T_3 \), which reflects the long-range interactions of the Gibbs process \( \xi \), does not appear in [6, Theorem 4.1]. This is due to the independence properties of the Poisson process.
Remark 9.3. Using the contractivity properties of the KR-distance, Theorem 9.1 can be generalized as follows. Let $X'$ be another complete separable metric space and let $f: X \to X'$ be measurable. Define (for a given compact set $W \subset \mathbb{R}^d$), the mapping $\Gamma_{f,W}: \mathcal{N}(X) \to \mathcal{N}(X')$ by

$$\Gamma_{f,W}(\mu)(\cdot) := \int \mathbf{1}\{f(x) \in \cdot, x \in W\} g(x, \mu - \delta_x) \mu(dx), \quad \mu \in \mathcal{N}(X).$$

(9.3)

Let $\nu$ be a Poisson process as in Theorem 9.1 and let $\nu_f$ be the (finite) Poisson process on $X'$ given by $\nu_f := \nu(f^{-1}(\cdot))$. Then $d_{KR}(\Gamma_{f,W}(\xi), \nu_f)$ can still be bounded by the right-hand side of (9.2).

Proof of Theorem 9.1. Assume for each $(x, r) \in X$ that $\chi^{x,r}$ is a point process with the Palm distribution $\mathbb{P}^{\Gamma(\xi)||\Gamma(\xi)}$ and such that $(\omega, x, r) \mapsto \chi^{x,r}(\omega)$ is measurable. (Later we shall choose $\chi^{x,r}$ in a specific way.) From [6, Theorem 3.1] we have that

$$d_{KR}(\Gamma(\xi)_W, \nu)$$

$$\leq d_{TV}(\mathbb{E}[\Gamma(\xi)_W], \mathbb{E}[\nu]) + 2 \int \mathbb{E}[\|\Gamma(\xi)_W - (\chi^{x,r}_W - \delta_{(x,r)})\|] \mathbb{E}[\Gamma(\xi)_W](d(x,r)),$$

(9.4)

where $\|\rho\|(B) := \|\rho\|(B \times Y)$ denotes the total variation measure of a (finite) signed measure $\rho$ on $X$ evaluated at $B \times Y$. We now turn to the integral on the right-hand side of (9.4). It equals

$$I := \int \mathbb{E}[\|\Gamma(\xi)_W - (\chi^{x,r}_W - \delta_{(x,r)})\|] \mathbb{E}[\Gamma(\xi)_W](d(x, r)).$$

Obviously,

$$I \leq \int \mathbb{E}[\Gamma(\xi)_W](d(y, s)) \mathbb{E}[\Gamma(\xi)](d(x, r))$$

$$+ \int \mathbb{E}[\chi^{x,r} - \delta_{(x,r)}](d(y, s)) \mathbb{E}[\Gamma(\xi)](d(x, r))$$

$$+ \mathbb{E} \int \mathbb{E}[\Gamma(\xi)_W - (\chi^{x,r}_W - \delta_{(x,r)})](d(y, s)) \mathbb{E}[\Gamma(\xi)](d(x, r)),$$

where here and in the following the integration with respect to $x$ and $y$ is always restricted to $W$. Using the GNZ equations, we find that the first term on the above right-hand side is given by

$$\int \mathbb{E}[g(x, r, \xi) \kappa(x, r, \xi)] \mathbb{E}[g(y, s, \xi) \kappa(y, s, \xi)] \mathbf{1}\{S(x) \cap S(y) \neq \emptyset\} Q^2(d(r, s)) d(x, y).$$

Since $\chi^{x,r}$ is a point process with the Palm distribution $\mathbb{P}^{\Gamma(\xi)||\Gamma(\xi)}$, we obtain from (8.1) and (2.2) that the second term on the right-hand side of (9.5) is given by

$$\int \mathbb{E}[g(x, r, \xi + \delta_{y,s}) \kappa(x, r, \xi + \delta_{y,s}) g(y, s, \xi + \delta_{x,r}) \kappa(y, s, \xi)]$$

$$\times \mathbf{1}\{S(x) \cap S(y) \neq \emptyset\} Q^2(d(r, s)) d(x, y).$$
To treat the third term on the right-hand side of (9.5) we use disagreement coupling. Assume for each \( (x, r) \in X \) that \( \xi^{x,r} \) is a point process whose distribution is the Palm distribution \( \mathbb{P}^{(x,r)}_{\xi(x,r)} \) and such that \( (\omega, x, r) \mapsto \xi^{x,r}(\omega) \) is measurable. It is a straightforward task to check that

\[
\chi^{x,r} \overset{d}{=} \Gamma(\xi^{x,r}), \quad (x, r) \in X.
\]

Let \( U(x) := (W + S) \setminus R(x) \). Note that since \( R \subset S \) and \( o \in S \), every pair of point \( x, y \in W \) with \( S(x) \cap S(y) = \emptyset \) satisfies \( y \in U(x) \). By the DLR equations (2.7), Lemma 8.1 and (9.1) we have that

\[
\mathbb{P}(\xi^{x,r}_{U(x)}) \in \cdot | \xi^{x,r}_{U(x)} \}
\]

is the distribution of a Gibbs process with PI \( \kappa_{U(x) \times Y, \xi^{x,r}_{U(x)} + \delta_{x,r}} \), while

\[
\mathbb{P}(\xi^{x,r}_{U(x)}) \in \cdot | \xi^{x,r}_{U(x)} \}
\]

is the distribution of a Gibbs process with PI \( \kappa_{U(x) \times Y, \xi^{x,r}_{U(x)}} \). By Lemma 5.3 there exist point processes \( \eta, \tilde{\xi} \) and \( \tilde{\xi}^{x,r} \) such that \( \eta \) is a Poisson process on \( X \) with intensity measure \( \alpha \lambda \otimes \mathbb{Q} \), \( \tilde{\xi} \overset{d}{=} \xi \), \( \tilde{\xi}^{x,r} \overset{d}{=} \xi^{x,r} \) and \( \tilde{\xi} \leq \eta, \tilde{\xi}^{x,r} \leq \eta \) almost surely. Let \( \Phi \) be an independent marking of \( \eta \) with the marks uniformly distributed on the interval \([0, \alpha]\). Then \( \Phi \) is a Poisson process on \( X \times [0, \alpha] \), whose intensity measure is the product of \( \lambda \) and Lebesgue measure on \([0, \alpha]\). We now consider the disagreement coupling

\[
\psi := T_{U(x) \times Y, \xi^{x,r}_{U(x)}}(\Phi_{U(x) \times Y \times [0, \alpha]}), \quad \psi^{x,r} := T_{U(x) \times Y, \xi^{x,r}_{U(x)} + \delta_{x,r}}(\Phi_{U(x) \times Y \times [0, \alpha]}),
\]

where we refer to the notation introduced in the first paragraph of Section 6. From Theorem 5.1, Remark 5.2 and the DLR equations we find that

\[
\xi \overset{d}{=} \psi + \tilde{\xi}_{U(x)}, \quad \xi^{x,r} \overset{d}{=} \psi^{x,r} + \tilde{\xi}^{x,r}_{U(x)}.
\]

Using this coupling we can rewrite the third term on the right-hand side of (9.5), \( I_3 \) say, as

\[
I_3 = \mathbb{E} \int \int 1\{S(x) \cap S(y) = \emptyset\} \|\Gamma(\psi + \tilde{\xi}_{U(x)}) - \Gamma(\psi^{x,r} + \tilde{\xi}^{x,r}_{U(x)})\|(d(y, s)) \mathbb{E}[\Gamma(\xi)](d(x, r)).
\]

Since a point \( y \in W \) with \( S(x) \cap S(y) = \emptyset \) satisfies \( y \in U(x) \), it can only contribute to the total variation \( \|\Gamma(\psi + \tilde{\xi}_{U(x)}) - \Gamma(\psi^{x,r} + \tilde{\xi}^{x,r}_{U(x)})\| \) if \( (y, s) \in \psi + \psi^{x,r} \) and if one of the following three cases occurs. In the first case we have that \( (y, s) \notin \psi^{x,r} \) and \( g(y, s, \psi + \tilde{\xi}_{U(x)}) = 1 \).

By (9.1) and the definition of \( U(x) \) we have that \( g(y, s, \psi + \tilde{\xi}_{U(x)}) = g(y, s, \psi) \). The second case is \( (y, s) \notin \psi \) and \( g(y, s, \psi^{x,r}) = 1 \). The third case is \( (y, s) \notin \psi \) and \( |g(y, s, \psi) - g(y, s, \psi^{x,r})| = 1 \). By (9.1), the third case can only occur if \( \|\psi - \psi^{x,r}\|(R(y) \times Y) \neq 0 \). The latter inequality holds in the other two cases as well. Therefore,

\[
\int 1\{S(x) \cap S(y) = \emptyset\} \|\Gamma(\psi + \tilde{\xi}_{U(x)}) - \Gamma(\psi^{x,r} + \tilde{\xi}^{x,r}_{U(x)})\|(d(y, s)) \leq \int 1\{S(x) \cap S(y) = \emptyset\} 1\{|\psi - \psi^{x,r}|(R(y) \times Y) \neq 0\} (\psi + \psi^{x,r})(d(y, s)).
\]
By Theorem 6.3, ψ + ψ^{x,r} is dominated by the restriction of η to U(x) × Y. Moreover, if ∥ψ − ψ^{x,r}∥(R(y) × Y) ≠ 0, then there exists (z, u) ∈ η with z ∈ R(y) such that (z, u) is connected via η_{(z,u)} := η − δ(z,u) to Ξ_{U(x)} + Ξ_{U(x)}. Since Ξ ≤ η, Ξ^{x,r} ≤ η we have that supp(Ξ^{x,r} + Ξ) ⊆ supp(η) almost surely. This implies that (z, u) is connected via η_{(z,u)} to η_{U(x)} and, hence, that R(y) ∋ η_{(y,u)} U(x)^c. Therefore we can bound the right-hand side of (9.9) by

\[ \int 1\{S(x) ∩ S(y) = \emptyset\} 1\{R(y) \leftrightarrow U(x)^c\} \eta(d(y, s)). \]

Taking the expectation and using the Mecke formula yields

\[ \mathbb{E} \int 1\{S(x) ∩ S(y) = \emptyset\} \|\Gamma(ψ + Ξ_{U(x)^c}) − \Gamma(ψ^{x,r} + Ξ_{U(x)^c})\| (d(y, s)) \]

\[ \leq α \int \int 1\{S(x) ∩ S(y) = \emptyset\} \mathbb{P}(R(y) \leftrightarrow U(x)^c) Q(ds) dy. \]

Therefore

\[ I_3 ≤ α \int \int 1\{S(x) ∩ S(y) = \emptyset\} \mathbb{P}(R(y) \leftrightarrow (W + S)^c ∪ R(x)) \]

\[ \times \mathbb{E}[g(x, r, ξ)κ(x, r, ξ)] Q(dr) d(x, y). \]

This finishes the proof. \hfill \Box

Next we provide a lemma which helps us control the term T_3 on the right-hand side of (9.2) in the case where R(x) and S(x) are balls; see Corollary 9.7. The lemma bounds the one-arm probabilities, a well-studied object in the theory of continuum percolation (see e.g. [29]). It is convenient to introduce a random variable Y with distribution Q which is independent of η. Let o be the origin in \( \mathbb{R}^d \). Then \( C(\langle o, Y \rangle, η + δ_{\langle o, Y \rangle}) \) can be interpreted as the cluster containing the typical point of η, at least if the relation ∼ is translation invariant. By this we mean that for any given \( (x, r), (y, s) \in \mathbb{R}^d \) we have that \( (x, r) \sim (y, s) \) implies \( (x + z, r) \sim (y + z, s) \) for each \( z \in \mathbb{R}^d \).

**Lemma 9.4.** Let η be a Poisson process on \( \mathbb{R}^d \) with intensity measure αλ_d ⊗ Q. Assume that ∼ is translation invariant and let u, v > 0. Then

\[ \mathbb{P}(B(o, u) \leftrightarrow B(o, u + v)^c) ≤ αλ_d(B(o, 1)) u^d \mathbb{P}(\langle o, Y \rangle \leftrightarrow B(o, v)^c > 0). \quad (9.10) \]

**Proof.** We have

\[ 1\{B(o, u) \leftrightarrow B(o, u + v)^c\} \]

\[ ≤ \int 1\{x ∈ B(o, u), C(x, r, η)(B(o, u + v)^c × Y) > 0\} η(d(x, r)). \]

Taking expectations and using the Mecke equation yields

\[ p := \mathbb{P}(B(o, u) \leftrightarrow B(o, u + v)^c) \]

\[ ≤ α \int \int 1\{x ∈ B(o, u)\} \mathbb{P}(C(x, r, η + δ_{\langle x, r \rangle})(B(o, u + v)^c × Y) > 0) dx Q(dr). \]

33
By translation invariance of $\sim$, 

$$C(x, r, \eta + \delta_{(x, r)})(B(o, u + v)^c \times \mathbb{Y}) = C(o, r, \theta_x \eta + \delta_{(o, r)})(B(-x, u + v)^c \times \mathbb{Y}),$$

where $\theta_x \eta$ is the point process on $\mathbb{X}$ defined by $\theta_x \eta(B \times C) := \eta((B + x) \times C)$ for measurable sets $B \subset \mathbb{R}^d$ and $C \subset \mathbb{Y}$. Since $\theta_x \|d \eta = \eta$ we obtain that

$$p \leq \alpha \int 1\{x \in B(o, u)\} \mathbb{P}(C(o, Y, \eta + \delta_{(o, Y)})(B(-x, u + v)^c \times \mathbb{Y}) > 0) \, dx.$$

If $x \in B(o, u)$ then $B(o, v) \subset B(-x, u + v)$. This implies the asserted inequality. \hfill \Box

**Example 9.5.** Let $R \in (0, \infty)$ and assume that $\mathbb{Y} = [0, R] \times \mathbb{Y}'$ for some complete separable metric space $\mathbb{Y}'$. For $(x, r, w)$ we interpret $x$ as the center of a ball with radius $r$ and mark $w$. Define the translation invariant relation $\sim$ on $\mathbb{R}^d \times \mathbb{Y}$ by $(x, r, w) \sim (y, s, z)$ if $\|x - y\| \leq r + s$ and $w \sim' z$, where $\sim'$ is a measurable symmetric relation on $\mathbb{Y}'$; see also (7.11). Then $\eta := \eta(\cdot \times \mathbb{Y}')$ is a Poisson process on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\alpha \lambda_d \otimes Q$, where $Q := Q(\cdot \times \mathbb{Y}')$. The associated Boolean model is given as

$$Z := \bigcup_{(x, r) \in \eta} B(x, r). \quad (9.11)$$

Let us define the critical intensity

$$\alpha_c := \sup \{ \beta > 0 : \Pi_{\beta \lambda_d \otimes Q}(\{\mu \in \mathbb{N}(\mathbb{R}^d \times \mathbb{R}_+) : |C(o, r, \mu)| = \infty\}) \, Q(dr) = 0 \}, \quad (9.12)$$

where $\Pi_{\beta \lambda_d \otimes Q}$ for $\beta > 0$ is the distribution of a Poisson process with intensity measure $\beta \lambda_d \otimes Q$ and the cluster $C(o, r, \mu)$ is defined with respect to the relation (7.11). This example can be generalized by replacing $[0, R]$ by the space of all particles contained in some fixed ball; see Example 7.4.

Assume that

$$\mathbb{P}((o, Y) \xrightarrow{\eta + \delta_{(o, Y)}} B(o, v)^c > 0) \leq c_1 e^{-c_2 v}, \quad v > 0, \quad (9.13)$$

for constants $c_1, c_2 > 0$ (depending only on the dimension and on $\alpha$). This implies that the graph $G(\eta)$ (introduced at the beginning of this section) does not percolate, that is, it does not have an unbounded connected component. (We refer to [29] for an extensive discussion of some standard models of continuum percolation.)

**Remark 9.6.** Consider the setting of Example 9.5 and assume that $\alpha < \alpha_c$. Then we obtain from [40, (3.7)] that (9.13) holds.

Let $R := B(o, u)$ and $S := B(o, u + v)$ for some $u, v > 0$. Then we obtain from Lemma 9.4 and the definition of $\Gamma$ that the term $T_3$ in (9.2) is bounded by

$$c_1 u^d e^{-c_2 v} \int \int 1\{x, y \in W\} \mathbb{E}[g(x, r, \xi) \kappa(x, r, \xi)] Q(dr) d(x, y) = c_1 \mathbb{E}[\Gamma(\xi)(W)] \lambda_d(W) u^d e^{-c_2 u}$$

for some constants $c_1, c_2 > 0$. This gives the following corollary of Theorem 9.1.
Corollary 9.7. Let $\xi$ be a Gibbs process on $\mathbb{R}^d \times \mathbb{Y}$ with a PI $\kappa$ satisfying (Loc1) and (Dom2) for some $\alpha > 0$ and assume that (9.13) holds. We define $\Gamma$ by (8.3), where $g$ is assumed to satisfy (8.2). Moreover, we assume that there is some $u > 0$ such that

$$g(x, r, \mu) = g(x, r, \mu_{B(x, u)}),$$

(9.14)

for all $(x, r, \mu) \in \mathbb{R}^d \times \mathbb{Y} \times \mathbb{N}(x)$. Let $W \subset \mathbb{R}^d$ be a compact set and let $\nu$ be a Poisson process with finite intensity measure $\mathbb{E}[\nu]$. Then we have for all $v > 0$

$$d_{KR}(\Gamma(\xi)_{W}, \nu) \leq d_{TV}(\mathbb{E}[\Gamma(\xi)_{W}], \mathbb{E}[\nu]) + F_1 + F_2 + c_1 \mathbb{E}[\Gamma(\xi)(W)] \lambda_d(W) u^d e^{-cv},$$

(9.15)

where

$$F_1 := 2 \int \int \mathbb{1}\{x, y \in W\} \mathbb{E}[g(x, r, \xi) \kappa(x, r, \xi)] \mathbb{E}[g(y, s, \xi) \kappa(y, s, \xi)]$$

$$\times \mathbb{1}\{\|x - y\| \leq 2(u + v)\} \mathbb{Q}^2(d(r, s)) d(x, y),$$

$$F_2 := 2 \int \int \mathbb{1}\{x, y \in W\} \mathbb{E}[g(x, r, \xi + \delta_{(y,s)}) \kappa(x, r, \xi + \delta_{(y,s)})] g(y, s, \xi + \delta_{(x,r)}) \kappa(y, s, \xi)$$

$$\times \mathbb{1}\{\|x - y\| \leq 2(u + v)\} \mathbb{Q}^2(d(r, s)) d(x, y).$$

The constants $c_1, c_2 > 0$ depend only on the dimension and on $\alpha$.

Remark 9.8. Assume that the probability measure $\mathbb{Q}$ in the Strauss process, the continuum random cluster model or the Widom-Rowlinson model is supported on a bounded subset of $[0, \infty)$. Then we can apply Remark 9.6 to see that Corollary 9.7 holds for $\alpha < \alpha_c$, where $\alpha_c$ is the percolation threshold of a spherical Boolean model with radius distribution $\mathbb{Q}$. Corollary 9.7 also holds for the area interaction process if additionally $\mathbb{Q}([r_1, \infty)) = 1$ for some $r_1 > 0$ (so that (Dom2) holds).

For a Gibbs process with a pair potential, assume that $X = \mathbb{R}^d$ and that $U(x, y) = U^*(\|x - y\|)$ for some measurable $U^* : \mathbb{R}^+ \to [0, \infty]$. Then (Dom2) holds and in order to apply Theorem 9.1 we assume that there exists $r_0 \geq 0$ such that $U^*(r) = 0$ for all $r \geq r_0$. We can then apply Remark 9.6 with $\mathbb{Q} = \delta_{r_0/2}$ to see that Corollary 9.7 holds for $\alpha < \alpha_c$, where $\alpha_c$ is the percolation threshold of a spherical Boolean model with deterministic radius $r_0/2$.

10 Matérn type I thinnings of Gibbs processes

In this section we apply Theorem 9.1 to scaled Matérn type I thinning of a Gibbs process $\xi$ on $\mathbb{R}^d \times \mathbb{Y}$. As in Example 9.5 we take $\mathbb{Y} := [0, R] \times \mathbb{Y}'$ equipped with a probability measure $\mathbb{Q}$ and the relation $\sim$ defined there. We assume that the PI $\kappa$ satisfies (Loc1) and (Dom2) for some $\alpha > 0$. Assume that

$$\kappa(x, r, 0) = \kappa(o, r, 0), \quad x \in \mathbb{R}^d, \ r \in \mathbb{Y}.$$ 

(10.1)

Condition (10.1) holds for Gibbs processes with pair potential (see Remark 9.8), for the Strauss process, the area interaction process, the continuum random cluster model and the Widom-Rowlinson model.
Let $c > 0$. We can find for each $n \in \mathbb{N}$ and each $x \in [0, n^{1/d}]^d$ a number $u_n(x)$ such that

$$
\lim_{n \to \infty} \sup_{x \in [0,1]^d} \left| c - n \mathbb{P}(\xi(B(n^{1/d}x, u_n(n^{1/d}x)) \times \mathcal{Y}) = 0) \right| = 0.
$$

(10.2)

In fact, we can choose

$$
u_n(x) := \sup \{ u \geq 0 : \mathbb{P}(\xi(B(x, u) \times \mathcal{Y}) = 0) \geq c/n \}
$$

with the convention $\sup \emptyset := 0$. Since the measure $\mathbb{E}\xi(\cdot \times \mathcal{Y})$ is absolutely continuous (it is dominated by $\alpha \lambda_d$), the probability $\mathbb{P}(\xi(B(x, u) \times \mathcal{Y}) = 0)$ is a continuous function of $u$. (The probability that $\xi(\cdot \times \mathcal{Y})$ has mass on the boundary of a ball vanishes.) Hence it follows straight from the definition, that (10.2) holds even without the limit, provided that $c < n$.

By (7.1) and Theorem 7.1 (see also Example 7.4) we have that

$$
\limsup_{n \to \infty} \frac{1}{\log n} \sup_{x \in [0,1]^d} u_n(n^{1/d}x)^d < \infty, \quad \liminf_{n \to \infty} \frac{1}{\log n} \inf_{x \in [0,1]^d} u_n(n^{1/d}x)^d > 0.
$$

(10.3)

We consider the process

$$
\chi_n := \sum_{(x,r) \in \xi} \mathbf{1}\{x \in [0,n^{1/d}]^d\} \mathbf{1}\{(\xi - \delta(x,r))(B(x,u_n(x)) \times \mathcal{Y}) = 0\} \delta_{(n^{-1/d}x,r)}.
$$

(10.4)

**Theorem 10.1.** Let $\xi$ be a Gibbs process on $\mathbb{R}^d \times [0,R] \times \mathcal{Y}$ with PI $\kappa$ satisfying (Loc1) and (Dom2) for some $\alpha \in (0,\alpha_c)$, where $\alpha_c$ is introduced at (9.12). Assume that (10.1) holds. Let $\nu$ be a Poisson process on $\mathbb{R}^d \times \mathcal{Y}$ with intensity measure

$$
ck(o,r,0)\mathbf{1}\{x \in [0,1]^d\} dx \mathcal{Q}(dr),
$$

where $c$ is the constant from (10.2). Then there exist $b > 0$ and $C > 0$ such that

$$
d_{KR}(\chi_n, \nu) \leq C \left( n^{-b} + \sup_{x \in [0,1]^d} \left| c - n \mathbb{P}(\xi(B(n^{1/d}x, u_n(n^{1/d}x)) \times \mathcal{Y}) = 0) \right| \right), \quad n \geq 1.
$$

Proof. For fixed $n \geq 2$ we shall apply Corollary 9.7 (combined with Remark 9.3) with $W_n := [0,n^{1/d}]^d$, $u_n := \sup_{x \in [0,1]^d} u_n(n^{1/d}x)$ and $v_n := 2c_2^{-1} \log n$, where $c_2$ is the constant from Corollary 9.7. For all $(x, r, \mu) \in \mathbb{R}^d \times \mathcal{Y} \times \mathcal{N}$ let

$$
g_n(x, r, \mu) \equiv g_n(x, \mu) = \mathbf{1}\{\mu(B(x,u_n(x)) \times \mathcal{Y}) = 0\}
$$

(10.5)

and $f_n(x, r) := (n^{-1/d}x, r)$. Obviously, $g$ satisfies the hereditary property (8.2) as well as (9.1). As Remark 9.6 shows, (9.13) holds.

For all $n$ with $\inf_{x \in [0,1]^d} u_n(n^{1/d}x) > 2R$ we find from (Loc1) and (10.1) that for all $B \in \mathcal{X}$,

$$
\mathbb{E}[\chi_n](B) = \int \int 1\{(n^{-1/d}x, r) \in B \cap ([0,1]^d \times \mathcal{Y})\} \mathbb{E}[g_n(x, r, \xi)] \kappa(x, r, 0) dx \mathcal{Q}(dr)
$$

$$
= n \int \int 1\{(x, r) \in B \cap ([0,1]^d \times \mathcal{Y})\} \mathbb{E}[g_n(n^{1/d}x, r, \xi)] \kappa(o, r, 0) dx \mathcal{Q}(dr).
$$

36
Hence, for those $n$ it holds that
\[
d_{TV}(E[\chi_n], E[\nu]) \leq \sup_{x \in [0, 1]^d} |c - n P(\xi(B(n^{1/d}x, u_n(n^{1/d}x)) \times Y) = 0)| \int \kappa(o, r, 0) Q(dr).
\]

Since we assumed that (Dom2) holds for some constant $\alpha > 0$, we find from (10.2) that for all $n$ large enough,
\[
E[g_n(x, r, \xi) \kappa(x, r, \xi)] \leq 2\alpha cn^{-1}.
\]
Hence, the term $F_1$ in Corollary 9.7 is for those $n$ bounded by
\[
8(\alpha c)^2 n^{-2} \int 1\{|x - y| \leq 2u_n + 4c_2^{-1} \log n\} d(x, y)
\leq \frac{8(\alpha c)^2 \kappa_d(2u_n + 4c_2^{-1} \log n)^d}{n}.
\]

For $F_2$ we obtain the bound
\[
2\alpha^2 \int \mathbb{P}((\xi + \delta_{y, t})(B(x, u_n(x)) \times Y) = 0, (\xi + \delta_{x, t})(B(y, u_n(y)) \times Y) = 0)
\times 1\{|x - y| \leq 2u_n + 4c_2^{-1} \log n\} d(x, y). \quad (10.6)
\]

Note that the probability in the integrand of $F_2$ is zero if $y \in B(x, u_n(x))$. For $y \in B(x, u_n(x))^c$ it is given by
\[
\mathbb{P}(\xi(B(x, u_n(x)) \cup B(y, u_n(y))) = 0). \quad (10.7)
\]

Since the ball $B(y + \frac{u_n(y)}{2} \frac{y - x}{\|y - x\|}, \frac{u_n(y)}{2})$ is contained in $B(y, u_n(y))$, (10.7) is bounded by
\[
\mathbb{P}(\xi(B(x, u_n(x))) = 0, \xi \left(B \left(y + \frac{u_n(y)}{2} \frac{y - x}{\|y - x\|}, \frac{u_n(y)}{2}\right) = 0\right)
\leq \mathbb{E}[g_n(x, r, \xi)] \mathbb{P}(\xi \left(B \left(y + \frac{u_n(y)}{2} \frac{y - x}{\|y - x\|}, \frac{u_n(y)}{2}\right) = 0\right| \xi(B(x, u_n(x))) = 0). \quad (10.8)
\]

Since the balls $B(y + \frac{u_n(y)}{2} \frac{y - x}{\|y - x\|}, \frac{u_n(y)}{2})$ and $B(x, u_n(x))$ are disjoint for $y \in B(x, u_n(x))^c$, we obtain from Theorem 7.1 and Theorem 4.3 that the conditional probability in (10.8) is bounded from above by
\[
e^{-c_0(u_n(y)/2-2R)^d} \leq e^{-c_0(2R)^d} \mathbb{E}[g_n(y, r, \xi)]^{c_0/(2^d \kappa_d)},
\]
where the constant $c_0$ is given at (7.6). Since $\mathbb{E}[g_n(y, r, \xi)] \leq 2\alpha n^{-1}$ by (10.2) for $n$ large enough, we find from (10.6) and (10.8) that
\[
F_2 \leq 2\alpha^2 e^{-c_0(2R)^d} \kappa_d(2u_n + 4c_2^{-1} \log n)^d n(2\alpha n^{-1})^+ c_0/(2^d \kappa_d)
\]
for all $n$ large enough. Now the assertion follows from (10.3) and Corollary 9.7. □

**Acknowledgments:** We wish to thank Steffen Betsch for making several useful comments. We also thank the referees for their very careful reading and for making several helpful suggestions.
References


