A UNIFIED APPROACH TO LINEAR-QUADRATIC-GAUSSIAN MEAN-FIELD TEAM: HOMOGENEITY, HETEROGENEITY AND QUASI-EXCHANGEABILITY

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This paper aims to systematically solve stochastic team optimization of large-scale system, in linear-quadratic-Gaussian framework. Concretely, the underlying large-scale system involves considerable weakly-coupled cooperative agents for which the individual admissible controls: (i) enter the diffusion terms, (ii) are constrained in some closed-convex subsets, and (iii) subject to a general partial decentralized information structure. A more important but serious feature: (iv) all agents are heterogenous with continuum instead of finite diversity. Combination of (i)-(iv) yields a quite general modeling of stochastic team-optimization, but on the other hand, also fails current existing techniques of team analysis. In particular, classical team consistency with continuum heterogeneity collapses because of (i). As the resolution, a novel unified approach is proposed under which the intractable continuum heterogeneity can be converted to a more tractable homogeneity. As a trade-off, the underlying randomness is augmented, and all agents become (quasi) weakly-exchangeable. Such approach essentially involves a subtle balance between homogeneity v.s. heterogeneity, and left (prior-sampling) v.s. right (posterior-sampling) information filtration. Subsequently, the consistency condition (CC) system takes a new type of forward-backward stochastic system with double-projections (due to (ii), (iii)), along with spatial mean on continuum heterogenous index (due to (iv)). Such system is new in team literature and its well-posedness is also challenging. We address this issue under mild conditions. Related asymptotic optimality is also established.

1. Introduction. The starting point of present work is the well-studied mean-field team (MT). In its standard form, a MT involves a large-scale system with considerable weakly-interactive but cooperative agents \( \{A_i\}_{i=1}^N \). All agents are endowed with an individual (principal) state, cost functional and admissible decision set respectively in the following manner. The individual state dynamic of \( A_i \) is formulated by a controlled Itô-type linear stochastic differential equation (LSDE):

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= [A(t)x_i(t) + B(t)u_i(t) + F(t)x^{(N)}(t) + f_i]dt + \sigma_i dW_i(t), \\
x_i(0) &= \xi \in \mathbb{R}^D, \quad 1 \leq i \leq N,
\end{align*}
\]

where \( x^{(N)} := \frac{1}{N} \sum_{i=1}^N x_i \) is the weakly-coupled state-average across all agents, \( W_i \) is a Brownian Motion (BM) that might be vector-valued (e.g., with a common noise). For each \( A_i \), its principal cost \( J_i \) (while we may call \( \{J_j\}_{j \neq i} \) the marginal costs for \( A_i \)) is measured.
by the following quadratic functional:
\[
J_i(\mathbf{u}(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle Q(t)(x_i(t) - H(t)x^{(N)}(t)), x_i(t) - H(t)x^{(N)}(t) \rangle + \langle R(t)u_i(t), u_i(t) \rangle \right] dt,
\]
with admissible team strategy \( \mathbf{u}(\cdot) = (u^T_1(\cdot), \cdots, u^T_N(\cdot))^T \). Note individual admissible \( u_i(\cdot) \in \mathcal{U}^d, f_{i,op} = L^2_{\mathbb{F};\mathbb{F}}(0, T; \mathbb{R}^m) \) with filtration \( \mathbb{F}^i \) defined later, representing the decentralized open-loop information of \( A_i \).

A subtle point here is the distinction between centralized (\( \mathcal{U}^{c,f} \)), and decentralized (\( \mathcal{U}^{d, f}_{i,op} \)) but of full information. This makes team-optimization differing from classical vector optimization/control. Superscripts “cl”, “ol” denote the closed-loop and open-loop and “f” the full-information. We will address this point more detailed in Section 2. Hereafter, we may exchange the usage of \( \mathbf{u} = (u_1, \cdots, u_N) \in \mathbb{R}^{m \times N}, \mathbf{u} = (u^T_1, \cdots, u^T_N) \in \mathbb{R}^m \) and \( \mathbf{u} = (u_i, u_{-i}) \in \mathbb{R}^{m \times N} \) with \( u_{-i} = (u_1, \cdots, u_{i-1}, u_{i+1}, u_N) \in \mathbb{R}^{m \times (N-1)} \) by noting all of them represent team profile among all agents, but only differ in formations. For simplicity, we focus on Lagrange problem only, and no essential difficulty to Bolza problem extension.

By mean-field “team” (MT), we refer all weakly-coupled agents \( \{A_i\}_{i=1}^N \) are cooperative aiming to optimize the following social (or, team) cost functional (the related optimal functional is called mean-field team): \( J^{(N)}_{soc}(\mathbf{u}(\cdot)) = \sum_{i=1}^N J_i(\mathbf{u}(\cdot)) \). Due to the new framework, MT is different from mean-field control (MC) problem and mean-field games (MFG).

**MT vs. MC.** (i) MT aims to analyze a complex large-scale system including many cooperative coupled agents, while MC only concerns single agent with state distribution (or, mean) entering dynamics or cost. So, essentially, MT is for multi-agent system with decentralized information but MC only for single-agent with (of course) centralized information. Consequently, MT seeks some (joint) strategy but without information compilation across team members; by contrast, MC only involves single agent so naturally seeks control by own centralized information. (ii) Owning to information distinction above, analysis of MT and MC also proceed very differently. For MT, two crucial steps are variational decomposition and duality procedure to construct auxiliary problem for a representative agent. By comparison, MC analysis is rather straightforward, no need to invoke variation and duality since it involves single-agent and central-information only. In addition, MT essentially invokes some fixed-point argument but not needed in MC. (iii) Although in context of homogeneous model (i.e., all agents are symmetric), there exists some connection between MT and MC (to partial content) in analysis. However, such connection will no longer be valid for heterogenous model in presence of non-symmetric agents, especially with continuum heterogeneity.

**MT vs. MFG.** Furthermore, MT is also quite different from MFG. (i) Concept difference. Although both for large population system, MC is for cooperative agents towards a social-optima (Pareto) while MFG for non-cooperative agents to an Nash equilibrium. (ii) Analysis difference. By (i), MFG and MT analysis are very distinctive, especially for fixed-point arguments. In MFG, we can directly freeze state-average limit \( \lim_{N \rightarrow +\infty} x^{(N)} \) to construct cost functional of auxiliary problem, and derive consistency condition (CC) to complete fixed point argument. However, for MT, we cannot freeze \( x^{(N)} \) directly as in MFG. Instead, MT auxiliary functional must be specified in an indirect way. Roughly speaking, we should apply variational decomposition and weak duality, then auxiliary cost based on it, then fixed-point argument. Noting such pivotal variational decomposition is not needed at all for MFG because of non-cooperative nature. (iii) Moreover, verifications of above MT asymptotic social-optima and MFG asymptotic Nash equilibrium are also very distinctive. For example, due to cooperative structure, all agents \( A_1, \cdots, A_N \) in MT cooperate to minimize \( J^{(N)}_{soc}(\mathbf{u}(\cdot)) \), and \( u_{-i} \) (i.e., all decisions except \( A_i \)) cannot be viewed as endogenous terms as in MFG.
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Please refer [3, 5, 9, 11, 13, 31, 30] for some recent work on MFG and refer [8, 10, 18, 29, 35] for the limit relation between MFG and non-cooperate $N$-player games. The interested readers may refer e.g., [27, 34, 38], for detailed analysis comparison between MFG and MT, and [37, 39] for some recent study from various perspectives with different modeling variants. In particular, see [23] for MT with volatility uncertainty; [25] for linear-quadratic-Gaussian (LQG) mean-field social optimization with a major player; [30] for MFG with optimal investment under relative performance criteria; [33] for LQG games with a major player and continuum-parametrized minor players; and [40] for mean-field team in LQG models with Markov jump parameters.

Our work distinguishes itself from all above MT literature by the following fairly (even not the most) general formulation, in LQG context. Unlike (1), the individual dynamic of agent $A_i$ now takes:

$$\begin{align*}
\begin{cases}
    dx_i(t) & = [A_{\Theta_i}(t)x_i(t) + B(t)u_i(t) + F(t)x_i^{(N)}(t)]dt \\
    + [C(t)x_i(t) + D_{\Theta_i}(t)u_i(t) + \tilde{F}(t)x_i^{(N)}(t)]dW_i(t),
\end{cases}
\end{align*}$$

where $\{\Theta_i\}_{i=1}^N$ is a sequence of independent random variables which are also independent of $\{W_i(s), s \geq 0\}_{i=1}^N$ to represent diversity. The range of $\{\Theta_i\}_{i=1}^N$ is a (possibly continuum) subset in $\mathbb{R}^k$, hence our framework includes both finite diversity and continuum diversity. Please refer to Section 2 for more information. The admissible strategy set for $A_i$ is

$$U_i^{d,p} = \{u_i(\cdot) | u_i(\cdot) \in L^2_{G_i}(0, T; \Gamma) \}$$

where $G_i \subseteq F_i$ or $G_i \subseteq \mathbb{H}^i$ is a sub-filtration representing the partial information; $\Gamma \subseteq \mathbb{R}^m$ is a nonempty closed convex set representing the input constraint.

There are four main modeling features in formulation (3), (4):

(i) **Weakly-coupled controlled-diffusion.** It is remarkable that in (3), when $D_{\Theta_i} \neq 0$ so control process enters diffusion terms of LSDE, and when $\tilde{F} \neq 0$ so all individual states are weakly-coupled in diffusion terms also. In this case, we may call (3) to be diffusion-controlled and weakly-coupled. This differs from [27] in modeling that is only drift-controlled and weakly-coupled. Such modeling difference also brings considerable analysis distinctions, for example, on the relevant study of Hamiltonian systems, as well as consistency condition (CC) (see more comparison details in Section 3 and Section 6). Without loss of generality, no forcing terms such as $f, \sigma$ involve in (3).

(ii) **Random diversity.** Recall that (1) is homogenous since all agents are endowed with identical parameters thus they become symmetric. Subsequently, the (decentralized) optimal strategy and states, still denoted as $\{u_i\}_{i=1}^N$ and $\{x_i\}_{i=1}^N$, should turn to be exchangeable. By contrast, in (3), a random index $\Theta_i$ is introduced in parameter $A, D$ (also possible to be equipped on other parameters including the cost) to model the diversity across underlying large-scale system. All agents thereby become heterogenous. Although heterogenous large-scale system is already addressed in such as [21, 25], we point out in these works, the heterogenous index is technically treated as some realization after random sampling, along with necessary ordinal arrangements within each sub-classes. Thus, essentially the index therein is some deterministic realization. This differs substantially from our random index treatment here along with related analysis, to be highlighted later. In addition, our index $\Theta_i$ can assume a continuum support that distinguishes from most heterogenous literature with only finite/discrete support (see., e.g., [21, 25]). Moreover, although continuum heterogeneity is also discussed in e.g., [33], but analysis therein heavily relies upon the LQ structure with full input and resultant explicit representation. Such analysis collapses in current formulation...
(3), due to the intrinsic diffusion-controlled weakly-coupled feature introduced before, and an input constraint feature to be introduced below.

(iii) Input constraint. Note that a convex-closed set $\Gamma$ is introduced in (4) denoting some point-wise constraint in control input. Recall that such point-wise input constraint is well documented in e.g., [14, 16, 22, 32]. A typical example is $\Gamma = \mathbb{R}^+$ representing the positive control, or no-shorting constraint in portfolio selection ([32]). Other examples may include subspace ([16]) or a general convex cone ([22]). We remark that point-wise input constraint is also studied in large-scale/large-population context such as [20] but in competitive MFG setup, which differs from our cooperative MT here.

(iv) Partial information. Last but not least, the admissible control set is confined on a partial information set $L^2_G, (0, T; \Gamma)$. LQG control with partial information is also well documented (e.g., [41]). Also, partial information for large population system is also addressed recently (see [6, 7, 17, 24] for partial information/observation mean-field game). However, to our best knowledge, it is the first time to address partial information in mean-field team context. Notice that the partial information setting differs from that of partial observation ([4]) for which some filtering method with innovation process should be invoked. We defer more detailed information structure in Section 2 after more rigorous formulation.

To certain content, our aim in current work is to solve LQG MT problem in a rather general setup, by combining aforementioned features (i)-(iv) together. Although we admit various effective techniques have been already proposed to tackle these features individually, however their combination brings much more technical hurdles, and makes the associated analysis rather challenging. For example, the continuum heterogeneous large-scale system is well studied by [33] in mean-field game setup. Nevertheless, its parallel analysis variant to MT fails to work in current formulation because of the following reasoning. Due to controlled-diffusion feature (i), the related CC does not admit direct characterization because the adjoint process of some backward SDE should enter CC dynamics. Therefore, the direct augmented method in [39] fails to work here. Instead, some indirect embedding method [21, 37] becomes necessary in the presence of (i). Nevertheless, due to continuum heterogeneous feature (ii), the classical embedding CC in [21, 37] no longer works since we have to construct an infinite-dimensional Brownian motion-driven system (on continuum-valued space) to replicate the empirical distribution generated by controlled large-scale system. Meanwhile, the method in [37] is also not infeasible since it mainly relies on some closed-form representation of optimal state/cost. This becomes unavailable because of the input constraint (iii) imposed above.

In a nutshell, in case (i) or (iii) not combined together, we may still handle continuum heterogeneous MT with (ii) by modifying existing methods in e.g., [37]. However, combination of (i), (ii), (iii) together makes all such existing methods no longer workable. Other examples include the person-by-person procedure due to continuum heterogeneous (ii), and tailor-made decentralized strategy in presence of both point-wise constraint (iii) and partial information constraint (iv). To circumvent these difficulties, we propose some novel analysis techniques such as weak duality and modified embedding representation, etc. More analysis details are illuminated in Section 3 and Section 4.3.

Our main contributions can be sketched as follows: (1) First, we devise a new framework to unify homogenous and heterogenous (discrete or continuum) setups in large-scale system. In particular, it is enabled to transform heterogenous setup into a homogenous one, with the tradeoff of an augmented randomness. (2) Second, under such new framework, we derive a modified embedding representation of CC system (a crux in MT analysis) to accommodate the continuum diversities. (3) Third, the input constraint and partial information constraint are tackled both, and a CC system with double projection operator is derived. Specifically, the CC system takes a coupled mean-field type forward backward stochastic differential equations (FBSDEs) involving both projection mapping and conditional expectation. This seems quite
novel in large-scale literature. (4) Last, the well-posedness of CC system and asymptotic team optimality are established under mild conditions. Please refer Section 6 for detailed literature comparisons and discussions on homogeneity and heterogeneity.

The remaining of this paper is organized as follows. In Section 2, we give the formulation of LQG heterogeneous agents problem with input constraints and partial information pattern. In Section 3, we apply variational decomposition and weak duality to find the auxiliary control problem of the individual agent. The decentralized strategy and well-posedness of consistency condition is established in Section 4. Section 5 studies the asymptotic optimality of decentralized strategy. We give a synthetic analysis on homogeneity and heterogeneity and compare our framework with those in the current literature in Section 6.

2. Problem formulation. We first introduce some standard notations used throughout this paper. Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with the inner product denoted by \( \langle \cdot , \cdot \rangle \). \( \mathbb{R}^{n \times m} \) is the space of all \((n \times m)\) matrices, endowed with the inner product \( \langle M_1, M_2 \rangle = tr[M_1^T M_2] \), where \( x^T \) denotes the transpose of a matrix (or vector) \( x \) and \( tr \) is the trace of a matrix. \( M \in \mathbb{S}^n \) denotes the set of symmetric \( n \times n \) matrices with real elements. \( M > (\geq) 0 \) denotes that \( M \in \mathbb{S}^n \) which is positive (semi)definite, while \( M \gg 0 \) denotes that, \( M - \varepsilon I \geq 0 \) for some \( \varepsilon > 0 \).

Assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space on which \( \{W_i(t), 0 \leq t \leq T\}_{i=1}^N \) is a \( N \)-fold Brownian motion (note here \( W_i \) might be vector-valued, say, including a common noise component \( W_0 \)) and \( \{\Theta_i\}_{i=1}^N \) is a sequence of independent random variables to represent diversity. In some sense, we may interpret \( \{\Theta_i\} \) as some endogenous randomness, while \( \{W_i\} \) some exogenous randomness for generic agent \( A_i \). Moreover, we assume \( \{\Theta_i\}_{i=1}^N \) are also independent of \( \{W_i(s), s \geq 0\}_{i=1}^N \). Let \( \{\mathcal{F}_t^W\}_{0 \leq t \leq T} \) be the filtration generated by \( \{W_i(s), 0 \leq s \leq t\}_{i=1}^N \) and define \( \mathcal{F}_t^{W, \Theta} = \sigma(\Theta_i, 1 \leq i \leq N) \lor \mathcal{F}_t^W \). The set of null sets on \( \Omega \) is defined by \( \mathcal{N}_\mathbb{P} = \{M \in \Omega | \exists G \in \mathcal{F}_\infty^{W, \Theta} \text{ with } M \subset G \text{ and } \mathbb{P}(G) = 0\} \). Consider the augmented filtration \( \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T} \) with \( \mathcal{F}_t = \sigma(\mathcal{F}_t^{W, \Theta} \cup \mathcal{N}_\mathbb{P}) \). Then \( \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T} \) represents the centralized information including all Brownian motions (BMs) and diversity index components across all agents (principal and marginals).

For any Euclidean space \( \mathbb{V}, 1 \leq p < \infty \), and \( T > 0 \), introduce the following spaces:

- \( L^p_{\mathbb{F}, \mathcal{V}}(\Omega; \mathbb{V}) := \{\eta : \Omega \rightarrow \mathbb{V} | \eta \text{ is } \mathcal{F}_T \text{-measurable such that } \mathbb{E}[|\eta|^p] < \infty\} \).
- \( L^{\infty}(0, T; \mathbb{V}) := \{\varphi(\cdot) : [0, T] \rightarrow \mathbb{V} | \text{esssup}_{0 \leq s \leq T} |\varphi(s)| < \infty\} \).
- \( L^p_p(0, T; \mathbb{V}) := \{\varphi(\cdot) : \Omega \times [0, T] \rightarrow \mathbb{V} | \text{progressively measurable such that } \mathbb{E} \int_0^T |\varphi(s)|^p ds < \infty\} \).

We consider a weakly coupled large population system of heterogeneous agents \( \{A_i : 1 \leq i \leq N\} \) with the dynamics of the agents given in (3), and cost functional (2). For sake of presentation, we restate them as follows:

\[
\begin{align*}
    dx_i &= [A_{\Theta,i}x_i + Bu_i + Fx^{(N)}]dt + [C_{\Theta,i}u_i + \bar{F}x^{(N)}]dW_i, x_i(0) = \xi \in \mathbb{R}^n, \\
    J_i(u(\cdot)) &= \frac{1}{2} \mathbb{E} \int_0^T \left( \langle Q(x_i - Hx^{(N)}), x_i - Hx^{(N)} \rangle + \langle Ru_i, u_i \rangle \right) dt, 1 \leq i \leq N.
\end{align*}
\]

As mentioned before, state (3) and functional (2) formulate a weakly coupled large-scale system with heterogeneous agents. The aggregate team functional of \( N \) agents is \( J^{soc}_{\Theta}(u(\cdot)) = \sum_{i=1}^N J_i(u(\cdot)) \). In (5), \( A_{\Theta,i}(\cdot), B(\cdot), C(\cdot), D_{\Theta,i}(\cdot), F(\cdot), \bar{F}(\cdot) \) are called the state-coefficient datum, while \( \langle Q(\cdot), H(\cdot), R(\cdot) \rangle \) the cost weight datum. We explain more details for above datum. \( F, \bar{F} \) are weakly-coupling coefficients on state-drift and state-diffusion respectively; \( H \) is weakly-coupling coefficient on functional; \( C, D_{\Theta,i} \) are diffusion state-dependence and diffusion dependence coefficients respectively. Note that \( D_{\Theta,i} \neq 0 \) represents the case
when control enters diffusion alike the risky portfolio selection (e.g., [22, 32, 43]); $F, \tilde{F} \neq 0$ denotes the agents are coupled in the dynamics such as the price formation problem (e.g., [19, 28]); $H \neq 0$ denotes the relative performance formulation (e.g., [16]).

Unlike state (1), we introduce $\{\Theta_i\}_{i=1}^N$ in (3) as some diversity index to characterize the possible heterogenous features among all agents. We point out that $\Theta_i$ may be vector-valued on a Cartesian grid space, say $[a_1, b_1] \times [a_2, b_2]$ or $[a_1, b_1] \times \{1, \cdots, K\}$, to represent various feature dimensions, either in continuum space or discrete space, or in a hybrid manner.

For simplicity, we only assume that the coefficients $A$ and $D$ to be dependent on $\Theta_i$. Similar analysis can be generalized to the case when all other coefficients are also $\Theta_i$-dependent. Besides, in what follows the time variable $t$ will usually be suppressed if no confusion occurs.

We now introduce the following assumption on distribution and coefficient datum set:

(A1) For $i = 1, \cdots, N$, $\Theta_i : \Omega \to S$ are independent identically distributed (i.i.d) with the distribution function $\Phi(\theta)$, i.e., $\int_S d\Phi(\theta) = 1$, where $S$ is a (possibly continuum) subset in Cartesian space $\mathbb{R}^k$. Note that the discrete set, i.e., finite diversity, is a special example.

(A2) For any $\theta \in S$, $A_\theta(\cdot), F(\cdot), C(\cdot), \tilde{F} \in L^\infty(0, T; \mathbb{R}^{n \times n})$, $B(\cdot), D_\theta(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times m})$, $Q(\cdot) \in L^\infty(0, T; \mathbb{P}^n)$, $H(\cdot) \in L^\infty(0, T; \mathbb{S}^n)$, $R(\cdot) \in L^\infty(0, T; \mathbb{S}^m)$.

(A3) $Q(\cdot) \geq 0$, $R(\cdot) \gg 0$.

**Remark 2.1.** We remark that discrete- or finite-valued $\Theta_i$ might be transformed into continuum one by assigning uniform distribution on compact interval along with given partitions. Indeed, this is equivalent to simulate a given discrete random variable using quantile method by uniform distribution. Thus, hereafter we focus on vector-valued index $\Theta_i$ on $\mathbb{R}^k$.

Under assumptions (A1)-(A2), the state (3) admits a unique strong solution $x(\cdot) = (x_1(\cdot), \cdots, x_N(\cdot)) \in L^2_\mathbb{P}(0, T; \mathbb{R}^{N \times n})$, and the cost functional is well defined for each admissible control strategy $u(\cdot)$ on appropriate admissible space, to be detailed soon. Moreover, under assumption (A3), the cost functional is uniform convex, that is, there exists some $\delta > 0$ such that $\mathcal{J}^{(N)}(u) \geq \delta \mathbb{E} \int_0^T |u(s)|^2 ds$.

Given state (3) and functional (2), we can specify the associated information structures. Because of interactive coupling by state-average $x^{(N)} := \frac{1}{N} \sum_{i=1}^N x_i$, $\mathcal{F}_i(u_i, u_{-i})$ depends on total team-decision $u = (u_i, u_{-i})$. In this sense, (3) exhibits the so-called weakly interactive coupling in decision when $N \to +\infty$. Again, by such interactive coupling, information structure of (3) becomes more involved:

- **Decentralized, open-loop information:** consider the filtration $\mathcal{F}^{W_i} = \sigma(W_i(s), 0 \leq s \leq t)$, $\mathcal{F}^{W_i, \Theta_i} = \sigma(\Theta_i) \vee \mathcal{F}^{W_i}$, $0 \leq t < \infty$, and the set of null sets $\mathcal{N} = \{M \in \Omega \mid \exists G \in \mathcal{F}^{W_i, \Theta_i}$ with $M \subset G$ and $\mathbb{P}(G) = 0\}$, and create the augmented filtration $\mathbb{F}^i = \{\mathcal{F}^i_t\}_{0 \leq t \leq T}$ with $\mathcal{F}^i_t = \sigma(\mathcal{F}^{W_i, \Theta_i} \cup \mathcal{N}^i)$. Note that $\{\mathcal{F}^i_t\}$ only depends on $W_i$ and $\Theta_i$ instead of state $x_i$ itself, thus we call it open-loop (although it also differs from classical open-loop due to mean-field nature) information since it depends directly on underlying randomness.

- **Decentralized, closed-loop information:** denote by $\{\mathcal{H}^i_t\}_{0 \leq t \leq T}$ the filtration of individual state $x_i$ augmented by $\mathcal{N}^i$, i.e., $\mathcal{H}^i_t = \sigma\{x_i(s), 0 \leq s \leq t\} \cup \mathcal{N}^i$. Note that $\{\mathcal{H}^i_t\}$ only depends on underlying principal state $x_i$ itself, thus we call it closed-loop (although it also differs from classical closed-loop due to mean-field nature). We remark that $x_i$ is not adapted to $\mathbb{F}^i$ due to weakly coupling.

- **Decentralized, partial information:** Let $\mathcal{G}^i \subseteq \mathcal{F}^i_t$ be a sub-$\sigma$-field of $\mathcal{F}^i_t$ (or, $\mathcal{G}^i \subseteq \mathcal{H}^i_t$ be a sub-$\sigma$-field of $\mathcal{H}^i_t$), then $\mathbb{G}^i = \{\mathcal{G}^i_t\}_{0 \leq t \leq T}$ represents the decentralized open-loop (or closed-loop) partial information available to $A_i$. 
Remark 2.2. For decentralized, partial information pattern, $G_i^t$ is a given filtration representing the information available to $A_i$ at time $t$. For example, $G_i^t = F_{i(t)}^{(t-\delta)+}$, or $G_i^t = H_{i(t-\delta)+}^i$, $t \in [0,T]$, where $\delta > 0$ denotes the fixed delay of information. In this case, $G_i^t$ represent the partial information in open-loop or closed-loop sense, respectively. Another example is that $W_i = (\tilde{W}_i, \tilde{W}_0)$ takes vector-valued Brownian motion including a common noise component $\tilde{W}_0$, then $G_i^t = \sigma\{\tilde{W}_i(s), \Theta_i, 0 \leq s \leq t\}$ denotes the partial information in open-loop. Also, in case $\Theta_i = (\Theta_{i1}, \Theta_{i2})$, then $G_i^t = \sigma\{W_i(s), \Theta_i, 0 \leq s \leq t\}$ denotes the partial information to underlying diversity.

Therefore, $B_i^t = F_i^t \bigcup H_i^t$ and $B_i^t := \{B_i^t\}_{0 \leq t \leq T}$ represents (full) decentralized information. Then we have the following structure inclusion chart:

\[
G^i \subset \{F^i(\text{decentralized open-loop}), \ H^i(\text{decentralized closed-loop})\} \subset B^i(\text{decentralized}) \subset F(\text{full}).
\]

Noticing due to state-average $x^{(N)}_i$, $x_i(t) \notin F_i^t$, thus, NO inclusion relations between open-loop $F = \{F_i^t\}_{0 \leq t \leq T}$ and closed-loop $H^i = \{H_i^t\}_{0 \leq t \leq T}$. This is different to classical control where the open-loop information includes closed-loop information. Given information structure, we are ready to formulate the relevant admissible control sets:

- Centralized full-information set: $U_{i}^{c,f} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{F}}(0, T; \Gamma)\}$.
- Decentralized full-information open-loop set: $U_{i, op}^{d,f} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{F}}(0, T; \Gamma)\}$.
- Decentralized full-information closed-loop set: $U_{i, cl}^{d,f} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{H}}(0, T; \Gamma)\}$.
- Decentralized partial-information set: $U_{i}^{d,p} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathbb{G}^i}(0, T; \Gamma)\}$.

We point out here $G^i$ is general to include both open-loop or closed-loop partial information. Now we propose the following optimization problem:

**Problem LQG-MT.** Find a team strategy set $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \ldots, \bar{u}_N(\cdot))$ where $\bar{u}_i(\cdot) \in U_i^{c,f}$, $1 \leq i \leq N$, such that

\[
J_{soc}^{(N)}(\bar{u}(\cdot)) = \inf \limits_{u_i \in U_i^{c,f}, 1 \leq i \leq N} J_{soc}^{(N)}(u_1(\cdot), \ldots, u_i(\cdot), \ldots, u_N(\cdot)).
\]

Under some mild conditions on datum $(Q, R)$ (e.g., (A3)), it is possible to ensure the existence and uniqueness of optimal mean-field team strategy in a centralized sense. This can be proceeded by classical vector-optimization or control method but in a high-dimension setting because of the existence of large number of weakly-coupled team agents. However, such strategy, from a computational viewpoint, turns to be intractable because of the information requirement to collect all agents’ states simultaneously. Instead, it is more tractable to consider some decentralized strategy for which only the local (distributed) information for given agent is needed. Moreover, considering the partial information pattern, we introduce the following definition on asymptotic social optimality.

**Definition 2.3.** A strategy set $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \ldots, \bar{u}_N(\cdot))$ with $\{\bar{u}_i \in U_i^{d,p}\}_{i=1}^N$ is said to be $\varepsilon$-social optimal if there exists $\varepsilon = \varepsilon(N) > 0$, $\lim \limits_{N \to +\infty} \varepsilon(N) = 0$ such that

\[
\frac{1}{N} (J_{soc}^{(N)}(\bar{u}(\cdot)) - \inf \limits_{u \in U_i^{c,f}} J_{soc}^{(N)}(u(\cdot))) \leq \varepsilon.
\]

Remark 2.4. In Remark 2.2, we emphasize $W_i$ might be vector-valued Brownian motion including a common noise component. For simplicity, in the following we assume that $W_i$, $i = 1, \ldots, N$ are independent one-dimensional Brownian motions. Note that for the case
\[ W_i = (\tilde{W}_i, \tilde{W}_0) \] takes vector-valued Brownian motion including a common noise component \( \tilde{W}_0 \) and \( \tilde{W}_i, i = 1, \ldots, N \) being independent one-dimensional Brownian motions, the procedures in Section 3 and Section 4 are still workable. However, in this case \( \mathbb{E}\alpha \) in (26) should be the conditional expectation \( \mathbb{E}[\alpha | F^0_1] \) where \( \{ F^0_t \} \) is the filtration generated by the common noise \( \tilde{W}_0 \). For this kind of consistency system, please refer [21] for more information.

As discussed before, centralized strategy based on classical vector optimization/control turns out to be ineffective to handle weakly-coupled but highly complex LQG-MT. Alternatively, it is more amenable to construct some decentralized strategy on distributional information only. Such strategy construction can be implemented using mean-field team analysis through the following steps:

- **Step 1 (§3.1):** applying variational decomposition for generic agent;
- **Step 2 (§3.2):** applying weak duality to construct auxiliary control (AC) problem;
- **Step 3 (§4):** solving AC to determine limiting state-average by consistency condition (CC);
- **Step 4 (§5):** verifying the asymptotic social optimality of derived decentralized team strategy.

We now proceed step by step to construct the distributed LQG-MT strategy.

### 3. Mean-field team analysis.

Our current work focuses on team optimization, so a variational analysis becomes unavoidable to calibrate response of related componentwise Fréchet differentials for a generic agent, say \( A_i \). Such an analysis is not required in MFG as all agents there are non-cooperative. Hence, unlike MFG, we need to quantify the total variation of social cost \( \delta \mathcal{J}^{(N)}_{soc}(\delta u_i) \) triggered by individual variation \( \delta u_i \) of \( A_i \).

#### 3.1. Variational decomposition.

In order to quantify (total) variation \( \delta \mathcal{J}^{(N)}_{soc}(\delta u_i) \) owning to basic \( \delta u_i \) by a generic \( A_i \), we need to compute variation of social cost when \( A_i \) adopts an alternative strategy while all others’ decisions keep unchanged. Subsequently, in Step 1, we would like to figure out a variational decomposition for original (5) around centralized strategy (although we prefer to circumvent its direct but high-dimensional computation). The variational decomposition consists of three sub-steps, as detailed below.

**3.1.1. Decomposition of \( \delta \mathcal{J}^{(N)}_{soc}(\delta u_i) \).**

First we will obtain \( \delta \mathcal{J}^{(N)}_{soc}(\delta u_i) \) when \( A_i \) uses an alternative strategy. Let \( \{ \tilde{u}_i \in \mathcal{U}_i \} \) be centralized optimal team strategy (its existence can be ensured under some mild convexity conditions). But, as discussed above, such strategies are intractable for real computation purpose because of “curse of dimensionality”). Now consider the perturbation for given benchmark agent, say, \( A_i \) use the alternative strategy \( u_i \in \mathcal{U}_i \) and all other agents still apply the strategy \( \tilde{u}_{-i} = (\tilde{u}_1, \ldots, \tilde{u}_{i-1}, \tilde{u}_{i+1}, \ldots, \tilde{u}_N) \). The realized state (3) corresponding to \( (u_i, \tilde{u}_{-i}) \) and \( (\tilde{u}_i, \tilde{u}_{-i}) \) are denoted by \( (x_1, \ldots, x_N) \) and \( (\tilde{x}_1, \ldots, \tilde{x}_N) \), respectively. We denote agent index set as \( I = \{ 1, \ldots, N \} \). To start the variational decomposition, it is helpful to present the following causal-relation flow-chart first:

\[
\begin{align*}
\delta u_i &= u_i - \tilde{u}_i & \Rightarrow \quad \delta x_i &= x_i(u_i) - \tilde{x}_i(\tilde{u}_i) & \Rightarrow \quad \delta x_{j,i} &= x_{j,i}(x_i) - \tilde{x}_{j,i}(\tilde{x}_i) \\
\text{principal basic variation} & & \text{principal intermediate variation} & & \text{marginal variation} \\
\Rightarrow \quad \delta \mathcal{J}_{soc}(\delta u_i) &= \mathcal{J}_j(u_i, \tilde{u}_{-i}) - \mathcal{J}_j(\tilde{u}_i, \tilde{u}_{-i}), j = 1, \ldots, N, \\
\text{marginal cost variation} & & & & \\
\Rightarrow \quad \delta \mathcal{J}^{(N)}_{soc}(\delta u_i) &= \mathcal{J}^{(N)}_{soc}(u_i, \tilde{u}_{-i}) - \mathcal{J}^{(N)}_{soc}(\tilde{u}_i, \tilde{u}_{-i}),
\end{align*}
\]

where \( \mathcal{J}_j \) is the cost functional with respect to the component \( j \).
where \( \delta u_i \) is the most basic variation “block” for other variation structures; we write \( x_i(u_i) \) to emphasize its dependence of \( x_i \) on \( u_i \), and similar for \( \bar{x}_i(u_i) \); we call \( \delta x_i \) the principal intermediate variation as it depends indirectly on basic \( \delta u_i \) via principal state; similarly, \( \delta x_{j,i} \) marginal variations from point of \( A_i \); also \( x_i(x_i) \) depends on \( x_i \) via weak-coupling \( x^{(N)} \), similar to \( \bar{x}_j(x_i) \). Note that the subscripts of \( \delta x_{j,i} \) means that \( x_i \) is the principal state while \( x_j, j \neq i, \) are marginal ones, from viewpoint of \( A_i \). Moreover, from standpoint of \( A_i \), the variational equations for principal state \( x_i \), and marginal states \( \{ x_j \}_{j \neq i} \) satisfy:

\[
\begin{align*}
(6) \quad d\delta x_i &= [A \Theta, \delta x_i + B \delta u_i + F \delta x^{(N)}]dt + [C \delta x_i + D \Theta, \delta u_i + \bar{F} \delta x^{(N)}]dW_i, \delta x_i(0) = 0, \\
(7) \quad j \neq i, \quad d\delta x_{j,i} &= [A \Theta, \delta x_{j,i} + F \delta x^{(N)}]dt + [C \delta x_{j,i} + \bar{F} \delta x^{(N)}]dW_j, \delta x_{j,i}(0) = 0.
\end{align*}
\]

Denote \( \delta x_{-i} = \sum_{j \neq i} \delta x_{j,i} \), the aggregate variation of marginal agents (benchmark to \( A_i \)), so applying linear state-aggregation,

\[
(8) \quad d\delta x_{-i} = \sum_{j \neq i} A \Theta, \delta x_{j,i} + (N - 1) F \delta x^{(N)}]dt + \sum_{j \neq i} [C \delta x_{j,i} + \bar{F} \delta x^{(N)}]dW_j, \delta x_{-i}(0) = 0.
\]

Similarly, we can obtain the variation of cost functionals as follows. For principal cost of \( A_i \):

\[
\delta J_i(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q(\bar{x}_i - H\bar{x}^{(N)}), \delta x_i - H\delta x^{(N)} \rangle + \langle R\bar{u}_i, \delta u_i \rangle \right] dt.
\]

For marginal costs of \( A_j \):

\[
\delta J_{j,i}(\delta u_i) = \mathbb{E} \int_0^T \langle Q(\bar{x}_j - H\bar{x}^{(N)}), \delta x_{j,i} - H\delta x^{(N)} \rangle dt, \quad j \neq i.
\]

Therefore, the total variation of social cost, from person-by-person variation of \( A_i \), becomes

\[
(9) \quad \delta J_{soc}^{(N)}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q(\bar{x}_i - H\bar{x}^{(N)}), \delta x_i - H\delta x^{(N)} \rangle + \sum_{j \neq i} \langle Q(\bar{x}_j - H\bar{x}^{(N)}), \delta x_{j,i} - H\delta x^{(N)} \rangle \right.
\]

\[
+ \langle R\bar{u}_i, \delta u_i \rangle \right] dt
\]

\[
= \mathbb{E} \int_0^T \left[ \langle Q(\bar{x}_i, \delta x_i) - \langle QH\bar{x}^{(N)}, \delta x_i \rangle - \langle QN\bar{x}^{(N)}, H\delta x^{(N)} \rangle + \langle QH\bar{x}^{(N)}, H\delta x^{(N)} \rangle \right.
\]

\[
+ \sum_{j \neq i} \langle Q(\bar{x}_j, \delta x_{j,i}) - \sum_{j \neq i} \langle QH\bar{x}^{(N)}, \delta x_{j,i} \rangle + (N - 1) \langle QH\bar{x}^{(N)}, H\delta x^{(N)} \rangle + \langle R\bar{u}_i, \delta u_i \rangle \rangle \right] dt
\]

\[
= \mathbb{E} \int_0^T \left[ \langle Q(\bar{x}_i, \delta x_i) - \langle (QH + HQ - HQH)\bar{x}^{(N)}, \delta x_i \rangle \right.
\]

\[- \langle (QH + HQ - HQH)\bar{x}^{(N)}, \sum_{j \neq i} \delta x_{j,i} \rangle + \sum_{j \neq i} \langle Q(\bar{x}_j, \delta x_{j,i}) + \langle R\bar{u}_i, \delta u_i \rangle \rangle \right] dt
\]

\[=: I_1 + I_2 + I_3 + I_4 + I_5. \]

There arise five decomposition terms in (9). Among them, \( I_5 \) depends directly on the principal basic variation \( \delta u_i \), whereas \( I_1, I_2 \) depend on principal intermediate variation \( \delta x_i \) that further depends on the basic \( \delta u_i \). Moreover, \( I_3, I_4 \) depend on the marginal variations \( \{ \delta x_{j,i} \}_{j \neq i} \) that further depends on the principal ones \( \delta x_i, \delta u_i \). We denote \( ||\delta x_i||_{L^2} = \)
(\mathbb{E} \int_0^T |\delta x_i|^2 ds)^{1/2}$. By standard SDE estimation, $||\delta x_i||_{L^2} \leq (K + O(N^{-\frac{1}{2}}))||\delta u_i||_{L^2}$ where $K$ is independent on $N$, and only depends on coefficients of (3). Moreover, $||\delta x_{j,i}||_{L^2} = O(N^{-\frac{1}{2}})||\delta u_i||_{L^2}$ for $j \neq i$. Also, note that in general, it is not true $||\delta u_i||_{L^2} = O(||\delta x_i||_{L^2})$.

Noting in (9), only $I_1$ and $I_5$ directly depend on (basic) principal variations $\delta u_i, \delta x_i$ whereas $I_2, I_3, I_4$ are intermediate with indirect dependence ($\tilde{x}^{(N)}$, $\delta x_{j,i}$). Thus, the reformulation below is invoked, in spirit of mean-field approximation, to get rid of such implicit dependence in $I_2, I_3, I_4$ progressively.

3.1.2. Reformulation of $I_2, I_3$. For $I_2, I_3$, we need to approximate the empirical state-average $\bar{x}^{(N)}$ by its mean-field limit using heuristic reasoning. Therefore, replacing $\bar{x}^{(N)}$ of $I_2, I_3$ in (9) by state-average limit $\bar{x}$ (to be determined later in Step 3) will yield

$$
\delta J_{soc}^{(N)}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q \bar{x}_i, \delta x_i \rangle - \langle (HQ + HQH) \bar{x}, \delta x_i \rangle - \langle (HQ + HQH) \bar{x}, \delta x_i \rangle \right] dt + \epsilon_1
$$

(10) : $I_1 + \tilde{I}_2 + \tilde{I}_3 + I_4 + I_5 + \epsilon_1$,

where

$$
\epsilon_1 = \mathbb{E} \int_0^T \langle (HQ + HQH) \bar{x}, N \delta x^{(N)} \rangle dt.
$$

Note that terms $I_1, \tilde{I}_2, I_5$ in (10) already depend on the principal variations $\delta u_i$ or $\delta x_i$. Thus, we only need to analyze the limiting behavior for remaining term $\tilde{I}_3$ and $I_4$. It is remarkable that $\tilde{I}_3, I_4$ respectively involve components: $\delta x_{-i}$ and $\frac{1}{\sqrt{N}} \sum_{j \neq i} \langle Q \bar{x}_j, N \delta x_{j,i} \rangle$ that both depend on principal basic $\delta u_i$ in rather implicit manner.

3.1.3. Reformulation of $\tilde{I}_3, I_4$. Note that for $j \neq i$, $||\delta x_{j,i}||_{L^2} = O(N^{-\frac{1}{2}})||\delta u_i||_{L^2}$, so we need to introduce some limiting term $x_j^*$ to replace the re-scaled $N \delta x_{j,i}$ in rate $||x_j^* - N \delta x_{j,i}|| = O(N^{-\frac{1}{2}})||\delta u_i||_{L^2}$. This helps us to deal with variation of $I_4$. Furthermore, we introduce limiting term $x_{**}^* = \int_S x_{**}^* d\Phi(\theta)$ to replace $\delta x_{-i}$ in rate $||x_{**}^* - \delta x_{-i}|| = O(N^{-\frac{1}{2}})||\delta u_i||_{L^2}$. This will help us to deal with $\tilde{I}_3$. Moreover, by the independence between $\{\Theta_j\}, \{W_j\}$ and heuristic mean-field arguments, we construct the following coupled system:

$$
\left\{
\begin{aligned}
&dx_j^* = [A_{j\Theta} x_j^* + F \delta x_i + F \int_S x_{**}^* d\Phi(\theta)] dt + [C x_j^* + \overline{F} \delta x_i + \overline{F} \int_S x_{**}^* d\Phi(\theta)] dW_j, \\
&dx_{\theta}^* = [A_{\theta} x_{\theta}^* + F \delta x_i + F x_{**}^*] dt, \quad x_{**}^*(0) = 0, \\
&x_j^*(0) = 0, \quad j \neq i, \quad \theta \in S.
\end{aligned}
\right.
$$

(11)
where
\[
\noting \tilde{x}_i \notas \text{number to identify the related average. We present some weak duality approach to break down dependence on some tail sigma-algebra. Also, it is observable that such tail sigma-algebra (12) connects to a sequence of exchangeable random variables \( \{ f_0^T Q \tilde{x}_j, x^*_j \} dt \) brings forth the following adjoint equations
\[
\begin{cases}
\delta J_{soc}^{(N)} (\delta u_i) = \mathbb{E} \int_0^T \left( \langle Q \tilde{x}_i, \delta x_i \rangle - \langle (Q H + H Q H) \dot{x}, \delta x_i \rangle - \langle (Q H + H Q) \dot{x}, x^* \rangle + \frac{1}{N} \sum_{j \neq i} \langle Q \tilde{x}_j, x_i^* \rangle + \langle R u_i, \delta u_i \rangle \right) dt + \sum_{l=1}^3 \varepsilon_l \\
= I_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + I_5 + \sum_{l=1}^3 \varepsilon_l,
\end{cases}
\]
\]
(12) where
\[
\varepsilon_2 = \mathbb{E} \int_0^T \langle (Q H + H Q - H Q H) \dot{x}, x^* - \delta x_{-i} \rangle dt,
\]
\[
\varepsilon_3 = \mathbb{E} \int_0^T \frac{1}{N} \sum_{j \neq i} \langle Q \tilde{x}_j, N \delta x_{j,i} - x^*_j \rangle dt.
\]
Noting \( \tilde{I}_4 \) of (12) connects to a sequence of exchangeable random variables \( \{ f_0^T Q \tilde{x}_j, x^*_j \} dt \) \( \in L^1_{F_{\tilde{I}_4}} (\Omega; \mathbb{R}) \). By de Finetti theorem, they are conditionally independent identically distributed with respect to some tail sigma-algebra. Also, it is observable that such tail sigma-algebra should depend on \( \delta x_i \) in rather implicit way. Then, we may apply conditional law of large number to identify the related average. We present some weak duality approach to break away \( \delta J_{soc}^{(N)} (\delta u_i) \) from dependence on \( x^*_j \) and \( x^{**} \).

3.2. Weak duality. Although (12) is free of the marginal \( \{ \delta x_{j,i} \}_{j \neq i} \) but a trade-off is the raised limiting quantities \( x^*_j \) and \( x^{**} \) that are still intermediate. For auxiliary construction, we need to further eliminate them via some duality. Due to continuum heterogeneity and state-coupling dynamics, pertinent duality will take a fairly complex argument, and heavily depend on some weak equivalence in distributional rather than (strong) pathwise sense. Thus, we term it as “weak” duality procedure, as detailed below. More specifically, in order to break away \( \delta J_{soc}^{(N)} (\delta u_i) \) of (12) from direct dependence on \( x^*_j \) and \( x^{**} \) (see \( \tilde{I}_3, \tilde{I}_4 \)), we introduce the following adjoint equations \( \{ y_1^j \}_{j \neq i} \) and \( y_2^\theta \) satisfying:
\[
\begin{cases}
dy_1^j = \alpha_1^j dt + \beta_1^{jj} dW_j + \sum_{l=1, l \neq j}^N \beta_1^{jl} dW_l, \quad y_1^j (T) = 0, \quad j \neq i, \\
dy_2^\theta = \alpha_2^\theta dt, \quad y_2^\theta (T) = 0, \quad \theta \in S,
\end{cases}
\]
(13) where \( \{ W_l \}_{l \neq i} \) are some Brownian motion copies matching all marginal agents in large-scaled system, from the benchmark point of \( A_i \). We remark that \( y_2^\theta \) is parametrized by diversity index in continuum support: \( \theta \in S \), while \( y_1^j \) is parametrized by marginal agent index \( j \in T \setminus \{ i \} \). Accordingly, the duality below should be in distributional and agent-wise sense, respectively indexed by \( \theta \in S \) and \( j \in T \). To start, first apply Itô’s formula to \( \langle y_1^j, x^*_j \rangle \) for each marginal agent index \( j \neq i \), and take expectation, by countable agent-wise addition for all \( j \in T \setminus \{ i \} \),
\[
0 = \mathbb{E} \int_0^T \left[ \frac{1}{N} \sum_{j \neq i} \langle \alpha_1^j + A_0^T y_1^j + C^T \beta_1^{jj}, x^*_j \rangle + \frac{1}{N} \sum_{j \neq i} \langle \bar{F}^T y_1^j + \bar{F}^T \beta_1^{jj}, x^{**} \rangle + \frac{1}{N} \sum_{j \neq i} \langle \bar{F}^T y_1^j + \bar{F}^T \beta_1^{jj}, \delta x_i \rangle \right] dt.
\]
(14)
Similarly, by distributed integral on all $\theta \in S$,

$$0 = \int_0^T \left[ \int_S (\alpha_1^\theta y_1^\theta + A_\delta^\theta y_2^\theta + F^\top y_2^\theta, x_0^{**}) d\Phi(\theta) + \int_S (F^\top y_2^\theta, \delta x_i) d\Phi(\theta) \right] dt. \quad (15)$$

Combing (14) and (15) with (12)

$$\delta J_{soc}^{[N]}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q \tilde{x}_i, \delta x_i \rangle - \langle (QH + HQ - HQH) \tilde{x}, \delta x_i \rangle - \frac{1}{N} \sum_{j \neq i} \langle F^\top y_1^j + F^\top \beta_1^{ij}, \delta x_i \rangle 
$$

$$- \int_S (F^\top y_2^\theta, \delta x_i) d\Phi(\theta) + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt + \mathbb{E} \int_0^T \left[ \frac{1}{N} \sum_{j \neq i} \langle Q \tilde{x}_j - \alpha_1^j - A_{\Theta} y_1^j - C^\top \beta_1^{ij}, x_0^{**} \rangle \right] dt
$$

$$- \mathbb{E} \int_0^T \int_S \langle (QH + HQ - HQH) \tilde{x} - F^\top E y_1^1 - F^\top F y_2^1, \delta x_i \rangle d\Phi(\theta) dt$$

$$+ \sum_{l=1}^3 \varepsilon_l. \quad (16)$$

Let

$$\begin{cases} 
\alpha_1^1 = Q \tilde{x}_j - A_{\Theta}^\top y_1^j - C^\top \beta_1^{ij}, \\
\alpha_2^j = -(QH + HQ - HQH) \tilde{x} - F^\top E y_1^j - F^\top F y_2^j - A_{\Theta} y_2^j - F^\top y_2^j,
\end{cases}$$

hence we reach the following weak duality adjoint process:

$$\begin{cases} 
y_1^j = \left( Q \tilde{x}_j - A_{\Theta}^\top y_1^j - C^\top \beta_1^{ij} \right) dt + \beta_1^{ij} dW_j + \sum_{l=1, l \neq j}^N \beta_l^{ij} dW_l, \\
y_2^j = -\left( (QH + HQ - HQH) \tilde{x} - (F^\top E y_1^j + F^\top F y_2^j) - A_{\Theta} y_2^j - F^\top y_2^j \right) dt,
\end{cases} \quad (17)$$

We point out that above system can be rewritten as

$$\begin{cases} 
y_1^j = \left( Q \tilde{x}_j - A_{\Theta}^\top y_1^j - C^\top \beta_1^{ij} \right) dt + \beta_1^{ij} dW_j + \sum_{l=1, l \neq j}^N \beta_l^{ij} dW_l, \\
y_2^j = -\left( (QH + HQ - HQH) \tilde{x} - (F^\top E y_1^j + F^\top F y_2^j) - A_{\Theta} y_2^j - F^\top y_2^j \right) dt,
\end{cases}$$

$$y_1^j(T) = 0, j \neq i, y_2^j(T) = 0, \theta \in S.$$ 

We remark that $y_2^\Theta$ is a degenerate BSDE by noting $\Theta \in F_0$. Also, it is not necessary to specify any dependence assumption between $\Theta$ and $\Theta$ since $y_1^j$ and $y_2^\Theta$ get coupled only through expectation operator. In other words, the coupling and associated consistency condition only concern their expectations. Still, we may term the resulting duality as weak duality. Substituting (17) into (16), we have

$$\delta J_{soc}^{[N]}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q \tilde{x}_i, \delta x_i \rangle - \langle (QH + HQ - HQH) \tilde{x}, \delta x_i \rangle - \frac{1}{N} \sum_{j \neq i} \langle F^\top y_1^j + F^\top \beta_1^{ij}, \delta x_i \rangle 
$$

$$- \int_S (F^\top y_2^\theta, \delta x_i) d\Phi(\theta) + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt + \sum_{l=1}^4 \varepsilon_l,$$
where
\[
\varepsilon_4 = \mathbb{E} \int_0^T \left( F^\top (\mathbb{E}[y'_1]) - \frac{1}{N} \sum_{j \neq i} y'_1(j) \right) + \bar{F}^\top (\mathbb{E}[\beta_i^{jj}]) - \frac{1}{N} \sum_{j \neq i} \beta_i^{jj}, \hat{x}(s) \right) dt.
\]

We observe that the initial terms such as \( \langle Q\tilde{x}_i, x^*_j \rangle \) in (12), is now reformulated with some inner product between principal intermediate variation \( \delta x_i \) and some quantities in terms by \( y_0^i \) and \( y_1' \) in an agent-wise (i.e., \( j \neq i \)) manner. Then, we can identify the tail filtration for exchangeable \( \{ \int_0^T \langle Q\tilde{x}_j, x^*_j \rangle dt \}_{j \neq i} \) based on \( \delta x_i \) with a degenerated filtration. So, applying conditional law of large number, and noticing \( \{ y'_1, j \neq i \} \) are identical distributed, we reach the following representation with expectation operator:

\[
\delta J_{soci}^{(\mathcal{N})}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q\tilde{x}_i, \delta x_i \rangle - \langle (QH + HQ - HQH)\hat{x} + F^\top \mathbb{E}[y_1] + \bar{F}^\top \mathbb{E}[\beta_i] \right]
+ F^\top \int_S y_2^0 d\Phi(\theta), \delta x_i \rangle + \langle R\tilde{u}_i, \delta u_i \rangle \right] dt + \sum_{l=1}^5 \varepsilon_l,
\]

where \( y_1 \) (depending on \( \tilde{x}_1 \), that is the optimized state for generic \( y_1' \)) is some copy with same distribution for generic \( y_1' \):\]

\[
dy_1 = [Q\tilde{x}_i - A^\top \mathbb{E}[y_1] - C^\top \beta_i] dt + \beta_i^1 dW_i + \sum_{l=1,l \neq i}^N \beta_i^l dW_l,
\]

\[
dy_2 = [-(QH + HQ - HQH)\hat{x} - (F^\top \mathbb{E}[y_1] + \bar{F}^\top \mathbb{E}[\beta_i]) - A^\top \mathbb{E}[y_1^0] - F^\top \mathbb{E}[y_2^0], dt,
\]

\[
y_1(T) = 0, \quad y_2(T) = 0, \quad \theta \in \mathcal{S},
\]

and

\[
\varepsilon_5 = \mathbb{E} \int_0^T \left( F^\top (\mathbb{E}[y_1]) - \frac{1}{N} \sum_{j \neq i} y'_1(j) \right) + \bar{F}^\top (\mathbb{E}[\beta_i]) - \frac{1}{N} \sum_{j \neq i} \beta_i^{jj}, \delta x_i \right) dt.
\]

We remark that \( y_1 \) has the same distribution with generic \( y_1' \). This again explains why we term above procedure as "weak" duality. We point out all variations terms in (18), are now directly depending only on principal (basic, or intermediate) variations. Thus, we now formulate a decentralized auxiliary cost differential \( \delta J_i(\delta u_i) \):

\[
\delta J_i(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q\tilde{x}_i, \delta x_i \rangle - \langle (QH + HQ - HQH)\hat{x} + F^\top \tilde{y}_1 + \bar{F}^\top \beta_i \right]
+ F^\top \int_S y_2^0 d\Phi(\theta), \delta x_i \rangle + \langle R\tilde{u}_i, \delta u_i \rangle \right] dt.
\]

**Remark 3.1.** There are four undetermined terms in (20) respectively : \( \hat{x} \) by (10) is the state-average limit; \( \tilde{y}_1 = \mathbb{E}[y_1], \beta_i = \mathbb{E}[\beta_i], y_2^0 \) is from (19) because of the weak duality procedure. All these terms, especially \( \hat{x} \), will be determined by CC in Section 4.

**Remark 3.2.** In (20), we introduce the first variation of auxiliary cost functional \( \delta J_i(\delta u_i) \) and ignore the error term \( \varepsilon_l, l = 1, \cdots, 5 \). The convergence rate estimation of these terms and the rigorous proofs will be given in Section 5.
4. Auxiliary control problem and consistency condition. This section aims to complete Step 3 concerning auxiliary problem, which has a double-fold role in its formulation and solvability. By weakly-coupling of MT, centralized strategy is infeasible due to curse of dimensionality. Alternatively, decentralized one is more desirable that can be derived by formulating an auxiliary cost with frozen state-average limit. Counterpart formulation in MFG is quite straightforward because of competitive feature. However, for MT, such auxiliary formulation becomes more complicated because each agent must take into account social cost on others. Subsequently, formulation of auxiliary problem, together with earlier variational decomposition and weak duality, will jointly complete above complex freezing procedure. Next, solvability of auxiliary problem enables us to design decentralized MT strategies with asymptotic optimality. Now, we present more role details.

4.1. Auxiliary control with double-projection. (20) contains only the principal terms $\delta x_i$ and $\delta u_i$, thus it links to an optimal control problem using local information of $A_i$ only. Now we can introduce the following auxiliary control (AC) problem for a generic $A_i$:

\[ \begin{align*}
\text{(AC)}: \quad \text{Minimize} & \quad J_i(u_i(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle Q x_i, x_i \rangle - 2 \langle \Xi, x_i \rangle + \langle Ru_i, u_i \rangle \right] dt, \\
\text{subject to} & \quad dx_i(t) = [A_{\Theta_i} x_i + B u_i + F \hat{x}] dt + [C x_i + D_{\Theta_i} u_i + F \bar{F} \hat{x}] dW_i(t), x_i(0) = \xi, \\
\end{align*} \]

with

\[ \Xi(t; \hat{x}, y^\theta, \hat{y}_i, \hat{\beta}_1) = (QH + HQ - HQH) \hat{x} + F^T \hat{y}_1 + F \bar{F} \hat{\beta}_1 + F^T \int_G y^\theta d\Phi(\theta), \]

where $\hat{x}$ is the limiting state-average term introduced in (10); $(y^\theta, \hat{y}_i, \hat{\beta}_1)$ depends on $\hat{x}$ satisfying dynamics (19). Also, we remark that $\hat{y}_1$ depends on optimal state $x_j$.

We will apply stochastic maximum principle to study Problem (AC). To this end, we introduce the following first-order adjoint equation:

\[ dp_i(t) = -[A^\top_{\Theta_i} p_i + Q x_i - \Xi + C^\top q_i] dt + q_i dW_i(t), \quad p_i(T) = 0. \]

Let $u^*_i$ be the optimal control and $(x^*_i, p^*_i, q^*_i)$ the corresponding state and adjoint state. For any $u_i \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ such that $u^*_i + u_i \in U_i^{d,p}$, we have $u^*_i := u^*_i + \epsilon u_i \in U_i^{d,p}$. The corresponding state and adjoint state with respect to $u^*_i$ are denoted by $(x^*_i, p^*_i, q^*_i)$. Introduce the following variational equation

\[ dy_i(t) = [A_{\Theta_i} y_i + B u_i] dt + [C y_i + D_{\Theta_i} u_i] dW_i(t), \quad y_i(0) = 0. \]

Applying Itô’s formula to $\langle p_i, y_i \rangle$, by the optimality of $u^*_i$ (i.e., $J_i(u^*_i) - J_i(u_i) \geq 0$), we have $\mathbb{E} \int_0^T \langle Ru^*_i + B^T p_i + D_{\Theta_i}^T q_i, u_i \rangle ds \geq 0$. For any $0 \leq t \leq T$ and $\mathcal{G}_t^i$-measurable random variable $\eta_i$, let

\[ u^*_i(s) + u_i(s) = \begin{cases} u^*_i(s), & s \notin [t, t + \epsilon]; \\ \eta_i, & s \in [t, t + \epsilon]. \end{cases} \]

Therefore, $\frac{1}{\epsilon} \mathbb{E} \int_t^{t+\epsilon} (Ru^*_i + B^T p_i + D_{\Theta_i}^T q_i, \eta_i - u^*_i) ds \geq 0$. Let $\epsilon \to 0$, we have $\mathbb{E}\{(R(t)u^*_i(t) + B^T(t)p^*_i(t) + D_{\Theta_i}^T(t)q^*_i(t), \eta_i - u^*_i(t)) \geq 0, t \in [0, T]\}$. For any $v \in \Gamma$ and $A \in \mathcal{G}_t^i$, define $\eta_i = v I_A + u^*_i(t) I_{\bar{A}}$, we have $\mathbb{E}\{(R(t)u^*_i(t) + B^T(t)p^*_i(t) + D_{\Theta_i}^T(t)q^*_i(t), v - u^*_i(t)) I_A \geq 0, t \in [0, T]\}$. Since $A \in \mathcal{G}_t^i$ is arbitrary, we have $\mathbb{E} \{(R(t)u^*_i(t) + B^T(t)p^*_i(t) + D_{\Theta_i}^T(t)q^*_i(t), v - u^*_i(t)) | \mathcal{G}_t^i \} \geq 0, t \in [0, T], \mathbb{P} - a.s., i.e.,$

\[ \langle -R(t)u^*_i(t) + \mathbb{E}[-B^T(t)p^*_i(t) - D_{\Theta_i}^T(t)q^*_i(t) | \mathcal{G}_t^i], v - u^*_i(t) \rangle \leq 0, t \in [0, T], \mathbb{P} - a.s. \]
Since \( v \in \Gamma \) is arbitrary and \( \Gamma \) is a closed convex set, it follows from the well-known results of convex analysis that (22) is equivalent to

\[
\begin{align*}
    u^*_i(t) = P_{\Gamma}[R^{-1}\mathbb{E}[\pi_i^*(t) - D_{\pi_i^*}(t)|G_t^i]], \quad a.e. \ t \in [0,T], \ P - a.s.,
\end{align*}
\]

where \( P_{\Gamma}[\cdot] \) is the projection mapping from \( \mathbb{R}^m \) to its closed convex subset \( \Gamma \) under the norm \( \|v\|^2_R := \langle R^2v, R^2v \rangle \). Note that there involve two projections in (23), because of the input constraint and partial information constraint. This differs from [20, 21] which include only input constraint. Furthermore, the two projections are non-commutative due to above maximum principle arguments. In this case, the related Hamiltonian system for (AC) becomes

\[
\begin{align*}
    dx^*_t & = \begin{bmatrix} A_{\Theta}x^*_i + BP_{\Gamma}[R^{-1}\mathbb{E}[\pi_i^*(t) - D_{\pi_i^*}(t)|G_t^i]] + F \bar{x} \end{bmatrix} dt \\
        & + \begin{bmatrix} Cx^*_i + D_{\Theta}P_{\Gamma}[R^{-1}\mathbb{E}[\pi_i^*(t) - D_{\pi_i^*}(t)|G_t^i]] + \bar{F}\bar{x} \end{bmatrix} dW_i(t), \\
    dp^*_i & = -[A_{\Theta}p^*_i + Qx^*_i - \Xi + C^Tq^*_i]dt + q^*_i dW_i(t), \\
    x^*_i(0) & = \xi, \quad p^*_i(T) = 0,
\end{align*}
\]

which is a fully-coupled FBSDEs with double-projection: the mapping on input convex-closed set, and the filtering for partial information (i.e., conditional expectation on sub-space).

### 4.2. Consistency condition

Note the optimal strategy for auxiliary control problem involves some undetermined terms \((\bar{x}, \bar{y}_i, \bar{\beta}_1)\). In this section, we will characterize the undetermined processes, especially state-average limit \( \bar{x} \), in (21) via some consistency matching scheme. Given the Hamiltonian system by (24), all agents should apply some exchangeable team decisions \( \{u^*_i\}_{i=1}^N \) and the realized states should be as follows:

\[
\begin{align*}
    dx^*_i & = \begin{bmatrix} A_{\Theta}x^*_i + BP_{\Gamma}[R^{-1}\mathbb{E}[\pi_i^*(t) - D_{\pi_i^*}(t)|G_t^i]] + F \bar{x}^{*,(N)} \end{bmatrix} dt \\
        & + \begin{bmatrix} Cx^*_i + D_{\Theta}P_{\Gamma}[R^{-1}\mathbb{E}[\pi_i^*(t) - D_{\pi_i^*}(t)|G_t^i]] + \bar{F}\bar{x}^{*,(N)} \end{bmatrix} dW_i(t), \\
    x^*_i(0) & = \xi,
\end{align*}
\]

where \( x^{*,(N)} = \frac{1}{N} \sum_{i=1}^N x^*_i \) and \((p^*_i, q^*_i)\) is the solution of (24). Making all such exchangeable strategies aggregated, and applying de Finetti theorem, we can obtain the limiting system by identifying \( \bar{x} = \mathbb{E}\bar{x}^{*,(N)} \),

\[
\begin{align*}
    d\bar{x} & = \begin{bmatrix} A_{\Theta}\bar{x} + BP_{\Gamma}[R^{-1}\mathbb{E}[\pi(t) - D_{\pi}(t)|G_t]] + F\mathbb{E}\bar{x} \end{bmatrix} dt \\
        & + \begin{bmatrix} C\bar{x} + D_{\Theta}P_{\Gamma}[R^{-1}\mathbb{E}[\pi(t) - D_{\pi}(t)|G_t]] + \bar{F}\mathbb{E}\bar{x} \end{bmatrix} dW(t), \\
    d\bar{p} & = -[A_{\Theta}\bar{p} + Q\bar{x} - (QH + HQ - HQH)\mathbb{E}\bar{x} - F^T\bar{y}_1 - \bar{F}^T\bar{\beta}_1 \int_{S} y^2 d\Phi(\theta) + C^T\bar{q}] dt + \bar{q} dW(t), \\
    \bar{x}(0) & = \xi, \quad \bar{p}(T) = 0,
\end{align*}
\]

where \( \Theta \) is a random variable with distribution defined in (A1), \( W(t) \) is a generic Brownian motion independent of \( \Theta \), \( G \) is sub-filtration representing the partial information and \((\bar{y}_1 = \mathbb{E}[y_1], \bar{\beta}_1 = \mathbb{E}[\beta_1], y^2_\theta)\) is from (19). Note that we suppress subscript \( i \) in (25) as all agents are statistically identical in the distribution sense. Combining with (19), we will obtain consistency condition (CC) of Problem LQG-MT. For simplicity, define \( \mathcal{E}_t[-B^T\gamma - D_{\Theta}^T\vartheta] = \mathbb{E}[-B^T\gamma - D_{\Theta}^T\vartheta|G_t] \). Hence we have the following result.
Proposition 4.1. The undetermined parameters of (21) can be determined by
\[
(\hat{x}, \hat{y}_1, \hat{\beta}_1, \hat{y}_2) = (E_{\alpha}, E_{\hat{y}_1}, E_{\hat{\beta}_1}, E_{\hat{y}_2}),
\]
where \((\alpha, \gamma, \vartheta, \hat{y}_1, \hat{\beta}_1, \hat{y}_2)\) is the solution of the consistency condition of Problem LQG-MT:
\[
\begin{align*}
\alpha'(t) &= [A_{\Theta}\alpha + B\mathbf{P}_t[R^{-1}\mathcal{E}_t(-B^T\gamma - D_{\Theta}\vartheta)]]dt + F\mathbb{E}[\alpha]dt
\quad + [C\alpha + D_{\Theta}\mathbf{P}_t[R^{-1}\mathcal{E}_t(-B^T\gamma - D_{\Theta}\vartheta)]]dW,
\gamma'(t) &= -Q\alpha + (QH + HQ - HQH)\mathbb{E}\alpha - A_{\Theta}\gamma + F^T\hat{y}_2 d\Phi(\theta) + F^T\mathbb{E}\hat{y}_1
\quad - C^T\vartheta + F^T\mathbb{E}\hat{\beta}_1 dt + \vartheta dW,
\hat{y}_1'(t) &= [Q\alpha - A_{\Theta}\hat{y}_1 - C^T\hat{\beta}_1]dt + \hat{\beta}_1 dW,
\hat{y}_2'(t) &= [-(QH + HQ - HQH)\mathbb{E}\alpha - F^T\mathbb{E}\hat{y}_1 - F^T\mathbb{E}\hat{\beta}_1 - A_{\Theta}\hat{y}_2 - F^T\hat{y}_2]dt,
\alpha(0) &= \xi, \quad \gamma(T) = 0, \quad \hat{y}_1(T) = 0, \quad \hat{y}_2(T) = 0, \quad \theta \in S.
\end{align*}
\]

Remark 4.2. (26) is a new type of fully coupled FBSDEs with double-projection (projection mapping on the convex-closed subset and partial-information sub-space). Moreover, both temporal variable \(t\) and spatial variable \(\theta\) appear in (26). Considering this, we can rewrite (26) in the following more compact form:
\[
\begin{align*}
\alpha'(t) &= [A_{\Theta}\alpha + B\mathbf{P}_t[R^{-1}\mathcal{E}_t(-B^T\gamma - D_{\Theta}\vartheta)]]dt + F\mathbb{E}[\alpha]dt
\quad + [C\alpha + D_{\Theta}\mathbf{P}_t[R^{-1}\mathcal{E}_t(-B^T\gamma - D_{\Theta}\vartheta)]]dW,
\gamma'(t) &= -Q\alpha + (QH + HQ - HQH)\mathbb{E}\alpha - A_{\Theta}\gamma + F^T\hat{y}_2 d\Phi(\theta) + F^T\mathbb{E}\hat{y}_1 + C^T\vartheta
\quad + F^T\mathbb{E}\hat{\beta}_1 dt + \vartheta dW(t),
\hat{y}_1'(t) &= [Q\alpha - A_{\Theta}\hat{y}_1 - C^T\hat{\beta}_1]dt + \hat{\beta}_1 dW,
\hat{y}_2'(t) &= [-(QH + HQ - HQH)\mathbb{E}\alpha - F^T\mathbb{E}\hat{y}_1 - F^T\mathbb{E}\hat{\beta}_1 - A_{\Theta}\hat{y}_2 - F^T\hat{y}_2]dt,
\alpha(0) &= \xi, \quad \gamma(T) = 0, \quad \hat{y}_1(T) = 0, \quad \hat{y}_2(T) = 0.
\end{align*}
\]

Note that by the independence between \(\Theta\) and \(W\), (27) can be viewed as defined on the product space \(\Omega_1 \times \Omega_2 \rightarrow S \times \mathbb{R}^n\). This is a general system which includes many framework in current literature as special cases. For more information, please refer to Section 6.1.

4.3. Wellposedness of consistency condition. This subsection continues to complete (Step 3) by establishing some well-posedness to CC derived in Section 4.2. Note that (26) is fully coupled FBSDEs involved with double projections whose well-posedness cannot be guaranteed by current literature. Moreover, as explained in Section 6.2, (26) is obtained by converting system with continuum heterogeneity to a homogenous one but with augmented randomness \((\{\Theta_i, W_i\}_{i=1}^N)\) as trade-off. Based on this, we will apply the discounting method to study (26) which would provide some mild conditions to ensure the existence and uniqueness of fully coupled FBSDEs as (26). Define \(X = \alpha, \ Y = (\gamma^T, \hat{y}_1^T, (\hat{y}_2^T)^T)^T\) and \(Z = (\hat{y}_1^T, \hat{\beta}_1^T, 0)^T\). For simplicity, let \(\mathcal{E}_t[Y] = \mathbb{E}[Y|\mathcal{G}_t]\) and \(\mathcal{E}_t[Z] = \mathbb{E}[Z|\mathcal{G}_t], \ \mathbb{E}[Y] = \)
\[
(f_\Sigma \gamma d\Phi(\theta))^T, (f_\Sigma \tilde{y}_1 d\Phi(\theta))^T, (f_\Sigma \tilde{y}_2 d\Phi(\theta))^T, \text{ then (26) takes the following form: (28)}
\]
\[
\begin{cases}
    dX = [A_\Theta X + FE[X] + B_1(Y, Z)]dt + [CX + \tilde{F}E[X] + D_\Theta(Y, Z)]dW, \\
    dY = [A_2X + \tilde{A}_2E[X] + B_2Y + \tilde{B}_2E[Y] + \tilde{B}_2E[Z] + C_2Z + \tilde{C}_2E[Z]]dt + ZdW,
\end{cases}
\]

\[
X(0) = \xi, \quad Y(T) = (0, \cdots, 0)^T,
\]

where
\[
\begin{align*}
B_1(Y, Z) &= BP_T[R^{-1}E([-B^T\gamma - D_\Theta \psi])] \\
&= BP_T[R^{-1}((-B^T, 0, \cdots, 0)E[Y] + (-D_\Theta, 0, \cdots, 0)E[Z])], \\
D_\Theta(Y, Z) &= D_\Theta P_T[R^{-1}E([-B^T\gamma - D_\Theta \psi])] \\
&= D_\Theta P_T[R^{-1}((-B^T, 0, \cdots, 0)E[Y] + (-D_\Theta, 0, \cdots, 0)E[Z])], \\
A_2 &= 0, \quad \tilde{A}_2 = \left( \begin{array}{cccc}
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
\end{array} \right), \\
B_2 &= \left( \begin{array}{cccc}
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
\end{array} \right), \quad \tilde{B}_2 = \left( \begin{array}{cccc}
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
\end{array} \right), \\
C_2 &= \left( \begin{array}{cccc}
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & \cdots \\
\end{array} \right).
\end{align*}
\]

and 0 denotes the zero vector or zero matrix with suitable dimensions. Note that in (28), \( \tilde{B}_2E[Y] = \tilde{E}[\tilde{B}_2Y] = \tilde{E}[\tilde{B}_2Y]. \) To start, we first give some results for general nonlinear mean-field forward-backward system with double projections:

\[
\begin{cases}
    dX = b(t, X, \tilde{E}[X], \tilde{E}[Y], \tilde{E}[Z])dt + \sigma(t, X, \tilde{E}[X], \tilde{E}[Y], \tilde{E}[Z])dW, \quad X(0) = x, \\
    dY(t) = -f(t, X, \tilde{E}[X], Y, \tilde{E}[Y], \tilde{E}[Z])dt + ZdW, \quad Y(T) = 0,
\end{cases}
\]

where \( \tilde{E}[\tilde{E}[Y]] = \tilde{E}[Y] \) and the coefficients satisfy the following conditions:

\begin{itemize}
    \item[(H1)] There exist \( \rho_1, \rho_2 \in \mathbb{R} \) and positive constants \( k_i, i = 1, \cdots, 8 \) such that for all \( t \in [0, T] \), \( x, x_1, x_2, \tilde{x}, \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^n, y, y_1, y_2, \tilde{y}, \tilde{y}_1, \tilde{y}_2 \in \mathbb{R}^m, z, z_1, z_2, \tilde{z}, \tilde{z}_1, \tilde{z}_2, \tilde{z} \),
\end{itemize}

\[
\begin{align*}
    &|b(t, x, \tilde{x}, y, \tilde{y}, z, \tilde{z}) - b(t, x_1, \tilde{x}_1, y_1, \tilde{y}_1, z_1, \tilde{z}_1)| \leq \rho_1 |x_1 - x_2|^2, \\
    &|b(t, x, \tilde{x}_1, y_1, z_1, \tilde{z}_1) - b(t, x, \tilde{x}_2, y_2, z_1, \tilde{z}_2)| \\
    &\leq k_1 |\tilde{x}_1 - \tilde{x}_2| + k_2 |y_1 - y_2| + k_2 |y_1 - \tilde{y}_2| + k_2 |z_1 - z_2| + k_2 |\tilde{z}_1 - \tilde{z}_2|, \\
    &|f(t, x, \tilde{x}, y_1, \tilde{y}, z, \tilde{z}) - f(t, x_1, \tilde{x}_1, y_1, \tilde{y}_1, z_1, \tilde{z}_1)| \leq \rho_2 |y_1 - y_2|^2, \\
    &|f(t, x, \tilde{x}_1, y, \tilde{y}_1, z_1, \tilde{z}_1) - f(t, x, \tilde{x}_2, y_2, \tilde{y}_2, z_2, \tilde{z}_2)| \\
    &\leq k_2 |x_1 - x_2| + k_2 |\tilde{x}_1 - \tilde{x}_2| + k_2 |y_1 - y_2| + k_2 |\tilde{y}_1 - \tilde{y}_2| + k_2 |z_1 - z_2| + k_2 |\tilde{z}_1 - \tilde{z}_2|, \\
    &|\sigma(t, x, \tilde{x}_1, y_1, \tilde{y}_1, z_1, \tilde{z}_1) - \sigma(t, x_2, \tilde{x}_2, y_2, \tilde{y}_2, z_2, \tilde{z}_2)|^2 \\
    &\leq k_2 |x_1 - x_2|^2 + k_2 |\tilde{x}_1 - \tilde{x}_2|^2 + k_2 |y_1 - y_2|^2 + k_2 |\tilde{y}_1 - \tilde{y}_2|^2 + k_2 |z_1 - z_2|^2 + k_2 |\tilde{z}_1 - \tilde{z}_2|^2.
\end{align*}
\]

\begin{itemize}
    \item[(H2)]
\end{itemize}

\[
\mathbb{E} \int_0^T \left[ |b(t, 0, 0, 0, 0)|^2 + |\sigma(t, 0, 0, 0, 0)|^2 + |f(t, 0, 0, 0, 0, 0)|^2 \right] dt < \infty.
\]

Similar to [21] and [36], we have the following result of the solvability of (29). For the readers’ convenience, we give the proof in the appendix.
Theorem 4.3. Suppose (H1) and (H2) hold. There exists a constant $\delta_1 > 0$ depending on $\rho_1, \rho_2, T, k_i, i = 1, 3, 4, 5, 6, 7, 8$ such that if $k_2 \in [0, \delta_1)$, FBSDEs (29) admits a unique adapted solution $(X, Y, Z) \in L^2_T((0, T; \mathbb{R}^n) \times L^2_T((0, T; \mathbb{R}^m)) \times L^2_T((0, T; \mathbb{R}^m))$. Furthermore, if $2\rho_1 + 2\rho_2 < -2k_1 - 2k_3 - 2k_4 - k_5 - k_6^2 - k_7^2 - k_8^2$, there exists a constant $\delta_2 > 0$ depending on $\rho_1, \rho_2, k_2, i = 1, 3, 4, 5, 6, 7, 8$ such that if $k_2 \in [0, \delta_2)$, FBSDEs (29) admits a unique adapted solution $(X, Y, Z) \in L^2_T((0, T; \mathbb{R}^n) \times L^2_T((0, T; \mathbb{R}^m)) \times L^2_T((0, T; \mathbb{R}^m))$.

Let $\rho_1^* = \text{ess sup}_{0 \leq s \leq T} \text{ess sup}_{p \in S} \Lambda_{\max}(-\frac{1}{2}(A_{\theta}(s) + A_{\theta}(s)^T))$ and $\rho_2^* = \text{ess sup}_{0 \leq s \leq T} \Lambda_{\max}\left(-\frac{1}{2}(B(s) + B(s)^T)\right)$, where $\Lambda_{\max}(M)$ is the largest eigenvalue of the matrix $M$. For $M(\cdot) \in L^\infty_T((0, T; \mathbb{R}^{n \times n})$, $\|M(\cdot)\| \triangleq \text{ess sup}_{0 \leq s \leq T} \text{ess sup}_{\omega \in \Omega} \|M(s)\|$. Comparing (29) with (28), we can check that the parameters of (H1) and (H2) can be chosen as follows:

$$
k_1 = \|F\|, \quad k_2 = \|\bar{B}_2\|, \quad k_4 = \|\bar{B}_2\|, \quad k_5 = \|C_2\|, \quad k_6 = \|\bar{C}_2\|, \quad k_7 = \sqrt{3}\|C\|, \quad k_8 = \sqrt{3}\|\bar{F}\|,
$$

$$
k_2 = \max\{\|B\|^2 R^{-1}, \|B\| R^{-1}\|D_{\Theta}\|, \|A_{\theta}\|, \|\bar{A}_{\theta}\|, \sqrt{6}\|D_{\Theta}\| R^{-1}, \sqrt{6}\|D_{\Theta}\| R^{-1}\|D_{\Theta}\|\}.
$$

Now we introduce the following assumption:

(A4) $2\rho_1^* + 2\rho_2^* < -2k_1 - 2k_3 - 2k_4 - k_5^2 - k_6^2 - k_7^2 - k_8^2$.

It follows from Theorem 4.3 that

Proposition 4.4. Under (A4), there exists a constant $\delta_3 > 0$ depending on $\rho_1^*, \rho_2^*, k_2, i = 1, 3, 4, 5, 6, 7, 8$ such that if $k_2 \in [0, \delta_3)$, FBSDEs (28) admits a unique adapted solution $(X, Y, Z) \in L^2_T((0, T; \mathbb{R}^n) \times L^2_T((0, T; \mathbb{R}^{3m})) \times L^2_T((0, T; \mathbb{R}^{3m}))$.

5. Asymptotic $\epsilon$-optimality. This section aims to complete (Step 4) so as to verify the asymptotic optimality of mean-field team strategy derived in Section 4. Here we proceed our verification based on the assumption in §4.3, i.e., (A4). Contrary to MFG entailing only one-side perturbation for single agent to asymptotic Nash equilibrium, MT must take into account team (integrated) perturbations upon all agents. Meanwhile, (cooperative) social cost is more intertwined with individual one of single agent, so a quadratic functional representation, as formalized below, will greatly facilitate our targeted analysis. For sake of clear presentation, we divide related analysis into four sub-steps in separated subsections.

5.1. Quadratic representation of social cost. We first give a quadratic representation of team functional that gives a tractable fortiori formulation of Fréchet differentials of social cost. Rewrite the large-population system (3) as follows:

$$
dx = (Ax + Bu)dt + \sum_{i=1}^{N} (C_i x + D_i u) dW_i, \quad x(0) = \bar{x},$$

where

$$A = \begin{pmatrix}
A_{\theta_1} + \frac{E}{N} & \frac{E}{N} & \cdots & \frac{E}{N} \\
\frac{E}{N} & A_{\theta_2} + \frac{E}{N} & \cdots & \frac{\bar{E}}{N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{E}{N} & \frac{E}{N} & \cdots & A_{\theta_N} + \frac{E}{N}
\end{pmatrix}, \quad B = \begin{pmatrix}
B_0 & 0 & \cdots & 0 \\
0 & B_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_0
\end{pmatrix}, \quad u = \begin{pmatrix}
u_1 \\
\vdots \\
u_N
\end{pmatrix},$$

$$C_i = \begin{pmatrix}
\frac{E}{N} & \frac{E}{N} & \cdots & \frac{E}{N} \\
\frac{E}{N} & \frac{E}{N} & \cdots & \frac{\bar{E}}{N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{E}{N} & \frac{E}{N} & \cdots & A_{\theta_N} + \frac{E}{N}
\end{pmatrix}, \quad D_i = \begin{pmatrix}
D_{\theta_1} & 0 & \cdots & 0 \\
0 & D_{\theta_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_{\theta_N}
\end{pmatrix}, \quad \tilde{\xi} = \begin{pmatrix}
\xi_1 \\
\vdots \\
\xi_N
\end{pmatrix}.$$
Similarly, the social cost takes the following form:

\[ J^{(N)}_{soc}(u) = \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle Qx, x \rangle + \langle Ru, u \rangle \right] dt, \]

where

\[ Q = \begin{pmatrix} Q+\mathcal{H}(H,Q) & \mathcal{H}(H,Q) & \cdots & \mathcal{H}(H,Q) \\ \mathcal{H}(H,Q) & Q+\mathcal{H}(H,Q) & \cdots & \mathcal{H}(H,Q) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}(H,Q) & \mathcal{H}(H,Q) & \cdots & Q+\mathcal{H}(H,Q) \end{pmatrix}, \]

\[ \mathcal{H}(H,Q) = \frac{1}{N}(H^T QH - QH - H^T Q), \]

\[ R = \begin{pmatrix} 0 & \cdots & 0 \\ R & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R \end{pmatrix}. \]

Next, by the variation of constant formula, the strong solution of (30) becomes

\[ x(t) = \Phi(t) \xi + \Phi(t) \int_0^t \Phi(s)^{-1}[(B - \sum_{i=1}^N C_i D_i)u(s)]ds + \sum_{i=1}^N \Phi(t) \int_0^t \Phi(s)^{-1} D_i u(s)dW_i(s), \]

where

\[ d\Phi(t) = A\Phi(t)dt + \sum_{i=1}^N C_i \Phi(t)dW_i(t), \]

where \( \Phi(t) = I \). Define the following operators

\[ \phi(u)(\cdot) := \Phi(\cdot)\left\{ \int_0^t \Phi(s)^{-1}[(B - \sum_{i=1}^N C_i D_i)u(s)]ds + \sum_{i=1}^N \int_0^t \Phi(s)^{-1} D_i u(s)dW_i(s) \right\}, \]

\[ \tilde{\phi}(u) := \phi(u)(T), \quad S(y)(\cdot) := \Phi(\cdot)\Phi^{-1}(0)\xi, \quad \tilde{S}(y) := S(y)(T), \]

then for any admissible control \( u \), we have \( x(\cdot) = \phi(u)(\cdot) + S(y)(\cdot) \), \( x(T) = \tilde{\phi}(u) + \tilde{S}(y) \). Note that \( \phi(\cdot) : (L^2_F(0,T;\Gamma), \cdots, L^2_F(0,T;\Gamma)) \to (L^2_F(0,T;\mathbb{R}^n), \cdots, L^2_F(0,T;\mathbb{R}^n)) \) is a bounded linear operator, thus there exists a unique bounded linear operator \( \phi^*(\cdot) : (L^2_F(0,T;\mathbb{R}^n), \cdots, L^2_F(0,T;\mathbb{R}^n)) \to (L^2_F(0,T;\Gamma), \cdots, L^2_F(0,T;\Gamma)) \) such that for any \( u(\cdot) \in (L^2_F(0,T;\mathbb{R}^n), \cdots, L^2_F(0,T;\mathbb{R}^n)) \) and \( x(\cdot) \in (L^2_F(0,T;\Gamma), \cdots, L^2_F(0,T;\Gamma)) \),

\[ \mathbb{E} \int_0^T \langle \phi(u)(t), x(t) \rangle dt = \mathbb{E} \int_0^T \langle u(t), \phi^*(x)(t) \rangle dt. \]

Hence, we can rewrite the cost functional as follows:

\[ 2J^{(N)}_{soc}(u) = \mathbb{E} \int_0^T \left[ \langle \phi^*Q\phi + R u, u \rangle + 2\langle \phi^*QS(y), u \rangle + \langle QS(y), S(y) \rangle \right] dt \]

\[ = \langle M_2(u)(\cdot), u(\cdot) \rangle + 2\langle M_1, u(\cdot) \rangle + M_0, \]

where we have used \( \langle \cdot, \cdot \rangle \) as inner products in different Hilbert spaces. Note that, \( M_2(\cdot) \) is a bounded self-adjoint semi-definite linear operator.

### 5.2. Agent \( A_i \) perturbation

This subsection gives a perturbation for single agent \( A_i \) that further triggers a team perturbation across the population, (see §5.4). Let \( \bar{u} = (\bar{u}_1, \cdots, \bar{u}_N) \) be decentralized strategy given by

\[ \bar{u}_i(t) = \varphi_{\Theta_i}(p_i(t), q_i(t)) := P_T[R(t)^{-1}E[B(t)^T p_i(t) + D\Theta_i(t)^T q_i(t)|G_t_i]], \]

where \( (p_i, q_i) \) is the solution of

\[
\begin{align*}
    dx_i &= [A\Theta_i x_i + B\varphi_{\Theta_i}(p_i, q_i) + F\Theta_i]dt + [C x_i + D\varphi_{\Theta_i}(p_i, q_i) + \bar{F}\Theta_i]dW_i(t), \\
    dp_i &= [-Q x_i + (QH + HQ - HQH)\Theta_i - A^T_{\Theta_i}p_i + F^T \int_S \theta \bar{g}_2 d\Phi(\theta) + F^T \bar{E}\bar{\gamma}_1 \\
    & \quad - C^T q_i + \bar{F}^T \bar{E}\bar{\beta}_1]dt + q_i dW_i(t), \\
    x_i(0) &= \xi, \quad p_i(T) = 0, \quad i = 1, \cdots, N.
\end{align*}
\]
Here, \((\alpha, \tilde{y}_i, \tilde{y}_i^0)\) is the solution of (26). Correspondingly, the realized decentralized states \((\tilde{x}_1, \ldots, \tilde{x}_N)\) satisfy
\[
\begin{cases}
dx_i = [A_{\Theta} \tilde{x}_i + B \varphi_{\Theta}(p_i, q_i) + F \tilde{x}(N)]dt + [C \tilde{x}_i + D_{\Theta} \varphi_{\Theta}(p_i, q_i) + \tilde{F} \tilde{x}(N)]dW_i,
\tilde{x}_i(0) = \xi,
\end{cases}
\]
and \(\tilde{x}(N) = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i\). Let us consider the case that the agent \(A_i\) (without loss of generality, assume \(i > 1\)) uses an alternative strategy \(u_i \in U_i^{i-}\) while the other agents \(A_j, j \neq i\) use the strategy \(\tilde{u}_{-i}\). The realized state with the \(i\)-th agent’s perturbation is
\[
\begin{cases}
\dot{x}_i = [A_{\Theta} \dot{x}_i + Bu_i + F \dot{x}(N)]dt + [C \dot{x}_i + D_{\Theta} u_i + \dot{F} \dot{x}(N)]dW_i,
\dot{x}_j = [A_{\Theta} \dot{x}_j + B \varphi_{\Theta}(p_j, q_j) + F \dot{x}(N)]dt + [C \dot{x}_j + D_{\Theta} \varphi_{\Theta}(p_j, q_j) + \dot{F} \dot{x}(N)]dW_j,
\dot{x}_i(0) = \xi, \quad \dot{x}_j(0) = \xi, \quad 1 \leq j \leq N, \quad j \neq i,
\end{cases}
\]
where \(\dot{x}(N) = \frac{1}{N} \sum_{i=1}^{N} \dot{x}_i\). For \(j = 1, \ldots, N\), denote the perturbation \(\delta x_i = u_i - \tilde{u}_i\), \(\delta x_{j,i} = \dot{x}_j - \dot{x}_i\). \(\delta J_{i-} = J_j(\tilde{u}_{i}, \tilde{u}_{-i}) - J_j(\tilde{u}_i, \tilde{u}_{-i})\). Introducing the following frozen states
\[
\begin{cases}
\dot{l}_i(0) = \xi, \quad j = 1, \ldots, N,
\end{cases}
\]
and
\[
\begin{cases}
\dot{l}_i = [A_{\Theta} \dot{l}_i + Bu_i + F \dot{E} \alpha]dt + [C \dot{l}_i + D_{\Theta} u_i + \dot{F} \dot{E} \alpha]dW_i,
\dot{l}_j = [A_{\Theta} \dot{l}_j + B \varphi_{\Theta}(p_j, q_j) + F \dot{E} \alpha]dt + [C \dot{l}_j + D_{\Theta} \varphi_{\Theta}(p_j, q_j) + \dot{F} \dot{E} \alpha]dW_j,
\dot{l}_i(0) = \xi, \quad \dot{l}_j(0) = \xi, \quad 1 \leq j \leq N, \quad j \neq i.
\end{cases}
\]
Similar to the computations in Section 3.1, we have
\[
\delta J_{soc}^{(N)} = \mathbb{E} \int_0^T \left[ \langle Q \tilde{l}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt + \sum_{l=1}^{7} \epsilon_l,
\]
where
\[
\begin{align*}
\epsilon_1 &= \mathbb{E} \int_0^T \langle (QH + HQ - HQH)(\dot{E} \alpha - \tilde{x}(N)), N \delta x(N) \rangle dt, \\
\epsilon_2 &= \mathbb{E} \int_0^T \langle (QH + HQ - HQH)\dot{E} \alpha, x^{**} - \delta x_{-i} \rangle dt, \\
\epsilon_3 &= \mathbb{E} \int_0^T \frac{1}{N} \sum_{j \neq i} \langle Q \tilde{x}_j, N \delta x_{j,i} - x_{j,i}^{**} \rangle dt, \\
\epsilon_4 &= \mathbb{E} \int_0^T \langle F^T (\mathbb{E}[y_i^1] - \frac{1}{N} \sum_{j \neq i} y_j^1) + F^T (\mathbb{E}[\beta_{j}^{11}] - \frac{1}{N} \sum_{j \neq i} \beta_{j}^{11}), \delta x_i \rangle dt, \\
\epsilon_5 &= \mathbb{E} \int_0^T \langle F^T (\mathbb{E}[y_i^1] - \frac{1}{N} \sum_{j \neq i} y_j^1) + F^T (\mathbb{E}[\beta_{j}^{11}] - \frac{1}{N} \sum_{j \neq i} \beta_{j}^{11}), x^{**} \rangle dt, \\
\epsilon_6 &= \mathbb{E} \int_0^T \langle \dot{l}_i - \dot{x}_i, \Xi \rangle + \langle \tilde{l}_i - \tilde{x}_i, \Xi \rangle dt, \\
\epsilon_7 &= \mathbb{E} \int_0^T \langle [Q(\tilde{x}_i - \tilde{l}_i), \delta x_i] + [Q\tilde{l}_i, \dot{x}_i - \tilde{l}_i] + [Q\tilde{l}_i, \tilde{x}_i - \tilde{l}_i] \rangle dt.
\end{align*}
\]
5.3. Preliminary estimations. By (34), in order to establish asymptotic optimality of decentralized strategies, we need to rely on some estimates on $\varepsilon_1, \cdots, \varepsilon_7$, based on structural estimations of variational equations (6), (7), (8) and mean-field approximations in §3.1. More elaborate estimates are thereby needed considering continuum heterogeneity. So this subsection will first study the properties of involved variational equations and mean-field approximations. In below, $L$ denotes a generic constant whose value may change from line to line.

Applying the same technique as in [21, Lemma 5.1], we have

**Lemma 5.1.** There exist a constant $L$ independent of $N$ such that

$$
E \sup_{0 \leq t \leq T} \left( |\alpha|^2 + |\gamma|^2 + |\bar{y}_1|^2 + |\bar{y}_2|^2 \right) + \sum_{j=1}^{N} E \sup_{0 \leq t \leq T} \left( |x_j|^2 + |p_j|^2 \right) + E \int_0^T \left( |\vartheta|^2 + |\bar{\vartheta}_1|^2 \right) dt + \sum_{j=1}^{N} E \int_0^T \left( |q_j|^2 + |\varphi_{\Theta_j}(p_j, q_j)|^2 \right) dt \leq L,
$$

and

$$
\sup_{1 \leq j \leq N} E \sup_{0 \leq t \leq T} |\bar{x}_j(t)|^2 + \sup_{1 \leq j \leq N} E \sup_{0 \leq t \leq T} |\bar{l}_j(t)|^2 \leq L.
$$

Next we give some estimations on variational equations (6), (7) and (8).

**Lemma 5.2.** There exists a constant $L$ independent of $N$ such that

$$
E \sup_{0 \leq s \leq t} |\delta x^{(N)}|^2 + \sup_{1 \leq j \leq N, j \neq i} E \sup_{0 \leq t \leq T} |\delta x_{j,i}|^2 \leq \frac{L}{N^2}.
$$

**Proof.** Recall the equations (6), (7) and (8), we have

$$
E \sup_{0 \leq s \leq t} |\delta x_i|^2 \leq L E \int_0^t |\delta x_i|^2 ds + L E \int_0^t |\delta x^{(N)}|^2 ds,
$$

$$
E \sup_{0 \leq s \leq t} |\delta x_{j,i}|^2 \leq L E \int_0^t |\delta x_{j,i}|^2 ds + L E \int_0^t |\delta x^{(N)}|^2 ds,
$$

$$
E \sup_{0 \leq s \leq t} |\delta x_{-i}|^2 \leq L E \int_0^t |\delta x_{-i}|^2 ds + L E \int_0^t |\delta x^{(N)}|^2 ds.
$$

Note that $\delta x^{(N)} = \frac{1}{N} \delta x_i + \frac{1}{N} \delta x_{-i}$, we have

$$
E \sup_{0 \leq s \leq t} |\delta x_i|^2 \leq L E \int_0^t |\delta x_i|^2 ds + \frac{L}{N^2} E \int_0^t |\delta x_{-i}|^2 ds,
$$

$$
E \sup_{0 \leq s \leq t} |\delta x_{-i}|^2 \leq L E \int_0^t |\delta x_{-i}|^2 ds + L E \int_0^t |\delta x_i|^2 ds.
$$

Therefore, it follows from Gronwall inequality that we have

$$
\sup_{1 \leq j \leq N, j \neq i} E \sup_{0 \leq s \leq t} |\delta x_{j,i}|^2 \leq \frac{L}{N^2}.
$$

$\blacksquare$
Now we study mean-field approximations: due to continuum heterogeneous setting, some new estimates are thus required with their own interests. Specifically, Lemma 5.3 is not standard SDE estimate, thus some specific techniques are invoked in its proof.

**Lemma 5.3.** There exists a constant $L$ independent of $N$ such that
\[
\sup_{0 \leq t \leq T} E|\bar{x}^{(N)}(t) - E\alpha|^2 \leq \frac{L}{N}.
\]

**Proof.** First, for any $\theta \in S$, let
\[
\begin{aligned}
&\begin{cases}
  d\bar{x}_{\theta,j} = [A_\theta \bar{x}_{\theta,j} + B_{\theta}(p_j, q_j) + F_{\theta}(x_{\theta,j})] dt + [C_{\theta} \bar{x}_{\theta,j} + D_{\theta}\tilde{\varphi}_\theta(p_j, q_j) + \tilde{F}_{\theta}^{\alpha}(x_{\theta,j})] dW_j(t), \\
  \bar{x}_{\theta,j}(0) = \xi,
\end{cases} \\
&\begin{cases}
  d\tilde{l}_{\theta,j} = [A_\theta \tilde{l}_{\theta,j} + B_{\theta}(p_j, q_j) + F_{\theta|\alpha}(\alpha_{\theta})] dt + [C_{\theta} \tilde{l}_{\theta,j} + D_{\theta}\varphi_{\theta}(p_j, q_j) + \tilde{F}_{\theta}(\alpha_{\theta})] dW_j(t), \\
  \tilde{l}_{\theta,j}(0) = \xi,
\end{cases}
\end{aligned}
\]
where $\bar{x}_{\theta,j} = \frac{1}{N} \sum_{j=1}^{N} \bar{x}_{\theta,j}$ and $\alpha_{\theta}$ is the solution of (26) corresponding to $\Theta = \theta$. By Cauchy-Schwarz inequality and Burkholder-Davis-Gundy inequality, we have
\[
E \sup_{0 \leq s \leq t} |\bar{x}_{\theta,j}(s) - \tilde{l}_{\theta,j}(s)|^2 \leq L E \int_0^t |\bar{x}_{\theta,j}(s) - \tilde{l}_{\theta,j}(s)|^2 + |\bar{x}_{\theta}(s) - E\alpha_{\theta}(s)|^2 ds.
\]

By Gronwall inequality, we have
\[
E \sup_{0 \leq s \leq t} |\bar{x}_{\theta,j}(s) - \tilde{l}_{\theta,j}(s)|^2 \leq L E \int_0^t |\bar{x}_{\theta,j}(s) - \tilde{l}_{\theta,j}(s)|^2 + |\bar{x}_{\theta}(s) - E\alpha_{\theta}(s)|^2 ds.
\]

Next, recalling the state equations (32) and (33), similarly we have
\[
E \sup_{0 \leq s \leq t} |\bar{x}_{j}(s) - \tilde{l}_{j}(s)|^2 \leq L E \int_0^t |\bar{x}^{(N)}(s) - E\alpha(s)|^2 ds.
\]

Note that for any $t \in [0, T]$, \[E|\bar{x}^{(N)}(t) - E\alpha(t)|^2 \leq 2E\left|\frac{1}{N} \sum_{j=1}^{N} \bar{x}_{j}(t) - \frac{1}{N} \sum_{j=1}^{N} \int S \bar{x}_{\theta,j}(t) d\Phi(\theta)\right|^2 \]
\[+ 2E\left|\frac{1}{N} \sum_{j=1}^{N} \int S \bar{x}_{\theta,j}(t) d\Phi(\theta) - \int S E[\alpha(t)|\Theta = \theta] d\Phi(\theta)\right|^2 \]
\[\leq \frac{6}{N} \sum_{j=1}^{N} E|\bar{x}_{j}(t) - \tilde{l}_{j}(t)|^2 + \frac{6}{N^2} \sum_{j=1}^{N} E|\tilde{l}_{j}(t)|^2 - \int S E[|\alpha(t)|^2 \Theta = \theta] d\Phi(\theta) \]
\[+ \frac{12}{N^2} \sum_{1 \leq j \neq k \leq N} (E(\tilde{l}_{j}(t) - \int S \tilde{l}_{\theta,j}(t) d\Phi(\theta)), E(\tilde{l}_{k}(t) - \int S \tilde{l}_{\theta,k}(t) d\Phi(\theta))) \]
\[+ 6E\left|\frac{1}{N} \sum_{j=1}^{N} \int S \tilde{l}_{\theta,j}(t) d\Phi(\theta) - \frac{1}{N} \sum_{j=1}^{N} \int S \bar{x}_{\theta,j}(t) d\Phi(\theta)\right|^2 \]
\[+ 2 \int S E\left|\frac{1}{N} \sum_{j=1}^{N} \bar{x}_{\theta,j}(t) d\Phi(\theta) - E[\alpha(t)|\Theta = \theta]\right|^2 d\Phi(\theta).\]
Similar to Lemma 5.1, there exists a constant $L$ such that $\sup_{\theta \in S} \sup_{1 \leq j \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_{\theta,j}(t)|^2 \leq L$. Consequently,

$$
\frac{6}{N^2} \sum_{j=1}^{N} \mathbb{E} |\bar{l}_j(t)| - \int_{\mathcal{S}} |\bar{t}_{\theta,j}(t)| d\Phi(\theta)|^2 \leq \frac{L}{N}.
$$

From $\mathbb{E} \alpha = \int_{\mathcal{S}} \mathbb{E} \alpha(t) d\Phi(\theta)$ and $\mathbb{E} (A_{\theta,j} \bar{l}_j) = \int_{\mathcal{S}} \mathbb{E} (A_{\theta} \bar{t}_{\theta,j}) d\Phi(\theta)$, we have

$$
\mathbb{E} (\bar{l}_j(t) - \int_{\mathcal{S}} \bar{t}_{\theta,j}(t) d\Phi(\theta)) = 0.
$$

It is easy to see that

$$
\mathbb{E} \left| \frac{1}{N} \sum_{j=1}^{N} \int_{\mathcal{S}} \bar{t}_{\theta,j}(t) d\Phi(\theta) - \frac{1}{N} \sum_{j=1}^{N} \int_{\mathcal{S}} \bar{x}_{\theta,j}(t) d\Phi(\theta) \right|^2
$$

$$
= \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^{N} \int_{\mathcal{S}} (\bar{t}_{\theta,j}(t) - \bar{x}_{\theta,j}(t)) d\Phi(\theta) \right|^2
$$

$$
\leq \frac{1}{N} \sum_{j=1}^{N} \int_{\mathcal{S}} \mathbb{E} |\bar{t}_{\theta,j}(t) - \bar{x}_{\theta,j}(t)|^2 d\Phi(\theta).
$$

Substituting (35), (36), (38), (39), and (40) into (37), we have

$$
\mathbb{E} |\bar{x}^{(N)}(t) - \mathbb{E} \alpha(t)|^2
$$

$$
\leq L \mathbb{E} \int_{0}^{t} |\bar{x}^{(N)}(s) - \mathbb{E} \alpha(s)|^2 ds + \frac{L}{N} \sum_{j=1}^{N} \int_{\mathcal{S}} \mathbb{E} \int_{0}^{t} |\bar{x}_{\theta,j}^{(N)}(s) - \mathbb{E} \alpha(s)|^2 ds d\Phi(\theta)
$$

$$
+ 2 \int_{\mathcal{S}} \mathbb{E} \left| \frac{1}{N} \sum_{j=1}^{N} \bar{t}_{\theta,j}(t) - \mathbb{E} [\alpha(t)|\Theta = \theta]|^2 d\Phi(\theta).
$$

Applying similar method as homogeneous case (e.g. [37, Lemma 6.3]), we have $\mathbb{E} \left| \frac{1}{N} \sum_{j=1}^{N} \bar{t}_{\theta,j}(t) - \mathbb{E} [\alpha(t)|\Theta = \theta]|^2 \leq \frac{L}{N}$, and $\mathbb{E} \int_{0}^{t} |\bar{x}_{\theta,j}^{(N)}(s) - \mathbb{E} \alpha(s)|^2 ds \leq \frac{L}{N}$. Therefore, there exists a constant $L$ independent of $t$ such that $\mathbb{E} |\bar{x}^{(N)}(t) - \mathbb{E} \alpha|^2 \leq L \mathbb{E} \int_{0}^{t} |\bar{x}^{(N)}(s) - \mathbb{E} \alpha(s)|^2 ds + \frac{L}{N}$. By Gronwall inequality, we have

$$
\mathbb{E} |\bar{x}^{(N)}(t) - \mathbb{E} \alpha|^2 \leq \frac{L}{N} e^{L t}.
$$

\[\square\]

**Lemma 5.4.** There exist some constant $L$ independent of $N$ such that

$$
\sup_{0 \leq t \leq T} \mathbb{E} |x^{**} - \delta x_{-i}|^2 \leq \frac{L}{N},
$$

$$
\mathbb{E} \sup_{0 \leq t \leq T} |N \delta x_{j,i} - x_{j}^*|^2 \leq \frac{L}{N}, \ j \neq i.
$$
PROOF. Introduce the following equations

\[
\begin{align*}
\delta x_i &= \left[A_\theta \delta x_i + B \delta u_i + \frac{F}{N} \delta x_i + \frac{F}{N} x^{**}\right] dt + [C \delta x_i + D_\theta \delta u_i + \tilde{F} \delta x_i + \tilde{F} x^{**}] dW_i,
\delta x_j &= \left[A_\theta \delta x_j + B \delta u_i + \frac{F}{N} \delta x_j + \frac{F}{N} x^{**}\right] dt + [C \delta x_j + D_\theta \delta u_i + \tilde{F} \delta x_j + \tilde{F} x^{**}] dW_j, 
\delta \bar{x}_i(0) = 0, \delta \bar{x}_j(0) = 0.
\end{align*}
\]

Recalling (7), by Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality, and Gronwall inequality, we have

\[
E \sup_{0 \leq s \leq t} |\delta x_{j,i}(s) - \delta \bar{x}_j(s)|^2 \leq \frac{L}{2} E \int_0^t |\delta \bar{x}_{-i}(s) - x^{**}(s)|^2 ds.
\]

For any \( \theta \in S \), let

\[
\begin{align*}
\delta \bar{x}_{\theta,i} &= \left[A_\theta \delta x_{\theta,i} + B_\theta \delta u_i + F_\theta \delta x_{\theta}^{(N)}\right] dt + [C_\theta \delta x_{\theta,i} + D_\theta \delta u_i + \tilde{F}_\theta \delta x_{\theta} + \tilde{F}_\theta x^{**}] dW_i, \delta x_{\theta,i}(0) = 0, \\
\delta \bar{x}_{\theta,j} &= \left[A_\theta \delta x_{\theta,j} + B_\theta \delta u_i + F_\theta \delta x_{\theta}^{(N)}\right] dt + [C_\theta \delta x_{\theta,j} + D_\theta \delta u_i + \tilde{F}_\theta \delta x_{\theta} + \tilde{F}_\theta x^{**}] dW_j, \delta x_{\theta,j}(0) = 0,
\end{align*}
\]

\[
\begin{align*}
\delta \bar{x}_{\theta,i} &= \left[A_\theta \delta x_{\theta,i} + B_\theta \delta u_i + F_\theta \delta x_{\theta,i} + \frac{F}{N} x^{**}\right] dt + [C_\theta \delta x_{\theta,i} + D_\theta \delta u_i + \tilde{F}_\theta \delta x_{\theta,i} + \tilde{F}_\theta x^{**}] dW_i,
\delta \bar{x}_{\theta,j} &= \left[A_\theta \delta x_{\theta,j} + B_\theta \delta u_i + F_\theta \delta x_{\theta,j} + \frac{F}{N} x^{**}\right] dt + [C_\theta \delta x_{\theta,j} + D_\theta \delta u_i + \tilde{F}_\theta \delta x_{\theta,j} + \tilde{F}_\theta x^{**}] dW_j,
\delta \bar{x}_{\theta,i}(0) = 0, \delta \bar{x}_{\theta,j}(0) = 0, j \neq i,
\end{align*}
\]

where \( \delta x_{\theta}^{(N)} = \frac{1}{N} \sum_{j=1}^N \delta x_{\theta,j} \). Similarly,

\[
E \sup_{0 \leq s \leq t} |\delta x_{\theta,j}(s) - \delta \bar{x}_{\theta,j}(s)|^2 \leq \frac{L}{2} E \int_0^t |\sum_{j \neq i} \delta x_{\theta,j}(s) - x_{\theta}^{**}(s)|^2 ds.
\]

For any \( t \in [0, T] \),

\[
E |x^{**}(t) - \delta x_{-i}(t)|^2
\leq 6(N - 1) \sum_{j \neq i} E |\delta x_j - \delta \bar{x}_j|^2 + 6 \sum_{j \neq i} E |\delta \bar{x}_j - \int_S \delta \bar{x}_{\theta,j} d\Phi(\theta)|^2
+ 12 \sum_{1 \leq j \neq k \leq N, j \neq i} E \langle \delta \bar{x}_j - \int_S \delta \bar{x}_{\theta,j} d\Phi(\theta), \delta \bar{x}_k - \int_S \delta \bar{x}_{\theta,k} d\Phi(\theta) \rangle
+ 6(N - 1) \sum_{j \neq i} \int_S E |\delta \bar{x}_{\theta,j} - \delta x_{\theta,j}|^2 d\Phi(\theta) + 2 \int_S E \sum_{j \neq i} \delta x_{\theta,j} - x_{\theta}^{**}|^2 d\Phi(\theta).
\]

Similar to Lemma 5.3, we have

\[
E |x^{**}(t) - \delta x_{-i}(t)|^2
\leq L \int_0^t |\delta x_{-i}(s) - x^{**}(s)|^2 ds + \frac{L}{N} + L \int_S E \int_0^t \sum_{j \neq i} |\delta x_{\theta,j}(s) - x_{\theta}^{**}(s)|^2 ds d\Phi(\theta)
+ 2 \int_S E \sum_{j \neq i} \delta x_{\theta,j} - x_{\theta}^{**}|^2 d\Phi(\theta).
\]
Applying similar technique as homogeneous case (e.g., pp. 29 in [37]), we have 
$$
\mathbb{E}\sup_{0 \leq t \leq T} \sum_{j \neq i} |\delta x_{i,j}(s) - x_{i,j}^*(s)|^2(s) \leq \frac{L}{N}.
$$
Therefore, there exists a constant $L$ independent of $t$ such that 
$$
\mathbb{E}|x_{i,j}^*(t) - \delta x_{i,j}(t)|^2 \leq L\mathbb{E} \int_0^t |\delta x_{i,j}(s) - x_{i,j}^*(s)|^2 ds + \frac{L}{N}.
$$
By Gronwall inequality, we have 
$$
\mathbb{E}|x_{i,j}^*(t) - \delta x_{i,j}(t)|^2 \leq \frac{L}{N}e^{L \xi}.
$$
Hence (41) follows. Note that 
$$
\begin{align*}
\left\{ d(x_j^* - N\delta x_{j,i}) = & [A_{\Theta}(x_j^* - N\delta x_{j,i}) + F(x_{i,j}^* - \delta x_{i,j})]dt \\
& + [C(x_j^* - N\delta x_{j,i}) + F(x_{i,j}^* - \delta x_{i,j})]dW_j,
\end{align*}
$$
where 
$$
\begin{align*}
(\tilde{x}_j - \delta x_{j,i})(0) = 0.
\end{align*}
$$
By (41), we have (42).

The following result follows directly by Lemma 5.3 together with the common Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality and Gronwall inequality.

**Lemma 5.5.** There exists a constant $L$ independent of $N$ such that 
$$
\sup_{1 \leq j \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\tilde{\mu}_j - \tilde{x}_j|^2 \leq \frac{L}{N}.
$$

5.4. Asymptotic optimality. *In view of §5.1-§5.3, we are now ready to complete Step 4, i.e., to establish the asymptotic optimality of $\tilde{\mu} = (\tilde{u}_1, \ldots, \tilde{u}_N)$. In order to prove asymptotic optimality, it suffices to consider the perturbations $u_i \in U^c$ such that $J_{soc}^{(N)}(u_1, \ldots, u_N) \leq J_{soc}^{(N)}(\tilde{u}_1, \ldots, \tilde{u}_N)$. It is easy to check that $J_{soc}^{(N)}(\tilde{u}_1, \ldots, \tilde{u}_N) \leq LN$, where $L$ is a constant independent of $N$. Therefore, in the following we only consider perturbations $u_i \in U^c$ satisfying $\sum_{i=1}^N \mathbb{E} \int_0^T |u_i|^2 dt \leq LN$. Therefore, similar to Lemma 5.3 and Lemma 5.5, we have*

**Lemma 5.6.** There exist a constant $L$ independent of $N$ such that 
$$
\mathbb{E} \sup_{0 \leq t \leq T} |\dot{x}^{(N)}(t) - \mathbb{E}\alpha|^2 \leq \frac{L}{N}, \quad \sup_{1 \leq j \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\dot{x}_j|^2 \leq \frac{L}{N}.
$$

Let $\delta u_i = u_i - \tilde{u}_i$, and consider a perturbation $u = \tilde{u} + (\delta u_1, \ldots, \delta u_N) := \tilde{u} + \delta u$. Then by Section 5.1, we have 
$$
2J_{soc}^{(N)}(\tilde{u} + \delta u) = \langle M_2(\tilde{u} + \delta u), \tilde{u} + \delta u \rangle + 2\langle M_1, \tilde{u} + \delta u \rangle + M_0
$$
$$
= 2J_{soc}^{(N)}(\tilde{u}) + 2\sum_{i=1}^N \langle M_2(\tilde{u}) + M_1, \delta u_i \rangle + \langle M_2(\delta u), \delta u \rangle,
$$
where $M_2(\tilde{u}) + M_1$ is the Fréchet differential of $J_{soc}^{(N)}$ on $\tilde{u}$.

**Theorem 5.7.** *Under the assumptions (A1)-(A5), $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N)$ defined in (31) is a $\left(\frac{1}{\sqrt{N}}\right)$-social optimal strategy for the agents.*

**Proof.** From Section 5.2, we have 
$$
\langle M_2(\tilde{u}) + M_1, \delta u_i \rangle = \mathbb{E} \int_0^T \left[ \langle Q\tilde{L}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R\tilde{u}_i, \delta u_i \rangle \right] dt + \sum_{l=1}^7 \varepsilon_l.
$$
From the optimality of $\tilde{u}$, we have \( \mathbb{E} \int_0^T \left[ \langle Q \tilde{I}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt \geq 0 \). Suppose this is not true, then for $u_i$ such that $\tilde{u}_i + u_i \in \mathcal{U}^{d,p}$, we have $\tilde{u}_i + \rho u_i \in \mathcal{U}^{d,p}$, $0 < \rho < 1$, and $\lim_{\rho \to 0} J_i(\tilde{u}_i, \rho u_i, \tilde{u}_{\rho} - 1) < 0$. Thus, $J_i(\tilde{u}_i, \rho u_i, \tilde{u}_{\rho} - 1) < J_i(\tilde{u}_i, \tilde{u}_{\rho})$ for sufficiently small $\rho$, which is a contradiction with the optimality of $\tilde{u}_i$. Moreover, combing Lemmas 5.3-5.6 with iteration analysis (e.g., [23]), we have $\sum_{i=1}^7 \xi_i = O\left( \frac{1}{\sqrt{N}} \right)$. Therefore,

$$
\mathcal{J}^{\text{soc}}(\tilde{u} + \delta u) = \mathcal{J}^{\text{soc}}(\tilde{u}) + \sum_{i=1}^N \mathbb{E} \int_0^T \left[ \langle Q \tilde{I}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt + \sum_{i=1}^N \sum_{l=1}^5 \varepsilon_i + \frac{1}{2} \langle M_2(\delta u), \delta u \rangle.
$$

Note that $\sum_{i=1}^N \mathbb{E} \int_0^T \left[ \langle Q \tilde{I}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt + \frac{1}{2} \langle M_2(\delta u), \delta u \rangle \geq 0$, and $\sum_{i=1}^N \sum_{l=1}^7 \varepsilon_i = O(\sqrt{N})$, there exists a constant $L$ independent of $N$ such that

$$
\frac{1}{N} \left( \mathcal{J}^{\text{soc}}(\tilde{u}) - \inf_{u \in \mathcal{U}^C} \mathcal{J}^{\text{soc}}(u) \right) \leq \frac{L}{\sqrt{N}}.
$$

\[ \square \]

6. Synthetic analysis on homogeneity and heterogeneity.

6.1. Literature comparison. We now present comparisons to some relevant mean-field literature.

6.1.1. Homogeneous case without diversity. For the homogeneous case with $S = \{s_1\}$ being singleton set, we have $A_{\Phi_{s_1}} = A_{s_1} := A$ and $D_{\Phi_{s_1}} = D_{s_1} := D$ for $i = 1, \ldots, N$. In this case, we do not need to introduce $x^{\text{ss}}$ as in (11) when applying variational decomposition. We only need to introduce $x^{\text{ss}}$ to replace $\delta x_{-i}$. In fact, in current case, $x^{\text{ss}}$ satisfies the

$$
dx^{\text{ss}} = [(A + F)x^{\text{ss}} + F\delta x_i]dt, \quad x^{\text{ss}}(0) = 0.
$$

Moreover, CC in homogeneous case becomes

$$
\begin{aligned}
d\alpha &= [A\alpha + B\Gamma_{\mathbb{R}^m}[R^{-1}\varepsilon_t[-B^\top \gamma - D^\top \theta]] + F\mathbb{E} \alpha] dt \\
&\quad + [C\alpha + D\Gamma_{\mathbb{R}^m}[R^{-1}\varepsilon_t[-B^\top \gamma - D^\top \theta]] + F\mathbb{E} \alpha] dW, \\
d\gamma &= [-Q\alpha + (QH + HQ - HQH)\mathbb{E} \alpha - A^\top \gamma + F^\top \tilde{y}_2 + F^\top \mathbb{E} \tilde{y}_1 - C^\top \theta] \\
&\quad + F^\top \mathbb{E} \tilde{y}_1] dt + \theta dW(t), \\
d\tilde{y}_1 &= [Q\alpha - A^\top \tilde{y}_1 - C^\top \tilde{y}_1] dt + \tilde{\beta}_1 dW, \\
d\tilde{y}_2 &= [-QH + HQ - HQH]\mathbb{E} \alpha - F^\top \mathbb{E} \tilde{y}_1 - F^\top \mathbb{E} \tilde{y}_1 - A^\top \tilde{y}_2 - F^\top \mathbb{E} \tilde{y}_1] dt, \\
\alpha(0) &= \xi, \quad \gamma(T) = 0, \quad \tilde{y}_1(T) = 0, \quad \tilde{y}_2(T) = 0.
\end{aligned}
$$

(47)

This is the special case of (26) with $\Phi(\theta)$ being a Dirac distribution. Subsequently, our framework covers the homogeneous case as its special case. Furthermore, in case $C = D = F = \bar{F} = 0$, $\Gamma = \mathbb{R}^m$ and $\mathbb{C}^2 = \mathbb{R}^2$, by taking expectation, $\tilde{\alpha} = \mathbb{E} \alpha$ and $\tilde{\gamma} = \mathbb{E} \gamma$ satisfy

$$
\begin{aligned}
d\tilde{\alpha} &= [A\tilde{\alpha} - BR^{-1}B^\top \gamma] dt, \quad \tilde{\alpha}(0) = \xi, \\
d\tilde{\gamma} &= [(QH + HQ - HQH)\tilde{\alpha} - A^\top \tilde{\gamma}] dt, \quad \tilde{\gamma}(T) = 0.
\end{aligned}
$$

(48)

This is just the special case discussed in pp. 1742 of [27] (see (42),(43) therein). The only difference is that (48) is of open-loop ($\tilde{\gamma}$ is the adjoint process) while (42) and (43) in [27] are of closed-loop ($\Pi \bar{x} + s$ is of feedback form).
6.1.2. Heterogeneous case with finite diversities. Specifically, we assume that $\Theta_i$ is deterministic (post-sampling) and assumes values in a finite discrete set $S = \{1, 2, \cdots, K\}$. For $1 \leq k \leq K$, introduce $\mathcal{I}_k = \{i|\Theta_i = k, 1 \leq i \leq N\}$, $\mathcal{N}_k = |\mathcal{I}_k|$, where $\mathcal{N}_k$ is the cardinality of index set $\mathcal{I}_k$ (i.e., cardinality of set of $k$-type agents). For $1 \leq k \leq K$, let $\pi^{(N)}_k = \frac{\mathcal{N}_k}{N}$, then $\pi^{(N)} = (\pi^{(N)}_1, \cdots, \pi^{(N)}_K)$ is a probability vector representing the empirical distribution of $\Theta_1, \cdots, \Theta_N$. Suppose there exists a probability mass vector $\pi = (\pi_1, \cdots, \pi_K)$ such that
\[
\lim_{N \to +\infty} \pi^{(N)} = \pi \quad \text{and} \quad \min \pi_k > 0.
\]
Under these assumptions, the variational decomposition procedure still proceeds as in Section 3.1. Let $\delta x_{(k)} = \sum_{j \in \mathcal{I}_k, j \neq i} \delta x_{j,i}$. By exchangeability of agents within same type, we only need to consider a representative agent in each type when using a limit to approximate $\delta x_{(k)}$. Therefore, for $k = 1, \cdots, K$, we should introduce the term $x^{**}_k$ to replace $\delta x_{(k)}$, where $x^{**}_k$ satisfies the following dynamics:
\[
dx^{**}_k = \left[ A_k x^{**}_k + F_k \delta x_i + F_k \pi_k \sum_{l=1}^{K} x^{**}_l \right] dt, \quad x^{**}_k(0) = 0, \quad k = 1, \cdots, K.
\]
Furthermore, if $\mathcal{G}^i = \mathcal{F}$, CC of heterogeneous case with finite diversities becomes:
\[
\begin{align*}
d\alpha_k &= [A_k \alpha_k + B \mathbf{P}_T[R_k^{-1}(B^T \gamma_k + D_k^T \vartheta_k)] + F \sum_{l=1}^{K} \pi[l] \mathcal{E}\alpha_l] dt \\
&\quad + [C \alpha_k + D_k \mathbf{P}_T[R_k^{-1}(B^T \gamma_k + D_k^T \vartheta_k)] + \tilde{F} \sum_{l=1}^{K} \pi[l] \mathcal{E}\alpha_l] dW_k(t), \\
d\gamma_k &= [-Q \alpha_k + (QH + HQ - HQH) \sum_{l=1}^{K} \pi[l] \mathcal{E}\alpha_l - A_k^T \gamma_k + F^T \sum_{l=1}^{K} \pi[l] \mathcal{E}\gamma_l] dt \\
&\quad - C^T \vartheta_k + \tilde{F}^T \sum_{l=1}^{K} \pi[l] \mathcal{E}\vartheta_l] dt + \vartheta_k dW_k(t), \\
d\tilde{y}_k^1 &= [Q \alpha_k - A_k^T \tilde{y}_k^1 - C^T \tilde{\beta}_k^1] dt + \tilde{\beta}_k^1 dW_k, \\
d\tilde{y}_k^2 &= -(QH + HQ - HQH) \sum_{l=1}^{K} \pi[l] \mathcal{E}\alpha_l - \sum_{l=1}^{K} \pi[l] (F^T \mathcal{E}\tilde{y}_l^1 + \tilde{F}^T \mathcal{E}\beta_l^1) - A_k^T \tilde{y}_k^2 - F^T \sum_{l=1}^{K} \pi[l] \mathcal{E}\tilde{y}_l^2) dt,
\end{align*}
\]
(49) is similar to the CC in [21] (see (2.15) therein). [21] deals with MFG with heterogeneous case with finite diversities, hence the CC only involves the Hamiltonian system of the auxiliary control problem. While for LQG-MT, besides the Hamilton system (25), CC also includes (19) by the weak duality procedure.

6.1.3. Heterogeneous case with continuum diversities but without state-coupling. When $F = \tilde{F} = 0$, i.e., there is no weakly-coupling in state, by (7) we have $\delta x_{j,i} \equiv 0, j \neq i$, thus $x_j^\ast, x_\theta^\ast$ both vanish in (11). The resulting (12) takes a rather simple form than (9),
\[
\begin{align*}
\delta J_{soc}^{(N)} = &E \int_0^T \left[ \langle Q \bar{x}, \delta x_i \rangle + \langle (QH + HQ - HQH) \dot{x}, \delta x_i \rangle + \langle R \bar{u}_i, \delta u_i \rangle \right] dt + \varepsilon_1, \\
\end{align*}
\]
where
\[
\varepsilon_1 = E \int_0^T \langle (QH + HQ - HQH)(\dot{x} - \bar{x}^{(N)}), N \delta x^{(N)} \rangle dt.
\]
From (50) we can obtain the auxiliary control problem directly, i.e., it becomes unnecessary to introduce the limit terms (11) and adjoint processes (13). This is similar to the case in Section IV.A of [27]. Note that in [27], there is no point-wise constraint or partial information constraint on the admissible control, hence the main focus is to find the optimal closed-loop control for the auxiliary control problem (see (32) therein). While with the above two constraints, we will obtain the optimal open-loop control for the auxiliary control problem (see (23)). In this case, (26) reduces to

\[
\begin{align*}
\frac{d\alpha}{dt} &= [A_\Theta \alpha + B_\Omega P_T [R^{-1} E_t \{-B^\top \gamma - D_\Theta^\top \vartheta\}]]dt \\
&\quad + [C_\alpha + D_\Theta P_T [R^{-1} E_t \{-B^\top \gamma - D_\Theta^\top \vartheta\}]]dW, \alpha(0) = \xi,
\end{align*}
\]

\[d\gamma = [-Q_\alpha + (QH + HQ - HQH)E\alpha - A_\Theta^\top \gamma - C_\Theta^\top \vartheta]dt + \vartheta dW(t), \gamma(T) = 0,
\]

for which the well-posedness is much more easily to establish. Furthermore, if \(C = D_\Theta = 0\), \(\Gamma = R^m\) and \(G_i = F_i\), by taking expectation to (51), the derived FBSDEs reduces to the case on pp. 1740 of [27].

By contrast, when \(F, \tilde{F} \neq 0\), variation functional \(\delta \mathcal{J}_{soc}^{(N)}(\delta u_i)\) of (12) becomes rather involved depending on \(x_j^*\) and \(x^{**}\) both. Those two terms are some intermediate variation limits related to basic variation term \(\delta x_i\) in an indirect manner. Thus, the current representation (12) cannot lead a direct construction to an auxiliary control. Some duality method are required to remove dependence on these intermediate variations.

6.1.4. Other cases. For homogeneous case, [20] studies linear-quadratic mean-field games with control process constrained in a closed convex subset of full space \(R^m\); [24] studies backward mean-field linear-quadratic games with partial information. When there involves only constraints on the control or only partial information, our framework is the extension of [20] and [24] for mean-field team case.

6.2. Homogeneity and heterogeneity: a unified quasi-exchangeable approach. Recall that the mean-field theory has been extensively applied to study the large-scale weakly-coupled system along both (competitive) game and (cooperative) team directions, see e.g., [5, 11, 21, 25, 26, 31] for recent relevant studies for game; and [37, 39] for team. Essentially, such mean-field analysis is build on some exchangeability among individual weakly-coupled agents. It can be proved that any exchangeable sequences should be conditional independent with respect to some tail-sigma algebra. Thus, applying de Finetti theorem, the original complex weakly-coupling structure can be replaced by a deterministic- or common-noise-driven process as agent number \(N\) tends to infinity. By this, all agents thus become asymptotically decoupled along with chaos propagation. Subsequently, original game or team can be reduced to low dimensional single agent optimization problem with some off-line quantities via consistency condition that matches the above exchangeable reasoning. In this sense, mean-field analysis connects closely to exchangeable game/team in random context, and further to symmetric game/team ([15]) in deterministic context. We remark that all agents in symmetric game are endowed with same underlying parameters and so become identical in analysis. So, the primal high-dimensional computation can be greatly reduced using “mirror” argument among all symmetric agents.

Regarding large-scale system, there exist three progressive levels of diversity relevant to aforementioned exchangeability: homogeneous, heterogenous with finite/discrete diversity, and heterogenous with continuum diversity. Among them, homogenous case is most special but tractable one because all agents are statistical identical and the designed optimal team strategies should also be exchangeable. Consequently, the resulting optimized states are thus exchangeable. We refer [37] for recent studies in such case for team, and [20] for game.
Compared with homogenous case, heterogenous case with finite/discrete diversity is more realistic. Virtually, most systems in reality demonstrate some diversities in their random behaviors. In this case, all agents, from whole system scale, are no longer identical because they are endowed with diversified parameters. However, all agents inside a sub-system with same diversity index, are still exchangeable in small scale. Thus, we can treat the large-scale system as some mixed combination of finite exchangeable sub-systems. The previous mean-field analysis to homogenous can be suitably modified to tackle such case, with some technical but straightforward arguments. We refer [2] for recent studies in such case for team in discrete time setup, and [25, 21] for game, where a similar partial exchangeability is introduced.

The heterogenous case with continuum diversity, as discussed in [27, 33], should be most realistic setup for practical large-scale system. Indeed, it is less possible that the diversity of real system, can only be limited on a finite or discrete support set. Instead, considerable statistical diversity demonstrates its support on a continuum set such as compact closed interval. On the other hand, such heterogenous case should be most difficult to be handled. One reason for the continuum heterogeneity to be analytically intractable, is that the sub-class exchangeability featured in finite heterogeneity case, will shrink to zero mass along with the continuum diversity support. For this reason, the relevant results for continuum heterogeneity seems few compared with homogeneous- or finite-heterogenous-case.

We remark [33] discussed mean-field analysis with continuum diversity in game setup, and [27] in team setup, using a direct state-aggregating method. However, the setting in both works are relatively simple, in particular, its weakly-coupled dynamics is only drift-controlled. This corresponds to our model with $C = D = \tilde{F} = 0$, and cannot cover various applications such as portfolio selection with relative performance. Our setup is more general (diffusion-controlled and -coupled) and above aggregation method no longer works. Meanwhile, due to continuum diversity, we cannot apply the weak embedding representation method used in [20, 21, 37] when tackling diffusion controlled system but of finite diversities only. Indeed, the analysis of [21] replies on a construction of $K$ independent copies of optimized states with individual BMs, where $K$ is the finite cardinality of diversity. This becomes impossible for current case in presence of continuum diversities.

As resolution, this paper proposes some unified approach to homogenous-, and heterogenous-case using a quasi-exchangeable method. The main idea is as follows: first, note that

$\begin{align*}
  &dx_i = [A_{\Theta}x_i + Bu_i + Fx^{(N)}}]dt + [Cx_i + D_{\Theta}u_i + \tilde{F}x^{(N)}]dW_i, \\
  &x_i(0) = \xi \in \mathbb{R}^n, \quad 1 \leq i \leq N,
\end{align*}$

can be reformulated as follows:

$\begin{align*}
  &dx_i = [A(z_i(t), t)x_i + Bu_i + Fx^{(N)}]dt + [Cx_i + D(z_i(t), t)u_i + \tilde{F}x^{(N)}]dW_i, \\
  &dz_i(t) \equiv 0, \\
  &x_i(0) = \xi \in \mathbb{R}^n; \quad z_i(0) = \Theta, \quad 1 \leq i \leq N,
\end{align*}$

that can be further written with some augmented state as

$dx_i = [A(x_i)x_i + Bu_i + Fx^{(N)}]dt + [Cx_i + D(x_i)u_i + \tilde{F}x^{(N)}]dW_i, \quad x_i(0) = (\xi^T, \Theta^T)^T.$

In other words, initial weakly-coupled system with continuum diversity can be viewed as some quasi-linear SDE with augmented state $x_i = (x_i^T, z_i^T)^T$ and random initial conditions $x_i(0)$ (noting $\Theta \in \mathcal{F}_0$, although $\xi$ might be deterministic).

To proceed, we introduce the following three systems. To ease notation, we are inclined to adopt symbols like $A(x)$ instead $A(x)$ when no confusion occurs. The first system is a McKean-Vlasov SDE with random initials:

$\mathcal{P}_1: \quad dx = [A(x)x + Bu + FEx]dt + [Cx + D(x)u + \tilde{F}Ex]dW, \quad x(0) = (\xi^T, \Theta^T)^T.$
For sake of illustration, we set $\Theta \in \Lambda = \{\theta_1, \theta_2, \cdots, \theta_K\}$ with the mass $m_1, \cdots, m_K$ to admit finite $K$ diversity classes. Later, we will illustrate its possible extension to infinite continuum diversities. The second system is a stochastic mixture: $\bar{x} = \sum_{j=1}^{K} m_j \bar{x}_j$ but driven by identical noise $W$:

$$P_2: \quad d\bar{x}_j = [A_{\theta_j} \bar{x}_j + Bu + F\bar{x}] dt + [C\bar{x}_j + D_{\theta_j} u + \bar{F}\bar{x}] dW, \quad \bar{x}_j(0) = (\xi^T, \theta_j^T)^T.$$  

By contrast, the third system is also a stochastic mixture $\bar{x} = \sum_{j=1}^{K} m_j \bar{x}_j$ but driven by $K$ i.i.d noises $\{W_j\}_{j=1}^{K}$:

$$P_3: \quad d\bar{x}_j = [A_{\theta_j} \bar{x}_j + Bu + F\bar{x}] dt + [C\bar{x}_j + D_{\theta_j} u + \bar{F}\bar{x}] dW_j, \quad \bar{x}_j(0) = (\xi^T, \theta_j^T)^T.$$  

It is obvious that above three systems: $x$, $\bar{x}$ and $\hat{x}$ are of the not same distributions. Actually, $x$ has different initial distribution at $t = 0$ with $\bar{x}$, $\hat{x}$, whereas $\bar{x}$ is driven by different noise with $x$, $\bar{x}$. However, they have same expectation dynamics, as verified using tower property of conditional expectation, $\forall t \in [0, T]: \mathbb{E}(x(t)) = \mathbb{E}(\mathbb{E}(x(t) | \Theta)) = \sum_{j=1}^{K} m_j \bar{x}_j(t) = \bar{x}(t) = \sum_{j=1}^{K} m_j \bar{x}_j(t) = \mathbb{E}(\hat{x}(t))$. Besides, all three systems have different second-moment function, and other finite-dimensional distributions. For example,

$$\mathbb{E}[x(t)]^2 = \mathbb{E}(\mathbb{E}(|x(t)|^2 | \Theta)) = \sum_{j=1}^{K} m_j \mathbb{E}[\bar{x}_j(t)]^2,$$

$$\mathbb{E}[\bar{x}(t)]^2 = \sum_{j=1}^{K} m_j^2 \mathbb{E}[\bar{x}_j(t)]^2 + \sum_{1 \leq j < l \leq K} m_j m_l \mathbb{E}[\bar{x}_j(t) \bar{x}_l(t)],$$

$$\mathbb{E}[\hat{x}(t)]^2 = \sum_{j=1}^{K} m_j^2 \mathbb{E}[\bar{x}_j(t)]^2 + \sum_{1 \leq j < l \leq K} m_j m_l \mathbb{E}[\bar{x}_j(t) \bar{x}_l(t)] = \sum_{j=1}^{K} m_j^2 \mathbb{E}[\bar{x}_j(t)]^2.$$

Noticing above expectation equivalence is special degenerated version of Jensen inequality, thanks to the underlying LQG context. Such property cannot be extended to nonlinear moments hence $x$, $\bar{x}$ and $\hat{x}$ are of the not same expectation but different distributions.

Corresponding to $P_1$, $P_2$, $P_3$, we may construct three weakly-coupled systems $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$:

$$\mathcal{M}_1: \quad dx_i = [A(x_i)x_i + Bu_i + Fx_i(N)] dt + [C(x_i) + D(x_i)u_i + \bar{F}x_i(N)] dW_i, \quad x_i(0) = (\xi^T, \Theta)^T,$$

where $x_i(N) = \frac{1}{N} \sum_{i=1}^{N} x_i$. Another is system $\mathcal{M}_2: \{\bar{x}_i\}_{i=1}^{N}$ with $\bar{x}_i = \sum_{j=1}^{K} m_j \bar{x}_{i,j}$.

$$\mathcal{M}_2: \quad d\bar{x}_{i,j} = [A_{\theta_j} \bar{x}_{i,j} + Bu_i + F\bar{x}_i(N)] dt + [C\bar{x}_{i,j} + D_{\theta_j} u_i + \bar{F}\bar{x}_i(N)] dW_i, \quad \bar{x}_{i,j}(0) = (\xi^T, \theta_j^T)^T,$$

where $\bar{x}_i(N) = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i$. For $1 \leq j \leq K$, we can introduce $\bar{\mathcal{M}}_j: \{\bar{x}_{i,j}\}_{i=1}^{N}$ that is a homogeneous weakly-coupled system indexed by $\theta_j$. Abusing notation, we may write informally that $\mathcal{M}_2 = \sum_{j=1}^{K} m_j \bar{\mathcal{M}}_j$, in other words, $\mathcal{M}_2$ is a finite mixture of homogeneous systems $\{\bar{\mathcal{M}}_j\}_{j=1}^{K}$. Noticing for $\bar{\mathcal{M}}_j$, the driving BMs become $\{W_i\}_{i=1}^{N}$ which are same to that of $\bar{\mathcal{M}}_j$ for $j \neq j'$. Thus, totally there involve $N$ independent BMs for $\mathcal{M}_2$. Moreover, if we introduce a sampling sequence from $\{1, \cdots, K\}$ with $I_j = \{\theta_i = j, 1 \leq i \leq N\}$ and $\lim_{N \to \infty} \frac{\text{Card} I_j}{N} = m_j, \quad 1 \leq j \leq K$. Then, $\mathcal{M}_2$ is equivalent in weak sense to stochastic $K$—heterogenous weakly-coupled system introduced in [21, 25].

The third system is $\mathcal{M}_3: \{\bar{x}_i\}_{i=1}^{N}$ with $\bar{x}_i = \sum_{j=1}^{K} m_j \bar{x}_{i,j}$.

$$\mathcal{M}_3: \quad d\bar{x}_{i,j} = [A_{\theta_j} \bar{x}_{i,j} + Bu_i + F\bar{x}_i(N)] dt + [C\bar{x}_{i,j} + D_{\theta_j} u_i + \bar{F}\bar{x}_i(N)] dW_{i,j}, \quad \bar{x}_{i,j}(0) = (\xi^T, \theta_j^T)^T,$$
where $\hat{x}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \hat{x}_i$. For $1 \leq j \leq K$, we can introduce $\hat{\mathcal{M}}_3^j : \{\hat{x}_i,j\}_{i=1}^{N}$ that is a homogeneous weakly-coupled system indexed by $\theta_j$. Noticing for $\hat{\mathcal{M}}_3^j$, the driving BMs become $\{W_{i,j}\}_{i=1}^{N}$. So, totally there arise $N \times K$ independent BMs for $\hat{\mathcal{M}}_3$, or re-scale to $N$ BMs for each sub-system $\hat{\mathcal{M}}_3^j, 1 \leq j \leq K$. This is not problematic when $K$ is finite. Again, $\hat{\mathcal{M}}_3$ is finite mixture of homogeneous system $\{\hat{\mathcal{M}}_3^j\}_{j=1}^{K}$. We remark that $\hat{\mathcal{M}}_3^i$ and $\hat{\mathcal{M}}_2^i$ are driven by different BMs, but they are equivalent weak-coupled homogeneous system in weak sense. This is because they share have same state-average limit by law of large numbers, although they are driven by different BMs systems.

Moreover, we can introduce an augmented state $y_i = (\hat{x}_{i,1}^T, \cdots, \hat{x}_{i,K}^T)^T$ and $\hat{x}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \hat{x}_i$, it follows that

$$dy_i = [\hat{A}y_i + \hat{B}\hat{u}_i + \hat{F}y^{(N)}]dt + \sum_{j=1}^{K} (\hat{C}_jy_i + \hat{D}_j\hat{u}_i + \hat{F}_jy^{(N)})dW_{i,j}, \ y_i(0) = (\xi^T, \theta_1^T, \cdots, \xi^T, \theta_K^T)^T;$$

where

$$\hat{A} = \begin{pmatrix} A_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{nK} \end{pmatrix}_{(nK \times nK)}, \quad \hat{B} = \begin{pmatrix} B \\ \vdots \\ 0 \end{pmatrix}_{(nK \times mK)}, \quad \hat{u}_i = \begin{pmatrix} u_i \\ \vdots \\ u_i \end{pmatrix}_{(mK \times 1)};$$

$$\hat{F} = \begin{pmatrix} F_{m_1} & \cdots & F_{m_K} \\ \vdots & \ddots & \vdots \\ F_{m_1} & \cdots & F_{m_K} \end{pmatrix}_{(nK \times nK)}, \quad \hat{C}_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ j & \cdots & \cdots & 0 \\ K & 0 & \cdots & 0 \end{pmatrix}_{(nK \times nK)};$$

$$\hat{D}_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ j & \cdots & \cdots & 0 \\ K & 0 & \cdots & 0 \end{pmatrix}_{(nK \times nK)}; \quad \hat{F}_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ j & \cdots & \cdots & 0 \\ K & 0 & \cdots & 0 \end{pmatrix}_{(nK \times nK)}.$$
they share the same state-average limit (in formulation, and Step 1 for decomposition) and expectation operator (in Step 3 for CC).

To recap, we present the following diagram where “⇐⇒” represents the equivalent expectation operator in first line, while asymptotic state-average operator in second line:

(52)

\[
\begin{align*}
\text{single-agent: } & P_1 \iff P_2 \iff P_3, \\
\text{weakly-coupled agents: } & M_1 \iff M_2 \iff M_3 \iff M \text{ (stochastic } K\text{-heterogenous system),} \\
M_1 & : \text{homogenous but with random diversity index } \Theta, \text{ augmented randomness, pre-sampling} \\
M_2 & : \text{mixture of } K \text{ homogenous system, pre-sampling} \\
M_3 & : \text{homogenous system with (augmented) mixture of states, post-sampling} \\
M & : K \text{ heterogenous system defined by relative frequency of diversity sequence, post-sampling.}
\end{align*}
\]

Above arguments in (52) are on the basis that \( \Theta \) is finite-valued only. Now we present its generalization to case when \( \Theta \) has continuum diversity support. In this case, we have

\[
\begin{align*}
M_1^c & : \text{homogenous but with random diversity index } \Theta, \text{ augmented randomness, pre-sampling} \\
M_2^c & : \text{mixture of continuum homogenous system, pre-sampling} \\
M_3^c & : \text{homogenous system with (augmented) mixture of states, post-sampling} \\
M^c & : \text{continuum heterogenous system defined by empirical distribution of diversity sequence, post-sampling.}
\end{align*}
\]

\( M_1^c \) is still well-defined and we have already proceeded the analysis as in Section 3. On the other hand, \( M_3^c \) is no longer well defined since now we have to introduce continuum-valued BMs for \( \hat{M}_3^{\theta,c} \) to model the diversity. By contrast, \( M_2^c \) is still well defined since we need still only to formulate countable BMs for each \( \hat{M}_2^{\theta,c} \), \( \theta \in S \), and in total, only countable BMs are still invoked. In this case, we may further set \( \hat{x}_i = \int_S \hat{x}_{i,\theta} d\Phi(\theta) \) and proceed the classical mean-field analysis as in [27]. However, classical mean-field analysis only works on \( M_2^c \) with \( C = D = F = \tilde{F} = 0 \). In general case with \( F, \tilde{F} \neq 0 \), such classical analysis fails because its CC system should invoke an embedding representation (see e.g., [23]), and a continuum-valued BMs system will be required to replicate the distribution for a generic agent who is still continuum-heterogenous (diversified). Moreover, in [26], the continuum heterogeneity is defined through some limiting empirical distribution by Glivenko-Cantelli Lemma. Note that the continuum set therein is required to be compact when using Glivenko-Cantelli arguments, while in our framework of \( M_1^c \), such compactness is not required. Consequently, this paper can deal with general continuum diversity based on \( M_1^c \), as summarized as follows.

First, we can verify that \( M_1^c, M_2^c \) as well as \( M^c \) (note that \( M_3^c \) becomes infeasible to be defined) are still of the same asymptotic state-average limit. In this sense, the generic agents in \( M^c \) are quasi-exchangeable because although they are not exchangeable after diversity sampling, \( M^c \) shares the same expectation and asymptotic state-average limit with \( M_1^c, M_2^c \), and all agents of \( M_1^c \) are exchangeable before the sampling. Second, given such quasi-exchangeable property, the original \( M^c \) or \( M_1^c \) system with continuum heterogeneity can be converted to \( M_1^c \) that is a homogenous one but with augmented randomness (\( \{\Theta_i, W_i\}_{i=1}^{N} \)) as trade-off. Third, as discussed in Section 3, some new type variation-decomposition and auxiliary control problem can thus be constructed, and CC can be represented via the construction on continuum diversity support as in Proposition 4.1.
APPENDIX

First, for any given \((Y, Z) \in L^2_F(0, T; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^m)\) and \(0 \leq t \leq T\), the following SDE has a unique solution:

\[
X(t) = x + \int_0^t b(s, X, \mathbb{E}[X], \mathcal{E}_t[Y], \mathcal{E}_t[Z])ds + \int_0^t \sigma(s, X, \mathbb{E}[X], \mathcal{E}_t[Y], \mathcal{E}_t[Z])dW(s).
\]

Therefore, we can introduce a map \(\mathcal{M}_t : L^2_F(0, T; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^m) \rightarrow L^2_F(0, T; \mathbb{R}^n)\). Moreover, by the standard estimations of SDE, we have the following result:

**Lemma A.1.** Let \(X_i\) be the solution of (53) corresponding to \((Y_i, Z_i)\), \(i = 1, 2\) respectively. Then for all \(\rho \in \mathbb{R}\) and some constants \(l_1 > 0\), we have

\[
\mathbb{E}e^{-\rho t}|X(t)|^2 + \rho_1 \mathbb{E} \int_0^t e^{-\rho s}|X(s)|^2 ds \\
\leq (k_2l_1 + k_2^2) \mathbb{E} \int_0^t e^{-\rho s}|Y(s)|^2 ds + (k_2 l_1 + k_2^2) \mathbb{E} \int_0^t e^{-\rho s}|\hat{Z}(s)|^2 ds,
\]

where \(\rho_1 = \rho - 2 \rho_1 - 2k_1 - 2k_2 l_1 - k_2^2 - k_2^2\) and \(\hat{\Phi} := \Phi_1 - \Phi_2\). Moreover,

\[
\mathbb{E} \int_0^T e^{-\rho t}|X(t)|^2 dt \leq (k_2 l_1 + k_2^2) \mathbb{E} \int_0^T e^{-\rho t}|Y(t)|^2 + |\hat{Z}(t)|^2 dt,
\]

\[
e^{-\rho T} \mathbb{E} \int_0^T e^{-\rho t}|\hat{X}(t)|^2 dt \leq (1 \vee e^{-\rho T}) \left\{ (k_2 l_1 + k_2^2) \mathbb{E} \int_0^T e^{-\rho t}|Y(t)|^2 + |\hat{Z}(t)|^2 dt \right\}.
\]

Specially, if \(\rho_1 > 0\),

\[
e^{-\rho T} \mathbb{E} |X(T)|^2 \leq (k_2 l_1 + k_2^2) \mathbb{E} \int_0^T e^{-\rho t}|Y(t)|^2 dt + (k_2 l_1 + k_2^2) \mathbb{E} \int_0^T e^{-\rho t}|\hat{Z}(t)|^2 dt.
\]

Next, for any given \(X \in L^2_F(0, T; \mathbb{R}^n)\), consider the following BSDE:

\[
Y(t) = \int_t^T f(s, X, \mathbb{E}[X], Y, \mathbb{E}[Y], Z, \mathbb{E}[Z])ds - \int_t^T Z(s)dW(s).
\]

**Proposition A.2.** (54) admits a unique solution \((Y, Z) \in L^2_F(0, T; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^m)\).

**Proof.** For any fixed \((y, z) \in L^2_F(0, T; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^m)\),

\[
Y(t) = \int_t^T f(s, X, \mathbb{E}[X], Y, \mathbb{E}[y], Z, \mathbb{E}[z])ds - \int_t^T Z(s)dW(s)
\]

admits a unique solution \((Y, Z) \in L^2_F(0, T; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^m)\). Hence we can introduce the mapping \(\mathcal{N} : (y, z) \rightarrow (Y, Z)\). For any \((y, z), (y', z') \in L^2_F(0, T; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^m)\), denote \((Y, Z) = \mathcal{N}(y, z)\) and \((Y', Z') = \mathcal{N}(y', z')\). Let \((\hat{y}, \hat{z}, \hat{Y}, \hat{Z}) = (y-y', z-z', Y-Y', Z-Z')\).
Applying Itô’s formula to $e^{\delta t} |\bar{Y}(s)|^2$, we have

$$e^{\delta t} |\bar{Y}(t)|^2 + \int_t^T e^{\delta s} |\bar{Z}(s)| ds + \int_t^T \delta e^{\delta s} |\bar{Y}(s)| ds$$

$$\leq \int_t^T e^{\delta s} (2\rho_2 + 4k_3^2 + 4k_4^2 + 4k_5^2 + 4k_6^2) |\bar{Y}(s)|^2 ds$$

$$+ \frac{1}{4} \int_t^T e^{\delta s} (E[|\bar{y}|^2] + E[|\bar{y}^2|] + |\bar{z}|^2 + E[|\bar{z}^2|]) ds + 2 \int_t^T e^{\delta s} \langle \bar{Y}(s), \bar{Z}(s)dW(s) \rangle.$$  

Note that $E[|\bar{y}|^2] = E[|\bar{y}^2|]$, letting $\delta = 2\rho_2 + 4k_3^2 + 4k_4^2 + 4k_5^2 + 4k_6^2$ and taking expectation, we have $E \int_t^T e^{\delta s} (|\bar{Y}(s)|^2 + |\bar{Z}(s)|^2) ds \leq \frac{1}{2} E \int_t^T e^{\delta s} (|\bar{y}(s)|^2 + |\bar{z}(s)|^2) ds$, i.e., $\mathcal{N}$ is a contraction mapping. Hence (54) admits a unique solution $(Y,Z) \in L^2(0,T;\mathbb{R}^m) \times L^2(0,T;\mathbb{R}^m)$. □

Thus, we can introduce another map $\mathcal{M}_2 : L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \rightarrow L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. By the standard estimation of BSDE, we have the following result:

**Lemma A.3.** Let $(Y_i, Z_i)$ be the solution of (54) corresponding to $X_i, i = 1, 2$, respectively. Then for all $\rho \in \mathbb{R}$ and some constants $l_1, l_2, l_3 > 0$, we have

$$E e^{-\rho t} |\bar{Y}(t)|^2 + \bar{\rho}_2 E \int_t^T e^{-\rho s} |\bar{Y}(s)|^2 ds + (1 - k_3l_2 - k_6l_3) E \int_t^T e^{-\rho s} |\bar{Z}(s)|^2 ds$$

$$\leq 2k_2l_1 E \int_t^T e^{-\rho s} |\bar{X}(s)|^2 ds,$$

$$E e^{-\rho t} |\bar{Y}(t)|^2 + (1 - k_2l_1 - k_3l_1) E \int_t^T e^{-\rho s} |\bar{Z}(s)|^2 ds \leq k_2l_1 E \int_t^T e^{-\bar{\rho}_2(s-t) - \rho s} |\bar{X}(s)|^2 ds,$$

where $\bar{\rho}_2 = -\rho - 2\rho_2 - 2k_3 - 2k_4 - 2k_2l_1 - k_3l_2 - k_6l_3$, and $\Phi := \Phi_1 - \Phi_2, \Phi = X, Y, Z$. Moreover,

$$E \int_0^T e^{-\rho t} |\bar{Y}(t)|^2 dt \leq \frac{1}{\bar{\rho}_2} \frac{e^{-\bar{\rho}_2 T}}{2k_2l_1} E \int_0^T e^{-\rho s} |\bar{X}(s)|^2 ds,$$

$$E \int_0^T e^{-\rho t} |\bar{Z}(t)|^2 dt \leq \frac{2k_2l_1(1 \lor e^{-\bar{\rho}_2 T})}{(1 - k_5l_2 - k_6l_3)(1 \lor e^{-\bar{\rho}_2 T})} E \int_0^T e^{-\rho s} |\bar{X}(s)|^2 ds.$$  

Specially, if $\bar{\rho}_2 > 0$,

$$E \int_0^T e^{-\rho t} |\bar{Z}(t)|^2 dt \leq \frac{2k_2l_1}{1 - k_5l_2 - k_6l_3} E \int_0^T e^{-\rho s} |\bar{X}(s)|^2 ds.$$  

**Proof of Theorem 4.3:** Define $\mathcal{M} := \mathcal{M}_2 \circ \mathcal{M}_1$, where $\mathcal{M}_1$ is defined by (53) and $\mathcal{M}_2$ is defined by (54). Thus $\mathcal{M}$ is a mapping from $L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ into itself. For $(U_i, V_i) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, let $X_i := \mathcal{M}_1(U_i, V_i)$ and $(Y_i, Z_i) := \mathcal{M}(U_i, V_i).$
Therefore,
\[
\mathbb{E} \int_0^T e^{-\rho t} |Y_1(t) - Y_2(t)|^2 dt + \mathbb{E} \int_0^T e^{-\rho t} |Z_1(t) - Z_2(t)|^2 dt \leq \left[ \frac{1 - e^{-\rho_2 T}}{\rho_2} + \frac{1}{1 - k_5 l_2 - k_6 l_3} \right] 2 k_2 l_1 \frac{1 - e^{-\rho_2 T}}{\rho_1} \left\{ (k_2 l_1 + k_2^2) \mathbb{E} \int_0^T e^{-\rho t} |U_1(t) - U_2(t)|^2 dt + (k_2 l_1 + k_2^2) \mathbb{E} \int_0^T e^{-\rho t} |V_1(t) - V_2(t)|^2 dt \right\}.
\]
Choosing suitable \(\rho\), we get that \(\mathcal{M}\) is a contraction mapping.
Furthermore, if \(2 \rho_1 + 2 \rho_2 < -2 k_1 - 2 k_3 - 2 k_4 - k_5^2 - k_5^2 - k_6^2 - k_6^2\), we can choose \(\rho \in \mathbb{R}\), \(0 < k_5 l_2 < \frac{1}{2}\) and \(0 < k_6 l_3 < \frac{1}{2}\) and sufficient large \(l_1\) such that \(\rho_1 > 0\), \(\rho_2 > 0\), \(1 - k_5 l_2 - k_6 l_3 > 0\). Therefore,
\[
\mathbb{E} \int_0^T e^{-\rho t} |Y_1(t) - Y_2(t)|^2 dt + \mathbb{E} \int_0^T e^{-\rho t} |Z_1(t) - Z_2(t)|^2 dt \leq \left[ \frac{1}{\rho_2} + \frac{1}{1 - k_5 l_2 - k_6 l_3} \right] \frac{1}{\rho_1} 2 k_2 l_1 (k_2 l_1 + k_2^2) \mathbb{E} \int_0^T e^{-\rho t} [|U_1(t) - U_2(t)|^2 + |V_1(t) - V_2(t)|^2]dt.
\]
Thus, the proof is complete. 

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