CONSTRUCTION OF CONTINUOUS-STATE BRANCHING PROCESSES IN VARYING ENVIRONMENTS

BY RONGJUAN FANG$^1$ AND ZENGHU LI$^2$

$^1$College of Mathematics and Informatics, Fujian Normal University, Fuzhou 350007, P.R. China, fangrj@fjnu.edu.cn

$^2$Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R. China, lizh@bnu.edu.cn

A continuous-state branching process in varying environments is constructed by the pathwise unique positive solution to a stochastic integral equation driven by time-space noises. The cumulant semigroup of the process is characterized in terms of a backward integral equation. We clarify the behavior of the process at its bottlenecks, which are the deterministic times when it arrives at zero almost surely by negative jumps. The process arises naturally as the scaling limit of Galton–Watson processes in varying environments.

1. Introduction. Continuous-state branching processes (CB-processes) are positive (= nonnegative) Markov processes often used to model the stochastic evolution of large populations with small individuals. The branching property means intuitively that different individuals in the population propagate independently of each other. The study of such processes was initiated by Feller (1951), who noticed that a diffusion process may arise in a limit theorem of rescaled Galton–Watson branching processes (GW-processes). The basic structures of general CB-processes were discussed in Jiřina (1958). It was proved in Lamperti (1967a) that the class of CB-processes with homogeneous transition semigroups coincides with that of scaling limits of classical GW-processes; see also Aliev and Shchurenkov (1982) and Grimvall (1974). A connection between CB-processes and time changed Lévy processes was established by Lamperti (1967b). We refer the reader to Kyprianou (2014) for systematic discussions of the trajectory properties of the CB-process based on the connection. A general existence theorem for homogeneous CB-processes was proved in Silverstein (1968); see also Watanabe (1969) and Ryzhov and Skorokhod (1970). The approach of stochastic equations for CB-processes without or with immigration has been developed by Bertoin and Le Gall (2006), Dawson and Li (2006, 2012), Fittipaldi and Fontbona (2012), Fu and Li (2010), Li (2011, 2020), Pardoux (2016) and many others.

There have also been some attempts at the understanding of inhomogeneous CB-processes. Let $X = \{X(t) : t \in I\}$ be a Markov process with state space $[0, \infty]$ and inhomogeneous transition semigroup $(Q_{r,t})_{t \geq r \in I}$, where $I \subset \mathbb{R}$ is an interval. We call $X$ a CB-process in varying environments (CBVE-process) if there is a family of continuous mappings $(v_{r,t})_{t \geq r \in I}$ on $(0, \infty)$ so that

$$
\int_{[0, \infty]} e^{-\lambda y} Q_{r,t}(x,dy) = e^{-xv_{r,t}(\lambda)}, \quad \lambda > 0, \, x \in [0, \infty]
$$

with $e^{-\lambda y} = 0$ for $y = \infty$ by convention. Clearly, the process has both 0 and $\infty$ as traps.

The function $\lambda \mapsto v_{r,t}(\lambda)$ in (1.1) is the exponential of the Laplace transform of an infinitely divisible probability measure on $[0, \infty]$. From this and the semigroup property of $(Q_{r,t})_{t \geq r \in I}$ it follows that:

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GW-processes in varying environments (GWVE-processes), where individuals in different environments (dom environments CB-processes in random environments) goes to zero instantaneously. The determination of the behavior of the CBVE-process at the bottlenecks was left open in Bansaye and Simatos (2015).

The purpose of this work is to give constructions of the CBVE-process under the basic assumptions of Bansaye and Simatos (2015), He et al. (2018), Helland (1981), Kurtz (1978), Li and Xu (2018), Palau et al. (2016), Palau and Pardo (2017, 2018) and the references therein for some progresses in the study. In particular, a very interesting scaling limit theorem for a sequence of GWVE-processes was proved by Bansaye and Simatos (2015), who provided general sufficient conditions for the weak convergence of the sequence and showed a CBVE-process indeed arises as scaling limits of GW-processes in varying environments (GWVE-processes), where individuals in different generations may have different reproduction distributions. A deep understanding of those models is important since they provide the bases of further study of CB-processes in random environments (CBRE-processes). The reader may refer to Bansaye et al. (2013, 2019), Bansaye and Simatos (2015), and rewrite (1.1) into

\[ v_{r,t}(\lambda) = a_{r,t} + h_{r,t} + \int_0^\infty (1 - e^{-\lambda y}) l_{r,t}(dy), \]

where \(a_{r,t} \geq 0, h_{r,t} \geq 0\) and \((1 \wedge y) l_{r,t}(dy)\) is a finite measure on \((0, \infty)\).

We say the parameters \((b_1, c, m)\) are weakly admissible provided:

- \(t \mapsto c(t)\) is increasing and continuous;
- \(t \mapsto b(t)\) is non-increasing and regularly bounded;
- \(m(s, dz) = m(s, dz)\) is a càdlàg function on \([0, \infty)\) and \(m(s, \{0\}) \leq 0\) for all \(s \geq 0\);
- \(m(s, \{0\}) = 0\) for all \(s \geq 0\);
• for every $s > 0$ we have

\begin{equation}
\delta(s) := \Delta b_1(s) + \int_0^1 zm(\{s\}, dz) \leq 1,
\end{equation}

where $\Delta b_1(s) = b_1(s) - b_1(s-)$. 

Given weakly admissible parameters $(b_1, c, m)$, we consider the backward integral evolution equation:

\begin{equation}
v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda) b_1(ds) - \int_r^t v_{s,t}(\lambda)^2 c(ds) - \int_r^t \int_0^\infty K_1(v_{s,t}(\lambda), z)m(ds, dz),
\end{equation}

where $K_1(\lambda, z) = e^{-\lambda z} - 1 + \lambda z 1_{\{z \leq 1\}}$. Let $J = \{s > 0 : \Delta b_1(s) = 1\}$ and $K = \{s \in J : m(\{s\} \times (0, \infty)) = 0\}$. We say the parameters $(b_1, c, m)$ are admissible if they are weakly admissible and $K$ is an empty set.

**Theorem 1.1.** Let $(b_1, c, m)$ be admissible parameters. Then for any $t \geq 0$ and $\lambda > 0$ there is a unique bounded positive solution $\{0, t] \ni r \mapsto v_{r,t}(\lambda)$ to the integral evolution equation (1.8), which satisfies $\inf_{0 \leq r \leq t} v_{r,t}(\lambda) > 0$. Moreover, a transition semigroup $(Q_{r,t})_{r \geq t}$ on $[0, \infty]$ is defined by (1.1).

**Theorem 1.2.** Let $(b_1, c, m)$ be admissible parameters. Then for any $t \geq 0$, $r \mapsto v_{r,t}(0) := \lim_{\lambda \downarrow 0} v_{r,t}(\lambda)$ is the largest positive solution to (1.8) with $\lambda = 0$.

For weakly admissible parameters $(b_1, c, m)$, we call any element $s \in K$ a bottleneck. Since $b_1$ is a càdlàg function, we can rearrange the bottlenecks into an increasing (finite or infinite) sequence $K := \{s_1, s_2, \cdots\}$. Let $s_0 = 0$ and let $I_i = [s_i, s_{i+1})$ for $i \geq 0$. For $t \geq 0$ let $\varphi(t) = \max\{s \in K : s \leq t\}$ with $\max \emptyset = 0$ by convention.

**Theorem 1.3.** Let $(b_1, c, m)$ be weakly admissible parameters. Then we have: (i) for any $t \geq 0$ and $\lambda > 0$ there is a unique bounded positive solution $\{\varphi(t), t] \ni r \mapsto v_{r,t}(\lambda)$ to the integral evolution equation (1.8), which satisfies $\inf_{\varphi(t) \leq r \leq t} v_{r,t}(\lambda) > 0$; (ii) for each $i \geq 0$ a transition semigroup $(Q_{r,t})_{r \geq t \in I_i}$ on $[0, \infty]$ is defined by (1.1); (iii) if $r \mapsto v_{r,t}(\lambda)$ extends to a bounded positive solution of (1.8) on $[0, t]$, we have $v_{r-,t}(\lambda) = 0$ for $r = \varphi(t)$.

The three theorems above give characterizations of the transition semigroup of the CBVE-process in terms of the backward integral equation (1.8). It is natural to call $\varphi(t)$ the last bottleneck before time $t \geq 0$. By Theorem 1.3, for any $t \geq 0$ and $\lambda > 0$, if $r \mapsto v_{r,t}(\lambda)$ is a bounded positive solution to (1.8) on $[0, t]$, then

\begin{equation}
\varphi(t) = \sup \left\{ s \leq t : \inf_{s \leq r \leq t} v_{r,t}(\lambda) = 0 \right\},
\end{equation}

which is also implied by Theorem 2.2 and Lemma 3.4 in Bansaye and Simatos (2015). Therefore our notion $\varphi(t)$ coincides with that of Bansaye and Simatos (2015).

To determine the sample path behavior of the CBVE-process, we need the more powerful tool of stochastic equations. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses. Let $W(ds, dw)$ be a time-space $(\mathcal{F}_t)$-Gaussian white noise on $(0, \infty)^2$ with intensity $2c(ds)du$. Let $M(ds, dz, du)$ be a time-space $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty)^3$ with intensity $m(ds, dz)du$. Denote by $M(ds, dz, du)$ the compensated
measure of $M(ds, dz, du)$. Given an $\mathcal{F}_0$-measurable random variable $X(0) \geq 0$, we consider the stochastic integral equation:

\begin{equation}
(1.10) \quad X(t) = X(0) + \int_0^t \int_0^s W(ds, du) + \int_0^t \int_0^1 \int_0^s zM(ds, dz, du),
\end{equation}

\[- \int_0^t X(s-)b_1(ds) + \int_0^t \int_1^\infty \int_0^s zM(ds, dz, du). \]

Let $\tau_0 = \inf\{t \geq 0 : X(t) = 0\}$ and $\tau_\infty = \lim_{k \to \infty} \tau_k$, where $\tau_k = \inf\{t \geq 0 : X(t) \geq k\}$. We say the process $\{X(t) : t \geq 0\}$ taking values in $[0, \infty]$ is a solution to (1.11) if the equation holds with $t$ replaced by $t \wedge \tau_k$ for every $t \geq 0$ and if $X(t) = 0$ for every $t \geq \tau_0$ and $X(t) = \infty$ for every $t \geq \tau_\infty$. We call $\tau_0$ and $\tau_\infty$ the extinction time and explosion time, respectively.

**Theorem 1.4.** Let $(b_1, c, m)$ be admissible parameters. Then there is a pathwise unique solution $\{X(t) : t \geq 0\}$ to (1.11) and the solution is a CBVE-process with transition semigroup $(Q_{r,t})_{t \geq r}$ defined by (1.1) and (1.8).

**Corollary 1.5.** In the situation of Theorem 1.4, for any $t \geq 0$ we have

\begin{equation}
(1.11) \quad P\{\tau_\infty \leq t\} = P\{X(t) = \infty\} = 1 - E(e^{-X(0)v_0},(0))
\end{equation}

and

\begin{equation}
(1.12) \quad P\{\tau_0 \leq t\} = P\{X(t) = 0\} = E(e^{-X(0)v_0},(\infty)).
\end{equation}

**Theorem 1.6.** Let $(b_1, c, m)$ be weakly admissible parameters. Then there is a pathwise unique solution $\{X(t) : t \geq 0\}$ to (1.11) and $\{X(t) : t \in I_0\}$ is a CBVE-process with transition semigroup $(Q_{r,t})_{t \geq r} \in I_0$ defined by (1.1) and (1.8). Moreover, we have $X(t) = 0$ for $t \geq s_1$ on the event $\{s_1 < \tau_\infty\}$ and $X(t) = \infty$ for $t \geq s_1$ on the event $\{s_1 \geq \tau_\infty\}$.

The behavior of the CBVE-process at the bottlenecks is clarified by Theorem 1.6. The results extend immediately to an arbitrary initial time $r_0 \geq 0$ as follows. Let $i \geq 0$ be such that $r_0 \in I_i = [s_i, s_{i+1})$. Then for any $\mathcal{F}_{r_0}$-measurable initial value $X_i(r_0) \geq 0$, there is a pathwise unique solution $\{X_i(t) : t \geq r_0\}$ to

\begin{equation}
(1.13) \quad X(t) = X_i(r_0) + \int_{r_0}^t \int_0^s W(ds, du) + \int_{r_0}^t \int_0^1 \int_0^s zM(ds, dz, du),
\end{equation}

\[- \int_{r_0}^t X(s-)b_1(ds) + \int_{r_0}^t \int_1^\infty \int_0^s zM(ds, dz, du). \]

The restriction of the solution to $J_i := [r_0, s_{i+1})$ is a CBVE-process with transition semigroup $(Q_{r,t})_{t \geq r} \in J_i$ defined by (1.1) and (1.8). Let $\tau_{i,k} = \inf\{t \geq r_0 : X_i(t) \geq k\}$ and let $\tau_{i,\infty} = \lim_{k \to \infty} \tau_{i,k}$ be the explosion time of $\{X_i(t) : t \geq r_0\}$. Then we have $X_i(t) = 0$ for $t \geq s_{i+1}$ on the event $\{s_{i+1} < \tau_{i,\infty}\}$ and $X_i(t) = \infty$ for $t \geq s_{i+1}$ on the event $\{\tau_{i,\infty} \leq s_{i+1}\}$.

We shall first treat special forms of (1.8) and (1.11) by imposing an integrability condition stronger than (1.6), which yields finite first moments of the CBVE-process. The existence of the cumulant semigroup is constructed by an iteration argument combined with an inhomogeneous nonlinear $h$-transformation. A suitably chosen transformation of this type reduces the construction of the solution of (1.11) to the martingale case and plays an important role in the establishment of the stochastic equation under the first moment assumption. The solutions
to the general equations (1.8) and (1.11) are then obtained by increasing limits. The Poisson random measure in (1.11) does not fit immediately into the framework of single valued point processes developed in standard references such as Ikeda and Watanabe (1989), Jacod and Shiryaev (2003) and Situ (2005). In fact, at a fixed discontinuity \( t > 0 \) the jump size \( \Delta X(t) \) of the CBVE-process is identified by a composite Lévy–Itô representation as the position at time \( X(t−) \) of a spectrally positive Lévy process constructed from the random measure \( M(\{t\}, dz, du) \), which typically has infinitely many atoms. This is essentially different from its homogeneous version discussed in Bertoin and Le Gall (2006) and Dawson and Li (2006, 2012), where \( M(\{t\}, dz, du) \) has no more than one atom. The complexity of jumps of the solution makes the treatment of (1.11) much more difficult than the homogeneous equations.

The probabilistic meaning of the quantity \( v_{\lambda,t}(0) \) is given by (1.4) and (1.11).

REMARK 1.7. We need to assume condition (1.7) to ensure that the solution of (1.8) stays positive, which is necessary for the existence of the CBVE-process. In fact, from the equation it follows that, for any \( 0 < \varepsilon \leq 1 \),

\[
v_{t−,t}(\lambda) = \lambda [1 - \Delta b_1(t)] - \int_0^{\infty} K_1(\lambda, z) m(\{t\}, dz) \\
\leq \lambda \left[ 1 - \Delta b_1(t) - \int_1^1 zm(\{t\}, dz) \right] + m(\{t\} \times (\varepsilon, \infty)).
\]

Then one would have \( v_{t−,t}(\lambda) < 0 \) for sufficiently large \( \lambda > 0 \) if condition (1.7) is not satisfied.

There are several equivalent formulations of the backward equation for the cumulant semigroup. In particular, following Bansaye and Simatos (2015), we may rewrite (1.8) equivalently as:

\[
(1.14) \quad v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda) b_2(ds) - \int_r^t v_{s,t}(\lambda)^2 c(ds) \\
- \int_r^t \int_0^{\infty} K_2(v_{s,t}(\lambda), z) m(ds, dz),
\]

where

\[
K_2(\lambda, z) = e^{-\lambda z} - 1 + \frac{\lambda z}{1 + z^2}
\]

and

\[
b_2(t) = b_1(t) + \int_0^t \int_0^{\infty} \left( 1_{\{z \leq 1\}} - \frac{1}{1 + z^2} \right) zm(ds, dz).
\]

In this setting, condition (1.7) is reexpressed as

\[
(1.15) \quad \delta(t) := \Delta b_2(t) + \int_0^{\infty} \left( \frac{z}{1 + z^2} \right) m(\{t\}, dz) \leq 1.
\]

Bansaye and Simatos (2015) obtained the equation by considering the scaling limit of a sequence of discrete models. They required that \( b_2 \) and \( c \) are càdlàg functions in their Assumption 2.1. From their Theorem 2.2, we understand that their assumption implies the function \( c \).
is continuous and increasing and the measure $m$ satisfies (1.6). Bansaye and Simatos (2015) did not introduce condition (1.15), but they pointed out that the weaker property $\Delta b_2(t) \leq 1$ follows from their assumption.

One may wonder if it is possible to derive our Theorems 1.1 and 1.3 from the result of Bansaye and Simatos (2015). That does not seem easy since one would then need to construct a discrete approximating sequence for $(b_1, c, m)$ and to check the properties (A1) and (A2) in their Assumption 2.1. In fact, it is the advantage of our approach to construct the cumulant semigroup and CBVE-process directly from the parameters $(b_1, c, m)$ so that we do not need to refer to the discrete approximations in our existence results.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we exploit the existence and uniqueness of solutions to some special cases of (1.8) under an additional integrability condition. The corresponding CBVE-process is constructed in Section 4 by solving a special form of (1.11). The general results for admissible or weakly admissible parameters are proved in Section 5.

2. Preliminaries. In this section, we present some preliminary results that will be used in the subsequent explorations. Some of the results are standard, but we could not find convenient references for them in the literature.

2.1. Forward and backward integral equations. We first discuss briefly integral equations driven by functions with locally bounded variations. The reader may see Appendix 6.4.1 of Pardoux and Răşcanu (2014) for similar discussions.

Given a càdlàg function $\alpha$ on $[0, \infty)$ with locally bounded variations and $\alpha(0) = 0$, we write $\Delta \alpha(t) = \alpha(t) - \alpha(t-)$ for the size of its jump at $t > 0$ and $||\alpha||(t)$ for the total variation of $\alpha$ on $[0, t]$. It is well-known the set $J_\alpha := \{ s > 0 : \Delta \alpha(s) \neq 0 \}$ is at most countable. The function $\alpha$ has the decomposition $\alpha(t) = \alpha_c(t) + \alpha_d(t)$, where $\alpha_d(t) := \sum_{0 < s \leq t} \Delta \alpha(s)$ is the jump part and $\alpha_c(t) := \alpha(t) - \alpha_d(t)$ is the continuous part.

**Proposition 2.1.** Suppose that $\alpha$ and $G$ are càdlàg functions on $[0, \infty)$ with locally bounded variations such that $\Delta \alpha(t) > -1$ for every $t > 0$. Let $\zeta$ be the càdlàg function on $[0, \infty)$ such that $\zeta_c(t) = \alpha_c(t)$ and $\Delta \zeta(t) = \log[1 + \Delta \alpha(t)]$ for every $t > 0$. Then we have:

(i) (Forward equation) There is a unique locally bounded solution to:

$$F(t) = G(t) + \int_0^t F(s-)\alpha(ds), \quad t \geq 0,$$

which is given by

$$F(t) = e^{\zeta(t) - \zeta(0)}G(0) + \int_0^t e^{\zeta(t) - \zeta(s)}G(ds).$$

(ii) (Backward equation) For every $t \geq 0$ there is a unique bounded solution to:

$$H(r) = G(r) + \int_r^t H(s)\alpha(ds), \quad r \in [0, t],$$

which is given by

$$H(r) = e^{\zeta(t) - \zeta(r)}G(t) - \int_r^t e^{\zeta(s) - \zeta(r)}G(ds).$$
Then of Gronwall’s inequalities and is left to the reader. By (2.2) and integration by parts, we have
\[ F(t) = G(0) + e^{-\zeta(0)}G(0) \int_0^t \alpha(s) \, ds + \int_0^t G(ds) + \int_0^t \alpha(s) \int_0^{s-} e^{-\zeta(v)}G(dv) \]
\[ = e^{-\zeta(0)}G(0) \int_0^t e^{\zeta(s)}d\zeta_c(s) + e^{-\zeta(0)}G(0) \sum_{s \in (0,t]} (e^{\zeta(s)} - e^{\zeta(s-)} + G(t) \]
\[ + \int_0^t e^{\zeta(s)}d\zeta_c(s) \int_0^{s-} e^{-\zeta(v)}G(dv) + \sum_{s \in (0,t]} (e^{\zeta(s)} - e^{\zeta(s-)} \int_0^{s-} e^{-\zeta(v)}G(dv) \]
\[ = e^{-\zeta(0)}G(0) \int_0^t e^{\zeta(s)-} \alpha(s) + G(t) + \int_0^t e^{\zeta(s)-} \alpha(s) \int_0^{s-} e^{-\zeta(v)}G(dv) \]
\[ = G(t) + \int_0^t F(s-) \alpha(s). \]

Then \( t \mapsto F(t) \) is a solution to (2.1). Similarly, by (2.4) and integration by parts,
\[ H(t) = H(r) + e^{\zeta(t)}G(t) \int_r^t e^{-\zeta(s)} \alpha(s) \, ds + \int_r^t G(ds) - \int_r^t e^{-\zeta(s)} \int_s^t e^{\zeta(v-)}G(dv) \]
\[ = H(r) - e^{\zeta(t)}G(t) \int_r^t e^{-\zeta(s)} \alpha(s) \, ds + \int_r^t G(ds) + \int_r^t e^{-\zeta(s)} \alpha(s) \int_s^t e^{\zeta(v-)}G(dv) \]
\[ = H(r) - \int_r^t H(s) \alpha(ds) + G(t) - G(r). \]

Then \( r \mapsto H(r) \) is a solution to (2.3) on \([0,t] \).

\[ \Box \]

\textbf{Corollary 2.2.} Let \( \alpha \) be a càdlàg function on \([0, \infty) \) with locally bounded variations such that \( \Delta \alpha(t) > -1 \) for \( t > 0 \). Then for \( \lambda \in \mathbb{R} \) we have:

(i) (Forward equation) There is a unique locally bounded solution to:
\[ \pi_t(\lambda) = \lambda + \int_0^t \pi_s(\lambda) \alpha(ds), \quad t \geq 0, \]
which is given by
\[ \pi_t(\lambda) = \lambda \prod_{s \in (0,t]} (1 + \Delta \alpha(s)) \exp\{\alpha_e(t) - \alpha_e(0)\}. \]

(ii) (Backward equation) For every \( t \geq 0 \) there is a unique bounded solution to:
\[ \pi_r(\lambda) = \lambda + \int_r^t \pi_s(\lambda) \alpha(ds), \quad r \in [0,t], \]
which is given by
\[ \pi_r(\lambda) = \lambda \prod_{s \in [r,t]} (1 + \Delta \alpha(s)) \exp\{\alpha_e(t) - \alpha_e(r)\}. \]
### 2.2. Upper and lower bounds for the solution

Let \((b_1, c, m)\) be admissible parameters. We here prove some useful upper and lower bounds for the solution to the integral evolution equation \((1.8)\). For \(\lambda > 0\) and \(t \geq r \geq 0\) let

\[
U_{r,t}(\lambda) = [\lambda + m((0, t] \times (1, \infty))] \exp\{\|b_1\|(t) - \|b_1\|(r)\}.
\]

By the admissibility of the parameters we have \(m(\{s\} \times (0, 1]) = 0\) and \(m(\{s\} \times (1, \infty)) > 0\) for \(s \in J\). For \(t \geq 0\) choose a sufficiently large constant \(\eta_t > 1\) so that \(m(\{s\} \times (1, \eta_t)) > 0\) when \(s \in (0, t] \cap J\). Let

\[
F_t(\lambda) = U_{0,t}(\lambda)^{-1}(1 - e^{-U_{0,t}(\lambda)}), \quad H_t(\lambda) = [\eta_t U_{0,t}(\lambda)]^{-1}(1 - e^{-\eta_t U_{0,t}(\lambda)}).
\]

Let \(\alpha(r) = \alpha(r, t, \lambda)\) be the càdlàg function on \([0, t]\) defined by

\[
\alpha(r) = -\frac{1}{2} U_{0,t}(\lambda) \int_0^r \int_0^{\varepsilon_t(\lambda)} z^2 m(ds, dz) + H_t(\lambda) \int_0^r \int_{\eta_t}^1 z m(ds, dz) - b_1(r) - U_{0,t}(\lambda) c(r) - [1 - F_t(\lambda)] \int_0^r \int_{\varepsilon_t(\lambda)}^1 z m(ds, dz),
\]

where \(\varepsilon_t(\lambda) = 1 \cap [U_{0,t}(\lambda)^{-1} F_t(\lambda)]\). Let

\[
l_{r,t}(\lambda) = \lambda \prod_{s \in (r, t]} [1 + (0 \wedge \Delta \alpha(s))] \exp\{\|\alpha\|(r) - \|\alpha\|(t)\}.
\]

**PROPOSITION 2.3.** Suppose that \(r \mapsto v_{r,t}(\lambda)\) is a bounded positive solution to \((1.8)\) with \(\lambda > 0\). Then we have

\[
v_{r,t}(\lambda) \leq \alpha(r, t, \lambda) \leq U_{r,t}(\lambda), \quad r \in [0, t].
\]

**PROOF.** The upper bound in \((2.12)\) follows by Gronwall’s inequality since \((1.8)\) implies

\[
v_{r,t}(\lambda) \leq \lambda + \int_0^t \int_1^{\infty} m(ds, dz) + \int_r^t v_{s,t}(\lambda) b_1\|ds\|.
\]

Let \(r \mapsto \pi_{r,t}(\lambda)\) be the solution to \((2.7)\) with \(\alpha\) given by \((2.10)\). Then we have

\[
v_{r,t}(\lambda) - \pi_{r,t}(\lambda) = \pi_{r,t}(\lambda) + \int_r^t [v_{s,t}(\lambda) - \pi_{s,t}(\lambda)] \alpha(ds),
\]

where

\[
G_{r,t}(\lambda) = \int_r^t v_{s,t}(\lambda) [U_{0,t}(\lambda) - v_{s,t}(\lambda)] c(ds) + \int_r^t \int_{\eta_t}^1 (1 - e^{-v_{s,t}(\lambda) z}) m(ds, dz)
\]

\[
+ \int_r^t \int_0^{\varepsilon_t(\lambda)} \left[ \frac{1}{2} U_{0,t}(\lambda) v_{s,t}(\lambda) z^2 - K(v_{s,t}(\lambda), z) \right] m(ds, dz)
\]

\[
+ \int_r^t \int_{\varepsilon_t(\lambda)}^1 [1 - e^{-v_{s,t}(\lambda) z} - F_t(\lambda) v_{s,t}(\lambda) z] m(ds, dz)
\]

\[
+ \int_r^t \int_{\eta_t}^1 [1 - e^{-v_{s,t}(\lambda) z} - H_t(\lambda) v_{s,t}(\lambda) z] m(ds, dz),
\]

where \(K(\lambda, z) = e^{-\lambda z} - 1 + \lambda z\). In view of \((1.7)\), for any \(s \in (0, t]\) we have

\[
\int_0^1 z m_d(\{s\}, dz) \leq 1 - \Delta b_1(s).
\]
It follows that
\[
\Delta \alpha(s) = -\frac{1}{2} U_{0,t}(\lambda) \int_0^{\varepsilon_s(\lambda)} z^2 m_d(\{s\}, dz) - \left[1 - F_t(\lambda)\right] \int_0^1 z m_d(\{s\}, dz) \\
- \Delta b_1(s) + H_t(\lambda) \int_1^{\eta_s} z m(\{s\}, dz) \\
\geq -\frac{1}{2} U_{0,t}(\lambda) \varepsilon_s(\lambda) \int_0^1 z m_d(\{s\}, dz) - \left[1 - F_t(\lambda)\right] \int_0^1 z m_d(\{s\}, dz) \\
- \Delta b_1(s) + H_t(\lambda) \int_1^{\eta_s} z m(\{s\}, dz) \\
\geq -\frac{1}{2} U_{0,t}(\lambda) \varepsilon_s(\lambda) \left[1 - \Delta b_1(s)\right] - \left[1 - F_t(\lambda)\right] \left[1 - \Delta b_1(s)\right] \\
- \Delta b_1(s) + H_t(\lambda) \int_1^{\eta_s} z m(\{s\}, dz).
\]

By the admissibility of the parameters we have \(1 - \Delta b_1(s) > 0\) when \(s \in (0, t] \setminus \mathcal{J}\) and \(m(\{s\} \times (1, \eta_s]) > 0\) when \(s \in (0, t] \cap \mathcal{J}\), so \(\Delta \alpha(s) > -1\) for each \(s \in (0, t]\). Then Proposition 2.1 applies to (2.13). Since \(r \mapsto G_{r,t}(\lambda)\) is a decreasing function, from (2.4) we see \(v_{r,t}(\lambda) - \pi_{r,t}(\lambda) \geq 0\). By comparing (2.8) and (2.11) we have the lower bound in (2.12).

2.3. A preliminary iteration construction. In this subsection, we construct a special type of cumulant semigroups by a simple iteration argument. This will be the basis for the construction of the general semigroups. Let \(\alpha\) be a càdlàg function on \([0, \infty)\) having locally bounded variations and satisfying \(\Delta \alpha(t) > -1\) for \(t > 0\). Let \(\mu(ds, dz)\) be a \(\sigma\)-finite measure on \((0, \infty)^2\) satisfying

\[
\int_0^t \int_0^\infty z \mu(ds, dz) < \infty, \quad t \geq 0.
\]

Given \(t \geq 0\) and \(\lambda \geq 0\), we first consider the backward integral evolution equation:

\[
u_{r,t}(\lambda) = \lambda + \int_r^t u_{s,t}(\lambda) \alpha(ds) + \int_r^t \int_0^\infty (1 - e^{-u_{s,t}(\lambda)z}) \mu(ds, dz),
\]

where \(r \in [0, t]\). This is the special case of (1.8) with

\[
c(t) = 0, \quad b_1(t) = -\alpha(t) - \int_0^t \int_0^1 z \mu(ds, dz)
\]

and \(m(ds, dz) = \mu(ds, dz)\).

**Proposition 2.4.** For \(t \geq 0\) and \(\lambda \geq 0\), there is a unique bounded positive solution \(r \mapsto u_{r,t}(\lambda)\) on \([0, t]\) to (2.15) and \((u_{r,t})_{t \geq r}\) is a conservative cumulant semigroup. Moreover, for \(\lambda \geq 0\) we have

\[
u_{r,t}(\lambda) \leq \lambda e^\|\rho\|(r,t) \leq \lambda e^\|\rho\|(t),
\]

where

\[
\rho(t) = \alpha(t) + \int_0^t \int_0^\infty z \mu(ds, dz).
\]

**Proof.** Step 1. Let \(r \mapsto u_{r,t}(\lambda)\) be a bounded positive solutions to (2.15). From the equation it is easy to see that

\[
u_{r,t}(\lambda) \leq \lambda + \int_r^t u_{s,t}(\lambda) \|\rho\|(ds).
\]
This yields (2.16) by Gronwall’s inequality. Suppose that \( r \mapsto w_{r,t}(\lambda) \) is also a bounded positive solution to (2.15). Then we have
\[
|u_{r,t}(\lambda) - w_{r,t}(\lambda)| \leq \int_0^t |u_{s,t}(\lambda) - w_{s,t}(\lambda)||\rho||\mu(ds).
\]
By Gronwall’s inequality we see |\( u_{r,t}(\lambda) - w_{r,t}(\lambda) | = 0 \) for every \( r \in [0, t] \).

**Step 2.** Consider the case where \( \alpha \) vanishes. Let \( t \geq 0 \) and \( \lambda \geq 0 \) be fixed. For \( r \in [0, t] \) set \( v_{r,t}^{(0)}(\lambda) = 0 \) and define \( v_{r,t}^{(k)}(\lambda) \) inductively by
\[
v_{r,t}^{(k+1)}(\lambda) = \lambda + \int_r^t \int_0^\infty (1 - e^{-v_{s,t}^{(k)}(\lambda)z}) \mu(ds,dz).
\]
By Proposition 4.2 of Silverstein (1968) one can use (2.17) to see inductively that each \( v_{r,t}^{(k)}(\lambda) \) has the Lévy–Khintchine representation (1.3). Moreover, we have \( v_{r,t}^{(k)}(0) = 0 \) and
\[
0 \leq v_{r,t}^{(k)}(\lambda) \leq v_{r,t}^{(k+1)}(\lambda) \leq \pi_{r,t}(\lambda),
\]
where \( r \mapsto \pi_{r,t}(\lambda) \) is the solution to (2.7) with
\[
\alpha(t) = \int_0^t \int_0^\infty z \mu(ds,dz).
\]
Then the limit \( v_{r,t}(\lambda) = \lim_{k \to \infty} v_{r,t}^{(k)}(\lambda) \) exists and the convergence is uniform in \( (r, \lambda) \in [0, t] \times [0, B] \) for every \( t \geq 0 \) and \( B \geq 0 \). In fact, setting \( u_k(r, t, \lambda) = \sup_{r \leq s \leq t} |v_{s,t}^{(k)}(\lambda) - v_{s,t}^{(k-1)}(\lambda)| \), we have
\[
\begin{align*}
\sum_{k=1}^\infty u_k(r, t, \lambda) & \leq Be^{\alpha([0,t])} < \infty. \quad \text{By Corollary 1.33 in Li (2011) we infer that } \\
v_{r,t}(\lambda) & \text{ has representation (1.3) with } v_{r,t}(0) = 0. \quad \text{By (2.17) and monotone convergence we see } r \mapsto v_{r,t}(\lambda) \text{ is a solution to (2.15) with } \alpha \equiv 0. \quad \text{The semigroup property (1.2) follows from the uniqueness of the solution. Then } (u_{r,t})_{t \geq r} \text{ is a conservative cumulant semigroup.}
\end{align*}
\]

**Step 3.** Let \( \zeta \) be the càdlàg function on \([0, \infty)\) such that \( \zeta_0(t) = \alpha_0(t) \) and \( \Delta \zeta(t) = \log[1 + \Delta \alpha(t)] \) for every \( t \geq 0 \). By the second step, there is a unique bounded positive solution \( r \mapsto u_{r,t}(\lambda) \) on \([0, t] \) to
\[
\begin{align*}
u_{r,t}(\lambda) = \lambda + \int_r^t \int_0^\infty (1 - e^{-u_{s,t}(\lambda)z}) e^{\zeta(s-)} \mu(ds,dz).
\end{align*}
\]
Moreover, the family \( (u_{r,t})_{t \geq r} \) is a conservative cumulant semigroup. Then we can define another conservative cumulant semigroup \( (v_{r,t})_{t \geq r} \) by \( v_{r,t}(\lambda) = e^{-\zeta(r)}u_{r,t}(e^{\zeta(t)}\lambda) \). Since \( u_{r,t}(\lambda) = \lambda \), by integration by parts we have \[
\begin{align*}
\lambda = e^{-\zeta(r)}u_{r,t}(e^{\zeta(t)}\lambda) + \int_r^t u_{s,t}(e^{\zeta(t)}\lambda)de^{-\zeta(s)} + \int_r^t e^{-\zeta(s-)}du_{s,t}(e^{\zeta(t)}\lambda)
\end{align*}
\]
Then \( r \mapsto v_{r,t}(\lambda) \) is a solution to (2.15) on \([0,t]\). \(\square\)

2.4. A composite Lévy–Itô representation. Suppose that \( \beta \in \mathbb{R} \) and \((z \wedge z^2)\gamma(dz)\) is a finite measure on \((0, \infty)\). Then we can define a function \( \psi \) on \([0, \infty)\) by

\[
\psi(\lambda) = \beta \lambda + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \gamma(dz), \quad \lambda \geq 0.
\]

Let \( \{Y_t : t \geq 0\} \) be a spectrally positive Lévy process defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that

\[
\mathbb{E}(e^{-\lambda Y_t}) = \exp\{t \psi(\lambda)\}, \quad \lambda \geq 0.
\]

By Lévy–Itô representation, there is a Poisson random measure \( G = G(dz, du) \) on \((0, \infty)^2\) with intensity \( \gamma(dz)du \) such that, for any \( t \geq 0 \),

\[
(2.19) \quad Y_t = -\beta t + \int_0^\infty \int_0^t zG(dz, du) = -\beta t + \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \int_0^t zG(dz, du),
\]

where the limit is taken in the a.s. sense; see, e.g., Theorem 19.2 in Sato (1999, p.120). Let \( \rho(z, u) = (z \wedge z^2)(1 + u^2)^{-1} \) for \( z > 0 \) and \( u > 0 \). Let \( M_\rho \) denote the space of all \( \sigma \)-finite Borel measures \( \nu \) on \((0, \infty)^2\) so that

\[
\int_0^\infty \int_0^\infty \rho(z,u)\nu(dz,du) < \infty.
\]

We may think of \( G \) as a random variable taking values in \( M_\rho \) equipped with the \( \sigma \)-algebra \( \mathcal{M}_\rho \) generated by the mappings \( \nu \mapsto \nu((a, \infty) \times B) \) for all \( a > 0 \) and bounded \( B \in \mathcal{B}(0, \infty) \).

Let \( \tilde{\nu}(dz, du) = \nu(dz, du) - \gamma(dz)du \) for any \( \nu \in M_\rho \). From (2.19) we see that the joint distribution \( P(t, dy, d\nu) \) of the random vector \((Y_t, G)\) on \( \mathbb{R} \times M_\rho \) is carried by

\[
A(t) := \{(y, \nu) \in \mathbb{R} \times M_\rho : y = -\beta t + \tilde{\nu}(t)\},
\]

where

\[
\tilde{\nu}(t) = \int_0^\infty \int_0^t z\tilde{\nu}(dz, du) := \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \int_0^t z\tilde{\nu}(dz, du).
\]

Let \( P_1(t, dy) \) and \( P_2(d\nu) \) denote the marginal distributions of \( Y_t \) and \( G \), respectively. Let \( \kappa_1(t, y, d\nu) \) be a regular conditional distribution of \( G \) given \( Y_t \). Then \( P(t, dy, d\nu) = P_1(t, dy)\kappa_1(t, y, d\nu) \). The next proposition establishes a composite Lévy–Itô representation, which generalizes the classical formula (2.19).

**Proposition 2.5.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with the sub-\( \sigma \)-algebra \( \mathcal{G} \subset \mathcal{F} \). Suppose that \((\xi, Y)\) is a random vector taking values in \([0, \infty) \times \mathbb{R}\) such that \( \xi \) is \( \mathcal{G} \)-measurable and

\[
(2.20) \quad \mathbb{E}(e^{-\lambda Y} | \mathcal{G}) = \exp\{\xi \psi(\lambda)\}, \quad \lambda \geq 0.
\]

Then on an extension of the probability space there exists a Poisson random measure \( N(dz, du) \) on \((0, \infty)^2\) with intensity \( \gamma(dz)du \) such that \( N \) is independent of \( \mathcal{G} \) and a.s.

\[
(2.21) \quad Y = -\beta \xi + \int_0^\infty \int_0^\xi z\tilde{N}(dz, du).
\]
Proof. We first define precisely the extension of the original probability space. Let \( \hat{\Omega} = \Omega \times M_\rho \) and \( \hat{F} = F \times M_\rho \). Let \( \hat{P} \) be the probability law on \( (\hat{\Omega}, \hat{F}) \) defined by \( \hat{P}(d\omega) = P(d\omega)\kappa_1(\xi(\omega), Y(\omega), d\nu) \), where \( \hat{\omega} = (\omega, \nu) \in \hat{\Omega} \). Given a random variable \( X \) on \((\Omega, F, P)\), we write \( X(\hat{\omega}) = X(\omega, \nu) \) for \( \hat{\omega} = (\omega, \nu) \in \hat{\Omega} \), which extends \( X \) to a random variable on \((\hat{\Omega}, \hat{F}, \hat{P})\). It is easy to see that \( \hat{G} := G \times \{0, M_\rho\} \subset \hat{F} \) and \( \xi \) is \( \hat{G} \)-measurable as a random variable on \((\hat{\Omega}, \hat{F}, \hat{P})\). Let \( N(\omega) = \nu \) for \( \hat{\omega} = (\omega, \nu) \in \hat{\Omega} \). From (2.20) we see, for any \( \lambda \geq 0 \),

\[
E(e^{-\lambda Y} | \xi) = E(e^{-\lambda Y} | G) = \exp\{\xi \psi(\lambda)\}.
\]

It follows that \( P(Y \in dy|G) = P(Y \in dy|\xi) = P_1(\xi, dy) \). Then a monotone class argument shows that, for any bounded measurable function \( f \) on \( \mathbb{R} \times M_\rho \),

\[
E[f(\xi, Y)] = E[f(\xi, Y)|G] = \int_\mathbb{R} f(\xi, y)P_1(\xi, dy).
\]

Let \( F \) be a bounded \( G \)-measurable random variable. Let \( g \) and \( h \) be bounded measurable functions on \( \mathbb{R} \) and \( M_\rho \), respectively. By (2.20) we have

\[
\tilde{E}[Fg(Y)h(N)] = E \left[ Fg(Y) \int_{M_\rho} h(\nu)\kappa_1(\xi, Y, d\nu) \right]
= E \left[ F \tilde{E} \left( g(Y) \int_{M_\rho} h(\nu)\kappa_1(\xi, Y, d\nu) \bigg| G \right) \right]
= E \left[ F \int_\mathbb{R} g(y)P_1(\xi, dy) \int_{M_\rho} h(\nu)\kappa_1(\xi, y, d\nu) \right]
= E \left[ F \int_{M_\rho} \int_\mathbb{R} g(y)h(\nu)P(\xi, dy, d\nu) \right].
\]

That shows \( \tilde{P}(Y \in dy, N \in d\nu|G) = P(\xi, dy, d\nu) \). It follows that \( \tilde{P}(N \in d\nu|G) = P_2(d\nu) \), implying that \( N \) under \( \hat{P} \) is a Poisson random measure on \((0, \infty)^2\) with intensity \( \gamma(dz)du \). Since \( P(\xi, dy, d\nu) \) is carried by \( A(\xi) \subset \mathbb{R} \times M_\rho \), the relation (2.21) a.s. holds. Taking \( g \equiv 1 \), we get

\[
\tilde{E}[Fh(N)] = E \left[ F \int_{M_\rho} h(\nu)P_2(\nu) \right] = \tilde{E}(F)\tilde{E}[h(N)].
\]

Then \( N \) is independent of \( G \) on \((\hat{\Omega}, \hat{F}, \hat{P})\).

\section{Conservative cumulant semigroups.} In this section, we give a construction of the cumulant semigroup under an extra integrability condition. Let \((b_1, c, m)\) be admissible parameters. Throughout this section, we assume:

\[
m(t) := \int_0^t \int_0^\infty (z \wedge z^2)m(ds, dz) < \infty, \quad t \geq 0.
\]

Then we can rewrite (1.8) equivalently into:

\[
v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda)b(ds) - \int_r^t v_{s,t}(\lambda)^2c(ds)
- \int_r^t \int_0^\infty K(v_{s,t}(\lambda), z)m(ds, dz),
\]

where \( K(\lambda, z) = e^{-\lambda z} - 1 + \lambda z \) and

\[
b(t) = b_1(t) - \int_0^t \int_1^\infty zm(ds, dz).
\]

(3.1)
Let $B[0, \infty)^+$ be the set of locally bounded positive Borel functions on $[0, \infty)$ and $M[0, \infty)$ the set of Radon measures on $[0, \infty)$. By a branching mechanism with parameters $(b, c, m)$ we mean the functional $\phi$ on $B[0, \infty)^+ \times M[0, \infty)$ defined by

$$\phi(f, B) = \int_B f(s)b(ds) + \int_B f(s)^2c(ds) + \int_B \int_0^\infty K(f(s), z)m(ds, dz),$$

where $f \in B[0, \infty)^+$ and $B \in \mathcal{B}[0, \infty)$. Using this notation, we can rewrite (3.2) equivalently into

$$v_{r,t}(\lambda) = \lambda - \phi(v_{r,t}(\lambda), (r, t]), r \in [0, t].$$

For any integer $n \geq 1$ we define the branching mechanism $\phi_n$ by

$$\phi_n(f, B) = \int_B f(s)b(ds) + 2n^2 \int_B \left( e^{-f(s)/n} - 1 + f(s)/n \right) c(ds)$$

$$+ \int_B \int_0^1 (e^{-f(s)z} - 1 + f(s)z)(1 \wedge nz)m(ds, dz)$$

$$+ \int_B \int_1^\infty (e^{-f(s)z} - 1 + f(s)z)m(ds, dz)$$

$$= - \int_B f(s)\alpha_n(ds) - \int_B \int_0^\infty (1 - e^{-f(s)z}) \mu_n(ds, dz),$$

where

$$\alpha_n(ds) = -b(ds) - 2nc(ds) - \int_0^1 z(1 \wedge nz)m(ds, dz)$$

$$- \int_1^\infty zm(ds, dz)$$

$$= -b_1(ds) - 2nc(ds) - \int_0^1 z(1 \wedge nz)m(ds, dz)$$

and

$$\mu_n(ds, dz) = 2n^2c(ds)\delta_{1/n}(dz) + 1_{\{z \leq 1\}}(1 \wedge nz)m(ds, dz)$$

$$+ 1_{\{z > 1\}}m(ds, dz).$$

Then $\Delta \alpha_n(t) > -1$ for every $t > 0$ since $(b_1, c, m)$ are admissible parameters.

**Lemma 3.1.** The branching mechanisms $\phi$ and $\phi_n$ have the following properties:

(i) For $t \geq r \geq 0$ and $f \in B[0, \infty)^+$, we have $\phi(f, (r, t]) = \lim_{n \uparrow \infty} \phi_n(f, (r, t])$;

(ii) For $t \geq s \geq r \geq 0$ and $f \leq g \in B[0, \infty)^+$,

$$\phi(f, (s, t]) - \phi_n(f, (s, t]) \leq \phi(g, (r, t]) - \phi_n(g, (r, t]);$$

(iii) For $t \geq s \geq r \geq 0$ and $f, g \in B[0, \infty)^+$,

$$|\phi(f, (s, t]) - \phi(g, (s, t])| \leq [C_1(t) + 1] \int_r^t |f(s) - g(s)|C_2(ds),$$

where $C_1(t) = \sup_{s \leq [0, t]}[f(s) + g(s)]$ and

$$C_2(ds) = \|b\|(ds) + c(ds) + \int_0^\infty (z \wedge z^2)m(ds, dz).$$
PROOF. By (3.5) we obtain immediately (i) and (ii). For any \( t \geq s \geq r \geq 0 \) and \( f, g \in B[0,\infty)^+ \), we have

\[
|\phi(f,(s,t)) - \phi(g,(s,t))| \leq \int_r^t |f(u) - g(u)||b|||b||du + C_1(t)\int_r^t |f(u) - g(u)| c(du) \\
+ C_1(t)\int_r^t \int_0^1 |f(u) - g(u)| z^2 m(du,dz) \\
+ \int_r^t \int_1^\infty |f(u) - g(u)| zm(du,dz).
\]

Then (iii) follows.

THEOREM 3.2. For every \( t \geq 0 \) and \( \lambda \geq 0 \) there is a unique bounded positive solution \( r \mapsto v_r(t) \) to the integral evolution equation (3.2) or (3.4) and \( (v_r(t))_{t \geq r} \) is a conservative cumulant semigroup.

PROOF. Let \( \phi_n \) be defined by (3.5). It is easy to see that \( \alpha_n \) and \( \mu_n \) satisfy the conditions of Proposition 2.4. In particular, for any \( s > 0 \) we have

\[
\Delta \alpha_n(s) \geq -\Delta b(s) - (1 - e^{-n}) \int_0^\infty zm(\{s\},dz) > -1.
\]

Then a conservative cumulant semigroup \( (v_{r,t}^{(n)})_{t \geq r} \) is defined by the evolution integral equation:

\[
v_{r,t}^{(n)}(\lambda) = \lambda - \phi_n(v_{s,t}^{(n)}(\lambda),(r,t)), \quad \lambda \geq 0, r \in [0, t].
\]

By (2.16) we have \( v_{r,t}^{(n)}(\lambda) \leq Ac|b|(t) \) for \( r \in [0, t] \) and \( \lambda \in [0, A] \). For \( n \geq k \geq 1 \) let

\[
D_{k,n}(r,t,\lambda) = \sup_{r \leq s \leq t} |v_{s,t}^{(n)}(\lambda) - v_{s,t}^{(k)}(\lambda)|.
\]

By Lemma 3.1, we have

\[
D_{k,n}(r,t,\lambda) \leq 2|\phi(Ac|b|(t),(0,t)) - \phi_k(Ac|b|(t),(0,t))| \\
+ [C_1(t) + 1] \int_r^t D_{k,n}(s,t,\lambda) C_2(ds),
\]

where \( C_1(t) = 2Ac|b|(t) \) and \( C_2(ds) \) is given by (3.6). By Gronwall’s inequality,

\[
D_{k,n}(r,t,\lambda) \leq 2|\phi(Ac|b|(t),(0,t)) - \phi_k(Ac|b|(t),(0,t))| e^{[C_1(t)+1]C_2(t)}.
\]

By Lemma 3.1 it is easy to see the limit \( v_{r,t}(\lambda) := \lim_{k \to \infty} v_{r,t}^{(k)}(\lambda) \) exists and convergence is uniform in \( (r,\lambda) \in [0,t] \times [0, A] \) for every \( A \geq 0 \). By Corollary 1.33 in Li (2011) we have the Lévy–Khintchine representation (1.3) with \( v_{r,t}(0) = 0 \). From (3.7) we get (3.2) by Lemma 3.1 and dominated convergence. The uniqueness of bounded positive solution to (3.2) follows by Lemma 3.1 and Gronwall’s inequality. The semigroup property (1.2) follows from the uniqueness of the solution. Then \( (v_{r,t})_{t \geq r} \) is a conservative cumulant semigroup.

PROPOSITION 3.3. Let \( r \mapsto v_{r,t}(\lambda) \) be the unique bounded positive solution to (3.2) on \([0, t] \). Then we have \( v_{r,t}(\lambda) \leq \pi_{r,t}(\lambda) \), where

\[
r \mapsto \pi_{r,t}(\lambda) := \lambda \prod_{r < s \leq t} (1 - \Delta b(s)) \exp\{b_c(r) - b_c(t)\}
\]

is the solution to (2.7) with \( \alpha(t) = -b(t) \).
The transformation of the cumulant semigroup used in the proof of Proposition 2.4 is an inhomogeneous nonlinear variation of the classical $h$-transformation and has been used in the study of CB-processes; see, e.g., Bansaye et al. (2013), He et al. (2018) and Li (2011, Section 6.1). A generalized form of the transformation is given below, which will be useful in the next section.

**Theorem 3.5.** Let $(v_{r,t})_{t \geq r}$ be the conservative cumulant semigroup defined by (3.2) or (3.4). Let $t \mapsto \zeta(t)$ be a locally bounded function on $[0, \infty)$. Then another conservative cumulant semigroup $(u_{r,t})_{t \geq r}$ is defined by:

\begin{equation}
(3.11) \quad u_{r,t}(\lambda) = e^{\zeta(t)} v_{r,t}(e^{-\zeta(t)} \lambda), \quad \lambda \geq 0.
\end{equation}

Moreover, if $\zeta$ is a càdlàg function on $[0, \infty)$ with locally bounded variations, then $[0, t] \ni r \mapsto u_{r,t}(\lambda)$ is the unique bounded positive solution to

\begin{equation}
(3.12) \quad u_{r,t}(\lambda) = \lambda - \int_{r}^{t} u_{s,t}(\lambda) d\beta(s) - \int_{r}^{t} \int_{0}^{\infty} K(u_{s,t}(\lambda), z) \phi(s-z) m(ds, e^{\phi(s)} dz) - \int_{r}^{t} u_{s,t}(\lambda) e^{-\Delta \zeta(s)} b(ds) - \int_{r}^{t} u_{s,t}(\lambda)^{2} e^{-\zeta(s)} c(ds),
\end{equation}

where

\begin{equation}
\beta(t) = \zeta_{c}(t) + \sum_{s \in (0,t]} (1 - e^{-\Delta \zeta(s)}).
\end{equation}

**Proof.** The arguments are generalizations of those in the last step of the proof of Proposition 2.4. Clearly, the family $(u_{r,t})_{t \geq r}$ defined by (3.11) is a conservative cumulant semigroup.
If \( \zeta \) is a càdlàg function with locally bounded variations, we can use integration by parts to get

\[
\lambda = e^{c(r)} u_{r,t}(e^{-\zeta(t)} \lambda) + \int_r^t v_{s,t}(e^{-\zeta(t)} \lambda) d\zeta(s) + \int_r^t e^{c(s)} dv_{s,t}(e^{-\zeta(t)} \lambda) \\
= u_{r,t}(\lambda) + \int_r^t v_{s,t}(e^{-\zeta(t)} \lambda) e^{\zeta(s)} \zeta_c(ds) + \sum_{s \in (0,t]} v_{s,t}(e^{-\zeta(t)} \lambda)(e^{\zeta(s)} - e^{\zeta(s-)}) \\
+ \int_r^t e^{c(s)} v_{s,t}(e^{-\zeta(t)} \lambda) b(ds) + \int_r^t e^{c(s)} v_{s,t}(e^{-\zeta(t)} \lambda)^2 e(ds) \\
+ \int_r^t \int_0^\infty e^{c(s)} K(v_{s,t}(e^{-\zeta(t)} \lambda), z)m(ds, dz)
\]

where we have used the continuity of \( s \mapsto c(s) \) for the last equality. Then \( r \mapsto u_{r,t}(\lambda) \) solves (3.12). The uniqueness of the solution holds by Theorem 3.2.

4. Stochastic equations for CBVE-processes. In this section, we give a construction of the CBVE-process under the extra integrability condition. Let \((b_1, c, m)\) be admissible parameters. Throughout this section, we assume:

\[
m(t) := \int_0^t \int_0^\infty (z \land z^2)m(ds, dz) < \infty, \quad t \geq 0.
\]

The above formula defines the increasing càdlàg function \( t \mapsto m(t) \) on the positive half line. The set \( J_m = \{ s > 0 : \Delta m(s) > 0 \} \) is at most a countable set. Let \( m_{d}(ds, dz) = 1_{J_m}(s)m(ds, dz) \) and \( m_{c}(ds, dz) = m(ds, dz) - m_{d}(ds, dz) \).

Suppose that \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) is a filtered probability space satisfying the usual hypotheses. Let \( W(ds, du) \) be a time-space \((\mathcal{F}_t)\)-Gaussian white noise on \((0, \infty)^2\) with intensity \(2c(ds)du\). Let \( M(ds, dz, du) \) be a time-space \((\mathcal{F}_t)\)-Poisson random measure on \((0, \infty)^3\) with intensity \(m(ds, dz)du\). One can see that \( M_c(ds, dz, du) := 1_{J_m}(s)M(ds, dz, du) \) and \( M_d(ds, dz, du) := 1_{J_m}(s)M(ds, dz, du) \) are \((\mathcal{F}_t)\)-Poisson random measures with intensities \(m_{c}(ds, dz)du\) and \(m_{d}(ds, dz)du\), respectively. Those random measures are independent of each other as they have disjoint supports. We can rewrite (1.11) equivalently into:

\[
X(t) = X(0) + \int_0^t \int_0^\infty W(ds, du) + \int_0^t \int_0^\infty \int_0^\infty z M_c(ds, dz, du) \\
- \int_0^t X(s-)(-b(ds) + \int_0^\infty \int_0^\infty z M_d(ds, dz, du),
\]

where \( b \) is defined by (3.3).
Let \( \{X(t) : t \geq 0\} \) be a solution to (4.2) satisfying \( \mathbb{E}[X(0)] < \infty \). Let \( \tau_k = \inf\{t \geq 0 : X(t) \geq k\} \) for \( k \geq 1 \). Then \( \tau_k \to \infty \) almost surely as \( k \to \infty \). Moreover, for \( t \geq 0 \) and \( k \geq 1 \) we have
\[
(4.3) \quad k \mathbb{P} \{ \tau_k \leq t \} \leq \mathbb{E}[X(0)]e^{[b(t)](t)} \quad \text{and} \quad \mathbb{E}[X(t)] \leq \mathbb{E}[X(0)]e^{[b(t)](t)}.
\]

We omit the proof of the above proposition, which is based on an application of Gronwall’s inequality. The comparison property of the solutions to (4.2) plays an important role in the analysis of the stochastic equation. In the proof, some special care has to be taken for the negative jumps brought about by the compensator of the Poisson random measure. For simplicity we only give a treatment of the property under a stronger integrability condition, which is sufficient for our applications in this work.

Proposition 4.2. The pathwise uniqueness of solution holds for (4.2) under the additional integrability condition
\[
(4.4) \quad \int_0^t \int_0^\infty z^2 m(ds, dz) < \infty, \quad t \geq 0.
\]

Moreover, under the above condition, if \( \{X_1(t) : t \geq 0\} \) and \( \{X_2(t) : t \geq 0\} \) are two solutions to (4.2) satisfying \( \mathbb{P}\{X_1(0) \leq X_2(0)\} = 1 \), then we have \( \mathbb{P}\{X_1(t) \leq X_2(t) \text{ for every } t \geq 0\} = 1 \).

Proof. It is sufficient to prove the second assertion. By passing to the conditional law \( \mathbb{P}(\cdot | \mathcal{F}_0) \) if it is necessary, we may assume both \( X_1(0) \) and \( X_2(0) \) are deterministic. Then the results of Proposition 4.1 are valid. For each integer \( n \geq 0 \) define \( a_n = \exp\{-n(n+1)/2\} \). Observe that \( a_n \to 0 \) decreasingly as \( n \to \infty \) and
\[
\int_{a_n}^{a_{n-1}} 2(nz)^{-1} \, dz = 2n^{-1} \log \left( \frac{a_{n-1}}{a_n} \right) = 2.
\]

Then there is a positive continuous function \( x \mapsto g_n(x) \) supported by \( (a_n, a_{n-1}) \) so that \( \int_{a_n}^{a_{n-1}} g_n(x) \, dx = 1 \) and \( g_n(x) \leq 2(nx)^{-1} \) for every \( x > 0 \). For \( n \geq 0 \) and \( z \in \mathbb{R} \) let
\[
f_n(z) = \int_0^z \int_0^y g_n(x) \, dx.
\]

It is easy to see that \( f_n(z) = 0 \) for \( z \leq 0 \) and \( f_n(z) \to z \) increasingly for \( z \geq 0 \) as \( n \to \infty \). Setting \( Y(t) = X_1(t) - X_2(t) \), from (4.2) we get
\[
(4.5) \quad Y(t) = Y(0) - \int_0^t Y(s-)b(ds) + \int_0^t \int_{X_s(s-)} q(s-)W(ds, du)
\]
\[
+ \int_0^t \int_{X_s(s-)} \int_{X_s(s-)} q(s-)(\bar{M}_c + \bar{M}_d)(ds, dz, du),
\]

where \( q(s-) = 1_{\{Y(s-) > 0\}} - 1_{\{Y(s-) < 0\}} \) and
\[
X_s(s-) = X_1(s-) \wedge X_2(s-), \quad X^*(s-) = X_1(s-) \vee X_2(s-).
\]

Let \( D_z f_n(x) = f_n(x + z) - f_n(x) - f_n'(x)z \). By Itô’s formula,
\[
(4.6) \quad f_n(Y(t)) = f_n(Y(0)) + \int_0^t f'_n(Y(s-))Y(s) + \frac{1}{2} \int_0^t f''_n(Y(s-))\,d[Y, Y]^c(s).
\]
\[ + \sum_{s \in [0,t]} D_{\Delta Y(s)} f_n(Y(s-)) \]
\[ = - \int_0^t Y(s-) f_n'(Y(s-)) b(ds) + \int_0^t |Y(s-)| f_n''(Y(s-)) c(ds) \]
\[ + \sum_{s \in [0,t]} D_{\Delta Y(s)} f_n(Y(s-)) + M(t), \]

where

\[
M(t) = \int_0^t \int_0^\infty X^*(s-) f_n'(Y(s-)) z q(s-)(\tilde{M}_c + \tilde{M}_d)(ds, dz, du)
\]
\[ + \int_0^t \int_{X^*(s-)} f_n'(Y(s-)) q(s-) W(ds, du). \]

By elementary calculations one sees

\[ E[M(t)^2] \leq 2E \left[ \int_0^t |Y(s-)| c(ds) \right] + E \left[ \int_0^t \int_0^\infty |Y(s-)| z^2 m(ds, dz) \right]. \]

Then \( \{ M(t) : t \geq 0 \} \) is a square-integrable martingale by Proposition 4.1. Let \( \text{supp}(M_c) \subset (0, \infty)^3 \) denote the countable support of the Poisson random measure \( M_c(ds, dz, du) \). Let \( J_M \) denote the projection of \( \text{supp}(M_c) \) to the temporal axis. Then \( J_M \cap (J_b \cup J_m) = \emptyset \) almost surely since \( m_c((J_b \cup J_m) \times (0, \infty)) = 0 \). In view of (4.5), we see that at time \( s \in J_M \) the process \( \{ Y(t) : t \geq 0 \} \) makes a jump of size \( \Delta Y(s) = \text{sgn}(Y(s-)) \) \( z 1_{X(s-) \leq u \leq X^*(s-)} \), where \( z, u > 0 \) satisfy \( (s, z, u) \in \text{supp}(M_c) \), and so \( \text{sgn}(\Delta Y(s)) = \text{sgn}(Y(s-)) \). At time \( s \in J_b \cup J_m \) the process \( \{ Y(t) : t \geq 0 \} \) jumps by

\[ \Delta Y(s) = \int_0^\infty \int_{X^*(s-)} z q(s-) \tilde{M}_d \{ s \}, dz, du - Y(s-) \Delta b(s) \]
\[ = \int_0^\infty \int_{X^*(s-)} z q(s-) M_d \{ s \}, dz, du - Y(s-) \Delta b(s) \]
\[ - [X^*(s-) - X_s(s-)] q(s-) \left( \int_0^\infty z m(s), dz \right) \]
\[ = \int_0^\infty \int_{X^*(s-)} z q(s-) M_d \{ s \}, dz, du - Y(s-) \delta(s), \]

where

\[ \delta(s) = \Delta b(s) + \int_0^\infty z m(s), dz = \Delta b_1(s) + \int_0^1 z m(s), dz \leq 1, \]

and so \( \text{sgn}(Y(s-)) + \Delta Y(s) = \text{sgn}(Y(s-)) \). In summary, we conclude that \( Y(s-) < 0 \) always implies \( Y(s-) + \Delta Y(s) \leq 0 \) and hence \( D_{\Delta Y(s)} f_n(Y(s-)) = 0 \). From (4.6) we get

\[ f_n(Y(t)) = - \int_0^t Y(s-) f_n'(Y(s-)) 1_{\{Y(s-) > 0\}} b(ds) + M(t) \]
\[ + \int_0^t |Y(s-)| f_n''(Y(s-)) 1_{\{Y(s-) > 0\}} c(ds) \]
\[ + \int_0^t \int_0^\infty \int_{X^*(s-)} D_z f_n(Y(s-)) 1_{\{Y(s-) > 0\}} M_c(ds, dz, du) \]
Now we let $D_{\Delta Y(s)}f_n(Y(s-))1_{\{Y(s-) > 0\}}1_{J_b \cup J_m}(s)$.

As in the proof of Theorem 8.2 in Li (2020) one sees $|Y(s-)f_n''(Y(s-))| \leq 2n^{-1}$ and $|Y(s-)D_z f_n(Y(s-))|1_{\{Y(s-) > 0\}} \leq n^{-1}z^2, \quad s, z > 0$.

In view of (4.7), for $s \in J_b \cup J_m$ we have $\Delta Y(s) = g_s(X_1(s-)) - g_s(X_2(s-))$, where

$$g_s(x) = \int_0^\infty \int_0^x zM_d(\{s\}, dz, du) - x\delta(s).$$

It is easy to see that $x \mapsto x + g_s(x)$ is an increasing function. By Lemma 3.1 in Li and Pu (2012) we have, for $s \in J_b \cup J_m$,

$$|D_{\Delta Y(s)}f_n(Y(s-))| \leq 2n^{-1}|Y(s-)^{-1}||\Delta Z(s)| + |Y(s-)\Delta b(s)|^2$$

$$\leq 4n^{-1}||Y(s-)^{-1}||\Delta Z(s)||^2 + |Y(s-)||\Delta b(s)||^2,$$

where

$$\Delta Z(s) = \int_0^\infty \int_0^{X^*(s-)} z\tilde{M}_d(\{s\}, dz, du),$$

Taking expectations in (4.8) gives

$$E[f_n(Y(t))] \leq E\left[\int_0^t Y(s-)1_{\{Y(s-) > 0\}}\|b\|(ds)\right] + 2n^{-1}c(t)$$

$$+ E\left[\int_0^t \int_0^{\infty} |Y(s-)D_z f_n(Y(s-))|1_{\{Y(s-) > 0\}}m_c(ds, dz)\right]$$

$$+ 4n^{-1}E\left[\sum_{s \in [0,t]} Y(s-)^{-1}|\Delta Z(s)|^21_{\{Y(s-) > 0\}}1_{J_m}(s)\right]$$

$$+ 4n^{-1}E\left[\sum_{s \in [0,t]} Y(s-)|\Delta b(s)|^21_{\{Y(s-) > 0\}}1_{J_m}(s)\right]$$

$$\leq \int_0^t E[(Y(s-) \lor 0)]\|b\|(ds) + 2n^{-1}c(t)$$

$$+ n^{-1} \int_0^t \int_0^\infty z^2m_c(ds, dz) + 4n^{-1} \int_0^t \int_0^\infty z^2m_d(ds, dz)$$

$$+ 4n^{-1} \sum_{s \in [0,t]} E[(Y(s-) \lor 0)]|\Delta b(s)|^21_{J_m}(s).$$

Now we let $n \to \infty$ and use Fatou’s lemma to obtain

$$E[Y(t) \lor 0] \leq \int_0^t E[Y(s-) \lor 0]|b|(ds).$$

Then an application of Gronwall’s inequality gives $E[Y(t) \lor 0] = 0$ for every $t \geq 0$, giving the desired comparison property.

The next proposition gives a characterization of the conditional distribution of the jump of the CBVE-process at any moment $t \in J_b \cup J_m$.

**Proposition 4.3.** The CBVE-process with transition semigroup $(Q_{r,t})_{t \geq r}$ given by (1.5) and (3.2) has a càdlàg semimartingale realization $(X(t), \mathcal{F}_t : t \geq 0)$ with the filtration satisfying the usual hypotheses. For such a realization and $t \in J_b \cup J_m$ we have

$$E(e^{-\lambda \Delta X(t)}|\mathcal{F}_{t-}) = e^{(\lambda-v_{t-},(\lambda))X(t-)}, \quad \lambda \geq 0,$$
where $\Delta X(t) = X(t) - X(t^-)$ and

\begin{equation}
\lambda - v_{t-,t}(\lambda) = \Delta b(t)\lambda + \int_0^\infty K(\lambda, z)m(\{t\}, dz).
\end{equation}

\textbf{Proof.} Let $\{(X(t), \mathcal{G}_t) : t \geq 0\}$ be a realization of the CBVE-process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In view of (3.2), for any $\lambda \geq 0$ we have $v_{r,t}(\lambda) \rightarrow \lambda$ as $t \downarrow r$. Then (1.5) implies $\lim_{t \uparrow r} Q_{r,t}(x, dy) = \delta_x(dy)$ by weak convergence and so $\lim_{t \uparrow r} Q_{r,t}(x, \{y \geq 0 : |y - x| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. By dominated convergence,

$$
\lim_{t \uparrow r} \mathbb{P}\{|X(t) - X(r)| > \varepsilon\} = \lim_{t \uparrow r} \mathbb{E}[Q_{r,t}(x, \{y \geq 0 : |y - x| \geq \varepsilon\})|_x=X(r)] = 0.
$$

Then $\{X(t) : t \geq 0\}$ is stochastically right continuous. From (3.2) we see $r \mapsto v_{r,t}(\lambda)$ is right-continuous on $[0, t]$, so $\{e^{-X(r)^v_{r,t}(\lambda)} : r \in [0, t]\}$ is stochastically right-continuous. Let $\mathcal{G}_t$ be the augmentation of $\mathcal{G}_t$ and let $\mathcal{F}_r = \mathcal{G}_t^\uparrow$ for $t \geq 0$. The Markov property implies

\begin{equation}
\mathbb{E}[e^{-\lambda X(t)}|^\mathcal{G}_r] = e^{-\lambda X(r)^v_{r,t}(\lambda)}, \quad t \geq r \geq 0.
\end{equation}

This means $\{e^{-X(r)^v_{r,t}(\lambda)} : r \in [0, t]\}$ is a positive bounded $(\mathcal{F}_r)$-martingale, so it has a càdlàg $(\mathcal{F}_r)$-martingale modification. Then $\{X(r)^v_{r,t}(\lambda) : r \in [0, t]\}$ has a càdlàg $(\mathcal{F}_r)$-semimartingale modification; see, e.g., Dellacherie and Meyer (1982, p.221). By Proposition 2.3 we have $v_{r,t}(\lambda) \geq l_0(\lambda) > 0$ for $\lambda > 0$. Then $\{X(r) : r \in [0, t]\}$ has a càdlàg $(\mathcal{F}_r)$-semimartingale modification; see, e.g., Dellacherie and Meyer (1982, p.219). It follows that $\{X(t) : t \geq 0\}$ has a càdlàg $(\mathcal{G}_r)$-semimartingale modification. Using such a modification we can replace $\mathcal{G}_r$ by $\mathcal{F}_r$ in (4.11). Then $\{(X(t), \mathcal{F}_t) : t \geq 0\}$ is a càdlàg semimartingale realization of the CBVE-process with the filtration satisfying the usual hypotheses. By letting $r \uparrow t$ in (3.2) and (4.11) we get

$$
\mathbb{E}[e^{-\lambda X(t)}|^\mathcal{F}_t] = e^{-X(t^-)^v_{t-,t}(\lambda)}, \quad \lambda \geq 0,
$$

where

$$
v_{t-,t}(\lambda) = (1 - \Delta b(t))\lambda - \int_0^\infty K(\lambda, z)m(\{t\}, dz).
$$

Those give (4.9) and (4.10). \hfill \square

In view of (4.9) and (4.10), for any $t \in J_d \cup J_m$ it is natural to expect that the jump $\Delta X(t)$ of the CBVE-process should be given by the position at time $X(t^-)$ of a spectrally positive Lévy process with Lévy measure $m(\{t\}, dz)$. This is justified by the next proposition. In the proof of the proposition, we first deal with the martingale case and then obtain the general result via an inhomogeneous nonlinear $h$-transformation.

\textbf{Proposition 4.4.} Suppose that (4.4) holds. Then the CBVE-process with transition semigroup $(Q_{r,t})_{t \geq r}$ defined by (1.5) and (3.2) is a weak solution to (4.2).

\textbf{Proof.} \textbf{Step 1.} Consider the case where $b(t) = 0$ for every $t \geq 0$. Let $(v_{r,t})_{t \geq r}$ be the conservative cumulant semigroup defined by (3.2) in this special case. Let $\{(X(t), \mathcal{F}_t) : t \geq 0\}$ be a realization of the corresponding CBVE-process provided by Proposition 4.3. Then the process is a martingale by Proposition 3.4. Let $N_0(ds, dz)$ be the optional random measure on $(0, \infty) \times \mathbb{R}$ by

$$
N_0(ds, dz) := \sum_{s>0} 1_{\{\Delta X(s) \neq 0\}} \delta(s, \Delta X(s))(ds, dz).
$$
Let $\tilde{N}_0(ds, dz)$ denote the predictable compensator of $N_0(ds, dz)$ and let $\hat{N}_0(ds, dz) = N_0(ds, dz) - \tilde{N}_0(ds, dz)$ be the compensated measure; see, e.g., Dellacherie and Meyer (1982, p.375). We can write

\begin{equation}
X(t) = X(0) + M(t) + \int_0^t \int_{\mathbb{R}} z \tilde{N}_0(ds, dz),
\end{equation}

where $\{M(t) : t \geq 0\}$ is a continuous local martingale. Let $\{C(t) : t \geq 0\}$ be its quadratic variation process. Let $f(x, \lambda) = e^{-x\lambda}$ for $x, \lambda \geq 0$. Then

$$f_1'(x, \lambda) = -\lambda f(x, \lambda), \quad f_2'(x, \lambda) = -xf(x, \lambda), \quad f_1''(x, \lambda) = \lambda f(x, \lambda).$$

By Itô’s formula, for $t \geq r \geq 0$ and $\lambda \geq 0$,

\begin{equation}
e^{-X(t)\lambda} = e^{-X(r)\nu_r(\lambda)} + \int_r^t f_2'(X(s-), v_{s-\lambda}(\lambda)) d\nu_{s-\lambda}(\lambda)
+ \frac{1}{2} \int_r^t f_1''(X(s-), v_{s-\lambda}(\lambda)) dC(s)
+ \sum_{s \in (r,t] \cap J_m} \left[ f(X(s), v_{s\lambda}(\lambda)) - f(X(s), v_{s-\lambda}(\lambda))
- f_1'(X(s-), v_{s-\lambda}(\lambda)) \Delta X(s) - f_2'(X(s-), v_{s-\lambda}(\lambda)) \Delta \nu_{s\lambda}(\lambda) \right]
+ \sum_{s \in (r,t] \setminus J_m} \left[ f(X(s), v_{s\lambda}(\lambda)) - f(X(s), v_{s\lambda}(\lambda))
- f_1'(X(s-), v_{s\lambda}(\lambda)) \Delta X(s) \right]
= e^{-X(r)\nu_r(\lambda)} - \int_r^t e^{-X(s)\nu_{s\lambda}(\lambda)} v_{s-\lambda}(\lambda) dX(s)
- \int_r^t e^{-X(s)\nu_{s\lambda}(\lambda)} X(s-) v_{s\lambda}(\lambda) dC(ds)
- \int_r^t \int_0^\infty e^{-X(s)\nu_{s\lambda}(\lambda)} X(s-) K(v_{s\lambda}(\lambda), z) m(ds, dz)
+ \frac{1}{2} \int_r^t e^{-X(s)\nu_{s\lambda}(\lambda)} v_{s\lambda}(\lambda) dC(ds)
+ \sum_{s \in (r,t] \cap J_m} \left[ e^{-X(s)\nu_{s\lambda}(\lambda)} + e^{-X(s)\nu_{s-\lambda}(\lambda)} v_{s-\lambda}(\lambda) \Delta X(s)
- e^{-X(s)\nu_{s\lambda}(\lambda)} + e^{-X(s)\nu_{s-\lambda}(\lambda)} X(s-) \Delta \nu_{s\lambda}(\lambda) \right]
+ \sum_{s \in (r,t] \setminus J_m} \left[ e^{-X(s)\nu_{s\lambda}(\lambda)} - e^{-X(s)\nu_{s\lambda}(\lambda)}
- e^{-X(s)\nu_{s\lambda}(\lambda)} v_{s\lambda}(\lambda) + e^{-X(s)\nu_{s-\lambda}(\lambda)} v_{s-\lambda}(\lambda) \Delta X(s) \right]
= e^{-X(r)\nu_r(\lambda)} + \int_0^r e^{-X(s)\nu_{s\lambda}(\lambda)} v_{s\lambda}(\lambda) dX(s)
+ \int_0^r e^{-X(s)\nu_{s\lambda}(\lambda)} X(s-) v_{s\lambda}(\lambda) dC(ds)
\[
\begin{align*}
&\int_{0}^{\infty} \int_{0}^{\infty} e^{-X(s)-v_{s,t}(\lambda)} X(s) K(v_{s,t}(\lambda), z) m(ds, dz) + Z(t) \\
&- \frac{1}{2} \int_{0}^{\infty} e^{-X(s)-v_{s,t}(\lambda)} v_{s,t}(\lambda)^2 C(ds) \\
&- \sum_{s \in (0,t] \cap J_m} \left[ e^{-X(s)} v_{s,t}(\lambda) + e^{-X(s)-v_{s,t}(\lambda)} X(s) \Delta v_{s,t}(\lambda) \right] \\
&- \sum_{s \in (0,t] \cap J_m} e^{-X(s)-v_{s,t}(\lambda)} \left[ e^{-\Delta X(s)} v_{s,t}(\lambda) - 1 + v_{s,t}(\lambda) \Delta X(s) \right],
\end{align*}
\]

where

\[
Z(t) = \int_{0}^{t} e^{-X(s)-v_{s,t}(\lambda)} v_{s,t}(\lambda) dX(s) - \int_{0}^{t} e^{-X(s)-v_{s,t}(\lambda)} X(s) K(v_{s,t}(\lambda), z) m(ds, dz) \\
+ \frac{1}{2} \int_{0}^{t} e^{-X(s)-v_{s,t}(\lambda)} v_{s,t}(\lambda)^2 C(ds) + \sum_{s \in (0,t] \cap J_m} \left[ e^{-X(s)} v_{s,t}(\lambda) - e^{-X(s)-v_{s,t}(\lambda)} \right] \\
+ e^{-X(s)-v_{s,t}(\lambda)} v_{s,t}(\lambda) \Delta X(s) + e^{-X(s)-v_{s,t}(\lambda)} X(s) \Delta v_{s,t}(\lambda) \\
+ \sum_{s \in (0,t] \cap J_m} e^{-X(s)-v_{s,t}(\lambda)} \left[ e^{-\Delta X(s)} v_{s,t}(\lambda) - 1 + v_{s,t}(\lambda) \Delta X(s) \right],
\]

Taking the conditional expectation in (4.13) we obtain

\[
\mathbb{E}[\cdot | \mathcal{F}_t] = \int_{0}^{r} e^{-X(s)-v_{s,t}(\lambda)} X(s) v_{s,t}(\lambda)^2 C(ds) + \text{mart.}
\]

\[
\begin{align*}
&\int_{0}^{r} \int_{0}^{\infty} e^{-X(s)-v_{s,t}(\lambda)} X(s) K(v_{s,t}(\lambda), z) m(ds, dz) \\
&- \frac{1}{2} \int_{0}^{r} e^{-X(s)-v_{s,t}(\lambda)} v_{s,t}(\lambda)^2 C(ds) \\
&- \sum_{s \in (0,t] \cap J_m} \left[ e^{-X(s)} v_{s,t}(\lambda) + e^{-X(s)-v_{s,t}(\lambda)} X(s) \Delta v_{s,t}(\lambda) \right] \\
&- \sum_{s \in (0,t] \cap J_m} e^{-X(s)-v_{s,t}(\lambda)} \left[ e^{-\Delta X(s)} v_{s,t}(\lambda) - 1 + v_{s,t}(\lambda) \Delta X(s) \right] \\
= \int_{0}^{r} e^{-X(s)-v_{s,t}(\lambda)} X(s) v_{s,t}(\lambda)^2 C(ds) + \text{mart.}
\end{align*}
\]
Then the uniqueness of canonical decompositions of martingales yields
\[
dC(s) = 2X(s-)c(ds) = 2c(ds) \int_0^\infty 1_{\{u \leq X(s-)\}}du
\]
and
\[
1_{J_m}(s)\tilde{N}_0(ds, dz) = 1_{J_m}(s)X(s-)m(ds, dz)
= m_c(ds, dz) \int_0^\infty 1_{\{u \leq X(s-)\}}du.
\]
By El Karoui and Méliédard (1990, Theorem III.6), on an extension of the original probability space there exists a Gaussian white noise \(W(ds, du)\) on \((0, \infty)^2\) with intensity \(2c(ds)du\) such that
\[
M(t) = \int_0^t \int_0^\infty 1_{\{u \leq X(s-)\}}W(ds, du) = \int_0^t \int_0^\infty X(s-)W(ds, du).
\]
By Kabanov et al. (1981, Theorem 1), on a further extension of the original probability space we can define a Poisson random measure \(M_c(ds, dz, du)\) with intensity \(m_c(ds, dz)du\) so that
\[
\int_0^t \int_0^\infty z1_{J_m}(s)\tilde{N}_0(ds, dz) = \int_0^t \int_0^\infty \int_0^\infty X(s-)z\tilde{M}_c(ds, dz, du);
\]
see also El Karoui and Lepeltier (1977). By (4.12) we see the process a.s. makes a jump at time \(s \in J_m\) with the representation:
\[
\Delta X(s) = \int_\mathbb{R} z\tilde{N}_0(\{s\}, dz).
\]
From Proposition 4.3 it follows that
\[
\mathbb{E}(e^{-\lambda \Delta X(s)} | \mathcal{F}_{s-}) = e^{(\lambda - v_{s-}(\lambda))X(s-)}, \quad \lambda \geq 0,
\]
where
\[
\lambda - v_{s-}(\lambda) = \int_0^\infty (e^{-\lambda z} - 1 + \lambda z)m_d(\{s\}, dz).
\]
By Proposition 2.5 we can make another extension of the probability space and define a Poisson random measure \(N_s(dz, du)\) on \((0, \infty)^2\) with intensity \(m_d(\{s\}, dz)du\) such that \(N_s\) is independent of \(\mathcal{F}_{s-}\) and
\[
\Delta X(s) = \int_0^\infty \int_0^\infty z\tilde{N}_s(dz, du).
\]
Let \(M_d(ds, dz, du)\) be the random measure on \((0, \infty)^3\) defined by
\[
M_d((0, t] \times A) = \sum_{s \in (0, t]\cap J_m} N_s(A), \quad t \geq 0, \quad A \in \mathcal{B}((0, \infty)^2).
\]
Then \(M_d(ds, dz, du)\) is a Poisson random measure with intensity \(m_d(ds, dz)du\). From (4.12) we see \(\{X(t) : t \geq 0\}\) is a solution to (4.2) with \(b = 0\). The independence of the noises follows by standard arguments; see, e.g., Ikeda and Watanabe (1989, pp.77-78).

**Step 2.** In the general case, let \((v_{r,t})_{t \geq r}\) be the conservative cumulant semigroup defined by (3.2). Let \(\zeta\) be the càdlàg function on \([0, \infty)\) such that \(\zeta(t) = -b(t)\) and \(\Delta \zeta(t) = \log[1 - b(t)]\) for every \(t \geq 0\). By Theorem 3.5 we can define another cumulant semigroup \((u_{r,t})_{t \geq r}\) by (3.11) and \(r \mapsto u_{r,t}(\lambda)\) is the unique bounded positive solution to
\[
u_{r,t}(\lambda) = \lambda - \int_r^t u_{s,t}(\lambda)^2 e^{-\zeta(s)}c(ds) - \int_r^t \int_0^\infty K(u_{s,t}(\lambda), z)e^{\zeta(s)}m(ds, e^{\zeta(s)}dz).
\]
Let $W(ds, du)$ and $M(ds, dz, du)$ be given as in the introduction. One can see that $W_0(ds, du) := e^{-\zeta(s)}W(ds, e^{\zeta(s)-}du)$ is a Gaussian white noise on $(0, \infty)^2$ with intensity $2e^{-\zeta(s)}c(ds)du$ and $M_0(ds, dz, du) := M(ds, e^{\zeta(s)}dz, e^{\zeta(s)-}du)$ is a Poisson random measure on $(0, \infty)^3$ with intensity $e^{\zeta(s)-}m(ds, e^{\zeta(s)}dz)du$. By Proposition 4.2 and the first step of the proof, we can construct a CBVE-process with cumulant semigroup $(u_{r,t})_{t \geq r}$ by the pathwise unique solution to

$$Z(t) = X(0) + \int_0^t \int_0^s Z(s-) W_0(ds, du) + \int_0^t \int_0^\infty \int_0^s z \tilde{M}_0(ds, dz, du).$$

It is easy to see that $t \mapsto X(t) := e^{\zeta(t)}Z(t)$ is a CBVE-process with cumulant semigroup $(v_{r,t})_{t \geq r}$. By integration by parts we have

$$X(t) = X(0) + \int_0^t Z(s-)d\zeta(s) + \int_0^t e^{\zeta(s)}dZ(s)
= X(0) - \int_0^t e^{\zeta(s)}Z(s-)b(ds) + \int_0^t e^{-\zeta(s)-}X(s-) e^{\zeta(s)}W_0(ds, du)
+ \int_0^t \int_0^\infty e^{-\zeta(s)-}X(s-) e^{\zeta(s)}z \tilde{M}_0(ds, dz, du)
= X(0) - \int_0^t X(s-)b(ds) + \int_0^t X(s-) e^{\zeta(s)}W_0(ds, e^{-\zeta(s)-}du)
+ \int_0^t \int_0^\infty X(s-) z \tilde{M}_0(ds, e^{-\zeta(s)}dz, e^{-\zeta(s)-}du).$$

Then $\{X(t) : t \geq 0\}$ solves (4.2). \hfill \Box

**THEOREM 4.5.** There is a pathwise unique solution $\{X(t) : t \geq 0\}$ to (4.2) and the solution is a CBVE-process with transition semigroup $(Q_{r,t})_{t \geq r}$ defined by (1.5) and (3.2).

**PROOF.** Step 1. Consider the special case where (4.4) holds. By Proposition 4.4, there is a weak solution to (4.2) and the solution is a CBVE-process with transition semigroup $(Q_{r,t})_{t \geq r}$ defined by (1.5) and (3.2). By Proposition 4.2 the pathwise uniqueness holds for (4.2). It follows by standard arguments that (4.2) has a pathwise unique strong solution; see, e.g., Situ (2005, p.104).

Step 2. Consider the general case. By the first step, for each $k \geq 1$ there is a pathwise unique solution $\{X_k(t) : t \geq 0\}$ to the stochastic equation

$$X(t) = X(0) - \int_0^t X(s-)b(ds) + \int_0^t \int_0^s X(s-) W(ds, du)
+ \int_0^t \int_0^\infty \int_0^s (z + k) \tilde{M}(ds, dz, du)
- \int_0^t \int_0^\infty X(s-)(z - k)m(ds, dz),$$

and the solution is a CBVE-process with cumulant semigroup $(v_{r,t}^{(k)})_{t \geq r}$ defined by

$$v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda)^2c(ds) - \int_r^t \int_0^\infty K(v_{s,t}(\lambda), z + k)m(ds, dz)
- \int_r^t v_{s,t}(\lambda)b(ds) - \int_r^t \int_k^\infty v_{s,t}(\lambda)(z - k)m(ds, dz).$$
We can rewrite (4.14) into the equivalent form:

\begin{equation}
X(t) = X(0) - \int_0^t X(s-)b(ds) + \int_0^t \int_0^t X(s-) W(ds, du) + \int_0^t \int_0^t X(s-) zM(ds, dz, du) - \int_0^t \int_0^\infty X(s-) zm(ds, dz) + \int_0^t \int_0^\infty X(s-) kM(ds, dz, du).
\end{equation}

This is obtained from (4.2) by using \( kM(ds, dz, du) \) to replace \( zM(ds, dz, du) \) in the last integral, which modifies the magnitudes of the large jumps. Let \( \zeta_{0,k} = 0 \) and for \( i \geq 0 \) inductively define

\[ \zeta_{i+1,k} = \inf \left\{ t > 0 : \int_0^{\zeta_i,k} \int_k^\infty \int_0^{X(s-)} M(ds, dz, du) \geq 1 \right\} . \]

Then \( \eta_{n,k} := \sum_{i=0}^n \zeta_{i,k} \) is the time when \( \{X_k(t) : t \geq 0\} \) has the \( n \)th modified jump. Since \( \{X_k(t) : t \geq 0\} \) is a càdlàg process, we have \( \eta_{n,k} \to \infty \) as \( n \to \infty \). The modification described above does not make any change of the solution at the time interval \( [0, \eta_{1,k}] \) and the process \( \{X_k(t) : t \geq 0\} \) takes its first modified jump at time \( \eta_{1,k} = \zeta_{1,k} \). For any \( k \geq 1 \) we have \( \eta_{1,k} \leq \eta_{1,k+1} \), which implies \( X_k(t) = X_{k+1}(t) \) for \( 0 \leq t \leq \eta_{1,k} \). From (4.16) it follows that

\[ \Delta X_k(\eta_{1,k}) = \int_0^{\eta_{1,k}} \int_0^{X_k(\eta_{1,k})} zM(\{\eta_{1,k}\}, dz, du) - \int_0^{\eta_{1,k}} X_k(\eta_{1,k}) \Delta b(\eta_{1,k}) + \int_0^{\eta_{1,k}} \int_0^{X_k(\eta_{1,k})} kM(\{\eta_{1,k}\}, dz, du) \]

and

\[ \Delta X_{k+1}(\eta_{1,k}) = \int_0^{\eta_{1,k+1}} \int_0^{X_k(\eta_{1,k})} zM(\{\eta_{1,k}\}, dz, du) - \int_0^{\eta_{1,k+1}} X_k(\eta_{1,k}) \Delta b(\eta_{1,k}) + \int_0^{\eta_{1,k+1}} \int_0^{X_k(\eta_{1,k})} (k+1)M(\{\eta_{1,k}\}, dz, du) \]

\[ = \int_0^{\eta_{1,k}} \int_0^{X_k(\eta_{1,k})} zM(\{\eta_{1,k}\}, dz, du) - \int_0^{\eta_{1,k}} X_k(\eta_{1,k}) \Delta b(\eta_{1,k}) + \int_0^{\eta_{1,k}} \int_0^{X_k(\eta_{1,k})} zM(\{\eta_{1,k}\}, dz, du) \]

\[ + \int_0^{\eta_{1,k}} \int_0^{X_k(\eta_{1,k})} (k+1)M(\{\eta_{1,k}\}, dz, du) \].

Then \( \Delta X_k(\eta_{1,k}) \leq \Delta X_{k+1}(\eta_{1,k}) \), and so \( X_k(\eta_{1,k}) \leq X_{k+1}(\eta_{1,k}) \). By applying Proposition 4.2 successively at the stopping times \( \eta_{n,k} \), \( n \geq 1 \) we infer that \( X_k(t) \leq X_{k+1}(t) \) for \( t \geq 0 \) and \( k \geq 1 \). For \( r, x, t \geq 0 \) let \( \{X_k(r, x, t) : t \geq r\} \) be the pathwise unique solution to:

\[ X(t) = x + \int_r^t X(s-) W(ds, du) + \int_r^t \int_r^t X(s-) (z \land k) \tilde{M}(ds, dz, du) - \int_r^t X(s-) b(ds) - \int_r^t \int_r^\infty X(s-)(z - k)m(ds, dz). \]

By the preceding arguments we have \( X_{k+1}(r, x, t) \geq X_k(r, x, t) \), which implies

\[ \nu_{r,t}^{(k+1)}(\lambda) = -\log \mathbb{E} \exp\{-\lambda X_{k+1}(r, 1, t)\} \]
\[ \geq -\log E \exp\{-\lambda X_k(r, 1, t)\} = v_{r,t}^{(k)}(\lambda). \]

Since \( v_{r,t}^{(k)}(\lambda) \leq U_{0,t}(\lambda) \) by Proposition 2.3, from (4.15) we see that \([0, t] \ni r \mapsto v_{r,t}(\lambda) := \lim_{k \to \infty} v_{r,t}^{(k)}(\lambda) \) is the unique bounded positive solution to (3.2). Observe that \( \eta_{1,k} = \xi_{1,k} \geq \tau_k = \inf\{t \geq 0 : X_k(t) \geq k\} \). By Proposition 4.1 we have \( \lim_{k \to \infty} \eta_{1,k} = \infty \). Consequently, there is a càdlàg process \( \{Y(t) : t \geq 0\} \) so that \( Y(t) = X_k(t) \) for \( 0 \leq t < \eta_{1,k} \) and \( k \geq 1 \). Moreover, for \( 0 \leq t < \eta_{1,k} \) we have

\[ \int_0^t \int_k^\infty X_k(s-) \ d M(ds, dz, du) = \int_0^t \int_k^\infty X_k(s-) \ kM(ds, dz, du) = 0, \]

so (4.16) implies

\[ Y(t) = X(0) + \int_0^t \int_0^\infty W(ds, du) + \int_0^t \int_0^k \int_s^\infty Y(s-) \ dM(ds, dz, du) \]
\[ - \int_0^t Y(s-) b(ds) - \int_0^t \int_k^\infty Y(s-) \ dM(ds, dz) \]
\[ + \int_0^t \int_0^\infty Y(s-) \ dM(ds, dz, du) \]
\[ = X(0) + \int_0^t \int_0^\infty W(ds, du) + \int_0^t \int_0^\infty Y(s-) \ dM(ds, dz, du) \]
\[ - \int_0^t Y(s-) b(ds). \]

Since \( \lim_{k \to \infty} \eta_{1,k} = \infty \), we see that \( \{Y(t) : t \geq 0\} \) is a solution to (4.2). On the other hand, it is clear that \( \{X_k(t) : t \geq 0\} \) increases to \( \{Y(t) : t \geq 0\} \) as \( k \to \infty \), so the later is a CBVE-process with cumulant semigroup \((v_{r,t})_{t \geq r}\). Similarly, for any solution \( \{X(t) : t \geq 0\} \) to (4.2) one can see \( X(t) = X_k(t) \) for \( 0 \leq t < \eta_{1,k} \) and \( k \geq 1 \). Then the pathwise uniqueness holds for (4.2). \( \square \)

5. Extensions to general parameters. In this section, we extend the results established in the last two sections to the general equations (1.8) and (1.11) with admissible or weakly admissible parameters.

**Proposition 5.1.** Let \((b_1, c, m)\) be admissible parameters. Then for any \( \lambda > 0 \) and \( t \geq 0 \) the uniqueness of bounded positive solutions holds for (1.8).

**Proof.** Suppose that both \( r \mapsto v_{r,t}(\lambda) \) and \( r \mapsto w_{r,t}(\lambda) \) are bounded positive solutions to (1.8). By Corollary 2.2 and Proposition 2.3, we have

\[ 0 < l_{0,t}(\lambda) \leq v_{r,t}(\lambda) \wedge w_{r,t}(\lambda) \leq v_{r,t}(\lambda) \vee w_{r,t}(\lambda) \leq U_{0,t}(\lambda). \]

It follows that

\[ |v_{r,t}(\lambda) - w_{r,t}(\lambda)| \leq \int_r^t |v_{s,t}(\lambda) - w_{s,t}(\lambda)||b_1||(ds) + \int_r^t |v_{s,t}(\lambda)|^2 - w_{s,t}(\lambda)^2|c|(ds) \]
\[ + \int_r^t \int_0^1 |K(v_{s,t}(\lambda), z) - K(w_{s,t}(\lambda), z)|m(ds, dz) \]
\[ + \int_r^t \int_0^{\infty} |e^{v_{s,t}(\lambda)z} - e^{-w_{s,t}(\lambda)z}|m(ds, dz) \]
\[ \leq \int_r^t |v_{s,t}(\lambda) - w_{s,t}(\lambda)||b_1||(ds) + 2U_{0,t}(\lambda) \int_r^t |v_{s,t}(\lambda) - w_{s,t}(\lambda)|c(ds) \]
Then Gronwall’s inequality implies $|v_{r,t}(\lambda) - w_{r,t}(\lambda)| = 0$ for $r \in [0,t]$. 

**Proof of Theorems 1.1 and 1.4.** By Theorem 3.2, for each $k \geq 1$ there is a cumulant semigroup $(v^{(k)}_{r,t})_{t \geq r}$ defined by the integral evolution equation:

\begin{equation}
(5.1) \quad v_{r,t}(\lambda) = \lambda - \int_r^t v_{s,t}(\lambda)^2 c(ds) - \int_r^t \int_0^\infty K(v_{s,t}(\lambda), z \wedge k)m(ds, dz) 
- \int_r^t v_{s,t}(\lambda)b_1(ds) + \int_r^t \int_1^\infty v_{s,t}(\lambda)(z \wedge k)m(ds, dz).
\end{equation}

By Theorem 4.5 we can construct a CBVE-process $\{X_k(t) : t \geq 0\}$ with cumulant semigroup $(v^{(k)}_{r,t})_{t \geq r}$ by the pathwise unique solution to:

\begin{align*}
X(t) &= X(0) + \int_0^t \int_0^\infty X(s-) W(ds, du) + \int_0^t \int_0^\infty \int_0^\infty X(s-) (z \wedge k) \tilde{M}(ds, dz, du) \\
&\quad - \int_0^t X(s-) b_1(ds) + \int_0^t \int_1^\infty X(s-) (z \wedge k)m(ds, dz).
\end{align*}

The above stochastic equation is equivalent to

\begin{equation}
(5.2) \quad X(t) = X(0) + \int_0^t \int_0^\infty X(s-) W(ds, du) + \int_0^t \int_0^1 \int_0^\infty X(s-) z \tilde{M}(ds, dz, du) \\
- \int_0^t X(s-) b_1(ds) + \int_0^t \int_1^\infty X(s-) (z \wedge k) M(ds, dz, du).
\end{equation}

Let $\zeta_{1,k}$ and $\tau_k$ be defined as in the last step of the proof of Theorem 4.5. By the arguments in that proof we have $X_{k+1}(t) = X_k(t)$ for $0 \leq t < \zeta_{1,k}$ and both $\{X_k(t) : t \geq 0\}$ and $(v^{(k)}_{r,t})_{t \geq r}$ are increasing in $k \geq 1$. By Proposition 2.3 we have $l_{0,t}(\lambda) \leq v^{(k)}_{r,t}(\lambda) \leq U_{0,t}(\lambda)$. Then for $\lambda > 0$ the limit $v^{(k)}_{r,t}(\lambda) := \lim_{k \to \infty} v^{(k)}_{r,t}(\lambda)$ exists and $\inf_{0 \leq r < t} v^{(k)}_{r,t}(\lambda) > 0$. By letting $k \to \infty$ in (5.1) we see $r \mapsto v_{r,t}(\lambda)$ is a solution to (1.8). The uniqueness of the solution is guaranteed by Proposition 5.1. Clearly, the family $(v^{(k)}_{r,t})_{t \geq r}$ is a cumulant semigroup. It is easy to see that $\lim_{k \to \infty} \zeta_{1,k} = \tau_\infty := \lim_{k \to \infty} \tau_k$. Let $\{X(t) : t \geq 0\}$ be the càdlàg process such that $X(t) = X_k(t)$ for $0 \leq t < \zeta_{1,k}$ and $X(t) = \infty$ for $t \geq \tau_\infty$. Then $\{X(t) : t \geq 0\}$ is a CBVE-process with cumulant semigroup $(v^{(k)}_{r,t})_{t \geq r}$. From (5.2) we see that $\{X(t) : t \geq 0\}$ is a solution to (1.11). The pathwise uniqueness for (1.11) follows from that for (5.2). 

**Proof of Theorem 1.2.** It is easy to see that $r \mapsto u_{r,t}(0)$ is indeed a bounded positive solution to (1.8) with $\lambda = 0$. Suppose that $r \mapsto u_{r,t}(0)$ is another positive solution to (1.8) with $\lambda = 0$ and $u_{r,t}(0) > 0$ for some $r \in [0,t]$. Let $t_0 = \inf\{r \in [0,t] : u_{r,t}(0) = 0\}$. We clearly have $u_{r,t}(0) = 0$ for $r \in [t_0, t]$, and hence $u_{r,t}(0) = 0$ by (1.8). Then for any $\lambda > 0$ we can choose $r_0 \in [0,t_0]$ so that $u_{r,t}(0) \leq l_{0,t}(\lambda) \leq u_{r,t}(\lambda)$ when $r \in [r_0, t_0]$. The definition of $t_0$ yields the existence of some $t_1 \in [r_0, t_0]$ so that $0 < u_{t_1,t}(0) \leq u_{t_1,t}(\lambda)$. For $r \in [0,t_1]$ we see from (1.8) that

\begin{align*}
\quad u_{r,t}(0) &= u_{t_1,t}(0) - \int_r^{t_1} u_{s,t}(0)b_1(ds) - \int_r^{t_1} u_{s,t}(0)^2c(ds) \\
+ 2U_{0,t}(\lambda) &\int_r^t \int_0^1 |v_{s,t}(\lambda) - w_{s,t}(\lambda)|z^2m(ds, dz) \\
+ \int_r^t \int_1^\infty |v_{s,t}(\lambda) - w_{s,t}(\lambda)|ze^{-\int_0^c(\lambda)z}m(ds, dz).
\end{align*}
We can define the transition semigroup
\[
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Then we have
\[ u_{r,t}(0) = v_{r,t}(u_{t,t}(0)) \leq v_{r,t}(v_{i,t}(\lambda)) = v_{r,t}(\lambda). \]
and
\[ u_{r,t}(0) \leq v_{r,t}(\lambda) \text{ for every } r \in [0,t], \]
for every \( r \in [0,t] \).

**Proof of Theorem 1.3.** Since the interval \( I_0 \) does not contain any bottleneck, by Theorem 1.1 for any \( t \in I_0 \) and \( \lambda > 0 \) there is a unique bounded positive solution \( [0,t] \ni r \mapsto v_{r,t}(\lambda) \) to (1.8), which satisfies \( \inf_{0 \leq r \leq t} v_{r,t}(\lambda) > 0 \). Moreover, a transition semigroup \((Q_{r,t})_{t \geq r \in I_0}\) on \([0,\infty)\) is defined by (1.1). For \( i \geq 0 \) let \( \beta_i \) be the function on \([0,\infty)\) defined by
\[
\beta_i(s) = \begin{cases} \ \ s s_i^{-1} b_1(s_i) & \text{for } 0 \leq s < s_i, \\ \ \ b_1(s) & \text{for } s \in I_i, \\ \ \ b_1(s_{i+1}) & \text{for } s \geq s_{i+1}. \end{cases}
\]
Then the parameters \((\beta_i,c,m)\), \( i \geq 0 \) are admissible. Now let \( t \in I_i = [s_i, s_{i+1}] \) for some \( i \geq 1 \), where \( s_i = \varphi(t) > 0 \). By Theorem 1.1, for any \( \lambda > 0 \) there is a unique bounded positive solution \( [0,t] \ni r \mapsto v_i(r,t,\lambda) \) to (1.8) with \( b_1 \) replaced by \( \beta_i \), which satisfies \( \inf_{0 \leq r \leq t} v_i(r,t,\lambda) > 0 \). Then \( [s_i, t] \ni r \mapsto v_{r,t}(\lambda) := v_i(r,t,\lambda) \) solves the original equation (1.8) and \( \inf_{s \leq r \leq t} v_{r,t}(\lambda) > 0 \). The uniqueness of the solution of (1.8) for \( r \in [s_i, t] \) is obvious since it is determined by the admissible parameters \((\beta_i,c,m)\). By Theorem 1.1, we can define the transition semigroup \((Q_{r,t})_{t \geq r \in I_i}\) on \([0,\infty]\) by (1.1). Now suppose that \( r \mapsto v_{r,t}(\lambda) \) extends to a bounded positive solution of (1.8) on \([0,t]\). Since \( \Delta b_1(s_i) = 1 \), we have \( \Delta v_{s_i,t}(\lambda) = v_{s_i,t}(\lambda) \Delta b_1(s_i) = v_{s_i,t}(\lambda) \), which implies \( v_{s_i,t}(\lambda) = 0 \) by (1.8).

**Proof of Theorem 1.6.** For \( i \geq 0 \) let \( \beta_i \) be the function on \([0,\infty)\) defined in the proof of Theorem 1.3. By Theorem 1.4, there is a pathwise unique solution \( \{X_0(t) : t \geq 0\} \) to (1.11) with \( b_1 \) replaced by \( \beta_0 \). Let \( \{X(t) : t \geq 0\} \) be a solution to the original equation (1.11). Then the processes \( \{X(t) : t \in I_0\} \) and \( \{X_0(t) : t \in I_0\} \) must be indistinguishable since both of them are defined by the admissible parameters \((\beta_0,c,m)\). Since \( \Delta b_1(s_i) = 1 \) and \( m(\{s_i\} \times (0,\infty)) = 0 \), on the event \( \{s_i < \tau_\infty\} \) we have \( \Delta X(s_i) = -X(s_i-1) \), implying \( X(s_i) = 0 \). By our convention, it follows that \( X(t) = 0 \) for all \( t \geq s_1 \) on \( \{s_1 < \tau_\infty\} \).

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