BREAKING A CHAIN OF INTERACTING BROWNIAN PARTICLES

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We investigate the behaviour of a finite chain of Brownian particles, interacting through a pairwise linear force, with one end of the chain fixed and the other end pulled away at slow speed, in the limit of slow speed and small Brownian noise.

We study the instant when the chain “breaks,” that is, the distance between two neighboring particles becomes larger than a certain threshold. There are three different regimes depending on the relation between the speed of pulling and the Brownian noise. We provide weak limit theorems for the break time and the break position for each regime.

1. Introduction and main results.

1.1. Introduction. Interacting Brownian particles are a popular model for various physical systems where a (possibly large) number of particles is subjected to inter-particle forces and ambient noise. An important example are ensembles of colloidal particles, i.e. mesoscopic particles suspended in a fluid. Approaches to model these particles through interacting Brownian motions go back at least to [14], and the investigation of increasingly complex situations continues until today, see e.g. [32]. Since the focus of this paper is on a detailed investigation of a specific mathematical model, we will not attempt to give an account of the vast physics literature on the topic, but rather refer to the introduction of [32] and to the review article [31].

The specific model that we are interested in consists of a chain of $d+1$ interacting Brownian particles, where nearest neighbor particles are coupled by a harmonic potential, the leftmost particle is fixed, and the rightmost particle is slowly pulled to the right. Furthermore, we assume that the chain breaks whenever the distance between two neighboring particles exceeds a certain threshold. Both the variance $\sigma^2$ of the noise and the speed of pulling $\varepsilon$ are assumed to be small, and we are interested in the position and time

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of breakage depending on the relative behaviour of \( \sigma \) and \( \varepsilon \) as both go to zero. We refer to equation (1) below for the precise definition. Models of this type have been studied in the physics literature in the context of rupture of polymer chains \([17, 18, 22, 30]\).

Our results are an essentially complete description of the behaviour of the chain in the following three regimes, which cover almost all possible behaviours of \( \sigma \) and \( \varepsilon \). We always assume that both \( \sigma \) and \( \varepsilon \) converge to zero. The case of fast pulling is characterised by the condition \( \sigma/\varepsilon \to 0 \). Then the chain behaves as if \( \sigma = 0 \), i.e. it breaks deterministically at the \( d \)-th bond, see Theorem 1. On the other end of the scale, the case of very slow pulling occurs when \( \sigma^2 |\ln \varepsilon| \to \infty \). For \( \varepsilon = 0 \), Lee [22] made (and numerically verified) the somewhat surprising prediction that the chain breaks at the first and the last bond with probability \( 1/(2(d-1)) \) each, and at any other bond with probability \( 1/(d-1) \). We prove that this prediction is indeed accurate, and that it extends to the whole regime of very slow pulling. Moreover, we find that the (properly rescaled) time of breakage follows an exponential distribution, see Theorem 3. The third regime, which we call moderately slow pulling, occurs when \( \varepsilon/\sigma \to 0 \) but \( \sigma^2 |\ln \varepsilon| \to 0 \). Here we find that in this regime, the breaking probabilities of the bonds are just like in the regime of very slow pulling, but the breakage time is distributed differently: its difference from the deterministic time \( t_\ast = d/\varepsilon \) follows a Gumbel distribution after proper rescaling. This shows that the mechanism that is responsible for the break event is slightly different in the cases of very slow and moderately slow pulling. While large deviations events are present in both regimes, in the case of very slow pulling one essentially deals with an exit event for a stationary process over a high constant level; while in the case of moderately slow pulling one encounters an exit over a level linearly varying in time. The appearance of the Gumbel distribution in exit problems is not new, see \([4, 5, 13]\). However, in all those cases it appears in the context of conditioning on an unlikely exit, while in our case, it appears in the context of the most likely exit. To our knowledge, this phenomenon has not been observed before.

Mathematically, our problem falls squarely into the class of exit problems for stochastic processes. This is a topic with a long tradition that started with the early non-rigorous works of Kramers [21] and Eyring [15], was significantly influenced by the works of Freidlin and Wentzell [16], and has been further refined by Berglund and Gentz using sample path techniques [6], and by Bovier and his collaborators using tools from potential theory, see Chapter 11 of [11]. We refer to the latter two monographs for further information on the available literature.

Let us discuss the technical differences between our approach and those
cited above: the method of potential theory relies on PDE connections and in particular the generator of the process. The obstacle when trying to apply these methods in our case is that one would have to deal with a PDE on a domain that changes in time. The theory of partial differential equations seems not to offer many useful quantitative results in these cases. While in the regime of very slow pulling the methods of [10] would presumably lead to at least partial results, the case of moderately slow pulling seems not tractable by these methods. So instead, we rely on detailed large deviation results for Gaussian processes, subdivision into suitable time intervals, and decoupling inequalities. This leads to sharp estimates on exit time distributions and is somewhat related to the ideas present in the approach pioneered by Berglund and Gentz [6], see also [8, 9] for applications in the context of fast-slow systems and mixed mode oscillations, and [7] for the passage through an unstable orbit. Again, our model differs from those mentioned above by the presence of the explicit time dependence, which prevents us from using large parts of the established theory; on the other hand, since we aim for less generality, our results are sharper than most. A more thorough discussion of our methods and their scope will be given at the end of Section 1.3.

The model (1) has been treated before by Allman and Betz [1], and Allmann, Betz and Hairer [2]. In the present paper, we significantly improve the results of [1] in several ways. First, we remove a logarithmic gap between the regimes of fast and moderately slow pulling that was present in [1]. Second, we treat the regime of very slow pulling, which was previously left out. Third, and most importantly, we are able to treat chains of arbitrary length, while the results in [1] were restricted to \( d = 2 \), i.e. only one mobile particle. The detailed weak limits theorems for the exit times are also new.

So apart from the interesting transitional regimes \( \sigma \approx \varepsilon \) and \( \sigma^2 \ln \varepsilon \approx 1 \) (here and elsewhere \( \approx \) means that the quotient of the expressions is bounded away from zero and infinity, while \( \sim \) means that the quotient of the two expressions tends to one) where we do not know what happens, the model (1) is now fairly well understood. However, the simplifying assumption that the potential is quadratic (or, more generally, convex), and the chain breaks at some pre-determined bond length, is not harmless: if switching to a more realistic potential (such as Lennard–Jones), the breaking behaviour of the chain in dependence \( \varepsilon \) and \( \sigma \) changes significantly, see [2] for the case \( d = 3 \). Nothing is known in these cases for chains of arbitrary length.

This paper is organized as follows. In Section 1.2, we introduce the model and the objects of our study. In Section 1.3, we state the main results, that is, we distinguish the three mentioned regimes and provide the limit theorems.
for the break time and break position in each regime.

Section 2 is devoted to some necessary technical tools; in particular, we provide some useful representations of the solution to our model. The proofs of the main results in the two basic regimes, fast pulling and very slow pulling are given in Sections 3 and 4, respectively. The proofs for the intermediate regime are skipped because they go along the same lines as those for very slow pulling. We refer the interested reader to [3] for the handling of this case.

1.2. The model. Let $d \geq 1$ be an integer. We consider a chain of $d + 1$ particles located on the real line. At time $t \geq 0$, the locations of the particles are denoted by $X^0_t, \ldots, X^d_t$, and we assume that they satisfy the following system of stochastic differential equations:

\[
\begin{aligned}
    &X^0_t = i \quad \text{for } i = 0, 1, \ldots, d; \\
    &X^0_t = 0 \quad \text{for } t \geq 0; \\
    &X^d_t = d + \varepsilon t \quad \text{for } t \geq 0; \\
    &dX^i_t = (X^{i+1}_t - X^{i-1}_t - 2X^i_t)dt + \sigma dB^i_t, \quad i = 1, \ldots, d - 1, t \geq 0,
\end{aligned}
\]

where $(B^i_t)_{t \geq 0}$ are independent Brownian motions, $i = 1, \ldots, d - 1$, $\sigma \geq 0$, and $\varepsilon \geq 0$.

This means that one border particle (number 0) is fixed at zero, another border particle (number $d$) is moving at a constant speed $\varepsilon$, and the neighboring particles interact according to the last (main) equation.

We study the behaviour of the system when $\sigma \to 0$ and $\varepsilon \to 0$. Our interest is focused on the study of the time when the chain of interacting particles “breaks” and on understanding at which place of the chain the break occurs.

Let us introduce the corresponding terminology and notation. We say that the chain breaks at time $t$ if the maximal distance between the neighboring particles attains 2 for the first time at $t$. Formally, for $i \in \{1, \ldots, d\}$ let

$$\tau^i := \inf\{t \geq 0 \mid X^i_t - X^{i-1}_t = 2\}$$

and define the break time as

$$\tau := \min_{1 \leq i \leq d} \tau^i = \inf\{t \geq 0 : \exists i \in \{1, \ldots, d\} : X^i_t - X^{i-1}_t = 2\}.$$

From the above formula for the $\tau^i$ one can see that with the term ‘neighboring particles’ we really mean particles with consecutive indices. This notion may differ from the geometric neighborhood in the case where the
ordering of the particles is not preserved by the dynamics prior to the first 
rupture event, i.e. where two particles switch places before the chain breaks. 
However, with probability tending to one, such ‘collisions’ do not occur be-
fore the break events, at least in the two most interesting regimes (called 
below fast pulling and intermediate slow pulling), cf. Section 3.3. Based on 
computer simulations and supported by heuristical reasonings, we conjec-
ture that also in the remaining regime (called very slow pulling), collisions 
do not play any role in the sense that the limiting distributions are the same 
if one works with a model where the vector \((X_1^t, \ldots, X_d^t)\) is replaced by its 
order statistics in the interaction (modelling really neighboring particles). 
A theoretical analysis of the order statistics model in the spirit of [19, 26] 
certainly merits further investigation in the future but it is by far beyond 
the scope and the size of this article.

From the geometric point of view based on the observation of the process 
\(X_t := (X_t^0, \ldots, X_t^d)^\top\), the break time simply means the exit time of \(X\) 
from a certain deterministic polytope. Accordingly, we call \(\tau^i\), \(\tau\), and other similar 
variables \textit{exit times}.

We are also interested in the asymptotics of the distribution of the position 
where break occurs, i.e. \((\mathbb{P}(\tau = \tau^i))_{i=1,\ldots,d}\).

A simple, but important observation is that for \(\varepsilon > 0\) there is the trivial 
deterministic bound due to the pulling:

\[\tau \leq t^* := d/\varepsilon.\]

1.3. \textit{Main results}. We have to distinguish three different regimes: fast 
pulling \(\sigma/\varepsilon \to 0\) (or \(\sigma = 0\)); intermediate slow pulling \(\varepsilon/\sigma \to 0\) and 
\(\sigma^2 |\ln \varepsilon| \to 0\); and very slow pulling \(\sigma^2 |\ln \varepsilon| \to \infty\) (or \(\varepsilon = 0\)).

\textit{Fast pulling regime and deterministic setting} (\(\sigma = 0\)). Let us first consider 
the fast pulling regime, where we assume that \(\sigma/\varepsilon \to 0\) (or \(\sigma = 0\)). In this 
case, the limiting exit distribution is concentrated on the last exit position 
and the exit time is deterministic up to lower order terms.

\textbf{Theorem 1.} \textit{Let} \(\varepsilon \to 0, \sigma/\varepsilon \to 0\) \textit{or let} \(\sigma = 0\). \textit{Then the asymptotic} 
\textit{probabilities of the exit positions are described by the formula}

\[\mathbb{P}(\tau = \tau^i) \to \begin{cases} 
1 & i = d, \\
0 & 1 \leq i \leq d - 1.
\end{cases}\]

Further,

\[\tau = t^* - \frac{(d-1)(2d-1)}{6} + o_P(1),\]
where \( o_P(1) \) is a sequence of random variables tending to zero in probability.

The proof of this theorem is given in Section 3.

**Intermediate slow pulling regime.** This regime is characterized by the conditions

\[
\sigma/\varepsilon \to \infty \quad \text{and} \quad \sigma^2|\ln \varepsilon| \to 0.
\]

Define the parameters

\[
\begin{align*}
v^2 &:= \frac{d-1}{2d}, \quad \gamma := \sqrt{2dv} = \sqrt{d(d-1)}, \\
A_1 &:= A_d := \frac{d}{d-1}, \quad A_i := \frac{2d}{d-1}, \quad i \in \{2, \ldots, d-1\}.
\end{align*}
\]

Recall that a random variable \( \chi \) is double exponential (or Gumbel) with parameters \( a, b > 0 \), if

\[
P(\chi \leq r) = \exp(-a \exp(-br)), \quad r \in \mathbb{R}.
\]

The main result in the intermediate regime is as follows.

**Theorem 2.** Assume that (2) holds. Then, as \( \varepsilon, \sigma \to 0 \),

\[
P(\tau = \tau^i) \to \begin{cases} 
\frac{1}{d-1} & i \in \{2, \ldots, d-1\}; \\
\frac{1}{2(d-1)} & i \in \{1, d\},
\end{cases}
\]

and we have the following weak limit theorems for the exit times:

\[
\sqrt{\ln(\sigma/\varepsilon)} \left( t_* - \tau^i - \gamma \frac{\sigma}{\varepsilon} \sqrt{\ln(\sigma/\varepsilon)} \right) \xrightarrow{d} \chi_i, \quad i \in \{1, \ldots, d\},
\]

and

\[
\sqrt{\ln(\sigma/\varepsilon)} \left( t_* - \tau - \gamma \frac{\sigma}{\varepsilon} \sqrt{\ln(\sigma/\varepsilon)} \right) \xrightarrow{d} \chi_0,
\]

where \( \chi_i \) is a double exponential random variable with parameters \( a_i := vdA_i/\sqrt{2\pi} \) for \( i \in \{1, \ldots, d\} \), \( a_0 := \sum_{i=1}^{d} a_i = 2vd^2/\sqrt{2\pi} \) and \( b := \sqrt{2/(vd)} \), while \( v, \gamma \), and the \( A_i \) are defined in (3).

The theorem says that the asymptotic distribution of the exit position is uniform on the points \( \{2, \ldots, d-1\} \), while the points \( \{1, d\} \) carry half the weight of the others. Heuristically, equation (4) has its origin in the fact that the end points are fixed, which induces less randomness in the differences between particles 0 and 1 (and \( d-1 \) and \( d \)) compared to the rest of the differences.
The asymptotic time scaling of the exit time is best expressed as the remaining time until $t_*$: Namely, asymptotically $t_* - \tau^i$ is given by a deterministic term $\gamma \sigma \sqrt{\ln(\sigma/\varepsilon)}$ and a stochastic term, $\sigma \epsilon \left( \sqrt{\ln(\sigma/\varepsilon)} \right) ^{-1} \chi_i$, with a double exponential random variable $\chi_i$ with parameters not depending on $\sigma$ and $\varepsilon$ anymore.

We remark that the condition (2) ensures that $t_* \gg \gamma \sigma \sqrt{\ln(\sigma/\varepsilon)}$.

**Very slow pulling regime.** Let us finally consider the case of very slow pulling characterized by the condition

$$\sigma^2 |\ln \varepsilon| \to \infty.$$  

The main result in this regime is as follows.

**Theorem 3.** Assume that (5) holds. Then, as $\varepsilon, \sigma \to 0$, the asymptotic distribution of exit positions is described by (4), while we have the following weak limit theorems for the exit times:

$$\tau^i \cdot (\sigma v)^{-1} \exp(-(\sigma v)^{-2}/2) \cdot \frac{A_i}{\sqrt{2\pi}} \overset{d}{\to} \mathcal{E}, \quad i \in \{1, \ldots, d\},$$

and

$$\tau \cdot (\sigma v)^{-1} \exp(-(\sigma v)^{-2}/2) \cdot \frac{2d}{\sqrt{2\pi}} \overset{d}{\to} \mathcal{E},$$

where $\mathcal{E}$ is a standard exponential random variable and $v$ and the $A_i$ are defined in (3).

We remark that the assertions of the last theorem also hold for $\varepsilon = 0$ and $\sigma \to 0$.

The asymptotic distribution of the exit position is the same as in the intermediate regime. However, the time scaling of the exits is quite different and in fact much smaller. Indeed, $\tau^i$ is of order $\exp((\sigma v)^{-2}/2)$ (up to lower order terms), which is of smaller order than $t_*$ due to (5).

The proof of this theorem is given in Section 4.

**Ideas of the proofs.** Let us give a quick sketch of the idea of the proofs. The case $\varepsilon = 0$ corresponds to the classical problem of a reversible diffusion process exiting from a fixed domain in the limit of small noise. In this case the general potential theoretic methods developed in [10] will certainly be applicable; also, the methods described in appendix C of [6] (which rely on deriving an integral equation for the density of the exit time distribution) could possibly be used. We do not, however, use either of these methods, but instead rely on an approach that is more direct and may seem less elegant on
first sight, but has the advantage of yielding very precise information on the exit times and at the same time being sufficiently robust to carry over to the cases of very slow and (with suitable modifications) moderately slow pulling. Furthermore, our method is not limited to Markov processes and works for all Gaussian processes satisfying suitable asymptotic variance and mixing conditions; see Lemma 16. As far as we know, the exit problem for general Gaussian processes has not been treated yet in the precision obtained here, so Lemma 16 may be of independent interest.

The basic idea we use is to identify a finite number of time intervals \((I_m), (J_m)\) of two different lengths, which appear in alternating order. Their lengths and number depend on \(\sigma\) (and later on \(\varepsilon\)) in a delicate way, and they are chosen such that the probability of the process exiting during \(\bigcup_m I_m\) converges to 1, but at the same time the intervals \(J_m\) are long enough to allow relaxation to the independent behaviour.

The picture that emerges is then that in each of the \(I_m\), the process makes an attempt to exit the set, that is, each of the correlated components tries to become large. Since the variance \(\sigma\) is small, the exit is a large deviation event; and therefore the process typically fails to exit. During the following \(J_m\), the process relaxes to the independent behaviour, and then gets another attempt during \(I_{m+1}\). We therefore deal with a sequence of (almost) independent trials with small success probability. The latter can be identified to very high precision thanks to sharp large deviation results available for the maxima of Gaussian processes.

In the situation of Theorem 3, the exit happens on a time scale way before the pulling significantly influences the dynamics. Therefore, as we will show, we are essentially in the situation \(\varepsilon = 0\) described above.

The situation of moderately slow pulling uses similar ideas, but with a different choice of the intervals \(I_m, J_m\) and their number. Also, the exit now happens much closer to the deterministic time \(t^*\), and the level that the process has to overcome changes in each of the \(I_m\). With these modifications, the proof works along the same lines as it does in the case of Theorem 3 and is therefore omitted in order to keep the paper at a reasonable length. Full details can however be found in [3].

Let us further remark that, in the intermediate and strong slow pulling regimes, the initial condition \(X_0^i = i, i = 0, \ldots, d\), has no influence on the results. One may very well start with different initial conditions, as long as they do not depend on \(\sigma\) and \(\varepsilon\) too much. In fact, one step in the proofs is to show that the resulting distributions of exit positions and exit times are the same as if the process were stationary.
Remarks on possible generalizations. First, let us say a few words on a regime intermediate between fast pulling and intermediate slow pulling, namely, $\sigma/\varepsilon \to c \in (0, \infty)$. Our technique applies in this case, too, but the relevant expressions become much less explicit. We did not push forward the research in this direction, but this regime is interesting, and we would like to make the following conjecture about it:

**Conjecture 4.** Let $\varepsilon \to 0$, $\sigma/\varepsilon \to c \in (0, \infty)$. Then there exist the asymptotic probabilities of the exit positions $p_i = p_i(c)$ such that $\sum_{i=1}^d p_i = 1$ and

$$\mathbb{P}(\tau = \tau^i) \to p_i, \quad 1 \leq i \leq d.$$ 

Further, there exists a positive random variable $\chi(c)$ such that

$$t_\ast - \tau \xrightarrow{d} \chi(c).$$

We do not expect that there are simple explicit expressions neither for the $p_i(c)$ nor for the law of $\chi(c)$. When $c$ runs from 0 to $\infty$, $p_i(c)$ should run from the values in Theorem 1 to those in Theorem 2, while the law of $\chi(c)$ should evolve from a degenerate law to the double exponential law.

There are various further ways in which our results can be generalized. It is natural to look at more general interaction potentials rather than the quadratic potential that gives the driving equation in (1). Even if the potential is quadratic, one can extend the interaction beyond the nearest neighbors. In this case, the discrete Laplacian $A$ appearing below needs to be replaced by a different matrix, and our techniques are quite robust with respect to the properties of this matrix (we only use that its eigenvalues are negative).

Finally, most of the results and methods should work when the model (1) is driven by some other Gaussian process instead of Brownian motion.

2. Preliminaries. In this section we reformulate the chain rupture problem as an exit problem of a stationary Gaussian process from a time-dependent domain, which we will then solve in the subsequent sections.

Let $\nu := (\nu^1, \ldots, \nu^{d-1})^\top$ with $\nu^i := i/d$ for all $i$, and write $A$ for the discrete Laplacian in one dimension with $d-1$ supporting points, i.e. $A$ is a $(d-1)$-dimensional square matrix such that $A_{i,j} = -2$ when $i = j$, $A_{i,j} = 1$ when $|i-j| = 1$, and $A_{i,j} = 0$ otherwise. Let $t \mapsto q_t = (q^1_t, \ldots, q_t^{d-1})^\top$ denote the unique solution of the ODE system

$$(6) \quad q' + \nu = Aq, \quad q(0) = 0,$$
and let \((Y_t)_{t \geq 0}\) with \(Y_t = (Y_t^1, \ldots, Y_t^{d-1})^\top\) be the \((d-1)\)-dimensional Ornstein–Uhlenbeck process given by the unique solution of the SDE system

\[
dY_t = AY_t dt + \sigma dB_t, \quad Y_0 = 0.
\]

Here \(B = (B^1, \ldots, B^{d-1})^\top\) is a \(d-1\)-dimensional Brownian motion. Define \(Y_0 \equiv Y^d \equiv 0 = q^0 \equiv q^d \equiv 0\).

The next lemma shows that one can decompose the solution \(X\) into a linear function (coming from the drift of the \(d\)-th particle); a deterministic perturbation; and an almost stationary process, which we shall analyze in the rest of the paper.

**Lemma 5.**

(i) The unique solution of equation (1) can be written in the form

\[
X^i_t := \frac{i}{d}(\varepsilon t + d) + \varepsilon q^i_t + Y^i_t, \quad i \in \{0, \ldots, d\}.
\]

(ii) The function \(q_t := (q^0, \ldots, q^d)\) admits the representation \(q^i_t = h_i + z^i_t\) with

\[
h_i := \frac{i(i^2 - d^2)}{6d}, \quad i \in \{0, \ldots, d\},
\]

and

\[
C^i_t := \sup_{t \geq 0} |z^i_t| < \infty, \quad \lim_{t \to \infty} z^i_t = 0, \quad i \in \{1, \ldots, d-1\}.
\]

**Proof:** The first claim is a direct computation. For the second claim, note that \(\nu^i = h_{i+1} + h_{i-1} - 2h_i\) for all \(i \in \{1, \ldots, d-1\}\), which rewrites \(Ah = \nu\). It follows that \(z_t := q_t - h\) solves the homogenous system \(z' = Az, z(0) = -h\), and since the matrix \(A\) is negative definite (see e.g. [25, formula (4)]), the function \(z\) decays exponentially for large \(t\). \(\square\)

Due to the form \(\tau^i = \inf\{t \geq 0 \mid X^i_t - X^{i-1}_t = 2\}\) of the chain’s break times, it is natural to consider differences of neighboring particles. Defining

\[
\sigma V^i_t := Y^i_t - Y^{i-1}_t, \quad \delta^i_t := h_i - h_{i-1} = \frac{1 - d^2}{6d} + \frac{i^2 - i}{2d},
\]

for \(i \in \{1, \ldots, d\}\), we have

\[
X^i_t - X^{i-1}_t = \frac{1}{d}(\varepsilon t + d) + \varepsilon \delta^i_t + \varepsilon (z^i_t - z^{i-1}_t) + \sigma V^i_t.
\]
A convenient way to represent the process $V_t := (V_t^1, \ldots, V_t^{d-1})^T$ is

$$V_t = \sigma^{-1} G Y_t,$$

where $G$ is the $(d-1) \times d$ with $G_{i,i} = 1$, $G_{i+1,i} = -1$ for all $i \in \{1, \ldots, d-1\}$, and $G_{i,j} = 0$ otherwise. We want to compare $V_t$ to its stationary version, and for this purpose we recall that the Ornstein–Uhlenbeck process $Y$ from (7) has the explicit representation

$$Y_t = \sigma e^{A t} \int_0^t e^{-A u} dB_u,$$

and that the corresponding stationary Ornstein–Uhlenbeck process is given by

$$\tilde{Y}_t := \sigma e^{A t} \left( (-2A)^{-1/2} \xi + \int_0^t e^{-A u} dB_u \right),$$

where $\xi$ is a vector of i.i.d. standard normal variables, independent of $B$.

We define

$$(11) \quad \tilde{V}_t := (\tilde{V}_t^1, \ldots, \tilde{V}_t^{d-1}):= \sigma^{-1} G \tilde{Y}_t$$

Let us write $v \otimes w := (v_i w_j)_{i,j=1,\ldots,d-1}$. In the next lemma, the covariance structure of the process $\tilde{V}$ is studied.

**Lemma 6.** The stationary Gaussian process $\tilde{V}$ has mean vector 0 and covariance function

$$(12) \quad \mathbb{E}[\tilde{V}_s \otimes \tilde{V}_t] = \frac{1}{2} G(-A)^{-1} e^{\left|t-s\right|A} G^\top.$$

As $t \to 0$, for all $i \in \{1, \ldots, d\}$ we have

$$(13) \quad \mathbb{E}[\tilde{V}_0^i \tilde{V}_t^i] = v^2 (1 - A_i |t| + o(t)),$$

where $v^2$ and the $A_i$ are defined in (3). Further,

$$(14) \quad \mathbb{E}[\tilde{V}_0^i \tilde{V}_0^j] = -\frac{1}{2d}, \quad i \neq j.$$

**Proof:** By stationarity, $\mathbb{E}[\tilde{V}_s^i \tilde{V}_t^i] = \mathbb{E}[\tilde{V}_0^i \tilde{V}_{t-s}^i]$. The representation

$$\mathbb{E}[\tilde{Y}_0 \otimes \tilde{Y}_t] = \frac{\sigma^2}{2} (-A)^{-1} e^{tA},$$

where $\sigma$ is defined in (6).
for the covariance function of the stationary Ornstein–Uhlenbek process is well known and can be verified readily. Therefore (12) follows by the transformation rules for Gaussian random variables under linear maps. A Taylor expansion now gives

\[
\mathbb{E}[\tilde{V}_0 \otimes \tilde{V}_t] = \frac{1}{2} G(-A)^{-1} G^\top - GG^\top t + o(t),
\]
as \(t \to 0\). By a direct computation, we obtain \((GG^\top)_{1,1} = (GG^\top)_{d,d} = 1, (GG^\top)_{i,i} = 2\) for all \(1 < i < d\), \(G_{i,j} = 1\) if \(|i - j| = 1\), and \(G_{i,j} = 0\) in all other cases. This identifies the linear term in the expansion (15).

Furthermore, we claim that

\[
G(-A)^{-1} G^\top = I - \frac{1}{d} P_\eta,
\]
where \(P_\eta\) is the projection onto the vector \(\eta = (1,1,\ldots,1)^\top \in \mathbb{R}^d\), i.e. \(P_\eta w = \langle \eta, w \rangle \eta\). Indeed, one readily computes that \(G^\top G = -A\), and thus

\[
G^\top (G(-A)^{-1} G^\top - I) = (G^\top G)(G^\top G)^{-1} G^\top - G^\top = 0,
\]
which means that the range of \(G(-A)^{-1} G^\top - I\) is contained in the null space of \(G^\top\). The latter is spanned by the vector \(\eta\), and it thus follows that

\[
G(-A)^{-1} G^\top = I + r P_\eta
\]
for some \(r \in \mathbb{R}\). The constant \(r\) is determined by applying the last equation to \(\eta\) and using that \(G^\top \eta = 0\) so that \(0 = \eta + r (\eta, \eta) \eta = \eta + rd \eta\), and so (16) follows.

Now, (16) says that the first term in the expansion (15), \(G(-A)^{-1} G^\top\), has diagonal elements \(1 - 1/d = (d-1)/d\) and off-diagonal elements \(-1/d\).

Next we investigate the difference between \(V\) and its stationary version \(\tilde{V}\). As above, let \(\xi\) be a vector of \(d-1\) i.i.d. standard normal random variables, and let \(\mu := 2(1 - \cos(\pi/d))\) be the minimum of the absolute values of the eigenvalues of \(A\). Define the nonnegative random variable

\[
\Xi := 2 \sqrt{\frac{d-1}{2\mu}} \|\xi\|_\infty.
\]

**Lemma 7.** We have

\[
\|V_t - \tilde{V}_t\|_\infty \leq e^{-\mu t} \Xi.
\]
Proof: Since \( \|Gx\|_\infty \leq 2\|x\|_\infty \) for all \( x \in \mathbb{R}^{d-1} \), we have \( \|V_t - \tilde{V}_t\|_\infty \leq 2\sigma^{-1}\|Y_t - \tilde{Y}_t\|_\infty \). Clearly, \( \tilde{Y}_t - Y_t = \sigma e^{At}(-2A)^{-1/2}\xi \). The operator norm of the matrix \( e^{At}(-2A)^{-1/2} \) is given by its largest eigenvalue, which by the spectral theorem is equal to \( (2\mu)^{-1/2}e^{-\mu t} \). Therefore \( \|\tilde{V}_t - V_t\|_\infty \leq 2\sigma^{-1}\|Y_t - \tilde{Y}_t\|_\infty \). Clearly, \( \tilde{Y}_t - Y_t = \sigma e^{At}(\delta_i - z_i t - z_{i-1} t) \). The operator norm of the matrix \( e^{At}(\delta_i - z_i t - z_{i-1} t) \) is given by its largest eigenvalue, which by the spectral theorem is equal to \( (2\mu)^{-1/2}e^{-\mu t} \). Therefore \( \|\tilde{Y}_t - Y_t\|_2 \leq \sqrt{d-1}\|x\|_\infty \) for vectors \( x \) in \( \mathbb{R}^{d-1} \). □

We are now in the position to replace our original exit problem by a pair of exit problems of the stationary Gaussian process \( \tilde{V} \) from two slightly different time-dependent domains. Namely, equation (10) implies that

\[
\tau^i = \inf \left\{ t \geq 0 : \sigma \tilde{V}^i_t \geq 1 - \frac{\varepsilon t}{d} - \varepsilon(\delta_i + z_i t - z_{i-1} t) \right\}.
\]

We set \( K^i := |\delta_i| + C^i_z + C^i_{z-1} \) (see Lemma 5 for the definition of the \( C^i_z \)), and define

\[
\tau^i_\pm := \inf \left\{ t \geq 0 : \sigma \tilde{V}^i_t \geq 1 - \frac{\varepsilon t}{d} \pm \varepsilon K^i \pm \sigma \Xi e^{-\mu t} \right\}.
\]

By Lemma 5 and Lemma 7 we have \( \tau^-_i \leq \tau^i \leq \tau^+_i \) for all \( i \), and thus the condition

\[
\sigma \tilde{V}^i_t \geq 1 - \frac{\varepsilon t}{d} - \varepsilon K^i - \sigma \Xi e^{-\mu t}
\]

for some \( t < t_0 \) is necessary for a break of the chain at position \( i \) before time \( t_0 \). On the other hand, the condition

\[
\sigma \tilde{V}^i_t \geq 1 - \frac{\varepsilon t}{d} + \varepsilon K^i + \sigma \Xi e^{-\mu t},
\]

for some \( t \leq t_0 \) is sufficient for the exit in the sense that (18) occurs before \( t_0 \) – of course, in this case one has to check separately that the chain has not broken at one of the other positions before.

There are two more ingredients for our proofs; the first is a mixing inequality that we will use to decouple the exit events of the process between subsequent time intervals. In view of Lemma 16 we state things in somewhat greater generality than would be necessary for the treatment of (1).

Let \( (W_t)_{t \in \mathbb{R}} \) be a stationary centered \( d \)-dimensional process with finite second moments. Consider the associated linear spaces

\[
\mathcal{W}_t := \text{span}(W^j_s, s \leq t, 1 \leq j \leq d)
\]
and
\[ W_t^+ := \text{span}(W_s^j, s \geq t, 1 \leq j \leq d). \]

The quantity
\[ r_W(\theta) := \sup \{ \text{corr}(Z_0, Z) : Z_0 \in W_0^-, Z \in W_\theta^+ \} \]

is called \textit{linear mixing coefficient} of the process \( W = (W_t) \). Here and elsewhere, we let \( \text{corr}(X,Y) \) denote the correlation coefficient of the random variables \( X,Y \). On the other hand, the quantity
\[ \text{mix}_W(\theta) := \sup_{A \in \sigma(W_t, t \leq 0), B \in \sigma(W_t, t \geq \theta)} |P(A \cap B) - P(A) \cdot P(B)| \]

is called \textit{strong mixing coefficient} (or \( \alpha \)-mixing coefficient). We refer to [12] for a survey on notions of mixing and their relations.

As an important tool in the proofs, we shall use the following classical decoupling result from the mixing theory.

**Lemma 8.** Let \((W_t)_{t \in \mathbb{R}}\) be a stationary centered \( d \)-dimensional Gaussian process. Then for every \( \theta > 0 \) we have \( \text{mix}_W(\theta) \leq r_W(\theta) \).

The lemma tells us that the dependence between two events that use information on \((W_t)\) for instants that are separated in time by at least \( \theta \) can be evaluated via covariance characteristics.

**Proof:** It is true that \( \text{mix}_W(\theta) \leq \rho_W(\theta) = r_W(\theta) \). Here, \( \rho_W \) is the \( \rho \)-mixing coefficient (see [12]). Indeed, the first inequality is always true (see (1.12) in [12]). The equality between \( \rho_W \) and \( r_W \) is true because we deal with Gaussian processes, and it follows from [20, Theorem 1] (also see (7.1) in [12]). \( \square \)

In order to apply Lemma 8, one needs estimates for \( r_{\tilde{V}}(\theta) \) for the process \( \tilde{V} \), which we provide now.

**Lemma 9.** The process \( \tilde{V} \) defined in (11) satisfies the relation
\[ r_{\tilde{V}}(\theta) \leq e^{-\mu \theta} \]
for all \( \theta > 0 \) and \( \mu = 2(1 - \cos(\pi/d)) \).

Basically, the claim follows by the fact that the generator of the process \( \tilde{V} \) has a spectral gap of size \( \mu \). We skip the details of the proof. A detailed derivation can be found in [3].
Our final ingredient is the following result that identifies the exact asymptotics of the large deviation probability of a stationary Gaussian process. It is usually called the Pickands lemma [27], [24], also see Theorem 2.1 in [29] (with a slightly differently defined constant $H_\alpha$) and Theorem 9.15 in [28]. The assertion with the varying time interval length that we use here is due to V. Piterbarg.

**Lemma 10.** Let $(W_t)_{t \in \mathbb{R}}$ be a stationary centered Gaussian process with covariance expansion

$$\text{cov}(W_t, W_0) = v^2 (1 - A|t|^{\alpha} + o(|t|^{\alpha})),$$

as $t \to 0$, for some $v > 0$, $A > 0$, and $0 < \alpha \leq 2$. Assume that

$$\limsup_{t \to \infty} \text{corr}(W_t, W_0) < 1.$$

Then

$$\mathbb{P}(\max_{s \in [0,t]} W_s > x) \sim \frac{A^{1/\alpha}H_\alpha}{\sqrt{2\pi}} \cdot t \cdot (x/v)^{2/\alpha - 1} \exp\left(-\frac{(x/v)^2}{2}\right),$$

for any $x$ and $t$ such that the right-hand side tends to zero, where $H_\alpha$ is Pickands constant (in particular, $H_1 = 1$).

3. Deterministic regime ($\sigma = 0$) and very fast pulling regime.

In this section, we show that in the deterministic regime (with $\varepsilon \to 0, \sigma = 0$) the exit occurs at the last position, i.e. $\tau = \tau^d$. Then we extend this result to stochastic systems satisfying $\varepsilon \to 0, \sigma/\varepsilon \to 0$. It is therefore natural to call the latter regime quasi-deterministic.

3.1. Deterministic regime.

**Proposition 11.** Let $\varepsilon \to 0, \sigma = 0$. Then the exit times are described by the formula

$$\tau^i = t_* + \frac{d^2 - 1}{6} - \frac{i(i-1)}{2} + o(1), \quad \text{as } \varepsilon \to 0, \quad 1 \leq i \leq d.$$

In particular,

$$\tau := \min_{1 \leq i \leq d} \tau^i = \tau^d = t_* - \frac{(d-1)(2d-1)}{6} + o(1), \quad \text{as } \varepsilon \to 0.$$
Proof of Proposition 11: Observe that in the deterministic regime the stochastic term \( \sigma V^i_t \) vanishes from (18) and we have
\[
\tau^i = \inf \left\{ t \geq 0 \mid \frac{t^*_s - t}{t^*_s} = \varepsilon (\delta^i + z^i_t - z^{i-1}_t) \right\}
\]
where \( t^*_s = \frac{d}{\varepsilon} \) and \( \delta^i := \frac{1 - d^2}{6d} + \frac{i(i-1)}{2d} \). Using (9) we obtain
\[
\begin{align*}
\tau^i & = t^*_s - d(\delta^i + z^i_{\tau^i} - z^{i-1}_{\tau^i}) \\
& = t^*_s - d(\delta^i + o(1)) \\
& = t^*_s + \frac{d^2 - 1}{6} - \frac{i(i-1)}{2} + o(1),
\end{align*}
\]
which is the claim of the proposition. \( \square \)

3.2. Quasi-deterministic regime. We now extend the results for the deterministic regime to the very fast pulling (quasi-deterministic) regime.

Proof of Theorem 1: For identifying the end of the chain as the exit position, it is sufficient to prove that for each \( i \) such that \( 1 \leq i < d \) one has
\[
P(\tau^i = \tau^i \leq \tau^d) \to 0.
\]
Let us fix an \( i < d \) and some \( M > 0 \). The following inequality is the starting point for proving (22),
\[
(23) \quad P(\tau = \tau^i \leq \tau^d) \leq P(\tau^i \leq t^*_s - M) + P(t^*_s - M \leq \tau^i \leq t^*_s, \tau^i \leq \tau^d).
\]

Step 1: Let us proceed with the evaluation of the second probability in (23). Assuming that \( \tau^i \leq \tau^d \), letting \( t := \tau^i \), and using expression (10) we transform the inequality
\[
X^i_t - X^{i-1}_t = 2 \geq X^d_t - X^{d-1}_t
\]
into
\[
\sigma(V^i_t - V^d_t) \geq \varepsilon (\delta^d - \delta^i - z^i_t + z^{i-1}_t + z^d_t - z^{d-1}_t),
\]
at \( t = \tau^i \). Recall that \( t \geq t^*_s - M \to \infty \); by (9) we have
\[
\begin{align*}
z^i_t - z^{i-1}_t - z^d_t + z^{d-1}_t &= o(1).
\end{align*}
\]
We have thus seen that on \( \tau^i \leq \tau^d \)
\[
\sup_{s \in [t^*_s - M, t^*_s]} (V^i_s - V^d_s) \geq (V^i_t - V^d_t) \geq \frac{\varepsilon}{\sigma} (\delta^d - \delta^i + o(1)),
\]

and recall that $\delta_d > \delta_i$ for $i < d$.

Therefore, we obtain

$$
P(t_* - M \leq \tau^i \leq t_*, \tau^i \leq \tau^d)
\leq P \left( \sup_{s \in [t_* - M, t_*)} (V^i_s - V^d_s) \geq \frac{\varepsilon}{\sigma} (\delta_d - \delta_i + o(1)) \right)
\leq P \left( \sup_{s \in [t_* - M, t_*)} |V^i_s - V^d_s| \geq \frac{\varepsilon}{\sigma} (\delta_d - \delta_i + o(1)) \right)
\leq P \left( \sup_{s \in [t_* - M, t_*)} |\tilde{V}^i_s - \tilde{V}^d_s| \geq \frac{\varepsilon}{\sigma} (\delta_d - \delta_i + o(1)) \right)
= P \left( \sup_{s \in [0, M]} |\tilde{V}^i_s - \tilde{V}^d_s| \geq \frac{\varepsilon}{\sigma} (\delta_d - \delta_i + o(1)) \right) \to 0
$$

for every fixed $M$. Here we used a stationary version $\tilde{V}^i$ of $V^i$, applied the Anderson inequality (3rd step), then stationarity (4th step), and finally used that $\frac{\varepsilon}{\sigma} \to \infty$.

**Step 2:** Let us now evaluate the first probability in (23), assuming additionally that $M$ is chosen so large that $M \geq K^i := |\delta_i| + C^i_z + C^{i-1}_z$, where the constant $C^i_z$ is defined in (9).

Using the representation (10) for $t = \tau^i$ we see that the exit condition $X^i_t - X^{i-1}_t = 2$ is equivalent to

$$
\sigma V^i_t = \frac{t_* - t}{t_*} - \varepsilon (\delta_i + z^i_t - z^{i-1}_t) = \varepsilon \left( \frac{t_* - t}{d} - (\delta_i + z^i_t - z^{i-1}_t) \right),
$$

hence,

$$
V^i_t = \frac{\varepsilon}{\sigma} \left( \frac{t_* - t}{d} - (\delta_i + z^i_t - z^{i-1}_t) \right) \geq \frac{\varepsilon}{\sigma} \left( \frac{t_* - t}{d} - K^i \right).
$$

Notice that the right hand side is positive for all $t \leq t_* - M$ due to the choice of $M$. Therefore,

$$
P(\tau^i \leq t_* - M) \leq P \left( \exists t \in [0, t_* - M] : V^i_t \geq \frac{\varepsilon}{\sigma} \left( \frac{t_* - t}{d} - K^i \right) \right)
\leq P \left( \exists t \in [0, t_* - M] : |V^i_t| \geq \frac{\varepsilon}{\sigma} \left( \frac{t_* - t}{d} - K^i \right) \right)
\leq P \left( \exists t \in [0, t_* - M] : |\tilde{V}^i_t| \geq \frac{\varepsilon}{\sigma} \left( \frac{t_* - t}{d} - K^i \right) \right).
$$
Here we used again the stationary version $\tilde{V}^i$ of $V^i$ and applied the Anderson inequality. Furthermore, by stationarity, and using $M^2 \geq K^i$, we have

$$P\left( \exists t \in [0, t^*_s - M] : |\tilde{V}^i_t| \geq \frac{\varepsilon}{\sigma} \left( \frac{t^*_s - t}{d} - K^i \right) \right)$$

$$\leq P\left( \exists s \in [M, \infty) : |\tilde{V}^i_s| \geq \frac{\varepsilon}{\sigma} \left( \frac{s}{d} - K^i \right) \right)$$

$$\leq P\left( \exists s \in [M, \infty) : |\tilde{V}^i_s| \geq \frac{\varepsilon}{\sigma} \frac{s}{2d} \right) \to 0,$$

since $\frac{\varepsilon}{\sigma} \to \infty$. We conclude that

$$P(\tau^i \leq t^*_s - M) \to 0, \quad \text{as } \frac{\varepsilon}{\sigma} \to \infty \text{ and } M^2 \geq K^i.$$

Now (23) yields the result for the exit position.

**Step 3:** We finally prove the statement about the exit time. Recall that for large enough $M$ we already proved $P(\tau^d < t^*_s - M) \to 0$.

Fix a small $\kappa > 0$. Recall that by (9) the term $z_t^d - z_t^{d-1}$ tends to zero, as $t \to \infty$, so that for $t > t^*_s - M$ we have, say,

$$|z_t^d - z_t^{d-1}| \leq \frac{\kappa}{2d}.$$

Therefore, we obtain from (10)

$$P(\tau^d \leq t^*_s - d\delta_d - \kappa)$$

$$\leq P(\exists t \in [t^*_s - M, t^*_s - d\delta_d - \kappa] : V_t^d \geq \frac{\varepsilon}{\sigma} \left( \frac{t^*_s - t}{d} - \delta_d - (z_t^d - z_t^{d-1}) \right))$$

$$\leq P(\exists t \in [t^*_s - M, t^*_s - d\delta_d - \kappa] : V_t^d \geq \frac{\varepsilon}{\sigma} \left( \frac{\kappa}{d} \right) \to 0,$$

as above. Since $\kappa$ can be chosen arbitrarily small, this estimate shows that $\tau^d \geq t^*_s - d\delta_d - o_P(1)$.

On the other hand, for any fixed small $\kappa > 0$ let $t = t(\varepsilon, \kappa) := t^*_s - d\delta_d + \kappa$.

Then, by using again (10) with $i = d$,

$$P(\tau^d \leq t) \geq P(V_t^d \geq \frac{\varepsilon}{\sigma} \left( \frac{t^*_s - t}{d} - \delta_d - (z_t^d - z_t^{d-1}) \right))$$

$$\geq P(V_t^d \geq \frac{\varepsilon}{\sigma} \left( \frac{t^*_s - t}{d} - \delta_d + \frac{\kappa}{2d} \right))$$

$$= P\left( V_t^d \geq -\frac{\varepsilon}{\sigma} \cdot \frac{\kappa}{2d} \right) \to 1,$$
because the law of $V_{t}^{d}$ converges to a non-degenerated Gaussian law, and the level tends to $-\infty$. Since $\kappa$ can be chosen arbitrarily small, this estimate shows that $\tau^{d} \leq t^{*} - d\delta_{d} + o_{P}(1)$. \hfill \Box

3.3. No collisions before break. In this section, we show that in the two regimes of fast pulling and intermediate slow pulling, with probability tending to one, no collision occurs before the chain break.

**Proposition 12.** Let $h \in (0, 1)$ and $T = T(h, \sigma) := \exp\left(\frac{h}{2(\sigma v)^{2}}\right)$. Then

$$
\mathbb{P}(\exists i \in \{1, \ldots, d\}, \exists t \in [0, T] : X_{t}^{i} = X_{t}^{i-1}) \to 0.
$$

**Proof.** Let us fix $i \in \{1, \ldots, d\}$. Assume that $X_{t}^{i} = X_{t}^{i-1}$ for some $t > 0$. Then using the representation in (10)

$$
0 = X_{t}^{i} - X_{t}^{i-1} = \frac{\varepsilon t + d}{d} - \varepsilon \delta_{i} + \varepsilon(z_{t}^{i} - z_{t}^{i-1}) + \sigma V_{t}^{i},
$$

hence, using (17)

$$
\sigma \tilde{V}_{t}^{i} \leq \sigma V_{t}^{i} + \sigma \Xi e^{-\mu t}
\leq \sigma \Xi e^{-\mu t} - \frac{\varepsilon t + d}{d} - \varepsilon \delta_{i} - \varepsilon(z_{t}^{i} - z_{t}^{i-1})
\leq \sigma \Xi - 1 + \varepsilon K_{i},
$$

where $K_{i}$ is defined after (18).

Let us fix $0 < \kappa < 1 - \sqrt{h}$ and let $\varepsilon$ be so small that $\varepsilon K_{i} < \kappa/2$. Then we have

$$
\mathbb{P}(\exists t \in [0, T] : X_{t}^{i} = X_{t}^{i-1}) \leq \mathbb{P}(\exists t \in [0, T] : \sigma \tilde{V}_{t}^{i} \leq -(1 - \kappa)) + \mathbb{P}(\sigma \Xi > \kappa/2)
= \mathbb{P}(\max_{t \in [0, T]} \tilde{V}_{t}^{i} \geq (1 - \kappa)\sigma^{-1}) + \mathbb{P}(\sigma \Xi > \kappa/2).
$$

The second probability obviously goes to zero, as $\sigma \to 0$. For the first probability, we apply Pickands lemma (Lemma 10) and obtain

$$
\mathbb{P}(\max_{t \in [0, T]} \tilde{V}_{t}^{i} \geq (1 - \kappa)\sigma^{-1}) \leq T \exp\left(-\frac{(1 - \kappa)^{2}}{2(\sigma v)^{2}} (1 + o(1))\right)
= \exp\left(\frac{h - (1 - \kappa)^{2}(1 + o(1))}{2(\sigma v)^{2}}\right) \to 0,
$$

because $h < (1 - \kappa)^{2}$. \hfill \Box
The last lemma shows that collisions, $X_i^t = X_i^{t-1}$, only happen after the time scale $T = e^{h(\sigma v)^{-2}/2}$. In the fast pulling regime and the intermediate regime, this time scale is larger than the time scale of the deterministic exit time $t^*_\star$, in particular, $T$ is larger than $\tau$, as we will show in the next corollary.

**Corollary 13.** If $\sigma^2 |\ln \varepsilon| \to 0$, then

$$\mathbb{P}(\exists i \in \{1, \ldots, d\}, \exists t \in [0, \tau] : X_i^t = X_i^{t-1}) \to 0.$$  

**Proof.** Indeed, in this case we have eventually

$$T = \exp \left( \frac{h}{2(\sigma v)^2} \right) = \exp \left( \frac{h|\ln \varepsilon|}{2v^2(\sigma^2|\ln \varepsilon|)} \right) \geq \exp \left( 2|\ln \varepsilon| \right) = \varepsilon^{-2} \geq \frac{d}{\varepsilon} = t^*_\star \geq \tau,$$

and Proposition 12 applies. \(\square\)

Note that this corollary covers the fast pulling and intermediate slow pulling regimes, but the approach fails for very slow pulling regime because in this case $\mathbb{P}(\tau > T) \to 1$ for every $h < 1$.

4. The case of very slow pulling and no pulling.

4.1. **Result for the stationary model.** We start with considering a stationary model that captures the main features of our (nonstationary) problem. We will make later a relatively easy passage from the stationary case to the non-stationary one.

Let $\tilde{V}_i^t$ be the stationary Gaussian processes defined in (11). Define the related exit times

$$\tilde{t}^i := \inf \left\{ t : \sigma \tilde{V}_i^t \geq 1 \right\}$$

and $\tilde{\tau} := \min_{1 \leq i \leq d} \tilde{t}^i$ (cf. (10)). The main result for stationary case is as follows.

**Proposition 14.** Let $\sigma \to 0$. Then

$$\mathbb{P}(\tilde{\tau} = \tilde{t}^i) \to \begin{cases} \frac{1}{d-1} & i \in \{2, \ldots, d-1\}; \\ \frac{1}{2^{d-1}} & i \in \{1, d\}. \end{cases}$$
Further, for any $i \in \{1, \ldots, d\}$,
\[ \tilde{\tau}_i \cdot (\sigma v)^{-1} \exp(-(\sigma v)^{-2}/2) \cdot \frac{A_i}{\sqrt{2\pi}} \xrightarrow{d} \mathcal{E}, \]
and
\[ \tilde{\tau} \cdot (\sigma v)^{-1} \exp(-(\sigma v)^{-2}/2) \cdot \frac{2d}{\sqrt{2\pi}} \xrightarrow{d} \mathcal{E}, \]
where $\mathcal{E}$ is a standard exponential random variable, $v$ and the $A_i$ are defined in (3).

Remark 15. At some point we will need a minor extension of Proposition 14 allowing a very mild flexibility of the exit times. Namely, let $\sigma_i^*$, $1 \leq i \leq d$, be so close to $\sigma$ that $\sigma_i^2 = \sigma^2 + o(1)$. Let
\[ \tilde{\tau}_i^* := \inf \{ t : \sigma_i^* \tilde{V}_t^i \geq 1 \} \]
and $\tilde{\tau}_* := \min_{1 \leq i \leq d} \tilde{\tau}_i^*$. Then we still have, as $\sigma \to 0$,
\[ \mathbb{P}(\tilde{\tau}_* = \tilde{\tau}_i^*) \to \begin{cases} 
\frac{1}{d-1} & i \in \{2, \ldots, d-1\}; \\
\frac{1}{2(d-1)} & i \in \{1, d\}. 
\end{cases} \]
\[ \tilde{\tau}_i^* \cdot (\sigma v)^{-1} \exp(-(\sigma v)^{-2}/2) \cdot \frac{A_i}{\sqrt{2\pi}} \xrightarrow{d} \mathcal{E}, \]
\[ \tilde{\tau}_* \cdot (\sigma v)^{-1} \exp(-(\sigma v)^{-2}/2) \cdot \frac{2d}{\sqrt{2\pi}} \xrightarrow{d} \mathcal{E}. \]

Proposition 14 will follow from the next lemma, which does not rely on the explicit form of the processes $\tilde{V}_t^i$ but captures their important features.

Lemma 16. Let $(\Upsilon_1^1, \ldots, \Upsilon_d^d)_{t \in \mathbb{R}}$ be a family of centered stationary Gaussian process with the following properties:

(V1) The variance of all processes is the same:
\[ \mathbb{E} [\Upsilon_i^i] = v^2 > 0, \quad i = 1, \ldots, d. \]
(V2) The local covariance behavior is as follows: there is an $\alpha \in (0, 2]$ such that for any $i$ there is $A_i > 0$ with
\[ \text{cov}(\Upsilon_t^i, \Upsilon_0^i) = v^2 (1 - A_i |t|^\alpha + o(|t|^\alpha)), \quad \text{as } t \to 0. \]
(V3) There is a non-degeneracy among the components:
\[ \sup_{s,t} \sup_{i,j \in \{1, \ldots, d\}, i \neq j} \text{corr}(\Upsilon_t^i, \Upsilon_s^j) < 1. \]
There is a mixing property: i.e. there is a number $\mu > 0$ such that

$$r(\theta) \leq e^{-\mu \theta}, \quad \theta > 0,$$

where $r$ is the linear mixing coefficient of $\Upsilon$ defined in (21).

Then, for $\tau^i := \min\{t > 0 : \sigma \Upsilon^i_t \geq 1\}$ and $\tau := \min_{i \in \{1, \ldots, d\}} \tau^i$ we have, as $\sigma \to 0$,

$$P(\tau = \tau^i) \to \frac{A_i^{1/\alpha}}{\sum_{j=1}^d A_j^{1/\alpha}},$$

where $\alpha$ and the $A_i$ are as in (25).

Further, as $\sigma \to 0$, we have the weak limit theorems

$$\tau^i \cdot (\sigma v)^{1-2/\alpha} \exp(-(\sigma v)^{-2}/2) \cdot \frac{A_i^{1/\alpha} \mathcal{H}_\alpha}{\sqrt{2\pi}} \to \mathcal{E},$$

for $i \in \{1, \ldots, d\}$ and

$$\tau \cdot (\sigma v)^{1-2/\alpha} \exp(-(\sigma v)^{-2}/2) \cdot \frac{\sum_{j=1}^d A_j^{1/\alpha} \mathcal{H}_\alpha}{\sqrt{2\pi}} \to \mathcal{E},$$

where $\mathcal{E}$ is a standard exponential random variable and $\mathcal{H}_\alpha$ is Pickands constant (cf. Lemma 10).

Remark 17. A similar proof works if the covariance expansion at zero (25) is given by $v^2(1 - A_i |t|^{\alpha_i} + \ldots)$ with some possibly distinct $\alpha_1, \ldots, \alpha_d \in (0, 2]$. The exit distribution is then concentrated on the positions $i$ where $\alpha_i = \min_{j \in \{1, \ldots, d\}} \alpha_j$. The same applies if the variances of the different components in (24) are different, say $=: v_i^2$. Then the exit distribution is concentrated on those positions $i$ where $v_i^2 = \max_{j \in \{1, \ldots, d\}} v_j^2$.

The idea of the proof of Lemma 16 was sketched after the statement of Theorem 3.


In order to prove (26) it is sufficient to show that, for all $i \in \{1, \ldots, d\}$,

$$\limsup_{\sigma \to 0} P(\tau = \tau^i) \leq \frac{A_i^{1/\alpha}}{\sum_{j=1}^d A_j^{1/\alpha}},$$

as the right-hand side sums up to one.
Fix $i$ for the rest of this proof and set

$$T_* := \sigma^{2/\alpha - 1} \exp((\sigma v)^{-2}/2), \quad \ell := B\sigma^{-2}, \quad L := \sigma^{-3}, \quad M := \frac{\mathcal{M}T_*}{L + \ell},$$

where $B > 0$ is a large constant chosen later and $\mathcal{M} > 0$ is fixed for a while but will be sent to infinity later.

Consider the intervals

$$I_m := [(L + \ell) (m-1), (L + \ell) (m-1) + L], \quad J_m := [(L + \ell) (m-1) + L, (L + \ell) m],$$

for $m \in \{1, 2, \ldots, M\}$; and note that the disjoint union of these intervals is the interval $[0, M T_*]$. For $m \in \{1, 2, \ldots, M\}$, define the events

$$E^i_m := \{ \max_{t \in I_m} \Upsilon^i_t \geq \sigma^{-1} \}, \quad N_m := \{ \max_{j \in \{1, \ldots, d\}} \max_{t \in I_m} \Upsilon^j_t < \sigma^{-1} \}.$$

The event $E^i_m$ means that the $i$-th component produces an exit (in the sense of the statement of the lemma) in the time interval $I_m$, while $N_m$ means that none of the components exits during the time interval $I_m$. By the stationarity of the processes $(\Upsilon^i_t)_{t \in \mathbb{R}}, i \in \{1, \ldots, d\}$,

$$\mathbb{P}(E^1_1) = \mathbb{P}(E^2_1) = \cdots = \mathbb{P}(E^i_M), \quad \mathbb{P}(N_1) = \mathbb{P}(N_2) = \cdots = \mathbb{P}(N_M).$$

The basic relation in order to prove (29) is the observation

$$\mathbb{P}(\tau = \tau^i) \leq \mathbb{P}(\tau^i \geq M T_*) + \mathbb{P}(\tau^i \in \bigcup_{m=1}^M J_m) + \sum_{m=1}^M \mathbb{P}(\tau^i \in I_m). \quad (30)$$

The first probability will tend to zero (first $\sigma \to 0$ then $\mathcal{M} \to \infty$), as an exit far beyond the critical time scale $T_*$ is unlikely. The second probability will also tend to zero, because the intervals $J_m$ are too short to produce an exit at all. The only contribution comes from the last term.

**Step 2: The second term in (30) tends to zero.**

Observe that

$$\mathbb{P}(\tau^i \in \bigcup_{m=1}^M J_m) \leq \sum_{m=1}^M \mathbb{P}(\tau^i \in J_m) \leq M\mathbb{P}(\max_{t \in J_1} \Upsilon^i_t \geq \sigma^{-1}).$$

Using stationarity again and the Pickands lemma (Lemma 10) with (V2),
we can estimate the last term as follows:

\[
MP(\max_{t \in J_1} \Upsilon_i^t \geq \sigma^{-1}) = MP(\max_{t \in [0,\ell]} \Upsilon_i^t \geq \sigma^{-1}) \\
\leq M \cdot 2 \frac{A_1^{1/\alpha}H_\alpha}{\sqrt{2\pi} \cdot \ell \cdot (\sigma v)^{1-2/\alpha} \exp(-(\sigma v)^{-2}/2)} \\
= C_{\alpha,A,v} \frac{MT_\ast}{L + \ell} \cdot \ell \cdot \sigma^{1-2/\alpha} \exp(-(\sigma v)^{-2}/2) \\
= C_{\alpha,A,v,M} \frac{\sigma^{2/\alpha - 1} e((\sigma v)^{-2}/2)}{L + \ell} \cdot \ell \sigma^{1-2/\alpha} \exp(-(\sigma v)^{-2}/2) \\
= C_{\alpha,A,v,M} \frac{\ell}{L + \ell} \to 0,
\]
as \sigma \to 0. We have thus seen that the second term in (30) tends to zero.

**Step 3:** The first term in (30) tends to zero, as first \(\sigma \to 0\) and then \(M \to \infty\).

Assumption (V2) and Lemma 10 yield

\[
P(E_i^1) = P(\max_{t \in [0,L]} \Upsilon_i^t \geq \sigma^{-1}) \sim \frac{A_1^{1/\alpha}H_\alpha}{\sqrt{2\pi} \cdot L \cdot (\sigma v)^{1-2/\alpha} \cdot \exp(-(\sigma v)^{-2}/2)},
\]

because the right-hand side indeed tends to zero, as the exponential term tends to zero faster than the increasing polynomial term \(L\sigma^{1-2/\alpha}\).

Further, we have to decouple the events \(E_i^m\). For this purpose, we use Assumption (V4), which makes Lemma 8 applicable: Namely, by applying iteratively Lemma 8 using that \(E_i^m \in \sigma(\Upsilon_i^t, t \in I_m)\) and the fact that the intervals \(I_m\) are separated by a distance of at least \(\ell\), we obtain

\[
\mathbb{P}(\tau^i \geq MT_\ast) \leq \mathbb{P}\left(\bigcap_{m=1}^M (E_i^m)^c\right) \\
\leq \mathbb{P}(E_i^1)^c \cdot \mathbb{P}\left(\bigcap_{m=2}^M (E_i^m)^c\right) + e^{-\mu \ell} \\
\leq \ldots \\
\leq \exp(-M\mathbb{P}(E_i^1)) + Me^{-\mu \ell}.
\]

We will show that both terms tend to zero, as first \(\sigma \to 0\) and then \(M \to \infty\).

Let us start with the second term. Here,

\[
Me^{-\mu \ell} = \frac{MT_\ast}{L + \ell} e^{-\mu B \sigma^{-2}} = \frac{M\sigma^{2/\alpha - 1} e((\sigma v)^{-2}/2)}{L + \ell} e^{-\mu B \sigma^{-2}},
\]
which tends to zero if $B$ is chosen sufficiently large ($\mu B > v^{-2}/2$; in fact, later we will choose $B$ such that even $\mu B > v^{-2}$).

For the first term in (32), let $c_{\alpha,A_i,v} := A_i^{1/\alpha} \mathcal{H}_\alpha v^{1-2/\alpha} / \sqrt{2\pi}$, and observe that by (31) we have that for $\sigma \to 0$

\begin{equation}
M \cdot \mathbb{P}(E_i^\tau \tau_i) \sim \frac{MT_*}{L + \ell} \cdot c_{\alpha,A_i,v} L \sigma^{1-2/\alpha} e^{-(\sigma v)^{-2}/2} = \frac{M\sigma^{2/\alpha-1} e^{-(\sigma v)^{-2}/2}}{L + \ell} \cdot c_{\alpha,A_i,v} L \sigma^{1-2/\alpha} e^{-(\sigma v)^{-2}/2} \sim M \cdot c_{\alpha,A_i,v}.
\end{equation}

(34)

We have thus seen that for any $M > 0$ we have

\begin{equation}
\limsup_{\sigma \to 0} \mathbb{P}(\tau \geq MT_*) \leq e^{-M c_{\alpha,A_i,v} \cdot v}.
\end{equation}

(35)

After letting first $\sigma \to 0$, then $M \to \infty$, the first term in (30) tends to zero.

**Step 4:** The last term in (30) gives (29).

We shall use again Lemma 8 and (V4). Namely, observe that since the intervals $I_k$, $k \in \{1, \ldots, m\}$ are separated by a distance of at least $\ell$, and that $E_m \in \sigma(Y_t, t \in I_m)$ and similarly for the $N_m$. We obtain for $m \in \{1, \ldots, M\}$

\begin{align*}
\mathbb{P}(\tau = \tau_i \in I_m) & \leq \mathbb{P}(E_m^i \cap \bigcap_{k=1}^{m-1} N_k) \\
& \leq \mathbb{P}(E_m^i) \cdot \mathbb{P}(\bigcap_{k=1}^{m-1} N_k) + e^{-\mu \ell} \\
& \leq \ldots \\
& \leq \mathbb{P}(E_1^i) \cdot \mathbb{P}(N_1)^{m-1} + M e^{-\mu \ell}.
\end{align*}

This yields

\begin{equation}
\sum_{m=1}^M \mathbb{P}(\tau = \tau_i \in I_m) \leq \mathbb{P}(E_1^i) \cdot \sum_{m=1}^M \mathbb{P}(N_1)^{m-1} + M e^{-\mu \ell} \leq \mathbb{P}(E_1^i) \cdot \frac{1}{1 - \mathbb{P}(N_1)} + M e^{-\mu \ell}.
\end{equation}

(36)

The last term is easily seen to tend to zero:

\begin{equation*}
M^2 e^{-\mu \ell} = \frac{M^2 T_*^2}{(L + \ell)^2} e^{-\mu B \sigma^{-2}} = \frac{M^2 \sigma^{4/\alpha-2} e^{2(\sigma v)^{-2}/2}}{(L + \ell)^2} e^{-\mu B \sigma^{-2}},
\end{equation*}
which tends to zero as $\sigma \to 0$, if we choose $B$ is sufficiently large ($\mu B > v^{-2}$).

In order to treat the first term in (36), first observe that
\begin{equation}
1 - \mathbb{P}(N_1) = \mathbb{P}(N_1^c) = \mathbb{P}\left(\bigcup_{j=1}^{d} E_j^1\right) \geq \sum_{j=1}^{d} \mathbb{P}(E_j^1) - \sum_{j_1, j_2 \in \{1, \ldots, d\}, j_1 \neq j_2} \mathbb{P}(E_{j_1}^j \cap E_{j_2}^j),
\end{equation}
where the inequality follows from the inclusion-exclusion principle (also called Bonferroni’s inequality).

Using again the Pickands lemma (Lemma 10), we have
\begin{equation}
\mathbb{P}(E_j^j) \sim A_j^{1/\alpha} H_{\alpha} \frac{1}{\sqrt{2\pi}} \cdot L \cdot (\sigma v)^{1-2/\alpha} \exp\left(-\frac{(\sigma v)^{-2}}{2}\right), \quad j \in \{1, \ldots, d\}.
\end{equation}

We shall further prove that for $j_1 \neq j_2, j_1, j_2 \in \{1, \ldots, d\}$,
\begin{equation}
\mathbb{P}(E_{j_1}^j \cap E_{j_2}^j) \leq \exp\left(-\eta(\sigma v)^{-2}/2(1 + o(1))\right),
\end{equation}
as $\sigma \to 0$, where $\eta > 1$ is some constant. Inserting this into (37) and using (38), we obtain
\begin{equation}
1 - \mathbb{P}(N_1) \geq \sum_{j=1}^{d} \mathbb{P}(E_j^j) (1 + o(1)),
\end{equation}
because the leading order of $\sum_{j=1}^{d} \mathbb{P}(E_j^j)$ is given by $\exp\left(-\frac{(\sigma v)^{-2}}{2}(1 + o(1))\right)$, by (38), while the second term in (37) is of order $\exp\left(-\eta(\sigma v)^{-2}/2(1 + o(1))\right)$ due to (39), and thus of lower order as $\eta > 1$.

Putting this together with (36) shows that, as $\sigma \to 0$,
\begin{equation}
\sum_{m=1}^{M} \mathbb{P}(\tau^i \in I_m) \leq \frac{A_1^{1/\alpha} H_{\alpha}}{\sqrt{2\pi}} \cdot L \cdot (\sigma v)^{1-2/\alpha} \exp\left(-\frac{(\sigma v)^{-2}}{2}\right) + o(1)
\end{equation}
\begin{equation}
= \frac{A_1^{1/\alpha} H_{\alpha}}{\sqrt{2\pi}} \cdot L \cdot (\sigma v)^{1-2/\alpha} \exp\left(-\frac{(\sigma v)^{-2}}{2}\right) + o(1)
\end{equation}
\begin{equation}
= \frac{A_1^{1/\alpha}}{\sum_{j=1}^{d} A_j^{1/\alpha}} + o(1),
\end{equation}
as required by (29). It remains to prove (39).

**Step 5: Proof of (39).**
As a comment, note that (39) means that the probability of "joint exits" of different components in one of the \( I_m \) intervals is of lower order (compared to the exit of only one component).

To simplify the notation, let \( j_1 = 1, \ j_2 = 2 \). By definition,

\[
E_1^1 \cap E_1^2 = \left\{ \max_{s\in[0,L]} \Upsilon_s^1 \geq \sigma^{-1}, \ \max_{t\in[0,L]} \Upsilon_t^2 \geq \sigma^{-1} \right\}.
\]

Obviously,

\[
E_1^1 \cap E_1^2 \subseteq \left\{ \max_{s,t\in[0,L]} (\Upsilon_s^1 + \Upsilon_t^2) \geq 2\sigma^{-1} \right\} = \left\{ \max_{s,t\in[0,L]} \Lambda(s,t) \geq 2\sigma^{-1} \right\},
\]

where \( \Lambda(s,t) := \Upsilon_s^1 + \Upsilon_t^2 \) is a centered Gaussian process with index set \( \mathcal{T} := [0, L]^2 \).

According to the general theory [23], the evaluation of the large values' probability for \( \Lambda \) requires a bound for its largest variance and the estimation of the corresponding covering numbers. By Assumption (V3), for all \( s, t \in \mathbb{R} \) we have the variance bound

\[
\mathbb{E} \Lambda(s,t)^2 = 2\nu^2 (1 + \mathbb{E} [\Upsilon_s^1 \Upsilon_t^2]) \leq 2\nu^2 (1 + \bar{\rho})
\]

with some \( \bar{\rho} < 1 \). Next, recall that for a Gaussian process \( (X_t, t \in T) \) the covering number \( \mathcal{N}(X, T, \varepsilon) \) is the minimal number of balls needed to cover the set \( T \) with respect to the Dudley semi-metric \( \rho_X(s,t)^2 := \mathbb{E} [(X_s - X_t)^2] \).

We refer to Section 14 of [23] for other relevant definitions.

Due to the property (V2) we have

\[
\rho_{\Upsilon_t}(t, t + h)^2 = 2\nu^2 A_i |h|^{\alpha} (1 + o(1)), \quad \text{as } h \to 0.
\]

It follows that \( \mathcal{N}(\Upsilon_t^i, [0, L], \varepsilon) \leq c_1 L \varepsilon^{-2/\alpha} \) for all \( \varepsilon > 0, L \geq 1 \). Hence, \( \mathcal{N}(\Lambda, T, 2\varepsilon) \leq c_1 L^2 \varepsilon^{-2/\alpha} \). Furthermore, for the Dudley integral we have the estimate

\[
\int_0^{2\nu} \sqrt{\ln \mathcal{N}(\Lambda, T, \varepsilon)} \, d\varepsilon \leq c_2 \sqrt{\ln L}, \quad L \geq 2.
\]

We can finally apply Corollary 2 from [23, Section 14, p.181], which involves the Dudley integral and shows in our case that (with \( \Phi \) the tail of the standard normal distribution) for small enough \( \sigma \)

\[
\mathbb{P} \left( \max_{(s,t)\in T} \Lambda(s,t) \geq 2\sigma^{-1} \right)
\]

\[
\leq \Phi((2\sigma^{-1} - c_3 \sqrt{\ln L})/(v \sqrt{2(1 + \bar{\rho}))})
\]

\[
\leq \exp\left( -\frac{\sigma^{-2}}{v^2(1 + \bar{\rho})} (1 + o(1)) \right)
\]

\[
= \exp\left( -\eta(\sigma v)^{-2}/2(1 + o(1)) \right),
\]
with $\eta := 2/(1 + \bar{\rho}) > 1$ as required in (39).

**Step 6**: We prove the limit theorem for $\tau^i$, i.e. (27).

By using (35) with $M := tc_{A_i,v}^{-1}$, we obtain that, for any $t > 0$,

$$\limsup_{\sigma \to 0} \mathbb{P}(\tau^i \geq tc_{A_i,v}^{-1} T_*) \leq \exp(-t).$$

This already shows one of the estimates needed for the limit theorem (27).

We now proceed with proving the lower bound in (27). For this purpose, let $M > 0$ and set $M$ (and all other variables) as in Step 1. Then

$$\mathbb{P}(\tau^i \geq M T_*) \geq \mathbb{P} \left( \bigcap_{m=1}^{M} (E_m^i)^c \cap \{ \tau^i \notin \bigcup_{m=1}^{M} J_m \} \right)$$

$$\geq \mathbb{P} \left( \bigcap_{m=1}^{M} (E_m^i)^c \right) - \mathbb{P}(\tau^i \in \bigcup_{m=1}^{M} J_m)$$

$$\geq (1 - \mathbb{P}(E_1^i)) M - Me^{-\mu_\ell} - \mathbb{P}(\tau^i \in \bigcup_{m=1}^{M} J_m),$$

by the decoupling argument from Lemma 8 together with (V4) and using stationarity. The term $Me^{-\mu_\ell}$ is seen to tend to zero with $\sigma \to 0$, as in (33), by the choice of the constant $B$. Further, $\mathbb{P}(\tau^i \in \bigcup_{m=1}^{M} J_m)$ tends to zero, as observed in Step 2.

Since $\mathbb{P}(E_1^i) \to 0$ as $\sigma \to 0$, we obtain

$$\mathbb{P}(\tau^i \geq M T_*) \geq e^{-M\mathbb{P}(E_1^i)(1+o(1))} - o(1).$$

We further know from (34) that $M\mathbb{P}(E_1^i) \sim M \cdot c_{A_i,v}$, which implies

$$\liminf_{\sigma \to 0} \mathbb{P}(\tau^i \geq M T_*) = \exp(-M \cdot c_{A_i,v}).$$

Using the specific choice $M := tc_{A_i,v}^{-1}$, we obtain that, for any $t > 0$,

$$\liminf_{\sigma \to 0} \mathbb{P}(\tau^i \geq tc_{A_i,v}^{-1} T_*) \geq \exp(-t),$$

which completes the proof of (27).

**Step 7**: We prove the limit theorem for $\tau$, i.e. (28).
The proof is very similar to that for $\tau^i$. First,

$$
\mathbb{P}(\tau \geq MT_*) \leq \mathbb{P}(\bigcap_{m=1}^{M} \bigcap_{j=1}^{d} (E_m^j)^c) \\
\leq \prod_{m=1}^{M} \mathbb{P}(\bigcap_{j=1}^{d} (E_m^j)^c) + Me^{-\mu t} \\
\leq \exp(-M\mathbb{P}(\bigcup_{j=1}^{d} E_1^j)) + Me^{-\mu t}.
$$

Again the second term is of lower order (cf. (33)), while for the first – by the argument in (37)-(40) – one can see that

$$
\mathbb{P}(\bigcup_{j=1}^{d} E_1^j) = \sum_{j=1}^{d} \mathbb{P}(E_1^j)(1 + o(1)).
$$

This shows (using (37) and the definition of $T_*$)

$$
\limsup_{\sigma \to 0} \mathbb{P}(\tau \geq MT_*) \leq \exp(-M \sum_{j=1}^{d} c_{\alpha,A,j,v}).
$$

Using the specific choice $M := t(\sum_{j=1}^{d} c_{\alpha,A,j,v})^{-1}$ shows the upper bound in (28).

For the lower bound, one can also proceed analogously:

$$
\mathbb{P}(\tau \geq MT_*) \\
\geq \mathbb{P}(\bigcap_{m=1}^{M} \bigcap_{j=1}^{d} (E_m^j)^c \cap \{ \forall j : \tau^j \not\in \bigcup_{m=1}^{M} J_m \}) \\
\geq \prod_{m=1}^{M} \mathbb{P}(\bigcap_{j=1}^{d} (E_m^j)^c) - \mathbb{P}(\exists j : \tau^j \in \bigcup_{m=1}^{M} J_m) \\
\geq \exp(-M\mathbb{P}(\bigcup_{j=1}^{d} E_1^j)(1 + o(1))) - Me^{-\mu t} - \sum_{j=1}^{d} \mathbb{P}(\tau^j \in \bigcup_{m=1}^{M} J_m).$$
Again the second and third term vanish (cf. (33) and Step 2). For the first expression, we can obtain
\[ P\left( \bigcup_{j=1}^{d} E_j^i \right) = P\left( \max_{j \in \{1, \ldots, d\}} \max_{t \in [0, L]} \Upsilon_t^j \geq \sigma^{-1} \right) \leq \sum_{j=1}^{d} P\left( \max_{t \in [0, L]} \Upsilon_t^j \geq \sigma^{-1} \right) , \]
and we apply the Pickands lemma (see (38)) to this giving
\[ \lim_{\sigma \to 0} \inf \mathbb{P}(\tau \geq \mathcal{M}T_\ast) \geq \exp(-\mathcal{M} \sum_{j=1}^{d} c_{\alpha, A_j, v}). \]
Again, using the specific choice \( \mathcal{M} := t(\sum_{j=1}^{d} c_{\alpha, A_j, v})^{-1} \) shows the lower bound in (28).

### 4.3. Proof of Proposition 14.

In this section, we prove Proposition 14. The goal is to apply Lemma 16 with \( \alpha = 1 \) and, accordingly, \( \mathcal{H}_{\alpha} = 1 \). We only have to show that the processes \((\tilde{V}_t^i)\) defined in (11) satisfy the assumptions (V1)-(V4) of Lemma 16.

Note that (V1) and (V2) follow directly from relation (13) in Lemma 6, where \( \alpha = 1 \) and \( v \) and the \( A_i \) are as in the statement of the proposition.

Let us prove (V3). First note that it is sufficient to consider \( s = 0 \) and \( t \geq 0 \), by stationarity. Fix \( i \neq j \). Note that
\[ |\mathbb{E}[\tilde{V}_t^i \tilde{V}_0^j]| \leq \sqrt{ \mathbb{E}[(\tilde{V}_t^i)^2] \cdot \mathbb{E}[(\tilde{V}_0^j)^2] } = v^2 , \]
with equality if and only if the two random variables are linearly dependent. In the latter case, we would have a.s. \( \tilde{V}_t^i = c \tilde{V}_0^j \), which has probability zero (for any \( t > 0 \)). Therefore, \( \mathbb{E}[\tilde{V}_t^i \tilde{V}_0^j] < v^2 \) for all \( t > 0 \).

At \( t = 0 \), by (14) in Lemma 6, we have \( \mathbb{E}[\tilde{V}_0^i \tilde{V}_0^j] = -1/(2d) \), so that for small \( t \) we can bound \( \mathbb{E}[\tilde{V}_t^i \tilde{V}_0^j] \) away from 0, by continuity. Further, \( \mathbb{E}[\tilde{V}_t^i \tilde{V}_0^j] = \sum_{k=1}^{d} a_k^i e^{\lambda_k t} \) is a sum of exponential terms, because each \( \tilde{V}_t^i \) is a linear combination of independent Ornstein–Uhlenbeck processes, so that, for large \( t \) it tends to zero. Summarizing, the function \( t \mapsto \mathbb{E}[\tilde{V}_t^i \tilde{V}_0^j] \) is continuous, negative for small \( t \), never reaches \( v^2 \), and tends to zero for \( t \to \infty \). Therefore, we can bound it as \( \mathbb{E}[\tilde{V}_t^i \tilde{V}_0^j] \leq \bar{\rho}_{i,j} v^2 \) with \( \bar{\rho}_{i,j} < 1 \). Taking \( \bar{\rho} := \max_{i \neq j} \bar{\rho}_{i,j} < 1 \) then satisfies (V3).

The fact that (V4) holds for \( (\tilde{V}_t^i)_{1 \leq i \leq d} \) with \( \mu := \min_{j} |\lambda_j| \) follows from Lemma 9.

### 4.4. Proof of Theorem 3.

We shall prove Theorem 3 by reducing it to the stationary case considered in Proposition 14.
4.4.1. Limit theorem for \( \tau^i \). First of all notice that the slow pulling condition (5) implies

\[
\varepsilon \ll \exp(-K\sigma^{-2}) \quad \forall K > 0.
\]

We will only need this (slightly weaker) condition.

Upper bound for the exit time. Recall that the sufficient condition for a break in the chain element \( i \) at time \( t \) is given in (20); it follows that the simpler condition \( \sigma \tilde{V}^i_t \geq 1 + \varepsilon K^i + \sigma \Xi e^{-\mu t} \) is also sufficient for the break.

Let us denote by \( \theta^i_o := \sigma v \exp((\sigma v)^{-2}/2) \frac{\sqrt{2\pi}}{A} \) the norming factor from our limit theorem. Then for all \( r, \delta, F > 0 \) we have

\[
\mathbb{P}(\tau^i \geq r\theta^i_o) \leq \mathbb{P}\left(\sigma \tilde{V}^i_t < 1 + \varepsilon K^i + \sigma \Xi e^{-\mu t}, 0 \leq t \leq r\theta^i_o\right) \\
\leq \mathbb{P}\left(\sigma \tilde{V}^i_t < 1 + \varepsilon K^i + \sigma \Xi e^{-\mu t}, 0 \leq t \leq r\theta^i_o\right) + \mathbb{P}(\Xi > F) \\
\leq \mathbb{P}\left(\sigma \tilde{V}^i_t < 1 + \varepsilon K^i + \sigma \Xi e^{-\mu \delta \theta^i_o}, \delta \theta^i_o \leq t \leq r\theta^i_o\right) + \mathbb{P}(\Xi > F) \\
= \mathbb{P}\left(\sigma \tilde{V}^i_t < 1, \delta \theta^i_o \leq t \leq r\theta^i_o\right) + \mathbb{P}(\Xi > F),
\]

where \( \sigma_{is} = \sigma(1 + \varepsilon K^i + \sigma \Xi e^{-\mu \delta \theta^i_o})^{-1} \). Using (41), notice that \( \varepsilon K^i + \sigma \Xi e^{-\mu \delta \theta^i_o} \ll \sigma^2 \), hence \( \sigma_{is}^{-2} = \sigma^{-2} + o(1) \), thus providing \( \theta^i_{is} \sim \theta^i_o \), and we obtain eventually

\[
\mathbb{P}(\tau^i \geq \theta^i_o r) \leq \mathbb{P}\left(\sigma_{is} \tilde{V}^i_t < 1, 2\delta \theta^i_{is} \leq t \leq (r - \delta) \theta^i_{is}\right) + \mathbb{P}(\Xi > F) \\
\leq \mathbb{P}\left(\sigma_{is} \tilde{V}^i_t < 1, 0 \leq t \leq (r - \delta) \theta^i_{is}\right) \\
+ \mathbb{P}\left(\exists t \in [0, 2\delta \theta^i_{is}] : \sigma_{is} \tilde{V}^i_t \geq 1\right) + \mathbb{P}(\Xi > F).
\]

By applying Proposition 14 with \( \sigma_{is} \) in place of \( \sigma \) we have that under (41)

\[
\limsup_{\varepsilon \to 0, \sigma \to 0} \mathbb{P}(\tau^i \geq r\theta^i_o) \leq e^{-(r - \delta)} + [1 - e^{-2\delta}] + \mathbb{P}(\Xi > F).
\]

Finally, by letting \( \delta \to 0, F \to +\infty \) we obtain under (41)

\[
\limsup_{\varepsilon \to 0, \sigma \to 0} \mathbb{P}(\tau^i \geq r\theta^i_o) \leq e^{-r}.
\]

Lower bound for the exit time. Recall that the necessary condition for the break is given in (19). Assuming as before that \( \Xi \leq F \), we obtain the necessary condition \( \sigma \tilde{V}^i_t \geq 1 - \frac{\varepsilon t}{d} - \varepsilon K^i - \sigma \Xi e^{-\mu t} \). We handle this condition

\[
\varepsilon \ll \exp(-K\sigma^{-2}) \quad \forall K > 0.
\]
differently for relatively small $t$ and for larger $t$. Namely, for $t \in [0, 3|\ln \sigma|/\mu]$ we use that (under (41))

$$\sup_{t \in [0, 3|\ln \sigma|/\mu]} \left[ \frac{\varepsilon t}{d} + \varepsilon K^i + \sigma Fe^{-\mu t} \right] \leq \frac{\varepsilon}{d} \cdot \frac{3|\ln \sigma|}{\mu} + \varepsilon K^i + \sigma F \to 0.$$  

Therefore, for $t \in [0, 3|\ln \sigma|/\mu]$ we obtain the necessary condition $\sigma \tilde{V}_t \geq \frac{1}{2}$. On the other hand, for $t \in [3|\ln \sigma|/\mu, r\theta^i]$ we obtain the necessary condition

$$\sigma \tilde{V}_t \geq 1 - \frac{\varepsilon r\theta^i}{d} - \varepsilon K^i - F \sigma^4,$$

which can be rewritten as $\sigma \tilde{V}_t \geq 1$, where $\sigma := \sigma \left( 1 - \frac{\varepsilon r\theta^i}{d} - \varepsilon K^i - F \sigma^4 \right)^{-1}$ again satisfies the requirement $\sigma \tilde{V}_t \geq \frac{1}{2}$ (under (41)), hence, $\sigma \tilde{V}_t \sim \theta^i$. We obtain

$$\mathbb{P}(\tau \geq r\theta^i)$$

$$\geq \mathbb{P}(\sigma \tilde{V}_t < 1, 0 \leq t \leq r\theta^i) - \mathbb{P}(\Xi > F) - \mathbb{P} \left( \sigma \sup_{t \in [0, 3|\ln \sigma|/\mu]} \tilde{V}_t \geq \frac{1}{2} \right)$$

$$\geq \mathbb{P}(\sigma \tilde{V}_t < 1, 0 \leq t \leq (1 + \delta)r\theta^i) - \mathbb{P}(\Xi > F)$$

$$- \mathbb{P} \left( \sigma \sup_{t \in [0, 3|\ln \sigma|/\mu]} \tilde{V}_t \geq \frac{1}{2} \right).$$

By stationarity and Gaussianity of $\tilde{V}_t$ we have

$$(42) \quad \lim_{\sigma \to 0} \mathbb{P} \left( \sigma \sup_{t \in [0, 3|\ln \sigma|/\mu]} \tilde{V}_t \geq \frac{1}{2} \right) = 0.$$  

Indeed, since the process $\tilde{V}_t$ is bounded, by [23, Theorem 1, Section 12] for every $h > 0$ and all $\sigma$ small enough one has

$$\mathbb{P} \left( \sigma \sup_{t \in [0, 1]} \tilde{V}_t \geq \frac{1}{2} \right) \leq \exp \left( -\frac{(\sigma v)^{-2}}{8(1 + h)} \right).$$

Hence, using stationarity, for every $L > 0$ we see that

$$\mathbb{P} \left( \sigma \sup_{t \in [0, L]} \tilde{V}_t \geq \frac{1}{2} \right) \leq (L + 1) \exp \left( -\frac{(\sigma v)^{-2}}{8(1 + h)} \right).$$

Applying this with $L = 3|\ln \sigma|/\mu$ we obtain (42).
By using (42) and applying Proposition 14 with \( \sigma_i \) in place of \( \sigma \) we have under (41)

\[
\liminf_{\epsilon \to 0, \sigma \to 0} P(\tau^i \geq r\theta^i_\sigma) \geq e^{-(1+\delta)r} - P(\Xi > F).
\]

Finally, by letting \( \delta \to 0, F \to +\infty \) we obtain that under (41)

\[
\liminf_{\epsilon \to 0, \sigma \to 0} P(\tau^i \geq r\theta^i_\sigma) \geq e^{-r}.
\]

4.4.2. Limit theorem for \( \tau \). Let us denote by \( \theta_\sigma := \sigma v \exp((\sigma v)^{-2}/2) \sqrt{2\pi} \)

the norming factor from our limit theorem.

**Upper bound for the total exit time.** Using the same argument as in the previous upper bound we obtain for all \( r, \delta, F > 0 \)

\[
P(\tau \geq r\theta_\sigma) = P(\tau^i \geq r\theta_\sigma, 1 \leq i \leq d)
\]

\[
\leq P\left(\sigma \tilde{V}^i_t < 1 + \epsilon K^i + \sigma \Xi e^{-\mu t}, 0 \leq t \leq r\theta_\sigma, 1 \leq i \leq d\right)
\]

\[
\leq P\left(\sigma \tilde{V}^i_t < 1 + \epsilon K^i + \sigma F e^{-\mu t}, 0 \leq t \leq r\theta_\sigma, 1 \leq i \leq d\right) + P(\Xi > F)
\]

\[
\leq P\left(\sigma \tilde{V}^i_t < 1 + \epsilon K^i + \sigma F e^{-\mu \theta_\sigma}, \delta\theta_\sigma \leq t \leq r\theta_\sigma, 1 \leq i \leq d\right) + P(\Xi > F)
\]

\[
\leq P\left(\sigma \tilde{V}^i_t < 1, \delta\theta_\sigma \leq t \leq r\theta_\sigma, 1 \leq i \leq d\right) + P(\Xi > F),
\]

where \( \sigma_* = \sigma(1 + \epsilon \max_{1 \leq i \leq d} K^i + \sigma F e^{-\mu \theta_\sigma})^{-1} \). Using (41), we obtain again \( \sigma_*^{-2} = \sigma^{-2} + o(1) \), providing \( \theta_* \sim \theta_\sigma \), and we obtain eventually

\[
P(\tau \geq r\theta_\sigma) \leq P\left(\sigma_* \tilde{V}^i_t < 1, 2\delta\theta_\sigma \leq t \leq (r - \delta)\theta_*^i, 1 \leq i \leq d\right) + P(\Xi > F)
\]

\[
\leq P\left(\sigma_* \tilde{V}^i_t < 1, 0 \leq t \leq (r - \delta)\theta_*^i, 1 \leq i \leq d\right) + P\left(\exists t \in [0, 2\delta\theta_\sigma, \sigma_*], \exists i \in \{1, \ldots, d\} : \sigma_* \tilde{V}^i_t \geq 1\right) + P(\Xi > F).
\]

By applying Proposition 14 with \( \sigma_* \) in place of \( \sigma \) we have

\[
\limsup_{\epsilon \to 0, \sigma \to 0} P(\tau \geq r\theta_\sigma) \leq e^{-(r-\delta)} + \left[1 - e^{-2\delta}\right] + P(\Xi > F).
\]

Finally, by letting \( \delta \to 0, F \to +\infty \) we obtain that under (41)

\[
\limsup_{\epsilon \to 0, \sigma \to 0} P(\tau \geq r\theta_\sigma) \leq e^{-r}.
\]
Lower bound for the total exit time. Using the same argument as in the previous lower bound we obtain for all \( r, \delta, F > 0 \)

\[
\mathbb{P}(\tau \geq r\theta) = \mathbb{P}(r^i \geq r\theta, 1 \leq i \leq d) \\
\geq \mathbb{P}(\sigma_{i^*} \bar{V}_t^i < 1, 0 \leq t \leq r\theta, 1 \leq i \leq d) - \mathbb{P}(\Xi > F) \\
- \mathbb{P}(\sigma \max \sup_{1 \leq i \leq d, t \in [0, 3|\ln |\sigma|/\mu]} \bar{V}_t^i \geq \frac{1}{2}) \\
\geq \mathbb{P}(\sigma \bar{V}_t^i < 1, 0 \leq t \leq (1 + \delta)r\theta, 1 \leq i \leq d) - \mathbb{P}(\Xi > F) \\
- \mathbb{P}(\sigma \max \sup_{1 \leq i \leq d, t \in [0, 3|\ln |\sigma|/\mu]} \bar{V}_t^i \geq \frac{1}{2})
\]

where \( \sigma_{i^*} := \max_{1 \leq i \leq d} \sigma_{i^*} \). Again we have

\[
\lim_{\sigma \to 0} \mathbb{P}(\sigma \max \sup_{1 \leq i \leq d, t \in [0, 3|\ln |\sigma|/\mu]} \bar{V}_t^i \geq \frac{1}{2}) \\
\leq \sum_{i=1}^{d} \lim_{\sigma \to 0} \mathbb{P}(\sigma \sup_{t \in [0, 3|\ln |\sigma|/\mu]} \bar{V}_t^i \geq \frac{1}{2}) = 0.
\]

By applying Proposition 14 with \( \sigma_{i^*} \) in place of \( \sigma \) we have that under (41)

\[
\liminf_{\epsilon \to 0, \sigma \to 0} \mathbb{P}(\tau \geq r\theta) \geq e^{-(1+\delta)r} - \mathbb{P}(\Xi > F).
\]

Finally, by letting \( \delta \to 0, F \to +\infty \) we obtain that under (41)

\[
\liminf_{\epsilon \to 0, \sigma \to 0} \mathbb{P}(\tau \geq r\theta) \geq e^{-r}.
\]

4.4.3. Limit theorem for the break position. Let us fix \( i \in \{1, \ldots, d\} \) and \( \delta, r, F > 0 \). Then by using the necessary and the sufficient conditions for exit, we have after some elementary calculations,

\[
\mathbb{P}(\tau_i) \\
\geq \mathbb{P}(\exists t \in [\delta\theta^i, r\theta^i] : \sigma_{s^i} \bar{V}_t^i \geq 1, \sigma_{s^j} \bar{V}_s^j < 1, \delta \theta^i \leq s \leq t, j \neq i) \\
- \mathbb{P}(\tau \leq \delta\theta^i) - \mathbb{P}(\Xi > F),
\]

where \( \sigma_{s^i} := \sigma(1 + \epsilon K^i + \sigma F e^{-\mu \delta \theta^i})^{-1} \) and \( \sigma_{s^j} = \sigma(1 - \frac{\epsilon r \theta^i}{d} - \epsilon K^j - \sigma F e^{-\mu \delta \theta^i})^{-1} \) for \( j \neq i \). By (41), \( \sigma_{s^j}^{-2} = \sigma^{-2} + o(1) \).
Introduce the corresponding exit times \( \tau^i_j = \inf \{ t : \sigma_{i,j} \tilde{V}_t^j \geq 1 \} \), \( 1 \leq j \leq d \), and \( \tau_* := \min_{1 \leq j \leq d} \tau^i_j \). Using this notation, we have

\[
\mathbb{P}(\exists t \in [\delta\theta^i_j, r\theta^i_j] : \sigma_{i,j} \tilde{V}_t^j \geq 1 \text{ and } \sigma_{i,j} \tilde{V}_s^j < 1, \delta\theta^i_j \leq s \leq t, j \neq i) \\
\geq \mathbb{P}(\tau_* = \tau^i_j \in [\delta\theta^i_j, r\theta^i_j]) \geq \mathbb{P}(\tau_* = \tau^i_j) - \mathbb{P}(\tau_* \notin [\delta\theta^i_j, r\theta^i_j]).
\]

By using Remark 15 for the exit times (first step) and the limit theorem for \( \tau \) that is already proved (second step), we have that under (41)

\[
\liminf_{\varepsilon \to 0, \sigma \to 0} \mathbb{P}(\tau = \tau^i_j) \\
\geq \lim_{\sigma \to 0} \mathbb{P}(\tau_* = \tau^i_j) - (1 - e^{-\delta} + e^{-r}) - \lim_{\varepsilon \to 0, \sigma \to 0} \mathbb{P}(\tau \leq \delta\theta^i_j) - \mathbb{P}(\Xi > F) \\
= \lim_{\sigma \to 0} \mathbb{P}(\tau_* = \tau^i_j) - (1 - e^{-\delta} + e^{-r}) - (1 - e^{-2d\delta/A}) - \mathbb{P}(\Xi > F).
\]

Then, letting \( \delta \to 0, r, F \to \infty \) and using Remark 15 for the exit positions we obtain the lower bound

\[
\liminf_{\varepsilon \to 0, \sigma \to 0} \mathbb{P}(\tau = \tau^i_j) \geq \lim_{\sigma \to 0} \mathbb{P}(\tau_* = \tau^i_j) = \begin{cases} \\
\frac{1}{d-1} & i \in \{2, \ldots, d-1\}; \\
\frac{1}{2(d-1)} & i \in \{1, d\}.
\end{cases}
\]

Since the right hand sides of the lower bound sum up in \( i \) to one, the corresponding upper bound follows automatically, which proves the result.

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**References.**


[31] Seifert, U. (2012), Stochastic thermodynamics, fluctuation theorems and molecular


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