The fractal cylinder process: existence and connectivity phase transitions

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Abstract

We consider a semi-scale invariant version of the Poisson cylinder model which in a natural way induces a random fractal set. We show that this random fractal exhibits an existence phase transition for any dimension $d \geq 2$, and a connectivity phase transition whenever $d \geq 4$. We determine the exact value of the critical point of the existence phase transition, and we show that the fractal set is almost surely empty at this critical point.

A key ingredient when analysing the connectivity phase transition is to consider a restriction of the full process onto a subspace. We show that this restriction results in a fractal ellipsoid model which we describe in detail, as it is key to obtaining our main results.

In addition we also determine the almost sure Hausdorff dimension of the fractal set.

1 Introduction

1.1 Background and motivation

The purpose of this paper is to introduce and analyse the fractal cylinder process which is a random fractal model lacking the so-called finite energy property. The concept of finite energy was introduced by Newman and Schulman in [1] as means to study the number of unbounded connected components in translation invariant percolation models on $\mathbb{Z}^d$.

Most classical models in statistical mechanics (such as Bernoulli percolation and the Random cluster model) possess the finite energy property. Informally put, this means that one can perform a local re-arrangement without affecting the global picture. The

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simplest example among the continuum percolation models is the Boolean percolation model with fixed radius (see for instance [2]), in which we start with a homogeneous Poisson process in $\mathbb{R}^d$ and then centre a ball of radius one at each of the points in the Poisson process. One can then re-sample the configuration within a bounded region of space without changing the configuration outside said region (ignoring the technical fact that balls centred close to the boundary overlap the boundary). Thus, this model does have finite energy, and this is essential when proving for instance uniqueness of unbounded connected components (see Theorem 3.6 in [2]). In turn, uniqueness can be used to analyse other properties of the model, e.g. the continuity of the percolation function (see Theorem 3.9 in [2]).

Recently, models that lack finite energy have received a lot of attention. A contemporary example of such a model is the random interlacements introduced by Sznitmann in [3], which in its simplest form is a Poisson process on the space of bi-infinite random walks on $\mathbb{Z}^d$. Since the objects (i.e. the bi-infinite random walks) are unbounded, removing or adding walks that intersect a bounded region will change the global configuration. Not long after the introduction of the random interlacements model, the analogue in $\mathbb{R}^d$ (called the Brownian interlacements model, see [4]) was introduced. A third example of a model lacking finite energy (and which is much closer to the current paper) is that of the Poisson cylinder model. This model is a Poisson process of bi-infinite cylinders (see Section 2.5 for the definition) which can be thought of as a development of the classical Boolean percolation model. Again, this model lacks the finite energy property since the cylinders are unbounded. Percolation aspects of this model was first analysed in [5] where the main result was that the vacant set undergoes a non-trivial percolative phase transition in dimensions $d \geq 4$. Later, this was completed in [6] where they showed the analogous result for when $d = 3$. The techniques used in these papers are fundamentally different from the classical techniques used for the corresponding result for the regular Boolean percolation model. Interestingly, the question whether the unbounded vacant component is unique is still not settled for the Poisson cylinder model as one cannot rely on finite energy. The occupied component for the Poisson cylinder model was later studied in [7], where the absence of a connectivity phase transition was proved. Clearly, this is very different from the Boolean percolation model, and in contrast, it was shown that for any $d \geq 2$ any two cylinders will be connected by using at most $d - 2$ other cylinders of the process. We mention that the Poisson cylinder model was previously studied in connection with stochastic geometry, see for example [8].

We see that when we move from a model with the finite energy property (i.e. the Boolean percolation model) to one lacking this property (i.e. the Poisson cylinder model) some phenomena remain intact while others change fundamentally. It is of interest to understand whether these changes are universal in the sense that they apply to all models with/without the finite energy property. It is also of interest to develop new proofs, such as for uniqueness, in the hope that these new techniques will further our understanding.

The main goal of the current paper is to take a step in this direction by introducing and analysing a semi-scale invariant version of the Poisson cylinder model (see Section 2 and in particular Subsection 2.6 for precise definitions). This can, and will, be contrasted...
to the fractal version of the Boolean percolation model (see for instance Section 8.1 of [2]). In particular we want to investigate which properties are unchanged and which are fundamentally different. Furthermore, we ask which proof ideas are robust enough to transfer into the new setting, and which need to be changed? Based on the results of the current paper it seems that some ideas are indeed robust enough to be useful after some (sometimes rather extensive) adjustment, while in other circumstances a fundamental rethink is necessary. Below, we will indicate what the status is for each of our main results. We also want to stress that random fractal models are of intrinsic interest and has been studied extensively before. In particular we mention [9] which was one of the first papers studying random fractals, and which introduced the Mandelbrot fractal percolation model. Of particular importance is also the Brownian loop soup introduced in by Lawler and Werner in [10]. We will give further references to papers dealing with random fractal models throughout the paper. The model of this paper is the first example of a Poissonian random fractal model that uses unbounded objects to generate the fractal.

1.2 Main results

There are many natural questions to ask about random fractals. In this paper we focus on the study of phase transitions and Hausdorff dimensions. In order to state our main results we first need to give an informal explanation of our model (as mentioned, see Section 2 where we give a formal definition with details).

Let \( A(d,1) \) denote the space of lines in \( \mathbb{R}^d \), and let \( \nu_d \) be the unique measure (up to scaling) on \( A(d,1) \) which is invariant under the isometries of \( \mathbb{R}^d \). In this paper we choose to normalize \( \nu_d \) so that the \( \nu_d \)-measure of the set of lines that intersect the unit ball in \( \mathbb{R}^d \) equals one (see further Section 2.3 and in particular (9)). Then, consider the space \( A(d,1) \times (0,1] \) and let

\[
\omega = \sum_{i \geq 1} \delta_{(L_i, r_i)}
\]

be a locally finite Poisson point process with intensity measure

\[
\lambda \nu_d \times I(0 < r \leq 1) r^{-d} dr, \quad \lambda > 0,
\]

and let \( \mathbb{P}_\lambda \) denote the corresponding law. Often, we will suppress \( \lambda \) from the notation and write \( \mathbb{P} \) instead of \( \mathbb{P}_\lambda \). Here \( \delta_{(L,r)} \) denotes point measure at \( (L,r) \in A(d,1) \times (0,1] \).

We let

\[
V = V(\omega) = \mathbb{R}^d \setminus \bigcup_{(L,r) \in \omega} L + B(o,r)
\]

denote the vacant set of the fractal cylinder process.

As the parameter \( \lambda > 0 \) varies, the random fractal model exhibits several phase transitions. The first that we shall consider is between the empty phase (i.e. where \( V = \emptyset \) a.s.) and the non-empty phase where \( \mathbb{P}(V \neq \emptyset) > 0 \). The critical value corresponding to this phase transition is denoted by \( \lambda_e \) and is defined by

\[
\lambda_e := \inf \{ \lambda > 0 : \mathbb{P}_\lambda(V = \emptyset) = 1 \}.
\]
We refer to this phase transition as the existence phase transition. We observe that one can also consider similar phase transitions of the model restricted to subspaces. Indeed, if we let $H_k := \mathbb{R}^k \times \{0\}^{d-k}$ and we define
\[ \lambda_e(d,k) := \inf \{ \lambda > 0 : \mathbb{P}_\lambda (V \cap H_k = \emptyset) = 1 \}, \]
then our main result concerning $\lambda_e(d,k)$ is as follows.

**Theorem 1.1.** For any $d \geq 2$ and $k \in \{1, \ldots, d\}$ we have that $\lambda_e(d,k) = k$ and
\[ \mathbb{P}_{\lambda_e(d,k)} (V \cap H_k = \emptyset) = 1. \]

**Remark 1.2.** Theorem 1.1 determines the value of $\lambda_e(d,k)$. Furthermore, it states that at the critical value of this phase transition, the model is in the empty phase.

The second phase transition that we will study is the so-called connectivity phase transition. Recall therefore that a set is said to be totally disconnected if the set does not contain any connected components larger than one point. The critical value of the connectivity phase transition is then defined by letting
\[ \lambda_c := \inf \{ \lambda > 0 : \mathbb{P}_\lambda (V \text{ is totally disconnected}) > 0 \}. \] (2)

Some of the earliest results related to the study of this phase transition was obtained in [9] and [13] where the so-called Mandelbrot fractal percolation model was studied. Among other results, it was proven that the critical parameter value was non-trivial. Later, this phase transition was studied for general Poissonian random fractal models with bounded generating sets in [14]. There, the main result was that at the critical threshold, the models were in the connected phase. However, the fact that we are working with unbounded generating objects makes the study of this phase transition much more complicated. Our main result concerning the connectivity phase transition is the following theorem.

**Theorem 1.3.** For $d \geq 4$, we have that $\lambda_c \in (0, \infty)$. When $d = 3$, then for any $\lambda > 0$, $V \cap H_2$ almost surely does not contain any non-trivial connected components while for $d = 2$, we have that $\lambda_c = 0$.

**Remark 1.4.** Obviously, Theorem 1.3 is incomplete in that we do not establish the full result when $d = 3$.

There is a difference between a set being topologically connected and being path connected. In this paper, whenever we refer to a set being connected, we mean in the
topological sense. However, we mention the paper [13] where it is proven that for the two
dimensional Mandelbrot fractal percolation model, the set is path connected whenever it
is topologically connected. It is an open question whether this also holds for our model.

The interesting question what happens at the critical point is an open question, see
Question 8.3 in Section 8. We note that the proof in [14] relies intimately upon the use
of bounded generating sets.

For Poissonian random fractals with bounded radii and non-empty interior, the ana-
logue of Theorem 1.3 can easily be deduced from the basic case of the fractal ball model
(see Theorem 8.1 of [2]). The idea in the case of bounded radii is simply to stochastically
compare the generating sets to balls. This cannot directly be done here, as an unbounded
cylinder cannot be inscribed within a finite ball. Instead, we consider the intersection
of our model with a lower dimensional subspace \( H_k = \mathbb{R}^k \times \{0\}^{d-k} \). This intersection
induces a fractal ellipsoid model on \( H_k \). However, since the intersection of the subspace
with a cylinder can be at any angle, the induced ellipsoid process does not have an upper
bound on the distribution of the diameter. In essence, one will occasionally encounter
thin but extremely long ellipsoids and these are hard to handle. This is why we spend
considerable effort to describe and analyse (Sections 5, 6 and most of the Appendix)
the induced ellipsoid process. The necessity of this sort of analysis is a consequence of
the use of unbounded generating objects. We note that the proof (again, see Theorem
8.1 in [2]) of the corresponding result for the fractal ball model uses a renormalization
argument. At the heart of this technique lies that the random fractal generated in well
separated regions are independent. Since this is not the case here, the idea behind this
proof cannot be adapted to our setting.

We anticipate that these induced processes will be useful for further studies of the
fractal cylinder process, and therefore we chose to state our main results concerning them
here. The result concerns both the fractal cylinder process (described above) as well as
the regular (i.e. non-fractal) cylinder process. This process is a random collection of
cylinders generated by first picking lines \( L \in A(d,1) \) using \( \lambda \nu_d \) as the intensity measure,
and then placing a cylinder \( c(L,r) \) of radius \( r \) around each such line. Here we let \( E_k^o \) denote the set of ellipsoids centred at the origin \( o \), and we let \( \ell_k \) denote \( k \)-dimensional
normalized Hausdorff measure (see further Section 2.1).

**Theorem 1.5.** Consider the Poisson cylinder model in \( \mathbb{R}^d \) with cylinder radius \( r \). The
restriction of this cylinder process to the subspace \( H_k \) where \( k \in \{1, \ldots, d-1\} \) is a Poisson
process of ellipsoids with intensity measure

\[
\lambda_k \times \xi_{k,r}.
\]

Here, \( \xi_{k,r} \) is a measure on \( E_k^o \) given by (17). Furthermore, the restriction of the fractal
cylinder process to \( H_k \) is a Poisson process with intensity measure

\[
\lambda_k \times \xi_k,
\]

where

\[
\xi_k(\cdot) = \int_0^1 \xi_{k,r}(\cdot)r^{-d}\,dr.
\]
Informally, Theorem 1.5 states that the induced model (3) can be described in the following way. The centres of the shapes are picked according to a Poisson process with Lebesgue measure on $\mathbb{R}^k$ as the intensity measure, while the actual shapes are then given by the measure $\xi_{k,r}$ which is supported on ellipsoids. Therefore, Theorem 1.5 determines the intuitive fact that the induced model is indeed an ellipsoid model, and it also gives an explicit description of this model. This description (i.e. 47) is postponed until Section 5, as defining it here would require too much space. Apart from its intrinsic value, Theorem 1.5 will also be used when proving Theorem 1.3.

It should be noted that although we write $\xi_{k,r}$, it also depends on the dimension $d$. The reason for not writing e.g. $\xi_{d,k,r}$ is that we think of $d$ as being fixed, and so adding it to the notation throughout the paper would be unnecessarily cumbersome. See also the comment just below (47).

The last of our main result concerns the Hausdorff dimension (see Section 2.1 for a short overview) of the random fractal set $\mathcal{V}$.

**Theorem 1.7.** For any $\lambda < k$ we have that
\[ P(\dim_H(\mathcal{V} \cap H_k) = k - \lambda) = 1. \]

**Remark 1.8.** As we will see, there is some overlap between Theorem 1.1 and Theorem 1.7. This is discussed in further details in Remark 4.2.

As is often the case when establishing Hausdorff dimensions, the overall approach to proving Theorem 1.7 is going through Frostman’s Lemma. However, while we do use an approach similar to the one used in [15], the technical details are different and we sometimes take a different route. In addition, we establish a 0-1 law (Lemma 2.3) which is only required since we are using unbounded generating objects.

As explained, in some places, we can prove our results by adapting existing approaches, while in other places this is not possible. Perhaps somewhat surprisingly, we never found use for techniques developed explicitly for the Poisson cylinder process developed in [5], [6] and [7].

### 1.3 Outline of the paper

The rest of the paper is structured as follows. In Section 2 we establish notation and define the models studied in this paper. In Section 3 we study the existence phase transition and prove Theorem 1.1 while in Section 4 we establish the almost sure Hausdorff dimension of the set $\mathcal{V}$ by proving Theorem 1.7. It turns out that while the standard representation (see Section 2.3) of the invariant line process is very useful for the study of the cylinder fractal model in dimension $d$, it is not suitable for the study of the restriction of said process onto $H_k$. This is why we in Appendix A.1 find an alternative representation which will be useful to us. This representation is then used in Section 5 to establish Theorem 1.5. The only main result that is left to prove is Theorem 1.3 concerning the connectivity phase transition. As mentioned above, in order to do this we will need to carefully analyse certain statistics of the induced ellipsoid model, and
this is done in Section 6. We then use this in Section 7 in order to prove Theorem 1.3. Finally, Section 8 contains some open questions concerning our model, while the Appendix contains a number of lengthy calculations removed from the main text as they may otherwise unnecessarily interrupt the flow of reading.

2 Models and definitions

In this section, we define the models we study in this paper, and in addition, we introduce much of the notation we shall use later.

2.1 Hausdorff measure and Fractal dimensions

In this subsection we will briefly discuss the concept of Hausdorff measures and Hausdorff dimension, see [16] for further details.

For $F \subset \mathbb{R}^d$, we define the $s$-dimensional Hausdorff measure of $F$ to be

$$\mathcal{H}^s(F) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^s : \{U_i\}_{i \geq 1} \text{ is a } \delta\text{-cover of } F \right\},$$

where $\{U_i\}_{i \geq 1}$ is a $\delta$-cover of $F$ if $\text{diam}(U_i) \leq \delta$ for every $i \geq 1$ and $F \subset \bigcup_{i=1}^{\infty} U_i$. We then define the normalized Hausdorff measure $\ell_k(\cdot) := c \mathcal{H}^k(\cdot)$ where the constant $c$ is chosen so that $\ell_k([0,1]^k \times \{0\}^{d-k}) = 1$. When constructed in this way, the measure $\ell_k$ is a measure on $\mathbb{R}^d$. However, it will be convenient to not reference this fact. Thus we will write $\ell_k([0,1]^k)$ or $\ell_{d-1}(S)$ if $S$ is a $d-1$-dimensional surface in $\mathbb{R}^d$ etc.

Next, the Hausdorff dimension of the set $F$ is defined to be

$$\dim_{\mathcal{H}}(F) := \inf\{s > 0 : \mathcal{H}^s(F) = 0\} = \sup\{s > 0 : \mathcal{H}^s(F) = \infty\}.$$

2.2 General notation

Throughout, if $A \subset \mathbb{R}^d$, we let $A^r$ be the closed Euclidean $r$-neighbourhood of $A$. Furthermore, the $L^2$-norm on $\mathbb{R}^d$ is denoted by $\|\cdot\|$ so that $B(x,r) = \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ is the closed ball centred at $x$ and with radius $r$. In a few places it will be important to emphasize the dimension $d$, and in those places we shall write $B^d(x,r)$ in place of $B(x,r)$.

Recall that for $k \in \{1, \ldots, d-1\}$ we define $H_k$ to be $\mathbb{R}^k \times \{0\}^{d-k}$. With a slight abuse of notation, we will routinely identify $H_k$ with $\mathbb{R}^k$. We let $e_1, \ldots, e_d$ denote the standard basis of $\mathbb{R}^d$.

The unit sphere in $\mathbb{R}^d$ is the boundary of $B(o,1)$, and we denote this by $\partial B(o,1)$. The following expression will surface often, and so we let

$$\psi_{d-1} := \ell_{d-1}(\partial B^d(o,1)) = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

where $\Gamma$ is the Gamma function.

We note for future reference that

$$\ell_d(B^d(o,1)) = \frac{\pi^{d/2}}{\Gamma(1+d/2)} = \frac{\psi_{d-1}}{d}.$$
and that
\[ \psi_{l+1} = \frac{2\pi}{l} \psi_{l-1}. \] (7)

We use the following convention for constants. With \( c \) and \( c' \) we denote strictly positive constants which might only depend on the dimensions \( d \) and \( k \leq d \). Its value might change from place to place. If a constant depend on other quantities than \( d \) or \( k \), this will be indicated. For example, \( c(\lambda) \) stands for a constant depending on \( k, d \) and \( \lambda \). Numbered constants \( C_1, \ldots, C_3 \) and \( c_1, \ldots, c_{10} \) are defined where they first appear and keep their values throughout the paper.

2.3 Lines and cylinders

Let \( A(d,1) \) be the set of all bi-infinite lines in \( \mathbb{R}^d \), and let \( G(d,1) \) be the set of of all bi-infinite lines in \( \mathbb{R}^d \) containing the origin. That is, \( A(d,1) \) is the set of all 1-dimensional affine subspaces of \( \mathbb{R}^d \), while \( G(d,1) \) is the set of all 1-dimensional linear subspaces of \( \mathbb{R}^d \). For any measurable set \( A \subset \mathbb{R}^d \) we let \( L_A \) denote the set of lines that intersects \( A \), that is \( L_A = \{ L \in A(d,1) : L \cap A \neq \emptyset \} \). For \( A, B \subset \mathbb{R}^d \), let \( L_{A,B} = L_A \cap L_B \) be the set of lines intersecting both \( A \) and \( B \). On \( A(d,1) \) there is a unique (up to constants) Haar measure, which we denote by \( \nu_d \). Furthermore, we shall assume that \( \nu_d \) is normalized so that \( \nu_d(L_{B(o,1)}) = 1 \).

There are several ways of explicitly expressing the measure \( \nu_d \) and we will give an informal description, starting from \( G(d,1) \), of one of the most frequently used. Clearly, any line \( L \in G(d,1) \) which is not parallel to \( H_{d-1} \) can be uniquely identified with the point of intersection between \( L \) and the “upper hemisphere” \( \partial B^+(o,1) := \{ x \in \partial B(o,1) : x_d > 0 \} \). We can then define an invariant measure on \( G(d,1) \) by letting the measure of any (measurable) \( L \subset G(d,1) \) be equal to
\[ \frac{2}{\psi_{d-1}} \ell_{d-1}(x \in \partial B^+(o,1) : \exists L \in L, x \in L). \]

Morally, the invariant measure on \( G(d,1) \) is then simply normalized surface measure on \( \partial B^+(o,1) \) and so with a slight abuse of notation we will write it as
\[ \frac{2d \ell_{d-1}(L)}{\psi_{d-1}}. \] (8)

Note that this measure on \( G(d,1) \) assigns zero measure to the set of lines parallel to \( H_{d-1} \) (see also the related discussion after the statement of Lemma 2.2), and note further that the total measure of \( G(d,1) \) is one.

The \( \nu_d \) measure of a set \( A \subset \mathbb{R}^d \) can then be obtained in the following way. Firstly, we integrate over \( G(d,1) \) using the measure from (8), and secondly, for every fixed line \( L \in G(d,1) \) we integrate over those \( y \in L^\perp \) (i.e. the hyperplane orthogonal to \( L \) containing the origin \( o \)) such that \( L + y \) intersects \( A \), using standard \( d-1 \)-dimensional Hausdorff measure on \( L^\perp \).
More formally, the following is a standard representation of the measure \( \nu_d \) (see [17] Theorem 13.2.12 p.588). For any measurable \( A \subset \mathbb{R}^d \) we have that

\[
\nu_d(\mathcal{L}_A) = \frac{4\pi}{\psi_d \psi_{d-1}} \int G_{(d,1)} \int_{L^+} I(L + y \in \mathcal{L}_A) d\ell_{d-1}(y) d\ell_{d-1}(L),
\]  

where here and in the future, \( I(\cdot) \) denotes an indicator function.

Note that if \( A = B^d(o,1) \), then for any fixed \( L \),

\[
\int_{L^+} I(L + y \in \mathcal{L}_{B^d(o,1)}) d\ell_{d-1}(y) = \ell_{d-1}(B^{d-1}(o,1)) = \frac{\psi_{d-2}}{d-1} = \frac{\psi_d}{2\pi},
\]

where we used (7). Combining this with (8) and (9), we see that \( \nu_d(\mathcal{L}_{B(o,1)}) = 1 \) as desired.

For \( L \in A(d,1) \) and \( r > 0 \), we let \( c(L,r) \) denote the open cylinder of base-radius \( r \) centred at \( L \):

\[
c(L,r) = \{ x \in \mathbb{R}^d : \| x - L \| < r \}.
\]

We now state two basic result that we shall make frequent use of in the rest of the paper. The proof of the following lemma is an elementary exercise using (9) and can be found in [18].

**Lemma 2.1.** Let \( r > 0 \) and \( A \subset \mathbb{R}^d \) be a measurable set. We have that

a) for any \( c > 0 \), \( \nu_d(\mathcal{L}_{cA}) = c^{d-1} \nu_d(\mathcal{L}_A) \),

b) \( \nu_d(\mathcal{L}_{B(x,r)}) = r^{d-1} \),

c) \( \nu_d(\mathcal{L}_{B(x,r),B(y,r)}) = r^{d-1} \nu_d(\mathcal{L}_{B(x',1),B(y',1)}) \),

where \((x',y')\) is any pair of points such that \( \| x - y \| = r \| x' - y' \| \).

The next Lemma gives us estimates for the measure of the set of lines intersecting two distant balls. The proof can be found in [5].

**Lemma 2.2.** Let \( x_1, x_2 \in \mathbb{R}^d \). Then there exists constants \( c_1 \) and \( c_2 \) depending only on \( d \) such that

\[
\frac{c_1}{\| x_1 - x_2 \|^{d-1}} \leq \nu_d(\mathcal{L}_{B(x_1,1),B(x_2,1)}) \leq \frac{c_2}{\| x_1 - x_2 \|^{d-1}},
\]

for every pair \( x_1, x_2 \) such that \( \| x_1 - x_2 \| \geq 4 \).

Next, we describe the parametrization of lines which we will use in this paper. Given \((a,p) \in (\mathbb{R}^{d-1} \times \{1\}) \times (\mathbb{R}^{d-1} \times \{0\})\), we let \( L(a,p) := \{ at + p : t \in \mathbb{R} \} \). Observe that \( L(\cdot, p) \) is a bijection between \((\mathbb{R}^{d-1} \times \{1\}) \times (\mathbb{R}^{d-1} \times \{0\})\) and \( \hat{A}(d,1) \), where \( \hat{A}(d,1) \) is the set of lines in \( A(d,1) \) not parallel to \( H_{d-1} \). Since \( \nu_d(\hat{A}(d,1) \setminus \hat{A}(d,1)) = 0 \), we can disregard lines in \( A(d,1) \setminus \hat{A}(d,1) \) and therefore the discrepancy between \( A(d,1) \) and \( \hat{A}(d,1) \) will hereafter be ignored. Note that the line \( L(a,p) \) intersects \( H_{d-1} \) at the point \( p \), and the vector \( a \) describes the direction of the line.
For reasons that will be clear later, we will often consider different parts of the
vectors $a$ and $p$ separately. First, we write $a = (a_1, \ldots, a_{d-1}, 1)$ and $p = (p_1, \ldots, p_{d-1}, 0)$. For $k \in \{1, \ldots, d - 1\}$, we then let $a^{(k)} := (a_1, \ldots, a_k)$, $a^{(k)} := (a_{k+1}, \ldots, a_{d-1})$, $p^{(k)} := (p_1, \ldots, p_k)$ and $p^{(k)} := (p_{k+1}, \ldots, p_{d-1})$. With a slight abuse of notation, we then have $a = (a^{(k)}, a^{(k)}, 1)$ and $p = (p^{(k)}, p^{(k)}, 0)$. Note that if $k = d - 1$, then both $a^{(k)}$ and $p^{(k)}$ are empty. Depending on the situation, we will write $L, L(a, p)$ or $L(a^{(k)}, a^{(k)}, p^{(k)})$.

Using the above described parametrization of lines, $\nu_d$ can be represented as follows. For any measurable $A \subset \mathbb{R}^d$,

$$\nu_d(L_A) = \Upsilon_d \int_{\{(a, p) : L(a, p) \in A\}} \frac{1}{\|a\|^{d+1}} \|a\|^{d+1} \|p\|^d \text{d}a^{(k)} \text{d}p^{(k)}$$

(10)

where

$$\Upsilon_d = \frac{4\pi}{\psi_d \psi_{d-1}}.$$  

(11)

The representation (10) is found on p.211 in [19] for the case $d = 3$. Since we could not locate a reference for the case of general $d \geq 3$, we provide a proof in Appendix A.1.

We note that (9) is useful when performing calculations in the full space $\mathbb{R}^d$. For example, it is used when proving results such as Lemmas 2.1 and 2.2. However, in the latter part of this paper we will focus on restrictions of the cylinder process to subspaces $H_k$, and for this, (10) will be more suitable. Indeed, it is essential when proving Theorem 1.5 and therefore also for the proof of Theorem 1.3.

### 2.4 Ellipsoids

The set of all open ellipsoids in $\mathbb{R}^k$ will be denoted by $\mathcal{E}^k$ while the subset of $\mathcal{E}^k$ consisting of ellipsoids centered at the origin will be denoted by $\mathcal{E}_0^k$. The letters $E$ and $\mathbf{E}$ will typically refer to an ellipsoid and a collection of ellipsoids respectively. For $E \in \mathcal{E}^k$, let $\text{cent}(E)$ be the center of $E$. Moreover, let $E_a = E - \text{cent}(E) \in \mathcal{E}_0^k$. Since an ellipsoid $E \in \mathcal{E}^k$ is uniquely determined by the pair $(\text{cent}(E), E_a) \in \mathbb{R}^k \times \mathcal{E}_0^k$, we will in what follows often identify $\mathcal{E}^k$ with $\mathbb{R}^k \times \mathcal{E}_0^k$. If $L \in \mathcal{L}_{H_k}$, then the intersection $c(L, r) \cap H_k$ induces an ellipsoid in $\mathbb{R}^k$ as follows. Define the map $E_k$ by

$$E_k(c, r) : \quad A(d, 1) \times (0, 1] \rightarrow \mathcal{E}^k \quad (L, r) \rightarrow c(L, r) \cap H_k.$$  

(12)

The fact that $E_k(L, r)$ is indeed an ellipsoid in $\mathbb{R}^k$ is proved in Lemma 5.1. For convenience, we will let $E_k(L, r) = \emptyset$ whenever $L \in \mathcal{L}_{H_k^c}$.

### 2.5 The Poisson cylinder model

As mentioned in the introduction, we will discuss our results in connection with the standard Poisson cylinder model in Section 6.3 so we give its definition before giving the
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definition of the fractal Poisson cylinder modell. We define the following space of point measures on $A(d,1)$:

$$\hat{\Omega} = \left\{ \hat{\omega} = \sum_{i \geq 1} \delta_{L_i} : L_i \in A(d,1), \hat{\omega}(L_K) < \infty, \text{ for every compact } K \subset \mathbb{R}^d \right\}.$$  \hspace{1cm} (13)

where $\delta_L$ denotes point measure at $L$. By a minor abuse of notation, we shall not distinguish the random measure $\hat{\omega} \in \hat{\Omega}$ and its support $\text{supp}(\hat{\omega}) \subset A(d,1)$. Similar comments apply below.

Let $\lambda \in [0,\infty)$ and let $\hat{P}_\lambda$ denote the law of a Poisson point process on $\hat{\Omega}$ with intensity measure $\lambda \nu_d$. In the Poisson cylinder model with intensity $\lambda$ and radius $r \geq 0$, we first choose $\hat{\omega}$ from $\hat{\Omega}$ according to $\hat{P}_\lambda$. Then, around each line in $\hat{\omega}$, we center a cylinder of base-radius $r$ and consider the random set $\hat{C}$ of $\mathbb{R}^d$ consisting of the union of all these cylinders:

$$\hat{C} = \hat{C}(\hat{\omega}) = \bigcup_{L \in \hat{\omega}} \epsilon(L,r),$$

and the corresponding vacant set $\hat{V} = \mathbb{R}^d \setminus \hat{C}$. In Section 6.3 we describe how to use results obtained in this paper to give an alternative proof of a result from [5] concerning percolation in $\hat{V}$.

2.6 The fractal Poisson cylinder model

We now introduce the object of main interest in this paper: the fractal Poisson cylinder model. Consider the space of point measures on $A(d,1) \times (0,1]$:

$$\Omega = \left\{ \omega = \sum_{i \geq 1} \delta_{(L_i,r_i)} \text{ where } (L_i,r_i) \in A(d,1) \times (0,1] \right. \hspace{1cm} \text{and} \hspace{1cm} \omega(L_K \times K') < \infty \text{ for all compact } K \subset \mathbb{R}^d, K' \subset (0,1]. \right\}$$

For $\lambda > 0$, we let $\hat{P}_\lambda$ denote the law of a Poisson process on $\Omega$ with intensity measure $\lambda \nu_d \times \varrho_s$ where $\varrho_s$ is a measure on $(0,1]$ defined by

$$d\varrho_s(r) = I(0 < r \leq 1)r^{-d}dr.$$ \hspace{1cm} (14)

Let $K \subset \mathbb{R}^d$, and consider the set $L_K \times (b,c]$ with $0 < b < c \leq 1$. For $\epsilon > 0$, we observe that it follows from [9] that $\nu_d(L_{\epsilon K}) = e^{d-1}\nu_d(L_K)$. Therefore, for $0 < \epsilon \leq 1/c$ we have that

$$\nu_d \times \varrho_s(L_{\epsilon K} \times (eb,ec]) = \nu_d(L_{\epsilon K}) \int_{eb}^{ec} r^{-d}dr = e^{d-1}\nu_d(L_K) \frac{e^{1-d}(c^{1-d} - b^{1-d})}{1 - d} = \nu_d(L_K) \int_b^c r^{-d}dr = \nu_d \times \varrho_s(L_K \times (b,c]),$$
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where we used Lemma 2.1 part a) in the second equality. It follows that \( \nu_d \times \varrho_s \) is a semi-scale invariant measure (it is semi-scale invariant rather than fully scale invariant since there is an upper cut off on the radius of the cylinders). As above, we will frequently abuse notation and write \( (L,r) \in \omega \) rather than \( (L,r) \in \operatorname{supp}(\omega) \).

Given \( \omega \) picked according to \( P_{\lambda} \), we then define the covered region as

\[
C(\omega) = \bigcup_{(L_i,r_i) \in \omega} c(L_i,r_i),
\]

and the vacant region as

\[
V(\omega) = \mathbb{R}^d \setminus C(\omega).
\]

We shall often write simply \( C \) in place of \( C(\omega) \) and similarly for \( V \). Note that it follows from the semi-scale invariance that \( P(o \in V) = 0 \), and so the set \( V \) is a (semi-)scale invariant fractal set.

2.7 A 0-1 law

Let \( T_t \) denote a shift in the direction of \( t \in \mathbb{R}^d \). Then, let \( S_t : \Omega \to \Omega \) be the induced transformation defined by the equation

\[
S_t \omega = \bigcup_{(L,r) \in \omega} \delta_{(L+t,r)}.
\]

Then, for any measurable event \( F \subset A(d,1) \times (0,1] \) we define \( S_t(F) := \bigcup_{\omega \in F} S_t(\omega) \). Furthermore, an event is called shift-invariant if for any \( t \in \mathbb{R}^d \), we have that \( S_t(F) = F \). We have the following result.

**Lemma 2.3.** For any shift invariant event \( F \) we have that

\[
P(F) \in \{0,1\}.
\]

The proof of Lemma 2.3 is given in Appendix A.2.

3 The existence phase transition

The purpose of this section is to prove Theorem 1.1. The general strategy of the proof is similar to the one used in [20], and we will prove Theorem 1.1 by considering a lower and upper bound on \( \lambda_e \) separately. The lower bound is given in Proposition 3.3 and and its proof uses a second moment method. The upper bound is dealt with in Proposition 3.5 and is basically a first moment approach. However, some extra steps are needed in order to establish that the random fractal dies out at the critical point.

We will now introduce some notation that is used in this section. First, let

\[
\chi_n^k := \left\{ x + [0,2^{-n}]^d : x \in \left(2^{-n}\mathbb{Z}^k \times \{0\}^{d-k}\right) \cap [0,1 - 2^{-n}]^d \right\},
\]
so that $\chi^k_n$ contains $2^{kn}$ $d$-dimensional sub-boxes of $[0,1]^k \times [0,2^{-n}]^{d-k}$ with side length $2^{-n}$ and with non-overlapping interior. For any $k$, we will call an element $X \in \chi^k_n$ a level $n$ box. Furthermore, let

$$\omega_n := \{(L,r) \in \omega : 2^{-n} \leq r \leq 1\}$$

and define

$$\mathcal{V}_n := \mathbb{R}^d \setminus \bigcup_{(L,r) \in \omega_n} c(L,r).$$

We immediately see that $\mathcal{V}_n \downarrow \mathcal{V}$. Furthermore, for $0 \leq m < n$ we define

$$\mathcal{V}_{m,n} := \mathbb{R}^d \setminus \bigcup_{(L,r) \in \omega_n \setminus \omega_m} c(L,r).$$

Next, let

$$m^k_n := \{X \in \chi^k_n : \not\exists (L,r) \in \omega_n \text{ such that } c(L,r) \cap X \neq \emptyset\},$$

be the random set of level $n$ boxes in $\chi^k_n$ that are untouched by the Poisson process $\omega_n$. Moreover, let

$$M^k_n := \{X \in \chi^k_n : \not\exists (L,r) \in \omega_n \text{ such that } X \subset c(L,r)\}$$

be the random set of level $n$ boxes in $\chi^k_n$ not covered by a single cylinder.

Our first result is a proposition which establishes a number of preliminary estimates. These will be useful when we prove Theorem 1.1 and also for proving Theorem 1.7.

**Proposition 3.1.** For all $1 \leq k \leq d$, $X \in \chi^k_n$ and $x,y \in [0,1]^d$ we have that

- $a) \ P(X \in m^k_n) \geq e^{-\lambda C_1} 2^{-\lambda n};$
- $b) \ P(x \in \mathcal{V}_n) = 2^{-\lambda n};$
- $c) \ P(x,y \in \mathcal{V}_n) \leq e^{\lambda C_2} 2^{-2\lambda n} \|x-y\|^{-\lambda};$
- $d) \ P(X \in M^k_n) \leq e^{\lambda C_3} 2^{-\lambda n}$ whenever $2^{-n} \sqrt{d} \leq 1.$

Here, $C_1, C_2$ and $C_3$ are constants depending on $d$ only.

**Proof.** We will prove the statements in order.

Part a): Begin by noting that for any $X \in \chi^k_n$ we have that $P(X \in m^k_n) = P([0,2^{-n}]^d \in m^k_n)$ by translation invariance. Moreover, $[0,2^{-n}]^d \subset B(o,2^{-n} \sqrt{d})$ since $B(o,2^{-n} \sqrt{d})$ is a closed ball. Therefore, if the box $[0,2^{-n}]^d$ is hit by a cylinder, then the ball $B(o,2^{-n} \sqrt{d})$ must also be hit by the same cylinder. Hence,

$$P([0,2^{-n}]^d \in m^k_n) \geq P\left(\left\{ (L,r) \in \omega_n : B(o,2^{-n} \sqrt{d}) \cap c(L,r) \neq \emptyset \right\} = \emptyset\right).$$
Now, by Lemma 2.1 part b),
\[
\mathbb{P}\left(\{(L,r) \in \omega_n : B(o,2^{-n}\sqrt{d}) \cap c(L,r) \neq \emptyset\} = \emptyset\right) = \exp\left(-\lambda \nu_d \times g_s\left((L,r) \in \omega_n : B(o,2^{-n}\sqrt{d}) \cap c(L,r) \neq \emptyset\right)\right) \\
= \exp\left(-\lambda \int_{2^{-n}}^1 \nu_d (L_{B(o,r+2^{-n}\sqrt{d})}) r^{-d} dr\right) \\
= \exp\left(-\lambda \int_{2^{-n}}^1 \left(r + 2^{-n}\sqrt{d}\right)^{\frac{d-1}{2}} r^{-d} dr\right).
\]

We have that
\[
\int_{2^{-n}}^1 \left(r + 2^{-n}\sqrt{d}\right)^{\frac{d-1}{2}} r^{-d} dr \\
= \int_{2^{-n}}^1 \sum_{i=0}^{d-1} \binom{d-1}{i} r^{d-i-1} \left(2^{-n}\sqrt{d}\right)^i r^{-d} dr \\
= \sum_{i=0}^{d-1} \binom{d-1}{i} \left(2^{-n}\sqrt{d}\right)^i \int_{2^{-n}}^1 r^{d-1-i} dr \\
= n \log 2 + \sum_{i=1}^{d-1} d/2 \binom{d-1}{i} \left(\frac{1}{i} - \frac{2^{-ni}}{i}\right) \\
\leq n \log 2 + \sum_{i=1}^{d-1} d/2 \binom{d-1}{i} = n \log 2 + C_1.
\]

Plugging this into (15) gives the desired result.

Part b): Using Lemma 2.1 part b),
\[
\mathbb{P}(x \in \mathcal{V}_n) = \exp\left(-\lambda \int_{2^{-n}}^1 \int_{A(d,1)} \mathbf{1}\{(L \in A(d,1) : x \in c(L,r))\} d\nu_d(L) d\varrho_s(r)\right) \\
= \exp\left(-\lambda \int_{2^{-n}}^1 \nu_d(L_{B(x,r)}) r^{-d} dr\right) = \exp\left(-\lambda \int_{2^{-n}}^1 r^{-d} dr\right) = 2^{-\lambda n}.
\]

Part c): Let \(\mathcal{P}_x^n = \{(L,r) \in A(d,1) \times (2^{-n},1] : x \in c(L,r)\}\) denote the set of cylinders with radii in \((2^{-n},1]\) that covers the point \(x\). We have that
\[
\mathbb{P}(x,y \in \mathcal{V}_n) = \mathbb{P}(\{(L,r) \in \omega_n : x \in c(L,r)\} = \emptyset, \{(L,r) \in \omega_n : y \in c(L,r)\} = \emptyset) \\
= \exp\left(-\lambda \nu_d \times g_s\left(\mathcal{P}_x^n \cup \mathcal{P}_y^n\right)\right) = \exp\left(-\lambda (2\nu_d \times g_s(\mathcal{P}_x^n) - \nu_d \times g_s(\mathcal{P}_x^n \cap \mathcal{P}_y^n))\right) \\
= 2^{-2\lambda n} \exp\left(\lambda \nu_d \times g_s(\mathcal{P}_x^n \cap \mathcal{P}_y^n)\right),
\]
where we used translation invariance and part b) of the current lemma. We now turn our attention to \( \nu_d \times g_s(\mathcal{P}_x^0 \cap \mathcal{P}_y^0) \). As before, note that

\[
\nu_d \times g_s(\mathcal{P}_x^0 \cap \mathcal{P}_y^0) = \int_{2^{-n}}^1 \int_{A(d,1)} I(\mathcal{P}_x^0 \cap \mathcal{P}_y^0) \, d\nu_d(L) \, d\eta_s(r)
\]

(19)

where we used Lemma 2.1 part c) and where \( x', y' \) are such that \( \|x' - y'\| = \|x - y\|/r \). Because of this, we have that for fixed \( x, y \) the “effective” distance \( \|x' - y'\| \) will be large whenever \( r \) is small. Furthermore, if \( r \leq \|x - y\|/4 \), then \( \|x' - y'\| = \|x - y\|/r \geq 4 \) and in this case we can use Lemma 2.2 to obtain that

\[
\nu_d (\mathcal{L}_B(x',1),B(y',1)) \leq c_2 \|x' - y'\|^{1-d}.
\]

It is therefore natural to split the integral on the right hand side of (19) into two parts. The first integral will be over small \( r \) so that \( x', y' \) are well separated, while the second integral part is over larger \( r \). Now, if \( \|x - y\| \geq 2^{-n} \) we can write

\[
\int_{2^{-n}}^1 r^{-1} \nu_d (\mathcal{L}_B(x',1),B(y',1)) \, dr
\]

(20)

\[
= \int_{2^{-n}}^{\|x - y\|/4} r^{-1} \nu_d (\mathcal{L}_B(x',1),B(y',1)) \, dr + \int_{\|x - y\|/4}^1 r^{-1} \nu_d (\mathcal{L}_B(x',1),B(y',1)) \, dr.
\]

By Lemma 2.2 we have that \( \nu_d (\mathcal{L}_B(x',1),B(y',1)) \leq c_2 \|x' - y'\|^{1-d} = c_2 \|x - y\|^{1-d} r^{-1} \) and then

\[
\int_{2^{-n}}^{\|x - y\|/4} r^{-1} \nu_d (\mathcal{L}_B(x',1),B(y',1)) \, dr \leq c_2 \|x - y\|^{1-d} \int_{2^{-n}}^{\|x - y\|/4} r^{-d-2} \, dr
\]

(21)

\[
= \frac{c_2}{d-1} \|x - y\|^{1-d} \left( \left( \frac{\|x - y\|}{4} \right)^{d-1} - 2^{-n(d-1)} \right) \leq \frac{c_2}{(d-1)^{d-1}}.
\]

We will now provide a bound to the second term in the right hand side of Equation (20). Start by noting that \( \mathcal{L}_B(x',1),B(y',1) \subset \mathcal{L}_B(x',1) \) and \( \nu_d (\mathcal{L}_B(x',1)) = 1 \) so that \( \nu_d (\mathcal{L}_B(x',1),B(y',1)) \leq 1 \). Therefore

\[
\int_{\|x - y\|/4}^1 r^{-1} \nu_d (\mathcal{L}_B(x',1),B(y',1)) \, dr \leq \int_{\|x - y\|/4}^1 r^{-1} \, dr = - \log(\|x - y\|/4).
\]

(22)

Using Equations (20), (21) and (22) we conclude that

\[
\int_{2^{-n}}^1 r^{-1} \nu_d (\mathcal{L}_B(x',1),B(y',1)) \, dr \leq \frac{c_2}{(d-1)^{d-1}} - \log(\|x - y\|/4).
\]

(23)
On the other hand, if \( \|x - y\| < 2^{2-n} \) then \( \|x - y\|/4 < 2^{-n} \), and so we can estimate (20) by

\[
\int_{2^{2-n}}^{1} r^{-1} \nu_d (L_{B(x',1),B(y',1)}) \, dr \\
\leq \int_{\|x-y\|/4}^{1} r^{-1} \nu_d (L_{B(x',1),B(y',1)}) \, dr \leq - \log(\|x - y\|/4),
\]

where the last inequality comes from (22). Thus, we see that (23) holds for all \( \|x - y\| > 0 \).

Combining (18), (19) and (23) we obtain

\[
P(x,y \in \mathcal{V}_n) \leq 2^{-2\lambda n} \exp \left( \lambda \left( \frac{C}{(d-1)(d-1)} - \log(\|x - y\|/4) \right) \right) = e^{\lambda C_3 2^{-2\lambda n} \|x - y\|^{-\lambda}},
\]

as desired.

Part d): We now assume that \( 2^{-n} \sqrt{d} \leq 1 \). Let \( p \) be the center point of the box \( X \). Clearly, if \( X \) is not covered by a single cylinder, then the ball \( B(p,2^{-n} \sqrt{d}) \) circumscribing it cannot be covered by a single cylinder. Furthermore, note that \( B(p,2^{-n} \sqrt{d}) \subset c(L,r) \) is equivalent to the condition that \( L \) intersects the ball \( B(p, r - 2^{-n} \sqrt{d}) \). Therefore,

\[
\mathbb{P}(X \in \mathcal{M}_n^k) \leq \mathbb{P} \left( \{(L,r) \in \omega_n : B(p,2^{-n} \sqrt{d}) \subset c(L,r)\} = \emptyset \right) \\
= \exp \left( -\lambda \int_{2^{-n} \sqrt{d}}^{1} r^{-1} \nu_d (L_{B(p, r - 2^{-n} \sqrt{d})}) r^{-d} \, dr \right) \\
= \exp \left( -\lambda \int_{2^{-n} \sqrt{d}}^{1} r^{-d} \, dr \right).
\]

Now, note that

\[
\int_{2^{-n} \sqrt{d}}^{1} r^{-d} \, dr \\
= \int_{2^{-n} \sqrt{d}}^{1} \left( \sum_{i=0}^{d-1} \binom{d-1}{i} r^{d-1-i} \left( -2^{-n} \sqrt{d} \right)^i \right) r^{-d} \, dr \\
= n \log 2 - \frac{\log d}{2} + \sum_{i=1}^{d-1} \binom{d-1}{i} \frac{(-1)^i}{i} (1 - 2^{-in} d^{i/2}) \\
\geq n \log 2 - \frac{\log d}{2} - \sum_{i=1, i \text{ odd}}^{d-1} \binom{d-1}{i} \frac{1}{i} \\
= n \log 2 - C_3.
\]

Thus we have that,

\[
\mathbb{P}(X \in \mathcal{M}_n^k) \leq \exp (-\lambda n \log 2 + \lambda C_3) = e^{\lambda C_3 2^{-\lambda n}},
\]

as desired.
Remark 3.2. Although we do not need to know the values of $C_1$, $C_2$ and $C_3$ for the results of this paper, we observe for possible future reference that

\[ C_1 = \sum_{i=1}^{d-1} \frac{d^{i/2}}{i} \left( \frac{d-1}{i} \right), \quad C_2 = \frac{e_2}{(d-1)4^{d-1}} + \log 4 \quad \text{and} \quad C_3 = \frac{\log d}{2} + \sum_{i=1}^{d-1} \left( \frac{d-1}{i} \right)^{1/2}. \]  

(25)

Throughout the rest of this section and Section 2.1, it will be convenient to use the notation $V^k := \mathcal{V} \cap ([0,1]^k \times \{0\}^{d-k})$ and $V^k_n := \mathcal{V}_n \cap ([0,1]^k \times \{0\}^{d-k})$. We will also let $V_{m,n}^k := \mathcal{V}_{m,n} \cap ([0,1]^k \times \{0\}^{d-k})$ when $m < n$.

We are ready to prove the following proposition which provides us with one direction of Theorem 1.1.

**Proposition 3.3.** For $d \geq 2$ and any $1 \leq k \leq d$, we have that $\lambda_c(d,k) \geq k$.

**Proof.** Start by noting that since the cylinders in the process are open sets, the sets $V^k_n$ are compact for all $n$. Moreover, we have that $V^k_n \supset V^k_{n+1}$ for every $n$. Therefore we conclude that if $m_n^k \neq \emptyset$ for infinitely many $n \geq 1$, then $V^k_n \neq \emptyset$ for every $n \geq 1$. It then follows by compactness that

\[ V^k = \bigcap_{n=1}^{\infty} V^k_n \neq \emptyset. \]

Thus, if we prove that for any $\lambda < k$,

\[ \mathbb{P}(|m_n^k| > 0 \text{ infinitely often}) \geq c > 0, \]  

(26)

where $c$ depends only on $\lambda$ and $d$, we would get that $V^k \neq \emptyset$ with positive probability, as desired (here $|\cdot|$ stands for cardinality).

Observe that by the reverse Fatou’s Lemma,

\[ \mathbb{P}(|m_n^k| > 0 \text{ infinitely often}) \geq \limsup \mathbb{P}(|m_n^k| > 0), \]

and so it suffices to show that there exists $c = c(\lambda) > 0$ such that

\[ \mathbb{P}(|m_n^k| > 0) \geq \frac{\mathbb{E}(|m_n^k|)^2}{\mathbb{E}(|m_n^k|^2)} \geq c \]

uniformly in $n$ (where we used the second moment method for the first inequality).

It follows from part a) of Proposition 3.1 that

\[ \mathbb{E}(|m_n^k|) = 2^{nk} \mathbb{P}([0,2^{-n}]^d \in m_n^k) \geq e^{-\lambda C_1} 2^{(k-\lambda)n}. \]  

(27)

We proceed to provide an upper bound to $\mathbb{E}(|m_n^k|^2)$. We have that

\[ \mathbb{E}(|m_n^k|^2) = \sum_{X_1, X_2 \in \chi_n^k} \mathbb{P}(X_1, X_2 \in m_n^k). \]
We split the sum into two parts:

\[
\sum_{X_1, X_2 \in \chi_n^k} \mathbb{P}(X_1, X_2 \in m_n^k) = \sum_{X_1, X_2 \in \chi_n^k \atop \|p_1 - p_2\| \leq 2^{-n+2}} \mathbb{P}(X_1, X_2 \in m_n^k) + \sum_{X_1, X_2 \in \chi_n^k \atop \|p_1 - p_2\| > 2^{-n+2}} \mathbb{P}(X_1, X_2 \in m_n^k),
\]

where \( p_i = p_i(X_i) \) is the center point of the box \( X_i \). This split will be necessary because the techniques used to bound \( \mathbb{P}(X_1, X_2 \in m_n^k) \) depends on the distance between the boxes. In order to provide an upper bound to \( \mathbb{P}(X_1, X_2 \in m_n^k) \) we only need to provide an upper bound to the probability that the center points \( p_1, p_2 \) of the boxes \( X_1, X_2 \) are not hit by any cylinders in \( \omega_n \). In fact, any pair of points \( (p_i, p_j) \in X_1 \times X_2 \) would do.

We will begin by providing a bound to the first sum on the right hand side of (28).

Note that

\[
\sum_{X_1, X_2 \in \chi_n^k \atop \|p_1 - p_2\| \leq 2^{-n+2}} \mathbb{P}(X_1, X_2 \in m_n^k) \leq \sum_{X_1 \in \chi_n^k} \sum_{X_2 \in \chi_n^k \atop \|p_1 - p_2\| \leq 2^{-n+2}} \mathbb{P}(X_2 \in m_n^k).
\]

For a fixed \( X_1 \) (and therefore fixed \( p_1 \)), there are at most \( 9^k \) different \( X_2 \in \chi_n^k \) (or centres \( p_2 \)) such that \( \|p_1 - p_2\| \leq 2^{-n+2} \) (there can be fewer if for instance \( X_1 = [0,2^{-n}]^k \times \{0\}^{d - k} \)). Therefore we can use Proposition 3.1 part b) to see that

\[
\sum_{X_2 \in \chi_n^k \atop \|p_1 - p_2\| \leq 2^{-n+2}} \mathbb{P}(X_2 \in m_n^k) \leq 9^k \mathbb{P}(X_2 \in m_n^k) \leq 9^k \mathbb{P}(p_2 \in V_n) = 9^k 2^{-\lambda n}.
\]

Hence,

\[
\sum_{X_1, X_2 \in \chi_n^k \atop \|p_1 - p_2\| \leq 2^{-n+2}} \mathbb{P}(X_1, X_2 \in m_n^k) \leq 9^k 2^{(k-\lambda)n},
\]

since we have \( 2^{kn} \) possible choices for \( X_1 \).

We now turn our attention to the second sum in the right hand side of (28). Observe therefore that for fixed \( X_1 \) we have that

\[
\sum_{X_2 \in \chi_n^k \atop \|p_1 - p_2\| > 2^{-n+2}} \mathbb{P}(X_1, X_2 \in m_n^k) \leq \sum_{l=2}^{n-1} \sum_{X_2 \in \chi_n^k \atop 2^l 2^{-n} \leq \|p_1 - p_2\| \leq 2^{l+1} 2^{-n}} \mathbb{P}(p_1, p_2 \in V_n),
\]

and that the number of \( X_2 \in \chi_n^k \) satisfying \( 2^l 2^{-n} \leq \|p_1 - p_2\| \leq 2^{l+1} 2^{-n} \) can be bounded by

\[
(2 \cdot 2^{l+1} + 1)^k - (2 \cdot 2^l + 1)^k \leq 3^k 2^{kl} \leq 9^k 2^{kl}.
\]
Therefore, we can use part c) of Proposition 3.1 to conclude that

$$\sum_{X_1, X_2 \in \chi^k_n} \mathbb{P}(X_1, X_2 \in m^k_n) \leq 2^{kn} \sum_{l=2}^{n-1} \sum_{X_2 \in \chi^k_n} \mathbb{P}(p_1, p_2 \in V_n)$$

(30)

Using (28), (29) and (30) we get that

$$\mathbb{E}(|m^k_n|^2) = \sum_{X_1, X_2 \in \chi^k_n} \mathbb{P}(X_1, X_2 \in m^k_n) \leq g^k 2^{(k-\lambda)n} + \frac{e^{\lambda C_2} g^k}{2^{k-\lambda - 1}} 2^{2(k-\lambda)n}.$$

We combine this with (27) to conclude that

$$\mathbb{P}(|m^k_n| > 0) \geq \frac{\mathbb{E}(|m^k_n|^2)}{\mathbb{E}(|m^k_n|^2)} \geq \frac{e^{-2\lambda C_1} 2^{2(k-\lambda)n}}{g^k 2^{(k-\lambda)n} + \frac{e^{\lambda C_2} g^k}{2^{k-\lambda - 1}} 2^{2(k-\lambda)n}} = \frac{e^{-2\lambda C_1} (g^k - 1)}{g^k 2^{(k-\lambda)n} (2^{k-\lambda - 1}) + e^{\lambda C_2} g^k}.$$

Since $\lambda < k$ we see that $\mathbb{P}(|m^k_n| > 0)$ is uniformly bounded away from zero.

We will now prove that $\lambda_0(d,k) \leq k$ and that we are in the empty phase at the critical point $\lambda_0 = k$. For that, we will need some additional results. Let $D_n \subset \chi^d_n$ be a minimal covering of $V_n^d$, that is, $V_n^d \subset \cup_{X \in D_n} X$ and let $D^k_n = D_n \cap \chi^k_n$. The choice of $D_n$ is not necessarily unique since a point belonging to the boundary of a box $X \in \chi_n$ can be covered by more than one box. We assume therefore that $D_n$ is picked according to some predetermined rule. Note also that for any $k$ we must have that $V_n^k \subset \cup_{X \in D_n^k} X$. Obviously, $D_n^k$ depends on the configuration $\omega_n$, and therefore we sometimes emphasise this by writing $D_n^k(\omega_n)$ or similar.

The next lemma relates the event $V^k \neq \emptyset$ with the limiting behaviour of $|D^k_n|$. The lemma is similar to Lemma 3.2 in [20] and Lemma 2.1 in [11], but the proof provided here is more detailed.

**Lemma 3.4.** For any $\lambda > 0$ and $1 \leq k \leq d$ we have that

$$\mathbb{P}(\{V^k \neq \emptyset\} \setminus \{\lim_{n \to \infty} |D^k_n| = \infty\}) = 0.$$

**Proof.** We will construct a sequence $(\eta_n)_{n \geq 1}$ in a specific way so that it has the same distribution as $(\omega_n)_{n \geq 1}$. The statement then follows from the details of this construction.

Fix $L < \infty$, and let $A_n$ be the event that $0 < |D^k_n| \leq L$. Note that there exists $\alpha = \alpha(\lambda)$ such that $\mathbb{P}(V^k = \emptyset) = \alpha > 0$. 

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The main idea is to prove that for any such \( L \) the events \( \mathcal{A}_n \) can only occur finitely many times. Intuitively, this is clear since every time \( \mathcal{A}_n \) occurs, there is a uniform positive probability to ”kill” the process.

Let \( \eta_1 \) be a Poisson process with intensity measure \( \lambda \nu_d \times \{2^{-1} < r \leq 1\} r^{-d} dr \), and observe that \( \eta_1 \) has the same distribution as \( \omega_1 \). Then, we let \( \eta_{1,2} \) be a Poisson process with intensity measure \( \lambda \nu_d \times \{2^{-2} < r \leq 2^{-1}\} r^{-d} dr \), and we pick this independent of \( \eta_1 \). Observe that \( \eta_2 := \eta_1 + \eta_{1,2} \) has the same distribution as \( \omega_2 \). If the event \( \mathcal{A}_n \) never occurs, we let \( X_1, X_2, \ldots \in \{0,1\} \) be an i.i.d. sequence of random variables such that \( P(X_1 = 1) = \alpha^L \) for \( i \in \{1,2,\ldots\} \). Otherwise, we continue the procedure of constructing \( \eta_n = \eta_{n-1} + \eta_{n-1,n} \) until the first time \( n_1 \) such that \( \mathcal{A}_n \) occurs.

Assume therefore that \( \mathcal{A}_{n_1} \) occurs. As before, let \( \eta_{n_1,n_1+1} \) be a Poisson process with intensity measure \( \lambda \nu_d \times \{2^{-n_1-1} < r \leq 2^{-n_1}\} r^{-d} dr \), and let

\[
X_1 = I(\gamma_{n_1+1}^k(\eta_{n_1} + \eta_{n_1,n_1+1}) \neq \emptyset).
\]

We note that for any \( \eta_1 \in \mathcal{A}_{n_1} \) we have that

\[
P(\gamma_{n_1+1}^k(\eta_{n_1} + \eta_{n_1,n_1+1}) = \emptyset|\eta_{n_1}) \geq P(\gamma_{n_1+1}^k(\eta_{n_1,n_1+1}) \cap D_{n_1}^k(\eta_{n_1}) = \emptyset|\eta_{n_1})
\]

\[
\geq \prod_{X \in D_{n_1}^k} P(\gamma_{n_1,n_1+1}(\eta_{n_1,n_1+1}) \cap X = \emptyset) = P(\gamma_{n_1}^k = \emptyset)^{|D_{n_1}^k|} = \alpha^{|D_{n_1}^k|} \geq \alpha^L,
\]

and so \( P(\gamma_{n_1}^k = \emptyset) \geq \alpha^L \). Here, the second inequality follows by the FKG inequality for Poisson processes, while the second to last equality follows by scale invariance.

We then proceed as follows. If \( \mathcal{A}_n \) never occurs again, we let \( X_2, X_3, \ldots \in \{0,1\} \) be an i.i.d. sequence such that \( P(X_i = 1) = \alpha^L \) for \( i \in \{2,3,\ldots\} \). Otherwise, we proceed with the construction until time \( n_2 \), which is the first time after \( n_1 \) such that \( \mathcal{A}_n \) occurs. Then, we construct \( X_2 \) analogously to how we constructed \( X_1 \) above. Note that the dependence between \( X_1 \) and \( X_2 \) is potentially complicated, but that for every \( \eta_{n_2} \in \mathcal{A}_{n_2} \) we must have (as above) that \( P(\gamma_{n_2+1}^k(\eta_{n_2} + \eta_{n_2,n_2+1}) = \emptyset|\eta_{n_2}) \geq \alpha^L \). Therefore,

\[
P(X_1 = 1, X_2 = 1) = E[P(X_2 = 1|X_1 = 1, \eta_{n_2})]P(X_1 = 1) \leq (1 - \alpha^L)^2.
\]

Proceeding in this manner we see that

\[
P(0 < |D_{n}^k| \leq L \text{ i.o.}) = \lim_{l \to \infty} P(X_1 = \cdots = X_l = 1) \leq \lim_{l \to \infty} (1 - \alpha^L)^l = 0.
\]

This implies that \( P(\lim_{n \to \infty} |D_{n}^k| \in \{0,\infty\}) = 1 \). Furthermore if \( \nu^k \neq \emptyset \), then \( \nu_{n}^k \neq \emptyset \) for all \( n \), which implies that \( |D_{n}^k| \neq 0 \) for all \( n \) and so we conclude that \( P(\lim_{n \to \infty} |D_{n}^k| = \infty|\nu^k \neq \emptyset) = 1 \).

\begin{proposition} \label{prop:fractal_cylinder_process}
For \( d \geq 2 \) and \( 1 \leq k \leq d \) we have that \( \lambda_e(d,k) \leq k \). Furthermore,

\[
\mathbb{P}_{\lambda_e(d,k)}(\nu = \emptyset) = 1.
\]
\end{proposition}
Proof. Observe that if \( X \in D^k_n \) then \( X \) is not covered by a single cylinder in the Poisson process and so \( |D^k_n| \leq |M^k_n| \). Therefore, if \( \lambda \) is such that \( \mathbb{P}_\lambda(\mathcal{V}^k \neq \emptyset) > 0 \) we see that by Lemma 3.4 we have that \( \mathbb{P}_\lambda(\lim_{n \to \infty} |D^k_n| = \infty) > 0 \) and so
\[
\lim_{n \to \infty} \mathbb{E}_\lambda(|M^k_n|) \geq \lim_{n \to \infty} \mathbb{E}_\lambda(|D^k_n|) = \infty. \tag{31}
\]

We will now provide an upper bound to \( \mathbb{E}_\lambda(|M^k_n|) \). From Proposition 3.1 part d) we have that
\[
\mathbb{P}_\lambda(X \in M^k_n) \leq e^{\lambda C_2} 2^{-\lambda n}
\]
and so
\[
\mathbb{E}_\lambda(|M^k_n|) \leq 2^{kn} \mathbb{P}_\lambda(X \in M^k_n) \leq e^{\lambda C_2} 2^{(k-\lambda)n}. \tag{32}
\]
The discussion above Equation (31) implies that
\[
\lim_{n \to \infty} e^{\lambda C_2} 2^{(k-\lambda)n} \geq \lim_{n \to \infty} \mathbb{E}_\lambda(|M^k_n|) = \infty,
\]
whenever \( \mathbb{P}_\lambda(\mathcal{V}^k \neq \emptyset) > 0 \). Clearly, this only holds if \( k > \lambda \). We conclude that \( \lambda_c(d,k) \leq k \) and that for \( \lambda = k \) we must have that \( \mathbb{P}_\lambda(\mathcal{V}^k \neq \emptyset) = 0 \).

**Proof of Theorem 1.7**. This follows from Propositions 3.3 and 3.5.

## 4 The Hausdorff dimension of \( \mathcal{V} \)

In this section we will prove Theorem 1.7 which establishes the almost sure Hausdorff dimension of \( \mathcal{V} \cap H_k \). We will prove Theorem 1.7 by establishing the upper and lower bound separately. As usual, it is easier to determine the upper bound, so we will do this first. Recall that the definitions of Hausdorff measure and fractal dimensions are found in Section 2.3.

**Theorem 4.1.** For \( \lambda < k \), we have that \( \dim_H(\mathcal{V} \cap H_k) \leq k - \lambda \) almost surely.

**Proof.** We will prove that \( \dim_H(\mathcal{V}^k) \leq k - \lambda \) almost surely. The statement then follows by tiling \( H_k \) with copies of \([0,1]^k \times \{0\}^{d-k}\) and using a countability argument.

Observe that the set \( M^k_n \) defined before Proposition 3.1 is a \( \sqrt{k} 2^{-n} \)-cover of \( \mathcal{V}^k \). Therefore \( \mathcal{H}^s(\mathcal{V}^k) \leq \sum_{X \in M^k_n} \mathrm{diam}(X)^s = |M^k_n| (\sqrt{k} 2^{-n})^s \). Then, by (32) we have that \( \mathbb{E}[\mathcal{H}^s(\mathcal{V}^k)] \leq (\sqrt{k} 2^{-n})^s \mathbb{E}(|M^k_n|) \leq e^{\lambda C_2} (\sqrt{k})^s 2^{(k-\lambda-s)n} \) for all \( n \). Thus if \( k - \lambda - s < 0 \) then \( \mathbb{E}[\mathcal{H}^s(\mathcal{V}^k)] = 0 \) and since \( \mathcal{H}^s(\mathcal{V}^k) \geq 0 \), we must have that almost surely \( \mathcal{H}^s(\mathcal{V}^k) = 0 \). By the definition of Hausdorff dimension, we conclude that \( \dim_H(\mathcal{V}^k) \leq k - \lambda \) almost surely.

**Remark 4.2.** We note that it does not follow from (an extension of) Theorem 4.1 that \( \lambda_c \leq k \). The reason for this is that for \( \lambda > k \), the set \( \mathcal{V} \cap H_k \) could be non-empty while still having Hausdorff dimension 0. However, it is the case that our next result, Theorem 4.3 implies Proposition 3.3 (which states that \( \lambda_c(d,k) \geq k \)). The reason for providing the proof of Proposition 3.3 is that it is done from first principles while the proof of Theorem 4.3 is much more involved.
The next step is to find a lower bound on the Hausdorff dimension of \( V \cap H_k \). We will do this by establishing that for any \( \lambda < k \), \( \dim_H(V^k) \geq k - \lambda \) with positive probability, and then use Lemma 2.3 to conclude that \( \dim_H(V \cap H_k) \geq k - \lambda \) almost surely. The aim of the rest of this section is to prove the following theorem.

**Theorem 4.3.** For any \( \lambda < k \) we have that
\[
\mathbb{P}(\dim_H(V^k) \geq k - \lambda) > 0.
\]

The proof of Theorem 4.3 will proceed through a number of lemmas and the overall approach is inspired by [15]. As is common when proving lower bounds on Hausdorff dimensions we will be utilizing Frostman’s Lemma (see for example Theorem 4.13 of [16]). This lemma states that if there is a random measure \( \zeta \) supported on \( V^k \) and satisfying \( 0 < \zeta(V^k) < \infty \) with finite \( r \)-energy, that is,
\[
\mathcal{I}_r(\zeta, [0,1]^k) = \int_{[0,1]^k \times [0,1]^k} \frac{d\zeta(x)d\zeta(y)}{\|x - y\|^r} < \infty, \tag{33}
\]
then \( \dim_H(V^k) \geq r \). When useful we will emphasize the dependence of \( \zeta \) on \( \omega \) by writing \( \zeta(\omega) \). Observe that we allow a small abuse of notation by considering \( \zeta \) to be a measure on \( [0,1]^k \) rather than on \([0,1]^k \times \{0\}^{d-k}\). A similar comment applies to many places in this section. The objective is therefore to find a suitable random measure \( \zeta \) as described. This will allow us to conclude that \( \mathbb{P}(\dim_H(V^k) \geq r) \) is uniformly bounded away from 0 for \( r < \lambda - k \). It then follows that also \( \mathbb{P}(\dim_H(V^k) \geq \lambda - k) > 0 \).

The measure \( \zeta \) will be obtained as a limit of a sequence of random measures \((\zeta_n)_{n \geq 1}\). Therefore, we let \( \zeta_n = \zeta_n(\omega) \) be a measure on \([0,1]^k\) defined by
\[
d\zeta_n(x) = 2^{\lambda n}I(x \in V_n^k)dx. \tag{34}
\]
We have the following lemma.

**Lemma 4.4.** Let \( f : [0,1]^k \rightarrow \mathbb{R} \) be a continuous and non-negative function with compact support, and let \( X_n(f) = \int f d\zeta_n \). Then \((X_n(f))_{n \geq 1}\) is a non-negative martingale.

**Proof.** Clearly \( X_n(f) \geq 0 \). Since \( f \) is continuous and with compact support, it is also bounded, and so
\[
0 \leq \mathbb{E}[X_n(f)] = \mathbb{E} \left[ \int f d\zeta_n \right] \leq \sup(f) \mathbb{E} \left[ \int d\zeta_n \right] = \sup(f) 2^{\lambda n} \mathbb{E}[\ell_k(V_n^k)] < \infty,
\]
(recall that \( \ell_k \) denotes \( k \)-dimensional normalized Hausdorff measure). Now let \( 0 \leq m < n \). We have that
\[
\mathbb{E}[X_n(f)|\omega_m] = \mathbb{E} \left[ \int f d\zeta_n|\omega_m \right] = \mathbb{E} \left[ \int f(x) 2^{\lambda n}I(x \in V_n^k)dx|\omega_m \right] = \mathbb{E} \left[ \int f(x) 2^{\lambda n}I(x \in V_m^k)I(x \in V_{m,n}^k)dx|\omega_m \right] = \int f(x) 2^{\lambda n}E[I(x \in V_m^k)|\omega_m]E[I(x \in V_{m,n}^k)|\omega_m]dx = \int f(x) 2^{\lambda n}I(x \in V_m^k)\mathbb{P}(x \in V_{m,n}^k)dx,
\]
where we used the independence of $V_m^k$ and $V_{m,n}^k$ in the penultimate equality, while in
the last equality we used that $V_m^k$ is measurable with respect to the $\sigma$-algebra generated
by $\omega_m$ and that $V_{m,n}^k$ is independent of $\omega_m$. As in the proof of Proposition 3.1 part b)
we have that $\mathbb{P}(x \in V_{m,n}^k) = 2^{\lambda(m-n)}$ and so

$$\int f(x)2^{\lambda m}1(x \in V_m^k)\mathbb{P}(x \in V_{m,n}^k)dx = \int f(x)2^{\lambda m}1(x \in V_m^k)dx = \int f \zeta_m = X_m(f)$$

and the proof is complete.

\[ \square \]

**Corollary 4.5.** Let $f : [0,1]^k \to \mathbb{R}$ be a continuous (not necessarily non-negative) function
with compact support and define $X_n(f)$ as before. Then, there exists $X(f) < \infty$ such
that $X_n(f) \to X(f)$ almost surely.

**Proof.** Decompose the function in its positive and negative parts and apply Lemma 4.4
to each component. The result follows by martingale convergence.

Let $\mathcal{M}$ denote the set of continuous functions $f : [0,1]^k \to \mathbb{R}$, and for $f, g \in \mathcal{M}$ let
$\|f - g\|_u := \sup_{x \in [0,1]^k}|f(x) - g(x)|$ denote the supremum norm. As it is a standard
consequence of the Stone-Weierstrass Theorem that $(\mathcal{M}, \| \cdot \|_u)$ is separable, we leave the
proof of the following lemma to the reader.

**Lemma 4.6.** The space $\mathcal{M}$ contains a countable and dense (with respect to the norm $\| \cdot \|_u$) subset $\mathcal{M}_0$. Furthermore, for every such set $\mathcal{M}_0$, there exists $g \in \mathcal{M}_0$ such that
$1 \leq g(x) < \infty$ for every $x \in [0,1]^k$.

**Lemma 4.7.** Fix some $\mathcal{M}_0$ as in Lemma 4.6 and assume that $\lim \int f d\zeta_n(\omega)$ exists and is
finite for all $f \in \mathcal{M}_0$. Then there exists a Radon measure $\zeta(\omega)$ such that $\zeta_n(\omega) \to \zeta(\omega)$
weakly.

**Proof.** Fix any $\omega$ satisfying the assumption. Since it is fixed, we will now again suppress
it from our notation. The proof is based on extending the linear functional $l(f) = \lim_n \int f d\zeta_n$ initially defined for $f \in \mathcal{M}_0$ to all $f \in \mathcal{M}$, and then apply Riesz representation theorem. Let $g \in \mathcal{M}_0$ be as in Lemma 4.6 and note that $\lim \sup_n \zeta_n([0,1]^d) = \lim \sup_n \int d\zeta_n \leq \lim \sup_n \int g d\zeta_n = \lim_n \int g d\zeta_n < \infty$ since the limit exists and is finite
by assumption. We conclude that

$$\limsup_{n \to \infty} \zeta_n([0,1]^d) < \infty. \quad (35)$$

Fix $f \in \mathcal{M}$ and let $(f_i)_{i \geq 1} \subset \mathcal{M}_0$ be such that $\lim_{i \to \infty} \|f_i - f\|_u = 0$. Then,

$$\lim_{i,j \to \infty} \|l(f_i) - l(f_j)\| \leq \lim_{i,j \to \infty} \lim_{n \to \infty} \int |f_i - f_j| d\zeta_n \leq \lim_{i,j \to \infty} \|f_i - f_j\|_u \limsup_{n \to \infty} \zeta_n([0,1]^d) = 0,$$

where we used that $(f_i)_{i \geq 1}$ converges (and in particular is a Cauchy sequence) and (35).
We conclude that $(l(f_i))_{i \geq 1}$ is itself a Cauchy sequence and so converges, and we can
now define \( l(f) := \lim_i l(f_i) \). Moreover, we have that the limit \( \lim_n \int f d\zeta_n \) exists and equals \( l(f) \). Indeed,

\[
\left| \lim_i l(f_i) - \lim_n \int f d\zeta_n \right| \leq \lim_i \left| \lim_n \int f_i d\zeta_n - \lim_n \int f d\zeta_n \right| \\
\leq \lim_i \lim_n \int |f_i - f| d\zeta_n \leq \lim_i \|f_i - f\|_u \lim_n \sup \zeta_n([0,1]^d) = 0,
\]

again by (35) and by the fact that \( f_i \to f \) in the supremum norm.

Next, let \( f \in \mathcal{M} \) be a non-negative function and let \((f_i)_{i\geq 1} \subset \mathcal{M}_0\) be a sequence approximating \( f \). Then,

\[
l(f) = \lim_i l(f_i) = \lim_i \lim_n \int f_i d\zeta_n \\
\geq \lim_i \lim_n \int -\|f - f_i\|_u d\zeta_n \geq -\lim_i \|f - f_i\|_u \lim_n \sup \zeta_n([0,1]^d) = 0,
\]

where we used (35) one more time. We conclude that \( l \) is a positive linear functional on \( \mathcal{M} \) and so by Riesz representation theorem (Theorem 7.2 from [21]) there exists a Radon measure \( \zeta \) such that

\[
l(f) = \int f d\zeta = \lim_n \int f d\zeta_n,
\]

for all \( f \in \mathcal{M} \). We conclude that \( \zeta_n(\omega) \to \zeta(\omega) \) weakly.

We will now combine Corollary 4.5 and Lemma 4.7 to show the following result.

**Proposition 4.8.** There exists a random measure \( \zeta \) on \([0,1]^d\) such that for almost every \( \omega \), \( \zeta_n(\omega) \to \zeta(\omega) \) weakly.

**Proof.** Let \( \mathcal{M}_0 \) be as in Lemma 4.6. Corollary 4.5 implies that for every fixed \( f \in \mathcal{M}_0 \)

\[
\int f d\zeta_n = X_n(f) \to X(f)
\]

except possibly for a null-set of \( \omega \) that we will denote by \( N(f) \). Then, Lemma 4.7 implies that for every \( \omega \notin \bigcup_{f \in \mathcal{M}_0} N(f) \) we have that \( \lim \int f d\zeta_n(\omega) \to \int f d\zeta(\omega) \) for every \( f \in \mathcal{M} \). Since \( \bigcup_{f \in \mathcal{M}_0} N(f) \) is a countable union of nullsets we conclude that almost surely, \( \zeta_n \to \zeta \) weakly.

We can now prove Theorem 4.3.

**Proof of Theorem 4.3.** As discussed before, we will use Frostman’s Lemma. Proposition 4.8 states that the sequence \((\zeta_n)_{n\geq 1}\) of random measures defined in (34) converge weakly to a measure \( \zeta \). In order to apply Frostman’s Lemma, it suffices to show that with probability bounded away from 0 uniformly in \( r < k - \lambda \), the measure \( \zeta \) is supported on \( \mathcal{V}^k \), that \( 0 < \zeta(\mathcal{V}^k) < \infty \) and that \( \zeta \) has finite \( r \)-energy (as explained in (33)).
We will start by considering the support of $\zeta$. To that end, let $x \in [0,1)^k \setminus V^k$. Then we must have that for some $n$, $x \in [0,1)^k \setminus V_n^k$. Furthermore, since $V_n^k$ is a compact set, there exists some $\delta > 0$ such that $B(x,\delta) \subset [0,1)^k \setminus V_n^k \subset [0,1)^k \setminus V^k$. We conclude that $x$ is not in the support of $\zeta$, and so we must have that $\text{supp } \zeta \subset V^k$.

Second, we observe that by Proposition 4.8 and (35) we have that almost surely, $\zeta(V^k) = \lim \zeta_n([0,1]^d) < \infty$.

Our next step is to prove that $\zeta(V^k) > 0$ with positive probability. To that end, let

$$ \|\zeta_n\|_{TV} = \int_{[0,1]^k} 2^{\lambda n} I(x \in V^k_n)dx = 2^{\lambda n} \ell_k(V^k_n) $$

be the total variation of $\zeta_n$, and note that $\zeta(V^k) = \|\zeta\|_{TV} = \lim \|\zeta_n\|_{TV}$. In what follows, we will prove that $\mathbb{E}(\|\zeta\|_{TV}) = 1$ and therefore that $\|\zeta\|_{TV} > 0$ with positive probability. For that purpose, note that part a) of Proposition 3.1 implies that

$$ \mathbb{E}[\|\zeta\|_{TV}] = \mathbb{E} \left[ \int_{[0,1]^k} 2^{\lambda n} I(x \in V^k_n)dx \right] = \int_{[0,1]^k} 2^{\lambda n} \mathbb{P}(x \in V^k_n)dx = 1. \quad (36) $$

Then, observe that by Lemma 4.1 applied to the constant function $f = 1$ we have that $(\|\zeta_n\|_{TV})_{n \geq 1}$ is a martingale. By standard theory, if we can show that this martingale is bounded in $L^2$, it follows that $\mathbb{E}[\|\zeta\|_{TV}] = \lim_n \mathbb{E}[\|\zeta_n\|_{TV}] = 1$. Consider therefore

$$ \mathbb{E}[\|\zeta_n\|^2_{TV}] = \mathbb{E} \left[ \int_{[0,1]^k \times [0,1]^k} 2^{2\lambda n} I(x \in V^k_n)I(y \in V^k_n)dx dy \right] $$

$$ = \int_{[0,1]^k \times [0,1]^k} 2^{2\lambda n} \mathbb{P}(x,y \in V^k_n)dx dy \leq \int_{[0,1]^k \times [0,1]^k} \frac{2^{2\lambda n}e^{\lambda C_2} 2^{2\lambda n} \|x-y\|^\lambda dx dy}{\|x-y\|^\lambda} $$

$$ = e^{\lambda C_2} \int_{[0,1]^k \times [0,1]^k} \frac{1}{\|x-y\|^\lambda} dx dy, $$

where we used part c) of Proposition 3.1 in the inequality. Recall the notation $\psi_m$ defined in Section 2.3

$$ \int_{[0,1]^k} \left( \int_{[0,1]^k} \frac{1}{\|x-y\|^\lambda} dx \right) dy \leq \max_{y \in [0,1]^k} \int_{[0,1]^k} \frac{1}{\|x-y\|^\lambda} dx \quad (37) $$

$$ \leq \max_{y \in [0,1]^k} \left\{ \int_{\{x:d(x,y) \leq 1\}} \frac{1}{\|x-y\|^\lambda} dx + 1 \right\} $$

$$ = \psi_{k-1} \int_0^1 r^{-\lambda} r^{k-1} dr + 1 = \frac{\psi_{k-1}}{k-\lambda} + 1, $$

for all $\lambda < k$. We conclude that

$$ \mathbb{E}[\|\zeta_n\|^2_{TV}] \leq e^{\lambda C_2} \left( \frac{\psi_{k-1}}{k-\lambda} + 1 \right). \quad (38) $$
In particular, we see that \( \sup_n \mathbb{E}[\|\zeta_n\|_{TV}^2] < \infty \) and so (as explained above), \( \mathbb{E}[\|\zeta\|_{TV}] = \mathbb{E}[\zeta(\mathcal{V}^k)] = 1 \). Thus, \( \zeta(\mathcal{V}^k) > 0 \) with positive probability.

It remains to show that \( \zeta \) has finite \( r \)-energy for every \( r < k - \lambda \). For such \( r \), we use part \( c \) of Proposition 3.1 to obtain

\[
\mathbb{E}[I_r(\zeta_n)] = \mathbb{E} \left[ \int_{[0,1]^k \times [0,1]^k} \frac{d\zeta_n(x)d\zeta_n(y)}{\|x - y\|^r} \right] \quad (39)
\]

\[
= \mathbb{E} \left[ \int_{[0,1]^k \times [0,1]^k} 2^{2\lambda n} I(x \in \mathcal{V}_n^k)I(y \in \mathcal{V}_n^k) \frac{d\zeta_n(x)d\zeta_n(y)}{\|x - y\|^r} \right]
\]

\[
= \int_{[0,1]^k \times [0,1]^k} 2^{2\lambda n} \mathbb{P}(x,y \in \mathcal{V}_n^k) \frac{1}{\|x - y\|^r} dxdy \leq e^{\lambda C_2} \int_{[0,1]^k} \int_{[0,1]^k} \frac{1}{\|x - y\|^{r+\lambda}} dxdy \leq e^{\lambda C_2} \left( \frac{\psi_{k-1}}{k - r - \lambda} + 1 \right),
\]

where the last inequality follows in the same way as in (47). Furthermore, we observe that

\[
I_r(\zeta) = \int_{[0,1]^k \times [0,1]^k} \frac{d\zeta(x)d\zeta(y)}{\|x - y\|^r} \quad (40)
\]

\[
= \lim_{M \to \infty} \int_{[0,1]^k \times [0,1]^k} \left( \frac{1}{\|x - y\|^r} \wedge M \right) d\zeta(x)d\zeta(y)
\]

\[
= \lim_{M \to \infty} \liminf_{n \to \infty} \int_{[0,1]^k \times [0,1]^k} \left( \frac{1}{\|x - y\|^r} \wedge M \right) d\zeta_n(x)d\zeta_n(y)
\]

\[
\leq \liminf_{n \to \infty} \int_{[0,1]^k \times [0,1]^k} \frac{1}{\|x - y\|^r} d\zeta_n(x)d\zeta_n(y) = \liminf_{n \to \infty} I_r(\zeta_n)
\]

where we used the monotone convergence theorem in the second equality and the fact that \( \zeta_n \times \zeta_n \to \zeta \times \zeta \) weakly (since \( \zeta_n \to \zeta \) weakly) in the third equality. We combine (39) and (40) to conclude that

\[
\mathbb{E}[I_r(\zeta)] \leq \mathbb{E}[\liminf_{n \to \infty} I_r(\zeta_n)] \leq \liminf_{n \to \infty} \mathbb{E}[I_r(\zeta_n)] < \infty,
\]

and so \( I_r(\zeta(\omega)) < \infty \) almost surely.

Since both \( I_r(\zeta) < \infty \) and \( \zeta(\mathcal{V}^k) < \infty \) almost surely, we see that

\[
\mathbb{P}(\exists \zeta : 0 < \zeta(\mathcal{V}^k) < \infty, I_r(\zeta) < \infty) = \mathbb{P}(\exists \zeta : \zeta(\mathcal{V}^k) > 0).
\]

Furthermore, for any \( \omega \) such that there exists \( \zeta = \zeta(\omega) \) with \( 0 < \zeta(\mathcal{V}^k) < \infty \) and \( I_r(\zeta) < \infty \), it follows from Frostman’s lemma that \( \dim_H(\mathcal{V}^k) \geq r \) for every \( r < k - \lambda \). From this it then follows that for any such \( \omega \), we have \( \dim_H(\mathcal{V}^k) \geq k - \lambda \), and so we conclude that

\[
\mathbb{P}(\dim_H(\mathcal{V}^k) \geq k - \lambda) \geq \mathbb{P}(\exists \zeta : \zeta(\mathcal{V}^k) > 0) > 0.
\]
Remark 4.9. In the proof of the previous proposition we showed that the limiting measure has positive mass (i.e. \( \zeta(\mathcal{V}^k) > 0 \)) with positive probability, and from this we concluded that also \( \mathbb{P}(\dim_H(\mathcal{V}^k) \geq k - \lambda) > 0 \). Using Theorem 4.1 we then conclude that \( \mathbb{P}(\dim_H(\mathcal{V}^k) = k - \lambda) > 0 \). However, it is not far fetched to suspect that in fact \( \dim_H(\mathcal{V}^k) = k - \lambda \) as soon as \( \mathcal{V}^k \neq \emptyset \). That is, whenever the fractal survives within \([0,1]^k \times \{0\}^{d-k}\) it must have dimension \( k - \lambda \). Indeed, a similar result holds for the much simpler case of the Mandelbrot fractal percolation model (see [9]). While we are not currently able to prove this stronger statement for our model, we can provide an explicit lower bound on the probability of \( \zeta(\mathcal{V}^k) > 0 \) (and therefore also on \( \mathbb{P}(\dim_H(\mathcal{V}^k) \geq k - \lambda) \)).

Proposition 4.10. We have that

\[
\mathbb{P}(\zeta(\mathcal{V}^k) > 0) \geq \frac{(k - \lambda)e^{-C_2\lambda}}{\psi_{k-1} + k - \lambda},
\]

where \( C_2 \) is defined in (25).

Proof. Let \( 0 < \alpha < 1 \). By Proposition 4.8 we have that \( \zeta_n \to \zeta \), and so \( \zeta_n \to \zeta \) along any subsequence. Therefore, if \( \|\zeta_n(\omega)\|_{TV} \geq \alpha \) for infinitely many \( n \), there exists some subsequence \( (n_i)_{i \geq 1} \) (depending on \( \omega \)) such that \( \zeta(\mathcal{V}^k) = \|\zeta\|_{TV} = \lim \|\zeta_n\|_{TV} \geq \alpha \).

Next, we use the Paley-Zygmund inequality to see that for every \( n \geq 1 \),

\[
\mathbb{P}(\|\zeta_n\|_{TV} \geq \alpha) = \mathbb{P}(\|\zeta_n\|_{TV} \geq \alpha \mathbb{E}[\|\zeta_n\|_{TV}]) \geq (1 - \alpha)^2 \mathbb{E}[\|\zeta_n\|_{TV}]^2 \mathbb{E}[\|\zeta_n\|_{TV}]^2 \geq \frac{(1 - \alpha)^2 (k - \lambda)e^{-C_2\lambda}}{\psi_{k-1} + k - \lambda} > 0
\]

where we used (36) in the first equality, and (36) and (38) in the second inequality. By the reverse Fatou Lemma we then see that

\[
\frac{(1 - \alpha)^2 (k - \lambda)e^{-C_2\lambda}}{\psi_{k-1} + k - \lambda} \leq \lim \sup_n \mathbb{E}[I(\|\zeta_n\| \geq \alpha)] \leq \mathbb{E}[\lim \sup I(\|\zeta_n\| \geq \alpha)] = \mathbb{P}(\|\zeta_n\| > \alpha) \text{ i.o.},
\]

and so we conclude, using the discussion in the beginning of the proof, that

\[
\mathbb{P}(\zeta(\mathcal{V}^k) \geq \alpha) \geq \frac{(1 - \alpha)^2 (k - \lambda)e^{-C_2\lambda}}{\psi_{k-1} + k - \lambda}.
\]

Then finally, we see that

\[
\mathbb{P}(\zeta(\mathcal{V}^k) > 0) = \sup_{\alpha > 0} \mathbb{P}(\zeta(\mathcal{V}^k) \geq \alpha) \geq \sup_{\alpha > 0} \frac{(1 - \alpha)^2 (k - \lambda)e^{-C_2\lambda}}{\psi_{k-1} + k - \lambda} = \frac{(k - \lambda)e^{-C_2\lambda}}{\psi_{k-1} + k - \lambda}.
\]

We can now prove Theorem 1.7.

Proof of Theorem 1.7. Theorem 4.1 states that \( \dim_H(\mathcal{V} \cap H_k) \leq k - \lambda \) almost surely.

Furthermore, the event \( F := \{\dim_H(\mathcal{V} \cap H_k) \geq k - \lambda\} \) is clearly shift-invariant so that by Lemma 2.3 \( \mathbb{P}(F) \in \{0,1\} \), and since by Theorem 1.3 we have that \( \mathbb{P}(F) > 0 \) we conclude that \( \dim_H(\mathcal{V} \cap H_k) \geq k - \lambda \) almost surely.

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5 Cylinders intersecting subspaces

In order to prove Theorem 1.3, we will need a detailed understanding of the intersection 
\( C(\omega) \cap H_k \) for \( k \leq d - 1 \). The purpose of this section is to obtain basic results in this 
direction.

Our first result of this section describes the shape of the intersection \( c(L,r) \cap H_k \) for 
\( L \in \mathcal{L}_H \). Recall the notation introduced in Section 2.3.

**Lemma 5.1.** Fix \( k \in \{1, \ldots, d - 1\} \). For \( L = L(a,p) \in \mathcal{L}_H \), the set 
\( c(L,r) \cap H_k \) is an ellipsoid defined by the equation

\[(\text{cent}_k - x) \left( I_k \times_k - \|a\|^{-2} A \right) (\text{cent}_k - x)^T < r^2 - \|p(k)\|^2 + \frac{\langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2},\]

where

\[\text{cent}_k = p(k) - \frac{\langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} a(k) \tag{41}\]

and

\[A = a^T(k) a(k) = (a(k,i) a(k,j))_{1 \leq i, j \leq k},\]

is the outer product of the vector \( a(k) \) with itself. More concretely, the ellipsoid has one 
major axis given by

\[\frac{\|a\|}{\sqrt{1 + \|a(k)\|^2}} \left( r^2 - \|p(k)\|^2 + \frac{\langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} \right)^{1/2}\]

which extends in the direction \( a(k)/\|a(k)\| \) and the lengths of the remaining \( k - 1 \) axes 
are given by

\[\left( r^2 - \|p(k)\|^2 + \frac{\langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} \right)^{1/2}\]

and these extends in the directions orthogonal to \( a(k)/\|a(k)\| \) in \( H_k \).

The proof of Lemma 5.1 contains some quite lengthy calculations, and is therefore 
given in Appendix A.3.

As an immediate corollary we obtain the volume and the diameter of the ellipsoid.

**Corollary 5.2.** Let \( L \in \mathcal{L}_H \). The volume of the ellipsoid \( E_k(L,r) \) is given by

\[\text{Vol}(E_k(L,r)) = \frac{\psi_{k+1}}{2\pi} \frac{\|a\|}{\sqrt{1 + \|a(k)\|^2}} \left( r^2 - \|p(k)\|^2 + \frac{\langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} \right)^{k/2}.\]

Moreover, the diameter is given by

\[\text{diam}(E_k(L,r)) = 2 \frac{\|a\|}{\sqrt{1 + \|a(k)\|^2}} \left( r^2 - \|p(k)\|^2 + \frac{\langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} \right)^{1/2}.\]
Proof. The proof is immediate since the volume of the ellipse is given by \( \ell_k(B^k(o,1)) \prod_{i=1}^k l_i \), where \( l_i \) is the length of axis number \( i \). Furthermore from Section 2.2 we have that \( \ell_k(B^k(o,1)) = \frac{\sqrt{k+1}}{2^k} \). Finally, the diameter is immediate from Lemma 5.1. \( \square \)

5.1 The induced ellipsoid models

The aim of this subsection is to prove Theorem 1.5, i.e. to show that the random set \( C \cap H_k \) can be generated (in law) using a Poisson process of ellipsoids in \( \mathbb{R}^k \). Recall therefore the notation of Section 2.4.

Note that it follows from Lemma 5.1 that for any \( L = L(a_{(k)},a^{(k)},p_{(k)},p^{(k)}) \in \mathcal{L} L_k \) and \( p_{(k)} \),
\[
E_k(L(a_{(k)},a^{(k)},p_{(k)},p^{(k)}),r)_o = E_k(L(a_{(k)},a^{(k)},0_k,p^{(k)}),r)_o,
\]
where \( 0_k \) is the zero-vector of length \( k \). For \( k \in \{1, \ldots, d-1\} \) and \( r \in (0,1] \), define the measure \( \mu_{k,r} \) on \( \mathcal{E}^k \) by letting
\[
\mu_{k,r}(E) = \nu_d(L : E_k(L,r) \in E),
\]
for any measurable \( E \subset \mathcal{E}^k \). According to Lemma 5.1 if \( L = L(a_{(k)},a^{(k)},p_{(k)},p^{(k)}) \in \mathcal{A}(d,1) \), the cylinder \( c(L,r) \) intersects \( H_k \) if and only if
\[
\|p^{(k)}\|^2 - \frac{(a^{(k)}p^{(k)})^2}{1 + \|a^{(k)}\|^2} < r^2.
\]
Note that when \( k = d - 1 \) we have that \( p^{(k)} \) and \( a^{(k)} \) are empty, and so the above expression is not defined. However, in this case every cylinder intersects \( H_{d-1} \) and therefore we simply interpret the the condition to always be satisfied. (Of course, there is a technical issue with lines parallel to the subspace \( H_{d-1} \), but as explained in Section 2.3 these can be ignored).

By using (10), we see that we can write (43) in the following way
\[
\mu_{k,r}(E) = \mathcal{T}_d \int_{L : E_k(L,r) \in E} \left( \frac{\|p^{(k)}\|^2 - (a^{(k)}p^{(k)})^2}{1 + \|a^{(k)}\|^2} < r^2 \right) \frac{1}{\|a\|^{d+1}} da_{(k)} da^{(k)} dp_{(k)} dp^{(k)}. \tag{45}
\]
The measure \( \mu_{k,r} \) is a measure on ellipsoids induced by the cylinder process with fixed radius \( r \). For the scale invariant process we define
\[
\mu_k(E) = \int_0^1 \mu_{k,r}(E) dq_{(r)}. \tag{46}
\]
As noted in Section 2.4 an ellipsoid \( E \) is uniquely determined by the pair \( (E_o, \text{cent}(E)) \). We will show that \( \mu_k \) and \( \mu_{k,r} \) can be written as product measures, where one factor is a measure on \( \mathcal{E}_o^k \) (corresponding to shapes of ellipsoids) and the other factor is a measure on \( \mathbb{R}^k \) (corresponding to centres of ellipsoids).
To this end, we define the measures \( \xi_{k,r} \) and \( \xi_k \) on \( \mathcal{E}_o^k \) by
\[
\xi_{k,r}(\cdot) = \mathcal{Y}_d \int_{(E_k(L,r))_o \in \cdot} 1 \left( \frac{\|p(k)\|^2 - \langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} < r^2 \right) \frac{1}{\|a\|^{d+1}} \, da(k) \, dp(k) \quad (47)
\]
and
\[
\xi_k(\cdot) = \int_0^1 \xi_{k,r}(\cdot) \, dq_s(r) \quad (48)
\]
Observe that \( \mu_{k,r} \) depends on \( d \) through the constant \( \mathcal{Y}_d \) and through \( \|a\|^{-(d+1)} \) in the integrand. Furthermore, \( \mu_k \) depends on \( d \) further through the use of the measure \( q_s \) (see \( (14) \)).

We will now turn to the proof of Theorem 1.5. Let \( \omega \) be as before (i.e. chosen according to \( \mathbb{P}_\lambda \)) and define
\[
\omega_e := \sum_{(L,r) \in \omega} \delta_{(cent(E_k(L,r)), E_k(L,r)_o)}.
\]

**Proof of Theorem 1.5.** We will only prove (4), as (3) follows in the same way by considering a Poisson cylinder process with fixed radius \( r \) (in place of the full fractal model). We therefore need to prove that \( \omega_e \) is a Poisson processes with intensity measure given by \( (4) \), and we start by showing that \( \omega_e \) is a Poisson process. Suppose therefore that \( \mathcal{E}_1, \mathcal{E}_2 \subset \mathbb{C}^k = \mathbb{R}^k \times \mathcal{C}_o^k \) are disjoint collections of ellipsoids. Then, \( E_{k-1}^{-1}(\mathcal{E}_1) \) and \( E_{k-1}^{-1}(\mathcal{E}_2) \) are disjoint subsets of \( A(d,1) \times (0,1] \), and since \( \omega_e(\mathcal{E}_i) = \omega(E_{k-1}^{-1}(\mathcal{E}_i)) \) we see that \( \omega_e \) is indeed a Poisson process on \( \mathbb{C}^k \). Furthermore, the mean of \( \omega_e(\mathcal{E}_i) \) equals \( \lambda \mu_k(\mathcal{E}_i) \).

Our next step is to prove that \( \mu_k \) equals \( \ell_k \times \xi_{k,r} \). Consider any measurable sets \( A \subset \mathbb{R}^k \) and \( \mathcal{E}_o \subset \mathcal{C}_o^k \), and let
\[
\mathbf{E}(A, \mathcal{E}_o) = \{ E \in \mathbb{C}^k : E = (x, \mathcal{E}_o) \text{ for some } x \in A \text{ and } \mathcal{E}_o \in \mathcal{E}_o \} \subset \mathbb{C}^k.
\]
The statement follows if we can show that for any \( A \) and \( \mathcal{E}_o \) as above we have that
\[
\mu_k(\mathbf{E}(A, \mathcal{E}_o)) = \ell_k(A) \xi_k(\mathcal{E}_o).
\]
To this end, observe that by \( (45) \) and \( (46) \),
\[
\mu_k(\mathbf{E}(A, \mathcal{E}_o)) = \int_0^1 \nu_d(L : \text{cent}(E_k(L,r)) \in A, E_k(L,r)_o \in \mathcal{E}_o) \, dq_s(r)
\]
\[
= \mathcal{Y}_d \int_{\text{cent}(E_k(L,r)) \in A} \int_{0 < r < 1} 1 \left( \frac{\|p(k)\|^2 - \langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} < r^2 \right) \frac{1}{\|a\|^{d+1}} \, da(k) \, dp(k) \, dq_s(r)
\]
\[
= \mathcal{Y}_d \int_{\text{cent}(E_k(L,r)) \in A} \int_{0 < r < 1} \left( \frac{\|p(k)\|^2 - \langle a(k), p(k) \rangle^2}{1 + \|a(k)\|^2} < r^2 \right) \frac{1}{\|a\|^{d+1}} \, da(k) \, dp(k)
\]
\[
= \ell_k(A) \xi_k(\mathcal{E}_o),
\]
where we used (42) in the third equality and (41) in the fourth equality.

Our next step is to couple the fractal ellipsoid model with a fractal ball model. This fractal ball model will be of use in the upcoming sections, and in particular for the proof of Theorem 1.3. Informally, for any fixed ellipsoid in the ellipsoid process we shall replace it with a ball centred in the same point and with the same diameter as the replaced ellipsoid. More precisely, we will consider

$$\omega_b := \sum_{(L,r) \in \omega} \delta_{(\text{cent}(E_k(L,r)), \frac{\text{diam}(E_k(L,r))}{2})},$$

and then let

$$\mathcal{V}_b := \mathbb{R}^k \setminus \bigcup_{(x,R) \in \omega_b} B^k(x,R),$$

where we recall the notation $B^k(x,R)$ from Section 2.2. The fact that the ball $B^k(x,R)$ is closed (see Section 2.2) rather than open will not be important. Obviously it follows that

$$\mathcal{V}_b \subset \mathcal{V} \cap H_k,$$

since every ellipsoid is replaced by a larger ball. Let

$$g(\cdot) = \xi_k(E : \frac{\text{diam}(E)}{2} \in \cdot). \tag{49}$$

We have the following theorem which is an analogue of Theorem 1.5. Since the proof is almost identical, we only give the statement. The reason for using $\frac{\text{diam}(E)}{2}$ (rather than just $\text{diam}(E)$) in the definition of $g$, stems from the fact that it is customary to specify a ball by using the radius rather than the diameter.

**Theorem 5.3.** For $k \in \{1, \ldots, d-1\}$, $\omega_b$ is a Poisson process on $\mathbb{R}^k \times \mathbb{R}_+$ with intensity measure

$$\lambda \ell_k \times g.$$ 

### 6 Analysis of the ellipsoid model

The purpose of this section is mainly to prepare the grounds for the proof (see Section 7) of Theorem 1.3. This theorem will be proved by comparing the ellipsoid model on $H_k$ with the fractal ball model described at the end of Section 5.1. Then, we will relate this induced ball model to classical results about fractal ball models (see for instance [2] Section 8, and [14]). However, in order to obtain useful results when performing this comparison, we first need a better understanding of the induced processes.

Subsection 6.1 establishes some basic properties of the measure $\xi_{k,r}$ when $r > 0$ which are then used in Subsection 6.2 where we focus on the distribution of the diameters of the induced ellipsoid process, since that is what will be useful when studying the induced ball model. The proof of Lemma 6.2 in Subsection 6.1 and the proofs of lemmas 6.3 and
6.4 in Subsection 6.2 contain long and involved calculations of integrals. For this reason, the proofs of those lemmas are contained in Appendix A.4.

We point out that in the end we will not use the full results that we obtain below. For instance, it would be sufficient to restrict our attention to the case of $H_2$ when studying the connectivity phase transition (i.e. when proving Theorem 1.3). However, we still chose to study the properties of the induced ellipsoid process in its generality for several reasons. Firstly, we do not believe that the general case makes the arguments more complicated or much longer. Secondly, for any eventual future projects it will be useful to have the full results in hand. Finally, in Section 6.3 we demonstrate how to use our results in order to obtain an alternative proof of a result from [5] concerning the standard Poisson cylinder model.

6.1 Total variation of the ellipsoid measures

The purpose of this subsection is to calculate the total variation of the measure $\xi_{k,r}$ for $r > 0$. First, we will describe some useful changes of coordinates which will simplify our calculations. These will also be used in later sections.

In order to perform our calculations we will need to differentiate between three different cases, namely when $2 \leq k \leq d - 3$, $k = d - 2$ and $k = d - 1$. The first of these will be the most involved and it will be useful to introduce the following change of variables. First, let

$$T_1: \begin{cases} 
(\rho, \theta) := (||a(k)||, \|a(k)||/\|a(k)||) \in [0, \infty) \times S^{k-1} \\
(\kappa, \phi) := (||a(k)||, \|a(k)||/\|a(k)||) \in [0, \infty) \times S^{d-k-2} \\
(\gamma, \varphi) := (||p(k)||, \|p(k)||/\|p(k)||) \in [0, \infty) \times S^{d-k-2},
\end{cases}$$

which transforms $a(k), a(k)$ and $p(k)$ into a suitable set of spherical coordinates. Observe that if $k = d - 2$ or $k = d - 1$ then some of these changes does not really make sense. This is the reason why we need to divide into several cases as mentioned above. Our second transformation rescales the length $\|p(k)\| = \gamma$ in terms of the length of $\|a(k)\| = \kappa$ and the normalized angle between $a(k)$ and $p(k)$. This transformation depends on $r$ and so we emphasize this in the notation. Our second transformation is therefore

$$T_2: \quad t := \frac{\gamma}{r} \left( \frac{1 + \kappa^2}{1 + (1 - \langle \varphi, \phi \rangle^2)\kappa^2} \right)^{-1/2}. \quad (51)$$

We will of course make use of the composition $T^r := T_2 \circ T_1$. For clarity, we will write $\xi_{k,r} \circ T^r$ (or $\xi_k \circ T_1$ e.t.c.) when we work with the measure $\xi_{k,r}$ using these new coordinates.

Recalling the definition (i.e. (17)) of $\xi_{k,r}$, we see that

$$\frac{d(\xi_{k,r} \circ T_1)}{d\rho d\kappa d\gamma d\ell_{d-1}(\theta) d\ell_{d-k-2}(\phi) d\ell_{d-k-2}(\varphi)}$$

$$= \Upsilon_{dI} \left( \gamma^2 - \frac{\gamma^2 \kappa^2 \langle \varphi, \phi \rangle^2}{1 + \kappa^2} \right) < r^2 \frac{r^{k-1}(\kappa \gamma)^{d-k-2}}{(\rho^2 + \kappa^2 + 1)^{d+1/2}} \quad (52)$$
and that
\[
\frac{d(\xi_{k,r} \circ T^r)}{d\rho d\varphi d\ell_{k-1}(\theta) d\ell_{d-k-2}(\varphi) d\ell_{d-k-2}(\phi)} = \Upsilon dr (0 < t < 1) r^{d-k-1} \left( \frac{1 + \kappa^2}{1 + (1 - \langle \varphi, \phi \rangle)^2 \kappa^2} \right)^{(d-k-1)/2} \frac{\rho^{k-1}(\kappa t)^{d-k-2}}{(\rho^2 + \kappa^2 + 1)^{(d+1)/2}}.
\]  
It should be noted that the right hand side of (53) depends on \( r \) only through the factor \( r^{d-k-1} \) and so
\[
d(\xi_{k,r} \circ T^r) = r^{d-k-1} d(\xi_{k,1} \circ T^1)
\]
(this can also been seen by using the scale invariance of the Poisson cylinder model). Note also that both of the above Jacobians depends on \( \varphi \) and \( \phi \) only through the scalar product \( \langle \varphi, \phi \rangle \).

Given \( L \in \mathcal{L}_{H_k} \) one can also check that under \( T^r \), the formulas for the volume and diameter of \( E_k(L,r) \) from Corollary 5.2 are given by (recall from Section 2.2 that \( \ell_k(B^k(o,1)) = \psi_{k-1}/k = \psi_{k+1}/(2\pi) \))
\[
\begin{align*}
\text{Vol}(E_k(L,r)) \circ T^r &= \psi_{k-1} k^{(d^2 + \kappa^2 + 1)/(\kappa^2 + 1)} (1 - t^2)^{k/2}, \\
\text{diam}(E_k(L,r)) \circ T^r &= 2r \left( \frac{\kappa^2 + 1}{\kappa^2 + 1} \right)^{1/2} (1 - t^2)^{1/2}.
\end{align*}
\]  

Remark 6.1. It should be remarked that one can make sense of this for \( k = d - 2 \), but as mentioned, the coordinate maps \( T_1 \) and \( T_2 \) have to change accordingly. More precisely, \( T_1 \) is now just the identity on \( a^{(k)}, p^{(k)} \) since \( a^{(k)}, p^{(k)} \in \mathbb{R} \) and moreover since \( (a^{(k)}/|a^{(k)}|, p^{(k)}/|p^{(k)}|)^2 = 1 \) the denominator of \( T_2^r \) vanishes. Additionally the expressions for the volume and diameter do not change but the Jacobian is changed slightly. The case \( k = d - 1 \) is also different. In this case almost every line intersects \( H_k \), so for example the indicator functions appearing in (52) and (53) disappear.

In the next lemma, we calculate the total mass (i.e. total variation) of \( \xi_{k,r} \).

Lemma 6.2. For \( r \geq 0, d \geq 3 \) and \( 2 \leq k \leq d - 1 \), we have that the total mass of \( \xi_{k,r} \) is given by
\[
\|\xi_{k,r}\|_{TV} = \frac{r^{d-k-1} \psi_{d-k-1}}{\psi_{d-1}}.
\]

The proof of Lemma 6.2 can be found in Appendix A.4.

6.2 Properties of the diameter of random ellipsoids

By using Lemma 6.2 we see that
\[
\tilde{\xi}_{k,r}(\cdot) := \frac{1}{\|\xi_{k,r}\|_{TV}} \xi_{k,r}(\cdot),
\]
defines a probability measure.
The first goal of this section is to calculate the moments of \(\text{diam}(E)\) when \(E\) is chosen according to \(\xi_{k,r}\), see Lemma 6.3. Then, we study the behavior of \(\xi_k(E : \text{diam}(E) \geq \tau)\) as a function of \(\tau > 0\). The first such result is Lemma 6.4 which considers large values of \(\tau\). The second result is Lemma 6.6 which holds for all \(\tau > 0\), but will only be used for small values of \(\tau\).

**Lemma 6.3.** Suppose that \(d \geq 3\), \(2 \leq k \leq d - 1\) and \(n \in \mathbb{N}_+.\) Then for any \(r > 0\),

\[
\mathbb{E}_{\xi_{k,r}}[(\text{diam}(E))^n] = \begin{cases} 
2^n r^{n(2\pi \psi_{d+n-k} \psi_{c-d-n})} & \text{if } n < d - k + 1 \\
\infty & \text{if } n \geq d - k + 1.
\end{cases} \tag{56}
\]

For the proof of Lemma 6.3 we refer to Appendix A.4. For us, the most important case of Lemma 6.3 is when \(n = k\). Then the lemma implies that

\[
\mathbb{E}_{\xi_{k,r}}[(\text{diam}(E))^k] = \begin{cases} 
2^k r^{k(2\pi \psi_{d-k} \psi_{c-d-k})} & k < (d+1)/2 \\
\infty & k \geq (d+1)/2.
\end{cases} \tag{57}
\]

**Lemma 6.4.** Suppose that \(d \geq 3\) and \(2 \leq k \leq d - 1\). There are constants \(c_3, \ldots, c_6\) such that for all \(\tau \geq 4\) and \(r > 0\),

\[
-c_3 \tau^{-d+k-2} r^{2(d-k)} \leq \frac{d}{d\tau} \xi_{k,r}(E : \text{diam}(E) \geq \tau) \leq -c_4 \tau^{-d+k-2} r^{2(d-k)}, \tag{58}
\]

\[
c_5 \tau^{-d+k-1} r^{2(d-k)} \leq \xi_{k,r}(E : \text{diam}(E) \geq \tau) \leq c_6 \tau^{-d+k-1} r^{2(d-k)}. \tag{59}
\]

The proof of Lemma 6.4 is contained in Appendix A.4.

**Lemma 6.5.** Suppose that \(d \geq 3\) and \(2 \leq k \leq d - 1\). If \(k \leq d/2\), then there are constants \(c_7, \ldots, c_{10}\) such that for all \(\tau \geq 4\) and \(r > 0\),

\[
-c_7 \tau^{-d+k-2} \leq \frac{d}{d\tau} \xi_k(E : \text{diam}(E) \geq \tau) \leq -c_8 \tau^{-d+k-2}, \tag{60}
\]

\[
c_9 \tau^{-d+k-1} \leq \xi_k(E : \text{diam}(E) \geq \tau) \leq c_{10} \tau^{-d+k-1}. \tag{61}
\]

On the other hand, if \(k \geq (d+1)/2\), then for all \(\tau \geq 0\),

\[
\xi_k(E : \text{diam}(E) \geq \tau) = \infty. \tag{62}
\]

**Proof.** If \(k \leq d/2\) we get using (48), (14) and (58) that for \(\tau \geq 4\),

\[
\frac{d}{d\tau} \xi_k(E : \text{diam}(E) \geq \tau) = \int_0^1 \frac{d}{d\tau} \xi_{k,r}(E : \text{diam}(E) \geq \tau)r^{-d} dr \propto -\tau^{-d+k-2} \int_0^1 r^{-2k} dr \propto -\tau^{-d+k-2}.
\]
This proves (60), from which (61) follows easily. Next, if $k \geq (d+1)/2$, then

$$
\xi_k(E : \text{diam}(E) \geq \tau) = \int_0^1 \xi_k,r(E : \text{diam}(E) \geq \tau) r^{-d} dr \\
\geq c \tau^{-d+k-1} \int_0^1 r^{2(d-k)-d} dr = c \tau^{k-d-1} \int_0^1 r^{d-2k} dr = \infty,
$$

where we used (59) in the inequality.

In the next lemma and its proof, we will use the notation

$$
\xi_{k,r}(\text{diam}(E)^k ; \text{diam}(E) \geq \tau) = \|\xi_{k,r}\|_{TV} \mathbb{E}_{\tilde{\xi}_{k,r}}[\text{diam}(E)^k : \text{diam}(E) \geq \tau].
$$

**Lemma 6.6.** Suppose that $d \geq 4$ and $2 \leq k \leq d/2$. Then, for any $\tau > 0$ we have

$$
\xi_k(E : \text{diam}(E) \geq \tau) = \frac{1}{k} \xi_{k,1}(\text{diam}(E)^k ; \text{diam}(E) \geq \tau) - \frac{1}{k} \xi_{k,1}(E : \text{diam}(E) \geq \tau).
$$

(63)

Moreover,

$$
\frac{d}{d\tau} \xi_k(E : \text{diam}(E) \geq \tau) = -\frac{1}{\tau k+1} \xi_{k,1}(\text{diam}(E)^k ; \text{diam}(E) \geq \tau),
$$

(64)

and so for $0 < a \leq b$,

$$
\xi_k(E : a \leq \text{diam}(E) \leq b) = \int_a^b \frac{1}{\tau k+1} \xi_{k,1}(\text{diam}(E)^k ; \text{diam}(E) \geq \tau) d\tau.
$$

(65)

**Proof.** We first show (63). Observe that by (48) and (54) we have that

$$
\xi_{k,r}(E : \text{diam}(E) \geq \tau) \\
= \int 1 (2r \left( \frac{\rho^2 + \kappa^2 + 1}{\kappa^2 + 1} \right)^{1/2} (1 - t^2)^{1/2} \geq \tau) d(\xi_{k,r} \circ T^r) \\
= \int 1 (2 \left( \frac{\rho^2 + \kappa^2 + 1}{\kappa^2 + 1} \right)^{1/2} (1 - t^2)^{1/2} \geq \frac{\tau}{r}) r^{d-k-1} d(\xi_{k,1} \circ T^1) \\
= r^{d-k-1} \xi_{k,1}(E : \text{diam}(E) \geq \tau/r).
$$

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In fact, this equality also follows from scale invariance. We see that
\[
\xi_k(E : \text{diam}(E) \geq \tau) = \int_0^1 \xi_{k,r}(E : \text{diam}(E) \geq \tau)r^{-d}dr
\]
\[
= \int_0^1 \int I(\text{diam}(E) \geq \tau/r)I(r \geq \tau) d\xi_{k,1}r^{-k-1}dr
\]
proving (63). We move on to prove (64). It is readily shown that the second term on the right hand side of (63) is differentiable with respect to \(\tau\). We now consider the derivative of \(\xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq \tau)\). Observe that for \(h > 0\),
\[
\frac{1}{h} \left( \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq \tau + h) - \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq \tau) \right)
\]
\[
= -\frac{1}{h} \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \leq \tau + h)
\]
\[
\geq -(\tau + h)^k \frac{1}{h} \xi_{k,1}(\text{diam}(E) \leq \tau + h). \tag{66}
\]
This inequality can obviously be reversed if we replace \((\tau + h)^k\) with \(\tau^k\). Letting \(h \to 0\), we then see that
\[
\frac{d}{d\tau} \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq \tau) \tag{67}
\]
\[
= -\tau^k \frac{d}{d\tau} \xi_{k,1}(\text{diam}(E) \leq \tau) = \tau^k \frac{d}{d\tau} \xi_{k,1}(\text{diam}(E) \geq \tau).
\]
Hence, by (63)
\[
\frac{d}{d\tau} \xi_k(E : \text{diam}(E) \geq \tau)
\]
\[
= \frac{d}{d\tau} \left( \frac{1}{\tau^k} \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq \tau) - \frac{1}{k} \xi_{k,1}(\text{diam}(E) \geq \tau) \right)
\]
\[
= -\frac{1}{\tau^{k+1}} \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq \tau)
\]
\[
+ \frac{1}{k\tau^k} \frac{d}{d\tau} \xi_{k,1}(\text{diam}(E) \geq \tau) - \frac{1}{k} \frac{d}{d\tau} \xi_{k,1}(\text{diam}(E) \geq \tau)
\]
\[
= -\frac{1}{\tau^{k+1}} \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq \tau),
\]
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where we used (67) in the penultimate equality. This finishes the proof (64). Finally, we get (65) from (64) since

\[ \xi_k(E : a \leq \text{diam}(E) \leq b) = -\int_a^b \frac{d}{d\tau} \xi_k(E : \text{diam}(E) \geq \tau) d\tau. \]

6.3 A remark on the standard Poisson cylinder model

Recall the notation \( \hat{\Omega}, \hat{\omega} \) and \( \hat{V} \) concerning the standard (that is, non-fractal) Poisson cylinder model from Section 2.5. We will now describe how to obtain an alternative proof of one of the main results in [5] using our results above. We point out that our alternative proof is not shorter than the original proof, since we will use for example Lemma 6.3. Let \( \text{Perc}_2 \) denote the event that \( \hat{V} \cap H_2 \) contains unbounded connected components (that is, it percolates). The following theorem is Theorem 5.1 in [5].

**Theorem 6.7** ([5]). Let \( d \geq 4 \). There is a constant \( c = c(d,r) > 0 \) such that if \( \lambda \in [0, c] \), then \( \hat{P}_\lambda(\text{Perc}_2) = 1 \).

We will proceed by comparing \( \hat{V} \cap H_2 \) with the vacant set of a Poisson Boolean ball model and appealing to a theorem appearing independently both in [22] and [23]. Therefore we first recall the part which we will use from that theorem. Let \( f \) be a positive measure with finite mass on \( \mathbb{R}^+ \) and let \( \hat{\omega}_{pb} \) be a Poisson point process on \( \mathbb{R}^2 \times \mathbb{R}^+ \) with intensity measure \( \lambda \ell^2 \times f \). Then let

\[ \hat{V}_{pb} = \mathbb{R}^2 \backslash \bigcup_{(x,R) \in \hat{\omega}_{pb}} B^2(x,R), \]

be the vacant set in this Poisson Boolean ball model. Theorem 2 in [23] or alternatively Theorem 2 in [22] (applied to \( \mathbb{R}^2 \)) implies that if

\[ \int_0^\infty R^2 df(R) < \infty, \]

then there is a.s. percolation in \( \hat{V}_{pb} \) when \( \lambda > 0 \) is sufficiently small.

**Alternative proof of Theorem 6.7** Let

\[ \hat{\omega}_b = \sum_{L \in \hat{\omega}} \delta_{(\text{cent}(E_2(L,r)), \text{diam}(E_2(L,r))/2)}. \]

In the same way as Theorem 5.3 was argued, one shows that \( \omega_b \) is a Poisson point process on \( \mathbb{R}^2 \times \mathbb{R}^+ \) with intensity measure \( \lambda \ell^2 \times \hat{g} \), where

\[ \hat{g}() = \xi_{2,r}(E : \text{diam}(E)/2 \in \cdot). \]

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Moreover, if we let
\[ \hat{V}_b = \mathbb{R}^2 \setminus \bigcup_{(x,R) \in \hat{\omega}_b} B^2(x,R), \]
then clearly
\[ \hat{V}_b \subset \hat{\nu} \cap H_2. \]

Similar constructions were used in Section 5.1. We will now be done if we can show that condition (68) holds with \( \hat{g} \) in place of \( f \). We have that
\[
\int_0^\infty R^2 d\hat{g}(R) = \int_0^\infty R^2 \xi_{2,r}(E : \text{diam}(E)/2 \in dR) = \|\xi_{2,r}\|_{TV} \int_0^\infty R^2 \tilde{\xi}_{2,r}(E : \text{diam}(E)/2 \in dR) = \frac{\|\xi_{2,r}\|_{TV}}{4} E_{\xi_{2,r}}[\text{diam}(E)^2].
\]

Lemma 6.3 applied to \( k = 2 \) and \( n = 2 \) shows that the last expectation is finite whenever \( d \geq 4 \), finishing the proof. \( \Box \)

7 The fractal process: Connectivity phase and domination by the fractal ball model

We will split the proof of Theorem 1.3 into two separate statements as the proofs use different approaches.

7.1 Connectivity when \( d \geq 4 \).

The objective of this subsection is to show that whenever \( \lambda > 0 \) is small enough, the fractal model contains non-trivial connected components with probability one. This result is known to hold for the regular fractal ball model, and our strategy is to couple the fractal cylinder model with this ball model and infer the result from this coupling. Since the cylinders are unbounded we will need to consider the intersection of the cylinder process with a lower dimensional subspace \( H_k \), i.e. we will use the induced ellipsoid process of Sections 5 and 6.

Therefore, define the regular fractal ball model as follows. Consider a Poisson process \( \omega_{\text{reg}} \) on \( \mathbb{R}^k \times \mathbb{R}_+ \) with intensity measure
\[
\lambda \ell_k \times R^{-k-1}I(0 < R \leq 2)dR,
\]
and then let
\[ \nu_{\text{reg}}(\omega_{\text{reg}}) := \mathbb{R}^k \setminus \bigcup_{(x,R) \in \omega_{\text{reg}}} B^k(x,R). \]

The intensity measure in (69) corresponds to a scale invariant model with an upper cutoff (of 2) on the radius. It is well known that for \( \lambda > 0 \) small enough, \( \nu_{\text{reg}}(\omega_{\text{reg}}) := \nu_{\text{reg}}(\omega_{\text{reg}}) \cap [0,1]^k \times \{0\}^{d-k} \) contains non-trivial connected components with positive probability (see for example Theorem 2.4 in [14]). We note that it is customary to use
The fractal cylinder process Broman, Elias, Mussini, Tykesson

$I(0 < R \leq 1)$ in place of $I(0 < R \leq 2)$ in (69). However, because of scaling, this does not change the conclusion that $V^k_{\text{reg}}(\omega_{\text{reg}})$ contains non-trivial connected components with positive probability as long as $\lambda > 0$ is small enough.

We now informally explain how the coupling between $\omega_{\text{reg}}$ and the fractal cylinder model will be performed. There are three steps to the procedure. First of all, the cylinder process induces a Poisson ball model as described in Section 5 (see in particular Theorem 5.3). Secondly, we will argue that “large” balls of this induced process can be disregarded (this uses the results of Section 6), and so essentially the induced fractal ball model will have a cutoff similar to the regular fractal ball model. The third and final step will be to prove that the induced ball model with a cutoff can be suitably dominated by a regular fractal ball model, and thereby we obtain a comparison between $V^k$ and $V^k_{\text{reg}}$ (recall that $V^k = V \cap [0,1]^k \times \{0\}^{d-k}$).

We can now state and prove the following result.

Theorem 7.1. For $d \geq 4$, $\lambda_c \in (0, \infty)$.

Proof. Fix $d \geq 4$ and $k \leq d/2$. We note that it would suffice to let $k = 2$ throughout, but keeping $k$ in place does not change the proof. The proof will rely on the discussion above that for $\lambda > 0$ small enough, $V^k_{\text{reg}}$ contains non-trivial connected components with positive probability.

The first step is short as most of the work is already done. Recall the notation $\omega_b$, Theorem 5.3, and the fact that $V^b \subset V \cap H_k$, where $V^b$ is the induced fractal ball model as defined in Section 5.

Our second step will be to remove “large” balls. To that end, let

$$\tilde{\omega}_b := \bigcup_{(x,R) \in \omega_b} \delta_{(x,R)},$$

so that $\tilde{\omega}_b$ is obtained by taking $\omega_b$ and removing any ball with radius larger than 2. Then, let

$$\tilde{V}^b := \mathbb{R}^k \setminus \bigcup_{(x,R) \in \tilde{\omega}_b} B^k(x,R)$$

and observe that $\tilde{V}^b \supset V^b$. As before, we will use the notation $\tilde{V}^k_{\text{reg}} = \tilde{V}^b \cap [0,1]^k \times \{0\}^{d-k}$ and similar for $V^k_{\text{reg}}$. It is clearly the case that $\mathbb{P}(\omega_b = \tilde{\omega}_b) = 0$ since these are processes on the entire space $\mathbb{R}^k$. However, by restricting our attention to $[0,1]^k \times \{0\}^{d-k}$ this will not pose a problem. Therefore, let

$$\mathcal{H}_0 := \{ \exists (x,R) \in \omega_b : R > 2, B(x,R) \cap [0,1]^k \times \{0\}^{d-k} \neq \emptyset \}$$

be the event that no “large” balls from $\omega_b$ hits $[0,1]^k \times \{0\}^{d-k}$.

Observe that by the nature of Poisson processes, conditioned on $\mathcal{H}_0$, we have that $V^k_{\text{reg}}$ has the same distribution as $\tilde{V}^k_{\text{reg}}$. Therefore,

$$\mathbb{P}(V^k_{\text{reg}} \text{ contains non-trivial connected components}) \geq \mathbb{P}(V^k_{\text{reg}} \text{ contains non-trivial connected components}|\mathcal{H}_0)\mathbb{P}(\mathcal{H}_0)$$

$$= \mathbb{P}(\tilde{V}^k_{\text{reg}} \text{ contains non-trivial connected components})\mathbb{P}(\mathcal{H}_0).$$
Consider now $P(H_0)$ and note that by Theorem 5.3 and Lemma 6.5 (which holds when the diameter is at least 4) we have that

\begin{equation}
P(H_0) = \exp \left( -\lambda \ell_k \times g((x,R) : R > 2, B^k(x,R) \cap [0,1]^k \times \{0\}^{d-k} \neq \emptyset) \right) \quad (71)
\end{equation}

whenever $d \geq 4$ and $k \leq d/2$. Observe that \((70)\) and \((71)\) allow us to transfer the problem from $\mathcal{V}^k_b$ to one about $\hat{\mathcal{V}}^k_b$. This concludes the second step of the proof.

The third step will be to show that $\hat{\mathcal{V}}^k_b$ contains non-trivial connected components with positive probability, and this will be done by comparison with $\mathcal{V}^k_{reg}$. To that end we note that the intensity measure corresponding to $\hat{\mathcal{V}}^k_b$ is simply

$$
\lambda d\ell_k(x) \times dg(R)I(0 < R \leq 2),
$$

since there are no balls of radius larger than 2 in $\hat{\omega}_b$.

By Lemma 6.6 (in particular Equation (65)),

$$
dg(R) = 2^{-k} \xi_{k,1}(\text{diam}(E)^k; \text{diam}(E) \geq 2R)R^{-k-1}dR
$$

\begin{align*}
= 2^{-k} \|\xi_{k,1}\|_{TV} \mathbb{E}_{\xi_{k,1}} \left[ \text{diam}(E)^k; \text{diam}(E) \geq 2R \right] R^{-k-1}dR \\
= 2^{-k} \beta_R \|\xi_{k,1}\|_{TV} R^{-k-1}dR,
\end{align*}

where

$$
\beta_R := \mathbb{E}_{\xi_{k,1}} \left[ \text{diam}(E)^k; \text{diam}(E) \geq 2R \right] \leq \mathbb{E}_{\xi_{k,1}} \left[ \text{diam}(E)^k \right] (= \beta_0) < \infty.
$$

In \((72)\), the factor $2^{-k}$ appears since Lemma 6.6 is formulated for $\text{diam}(E)$ rather than $\text{diam}(E)^k$. The fact that $\beta_0 < \infty$ comes from Lemma 6.3 where we used $n = k$ and that $k \leq d/2$. Letting

$$
dg_0(R) := 2^{-k} \beta_0 \|\xi_{k,1}\|_{TV} R^{-k-1}I(0 < R \leq 2)dR,
$$

we let $\omega_{b,0}$ be a Poisson process on $\mathbb{R}^k \times \mathbb{R}_+$ with the intensity measure

$$
\lambda d\ell_k(x) \times dg_0(R).
$$
Clearly, this is the same intensity measure as in (69) but with a different constant. Letting
\[ V(\omega_{b,0}) = \mathbb{R}^k \setminus \bigcup_{(x,R) \in \omega_{b,0}} B^k(x,R), \]
it therefore follows that \( V(\omega_{b,0}) \) contains non-trivial connected components whenever \( \lambda > 0 \) is chosen small enough.

It is straightforward to couple \( \tilde{\omega}_b \) and \( \omega_{b,0} \) so that \( V(\omega_{b,0}) \subset \tilde{V}_b \).

Indeed, one can do this by considering a point process on the space \( \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^+ \) with intensity measure
\[ \lambda dx \times R^{-k-1}I(0 < R \leq 2) dR \times I(q > 0) dq, \]
and then for a given triple \( (x,R,q) \) we let \( (x,R) \in \omega_{b,0} \) iff \( q \leq 2^{-k} \beta_0 ||\xi_{k,1}\||_{TV} \), and \( (x,R) \in \tilde{\omega}_b \) iff \( q \leq 2^{-k} \beta_0 \parallel \xi_{k,1} \parallel_{TV} \). Letting \( \mathbb{P}_{b,0} \) denote the law of \( \omega_{b,0} \), we conclude that
\[ \mathbb{P}(\tilde{V}_b^k \text{ contains non-trivial connected components}) \geq \mathbb{P}_{b,0}(V^k(\omega_{b,0}) \text{ contains non-trivial connected components}) > 0 \]

By combining (70), (71) and (73) the statement follows.

7.2 Connectivity when \( d = 2,3 \).

We now turn to the final case of connectivity when \( d = 2,3 \). We will prove the following theorem.

**Theorem 7.2.** Let \( \lambda > 0 \).

a) For \( d = 2 \), there are almost surely only trivial connected components in \( V \).

b) For \( d = 3 \), there are almost surely only trivial connected components in \( V \cap H_2 \).

Before we proceed with the proof of Theorem 7.2, we need to establish some notation and auxiliary results. For \( s,t > 0 \), we define the rectangle \( K(s,t) := [-s/2,s/2] \times [-t/2,t/2] \subset H_2 \) and we write \( K(s) = K(s,s) \) for the square of side-length \( s \) centred at the origin. For \( \epsilon > 0 \), let \( LR = LR(\epsilon) \) denote the set of ellipsoids centred in the square \( K(\epsilon/2) \subset H_2 \) intersecting both the left and right-hand sides of the rectangle \( K(3\epsilon,\epsilon) \). That is,
\[ LR = \{ E \in \mathcal{C}^2 : \text{cent}(E) \in K(\epsilon/2), E \cap \{-3\epsilon/2\} \times [-\epsilon/2,\epsilon/2] \neq \emptyset, E \cap \{3\epsilon/2\} \times [-\epsilon/2,\epsilon/2] \neq \emptyset \}. \]

For any set \( R \subset \mathbb{R}^2 \) and \( A,B \subset \mathbb{R}^2 \) we define
\[ A \xrightarrow{\mathcal{R}} B \]
to be the event that there exists a connected component in $R$ intersecting both $A$ and $B$. Let
\[
\text{Arm}(\epsilon, R) = \left\{ K(\epsilon) \overset{R}{\leftrightarrow} \partial K(3\epsilon) \right\},
\]
be the event that there is a crossing in $R$ of the annulus $K(3\epsilon) \setminus K(\epsilon)$.

**Lemma 7.3.** Assume that $R$ is a random closed subset of $\mathbb{R}^2$ and that the law $P$ of $R$ is invariant under translations and rotations of $\mathbb{R}^2$. Assume further that for every $\epsilon \in (0, 1/5)$ we have
\[
P \left( \{-3\epsilon/2\} \times [-\epsilon/2, \epsilon/2] \overset{R \cap K(3\epsilon, \epsilon)}{\leftrightarrow} \{3\epsilon/2\} \times [-\epsilon/2, \epsilon/2] \right) = 1. \tag{75}
\]
Then $P(R \text{ is totally disconnected}) = 1$.

Observe that the event inside (75) is the existence of a connected component contained in $R^c \cap K(3\epsilon, \epsilon)$ connecting the right and left sides of the rectangle $K(3\epsilon, \epsilon)$.

**Proof.** Let $\epsilon \in (0, 1/5)$. First observe that if the four events
\[
\{-3\epsilon/2\} \times [\epsilon/2, 3\epsilon/2] \overset{R^c \cap K(3\epsilon, \epsilon)+(0, \epsilon)}{\leftrightarrow} \{3\epsilon/2\} \times [\epsilon/2, 3\epsilon/2],
\]
\[
\{-3\epsilon/2\} \times [-3\epsilon/2, -\epsilon/2] \overset{R^c \cap K(3\epsilon, \epsilon)-(0, \epsilon)}{\leftrightarrow} \{3\epsilon/2\} \times [-3\epsilon/2, -\epsilon/2]
\]
\[
[\epsilon/2, 3\epsilon/2] \times \{-3\epsilon/2\} \overset{R^c \cap K(3\epsilon, \epsilon)+(\epsilon, 0)}{\leftrightarrow} [\epsilon/2, 3\epsilon/2] \times \{3\epsilon/2\},
\]
\[
[-3\epsilon/2, -\epsilon/2] \times \{-3\epsilon/2\} \overset{R^c \cap K(3\epsilon, \epsilon)-(\epsilon, 0)}{\leftrightarrow} [-3\epsilon/2, -\epsilon/2] \times \{3\epsilon/2\},
\]
all occur, then $\text{Arm}(\epsilon, R)$ does not occur. Furthermore, since the aforementioned four events all have probability 1 (by assumption (75) and the rotational and translation invariance of the law of $R$), we get that
\[
P(\text{Arm}(\epsilon, R)) = 0. \tag{76}
\]
Now let $q \in \mathbb{Q}^2$, $\epsilon \in (0, 1/5) \cap \mathbb{Q}$ and $\text{Arm}_q(\epsilon, R)$ denote the event that there is a crossing in $R$ in the annulus
\[
q + K(3\epsilon) \setminus K(\epsilon).
\]
Then we have that
\[
\{R \text{ is totally disconnected} \}^c \subset \bigcup_{q \in \mathbb{Q}^2} \bigcup_{\epsilon \in (0, 1/5) \cap \mathbb{Q}} \text{Arm}_q(\epsilon, R),
\]
since if $R$ has a connected component there must exist some $q \in \mathbb{Q}^2$ and some $\epsilon \in (0, 1/5) \cap \mathbb{Q}$ such that there is a crossing of the annulus
\[
q + K(3\epsilon) \setminus K(\epsilon).
\]
Hence we have
\[
P(\{R \text{ is totally disconnected}\}^c) \leq \sum_{q \in \mathbb{Q}^2} \sum_{\epsilon \in (0,1/5) \cap \mathbb{Q}} P(\text{Arm}_q(\epsilon, R)) = 0,
\]
using (76) and translational invariance in the last equality. Thus
\[
P(\{R \text{ is totally disconnected}\}) = 1,
\]
as required.

To deal with \(V \cap H_2\) when \(d = 3\) we will need one additional result. We first recall some formulas used if \(d = 3\) and \(k = 2\). For \(d = 3\), the expression for the shape measure \(\xi_{2,r}\) is given by
\[
\xi_{2,r}(\cdot) = \int_{E_2(L,r)_e} \frac{1}{\left(1 + a_1^2 + a_2^2\right)^2} da_1 da_2,
\]
see Theorem (1.5) and Equation (47). Moreover, according to Corollary 5.2, the expression for the diameter of an ellipse \(E_2(L,r)\) is then given by
\[
diam(E_2(L,r)) = 2r \sqrt{1 + a_1^2 + a_2^2}.
\]
Recall also the notation
\[
\omega_e = \sum_{(L,r) \in \omega} \delta(\text{cent}(E_2(L,r)), E_2(L,r)_e).
\]
We have the following lemma.

**Lemma 7.4.** Let \(d = 3\) and \(\epsilon \in (0,1/5)\). Then for every \(\lambda > 0\),
\[
P(LR(\epsilon, \omega_e)) = 1. \quad (77)
\]

**Proof.** The proof essentially follows that of the lower bound of Proposition 5.1 in [24]. For an ellipse \(E \in \mathbb{E}^2\), let \(\arg(E) \in [-\pi/2, \pi/2)\) denote the angle between the \(e_1\) axis and the line containing the major axis of \(E\). It is easy to check that if \(E\) satisfies
\[
\text{cent}(E) \in K(\epsilon/2),
\]
\[
diam(E) \geq 10\epsilon, \ |\arg(E)| < 1/10,
\]
then \(E \in LR\). Therefore, if we let
\[
LR_1 = LR_1(\epsilon, \omega_e) = \{E \in \mathbb{E}^2 : \text{cent}(E) \in K(\epsilon/2), \text{diam}(E) \geq 10\epsilon, \ |\arg(E)| < 1/10\},
\]
then we have \(LR_1 \subset LR\) so that
\[
LR_1(\omega_e) \subset LR(\omega_e).
\]
Using Theorem 1.5 we have that the intensity measure of \( \omega_e \) is given by \( \lambda \ell_2 \times \xi_2 \) where
\[
\xi_2(\cdot) = \int_0^1 \xi_{2,r}(\cdot)r^{-3}dr.
\]
Hence
\[
\mathbb{P}(LR(\omega_e)) \geq \mathbb{P}(LR_1(\omega_e)) = 1 - e^{-\lambda \ell_2 \times \xi_2(LR_1)}.
\]
It remains to show that \( \ell_2 \times \xi_2(LR_1) = \infty \). We have that
\[
\ell_2 \times \xi_2(LR_1) = \ell_2(K(\epsilon/2)) \int_0^1 \int_{\text{diam}(E_2(L,r)) \geq 10 \epsilon} \frac{1}{(1 + a_1^2 + a_2^2)^2} r^{-3} da_1 da_2 dr
\]
\[
= \frac{\epsilon^2}{4} \int_0^1 \int_{\text{diam}(E_2(L,r)) \geq 10 \epsilon} \frac{1}{(1 + a_1^2 + a_2^2)^2} r^{-3} da_1 da_2 dr. \tag{78}
\]
We now change coordinates from \((a_1,a_2)\) to polar coordinates \((\rho,\theta)\) in \( \mathbb{R}_+ \times [-\pi,\pi) \). By the facts that \( \theta = \arg(E(r)) = \arctan(a_2/a_1) \) and \( \text{diam}(E(r)) = 2r/1 + a_1^2 + a_2^2 \), Equation (78) equals
\[
\frac{\epsilon^2}{4} \int_0^1 \int_{|\theta| < 1/10} \frac{\rho}{(1 + \rho^2)^2} r^{-3} d\rho d\theta dr = \frac{\epsilon^2}{20} \int_0^1 \int_{r \sqrt{1 + \rho^2} \geq 5 \epsilon} \frac{\rho}{(1 + \rho^2)^2} r^{-3} d\rho dr
\]
\[
= \frac{\epsilon^2}{20} \int_0^1 \left[ \frac{-1}{2(1 + \rho^2)} \right]^{\infty}_{\max(5\epsilon/r,1)} r^{-3} dr = \frac{\epsilon^2}{40} \int_0^1 \frac{1}{\max(5\epsilon/r,1)^2} r^{-3} dr
\]
\[
\geq \frac{\epsilon^2}{40} \int_0^{5\epsilon} \frac{1}{25\epsilon^2} r^{-1} dr = \infty, \tag{79}
\]
as required.

\[\square\]

\textit{Proof of Theorem 7.2.} We start with part a). Recall that in this case, we work with \( d = 2 \). The aim is to verify the assumption (75) of Lemma 7.3. For \( \epsilon > 0 \), define
\[
\text{Cross}(\epsilon) = L_{(-3\epsilon/2) \times [-\epsilon/2,\epsilon/2],(3\epsilon/2) \times [-\epsilon/2,\epsilon/2]} \subset A(2,1).
\]
We observe that for any \( \lambda > 0 \) and \( \epsilon > 0 \) we have that
\[
(\lambda \nu_2 \times \rho_\lambda)((L,r) : L \in \text{Cross}(\epsilon)) = \int_0^1 \nu_2(\text{Cross}(\epsilon)) r^{-2} dr = \infty,
\]
since \( \nu_2(\text{Cross}(\epsilon)) > 0 \) (this claim is easy to check and left to the reader). This implies that
\[
\mathbb{P}(\exists (L,r) \in \omega : L \in \text{Cross}(\epsilon)) = 1. \tag{80}
\]
Since
\[ \{\exists (L,r) \in \omega : L \in \text{Cross}(\epsilon)\} \subset \left\{ \{3\epsilon/2\} \times [-\epsilon/2,\epsilon/2] \right\}, \]
we get the result from (80) and Lemma 7.3.

We now move on to prove part b), for which most of the work has already been done. Let \( d = 3 \) and \( \lambda > 0 \). We have that
\[ \{LR(\epsilon,\omega_e)\} \subset \left\{ \{3\epsilon/2\} \times [-\epsilon/2,\epsilon/2] \right\} \]
and so the statement follows from Lemmas 7.4 and 7.3.

We can now prove Theorem 1.3.

**Proof of Theorem 1.3.** The statement for \( d \geq 4 \) is Theorem 7.1 while the statements for \( d = 2,3 \) are covered by Theorem 7.2.

\[ \square \]

## 8 A list of open questions

This section contains a brief list of some open questions. The first of these appears already in Remark 4.9 and so we refer to that remark for further discussion.

**Question 8.1.** Does \( V^k \neq \emptyset \) imply that almost surely \( \dim_H(V^k) = k - \lambda \)?

Our next question is related to Theorem 1.1 in [25]. For \( x \in \mathbb{R}^d \), let \( V_x \) denote the connected component containing \( x \) and partition \( V \) into the “dust” defined by
\[ V^{\text{dust}} = \{ x \in V : V_x = \{x\} \} \]
and the “connected” set \( V^{\text{conn}} = V \setminus V^{\text{dust}}. \)

**Question 8.2.** Is it the case that if \( V \neq \emptyset \) then
\[ \dim_H(V^{\text{conn}}) < \dim_H(V) \text{ almost surely?} \]

The question is then simply whether it is the dust that “carries” the dimension of the random fractal, as it was shown to do for the Mandelbrot fractal percolation model in [25].

In any model that exhibits a phase transition it is of interest to determine what happens at the critical point of said phase transition. In the current paper, we determine (in Theorem 1.1) that at the critical point of the existence phase transition, the fractal set is almost surely empty. For the Mandelbrot fractal percolation model and for every Poissonian random fractal with bounded objects, it was proven in [14] (see Theorem 2.4 and Corollary 2.6) that those random fractals were in the connected phase at the point of the connectivity phase transition.
Question 8.3. Is it the case that $\mathbb{P}_{\lambda_c}(\mathcal{V}(\omega) \text{ is totally disconnected}) = 0$?

An integral ingredient of the proof of Theorem 2.4 in [14] is that the fractal in two spatially well separated regions are independently generated. This is clearly not the case in the current paper since for any two (non-degenerate) regions, the probability that some cylinder will intersect both of these is in fact one.

Two additional questions related to the connectivity phase are given in and below Remark 1.4. The first addresses the fact that Theorem 1.3 is incomplete.

Question 8.4. Is it the case that $\lambda_c(3) > 0$?

The second question concerns a more subtle topological property of the random fractal.

Question 8.5. Consider any connected component of $\mathcal{V}^{\text{conn}}$. Is it the case that this component is also path connected?

Clearly, the model in this paper is just one out of many possible choices of (fractal) models that lack the finite energy property (see the discussion in the Introduction). One other such model can easily be created from the Brownian interlacement model introduced by Sznitman in [4]. It seems to us that the techniques of the proofs of the existence phase transition and the computation of the Hausdorff dimension are robust enough to handle a wide variety of models lacking the finite energy property. However, the analysis of the connectivity phase transition will need new ideas and tools. We attempt to formulate this into a question as follows.

Question 8.6. Construct a fractal Brownian interlacements model and study the connectivity properties of that model. Are there any qualitative differences compared to the fractal cylinder process?
Appendix A

The appendix is organised as follows. First, in part A.1 we show that the invariant measure \( \nu_d \) can be written as in \[10\]. In part A.2 we give the proof of our 0-1 law, Lemma 2.3 In part A.3 we give the proof of Lemma 5.1. Finally, in part A.4 we give the proofs that were omitted in Section 6.

A.1 The invariant measure on the space of lines

The aim of this part of the appendix is to derive the representation of the invariant measure \( \nu_d \) given in \[10\]. We will do this starting from a third representation given in \[26\].

We first recall the parametrization we use for lines in this paper. We write a line \( L \in A(d,1) \) as \( L = L(a,p) = \{ at + p : t \in \mathbb{R} \} \) where \( a = (a_1, \ldots, a_{d-1}, 1) \in \mathbb{R}^{d-1} \times \{ 1 \} \) and \( p = (p_1, \ldots, p_{d-1}, 0) \in \mathbb{R}^{d-1} \times \{ 0 \} \). We have the following result.

**Theorem A.1.** For \( d \geq 2 \), the Haar measure on \( A(d,1) \) is given by

\[
d\nu_d(L) = d\nu_d(a_1, a_2, \ldots, a_{d-1}, p_1, p_2, \ldots, p_{d-1}) = \frac{\Upsilon_d}{||a||^{d+1}}da_1da_2\ldots da_{d-1}dp_1\ldots dp_{d-1}. \tag{83}\]

Before the proof of Theorem A.1 we describe a slightly different parametrization of a line \( L \in A(d,1) \). Let (as in Section 2.3) \( \partial B(o,1)_+ = \partial B(o,1) \cap \{ x \in \mathbb{R}^d : x_d > 0 \} \) denote the upper hemisphere. We will write \( \alpha \in \partial B(o,1)_+ \) as \( \alpha = (\alpha_1, \ldots, \alpha_d) \) where \( \alpha_d = (1 - \sum_{i=1}^{d-1} \alpha_i^2)^{1/2} \). Any line \( L \in A(d,1) \setminus \hat{A}(d,1) \) can be uniquely written as \( L = L(\alpha, p) = \{ \alpha t + p : t \in \mathbb{R} \} \) where \( \alpha \in \partial B(o,1)_+ \) and \( p \) is as above. In this parametrization, \( p \) is again the intersection between \( L(\alpha,p) \) and \( H_{d-1} \), while \( \alpha \) describes the direction of the line.

According to Equation (1.7) in \[26\], the invariant measure \( d\nu_d(L) \) using the parametrization \( (\alpha, p) \) is given by

\[
d\nu_d(\alpha, p) = \Upsilon_d \sin(\theta)dp_1\cdots dp_{d-1}I(\alpha \in \partial B(o,1)_+)\ell_{d-1}(d\alpha), \tag{84}\]

where \( \theta = \theta(\alpha) \) is defined as the angle between \( L(\alpha, p) \) and \( H_{d-1} \). We note that the normalization used in \[26\] is different (see in particular (1.3) in that paper) from the one we use here, and this must be taken into account when arriving at (84).

Note that

\[
\sin(\theta) = \left(1 - \sum_{i=1}^{d-1} \alpha_i^2\right)^{1/2}. \tag{85}\]

Moreover, according to Equation A.3 in \[27\], we have that

\[
I(\alpha \in \partial B(o,1)_+)d\ell_{d-1}(\alpha) = \frac{d\alpha_1 \cdots d\alpha_{d-1}}{(1 - \sum_{i=1}^{d-1} \alpha_i^2)^{1/2}}. \tag{86}\]

By (84), (85) and (86) we get

\[
d\nu_d(\alpha, p) = \Upsilon_d d\alpha_1 \cdots d\alpha_{d-1}dp_1 \cdots dp_{d-1}. \tag{87}\]
We can now prove Theorem A.1.

**Proof of Theorem A.1.** Note that \( a \mapsto a/\|a\| \) is a bijection between \( \mathbb{R}^{d-1} \times \{1\} \) and the hemisphere \( \partial B(0,1) \). To go between the parametrization \((a,p)\) and the parametrization \((\alpha,p)\) we simply let \( \alpha = a/\|a\| \). Then \( L(a,p) \) is the same line as \( L(\alpha,p) \). From (87) we see that to obtain (83) from (84), it suffices to show that the determinant of the Jacobian corresponding to this change of coordinates is given by \( \|a\|^{-d-1} \).

We have that
\[
\alpha_i = \frac{a_i}{\|a\|} = \frac{a_i}{\sqrt{a_1^2 + \ldots + a_{d-1}^2 + 1}},
\]
for \( i = 1, \ldots, d \), where we recall that \( a_d = 1 \). Let \( M \) be the Jacobian corresponding to the change of variables from \( a \) to \( \alpha \). That is, \( M \) is the \((d-1) \times (d-1)\) square matrix with entries \( (m_{i,j}) \) where \( m_{i,j} = \frac{\partial \alpha_i}{\partial a_j} \). A straightforward calculation shows that
\[
m_{i,i} = \frac{\|a\|^2 - a_i^2}{\|a\|^3}, \quad \text{and that} \quad m_{i,j} = \frac{-a_i a_j}{\|a\|^3} \quad \text{whenever} \quad i \neq j.
\]

Let \( u = -(a_1, \ldots, a_{d-1}) \) so that
\[
M = \frac{1}{\|a\|^2} (\|a\|^2 I_{d-1} - u^T u).
\]

We then see that by the matrix determinant lemma,
\[
det(M) = \frac{1}{\|a\|^{2(d-1)}} \det(\|a\|^2 I_{d-1} - u^T u) \\
= \frac{1}{\|a\|^{2(d-1)}} \left( 1 - u(\|a\|^2 I_{d-1})^{-1} u^T \right) \det(\|a\|^2 I_{d-1}) \\
= \frac{1}{\|a\|^{d-1}} \left( 1 - \sum_{k=1}^{d-1} a_k^2 \right) = \frac{1}{\|a\|^{d-1}} \frac{1}{\|a\|^2} = \frac{1}{\|a\|^{d+1}},
\]
and the statement follows. \( \square \)

### A.2  A 0-1 law

**Proof of Lemma 2.3.** We will start by considering events depending only on some finite region of \( A(d,1) \times (0,1] \). To that end, for \( R \geq 1 \), let
\[
\Gamma_{x,R} = \{(L,r) \in A(d,1) \times (0,1] : L \in \mathcal{L}_{B(x,R)} \text{ and } r \in (R^{-1},1]\},
\]
and
\[
\omega|_{\Gamma_{x,R}} = \{(L,r) \in \omega : (L,r) \in \Gamma_{x,R}\}
\]
so that \( \omega|_{\Gamma_{x,R}} \) is the restriction of \( \omega \in \Omega \) to the finite region \( \Gamma_{x,R} \). When \( x = o \) we will simply write \( \Gamma_R \). The event \( F \) is determined by \( \Gamma_{x,R} \) if and only if for every \( \omega \in F \) and
any \( \tilde{\omega} \) such that \( \omega|_{\Gamma_{x,R}} = \tilde{\omega}|_{\Gamma_{x,R}} \) we have that \( \tilde{\omega} \in F \). Note that if \( F \) is determined by \( \Gamma_{x,R} \) then it is also determined by \( \Gamma_{x,R'} \) for any \( R' > R \).

We will now prove that, for any events \( F_i \) determined by \( \Gamma_{x,R} \) for \( i = 1, 2 \) where \( \|x_1 - x_2\| \geq 4R \) we have that

\[
|\mathbb{P}(F_1 \cap F_2) - \mathbb{P}(F_1)\mathbb{P}(F_2)| \leq 4 \left( 1 - \exp \left( -\frac{\lambda c_2 R^{2(d-1)} (R^{d-1} - 1)}{(d-1)\|x_1 - x_2\|^{d-1}} \right) \right). 
\] (88)

For simplicity, write \( \Gamma_{x_1,x_2,R} = \mathcal{L}_{B(x_1,R),B(x_2,R)} \times (R^{-1},1] \) and observe that

\[
\mathbb{P}(F_1 \cap F_2) = \mathbb{P}(F_1|\omega(\Gamma_{x_1,x_2,R}) = 0)\mathbb{P}(F_2|\omega(\Gamma_{x_1,x_2,R}) = 0) \mathbb{P}(\omega(\Gamma_{x_1,x_2,R}) = \omega(\Gamma_{x_1,x_2,R}') = 0),
\] (89)

since the events \( F_1 \) and \( F_2 \) are conditionally independent on the event that \( \omega(\Gamma_{x_1,x_2,R}) = 0 \). Furthermore, writing

\[
\mathbb{P}(F_i) = \mathbb{P}(F_i|\omega(\Gamma_{x_1,x_2,R}) = 0)\mathbb{P}(\omega(\Gamma_{x_1,x_2,R}) = 0) + \mathbb{P}(F_i|\omega(\Gamma_{x_1,x_2,R}) \neq 0)\mathbb{P}(\omega(\Gamma_{x_1,x_2,R}) \neq 0)
\]

for \( i = 1, 2 \) and using (89), a straightforward calculation gives us that

\[
|\mathbb{P}(F_1 \cap F_2) - \mathbb{P}(F_1)\mathbb{P}(F_2)| \leq 4\mathbb{P}(\omega(\Gamma_{x_1,x_2,R}) \neq 0).
\]

We will now bound \( \mathbb{P}(\omega(\Gamma_{x_1,x_2,R}) \neq 0) \). First we use Lemma 2.1 part c) to see that

\[
\mathbb{P}(\omega(\Gamma_{x_1,x_2,R}) \neq 0) = 1 - \exp \left( -\lambda \int_{R^{-1}}^1 \nu_d \left( \mathcal{L}_{B(x_1,R),B(x_2,R)} \right) r^{-d} dr \right)
\]

\[
= 1 - \exp \left( -\lambda \int_{R^{-1}}^1 R^{d-1} \nu_d \left( \mathcal{L}_{B(x_1',R),B(x_2',R)} \right) r^{-d} dr \right),
\]

where \( x_1' \) and \( x_2' \) is as in Lemma 2.1. Since \( \|x_1' - x_2'\| = R^{-1}\|x_1 - x_2\| \geq 4 \) we use Lemma 2.2 to see that

\[
\mathbb{P}(\omega(\Gamma_{x_1,x_2,R}) \neq 0)
\leq 1 - \exp \left( -\lambda \int_{R^{-1}}^1 c_2 \left( \frac{R}{\|x_1' - x_2'\|} \right)^{d-1} r^{-d} dr \right)
\]

\[
= 1 - \exp \left( -\lambda c_2 \left( \frac{R}{\|x_1' - x_2'\|} \right)^{d-1} \left( \frac{1}{1 - d} - \frac{R^{d-1}}{1 - d} \right) \right)
\]

\[
= 1 - \exp \left( -\lambda c_2 R^{d-1} (R^{d-1} - 1) \left( \frac{1}{(d-1)\|x_1' - x_2'\|^{d-1}} \right) \right)
\]

\[
= 1 - \exp \left( -\lambda c_2 R^{2(d-1)} (R^{d-1} - 1) \left( \frac{1}{(d-1)\|x_1 - x_2\|^{d-1}} \right) \right)
\]
which proves (88).

Consider now some arbitrary shift-invariant event \( F \). Define
\[
I_{F,x,R} := I \left( \omega \in \left\{ \mathbb{P}(F \mid \omega|_{\Gamma_{x,R}}) > 1/2 \right\} \right),
\]
and note that informally, we have that \( I_{F,x,R}(\omega) = 1 \) if \( \omega \) is such that knowledge of \( \omega|_{\Gamma_{x,R}} \), i.e. the configuration inside \( \Gamma_{x,R} \), makes the occurrence of \( F \) probable. The value \( 1/2 \) is somewhat arbitrary.

Note also that the event that \( I_{F,x,R} = 1 \) is determined by \( \Gamma_{x,R} \). Then, by Lévy’s 0-1 law,
\[
\lim_{R \to \infty} I_{F,o,R} = I_F \text{ a.s.}
\]
Using that \( F \) is invariant under isometries, it is straightforward to prove that the laws of \( (I_{F},I_{F,o,R}) \) and \( (I_{F},I_{F,x,R}) \) are the same for every \( x \), and so \( I_{F,R^t,e_1,R} \) converges in probability to \( I_{F} \). Thus, \( \lim_{R \to \infty} \mathbb{P}(I_{F,o,R} = I_{F,R^t,e_1,R} = I_F) = 1 \) and so
\[
\lim_{R \to \infty} \mathbb{P}(I_{F,o,R} = I_{F,R^t,e_1,R} = 1) = \mathbb{P}(F).
\] (90)

By using (88) we then see that
\[
\lim_{R \to \infty} \left| \mathbb{P}(I_{F,o,R} = I_{F,R^t,e_1,R} = 1) - \mathbb{P}(I_{F,o,R} = 1)\mathbb{P}(I_{F,R^t,e_1,R} = 1) \right| \leq \lim_{R \to \infty} 4 \left( 1 - \exp \left( -c \frac{R^{2(d-1)}(R^{d-1} - 1)}{R^{4(d-1)}} \right) \right) = 0.
\]

We therefore conclude that
\[
\lim_{R \to \infty} \mathbb{P}(I_{F,o,R} = I_{F,R^t,e_1,R} = 1) = \lim_{R \to \infty} \mathbb{P}(I_{F,o,R} = 1)\mathbb{P}(I_{F,R^t,e_1,R} = 1) = \mathbb{P}(F)^2,
\]
and by comparing this to (90) we conclude that \( \mathbb{P}(F) \in \{0,1\} \).

\section*{A.3 Proof of Lemma 5.1.}

\textbf{Proof of Lemma 5.1.} Recall that we write \( L(a,p) = \{at + p : t \in \mathbb{R} \} \), see Section 2.3. Let \( x_k \in \mathbb{R}^k \), and for convenience, write \( x = (x_k,0,\ldots,0) \in \mathbb{R}^d \). The squared distance from the point \( x \) to the point \( L_t := at + p \) is given by
\[
f(t) = \|L_t - x\|^2 = \|at + p - x\|^2 = \|a\|^2 t^2 + \|p - x\|^2 + 2t \langle a, p - x \rangle,
\]
and since \( f'(t) = 2t\|a\|^2 + 2\langle a, p - x \rangle \), we see that \( f(t) \) is minimised at \( t^* = -\frac{\langle a, p - x \rangle}{\|a\|^2} \). Furthermore,
\[
f(t^*) = \frac{(a,p - x)^2}{\|a\|^2} + \|p - x\|^2 - 2\langle a, p - x \rangle \frac{\langle a, p - x \rangle}{\|a\|^2} = \|p - x\|^2 - \frac{(a,p - x)^2}{\|a\|^2} = \|p(k) - x_k\|^2 + \|p(k)\|^2 - \left( \frac{\langle a(k),p(k) - x_k \rangle + \langle a(k),p(k) \rangle^2}{\|a\|^2} \right)
\]
\[
= \|p(k) - x_k\|^2 + \|p(k)\|^2 - \left( \frac{\langle a(k),p(k) - x_k \rangle^2 + \langle a(k),p(k) \rangle^2 + 2\langle a(k),p(k) - x_k \rangle \langle a(k),p(k) \rangle}{\|a\|^2} \right).
\]

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Let $A$ be as above, and let

$$q_k = (q_{k,1}, \ldots, q_{k,k}) = -\frac{\langle a^{(k)}_p, p^{(k)}_k \rangle}{\|a^{(k)}_p\|^2 + 1} a^{(k)}_p.$$  

Since $I - A/\|a\|^2$ is symmetric, we have that

$$\begin{align*}
(p^{(k)}_k + q_k - x_k) \left( I - \frac{1}{\|a\|^2} A \right) (p^{(k)}_k + q_k - x_k)^T & \quad (91) \\
= (p^{(k)}_k - x_k) \left( I - \frac{1}{\|a\|^2} A \right) (p^{(k)}_k - x_k)^T \\
+ 2q_k \left( I - \frac{1}{\|a\|^2} A \right) (p^{(k)}_k - x_k)^T + q_k \left( I - \frac{1}{\|a\|^2} A \right) q_k^T,
\end{align*}$$

and furthermore,

$$(p^{(k)}_k - x_k) A (p^{(k)}_k - x_k)^T$$

$$= (p^{(k)}_k - x_k) a^{(k)}_p a^{(k)}_p (p^{(k)}_k - x_k)^T = \langle a^{(k)}_p, p^{(k)}_k - x_k \rangle^2.$$

Therefore, the first term on the right hand side of (91) equals

$$(p^{(k)}_k - x_k) \left( I - \frac{1}{\|a\|^2} A \right) (p^{(k)}_k - x_k)^T = \|p^{(k)}_k - x_k\|^2 - \frac{\langle a^{(k)}_p, p^{(k)}_k - x_k \rangle^2}{\|a\|^2}.$$  

Continuing, we note that since $a^{(k)}_p = a^{(k)}_p a^{(k)}_p a^{(k)}_p = \|a^{(k)}_p\|^2 a^{(k)}_p$ we have that

$$\begin{align*}
q_k \left( I - \frac{1}{\|a\|^2} A \right) (p^{(k)}_k - x_k)^T \\
= -\frac{\langle a^{(k)}_p, p^{(k)}_k \rangle}{\|a^{(k)}_p\|^2 + 1} \left( a^{(k)}_p - \frac{1}{\|a\|^2} \|a^{(k)}_p\|^2 a^{(k)}_p \right) (p^{(k)}_k - x_k)^T \\
= \frac{\langle a^{(k)}_p, p^{(k)}_k \rangle}{\|a^{(k)}_p\|^2 + 1} \frac{\|a\|^2 - \|a^{(k)}_p\|^2}{\|a\|^2} (a^{(k)}_p, p^{(k)}_k - x_k) = -\frac{\langle a^{(k)}_p, p^{(k)}_k \rangle}{\|a\|^2} \langle a^{(k)}_p, p^{(k)}_k - x_k \rangle.
\end{align*}$$

For the third term note that in the same way,

$$\begin{align*}
q_k \left( I - \frac{1}{\|a\|^2} A \right) q_k^T \\
= -\frac{\langle a^{(k)}_p, p^{(k)}_k \rangle}{\|a\|^2} \langle a^{(k)}_p, q_k \rangle \\
= -\frac{\langle a^{(k)}_p, p^{(k)}_k \rangle}{\|a\|^2} \frac{\langle a^{(k)}_p, p^{(k)}_k \rangle}{\|a^{(k)}_p\|^2 + 1} a^{(k)}_p \\
= \frac{\langle a^{(k)}_p, p^{(k)}_k \rangle^2}{\|a\|^2 (\|a^{(k)}_p\|^2 + 1)} \langle a^{(k)}_p, a^{(k)}_p \rangle = \frac{\langle a^{(k)}_p, p^{(k)}_k \rangle^2 \|a^{(k)}_p\|^2}{\|a\|^2 (\|a^{(k)}_p\|^2 + 1)}.
\end{align*}$$
Inserting all of the above into (91) we arrive at

\[
(p(k) + q_k - x_k) \left( I - \frac{1}{\|a\|^2} A \right) (p(k) + q_k - x_k)^T = \|p(k) - x_k\|^2 - \frac{\langle a(k), p(k) - x_k \rangle}{\|a\|^2} - \frac{2 \langle a(k), p(k) \rangle}{\|a\|^2} (a(k) \cdot p(k) - x_k) + \frac{\langle a(k), p(k) \rangle^2}{\|a\|^2} \|a(k)\|^2 - \frac{\|a\|^2}{\|a\|^2} (\|a(k)\|^2 + 1)
\]

\[
= \|p(k) - x_k\|^2 + \|p(k)\|^2 - \frac{\langle a(k), p(k) - x_k \rangle}{\|a\|^2} - \frac{2 \langle a(k), p(k) \rangle}{\|a\|^2} (a(k) \cdot p(k) - x_k) + \frac{\langle a(k), p(k) \rangle^2}{\|a\|^2} \|a(k)\|^2 + 1
\]

\[
f(t^*) - \|p(k)\|^2 = f(t^*) - \|p(k)\|^2 + \frac{\langle a(k), p(k) \rangle^2}{\|a\|^2} \|a(k)\|^2 + 1.
\]

The intersection of \(e(L,r)\) with \(H_k\) is given by those \(x\) such that \(t^* = t^*(x)\) satisfies the inequality \(f(t^*) < r^2\). The boundary is therefore given by \(f(t^*) = r^2\) or equivalently

\[
(p(k) + q_k - x_k) \left( I - \frac{1}{\|a\|^2} A \right) (p(k) + q_k - x_k)^T = r^2 - \|p(k)\|^2 + \frac{\langle a(k), p(k) \rangle^2}{\|a\|^2} \|a(k)\|^2 + 1.
\]

It is well known that \((x - v)B(x - v)^T = r^2\) defines an ellipsoid centred at \(v\) and with axis along the eigenvectors of \(B\) whenever \(B\) is a positive definite matrix. Furthermore, the length of these axes are given by \(r\) times one over the square root of the corresponding eigenvalues.

Clearly we have that for any \(x \in \mathbb{R}^k\),

\[
x \left( I - \frac{1}{\|a\|^2} A \right) a_k = \|x\|^2 - \frac{x a_k^T a_k x^T}{\|a\|^2} = \|x\|^2 - \frac{\langle x, a_k \rangle^2}{\|a\|^2} \geq \|x\|^2 - \frac{\|a_k\|^2}{\|a\|^2} = \|x\|^2\left(1 - \frac{\|a_k\|^2}{\|a\|^2}\right),
\]

and so \(I - \frac{1}{\|a\|^2} A\) is a positive definite matrix. Furthermore, the center of the ellipsoid is given by

\[
p(k) + q_k = p(k) - \frac{\langle a(k), p(k) \rangle}{\|a\|^2} a(k).
\]

It remains to determine the eigenvectors and the corresponding eigenvalues. To that end, observe that

\[
(I - \frac{1}{\|a\|^2} A) a_k = a_k^T a_k = \|a_k\|^2 - \frac{\|a\|^2}{\|a\|^2} a_k = \|a(k)\|^2 + 1 a_k^T
\]

and so \(u = a_k\) is an eigenvector corresponding to the eigenvalue \(\frac{\|a(k)\|^2 + 1}{\|a\|^2}\). Furthermore, let

\[
u_1 = (-a(k),1,0,\ldots,0,a(k),1,0,\ldots,0) = (-a(k),1e_1^k + a(k),1e_k^k),
\]

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where \( e^k_l \) is the vector of length \( k \) consisting of all zeros except entry number \( l \) which equals one. We get that

\[
Au^T_l = (-a(k),1,a(k),l + a(k),1a(k),l, -a(k),2a(k),l + a(k),2a(k),l,1) = 0
\]

and so for every \( u_l, l = 2, \ldots, k \) we have that \( Au^T_l = 0 \) whence

\[
(I - \frac{1}{\|a\|^2}A)u^T_l = u^T_l,
\]

and so \( u_l \) is an eigenvector corresponding to the eigenvalue 1.

Comparing this to (93) we see that if

\[
r^2 := r^2 - \|p^{(k)}\|^2 + \frac{\langle a^{(k)}, p^{(k)} \rangle^2}{\|a^{(k)}\|^2 + 1},
\]

then the lengths of the axes are given by

\[
\left( \frac{\|a\|}{\sqrt{\|a^{(k)}\|^2 + 1}} \tilde{r}, \tilde{r}, \ldots, \tilde{r} \right).
\]

\( \square \)

### A.4 Proofs from Section 6

In this part of the appendix we will frequently deal with integrals over spheres and will repeatedly use the following coordinate change (see p.7 in [28]). Recall that \( e_1, \ldots, e_d \) is the standard basis on \( \mathbb{R}^d \). If \( d \geq 3 \), let \( \phi \in \partial B^{d}(o,1) \) be written as

\[
\phi = s e_d + \sqrt{1 - s^2} u, \quad s \in [-1,1], \ u \in \partial B^{d-1}(o,1).
\]

The corresponding surface element \( d\sigma_{d-1}(\phi) \) is then

\[
d\sigma_{d-1}(\phi) = (1 - s^2) \frac{d-2}{d} ds d\ell_{d-2}(u).
\]

We will use this change of coordinates when we need to perform integrals over spheres. One integral will in particular surface in many places, and so we study it separately in the following lemma. Recall the notation \( \psi_m = \ell_m(\partial B^{m+1}(o,1)) \) from Section 2.3. It will be convenient to write \( S^{d-1} = \partial B^d(o,1) \) so that \( \psi_m = \ell_m(S^m) \) here.

**Lemma A.2.** Suppose that \( k \in \{2, \ldots, d - 3\} \) and let

\[
I_{d,k} := \int_{(S^{d-2})^2} \int_0^\infty \frac{k^{d-k-2}}{1 + \kappa^2 (1 + (\langle \phi', \phi \rangle)^2) (d-k-1)/2} \, d\kappa d\ell_{d-2}(\phi') d\ell_{d-2}(\phi).
\]

We have that

\[
I_{d,k} = \frac{\psi_{d-k-1} \psi_{d-k-2}}{2}.
\]
In the next lemmas, we present two useful integrals which will be used frequently in what follows. The proofs will be kept to a minimum and rely on standard tables.

**Lemma A.3.** Suppose that \( b > 0, -2s + t < -1 \) and \( t > -1 \). Then
\[
\int_0^\infty (x^2 + b)^{-s}x^t \, dx = b^{(-2s+t+1)/2} \frac{\psi_{2s-1}}{\psi_t \psi_{2s-t-2}}. \tag{98}
\]

**Proof.** Recall from Section 2.2 that \( \psi_{d-1} = 2^{(d/2)} \) so that
\[
\int_0^\infty (x^2 + b)^{-s}x^t \, dx = \frac{1}{2} \int_0^\infty (y + b)^{-s}y^{t/2-1/2} \, dy
\]
\[
= \frac{b^{(-2s+t+1)/2} \Gamma \left( \frac{t+1}{2} \right) \Gamma \left( s - \frac{t}{2} - \frac{1}{2} \right)}{2 \Gamma(s)}
\]
\[
= b^{(-2s+t+1)/2} \frac{\Gamma \left( \frac{t+1}{2} \right) \Gamma \left( \frac{2s-t-1}{2} \right)}{2 \pi^{(t+1)/2} \Gamma(s)} \frac{2\pi^s}{\Gamma(s)} = b^{(-2s+t+1)/2} \frac{\psi_{2s-1}}{\psi_t \psi_{2s-t-2}},
\]
where the second equality can be found in \cite{29} (p 175, table 7.5, formula 14). \qed

**Lemma A.4.** Suppose that \( s, t > -1 \). Then
\[
\int_0^1 (1 - x^2)^s x^t \, dx = \frac{\psi_{2s+t+2}}{\psi_{2s+1} \psi_t}. \tag{99}
\]

**Proof.** We have that
\[
\int_0^1 (1 - x^2)^s x^t \, dx = \frac{1}{2} \int_0^1 (1 - y)^s y^{t/2-1/2} \, dy
\]
\[
= \frac{\Gamma(s+1) \Gamma \left( \frac{t+1}{2} \right)}{2 \Gamma \left( s + \frac{t}{2} + \frac{1}{2} \right)} = \frac{\Gamma(s+1) \Gamma((t+1)/2)}{2 \pi^{(2s+2)/2} \pi^{(t+1)/2}} \frac{2\pi^{2s+t+3/2}}{\Gamma((2s+t+3)/2)} = \frac{\psi_{2s+t+2}}{\psi_{2s+1} \psi_t},
\]
where the value of the last integral can be found in \cite{29} (p 174, table 7.5, formula 1). \qed

**Proof of Lemma A.2.** First, we assume that \( 2 \leq d \leq d - 4 \). By the change of variable \( \kappa = t/\sqrt{1 - (\phi', \phi)^2} \) and straightforward calculations it follows that
\[
I_{d,k} = \int_{(S^{d-k-2})^2} \int_0^\infty \frac{\ell_{d-k-2} - 1}{(1 + t^2)^{d-k-2} 1 - (\phi', \phi)^2 + t^2} \frac{dt \, d\ell_{d-k-2}(\phi') \, d\ell_{d-k-2}(\phi)}{(1 - (\phi', \phi)^2)^{d-k-3/2}}, \tag{100}
\]
and we now calculate \(100\). Since this integral only depends on \( \phi, \phi' \) through the inner-product, we can simply let \( \phi = e_{d-k-1} \) so that
\[
I_{d,k} = \int_{S^{d-k-2}} \int_0^\infty \frac{\ell_{d-k-2}(S^{d-k-2})}{(1 + t^2)^{(d-k-1)/2} 1 - (\phi', e_{d-k-1})^2 + t^2} \frac{dt \, d\ell_{d-k-2}(\phi')}{(1 - (\phi', e_{d-k-1})^2)^{(d-k-3)/2}}.
\]

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Since we want to integrate over $S^{d-k-2} = \partial B^{d-k-1}(o,1)$ and $d-k-1 \geq 3$, we can now we apply the coordinate change of Equation (95) where we write
\[ \phi' = s e_{d-k-1} + \sqrt{1-s^2}u, \quad s \in [-1,1], \quad u \in S^{d-k-3}, \]
so that the surface measure now equals
\[ d\ell_{d-k-2}(\phi') = (1-s^2)^{(d-k-4)/2} ds d\ell_{d-k-3}(u) \]
where $d\ell_{d-k-3}$ is the surface measure on $S^{d-k-3}$. We have that $(\phi', e_{d-k-1})^2 = s^2$ so that
\[
I_{d,k} = \int_{S^{d-k-3}} \int_0^\infty \int_{s=-1}^1 \frac{t^{d-k-2}}{(1+t^2)^{(d-k-1)/2}} \frac{\psi_{d-k-2}}{1-s^2+t^2} (1-s^2)^{(d-k-3)/2} ds dt d\ell_{d-k-3}(u) \\
= \psi_{d-k-2}^2 \psi_{d-k-3} \int_0^\infty \int_{s=-1}^1 \frac{t^{d-k-2}}{(1+t^2)^{(d-k-1)/2}} \frac{1}{1-s^2+t^2} \frac{1}{\sqrt{1-s^2}} ds dt.
\]
Now let $\alpha = \sin^{-1}(s)$ for $s \in [-1,1]$. Then $ds = \sqrt{1-s^2} \, d\alpha$ and $1-s^2 = \cos^2(\alpha)$ so that
\[
I_{d,k} = \psi_{d-k-2}^2 \psi_{d-k-3} \pi \int_0^\infty \int_{\alpha=-\pi/2}^{\pi/2} \frac{t^{d-k-2}}{(1+t^2)^{(d-k-1)/2}} \left[ \frac{1}{t \sqrt{1+t^2}} \arctan \left( \frac{t}{\sqrt{1+t^2}} \tan(\alpha) \right) \right]_{\alpha=-\pi/2}^{\pi/2} dt \\
= \psi_{d-k-2}^2 \psi_{d-k-3} \pi \int_0^\infty \frac{t^{d-k-3}}{(1+t^2)^{(d-k)/2}} dt \\
= \psi_{d-k-2}^2 \psi_{d-k-3} \pi \int_0^\infty \frac{t^{d-k-2}}{(d-k-2)(1+t^2)^{(d-k-2)/2}} dt \\
= \frac{\psi_{d-k-1}^2 \psi_{d-k-2} \pi}{d-k-2}.
\]

Here, the second and fourth equality follows by standard methods and the last equality follows from (7).

Now assume that $k = d-3$. In this case, to calculate (96), we need to integrate over $S^1 = \partial B^2(o,1)$ so we cannot use the coordinate change in (95). Instead, we see that (96) becomes
\[
I_{d,d-3} = \int_{S^1} \int_0^\kappa \frac{K}{1 + K^2} \frac{ds d\ell_1(\phi') d\ell_1(\phi)}{1 + (1 - (\phi', e_2)^2) K^2} \\
= \ell_1(S^1) \int_0^\kappa \frac{K}{1 + K^2} \frac{ds d\ell_1(\phi')}{1 + (1 - (\phi', e_2)^2) K^2}.
\]
Writing $\phi' = (\cos(\theta), \sin(\theta))$ we have $d\ell_1(\phi') = d\theta$ and $(\phi', e_2)^2 = \sin^2(\theta)$. Hence,
\[ I_{d,d-3} = \ell_1(S^1) \int_0^\infty \frac{\kappa}{1 + \kappa^2} \int_0^{2\pi} \frac{1}{1 + \cos^2(\theta) \kappa^2} d\theta d\kappa \]
\[ = 2\pi \ell_1(S^1) \int_0^\infty \frac{\kappa}{(1 + \kappa^2)^{3/2}} d\kappa = 2\pi \ell_1(S^1) = \psi_2 \psi_1 \]

where the second and third equality follows from standard methods, and the fourth equality follows since \( \psi_2/2 = \pi^{3/2}/\Gamma(3/2) = \pi^{3/2}/(\pi^{1/2}/2) = 2\pi \). Here, we used the formula (5).

We now prove Lemma 6.2 and subsequently 6.4

**Proof of Lemma 6.2.** We only prove the case \( k \leq d - 3 \). The reason for this is that (as in Lemma A.2) the cases \( k = d - 2 \) and \( k = d - 1 \) while somewhat easier, follow slightly different paths. The adjustments needed are outlined in Remark 6.1.

We start by calculating

\[ \| \xi_{k,r} \|_{TV} = \Upsilon_d \int \left( r^2 - \| p^{(k)} \|^2 + \frac{(a^{(k)} p^{(k)})^2}{1 + \| a^{(k)} \|^2} > 0 \right) \frac{1}{\| a \|^x} d\alpha(d) d\alpha(d) dp^{(k)}, \] (101)

where the integral is over \( a^{(k)} \in \mathbb{R}^k \), \( a^{(k)} \in \mathbb{R}^{d-k-1} \) and \( p^{(k)} \in \mathbb{R}^{d-k-1} \). Using (53) we get that (101) equals

\[ r^{d-k-1} \Upsilon_d \int_{\varphi, \phi \in S^{d-k-2}, \theta \in S^{k-1}} \frac{\rho^{k-1} \kappa^{d-k-2}}{(\rho^2 + \kappa^2 + 1)(d+1)/2} \times \left( \frac{1 + \kappa^2}{1 + (1 - \langle \varphi, \phi \rangle^2) / \kappa^2} \right)^{(d-k-1)/2} \int_0^{(d-k-2)(d) \rho \kappa} dl_{d-k-2}(\varphi) dl_{d-k-2}(\phi) dl_{k-1}(\theta) \]

\[ = \Upsilon_d \frac{r^{d-k-1} \psi_{k-1}}{d-k-1} \int_{\varphi, \phi \in S^{d-k-2}, \kappa \rho \geq 0} \frac{\rho^{k-1} \kappa^{d-k-2}}{(\rho^2 + \kappa^2 + 1)(d+1)/2} \times \left( \frac{1 + \kappa^2}{1 + (1 - \langle \varphi, \phi \rangle^2) / \kappa^2} \right)^{(d-k-1)/2} d\rho d\kappa dl_{d-k-2}(\varphi) dl_{d-k-2}(\phi), \]

where the equality follows from performing the integration over \( \theta \) and \( t \).

Integrating with respect to \( \rho \) we get by using (98)

\[ \int_0^\infty \frac{\rho^{k-1}}{(\rho^2 + \kappa^2 + 1)(d+1)/2} d\rho = \frac{1}{(1 + \kappa^2)^{(d-k+1)/2} \psi_{k-1} \psi_{d-k}} \]

and so (102) equals

\[ \frac{r^{d-k-1} \psi_{k-1}}{d-k-1} \frac{\psi_d}{\psi_{k-1} \psi_{d-k}} \Upsilon_d \int_{\varphi, \phi \in S^{d-k-2}, \kappa \rho \geq 0} \frac{\kappa^{d-k-2}}{(1 + (1 - \langle \varphi, \phi \rangle^2) / \kappa^2)^{(d-k-1)/2}} d\rho d\kappa dl_{d-k-2}(\varphi) dl_{d-k-2}(\phi), \] (103)
The integral on the right hand side is simply $I_{d,k}$ of Lemma \ref{lem:12}. Therefore, by combining \eqref{101}, \eqref{102} and \eqref{103} with \eqref{97} we see that

$$
\|\xi_{k,r}\|_{TV} = \sum_{d-k-1}^{d-k} \psi_{d-k-1} \psi_{d-k-2} \frac{2}{2} \frac{4\pi}{\psi_d \psi_{d-1}} = \frac{\psi_{d-k-1} \psi_{d-k}}{4\pi} = \frac{\psi_{d-k-1} \psi_{d-k}}{4\pi}
$$

where we used \eqref{7} in the second equality. \hfill \square

**Proof of Lemma \ref{lem:6.3.}.** For the same reasons as in Lemma \ref{lem:6.2} we only give the proof for the case $k \leq d - 3$. Using \eqref{53} and \eqref{55}, we see that $E_{\xi_{k,r}}[\text{diam}(E)^n]$ equals

$$
\int_{0 \leq \theta \in S^{n-1}, \phi, \varphi \in S^{d-2}} \frac{(2r)^n}{(r^2 + \kappa^2 + 1)^{n/2}} \left( \frac{1 + \kappa^2}{(1 - (\phi, \varphi)^2)^{n/2}} \right)^{(d-k-1)/2} \frac{\rho^k \kappa^k \kappa^k}{(\rho^2 + \kappa^2 + 1)^{n/2}} \frac{d\rho d\kappa d\ell_{d-k-2}(\theta) d\ell_{d-k-2}(\varphi)}{d\kappa d\ell_{d-k-2}(\varphi)}.
$$

After simplifying the integrand and integrating over $\theta$, we obtain

$$
E_{\xi_{k,r}}[\text{diam}(E)^n] = \sum_{d-k-1}^{d-k} \psi_{d-k-1} \psi_{d-k-2} \frac{2}{2} \frac{4\pi}{\psi_d \psi_{d-1}} \left( \frac{1 + \kappa^2}{(1 - (\phi, \varphi)^2)^{n/2}} \right)^{(d-k-1)/2} \frac{\rho^k \kappa^k \kappa^k}{(\rho^2 + \kappa^2 + 1)^{n/2}} \frac{d\rho d\kappa d\ell_{d-k-2}(\varphi)}{d\kappa d\ell_{d-k-2}(\varphi)}
$$

$$
\int_{t=0}^{1} (1 - t^2)^{n/2} t^{d-k-2} dt.
$$

The integral over $t$ is, using \eqref{99},

$$
\int_{t=0}^{1} (1 - t^2)^{n/2} t^{d-k-2} dt = \frac{\psi_{d-k-1}}{\psi_{d-k-2}}.
$$

Moreover, for $\kappa \in [0, \infty)$, we have by \eqref{98} that

$$
\int_{0}^{\infty} \frac{\rho^{d-k}}{(\rho^2 + \kappa^2 + 1)^{n/2}} \frac{d\rho}{\psi_{d-k} \psi_{d-k-1}} \left( \frac{1 + \kappa^2}{(1 - (\phi, \varphi)^2)^{n/2}} \right)^{(d-k-1)/2} \frac{\rho^k \kappa^k \kappa^k}{(\rho^2 + \kappa^2 + 1)^{n/2}} \frac{d\rho d\kappa d\ell_{d-k-2}(\varphi)}{d\kappa d\ell_{d-k-2}(\varphi)} = \begin{cases} \psi_{d-n} \frac{1 + \kappa^2}{n - \psi_{d-n-k} (1 + \kappa^2)^{n/2}} & n < d - k + 1, \\ \psi_{d-n} \frac{1 + \kappa^2}{n - \psi_{d-n-k} (1 + \kappa^2)^{n/2}} & n \geq d - k + 1. \end{cases}
$$

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Hence, we get that in the case $n < d - k + 1$,
\[
\int_{\rho,\kappa \geq 0, \phi, \varphi \in S^{d-k-2}} \frac{(\rho^2 + \kappa^2 + 1)^{d-d+n-1/2}}{(\kappa^2 + 1)^{d-n+k+1/2}} \frac{\rho^{k-1} \kappa^{d-k-2}}{(1 + (\langle \phi, \varphi \rangle)^2) \kappa^{(d-k-1)/2}} \, d\rho d\kappa d\ell_{d-k-2}(\phi) d\ell_{d-k-2}(\varphi)
\]
\[
= \frac{\psi_{d-n}}{\psi_{k-1} \psi_{d-n-k}} \int_{\rho,\varphi \in S^{d-k-2}} \frac{\kappa^{d-k-2}}{(\kappa^2 + 1)(1 + (\langle \phi, \varphi \rangle)^2) \kappa^{(d-k-1)/2}} \, d\kappa d\ell_{d-k-2}(\phi) d\ell_{d-k-2}(\varphi).
\]
(107)

The integral on the right hand side is $I_{d,k}$ of Lemma A.2. Using (97) we then obtain by plugging (105) and (107) into (104) that
\[
\mathbb{E}[\xi_{k,r}[\text{diam}(E)^n]] = \Upsilon_d \frac{2^{n+d+k-1} \psi_{k-1} \psi_{d-k-1} \psi_{d+n-k}}{\|\xi_{k,r}\|TV} \frac{\psi_{d-n}}{\psi_{k-1} \psi_{d-n-k}} I_{d,k}
\]
\[
= \frac{4\pi}{\psi_{d-n}} \frac{2^n r^{d+k-1} \psi_{d-n-k} \psi_{d-k-1}}{\psi_{d-k-2} \psi_{d-n-k} \psi_{d-n}} I_{d,k}
\]
\[
= 2^n r^{d+k-n} \psi_{d-n-k} \psi_{d-n}.
\]

Proof of Lemma 6.4. Again, as in Lemma 6.2 we only give the proof for the case $k \leq d - 3$.

We first prove (58), from which (59) easily follows. Fix $r \in (0,1)$ and $\tau \geq 4$. Observe that by (47), (53) and (55) we have that
\[
\xi_{k,r}(E : \text{diam}(E) \geq \tau) = \Upsilon_d r^{d-k-1} \int \left( \frac{1 + \kappa^2}{1 + (1 - (\langle \phi, \varphi \rangle)^2) \kappa^2} \right)^{(d-k-1)/2}
\]
\[
\times \int_{A_r} \frac{\rho^{k-1} (\kappa \tau)^{d-k-2}}{(\rho^2 + \kappa^2 + 1)^{(d+1)/2}} \, d\rho d\kappa d\ell_{d-k-1}(\theta) d\ell_{d-k-2}(\phi) d\ell_{d-k-2}(\varphi),
\]
(108)

where $A_r = \left\{ \rho : 2r \left( \frac{\rho^2 + \kappa^2 + 1}{\kappa^2 + 1} \right)^{1/2} \right\}$ and the first integral is over $\kappa \geq 0, t \in (0,1), \theta \in S^{d-1}$ and $\phi, \varphi \in S^{d-k-2}$.

Hence,
\[
\frac{d}{d\tau} \xi_{k,r}(E : \text{diam}(E) \geq \tau) = \Upsilon_d r^{d-k-1} \int \left( \frac{1 + \kappa^2}{1 + (1 - (\langle \phi, \varphi \rangle)^2) \kappa^2} \right)^{(d-k-1)/2}
\]
\[
\times \frac{d}{d\tau} \int_{A_r} \frac{\rho^{k-1} (\kappa \tau)^{d-k-2}}{(\rho^2 + \kappa^2 + 1)^{(d+1)/2}} \, d\rho d\kappa d\ell_{d-k-1}(\theta) d\ell_{d-k-2}(\phi) d\ell_{d-k-2}(\varphi).
\]
(109)

Now let $g(\tau, t, k, r) = \sqrt{1 + \kappa^2 \left( \frac{\rho^2 + \kappa^2 + 1}{\kappa^2 + 1}\tau^2 - 1 \right)}$. Observe that since $\tau \geq 4$ we have that $\frac{\rho^2 + \kappa^2 + 1}{\kappa^2 + 1}\tau^2 - 1 \geq 3$ and so $g(\tau, t, k, r)$ is real. In addition, let $f(\rho, \kappa) = \frac{\rho^{k-1}}{(\rho^2 + \kappa^2 + 1)^{(d+1)/2}}$. 58
Then,
\[
\frac{d}{dt} \int_{A_r} \rho^{k-1} \frac{d\rho}{\rho^2 + \kappa^2 + 1} = \frac{d}{dt} \int_{\rho=g(t,t,\kappa,r)}^\infty f(g(t,t,\kappa,r),\kappa) \frac{d\rho}{\rho} = -f(g(t,t,\kappa,r),\kappa) \frac{d}{dt} g(t,t,\kappa,r)
\]
We have that
\[
\frac{d}{dt} g(t,t,\kappa,r) = \frac{(1 + \kappa^2)^2}{2r^2(1 - t^2)^2} \left( (1 + \kappa^2) \frac{\tau^2}{4r^2(1 - t^2)} - 1 \right)^{-1/2} \tau (1 + \kappa^2)^{1/2} = \left( 4r^2(1 - t^2) \right)^{1/2} \left( \tau^2 - 4r^2(1 - t^2) \right)^{1/2},
\]
and that
\[
f(g(t,t,\kappa,r),\kappa) = \frac{(1 + \kappa^2)^{(k-1)/2}}{\left( \frac{(1 + \kappa^2)^2}{1 + (1 - t^2)} \right)^{(d+1)/2}} \left( \frac{d+1}{4r^2(1 - t^2)} \right)^{1/2} \tau^{-d-1} (1 + \kappa^2)^{(2d-k)/2} (4r^2(1 - t^2))^{-(d-k)/2} (\tau^2 - 4r^2(1 - t^2))^{(k-1)/2}.
\]
Hence,
\[
-f(g(t,t,\kappa,r),\kappa) \frac{d}{dt} g(t,t,\kappa,r) = -2^{d-k+1} \tau^{-d} (1 + \kappa^2)^{(2d-k-1)/2} (4r^2(1 - t^2))^{(d-k+1)/2} (\tau^2 - 4r^2(1 - t^2))^{(k-2)/2}.
\]
For \( \tau \geq 4 \) we have (keeping in mind that \( r,t \in (0,1) \)),
\[
\tau^2 / 2 \leq \tau^2 - 4r^2 (1 - t^2) \leq \tau^2.
\]
We will use the notation \( h_1 \sim h_2 \) whenever the two functions \( h_1,h_2 \) are such that \( h_1(x) \leq ch_2(x) \) and \( h_1(x) \geq ch_2(x) \) for some constants \( 0 < c,c' < \infty \). We then see that
\[
-f(g(t,t,\kappa,r),\kappa) \frac{d}{dt} g(t,t,\kappa,r) \gtrsim -\tau^{-d-k-2} (1 + \kappa^2)^{(2d-k-1)/2} \tau^{d-k+1} (1 - t^2)^{(d-k+1)/2}.
\]
Inserting (111) into (109) now gives, for \( \tau \geq 4 \),
\[
\frac{d}{dt} \xi_{k,r}(E : \text{diam}(E) \geq \tau) \gtrsim -\frac{\tau^{2(d-k)}}{\tau^d + d} \int \frac{(1 - t^2)^{(d-k+1)/2} \kappa^{d-k-2}}{(1 + (\varphi,\phi)^2)\kappa^2(1 + \kappa^2)^{(d-k-1)/2} \kappa^{d-k-2}} \frac{d\kappa}{2}\int \frac{(1 + (\varphi,\phi)^2)\kappa^2(1 + \kappa^2)^{(d-k-1)/2} \kappa^{d-k-2}}{(1 + (\varphi,\phi)^2)\kappa^2(1 + \kappa^2)^{(d-k-1)/2} \kappa^{d-k-2}} \frac{d\kappa}{2} \leq -I_{d,k} \frac{\tau^{d-k}}{d}.
\]
where we use (96) in the last equality. This proves (58). We now obtain (59) using the identity
\[
\xi_{k,r}(E : \text{diam}(E) \geq \tau) = -\int_\tau^\infty \frac{d}{ds} \xi_{k,r}(E : \text{diam}(E) \geq s) ds.
\]
The fractal cylinder process
Broman, Elias, Mussini, Tykesson

References


