AN EFFICIENT ALGORITHM FOR SOLVING ELLIPTIC PROBLEMS ON PERCOLATION CLUSTERS

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ABSTRACT. We present an efficient algorithm to solve elliptic Dirichlet problems defined on the cluster of $\mathbb{Z}^d$ supercritical Bernoulli percolation, as a generalization of the iterative method proposed by S. Armstrong, A. Hannukainen, T. Kuusi and J.-C. Mourrat. We also explore the two-scale expansion on the infinite cluster of percolation, and use it to give a rigorous analysis of the algorithm.

Figure 1. A simulation of 2D Bernoulli bond percolation with $p = 0.51$ in a cube $\Box$ of size $100 \times 100$. The cluster in blue is the maximal cluster $C_\ast(\Box)$ while the clusters in red are the other small ones.
1. Introduction

1.1. Motivation and main result. The main goal of this paper is to study a fast algorithm for computing the solution of Dirichlet problems with random coefficients on Bernoulli percolation clusters. For dimension $d \geq 2$, let $(\mathbb{Z}^d, E_d)$ be the Euclidean lattice, where $E_d$ denotes the set of (unoriented) nearest-neighbor bonds (or edges), that is, two-element sets $\{x, y\}$ with $x, y \in \mathbb{Z}^d$ satisfying $|x - y| = 1$. We also write $x \sim y$ whenever $\{x, y\} \in E_d$. Then we give ourselves a constant $\Lambda > 1$ and a random conductance $a : E_d \to \{0\} \cup [\Lambda^{-1}, 1]$ such that the random variables $\{a(e)\}_{e \in E_d}$ are independent and identically distributed. The Bernoulli percolation in this work is defined by the random conductance $\{a(e)\}_{e \in E_d}$: for every bond $e \in E_d$, we say that $e$ is an open bond if $a(e) > 0$, and that $e$ is a closed bond otherwise. The connected components on $(\mathbb{Z}^d, E_d)$ generated by the open bonds are called clusters, and we are interested in the supercritical percolation case, that is, we assume that $p := \mathbb{P}[a(e) > 0]$ is strictly larger than the critical percolation parameter, which we denote by $p_c(d)$. As a consequence, there exists a unique infinite percolation cluster $\mathcal{C}_\infty$ [40].

The configuration of clusters is random in a finite cube $d_m := \left(-\frac{3m}{2}, \frac{3m}{2}\right)^d \cap \mathbb{Z}^d$. However, under the supercritical percolation setting and when the cube $d_m$ is large, typically we will see a giant cluster $\mathcal{C}_s(d_m)$, which takes most of the volume in $d_m$, and the other clusters are very small. (See Figure 1 for an illustration.) We call this situation “$d_m$ is a good cube” and informally one can think of $\mathcal{C}_s(d_m)$ as the largest cluster of $\mathcal{C}_\infty \cap d_m$. The rigorous definitions of “$d_m$ is a good cube” and of the maximal cluster $\mathcal{C}_s(d_m)$ will be given in Definitions 2.2 and 2.5 below, and they are typical since there exists a positive constant $C(d, p)$ such that

$$\mathbb{P}[d_m \text{ is a good cube}] \geq 1 - C(d, p) \exp(-C(d, p)^{-1}3^m).$$

Our goal is to find an algorithm for solving Dirichlet problems on $\mathcal{C}_s(d_m)$. That is, given two functions $f, g : d_m \to \mathbb{R}$, we aim to define and study an efficient method for calculating the solution $u$ of

$$\begin{cases}
-\nabla \cdot a \nabla u = f & \text{in } \mathcal{C}_s(d_m), \\
u = g & \text{on } \mathcal{C}_s(d_m) \cap \partial d_m,
\end{cases}$$

where the divergence-form operator is defined as

$$-\nabla \cdot a \nabla u(x) := \sum_{y \sim x} a(x, y) (u(x) - u(y)).$$

Equation (1.1) is very natural to describe many models in applied mathematics and other disciplines. For example, one can think of the electric potential in a porous medium: a domain is made of two types of composites, represented respectively by the open bonds and the closed bonds on the lattice graph $(\mathbb{Z}^d, E_d)$, and only the open bonds are available for the current to flow, while the closed
bonds are insulating. See [61] for a comprehensive introduction and [20, 47, 50] for some examples of its applications in nanomaterials.

The complex geometry of the percolation cluster causes significant perturbations to the electric potential, and this makes efficient numerical calculations challenging. Naive finite-difference schemes will become very costly as the size of the domain is increased, and the perforated geometry and low regularity of solutions does not allow for simple coarsening mechanisms. As is well-known, using the effective conductance \( \bar{a} \), which is a constant matrix (in fact a scalar by the symmetries in our assumptions) whose definition will be recalled in eqs. (C.1) and (C.2), one can replace the heterogeneous operator \(-\nabla \cdot a \nabla\) by the constant-coefficient operator \(-\bar{a} \Delta\) defined by

\[
-\bar{a} \Delta u(x) := \bar{a} \sum_{y \sim x} (u(x) - u(y)),
\]

and thus obtain an approximation \( \bar{u} \) as the solution of a homogenized equation. This is a nice idea, but the gap between \( \bar{u} \) and \( u \) always exists: on small scales, the homogenized solution \( \bar{u} \) will typically be very smooth, while \( u \) has oscillations. Indeed, the homogenized solution \( \bar{u} \) can only approximate \( u \) in \( L^2 \), but not in \( H^1 \). Moreover, the \( L^2 \) norm of \( (u - \bar{u}) \) depends on the size of \( \Box_m \) and only goes to zero in the limit \( m \to \infty \). In other words, \( \bar{u} \) converges to \( u \) in \( L^2 \) only in the limit of “infinite separation of scales”.

The goal of the present work is to go beyond these limitations: we will devise an algorithm that produces a sequence of approximations which rapidly converges to \( u \) in \( H^1 \), in a regime of large but finite separation of scales. The main idea is to look for a way to use the homogenized operator as a coarse operator in a scheme analogous to a multigrid method. In fact, the algorithm here is at first proposed in [4] by Armstrong, Hannukainen, Kuusi and Mourrat for the same equation under uniform ellipticity condition on \( \mathbb{R}^d \), where the authors believe that their method can be extended to a more degenerate case like percolation model. This generalization is more challenging, since we have to figure out not only the coarse operator but also the projection operator; which comes from the perturbation of the geometry. We use a new idea of mask operator to resolve it, see Section 1.3 for more detailed discussions. Thus the present work also confirms the robustness of their algorithm by stating clearly how to adapt it on percolation clusters and giving rigorous analysis for the rate of convergence.

Let us introduce some more notations and state the main theorem. For any \( V \subset \mathbb{Z}^d \), the interior of \( V \) is defined as \( \text{int}(V) := \{ x \in V : y \sim x \Rightarrow y \in V \} \), and the boundary is defined as \( \partial(V) := V \setminus \text{int}(V) \). The function space \( C_0(V) \) is the set of functions with zero boundary condition.

![Figure 2](image-url)

**Figure 2.** The classical multigrid algorithm contains two main steps: coarsening and projection. The algorithm in [4] gives idea how to use homogenization as the coarse operator for random conductance, and this work proposes to use the mask operator as a counterpart of projection in degenerated random conductance case.
The $L^2$ integration of the gradient of $v$ on the percolation cluster is defined as
\[
\|\nabla v 1_{\{a \neq 0\}}\|_{L^2(V)} := \left( \frac{1}{2} \sum_{x,y \in V, x \sim y} (v(y) - v(x))^2 1_{\{a(x,y) \neq 0\}} \right)^{\frac{1}{2}}.
\]

For any $V \subset \mathbb{Z}^d$ we define its associated $\sigma$-algebra $\mathcal{F}(V) := \sigma(\{a(e)\}_{e \in V \setminus \emptyset})$ and $\mathcal{F}$ shorthand for $\mathcal{F}_{\mathbb{Z}^d}$. We denote the probability space by $(\{a(e)\}_{e \in E_d}, \mathcal{F}, \mathbb{P})$. For a random variable $X$, we use two positive parameters $s, \theta$, and the notation $O$ to measure its size by
\[
X \leq O_s(\theta) \iff \mathbb{E} [\exp(\theta^{-1} X)^s] \leq 2,
\]
where $(\theta^{-1} X)_+ := \max\{\theta^{-1} X, 0\}$. Roughly speaking, the statement $X \leq O_s(\theta)$ tells us that $X$ has a tail lighter than $\exp(-\theta^{-1} x^s)$. We also define, for each $\lambda > 0$, the mappings $\lambda_{\mathcal{E},m} : \mathbb{Z}^d \to \mathbb{R}$, and $\ell : \mathbb{R}^+ \to \mathbb{R}^+$ by
\[
\lambda_{\mathcal{E},m}(x) := \begin{cases} 
\lambda & \text{if } x \in \mathcal{E}_s(\square_m), \\
0 & \text{otherwise}.
\end{cases}
\quad \ell(\lambda) := \begin{cases} 
\log^\frac{1}{2}(1 + \lambda^{-1}) & \text{if } d = 2, \\
1 & \text{if } d > 2.
\end{cases}
\]

**Theorem 1.1** (Main theorem). There exist two finite positive constants $s := s(d, p, \Lambda), C := C(d, p, \Lambda, s)$, and for every integer $m > 1$ and $\lambda \in \left(\frac{\sqrt{s}}{3^m}, \frac{1}{3^m}\right)$, an $\mathcal{F}$-measurable random variable $Z$ satisfying
\[
Z \leq O_s \left( C \ell(\lambda) \frac{1}{2} \lambda^\frac{1}{2} m^{\frac{1}{2} + d} \right),
\]
such that the following holds. Let $f, g : \square_m \to \mathbb{R}, u_0 \in g + C_0(\square_m)$ and $u \in g + C_0(\square_m)$ be the solution of eq. (1.1). On the event that $\square_m$ is a good cube, for $u_1, u_2 \in C_0(\square_m)$ solving (with null Dirichlet boundary condition)
\[
\begin{cases}
(\lambda^2 - \nabla \cdot a \nabla) u_1 = f + \nabla \cdot a \nabla u_0 & \text{in } \mathcal{E}_s(\square_m) \setminus \partial \square_m, \\
- \nabla \cdot a \nabla \tilde{u} = \lambda_{\mathcal{E},m}^2 u_1 & \text{in } \text{int}(\square_m), \\
(\lambda^2 - \nabla \cdot a \nabla) u_2 = (\lambda^2 - \nabla \cdot a \nabla) \tilde{u} & \text{in } \mathcal{E}_s(\square_m) \setminus \partial \square_m,
\end{cases}
\]
and for $\tilde{u} := u_0 + u_1 + u_2$, we have the contraction estimate
\[
\|\nabla (\tilde{u} - u) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\square_m))} \leq Z \|\nabla (u_0 - u) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\square_m))}.
\]

We explain a little more why this theorem ensures the good performance of the algorithm. We are mainly interested in two aspects: the convergence rate of the algorithm and its numerical complexity. To better illustrate the typical size, we denote by $r = 3^m$ the diameter of the domain.

- **Convergence rate of the algorithm.** We start by an arbitrary guess $u_0 \in g + C_0(\square_m)$ as an approximation of $u$, and repeat the eq. (1.5) several rounds. At the end of every round, we use the $\tilde{u}$ just obtained in place of $u_0$ in the new round of iteration. The contraction rate of iteration has a bound $Z$, which is a random factor only depending on the conductance $a$, the choice of our regularization $\lambda$, and the size of the cube $\square_m$, but independent of the data $f, g, u_0$. We can choose $\lambda$ such that $\frac{1}{r} \ll \lambda \ll (\log r)^{-\left(\frac{1}{2} + d\right)}$, then
\[
Z \leq O_s \left( C \ell(\lambda) \frac{1}{2} \lambda^\frac{1}{2} (\log r)^{\frac{1}{2} + d} \right)
\]
tells us that $Z$ has large probability to be smaller than $1$ and implies a geometric rate of convergence.

- **Complexity analysis.** The numerical costs come from three parts: the iteration eq. (1.5) itself, the cost to calculate the homogenized coefficient $\tilde{a}$ and to determine the maximal cluster $\mathcal{E}_s(\square_m)$. Sometimes we also omit that the last two are already known, as they cost less numerical resources compared to the first one.

  - **The cost for the iteration.** We notice that if we solve a Dirichlet problem for $-\nabla \cdot a \nabla$ naively, for a precision $\varepsilon$ it requires total $O(r \log(\varepsilon^{-1}))$ iterations of conjugate gradient descent (CGD), while for the problem with regularization like the first and third equation in eq. (1.5), it can be reduced to $O(\lambda^{-1} \log(\varepsilon^{-1}))$ rounds of CGD. The second equation in eq. (1.5) can be solved by a standard multigrid algorithm with $O(\log(\varepsilon^{-1}))$ iterations
of CGD (see [17, Chapter 4]). Therefore, applying eq. (1.5) with more detailed choice of resolution in every iteration, it allows us to solve the problem for precision $\varepsilon$ with $O((\log r)^{(\frac{d}{2}+d)(\log(e^{-1}))^2})$ rounds of CGD; see [36, Section 1.2] for details.

- The homogenized coefficient $\bar{a}$. There exist many excellent methods to calculate $\bar{a}$ quickly, which can be naturally generalized to the percolation setting; see for example [26, 22, 53, 25, 37]. The result from [53, Proposition 1.1] tells us the best precision in a domain of size $r$ is $\varepsilon = r^{-\frac{d}{2}}$ with $O(r^d \log r)$ operations, which corresponds to about $O((\log r)^d)$ rounds of CGD.

- The maximal cluster $\mathcal{C}_\epsilon(\Box_m)$. This is a supplementary step compared to the problem on $\mathbb{R}^d$, and one can use the “UnionFind” algorithm [18, Chapter 21] which requires at most $O(r^d \log r)$ operations, which corresponds to about $O((\log r)^d)$ rounds of CGD.

In conclusion, from the discussion above we know the limit for the precision is about $\varepsilon = r^{-n}$ for $n \leq \frac{d}{2}$, thus our algorithm does reduce the numerical complexity.

The rest of this paper focuses more on the theoretical proof of Theorem 1.1 and we add two remarks to conclude the introduction part. Firstly, eq. (1.1) can be defined in a more general domain $\mathbb{Z}^d \cap U_r$, where $U$ is a convex domain with $C^{1,1}$ boundary, $r > 0$ is a length scale which we think of as being large, and $U_r := \{x | x \in U\}$. In this case $\mathcal{C}_\epsilon(U_r)$ can be informally thought as the largest cluster in $U_r$. Our iterative algorithm eq. (1.5) and its analysis can be adapted to this more general setting by following very similar arguments.

Secondly, in eq. (1.1) one can simply write $-\nabla \cdot \bar{a} \nabla$ as $-\bar{a} \Delta$ defined in eq. (1.3) as $\bar{a}$ is in fact a scalar coefficient. However, for some other models like inhomogeneous percolation (see [34, Chapter 11.9] and [35]), where $\bar{a}$ can be an effective matrix rather than a scalar. One example is in $\mathbb{Z}^2$, we choose two different parameters $p_1, p_2$ and $p_1 + p_2 > 1$, then let $\mathbb{P}[a(e) > 0] = p_1$ for the horizontal bonds and $\mathbb{P}[a(e) > 0] = p_2$ for the vertical bonds. We believe that our algorithm also works in these models by re-establishing all the quantitative homogenization theory from [3, 19] and repeating all the analysis in this paper. Thus, to state the algorithm more generally, we choose to use the notation of $\bar{a}$ as a matrix in Theorem 1.1 and in the rest of the paragraph, especially Appendix C.

1.2. Previous work. The homogenization theory was first developed for elliptic or parabolic equations with periodic coefficients, and then generalised to the case of random stationary coefficients. There exist many classical references such as [14, 45, 60, 38, 1]. Quantitative results in stochastic homogenization took a long time to emerge. The first partial results result were obtained by Yurinskii [62]. Recently, thanks to the work of Gloria, Neukamm and Otto [30, 31, 27, 28, 29], and Armstrong, Kuusi, Mourrat and Smart [9, 5, 10, 6], we understand better the typical size of the fundamental quantities in the stochastic homogenization of uniformly elliptic equations, which provides us with the possibility to analyze the performance of numerical algorithms in this context.

The homogenization of environments that do not satisfy a uniform ellipticity condition also drew attention. In [63], Zhikov and Piatnitski establish many results qualitatively and explain how to formulate the effective equation on various types of degenerate stationary environments. In [48], Lamacz, Neukamm and Otto obtain a bound of correctors on a simplified percolation model by imposing all the bonds in the first coordinate direction to be open. In [13], the Liouville regularity problem in a general context of random graphs is studied by Benjamini, Duminil-Copin, Kozma, and Yadin using the entropy method, and its complete description on infinite cluster of Bernoulli percolation is given by Armstrong and Dario in [3]. Dario also gives the moment estimate of the correctors of the same model in [19].

Homogenization has a natural probabilistic interpretation in terms of random walks in random environment, as a generalised central limit theorem. One fundamental work in this context is the paper [41] by Kipnis and Varadhan, where the case of general reversible Markov chains is studied. The case of random walks on the supercritical percolation cluster attracted particular interest, and the quenched central limit theorem was obtained at first by Sidoravicius and Szmit in [59] for
dimension $d \geq 4$, then generalized by Berger, Biskup, Mathieu and Piatnitski in [15, 52] for any dimension $d \geq 2$. We also refer to [16, 43, 46] for overviews of this line of research.

Finally, concerning the construction of efficient numerical methods, our algorithm is inspired by the one introduced in [4] by Armstrong, Hannukainen, Kuusi and Mourrat, which is designed to treat the same question in a uniform ellipticity context, and also [36] where a uniform estimate is obtained. Besides the fact that the problem we consider here is not uniformly elliptic, we stress that a fundamental issue we need to address relates to the fact that the geometry of the domain itself must be modified as we move from fine to coarse scales. Indeed, the fine scales must be resolved on the original, highly perforated domain, while the coarse scales are resolved in a homogeneous medium in which the wholes have been “filled up”. As far as I know, this is the first work proposing a practical and rigorous method for the numerical approximation of elliptic problems posed in rapidly oscillating perforated domains. For the homogenized coefficient $\bar{a}$, we have many excellent works like [26, 22, 53, 25, 37], which can be adapted naturally on the percolation setting. Alternative numerical methods for computing the solution of elliptic problems in non-perforated domains have been studied extensively; we refer in particular to [12, 11, 21, 32, 56, 51, 44, 55], as well as to [33, 42, 23, 24] where the concept of homogenization is used explicitly.

1.3. Ideas of the proof and main contributions. In this part, we introduce some key concepts underlying the analysis of the algorithm and the proof of Theorem 1.1. We also present our main contributions, including the mask operator and some other results like estimates on the flux and a quantitative version of the two-scale expansion on the cluster of percolation, which are of independent interest. Some notations are explained quickly in the statement and their rigorous definitions will be given in Section 2 or in the later part when they are used.

1.3.1. Main strategy. The main strategy of the algorithm is very similar to an algorithm proposed in the previous work [4, 36] where we study the classical Dirichlet problem in $\mathbb{R}^d$ setting with symmetric $\mathbb{R}^{d \times d}$-valued coefficient matrix $a$, which is random, stationary, of finite range correlation and satisfies the uniform ellipticity condition. We recall the idea in the previous work with a little abuse of notation that $\Box_m$ stands $\big(-\frac{3m}{2}, \frac{3m}{2}\big)^d$ in this paragraph: to solve a divergence-form equation $-\nabla \cdot a \nabla u = f$ in $\Box_m$ with boundary condition $g$, we propose to compute $(u_1, \bar{u}, u_2)$ with null Dirichlet boundary condition solving

\[
\begin{aligned}
\begin{cases}
(\lambda^2 - \nabla \cdot a \nabla) u_1 &= f + \nabla \cdot a \nabla u_0 & \text{in } \Box_m, \\
-\nabla \cdot \bar{a} \nabla \bar{u} &= \lambda^2 u_1 & \text{in } \Box_m, \\
(\lambda^2 - \nabla \cdot a \nabla) u_2 &= (\lambda^2 - \nabla \cdot \bar{a} \nabla) \bar{u} & \text{in } \Box_m.
\end{cases}
\end{aligned}
\]

In [4, 36] we proved that $\bar{u} := u_0 + u_1 + u_2$ satisfies

$$\|\bar{u} - u\|_{H^1(\Box_m)} \leq \mathcal{Z} \|u_0 - u\|_{H^1(\Box_m)},$$

with a random factor $\mathcal{Z}$ of size $\mathcal{Z} \leq O_s \left(C(\Lambda, s, d) \ell(\lambda)^{\frac{d}{2}} \lambda^{\frac{d}{2}} m^\frac{d}{2}\right)$ for any $s \in (0, 2)$ and independent of $u, u_0, f, g$.

The main ingredient in the proof is the two-scale expansion theorem: for $v, \bar{v}$ with the same boundary condition and satisfying

\[
(\mu^2 - \nabla \cdot a \nabla)v = (\mu^2 - \nabla \cdot \bar{a} \nabla)\bar{v} \quad \text{in } \Box_m,
\]

one can use $\bar{v} + \sum_{k=1}^d (\partial_{x_k} \bar{v}) \phi_{e_k}$ to approximate $v$ in $H^1$. Here $\{e_k\}_{1 \leq k \leq d}$ stands for the canonical basis in $\mathbb{R}^d$, and $\phi_{e_k}$ is the first order corrector associated with the direction $e_k$. In our algorithm eq. (1.7), combining the first equation, the second equation of eq. (1.7) and $-\nabla \cdot a \nabla u = f$, we can obtain that

$$-\nabla \cdot \bar{a} \nabla \bar{u} = -\nabla \cdot a \nabla (u - u_0 - u_1) \quad \text{in } \Box_m,$$
which is an equation of type eq. (1.8) with $\mu = 0$. Moreover, the third equation in eq. (1.7) also follows the form of eq. (1.8), this time with $\mu = \lambda$. Thus, we have

$$(u - u_0 - u_1) \simeq w := \tilde{u} + \sum_{k=1}^{d} (\partial_{x_k} \tilde{u}) \phi e_k \simeq u_2,$$

up to a small error, so we can estimate $|\tilde{u} - u|$ by studying

$$|\tilde{u} - u| = |u - (u_0 + u_1 + u_2)| \leq |(u - u_0 - u_1) - w| + |w - u_2|.$$

In [4] the error in the two-scale expansion theorem is made quantitative, and in [36] we refine this bound so that the contraction bound is uniform over the relevant data (most importantly: the bound is uniform over $u_0$, which guarantees that the algorithm can indeed be iterated).

1.3.2. Mask operator trick. In order to figure out the generalization of the algorithm on clusters, we recall the interpretation from multigrid method for eq. (1.7): the first equation in eq. (1.7) is the scheme in fine grid, which runs several rounds thanks to the regularization. The second step of eq. (1.7) is a coarse grid, where we use the homogenized matrix as a coarse operator. The third equation in eq. (1.7) is a post-treatment to project the error in the coarse grid back to the fine grid. However, in the $\mathbb{R}^d$ setting, the projection step is natural, but we should treat it more carefully in percolation setting, since the fine grid is defined on $\mathcal{C}_s(\Box_m)$, which is random and depends on the realization of $a$, while the coarse grid is defined on $\Box_m$; see Figure 2 for an illustration.

To resolve the problem of projection, we use an idea called mask operator, which is defined as

$$\lambda_{\varepsilon,m}(x) := \begin{cases} \lambda & \text{if } x \in \mathcal{C}_s(\Box_m), \\ 0 & \text{otherwise}. \end{cases}$$

We remark that some similar idea also appears in the early work [63], where they call it singular random measure in the degenerate ergodic environment. In our algorithm, the mask operator is already implicitly used in the third equation of eq. (1.5), but the following nice observation Proposition 1.1 shows all its power: it allows us to treat the problem on percolation as if it is on $\mathbb{Z}^d$.

**Proposition 1.1 (Arbitrary extension).** After an arbitrary extension of the function $u_0, u_1, u_2$ defined in eq. (1.5) on $\text{int}(\Box_m) \setminus \mathcal{C}_s(\Box_m)$, the functions $u_1, \tilde{u}, u_2$ also satisfy

$$(\lambda_{\varepsilon,m}^2 - \nabla \cdot a_{\varepsilon,m} \nabla) u_1 = f_{\varepsilon,m} + \nabla \cdot a_{\varepsilon,m} \nabla u_0 \quad \text{in } \text{int}(\Box_m),$$

$$-\nabla \cdot a \nabla \tilde{u} = \lambda_{\varepsilon,m}^2 u_1 \quad \text{in } \text{int}(\Box_m),$$

$$(\lambda_{\varepsilon,m}^2 - \nabla \cdot a_{\varepsilon,m} \nabla) u_2 = (\lambda_{\varepsilon,m}^2 - \nabla \cdot a \nabla) \tilde{u} \quad \text{in } \text{int}(\Box_m).$$

**Proof.** In the first equation of (1.9) the left hand side can be rewritten as

$$(\lambda_{\varepsilon,m}^2 - \nabla \cdot a_{\varepsilon,m} \nabla) u_1(x) = \lambda_{\varepsilon,m}^2 u_1(x) + \sum_{y \sim x} (a_{\varepsilon,m}(x,y)) (u_1(x) - u_1(y)),$$

while the right hand side equals

$$f_{\varepsilon,m}(x) + \nabla \cdot a_{\varepsilon,m} \nabla u_0(x) = f_{\varepsilon,m}(x) + \sum_{y \sim x} (a_{\varepsilon,m}(x,y)) (u_0(y) - u_0(x)).$$

If $x \in \mathcal{C}_s(\Box_m) \setminus \partial \Box_m$ the left hand side and the right hand side both equal to the first equation in eq. (1.5), so the equation is established. If $x \in \text{int}(\Box_m) \setminus \mathcal{C}_s(\Box_m)$, no matter what values $u_1, u_0$ takes on the extension, the factors and function $f_{\varepsilon,m}(x) = \lambda_{\varepsilon,m}(x) = a_{\varepsilon,m}(x,y) = 0$ make both left hand side and right hand side 0.

In the second equation, on the right hand side $\lambda_{\varepsilon,m}^2 u_1$ coincides with that in eq. (1.5) so the equation is also established.

The third equation is valid, if $x \in \mathcal{C}_s(\Box_m) \setminus \partial \Box_m$ for the similar reason as described in the first equation. If $x \in \text{int}(\Box_m) \setminus \mathcal{C}_s(\Box_m)$, the left hand side equals 0 since all the factors and conductance
are 0. The right hand side is also 0 thanks to a simple manipulation using the second equation of eq. (1.9)

\[
(\lambda^2_{\varphi,m} - \nabla \cdot \mathbf{a} \nabla) \bar{u}(x) = \lambda^2_{\varphi,m}(x) (\bar{u}(x) + u_1(x)) = 0,
\]

and this finishes the proof. □

The same idea also works for \( u \) defined in eq. (1.1), which can also be defined as the solution

\[
\begin{aligned}
-\nabla \cdot \mathbf{a}_{\varphi,m} \nabla u &= f & \text{in } \text{int}(\square_m), \\
u &= g & \text{on } \mathcal{C}_s(\square_m) \cap \partial \square_m,
\end{aligned}
\]

with an arbitrary extension outside \( \mathcal{C}_s(\square_m) \).

1.3.3. Two-scale expansion on \( \mathcal{C}_\infty \). Once we obtain the description of the algorithm eq. (1.9), we can repeat the argument for eq. (1.7) to explore the two-scale convergence theorem, which should define its left hand side on the cluster \( \mathcal{C}_s(\square_m) \) and its right hand side on the homogenized geometry \( \square_m \),

\[
(\lambda^2_{\varphi,m} - \nabla \cdot \mathbf{a}_{\varphi,m} \nabla) v = (\lambda^2_{\varphi,m} - \nabla \cdot \mathbf{a} \nabla) \bar{v} \quad \text{in } \text{int}(\square_m),
\]

and we hope to use a modified two-scale expansion

\[
w := \bar{v} + \sum_{k=1}^{d} \left( \mathcal{Y} \mathcal{D}_{\varepsilon_k} \bar{v} \right) \phi^{(\lambda)}_{\varepsilon_k},
\]

to approximate \( v \). Here \( \mathcal{D}_{\varepsilon_k} \bar{v}(x) := \bar{v}(x + \varepsilon_k) - \bar{v}(x) \) and \( \mathcal{Y} \) is a cut-off function supported in \( \square_m \), constant 1 in the interior and decreases to 0 linearly near the boundary defined as

\[
\mathcal{Y} := 1_{\{\square_m\}} \wedge \left( \frac{\text{dist}(\cdot, \partial \square_m) - \ell(\lambda)}{\ell(\lambda)} \right)_+,
\]

so the function \( \mathcal{Y} \) can help reduce the boundary layer effect of the two-scale expansion. The modified corrector \( \{\phi^{(\lambda)}_{\varepsilon_k}\}_{1 \leq k \leq d} \) is defined as

\[
\phi^{(\lambda)}_{\varepsilon_k} := \phi_{\varepsilon_k} - [\phi_{\varepsilon_k}]_\mathcal{Y} \ast \Phi_{\lambda^{-1}},
\]

where \( \Phi_{\lambda^{-1}} \) is a heat kernel of scale \( \lambda^{-1} \), i.e. \( \Phi_{\lambda^{-1}}(x) := \frac{1}{(4\pi \lambda^2)^{d/2}} \exp \left(-\frac{x^2}{4\lambda^{-2}}\right) \) and \( [\phi_{\varepsilon_k}]_\mathcal{Y} \) is a coarsened version of \( \phi_{\varepsilon_k} \), whose proper definition will be given in Definition 2.2. Although the corrector is only well-defined up to a constant, notice that eq. (1.14) is well-defined. Notice also that by (1.11), the function \( \bar{v} \) is discrete-harmonic outside of \( \mathcal{C}_s(\square_m) \).

In Section 4, we will prove the following quantitative two-scale expansion theorem as a main tool to prove the contraction estimate (Theorem 1.1). We remark here that we also add a technical condition \( \square_m \in \mathcal{P}_s \), which is defined in Definition 2.6 and it comes from the partition of the cluster \( \mathcal{C}_\infty \). It is stronger than “\( \square_m \) is good”, and means that “the cluster \( \mathcal{C}_s(\square_m) \) is indeed a subset of \( \mathcal{C}_\infty \).

**Theorem 1.2** (Two-scale expansion on percolation). There exist two positive constants \( s := s(d, p, \Lambda) \), \( C := C(d, p, \Lambda, s) \), and for every integer \( m > 1 \) such that \( \square_m \in \mathcal{P}_s \) and every \( \lambda \in \left( \frac{1}{m^2}, \frac{1}{2} \right) \), there exists a random variable \( \tilde{Z} \) controlled by

\[
\tilde{Z} \leq \mathcal{O}_s \left( C(d, p, \Lambda, s) \ell(\lambda) m^{\frac{1}{2} + d} \right),
\]

such that the following is valid: for any \( \mu \in [0, \lambda] \) and any \( v, \bar{v} \in C_0(\square_m) \) satisfying

\[
(\mu^2_{\varphi,m} - \nabla \cdot \mathbf{a}_{\varphi,m} \nabla) v = (\mu^2_{\varphi,m} - \nabla \cdot \mathbf{a} \nabla) \bar{v} \quad \text{in } \text{int}(\square_m),
\]
defining a two-scale expansion \( w := \tilde{v} + \sum_{k=1}^{d} (\mathcal{Y} D_{e_k} \tilde{v}) \phi_{e_k}^{(\lambda)} \), we have
\[
\| \nabla (w - v) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{C}_e(\underline{m}))} \leq \tilde{Z} \left( \left( 3^{-\frac{m}{2}} \varepsilon^{-\frac{1}{2}} (\lambda) + \mu \right) \| \nabla \tilde{v} \|_{L^2(\underline{m})} + \| \Delta \tilde{v} \|_{L^2(\text{int}(\underline{m}))} \right) + \| \nabla \tilde{v} \|_{L^2(\underline{m})} \| \Delta \tilde{v} \|_{L^2(\text{int}(\underline{m}))} \right).
\]
(1.16)

**Remark.** We add some more explanations for the technical condition \( \square_m \in \mathcal{P}_* \). In fact, when one implements the algorithm, the condition "\( \square_m \) is a good cube" is very natural, but this condition is \( \mathcal{F}_{\square_m+1} \)-measurable thus only uses local information, and it does not necessarily imply that the maximal cluster \( \mathcal{C}_*(\underline{m}) \) is part of the cluster \( \mathcal{C}_\infty \). On the other hand, the corrector theory is established for the infinite cluster \( \mathcal{C}_\infty \), so is two-scale expansion. We have to pay attention to this minor difference, although it is well-known that "the maximal cluster in a good cube is part of infinite cluster with high probability". One can remove the condition \( \square_m \in \mathcal{P}_* \) in Theorem 1.2 with more technical analysis, but we choose to fill this gap in the part analysis of algorithm in Section 5 with a very simple inequality eq. (5.3).

1.3.4. **Centered flux \( g_p \).** Another topic studied in detail in this paper is an object called centered flux defined for each \( p \in \mathbb{R}^d \) by
\[
g_p := a_x (\mathcal{D} \phi_p + p) - \mathbf{a} p,
\]
where \( a_x (x,y) := a(x,y) 1_{\{x,y \in \mathcal{C}_\infty \}} \). Together with the corrector \( \phi_p \), they are two quantities required for the proof of the two-scale convergence Theorem 1.2. Its physical interpretation is clear: we define \( l_p(x) := p \cdot x \) and recall that the harmonic function can be seen as an electric potential. Then \( (l_p + \phi_p) \) is the electric potential defined on \( \mathcal{C}_\infty \) with conductance \( a \) associated to the direction \( p \), while \( l_p \) is the one for the homogenized conductance \( \mathbf{a} \). We know that \( \phi_p \) as the difference between the electric potentials is small compared to \( l_p \), and heuristically, it should also be the case for the electric current. By Ohm’s law, the two electric currents are defined by \( a_x \nabla (\phi_p + l_p) \) and \( \mathbf{a} \nabla l_p \), so we expect indeed that \( g_p \) will be small. This is however only true in a weak sense, or equivalently, after a spatial convolution. In fact, we expect that \( g_p \) satisfies estimates that are very similar to those satisfied by \( \nabla \phi_p \), and we will indeed prove an analogue of the result of [19, Proposition 3.1].

Here we use the notation \([::]\) to represent the constant extension on every cube of the form \( z + (-\frac{1}{2}, \frac{1}{2})^d \) for some \( z \in \mathbb{Z}^d \).

**Proposition 1.2** (Spatial average). There exist two positive constants \( s := s(d,p,\Lambda), C := C(d,p,\Lambda, s) \) such that for every \( R \geq 1 \) and every kernel \( K_R : \mathbb{R}^d \to \mathbb{R}^+ \) integrable and satisfying
\[
\exists C_{K,R} < \infty, \quad \forall x \in \mathbb{R}^d, \quad K_R(x) \leq \frac{C_{K,R}}{R^d \left( |x| \sqrt{1} \right)^{\frac{d+1}{2}}},
\]
the quantity \( (K_R * [g_p]) (x) \) is well defined for every \( x \in \mathbb{R}^d, p \in \mathbb{R}^d \) and it satisfies
\[
|K_R * [g_p]) (x) \leq O_s(C_{K,R} |p| R^{-\frac{d}{2}}).
\]

1.4. **Organization of the paper.** In Section 2, we define all the notations precisely and restate some important theorems in previous work. Section 3 is devoted to the study of the centered flux \( g_p \) and to the proof of Proposition 1.2. Section 4 gives the proof of the two-scale expansion on the cluster of percolation (Theorem 1.2). In Section 5, we use the two-scale expansion to analyze our algorithm. Finally, in Section 6, we present numerical experiments confirming the usefulness of the algorithm.

2. Preliminaries

This part defines rigorously all the notations used throughout this article. We also record some important results developed in previous work.
2.1. Notations $O_s(1)$ and its operations. We recall the definition of $O_s$

$$X \leq O_s(\theta) \iff \mathbb{E} \left[ \exp \left( \theta^{-1} X^s \right) \right] \leq 2,$$

where $(\theta^{-1} X)_+^s$ means $\max\{\theta^{-1} X, 0\}$. One can use the Markov inequality to obtain that

$$X \leq O_s(\theta) \implies \forall x > 0, \mathbb{P}[X \geq \theta x] \leq 2 \exp(-x^s).$$

For $\lambda \in \mathbb{R}^+$, $X \leq O_s(\theta) \implies \lambda X \leq O_s(\lambda \theta)$. We list some results on the estimates of the random variables with respect of $O_s$ in [7, Appendix A]. For the product of random variables, we have

$$|X| \leq O_{s_1}(\theta_1), |Y| \leq O_{s_2}(\theta_2) \implies |XY| \leq O_{s_1 + s_2}(\theta_1 \theta_2).$$

By choosing $Y = 1$, one can always use the estimate above to get an estimate for smaller exponent, i.e. for $0 < s' < s$, there exists a constant $C_{s'} < \infty$ such that

$$X \leq O_s(\theta) \implies X \leq O_{s'}(C_{s'} \theta).$$

We have an estimate on the sum of a series of random variables: for a measure space $(E,S,m)$ and \{X(z)\}_{z \in E} a family of random variables, we have

$$\forall z \in E, X(z) \leq O_s(\theta(z)) \implies \int_E X(z) m(dz) \leq O_s \left( C_s \int_E \theta(z) m(dz) \right),$$

where $0 < C_s < \infty$ is a constant defined by

$$C_s = \begin{cases} \left( \frac{1}{s \log 2} \right)^{\frac{1}{2}} & s < 1, \\ 1 & s \geq 1. \end{cases}$$

Finally, we can also obtain the estimate of the maximum of a finite number of random variables, which is proved in [36, Lemma 3.2]: for all $N \geq 1$ and family of random variables \{X_i\}_{1 \leq i \leq N} satisfying that $X_i \leq O_{s_i}(1)$, we have

$$\max_{1 \leq i \leq N} X_i \leq O_s \left( \frac{\log(2N)}{\log(3/2)} \right)^{\frac{1}{2}}.$$

2.2. Discrete analysis. This part is devoted to introducing notations and some functional inequalities on graphs or on lattices. We take two systems of derivative in our setting: $\nabla$ on graph and the finite difference $\mathcal{D}$ on $\mathbb{Z}^d$. The notation $\nabla$ is more general, but it loses the sense of derivative with respect to a given direction, which is very natural in the system of $\mathcal{D}$.

2.2.1. Spaces and functions. For every $V \subset \mathbb{Z}^d$, we can construct two types of geometry $(V, E_d(V))$ and $(V, E_{d}^a(V))$. The set of edges $E_d(V)$ inherited from $(\mathbb{Z}^d, E_d)$ and $E_d^a(V)$ inherited from the open bonds of the percolation are defined as

$$E_d(V) := \{(x,y) | x,y \in V, x \sim y \}, \quad E_{d}^a(V) := \{(x,y) | x,y \in V, a(x,y) \neq 0 \}.$$  

The interior of $V$ with respect to $(V, E_d(V))$ and $(V, E_{d}^a(V))$ are defined

$$\text{int}(V) := \{ x \in V | y \sim x \implies y \in V \}, \quad \text{int}_a(V) := \{ x \in V | y \sim x, a(x,y) \neq 0 \implies y \in V \},$$

and the boundaries are defined as $\partial(V) := V \setminus \text{int}(V)$ and $\partial_a(V) := V \setminus \text{int}_a(V)$. For any $x, y \in \mathbb{Z}^d$, we say $x \xrightarrow{a} y$ if there exists an open path connecting $x$ and $y$.

We denote by $E_d^\rightarrow$ the oriented bonds of $(\mathbb{Z}^d, E_d)$, i.e. $E_d^\rightarrow := \{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d : |x - y| = 1 \}$, and for any $E \subset E_d$, we can associate it to a natural oriented bonds set $\overrightarrow{E}$. An (anti-symmetric) vector field $\overrightarrow{F}$ on $\overrightarrow{E}_d$ is a function $\overrightarrow{F} : \overrightarrow{E}_d \to \mathbb{R}$ such that $\overrightarrow{F}(x,y) = -\overrightarrow{F}(y,x)$. Sometimes we also write
\( \bar{F}(e) \) for \( e = \{x, y\} \in \mathbb{E}_d \) to give its value with an arbitrary orientation for \( e \), in the case it is well defined (for example \( |\bar{F}|(e) \)). The \textit{discrete divergence} of \( \bar{F} \) is defined as \( \nabla \cdot \bar{F} : \mathbb{R}^d \rightarrow \mathbb{R} \)
\[
\forall x \in \mathbb{Z}^d, \quad \nabla \cdot \bar{F}(x) := \sum_{y \sim x} \bar{F}(x, y).
\]

For any \( u : \mathbb{Z}^d \rightarrow \mathbb{R} \), we define the \textit{discrete derivative} \( \nabla u : \mathbb{E}_d^\rightarrow \rightarrow \mathbb{R} \) as a vector field
\[
\forall (x, y) \in \mathbb{E}_d^\rightarrow, \quad \nabla u(x, y) := u(y) - u(x),
\]
and \( a \nabla u, \nabla u 1_{\{a \neq 0\}} \) are vector fields \( \mathbb{E}_d^\rightarrow \rightarrow \mathbb{R} \) defined by
\[
a \nabla u(x, y) := a(x, y) \nabla u(x, y), \quad \nabla u 1_{\{a \neq 0\}}(x, y) := \nabla u(x, y) 1_{\{a(x, y) \neq 0\}}.
\]

Then, the \textit{a-Laplacian operator} \( -\nabla \cdot a \nabla \) is well defined and we have
\[
-\nabla \cdot a \nabla u(x) := \sum_{y \sim x} a(x, y)(u(x) - u(y)).
\]

2.2.2. \textit{Finite difference derivative.} We start by introducing the notation of translation: let \( B \) be a Banach space, then for any \( h \in \mathbb{Z}^d \) and \( u : \mathbb{Z}^d \rightarrow B \) a \( B \)-valued function, we define \( T_h \) as an operator
\[
\forall x \in \mathbb{Z}^d, \quad (T_h u)(x) = u(x + h).
\]

We also define the operator \( \mathcal{D}_h \) and its conjugate operator \( \mathcal{D}_h^* \) for any \( u : \mathbb{Z}^d \rightarrow \mathbb{R} \),
\[
\mathcal{D}_h u := T_h u - u, \quad \mathcal{D}_h^* u := T_{-h} u - u.
\]

It is easy to check \( \mathcal{D}_h^* = -T_{-h}(\mathcal{D}_h u) \) and for two functions \( f, g : \mathbb{Z}^d \rightarrow \mathbb{R} \), we have
\[
(2.7) \quad \mathcal{D}_h(fg) = (\mathcal{D}_h f)g + (T_h f)(\mathcal{D}_h g).
\]

In this system, we also define \textit{vector field} \( \bar{F} : \mathbb{Z}^d \rightarrow \mathbb{R}^d, \bar{F}(x) = (\bar{F}_1(x), \bar{F}_2(x), \ldots, \bar{F}_d(x)) \) and this can be distinguished with the one defined on \( \mathbb{E}_d \) by the context. We use \( (e_1, e_2, \ldots, e_d) \) to represent the \( d \) canonical directions in \( \mathbb{Z}^d \), and a discrete gradient \( \mathcal{D} u : \mathbb{Z}^d \rightarrow \mathbb{R}^d \) is a vector field
\[
\mathcal{D} u(x) := (\mathcal{D}_{e_1} u(x), \mathcal{D}_{e_2} u(x), \ldots, \mathcal{D}_{e_d} u(x)).
\]

Then the finite difference divergence operator is defined as the conjugate operator of \( \mathcal{D} \)
\[
\mathcal{D}^* \cdot \bar{F} := \sum_{j=1}^d \mathcal{D}^*_{e_j} \bar{F}_j.
\]

As convention, we use the notation \( a \mathcal{D}_{e_j} u \) and \( a \mathcal{D} u \) to represent
\[
a \mathcal{D}_{e_j} u(x) := a(x, x + e_j) \mathcal{D}_{e_j} u(x), \quad a \mathcal{D} u := (a \mathcal{D}_{e_1} u, \ldots, a \mathcal{D}_{e_d} u),
\]
and \( 1_{\{a \neq 0\}} \mathcal{D}_{e_j} u, 1_{\{a \neq 0\}} \mathcal{D} u, \) to represent
\[
1_{\{a \neq 0\}} \mathcal{D}_{e_j} u(x) := 1_{\{a(x, x + e_j) \neq 0\}} \mathcal{D}_{e_j} u(x), \quad 1_{\{a \neq 0\}} \mathcal{D} u := (1_{\{a \neq 0\}} \mathcal{D}_{e_1} u, \ldots, 1_{\{a \neq 0\}} \mathcal{D}_{e_d} u).
\]

Thus the \textit{a-Laplacian operator} \( -\nabla \cdot a \nabla \) can also be expressed by the finite difference \( \mathcal{D}^* \cdot a \mathcal{D} \). We can prove it by a simple calculation that
\[
(2.8) \quad -\nabla \cdot a \nabla u = \mathcal{D}^* \cdot a \mathcal{D} u.
\]
2.2.3. Inner product and norm. For $V \subset \mathbb{Z}^d$ and $E \subset E_d$, we define inner product $\langle \cdot, \cdot \rangle_V$ for any function $u, v : V \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_E$ for any vector field $\vec{F}, \vec{G} : E \to \mathbb{R}$

$$\langle u, v \rangle_V := \sum_{x \in V} u(x)v(x), \quad \langle \vec{F}, \vec{G} \rangle_E := \sum_{\{x,y\} \in E} \vec{F}(x,y)\vec{G}(x,y),$$

and this defines a norm $\|u\|_{L^2(V)} := \sqrt{\langle u, u \rangle_V}$ and $\|\vec{F}\|_{L^2(E)} := \sqrt{\langle \vec{F}, \vec{F} \rangle_E}$. We also abuse a little the notation to define $\langle \cdot, \cdot \rangle_V$ for vector field

$$\langle \vec{F}, \vec{G} \rangle_V := \langle \vec{F}, \vec{G} \rangle_{E_d(V)} = \sum_{\{x,y\} \in E_d(V)} \vec{F}(x,y)\vec{G}(x,y) = \frac{1}{2} \sum_{x,y \in V, y \sim x} \vec{F}(x,y)\vec{G}(x,y).$$

We use the notation $\langle \cdot, \cdot \rangle_{E^2_d(V)}$ to represent the inner product of the vector field on $(V, E^2_d(V))$. For two vector fields $\vec{F}, \vec{G} : V \to \mathbb{R}^d$, the inner product is defined as

$$\langle \vec{F}, \vec{G} \rangle_V := \sum_{x \in V} \sum_{j=1}^d \vec{F}_j(x)\vec{G}_j(x),$$

and similarly $\|\vec{F}\|_{L^2(V)} = \sqrt{\langle \vec{F}, \vec{F} \rangle_E}$ also defines a norm.

To define a general $L^p(V)$ norm for vector fields, we have to introduce its modules. For any $\vec{F} : E_d \to \mathbb{R}$ or $\vec{F} : \mathbb{Z}^d \to \mathbb{R}^d$, we write

$$|\vec{F}|(x) := \left( \frac{1}{2} \sum_{y \sim x} \vec{F}^2(x,y) \right)^{\frac{1}{2}}, \quad |\vec{F}|(x) := \left( \sum_{j=1}^d \vec{F}_j^2(x) \right)^{\frac{1}{2}}.$$

Then for $f$ (a function, an $\mathbb{R}^d$-valued vector field or a vector field on $E_d$)

$$\|f\|_{L^p(V)} := \left( \sum_{x \in V} |f|^p(x) \right)^{\frac{1}{p}}, \quad \|f\|_{L^p(V)} := \left( \frac{1}{|V|} \sum_{x \in V} |f|^p(x) \right)^{\frac{1}{p}}.$$

We recall $C_0(V)$ the space of functions supported on $V$ with null boundary condition. Then one can deduce integration by part formula: for any function $v \in C_0(V)$, $\vec{F} : E_d(V) \to \mathbb{R}$ and $\vec{F} : \mathbb{Z}^d \to \mathbb{R}^d$, one can check

$$\langle v, -\nabla \cdot \vec{F} \rangle_{\text{int}(V)} = \langle \nabla v, \vec{F} \rangle_V, \quad \langle v, \mathcal{D}^* \cdot \vec{F} \rangle_{\text{int}(V)} = \langle \mathcal{D}v, \vec{F} \rangle_V.$$

2.2.4. Some functional inequalities. Here are some discrete functional inequalities used throughout the article.

**Lemma 2.1** (Discrete functional inequality). (i) (A naive estimate) Given a $V \subset \mathbb{Z}^d$ and for a function $v : V \to \mathbb{R}$, we have

$$\langle \nabla v, \nabla v \rangle_V \leq 2d \langle v, v \rangle_V.$$

(ii) (Poincaré’s inequality) For every $v \in C_0(\square_m)$, we have

$$\|v\|_{L^2(\square_m)} \leq C(d)3^m \|\nabla v\|_{L^2(\square_m)}.$$

(iii) (H^2 interior regularity for discrete harmonic function) Given two functions $v, f \in C_0(\square_m)$ satisfying the discrete elliptic equation $(\Delta v = \nabla \cdot \nabla v)$

$$- \Delta v = f \quad \text{in int}(\square_m),$$

$$\langle \nabla v, \nabla v \rangle_V \leq \frac{1}{2} \langle v, v \rangle_V.$$
then we have an interior estimate

\begin{equation}
\|D_s^* D v\|_{L^2(\text{int}(\square_m))}^2 := \sum_{i,j=1}^d \|D_{e_i}^* D_{e_j} v\|_{L^2(\text{int}(\square_m))}^2 \leq d \|f\|_{L^2(\text{int}(\square_m))}^2.
\end{equation}

(iv) (Trace inequality) For every $u : \square_m \to \mathbb{R}$ and $0 \leq K \leq 3^m$, we have the following inequality

\begin{equation}
\|u 1_{\{\text{dist}(\cdot, \partial \square_m) \leq K\}}\|_{L^2(\square_m)}^2 \leq C(d)(K+1) \left(3^{-m} \|u\|_{L^2(\square_m)}^2 + \|u\|_{L^2(\square_m)} \|
abla u\|_{L^2(\square_m)} \right).
\end{equation}

The inequality (2.10) is very elementary, and the proof of eq. (2.11) is similar to the standard case, so we skip their proofs. The inequality (2.13) is also relatively standard, but involves a careful calculation. The argument for eq. (2.14) is more combinatorial and non-trivial. We provide their proofs in Appendix A.

2.3. Partition of good cubes. One difficulty to treat the function defined on the percolation clusters comes from its random geometry. To overcome this problem, [3] introduces a Calderón-Zygmund type partition of good cubes, and we recall it here.

We denote by $T$ the triadic cube and $\square_m(z)$ is defined by

$$\square_m(z) := \mathbb{Z}^d \cap \left( z + \left(-\frac{1}{2}3^m,\frac{1}{2}3^m\right) \right), z \in 3^m \mathbb{Z}^d, m \in \mathbb{N},$$

where center and size of the cube above is respectively $z$ and $3^m$, and we use the notation size($\cdot$) to refer to the size, i.e. size($\square_m(z)$) = $3^m$. In this paper, without further mention, we use the word “cube” for short of “triadic cube” and $\square_m$ for short of $\square_m(0)$. The collection of all the cubes of size $3^m$ is defined by $T_n$, i.e. $T_n := \{z + \square_m : z \in 3^n \mathbb{Z}^d\}$. Then we have naturally $T = \bigcup_{n \in \mathbb{N}} T_n$. Every cube of size $3^m$ can be divided into a partition of $3^{(m-n)}$ cubes in $T_n$, and two cubes in $T$ can be either disjoint or included one by the other. For each $\square \in T$, the predecessor of $\square$ is the unique triadic cube $\hat{\square} \in T$ satisfying $\square \subset \hat{\square}$, and $\frac{\text{size}(\square)}{\text{size}(\hat{\square})} = 3$, and reciprocally, we say $\square$ is a successor of $\hat{\square}$.

The distance between two points $x, y \in \mathbb{R}^d$ is defined to be $\text{dist}(x, y) = \max_{i \in \{1,2,\ldots,d\}} |x_i - y_i|$ and the distance for $U, V \subset \mathbb{Z}^d$ is $\text{dist}(U, V) = \inf_{x \in U, y \in V} \text{dist}(x, y)$. In particular, two $\square, \square'$ are neighbors if and only if $\text{dist}(\square, \square') = 1$ and one is included in the other if and only if $\text{dist}(\square, \square') = 0$.

2.3.1. General setting. We state at first the general setting of partition of good cubes.

**Proposition 2.1** (Proposition 2.1 of [3]). Let $\mathcal{G} \subset T$ a sub-collection of triadic cubes satisfying the following: for every $\square \subset \square_n \in T$, $\{\square \notin \mathcal{G}\} \in \mathcal{F}(\square + \square_{n+1})$, and there exist two finite positive constants $K, s$

$$\sup_{z \in 3^n \mathbb{Z}^d} \mathbb{P}[z + \square_n \notin \mathcal{G}] \leq K \exp(-K^{-1}3^{ns}).$$

Then, $\mathbb{P}$-almost surely there exists $S \subset T$ a partition of $\mathbb{Z}^d$ with the following properties:

1. Cubes containing elements of $S$ are good: for every $\square, \square' \in T$, $\square \subset \square'$, $\square \in S \implies \square' \in \mathcal{G}$.
2. Neighbors of elements of $S$ are comparable: for every $\square, \square' \in S$ such that $\text{dist}(\square, \square') \leq 1$, we have $\frac{1}{3} \leq \frac{\text{size}(\square)}{\text{size}(\square')} \leq 3$.
3. Estimate for the coarseness: we use $\square_S(x)$ to represent the unique element in $S$ containing a point $x \in Z^d$, then there exists a finite positive constant $C := C(s, K, d)$ such that, for every $x \in \mathbb{Z}^d$, $\text{size}(\square_S(x)) \leq O_s(C)$.
2.3.2. Case of well-connected cubes. The construction in Proposition 2.1 works for all collection of good cubes \( \mathcal{G} \), here we give the concrete definition of good cubes we use in our context of percolation, as appearing in the work [57], [58] and [2] of Antal, Pisztora and Penrose. We remark that in Definition 2.1 and Definition 2.2 we use “cube” exceptionally for a general lattice cube, and we will highlight explicitly “triadic cube” when using it. The notation \( \frac{3}{4} \Box \) indicates that we take the convex hull of the lattice cube, and then change its size by multiplying by \( \frac{3}{4} \) while keeping the center fixed.

**Definition 2.1** (Crossability and crossing cluster). We say that a cube \( \Box \) is crossable with respect to the open edges defined by \( \mathbf{a} \) if each of the \( d \) pairs of opposite \((d - 1)\)-dimensional faces of \( \Box \) can be joined by an open path in \( \Box \). We say that a cluster \( \mathcal{C} \subset \Box \) is a crossing cluster for \( \Box \) if \( \mathcal{C} \) intersects each of the \((d - 1)\)-dimensional faces of \( \Box \).

**Definition 2.2** (Well-connected cube and good cube, Theorem 3.2 of [58]). We say that \( \Box \in \mathcal{T} \) is well-connected if there exists a crossing cluster \( \mathcal{C} \) for \( \Box \) such that :

1. each cube \( \Box' \) with \( \frac{1}{10} \text{size}(\Box) \leq \text{size}(\Box') \leq \frac{1}{2} \text{size}(\Box) \) and \( \Box \cap \frac{3}{4} \Box \neq \emptyset \) is crossable.
2. every path \( \gamma \subset \Box' \) defined above with \( \text{diam}(\gamma) \geq \frac{1}{10} \text{size}(\Box) \) is connected to \( \mathcal{C} \) within \( \Box' \).

We say that \( \Box \in \mathcal{T} \) is a good cube if \( \text{size}(\Box) \geq 3 \), \( \Box \) is connected and all his \( 3^d \) successors are well-connected. Otherwise, we say that \( \Box \in \mathcal{T} \) is a bad cube.

The following estimates makes the construction defined in Proposition 2.1 work.

**Lemma 2.2** ((2.24) of [2]). For each \( p \in (p_c, 1] \), there exists a positive constant \( C := C(d, p) \) such that for every \( n \in \mathbb{N} \),

\[
\mathbb{P}[\Box_n \notin \mathcal{G}] \geq 1 - C \exp(-C^{-1}3^n).
\]

**Definition 2.3** (Partition of good cubes in percolation context). We let \( \mathcal{P} \subset \mathcal{T} \) be the partition \( \mathcal{S} \) of \( \mathbb{Z}^d \) obtained by applying Proposition 2.1 to the collection of good cubes defined in Definition 2.2

\[
\mathcal{G} := \{ \Box \in \mathcal{T} : \Box \text{ is good cube } \}.
\]

A direct application of Lemma 2.2 and Proposition 2.1 gives us:

**Corollary 2.1.** There exists a positive constant \( C(d, p) \), such that for every \( z \in \mathbb{Z}^d \), we have the two estimates

\[
\text{size}(\Box_p(z)) \leq O_1(C), \quad 1_{\{\text{size}(\Box_p(z)) \geq n\}} \leq O_1(C3^{-n}).
\]

The maximal cluster is well defined on every good cube by Definition 2.2.

**Definition 2.4** (Maximal cluster in good cubes). For every good cube \( \Box \), there exists a unique maximal crossing cluster in it, and we denote this cluster by \( \mathcal{C}_*(\Box) \).

Although \( \mathcal{C}_*(\Box) \) only uses local information, the next lemma shows that, for a \( \Box \in \mathcal{P} \) (stronger than \( \Box \) is good), its maximal cluster \( \mathcal{C}_*(\Box) \) must belong to the infinite cluster \( \mathcal{C}_\infty \).

**Lemma 2.3** (Lemma 2.8 of [3]). Let \( n, n' \in \mathbb{N} \) with \( |n - n'| \leq 1 \) and \( z, z' \in 3^n \mathbb{Z}^d \) such that

\[
\text{dist}(\Box_n(z), \Box_{n'}(z')) \leq 1.
\]

Suppose also that \( \Box_n(z) \) and \( \Box_{n'}(z') \) are all good cubes, then there exists a cluster \( \mathcal{C} \) such that

\[
\mathcal{C}_*(\Box_n(z)) \cup \mathcal{C}_*(\Box_{n'}(z')) \subset \mathcal{C} \subset \Box_n(z) \cup \Box_{n'}(z').
\]

This lemma helps us generalize the definition of maximal cluster in a general set \( U \subset \mathbb{Z}^d \), the idea is to define the union of the partition cubes that cover \( U \), and then find the maximal cluster in it.

**Definition 2.5** (Maximal cluster in general set). For a general set \( U \subset \mathbb{Z}^d \), we define its closure with respect to \( \mathcal{P} \) by

\[
\text{cl}_\mathcal{P}(U) := \bigcup_{z \in U} \Box_p(z),
\]

and \( \mathcal{C}_*(U) \) to be the cluster contained in \( \text{cl}_\mathcal{P}(U) \) which contains all the clusters of \( \mathcal{C}_*(\Box_p(z)), z \in U \).
One can check easily that Lemma 2.3 makes the definition \( C_s(U) \) well-defined. However, we do not have necessarily \( C_s(U) = \bigcup_{z \in U} C_s(\square p(z)) \). We provide with a detailed discussion of this point in Appendix B.

Since the cubes in \( \mathcal{T} \) can be either included in one another or disjoint, if one cube \( \square \in \mathcal{T} \) contains an element in \( \mathcal{P} \), then it can be decomposed as the disjoint union of elements in \( \mathcal{P} \) without enlarging the domain. Thus, we define:

**Definition 2.6** (Minimal scale for partition).

\[ \square \in \mathcal{T} : \exists \square' \subset \square \text{ and } \square' \in \mathcal{P} \].

The following observations are very useful and can be checked easily: for every \( \square \in \mathcal{T} \), we have

\[ \square \in \mathcal{P} \iff \text{cl}_p(\square) = \square, \quad \mathbf{1}_{\{\square \notin \mathcal{P} \}} \leq \mathbf{1}_{\{\text{size}(\square p(z)) > \text{size}(\square)\}} \leq O_1(\text{size}(\square)^{-1}). \]

2.3.3. **Mask operator and coarsened function.** To overcome the problem of the passage between the two geometries \((\mathbb{Z}^d, E_d)\) and \((\mathcal{C}_\infty, E^a_d)\), one useful technique is the mask operator.

**Definition 2.7** (Mask operator and local mask operator). For \( f : \mathbb{Z}^d \to \mathbb{R} \) and \( a : E_d \to \mathbb{R} \), we define a mask operator \( \cdot \circ \psi \) to restrict their support on \( C_\infty \) and \( E^a_d(\mathcal{C}_\infty) \) respectively

\[ f\circ \psi (x) := \begin{cases} f(x) & \text{if } x \in C_\infty, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad a\circ \psi (x,y) := \begin{cases} a(x,y) & \text{if } x,y \in C_\infty, \\ 0 & \text{otherwise.} \end{cases} \]

Moreover, we also define a local mask operator for \( \square_m \in \mathcal{G} \) as

\[ f_{\psi,m} (x) := \begin{cases} f(x) & \text{if } x \in C_\infty(\square_m), \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad a_{\psi,m} (x,y) := \begin{cases} a(x,y) & \text{if } x,y \in C_\infty(\square_m), \\ 0 & \text{otherwise.} \end{cases} \]

Then we call \( f_{\psi}(f_{\psi,m}), a_{\psi}(a_{\psi,m}) \) (local) masked function and (local) masked conductance.

Reciprocally, for a function only defined on the clusters, sometimes we have to extend them to the whole space. We can apply the technique of coarsening the function defined on the percolation cluster.

**Definition 2.8** (Coarsened function). Given \( \square \in \mathcal{P} \), we let \( \tilde{z}(\square) \) represent the vertex in \( C_s(\square) \) which is closest to its center. For a function \( u : C_\infty \to \mathbb{R} \), we define the coarsened function with respect to \( \mathcal{P} \) to be \( [u]_\mathcal{P} : \mathbb{Z}^d \to \mathbb{R} \) that

\[ [u]_\mathcal{P} (x) := u(\tilde{z}(\square p(x))). \]

We also use the notation \([\cdot]\) to mean doing constant extension on every cube, i.e. given \( v : \mathbb{Z}^d \to \mathbb{R} \), we define \([v] : \mathbb{R}^d \to \mathbb{R} \) such that for every \( z \in \mathbb{Z}^d \) and every \( x \in z + \left[ -\frac{1}{2}, \frac{1}{2} \right]^d \), \([v](x) := v(z)\).

The advantage of the coarsened function is that it allows to extend the support of function from \( C_\infty \) to the whole space, and constant in every cube by paying a small cost of errors.

**Proposition 2.2** (Lemmas 3.2 and 3.3 of [3]). For every \( 1 \leq s < \infty \), there exists a finite positive constant \( C(s,d,p) \), such that for every \( \square \in \mathcal{P}_*, \ u : C_\infty \to \mathbb{R} \), we have

\[ \sum_{x \in C_s(z)} |u(x) - [u]_\mathcal{P} (x)|^s \leq C^s \sum_{y,z} \text{size}(\square p(y))^{sd} |\nabla u|^s(y,z), \]

\[ \sum_{\{x,y\} \in E_d(z)} |\nabla [u]_\mathcal{P} (x,y)|^s \leq C^s \sum_{\{x,y\} \in E_d(z)} \text{size}(\square p(x))^{sd-1} |\nabla u|^s(x,y). \]

**Remark.** The main idea of coarsened function is to give function a constant value in every cube, but the value does not have to be of the one closest to the center. Following the same idea of proof of [3, Lemmas 3.2 and 3.3], one can prove that for \( \square \in \mathcal{P}_*, \ u \in C_0(\square) \)

\[ [u]_{\mathcal{P},\square} (x) = \begin{cases} [u]_\mathcal{P} (x) & \text{if } \text{dist}(\square p(x), \partial \text{cl}_p(\square)) \geq 1, \\ 0 & \text{if } \text{dist}(\square p(x), \partial \text{cl}_p(\square)) = 0, \end{cases} \]

we have the same inequality as eq. (2.21) and eq. (2.22) by putting \([u]_{\mathcal{P},\square}\) in the place of \([u]_\mathcal{P}\).
2.4. Harmonic functions on the infinite cluster. We define \( \mathcal{A}(U) \), the set of \( \alpha \)-harmonic functions on \( U \subset \mathbb{Z}^d \), by
\[
\mathcal{A}(U) := \{ v : \mathcal{C}_\infty \to \mathbb{R} | - \nabla \cdot \alpha \nabla v = 0, \forall x \in \text{int}_a(U) \},
\]
and \( \mathcal{A}(\mathcal{C}_\infty) \) the set \( \alpha \)-harmonic functions on \( \mathcal{C}_\infty \). The \( \alpha \)-harmonic function \( \mathcal{A}_k(\mathcal{C}_\infty) \) is the subspace of \( \alpha \)-harmonic functions which grows more slowly than a polynomial of degree \( k + 1 \):
\[
\mathcal{A}_k(\mathcal{C}_\infty) := \left\{ u \in \mathcal{A}(\mathcal{C}_\infty) \mid \limsup_{R \to \infty} R^{-(k+1)} \|u\|_{L^2(\mathcal{C}_\infty \cap B_R)} = 0 \right\}.
\]
Similarly, we can define the spaces \( \tilde{\mathcal{A}}, \tilde{\mathcal{A}}_k \) for harmonic functions on \( \mathbb{R}^d \). It is well-known that the space \( \tilde{\mathcal{A}}_k \) is a finite-dimensional vector space of polynomials. A recent remarkable result about \( \alpha \)-harmonic functions on the infinite cluster of percolation conjectured in [13] and proved in [3] is that the space \( \mathcal{A}_k(\mathcal{C}_\infty) \) also has this property, and in fact has the same dimension as \( \tilde{\mathcal{A}}_k \). Here we only recall the structure of \( \mathcal{A}_1(\mathcal{C}_\infty) \): for every \( \alpha \)-harmonic functions \( u \in \mathcal{A}_1(\mathcal{C}_\infty) \), there exists \( c \in \mathbb{R}, p \in \mathbb{R} \) such that
\[
\forall x \in \mathcal{C}_\infty, \quad u(x) = c + p \cdot x + \phi_p(x),
\]
where the functions \( \{ \phi_p \}_{p \in \mathbb{R}^d} \) are called the first order correctors. The first order correctors have sub-linear growth: there exists a positive exponent \( \delta(d,p,\Lambda) < 1 \) and a minimal scale \( M \leq O_s(C(d,p,\Lambda)) \) such that, for every \( r \geq M \) and \( p \in \mathbb{R}^d \),
\[
\tag{2.24}
\|\phi_p\|_{L^2(\mathcal{C}_\infty \cap B_r)} \leq C|p|r^{-\delta}.
\]
Combining eq. (2.24) and Cacciopoli’s inequality [3, Lemma 3.5], for every \( r \geq 2M \), we have
\[
\|\nabla(\phi_p + l_p)\|_{L^2(\mathcal{C}_\infty \cap B_r/2)} \leq \frac{1}{r} \|\phi_p + l_p\|_{L^2(\mathcal{C}_\infty \cap B_r)} \leq \frac{C}{r}(r^{-\delta} + r)|p| \leq C|p|,
\]
so it also implies that the estimate for the gradient of corrector that
\[
\|\nabla \phi_p\|_{L^2(\mathcal{C}_\infty \cap B_r/2)} \leq C|p|.
\]
The corrector plays an important role in the homogenization theory, and [19] gives a more precise description of these correctors. We recall that \( \Phi_R(x) := \frac{1}{(4\pi R^2)^{d/2}} \exp \left(-\frac{x^2}{4R^2}\right) \), and \( [\phi_p]^\eta_p := [\phi_p]_p \ast \eta \) where \( \eta \in C_0^\infty(B_1) \) is positive, and \( \eta \equiv 1 \) in \( B_\frac{3}{4} \).

**Proposition 2.3** (Local estimate and spatial average estimate, Proposition 3.1 of [19]). There exist two finite positive constants \( s := s(d,p,\Lambda), C := C(d,p,\Lambda) \) such that for each \( R \geq 1 \) and each \( p \in \mathbb{R}^d \),
\[
\tag{2.26}
\forall x \in \mathbb{Z}^d, \quad |\nabla \phi_p 1_{\{a \neq 0\}}(x)| \leq O_s(C|p|),
\]
\[
\tag{2.27}
\forall x \in \mathbb{R}^d, \quad |\nabla (\Phi_R \ast [\phi_p]^\eta_p)(x)| \leq O_s(C|p|R^{-\frac{4}{2}}).
\]

**Proposition 2.4** (Theorem 1 and 2 of [19], \( L^q \) estimates on \( \mathcal{C}_\infty \)). There exist three finite positive constants \( s := s(d,p,\Lambda), k := k(d,p,\Lambda) \) and \( C := C(d,p,\Lambda) \) such that for each \( q \in [1, \infty) \), \( R \geq 1 \) and \( p \in \mathbb{R}^d \),
\[
\tag{2.28}
\left( R^{-d} \int_{\mathcal{C}_\infty \cap B_R} |\phi_p - (\phi_p)_{\mathcal{C}_\infty \cap B_R}|^q \right)^{\frac{1}{q}} \leq \begin{cases} O_s(C|p|q^k \log^{\frac{1}{2}}(R)) & d = 2, \\ O_s(C|p|q^{\frac{k}{d}}) & d = 3, \end{cases}
\]
and for every \( x, y \in \mathbb{Z}^d \) and \( p \in \mathbb{R}^d \),
\[
|\phi_p(x) - \phi_p(y)|1_{\{x,y \in \mathcal{C}_\infty\}} \leq \begin{cases} O_s(C|p| \log^{\frac{1}{2}}|x-y|) & d = 2, \\ O_s(C|p|) & d = 3. \end{cases}
\]
3. Centered flux on the cluster

In this part, we will study an object $g_p : \mathbb{Z}^d \to \mathbb{R}^d$ called \textit{centered flux} defined by

$$g_p := \mathbf{a}_{\phi}(D\phi_p + p) - \bar{a}p,$$

where $\mathbf{a}_{\phi}$ is the masked conductance defined in eq. (2.19) and it is $\mathbf{a}$ restricted on the infinite cluster $\mathcal{C}_\infty$. Because $g_p$ satisfies $D^* \cdot g_p = 0$ on $\mathbb{Z}^d$, following the spirit of Helmholtz-Hodge decomposition, in the later part of this section we will also study another object $S_p : \mathbb{Z}^d \to \mathbb{R}^{d \times d}$ called \textit{flux corrector} such that $g_p = D^* \cdot S_p$ on $\mathbb{Z}^d$, in the sense $g_{pi} = \sum_{j=1}^{d} D_{ji}^* S_{pj}$.

The quantities $g_p$ and $S_p$ are fundamental to the quantitative analysis of the two-scale expansion, see for instance [30] and [7, Chapter 6]. Roughly speaking, $D\phi_p, g_p$ and $DS_p$ should satisfy similar estimates. The goal of this section is to study various quantities like spatial averages and $L^p$ and $L^\infty$ estimates on $S_p$, as a counterpart of the work [19] concerning $\phi_p$.

We can prove at first a very simple result.

\textbf{Proposition 3.1} (Local average). There exit two finite positive constants $s := s(d, p, \Lambda)$ and $C := C(d, p, \Lambda)$ such that

$$\forall x \in \mathbb{Z}^d, \quad |g_p(x) \leq O_s(C|p|).$$

\textbf{Proof.} We have, by eq. (2.26)

$$|g_p| = |\mathbf{a}_{\phi}(D\phi_p + p) - \bar{a}p| \leq |\mathbf{a}_{\phi}D\phi_p| + |\mathbf{a}_{\phi}p| + |\bar{a}p| \leq |\nabla \phi_p \mathbb{1}_{\{a\neq 0\}}| + 2|p| \leq O_s(C|p|).$$

\hfill \Box

3.1. Spatial average of centered flux. In this part, we focus on the spatial average quantity $K_R \ast |g_p|$ and prove Proposition 1.2. The spirit of the proof can go back to the spectral gap method (or Efron-Stein type inequality) in the work of Naddaf and Spencer [54], which is also employed in the work of Gloria and Otto [30, 31, 27]. Proposition 1.2 is more technical in two aspects:

- In the percolation context, the perturbation of the geometry of clusters has to be taken into consideration when applying the spectral gap method.
- The result stated with $O_s$ notation requires a stronger concentration analysis.

Our proof follows generally the main idea of eq. (2.27) appearing in [19, Proposition 3.1], and the main tool used in this proof is a variant of the Efron-Stein type inequality, combined with the Green’s function and Meyers’ inequality on $\mathcal{C}_\infty$.

\textbf{Proof of Proposition 1.2.} Without loss of generality we suppose that $|p| = 1$, and the proof is decomposed into 4 steps.

\textit{Step 1: Spectral gap inequality and double environment.} We introduce the Efron-Stein type inequality used for the proof, which is proved first in [8, Proposition 2.2] and also used in [19, Proposition 2.17]. (We remark kindly that there is a typo in the exponent in [8, Proposition 2.2], which should be $2-\beta$; see also [8, Appendix A] where the exponent is correct.)

\textbf{Proposition 3.2} (Exponential Efron-Stein inequality, Proposition 2.2 of [8]). Fix $\beta \in (0, 2)$ and let $X$ be a random variable defined in the random space $(\Omega, \mathcal{F}, \mathbb{P})$ generated by $\{a(e)\}_{e \in E_d}$, and we define

$$\mathcal{F}(E_d \setminus \{e\}) := \sigma \left(\{a(e')\}_{e' \in E_d \setminus \{e\}}\right),$$

$$X_e := \mathbb{E}[X | \mathcal{F}(E_d \setminus \{e\})], \quad \mathbb{V}[X] := \sum_{e \in E_d} (X - X_e)^2.$$ 

Then, there exists a positive constant $C := C(d, \beta)$ such that

$$\mathbb{E} \left[ \exp \left( |X - \mathbb{E}[X]|^\beta \right) \right] \leq C \mathbb{E} \left[ \exp \left( (C\mathbb{V}[X])^{\frac{\beta}{1-\beta}} \right) \right]^{\frac{2-\beta}{\beta}}.$$
In the proof of Proposition 3.1, we apply this inequality by posing $X := (K_R * [g_R]) (x)$ and we claim that it suffices to verify two conditions

\begin{align}
\mathbb{E}[X] &\leq C_1 R^{-\frac{d}{2}}, \\
\mathbb{V}[X] &\leq O_s(C_2 R^{-d}).
\end{align}

(3.5) \hspace{2cm} (3.6)

It is also very natural, because the two conditions say that the average and fluctuation of $X$ are of the order of $R^{-\frac{d}{2}}$. We choose a $s$ such that $\frac{s}{2-s} = s'$ where $s'$ is the exponent in eq. (3.6) and $C_3 := (C_1 \vee C_2) C(d, \beta)$ where $C(d, \beta)$ is the constant in eq. (3.4) and $C_1, C_2$ the one in eq. (3.5), eq. (3.6), then

\begin{align}
\mathbb{E} \left[ \exp \left( \frac{X}{C_3 R^{-\frac{d}{2}}} \right)^s \right] &\leq \mathbb{E} \left[ \exp \left( \frac{X - \mathbb{E}[X]}{C_3 R^{-\frac{d}{2}}} + \frac{\mathbb{E}[X]}{C_3 R^{-\frac{d}{2}}} \right)^s \right] \\
&\leq C \mathbb{E} \left[ \exp \left( \frac{|X - \mathbb{E}[X]|}{C_3 R^{-\frac{d}{2}}} \right)^s \right] \\
&\leq C \mathbb{E} \left[ \exp \left( \frac{\mathbb{V}[X]}{C_2 R^{-d}} \right)^{s'} \right]^{\frac{2-s}{s'}} \\
&\leq 2C.
\end{align}

Finally, we increase $C_3$ with respect to $s$ so that we get $X \leq O_s(C R^{-\frac{d}{2}})$.

We focus on the two conditions eq. (3.5), eq. (3.6). In fact, we can check the condition eq. (3.5) by proving $\mathbb{E}[a_e(D\phi_p + p)] = \tilde{a}p$, which is a well-known result in classic homogenization. In percolation context, it is also true by a careful check of the several equivalent definitions of $\tilde{a}$. We put its proof in Theorem C.1.

To prove the condition eq. (3.6), we use a useful technique in Efron-Stein type inequality of “doubling” the probability space: we sample a copy of random conductance \{$(\tilde{a}(e'))_{e' \in E_d}$\} with the same law but independent to \{$_a(e')_{e' \in E_d}$\}, and the two probability spaces generated by the two copies are denoted respectively by ($\Omega_a, \mathcal{F}_a, \mathbb{P}_a$), ($\tilde{\Omega}_a, \tilde{\mathcal{F}}_a, \tilde{\mathbb{P}}_a$). Then we put the two copies of random conductance together and make a larger probability space ($\Omega', \mathcal{F}', \mathbb{P}'$) = ($\Omega_a \times \tilde{\Omega}_a, \mathcal{F}_a \otimes \mathcal{F}_a, \mathbb{P}_a \otimes \tilde{\mathbb{P}}_a$), and we also use the notation $\mathcal{O}'_s$ to represent the same definition eq. (2.1) in the larger probability space ($\Omega', \mathcal{F}', \mathbb{P}'$). We also introduce another random environment \{$_a(e')_{e' \in E_d}$\}, obtained by replacing one conductance $a(e)$ by $\tilde{a}(e)$, i.e.

\begin{align}
_a(e') = \begin{cases} 
_a(e'), & e' \neq e, \\
\tilde{a}(e'), & e' = e.
\end{cases}
\end{align}

We use $X_e, \mathcal{D}_x, \phi_x$ to represent respectively the random variable, the infinite cluster and the corrector in the environment \{$_a(e')_{e' \in E_d}$\}. The definition of $\mathbb{V}[X]$ says that the variance comes from the fluctuation caused by the perturbation of every conductance, which suggests the following lemma:

**Lemma 3.1.** We have the following estimate

\begin{align}
\sum_{e \in E_d} (X - X_e)^2 &\leq O_s(C R^{-d}) \implies \mathbb{V}[X] \leq O_s(C R^{-d}).
\end{align}

(3.8)

**Proof.** We use the double environment trick to see that

$$X_e = \mathbb{E} [X | \mathcal{F}(E_d \setminus \{e\})] = \mathbb{E}_a [X^e],$$

\[\square\]
and Jensen’s inequality to reformulate at first the inequality
\[
E \left[ \exp \left( \left( \frac{\sqrt{\text{Var}[X]}}{C R^{-d}} \right)^s \right) \right] = \int_{\Omega} \exp \left( \left( \frac{\sum_{e \in E_d} (X - X_e)^2}{CR^{-d}} \right)^s \right) d\mathbb{P}_a(\omega)
\]
\[
= \int_{\Omega} \exp \left( \left( \frac{\sum_{e \in E_d} \left( \int_{\Omega_a} (X - X_e) d\mathbb{P}_a(\omega) \right)^2}{CR^{-d}} \right)^s \right) d\mathbb{P}_a(\omega)
\]
\[
\leq \int_{\Omega} \exp \left( \left( \int_{\Omega_a} \frac{\sum_{e \in E_d} (X - X_e)^2}{CR^{-d}} d\mathbb{P}_a(\omega) \right)^s \right) d\mathbb{P}_a(\omega)
\]
In the next step, we want to add a constant \( t_s \) to make \( \exp((\cdot + t_s)^s) \) convex, and then exchange the expectation and \( \exp((\cdot + t_s)^s) \) by Jensen’s inequality. We can choose \( t_s = 0 \) for \( s \geq 1 \), and \( t_s = \left( \frac{1-s}{s} \right)^{\frac{1}{2}} \) for \( 0 < s < 1 \). (The spirit is the same as eq. (2.4) and see [7, Lemma A.4] for details of this proof.)

\[
E \left[ \exp \left( \left( \frac{\sqrt{\text{Var}[X]}}{C R^{-d}} \right)^s \right) \right] \leq \int_{\Omega} \int_{\Omega_a} \exp \left( \left( \frac{\sum_{e \in E_d} (X - X_e)^2}{CR^{-d}} + t_s \right)^s \right) d\mathbb{P}_a(\omega) d\mathbb{P}_a(\omega)
\]
\[
\leq \tilde{C} \int_{\Omega} \int_{\Omega_a} \exp \left( \left( \frac{\sum_{e \in E_d} (X - X_e)^2}{CR^{-d}} \right)^s \right) d\mathbb{P}_a(\omega) d\mathbb{P}_a(\omega)
\]
\[
\leq 2\tilde{C}.
\]
In the last step we use the condition \( \sum_{e \in E_d} (X - X_e)^2 \leq \mathcal{O}'_s(CR^{-d}) \) and we reduce the constant \( C \) to get the desired result. \( \square \)

By Lemma 3.1, to prove eq. (3.6) it suffices to focus on the quantity \( \sum_{e \in E_d} (X - X_e)^2 \), and in our context it is
\[
(3.9) \quad \sum_{e \in E_d} \left| K_R * (\lfloor g_p \rfloor - \lfloor g_p^e \rfloor) \right|^2 (x) \leq \mathcal{O}'_s(CR^{-d}).
\]
Since we have
\[
\left| K_R * (\lfloor g_p \rfloor - \lfloor g_p^e \rfloor) \right|(x) \leq \left| K_R * (\lfloor g_p \rfloor - \lfloor g_p^e \rfloor) \right|(x) \mathbf{1}_{\{a^e(e) \leq a(e)\}} + \left| K_R * (\lfloor g_p \rfloor - \lfloor g_p^e \rfloor) \right|(x) \mathbf{1}_{\{a(e) \leq a^e(e)\}},
\]
and the two terms have the same law, without loss of generality, we suppose
\[
(3.10) \quad a^e(e) \leq a(e),
\]
is always valid in the following paragraphs in order to avoid the indicator function everywhere. We will then distinguish several cases and attack them one by one.

**Step 2:** Case \( \mathcal{C}_\infty \neq \mathcal{C}_e \), proof of \( g_p = g_p^e \). We have to consider the perturbation of the geometry between \( \mathcal{C}_\infty \) and \( \mathcal{C}_e \). We prove the following lemma, which has a typical realization in Figure 3.

**Lemma 3.2** (Pivot edge). Under the condition eq. (3.10) and in the case \( \mathcal{C}_\infty \neq \mathcal{C}_e \), we have:

1. The part \( \mathcal{C}_\infty \setminus \mathcal{C}_e \) is connected to \( \mathcal{C}_e \) by \( e \) (called the pivot edge), and \( |\mathcal{C}_\infty \setminus \mathcal{C}_e| < \infty \).
2. We denote by \( e := \{e_*, e^*\}, e_* \in \mathcal{C}_e \cap \mathcal{C}_\infty \) and \( e^* \in \mathcal{C}_\infty \setminus \mathcal{C}_e \), then the function \( \phi_p + l_p \) is constant on \( \mathcal{C}_\infty \setminus \mathcal{C}_e \) and equals to \( (\phi_p + l_p)(e_*) \).
3. The function \( \phi_p^e \) has a representation that \( \phi_p^e = \phi_p \mathbf{1}_{\{e_\infty\}} \) up to a constant and satisfies \( a_e \nabla (\phi_p + l_p) = a_e^e \nabla (\phi_p^e + l_p) \) on \( E_d \).

**Proof.** (1) It comes from the fact that \( a \) and \( a^e \) are different only by one edge, thus \( \mathcal{C}_e \subseteq \mathcal{C}_\infty \) means that \( a(e) > 0 \) in \( \mathcal{C}_\infty \) but \( a^e(e) = 0 \) in \( \mathcal{C}_e \) and makes one part disconnected from \( \mathcal{C}_\infty \).
It is well-known that in the supercritical percolation, almost surely there exists one unique infinite cluster, thus we have $|C_\infty \setminus C_\infty| < \infty$.

(2) We study the harmonic function $-\nabla \cdot a \nabla (\phi_p + l_p) = 0$ on the part $C_\infty \setminus C_\infty$. This is a non-degenerate linear system with $|C_\infty \setminus C_\infty| + 1$ variables, thus the solution is of 1 dimension and we know this constant is $(\phi_p + l_p)(e_\ast)$.

\[ \begin{align*}
\text{Figure 3. In the image the segment in red is the edge } e = \{e_\ast, e^*\} \text{ and the part in blue is the cluster } C_\infty \setminus C_\infty, \text{ where } a\text{-harmonic function } (\phi_p + l_p) \text{ is constant of value } (\phi_p + l_p)(e_\ast).
\end{align*} \]

(3) We prove that at first that $a \nabla (\phi_p + l_p) = a \nabla (\phi_p 1\{e_\ast\} + l_p)$ for every $e \in E_d$.

- For the edge $e'$ such that $a \nabla (e') = 0$, as $a \nabla (e') \leq a \nabla (e')$, the two functions $a \nabla (\phi_p + l_p)(e')$ and $a \nabla (\phi_p 1\{e_\ast\} + l_p)(e')$ are null.

- For the only pivot edge $e$ that $a \nabla (e) > 0, a \nabla (e) = 0$, thanks to the second term of Lemma 3.2, we have $\nabla (\phi_p + l_p)(e) = 0$. Thus, the equation also establishes.

- For the edge that $a \nabla (e') > 0, a \nabla (e') > 0$, we know that this implies that the two endpoints are on $C_\infty$ so that we have

\[ \begin{align*}
\nabla (\phi_p + l_p)(e') &= \nabla (\phi_p 1\{e_\ast\} + l_p)(e'),
\end{align*} \]

and $a \nabla (e') = a \nabla (e')$, so the equation is also established.

\[ a \nabla (\phi_p + l_p) = a \nabla (\phi_p 1\{e_\ast\} + l_p) \text{ implies directly that } -\nabla \cdot a \nabla (\phi_p 1\{e_\ast\} + l_p) = 0 \text{ on } \mathbb{Z}^d, \text{ therefore, by the Liouville regularity, we obtain that } \phi_p = \phi_p 1\{e_\ast\} \text{ on } C_\infty \text{ up to a constant.} \]

A direct corollary of the third part of Lemma 3.2 is that $g_p = g_p^e$ on $E_d$ when $C_\infty \neq C_\infty$, thus $K_R * ([g_p] - [g_p^e]) = 0$. So, it suffices to consider $\sum_{e \in E_d} |K_R * ([g_p] - [g_p^e])|^2 (x)$ under the condition $C_\infty = C_\infty^e$. Then, we can reformulate the quantity in eq. (3.9) as following:

\[ \begin{align*}
K_R * ([g_p] - [g_p^e]) (x) &= K_R * \left([a \nabla (\phi_p + l_p)] - [a \nabla (\phi_p + l_p)]^e \right) (x) 1\{e_\ast = e_\ast\} \\
&= K_R * \left((a - a^e) D (\phi_p + l_p) \right) (x) 1\{e_\ast = e_\ast\} + K_R * \left(a D (\phi_p - \phi_p^e) \right) (x) 1\{e_\ast = e_\ast\}. \\
:= &A_e(x) \\
&:= B_e(x)
\end{align*} \]

In order to prove eq. (3.9), we study $A_e(x)$ and $B_e(x)$ separately.
Step 3: Case $\mathcal{C}_\infty = \mathcal{C}_\infty^e$, proof of $\sum_{e \in E_d} |A_e(x)|^2 \leq O'_s(CR^{-d})$. Lemma 3.2 helps us simplify the discussion on the case $\mathcal{C}_\infty = \mathcal{C}_\infty^e$ and following lemma carries the convolution to the cluster $\mathcal{C}_\infty$.

**Lemma 3.3.** For a kernel $K_{R}$ as in Proposition 1.2 and every $x \in \mathbb{R}^d$, there exists a function $\Gamma_{K,R}^x : \mathbb{R}^d \to \mathbb{R}^+$ such that for every function $\xi$ supported on $\mathcal{C}_\infty$, we have

$$\forall \text{ and } (3.14) \quad (\Gamma_{K,R}^x \ast [\xi])(x) = \langle \Gamma_{K,R}^x \phi, \xi \rangle_{\mathcal{C}_\infty},$$

and we have the estimate

$$\forall \text{ and } (3.15) \quad (\Gamma_{K,R}^x \ast [\xi])(x) = \langle \Gamma_{K,R}^x \phi, \xi \rangle_{\mathcal{C}_\infty},$$

$$\Gamma_{K,R}^x(z) \leq \frac{2^d C_{K,R}}{R^d (|\frac{z-y}{R}| + 1)^{\frac{d+1}{2}}},$$

$$\forall \text{ and } (3.16) \quad (\Gamma_{K,R}^x \ast [\xi])(x) = \langle \Gamma_{K,R}^x \phi, \xi \rangle_{\mathcal{C}_\infty}.$$
Proof. This estimate is very easy when \( a^c(e) > 0 \), since it implies \( a^c(e) > \Lambda^{-1} \) and we obtain \( a^c(e) \leq \Lambda a^c(e) \) together with eq. (3.10). We then use Proposition 3.1 directly that
\[
| (a^c - a^c) D(\phi^c + l_p) 1_{e^c = e^c} | \leq (1 + \Lambda) | a^c D(\phi^c + l_p) 1_{e^c = e^c} | \leq O'_{\delta}(C).
\]
The less immediate part comes from the case \( a^c(e) = 0 \) while \( a^c(e) > 0 \), where \( a^c D(\phi^c + l_p) = 0 \) and cannot be used to dominate \( |\Theta(e)| \). We treat this case differently: we denote by \( e = \{ e_s, e^* \} \), \( e^c = e^c \) implies the existence of another open path \( \gamma \) in \( e^c \) connecting \( e_s \) and \( e^* \) (see Figure 4). This path can be chosen in \( e^c_s(\Box p^c(e_s)) \cup e^c_s(\Box p^c(e^*)) \) applying Lemma 2.3 to the partition cube \( P^e \).

\[
|\Theta(e_s, e^*)| 1_{e^c = e^c} = | (a^c - a^c) \nabla (\phi^c + l_p) | (e_s, e^*) 1_{e^c = e^c} \leq \sum_{e' \in e^c \cap e^c_s(\Box p^c(e_s)) \cup e^c_s(\Box p^c(e^*))} |\nabla (\phi^c + l_p) | (e') 1_{e^c = e^c} \leq 2 \sum_{e' \in e^c \cap e^c_s(\Box p^c(e_s)) \cup e^c_s(\Box p^c(e^*))} |\nabla (\phi^c + l_p) | (e') 1_{e^c = e^c} \leq C.
\]

This sum, we have to notice that not only the corrector \( \phi^c \) is random, but also the path \( \gamma \) and its length. This forbids us to use directly eq. (2.4) or eq. (2.6), so we apply the minimal scale argument and eq. (2.25) in the environment \( \{ a^e(e) \}_{e \in E_d} \): there exists \( s := s(d, p, \Lambda) > 0 \) and \( C := C(d, p, \Lambda) < \infty \) such that for any \( x \in e^c \), we have a random variable \( M^e(x) \leq O'_{\delta}(C) \) and for every \( r \geq M^e(x) \),
\[
|\nabla a^c | 1_{e^c \neq 0} |_{L^2(e^c \cap B_r(x))} \leq C.
\]

Then we take
\[
\tilde{M}(e) = \max \{ M^e(e^*), \text{size}(\Box p^c(e^*)) , \text{size}(\Box p^c(e_s)) \},
\]
and it is clear that \( \tilde{M}(e) \leq O'_{\delta}(C) \) and the ball \( B_{\tilde{M}(e)}(e^*) \) contains \( \Box p^c(e^*) \) and \( \Box p^c(e_s) \). Thus we can use eq. (3.18) and Cauchy-Schwarz inequality to control the sum over the path \( \gamma \)
\[
|\Theta(e_s, e^*)| 1_{e^c = e^c} \leq 2 \sum_{e' \in e^c \cap B_{\tilde{M}(e)}(e^*)} |\nabla (\phi^c + l_p) | (e') 1_{e^c = e^c} \leq (\tilde{M}(e))^d |\nabla (\phi^c + l_p) 1_{e^c = e^c} |_{L^2(e^c \cap B_{\tilde{M}(e)}(e^*))} \leq C(\tilde{M}(e))^d.
\]

Finally we use \((\tilde{M}(e))^d \leq O'_{\delta / d}(C)\) to conclude the proof of Lemma 3.4.\(\square\)

We conclude from eq. (3.14), eq. (3.12) and Lemma 3.4 that
\[
\sum_{e \in E_d} |A_e|^2(x) \leq \sum_{i=1}^d \sum_{z \in e^c} |\Gamma^x_{K,R}|^2(z) |\Theta|^2(z, z + e_i) 1_{e^c = e^c} \leq \sum_{i=1}^d \sum_{z \in E_d} 4^d C_{K,R}^2 \frac{2d}{R^2d} \frac|\Theta|^2(z, z + e_i) 1_{e^c = e^c} \leq O'_{\delta} \left( \frac{C^2_{K,R}}{R^d} C(d, p, \Lambda) \right) .
\]

Step 4: Case \( e^c = e^c \), proof of \( \sum_{e \in E_d} |B_e|^2(x) \leq O'_{\delta}(CR^{-d}) \). This step is similar to that for \( A_e \) but more technical. We define a space
\[
\tilde{H}^1(e^c) := \{ v : e^c \to \mathbb{R}, (\nabla v, \nabla v)_{E_d(e^c)} < \infty \},
\]
and use the Green’s function on \((e^c, E^a_d(e^c)) \) [19, Proposition 2.11]:

\[

\]
Figure 4. This image shows the case $C_\infty = C_e\infty$ and $a_e^\infty(e) = 0$ while $a_\infty(e) > 0$, so $e = \{e^*, e^e\}$ (the segment in red) is an open bond in $C_\infty$ but not in $C_e\infty$. The condition $C_\infty = C_e\infty$ ensures another open path $\gamma$ (the segment in green) in $C_e\infty$ connecting $e^*$ and $e^e$. The path $\gamma$ is contained in the union of the partition cube $\Box_{p^\infty}(e^e)$ and $\Box_{p^e}(e^e)$ (the cubes in yellow). To estimate the sum of $\nabla \phi^e$ over this path $\gamma$, we choose a minimal scale $\tilde{M}(e)$ and study the average in this scale (the ball in blue).

**Proposition 3.3** (Green’s function on $C_\infty$). Let $a \in \Omega$ be an environment with an infinite cluster $C_\infty$ and $x, y \in C_\infty$, then there exist a constant $C := C(d, \Lambda) < \infty$ and a Green’s function $G^{x,y} \in \dot{H}^1(C_\infty)$ such that

$$-\nabla \cdot a^C \nabla G^{x,y} = \delta_y - \delta_x \text{ on } C_\infty,$$

in the sense for any $v \in \dot{H}^1(C_\infty)$, we have

$$\langle \nabla G^{x,y}, a^C \nabla v \rangle_{E_d^a(C_\infty)} = v(y) - v(x).$$

In the case that $e = (x, y) \in \overrightarrow{E_d^a}(C_\infty)$, we note $G^{x,y} := G^e$. The Green’s function $G^{x,y}$ has the following properties

- **Symmetry:** For every $x, y, x', y' \in C_\infty$, we have $G^{x,y}(y') - G^{x,y}(x') = G^{x',y'}(y) - G^{x',y'}(x)$.
- **Representation:** For every $v \in \dot{H}^1(C_\infty)$, every vector field $\xi : \overrightarrow{E_d^a}(C_\infty) \to \mathbb{R}$, and $u_\xi \in \dot{H}^1(C_\infty)$ such that

$$\langle \nabla u_\xi, a^C \nabla v \rangle_{E_d^a(C_\infty)} = \langle \xi, \nabla v \rangle_{E_d^a(C_\infty)},$$

we have the representation

$$\nabla u_\xi = \sum_{e \in \overrightarrow{E_d^a}(C_\infty)} \xi(e) \nabla G^e.$$

In this formula, we give an arbitrary orientation for $e \in \overrightarrow{E_d^a}(C_\infty)$, and the equation is well-defined.

**Proof.** The proof of the existence and uniqueness up to a constant for the function $G^{x,y}$ comes from the Lax-Milgram theorem on the space $\dot{H}^1(C_\infty)$ where the conductance satisfies the quenched uniform ellipticity condition. The symmetry comes from testing the equation $-\nabla \cdot a^C \nabla G^{x,y} = \delta_y - \delta_x$
by $G^{x',y'}$ and testing the equation $-\nabla \cdot a\phi \nabla G^{x',y'} = \delta_{y'} - \delta_{x'}$ by $G^{x,y}$ that

$$G^{x,y}(y') - G^{x,y}(x') = \left\langle \nabla G^{x,y}, a\phi \nabla G^{x',y'} \right\rangle_{E_d^a(\mathcal{C}_\infty)} = G^{x',y'}(y) - G^{x,y}(x).$$

The final representation formula can be checked easily by the linear combination of the Green’s function.

The proof of $\sum_{e \in E_d} |B_e|^2(x) \leq O_e'(CR^{-d})$ can be divided in 4 steps.

**Step 4.1:** Identification of $\mathcal{D}(\phi_p - \phi_p^e)$ using Green’s function. We identify at first $\mathcal{D}(\phi_p - \phi_p^e)$ by using the Green’s function $G^e$ introduced in Proposition 3.3 and then estimate its size by this representation. We want to carry all the analysis on the geometry $(\mathcal{C}_\infty, E_d^a(\mathcal{C}_\infty))$ and to claim the following lemma:

**Lemma 3.5.** We denote $e := \{e_s, e^*\} \in E_d^a(\mathcal{C}_\infty)$, under the condition $\mathcal{C}_\infty = \mathcal{C}_\infty^e$, then we have the following representation for $(\phi_p^e - \phi_p)$, using Proposition 3.3 and the definition $\Theta$ in eq. (3.15),

$$(3.20) \quad \nabla (\phi_p^e - \phi_p)(\cdot) = \Theta(e_s, e^*) \nabla G^{e^*,e^*}(-\cdot) \text{ on } E_d^a(\mathcal{C}_\infty).$$

**Proof.** Using the a-harmonic equation and a-harmonic equation for their correctors, we have at first

$$-\nabla \cdot a\phi \nabla (\phi_p + l_p) = -\nabla \cdot a\phi \nabla (\phi_p^e + l_p) \text{ on } \mathbb{Z}^d,$$

then we obtain that

$$(3.21) \quad -\nabla \cdot a\phi \nabla (\phi_p - \phi_p^e) = -\nabla \cdot (a\phi^e - a\phi) \nabla (\phi_p^e + l_p) \text{ on } \mathbb{Z}^d.$$

Using the definition $\Theta(e_s, e^*) = (a\phi - a\phi^e) \nabla (\phi_p + l_p)(e_s, e^*)$ and under the condition $\mathcal{C}_\infty = \mathcal{C}_\infty^e$, the right hand side of eq. (3.21) equals to $\Theta(e)(\delta_{e^*} - \delta_e)$. Moreover, since $e_s, e^* \in \mathcal{C}_\infty$, eq. (3.21) can be seen restricted on the cluster $(\mathcal{C}_\infty, E_d^a(\mathcal{C}_\infty))$. Thus we solve the in $\tilde{H}^1(\mathcal{C}_\infty)$ the equation

$$-\nabla \cdot a\phi \nabla \tilde{w} = -\nabla \cdot (a\phi^e - a\phi) \nabla (\phi_p^e + l_p) \text{ on } \mathcal{C}_\infty,$$

and by Proposition 3.3, it has a unique solution up to a constant that $\tilde{w} = \Theta(e_s, e^*) G^{e^*, e^*}$.

Now we have $(\phi_p - \phi_p^e)$ and $\tilde{w}$ solving the same equation, but we do not yet know if $(\phi_p - \phi_p^e)$ belongs to $\tilde{H}^1(\mathcal{C}_\infty)$. We hope to identify that $\phi_p - \phi_p^e = \tilde{w}$ and the argument is to use the Liouville regularity theorem: notice that $(\phi_p - \phi_p^e - \tilde{w})$ is an a-harmonic function on $\mathcal{C}_\infty$ and $\langle \nabla \tilde{w}, \nabla \tilde{w} \rangle_{E_d^a} < \infty$ implies that $(\phi_p - \phi_p^e - \tilde{w}) \in A_1$. We claim that it is in fact in $A_0$ and prove by contradiction: suppose that $(\phi_p - \phi_p^e - \tilde{w}) \in A_1 \setminus A_0$, then by the Liouville regularity there exists $h \neq 0$ such that

$$\phi_p - \phi_p^e - \tilde{w} = l_h + \phi_h.$$

However, this implies that $\tilde{w} = \phi_p - \phi_p^e - \phi_h - l_h$, so $\tilde{w}$ has an asymptotic linear increment at infinity, which contradicts the fact that $\tilde{w} \in \tilde{H}^1(\mathcal{C}_\infty)$. In conclusion, we have $\phi_p - \phi_p^e - \Theta(e_s, e^*) G^{e^*, e^*} = c$ and we get eq. (3.20). \hfill \Box

**Step 4.2:** Carry the analysis on $(\mathcal{C}_\infty, E_d^a(\mathcal{C}_\infty))$. Observing that we do the sum of $a\phi \mathcal{D}(\phi_p - \phi_p^e)$, it suffices to do the sum over $E_d^a(\mathcal{C}_\infty)$ and with the help of Lemma 3.5, we have the formula

$$\sum_{e \in E_d} |B_e|^2(x) = \sum_{i=1}^d \sum_{e \in E_d^a(\mathcal{C}_\infty)} |K_R \ast [a\phi \mathcal{D}_e(\phi_p - \phi_p^e)]|^2(x) 1_{\mathcal{C}_\infty = \mathcal{C}_\infty^e}$$

$$(3.22) \quad = \sum_{i=1}^d \sum_{e \in E_d^a(\mathcal{C}_\infty)} |K \ast [a\phi \mathcal{D}_e G^e]|^2(x) \Theta^2(e) 1_{\mathcal{C}_\infty = \mathcal{C}_\infty^e}$$

$$= \sum_{i=1}^d \sum_{e \in E_d^a(\mathcal{C}_\infty)} \left| \langle \Gamma_{K,R}^e, a\phi \mathcal{D}_e G^e \rangle_{\mathcal{C}_\infty} \right|^2(x) \Theta^2(e) 1_{\mathcal{C}_\infty = \mathcal{C}_\infty^e}.$$
We analyze \( \langle \Gamma_{K,R}^x, a \ast D_{e_i} G^e \rangle_{E^d_1(\mathcal{C}_\infty)} \) by defining the notation that
\[
1\{E_d^1\}(e') = \begin{cases} 
1 & \text{if } \exists z \in \mathbb{Z}^d \text{ such that } e' = \{z, z + e_i\}, \\
0 & \text{Otherwise}
\end{cases}
\]
and a vector field \( \widetilde{\Gamma}_{K,R,i}^x : E^d_1(\mathcal{C}_\infty) \rightarrow \mathbb{R} \) that
\[
(3.23) \quad \widetilde{\Gamma}_{K,R,i}^x := \Gamma_{K,R}(e'^i, i) a \ast D_{e_i} G^e 1\{\mathcal{C}_\infty = \mathcal{C}_\infty^e\} , \quad e'^i, i \in \mathcal{C}_\infty \text{ such that } e' = \{e'^i, e'^i + e_i\}.
\]
Then, we can send \( \langle \Gamma_{K,R}^x, a \ast D_{e_i} G^e \rangle_{E^d_1(\mathcal{C}_\infty)} \) to the inner product of vector field on \( E^d_1(\mathcal{C}_\infty) \)
\[
(3.24) \quad \langle \Gamma_{K,R}^x, a \ast D_{e_i} G^e \rangle_{E^d_1(\mathcal{C}_\infty)} 1\{\mathcal{C}_\infty = \mathcal{C}_\infty^e\} = \langle \widetilde{\Gamma}_{K,R,i}^x, \nabla G^e \rangle_{E^d_1(\mathcal{C}_\infty)} 1\{\mathcal{C}_\infty = \mathcal{C}_\infty^e\}.
\]

**Step 4.3:** Apply once again the representation with the Green’s function. Since \( \widetilde{\Gamma}_{K,R,i}^x \) is defined on \( E^d_1(\mathcal{C}_\infty) \), we can apply Proposition 3.3 to define \( w_{\Gamma_{K,R,i}^x} \in \dot{H}^1(\mathcal{C}_\infty) \) the solution of the equation
\[
(3.25) \quad - \nabla \cdot a \ast \nabla w_{\Gamma_{K,R,i}^x} = - \nabla \cdot \widetilde{\Gamma}_{K,R,i}^x, \quad \text{on } \mathcal{C}_\infty \text{,}
\]
and it has a representation \( \nabla w_{\Gamma_{K,R,i}^x}(e) = \sum_{e' \in E^d_1(\mathcal{C}_\infty)} \widetilde{\Gamma}_{K,R,i}^x(e') \nabla G^{e'}(e) \). We use the symmetry \( \nabla G^{e'}(e) = \nabla G^e(e') \)
\[
(3.26) \quad \nabla w_{\Gamma_{K,R,i}^x}(e) = \sum_{e' \in E^d_1(\mathcal{C}_\infty)} \widetilde{\Gamma}_{K,R,i}^x(e') \nabla G^e(e') = \langle \widetilde{\Gamma}_{K,R,i}^x, \nabla G^e \rangle_{E^d_1(\mathcal{C}_\infty)}.
\]
We combine eq. (3.22) eq. (3.24) and eq. (3.26) together and obtain that
\[
(3.27) \quad \sum_{e \in E_d} |B_e|^2(x) \leq \sum_{i=1}^d \sum_{e \in E^d_1(\mathcal{C}_\infty)} |\nabla w_{\Gamma_{K,R,i}^x}|^2(\mathcal{C}_\infty^e) 1\{\mathcal{C}_\infty = \mathcal{C}_\infty^e\}.
\]

**Step 4.4:** Meyers’ inequality and minimal scale. From the eq. (3.25), we obtain a \( \dot{H}^1(\mathcal{C}_\infty) \) estimate using eq. (3.12)
\[
\langle \nabla w_{\Gamma_{K,R,i}^x}, a \ast \nabla w_{\Gamma_{K,R,i}^x} \rangle_{E^d_1(\mathcal{C}_\infty)} \frac{1}{2} = \langle \nabla w_{\Gamma_{K,R,i}^x}, \widetilde{\Gamma}_{K,R,i}^x \rangle_{E^d_1(\mathcal{C}_\infty)} \frac{1}{2}
\]
\[
\implies \left\| \nabla w_{\Gamma_{K,R,i}^x} \right\|_{L^2(E^d_1(\mathcal{C}_\infty))} \leq \Lambda \left\| \Gamma_{K,R,i}^x \right\|_{L^2(E^d_1(\mathcal{C}_\infty))} \leq \Lambda \left\| \Gamma_{K,R}^x \right\|_{L^2(E^d_1(\mathcal{C}_\infty))} \leq C_{K,R}^2 R^{-\frac{d}{2}}.
\]
Combining eq. (3.27) and the estimate on \( \Theta(e) \), one may want to argue that
\[
\sum_{e \in E_d} |B_e|^2(x) \leq \sum_{i=1}^d \sum_{e \in E^d_1(\mathcal{C}_\infty)} \Theta\left( |\nabla w_{\Gamma_{K,R,i}^x}|^2(\mathcal{C}_\infty) \right) \leq O\left( C_{K,R}^2 R^{-d} \right).
\]
However, this argument is not correct since eq. (2.4) does not work for our case where \( w_{\Gamma_{K,R,i}^x} \) is stochastic. A rigorous proof needs an argument as in [19, Lemma 3.7] using the minimal scale: We construct a collection of good cubes \( \mathcal{G}' \) such that not only Meyers’ inequality [19, Proposition 3.6] is established, but also there exists \( e(d, p, \Lambda) > 0 \) and \( C(d, p, \Lambda) < \infty \) for all \( \square \in \mathcal{G}' \)
\[
(3.28) \quad \frac{1}{|\square|} \left( \int_{E^d_1(\mathcal{C}_\infty \cap \square)} \Theta(e) \frac{2(|2+i|+e)}{e} \right)^{\frac{e}{2+e}} < C(d, p, \Lambda).
\]
Then we do the Calderón-Zygmund decomposition Proposition 2.1 for \( \mathcal{G}' \) to obtain a partition of cubes \( \mathcal{U} \), and apply Meyers’ inequality for eq. (3.27)
and satisfying the following equations:

\[
\sum_{e \in E_d} |B_e|^2(x) \leq \sum_{i=1}^{d} \sum_{\square \in \mathcal{U}} \sum_{e \in E_d^a(\epsilon_\infty \cap \square)} |\nabla w_{\Gamma_{K,R,i}}^\varepsilon|^2(e) \Theta^{2+\varepsilon}(e) 1_{\{\epsilon_\infty = \epsilon_\infty^c\}}
\]

\[
\leq \sum_{i=1}^{d} \sum_{\square \in \mathcal{U}} \left( \frac{1}{|\square|} \sum_{e \in E_d(\epsilon_\infty \cap \square)} |\nabla w_{\Gamma_{K,R,i}}^\varepsilon|^2(e) \right)^{\frac{1}{2+\varepsilon}} \left( \frac{1}{|\square|} \sum_{e \in E_d(\epsilon_\infty \cap \square)} \Theta^{2+\varepsilon}(e) \right)^{\frac{1}{2+\varepsilon}} 1_{\{\epsilon_\infty = \epsilon_\infty^c\}}
\]

Applying Meyers' inequality

\[
\leq C \sum_{i=1}^{d} \sum_{\square \in \mathcal{U}} \left( \frac{1}{|\square|} \sum_{e \in E_d^a(\epsilon_\infty \cap \square)} |\nabla w_{\Gamma_{K,R,i}}^\varepsilon|^2(e) \right)^{\frac{1}{2+\varepsilon}} \left( \frac{1}{|\square|} \sum_{e \in E_d^a(\epsilon_\infty \cap \square)} \tilde{\Gamma}_{K,R,i}^{\varepsilon}(e) \right)^{\frac{2}{2+\varepsilon}}
\]

The first term can be controlled by the \(H^1\) estimate for \(w_{\Gamma_{K,R,i}}\), that

\[
\left\| \nabla w_{\Gamma_{K,R,i}} \right\|^2_{L^2(E_d^a(\epsilon_\infty))} \leq \Lambda^2 \left\| \Gamma_{K,R} \right\|^2_{L^2(E_d^a(\epsilon_\infty))} \leq C_K^2 R^{-d}.
\]

While for the second term, we can now apply eq. (2.4) as \(\tilde{\Gamma}_{K,R,i}^{\varepsilon}\) is deterministic

\[
\sum_{i=1}^{d} \sum_{\square \in \mathcal{U}} \left( \frac{1}{|\square|} \sum_{e \in E_d^a(\epsilon_\infty \cap \square)} |\tilde{\Gamma}_{K,R,i}^{\varepsilon}(e) | \right)^{2} \leq \sum_{i=1}^{d} \sum_{\square \in \mathcal{U}} |\tilde{\Gamma}_{K,R,i}^{\varepsilon}(e) | \leq O'(C_K^2 R^{-d}).
\]

This concludes the proof.  \(\Box\)

### 3.2. Construction of the flux correctors.

In this part, we prove a Helmholtz-Hodge type decomposition for \(g_p\), which is another quantity \(S_p\) used in the further quantification of algorithm. We recall that we use \(g_{p,i}\) to represent the \(i\)-th component of the vector field \(g_p : \mathbb{Z}^d \to \mathbb{R}^d\) and the standard heat kernel is defined as \(\Phi_R(x) := \frac{1}{(4\pi R^2)^{d/2}} \exp \left( -\frac{x^2}{4R^2} \right)\).

**Proposition 3.4.** For each \(p \in \mathbb{R}^d\), almost surely there exists a vector field \(S_p : \mathbb{Z}^d \to \mathbb{R}^{d \times d}\) called flux corrector of \(g_p\), which takes values in the set of anti-symmetric matrices (that is, \(S_{p,ij} = -S_{p,ji}\)) and satisfying the following equations:

\[
\begin{cases}
D^* \cdot S_p = g_p, \\
-\Delta S_{p,ij} = D_{e_j} g_{p,i} - D_{e_i} g_{p,j},
\end{cases}
\]

where the first equation means that for every \(i \in \{1, 2 \cdots d\}\), \(\sum_{j=1}^{d} D^*_{e_j} S_{p,ij} = g_{p,i}\).

The quantity satisfies similar estimation as eq. (2.26) and eq. (2.27): there exist two positive constants \(s := s(d, p, \Lambda), C := C(d, p, \Lambda, s)\) such that

\[
\forall 1 \leq i, j \leq d, \quad \forall x \in \mathbb{Z}^d, \quad |\nabla e^x_{i,j} S_{p,ij}|(x) \leq O_s(C|p|),
\]

and for the heat kernel \(\Phi_R\), we have

\[
|\Phi_R * |\nabla e^x_{i,j} S_{p,ij}|(x) \leq O_s(C|p|R^{-\frac{d}{2}}).
\]
3.2.1. Heuristic analysis. The following discussion gives a little heuristic analysis before a rigorous
proof. In fact, if we define a field \( H_p : \mathbb{Z}^d \rightarrow \mathbb{R}^d \) such that
\[
(3.32) \quad -\Delta H_{p,i} = g_{p,i},
\]
where \(-\Delta := -\nabla \cdot \nabla = D^* \cdot D\) is the discrete Laplace and then we define \( S_p \) such that
\[
(3.33) \quad S_{p,ij} = D_{e_j} H_{p,i} - D_{e_i} H_{p,j}.
\]
We see that this definition gives us a solution of eq. (3.29) since
\[
-\Delta S_{p,ij} = -\Delta (D_{e_j} H_{p,i} - D_{e_i} H_{p,j}) = D_{e_j} (-\Delta H_{p,i}) - D_{e_i} (-\Delta H_{p,j}) = D_{e_j} g_{p,i} - D_{e_i} g_{p,j}.
\]
\[
(D^* \cdot S_p)_i = \sum_{j=1}^d D^*_{e_j} (D_{e_j} H_{p,i} - D_{e_i} H_{p,j}) = -\Delta H_{p,i} - D_{e_i} \sum_{j=1}^d D^*_{e_j} H_{p,j} = g_{p,i}.
\]
Here we use one property that \( H_{p,i} = (-\Delta)^{-1} g_{p,i} \) so that \( H_p \) is also divergence free. This idea works
on periodic homogenization problem [39, Lemma 3.1], but in our context, one key problem is to well
define eq. (3.32). In the present work, we apply an elementary probabilistic approach: Let \((S_t)_{k \geq 0}\)
defines a lazy discrete time simple random walk on \( \mathbb{Z}^d \) with probability \( \frac{1}{2} \) to stay unmoved and \( \frac{1}{4d} \)
to move towards one of the nearest neighbors on \( \mathbb{Z}^d \), and we use \((P_t)_{t \in \mathbb{N}}\) to define its semigroup, with the notation
\[
(3.34) \quad P_t(x,y) := \mathbb{P}[S_t = y - x] = \begin{cases} \mathbb{P}[S_t = y - x] & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}, \quad P_t f(x) := \sum_{y \in \mathbb{Z}^d} P_t(x,y) f(y) = ([P_t] * [f])(x), \quad \forall f \in L^1(\mathbb{Z}^d),
\]
where \([\cdot]\) denotes a constant extension on every \( z + (-\frac{1}{2}, \frac{1}{2})^d \). Using the representation of the solution
of harmonic function by a simple random walk
\[
H_{p,i}(x) = \frac{1}{4d} \sum_{t=0}^\infty (P_t g_{p,i})(x),
\]
and we deduce from the definition of \( S_p \) in eq. (3.33)
\[
S_{p,ij}(x) = \frac{1}{4d} \sum_{t=0}^\infty D_{e_j} (P_t g_{p,i})(x) - D_{e_i} (P_t g_{p,j})(x).
\]
If we believe that \( P_t \) is close to the heat kernel that \( P_t(x,y) \approx \frac{1}{(\pi t)^d/2} \exp \left( -\frac{|y-x|^2}{t} \right) \), and that
the operator \( D \) helps to gain another factor of \( t^{-\frac{d}{2}} \), then Proposition 1.2 would give us that
\[
|D_{e_j} (P_t g_{p,i})(x)| \lesssim O_s(t^{-\frac{d}{2}} - \frac{d}{2}).
\]
We expect that this upper bound is sharp in general, and the fact
that \( \sum_{t=1}^\infty t^{-\frac{1}{2}} = \infty \) prevents us from being able to define \( S_{p,ij} \) directly in dimension \( d = 2 \).
Nevertheless we can make sense of
\[
(3.35) \quad (D_{e_k} S_{p,ij})(x) = \frac{1}{4d} \sum_{t=0}^\infty D_{e_k} D_{e_j} (P_t g_{p,i})(x) - D_{e_k} D_{e_i} (P_t g_{p,j})(x),
\]
because differentiating \( P_t \) a second time will allow us to gain an extra factor of \( t^{-\frac{1}{2}} \), and thus give
us that that \( |D_{e_k} D_{e_j} P_t g_{p,i}| \lesssim O_s(C t^{-1 - \frac{d}{4}}) \). Then we can apply eq. (2.4) to say that \( D_{e_k} S_{p,ij} \) is
well-defined and prove other properties.
3.2.2. Rigorous construction of $\mathcal{D} S$.

**Proof of Proposition 3.4.** We will give a rigorous proof that eq. (3.35) gives a well-defined anti-symmetric valued vector field $S_e$. The proof can be divided into three steps.

**Step 1: Stochastic integrability of $\mathcal{D}_{e_k} \mathcal{D}_{e_j}(P_t^i g_{p,i})$.** In the first step, we prove that eq. (3.35) makes sense, that is the part $\mathcal{D}_{e_k} \mathcal{D}_{e_j}(P_t^i g_{p,i})(x) - \mathcal{D}_{e_k} \mathcal{D}_{e_j}(P_t^i g_{p,j})(x)$ is summable. In the heuristic analysis, we compare $P_t$ with the heat kernel, which can be reformulated carefully by the local central limit theorem.

**Lemma 3.6** (Page 61, Exercise 2.10 of [49]). We denote by $\bar{P}_t(x, y) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|y-x|^2}{t}\right)$, then there exists a positive constant $C(d)$, such that for all $t > 0$,

$$
\sup_{x \in \mathbb{R}^d} \left| \mathcal{D}_{e_k} \mathcal{D}_{e_j} \bar{P}_t - \mathcal{D}_{e_k} \mathcal{D}_{e_j} \tilde{P}_t \right|(x) \leq C(d) t^{-\frac{d+3}{2}}. 
$$

**Proof.** The proof follows the idea in [49, Theorem 2.3.5] and also relies on [49, Lemmas 2.3.3 and 2.3.4] where we have

$$
P_t(x) = \bar{P}_t(x) + V_t(x, r) + \frac{1}{(2\pi)^{d/2}} \int_{|\theta| \leq r} e^{-\frac{ix \cdot \theta}{\sqrt{t}}} e^{-\frac{|\theta|^2}{4} F_t(\theta)} d\theta,
$$

and there exits $\zeta > 0$ such that for every $0 < \theta, r \leq t^{\frac{1}{2}}, V_t(x, r), F_t(\theta)$ satisfy

$$
|F_t(\theta)| \leq \frac{|\theta|^4}{t}, \quad |V_t(x, r)| \leq c(d) t^{-\frac{d}{2}} e^{-\zeta r^2}.
$$

We apply $\mathcal{D}_{e_k} \mathcal{D}_{e_j}$ with respect to $x$ and obtain that

$$
\left| \mathcal{D}_{e_k} \mathcal{D}_{e_j} P_t - \mathcal{D}_{e_k} \mathcal{D}_{e_j} \tilde{P}_t \right|(x) = \left| \mathcal{D}_{e_k} \mathcal{D}_{e_j} V_t(x, r) + \frac{1}{(2\pi)^{d/2}} \int_{|\theta| \leq r} \mathcal{D}_{e_k} \mathcal{D}_{e_j} e^{-\frac{ix \cdot \theta}{\sqrt{t}}} e^{-\frac{|\theta|^2}{4} F_t(\theta)} d\theta \right|.
$$

We take $r = t^{\frac{1}{8}}$, then the term $V_t(x, r)$ has an error of exponential type

$$
\left| V_t(x, t^{\frac{1}{8}}) \right| \leq c(d) t^{-\frac{d}{2}} e^{-\zeta t^{\frac{1}{4}}} \leq c'(d) t^{-\frac{d+3}{2}}.
$$

So we focus on another part, by a simple finite difference calculus we have $\mathcal{D}_{e_k} \mathcal{D}_{e_j} e^{-\frac{ix \cdot \theta}{\sqrt{t}}} \leq \frac{2|\theta|}{\sqrt{t}}$.

Moreover, we apply $|F_t(\theta)| \leq \frac{|\theta|^4}{t}$ and have

$$
\left| \frac{1}{(2\pi)^{d/2}} \int_{|\theta| \leq r} \mathcal{D}_{e_k} \mathcal{D}_{e_j} e^{-\frac{ix \cdot \theta}{\sqrt{t}}} e^{-\frac{|\theta|^2}{4} F_t(\theta)} d\theta \right| = \left| \frac{1}{(2\pi)^{d/2}} \int_{|\theta| \leq r} |\theta|^4 e^{-\frac{|\theta|^2}{4} F_t(\theta)} d\theta \right| \leq C t^{-\frac{d+3}{2}}.
$$

This concludes the proof. \hfill \Box

We prove that eq. (3.35) is well defined by showing that

$$
P \text{-a.s. } \forall 1 \leq i, j, k \leq d, \forall x \in \mathbb{R}^d, \sum_{t=0}^{\infty} \left| \mathcal{D}_{e_k} \mathcal{D}_{e_j}(P_t g_{p,i}) \right|(x) < \infty.
$$

We break this term into two

$$
\sum_{t=0}^{\infty} \left| \mathcal{D}_{e_k} \mathcal{D}_{e_j}(P_t g_{p,i}) \right|(x) = \sum_{t=0}^{\infty} \left| \mathcal{D}_{e_k} \mathcal{D}_{e_j}(\bar{P}_t g_{p,i}) \right|(x) + \sum_{t=0}^{\infty} \left| \mathcal{D}_{e_k} \mathcal{D}_{e_j}(P_t - \bar{P}_t) g_{p,i} \right|(x).
$$

\text{(3.38)-a} \quad \text{eq. (3.38)-b}
\(P_t\) is better than \(P_t\) since it is a standard heat kernel and we can do explicit calculation. We observe that

\[
\forall t \geq 1, \forall y \in \mathbb{Z}^d, |D_{e_k} D_{e_j} \tilde{P}_t| (y) \leq \frac{C(d)}{t} \tilde{P}_{2t}(y) = \frac{C(d)}{t(2\pi t)^{d/2}} \exp \left( -\frac{|y|^2}{2t} \right) \leq \frac{C(d)}{t^{d+1}2^{d+1} \left( \frac{y}{\sqrt{t}} \vee 1 \right)^{d+1}}.
\]

Then \(K_{\sqrt{t}} := D_{e_k} D_{e_j} \tilde{P}_t\) is a kernel described in Proposition 1.2 with the constant \(C_{K_{\sqrt{t}}} := \frac{C(d)}{t}\), so we have

\[
|D_{e_k} D_{e_j} (\tilde{P}_t g_{p,i})| (x) = |K_{\sqrt{t}} \ast [g_{p,i}]| (x) \leq O_s(\sqrt{Ct^{-\frac{d}{2}}}).
\]

We put these estimates with Proposition 3.1 in the eq. (3.38)-a and get

eq. (3.38)-a \leq |g_{p,i}| (x) + \sum_{t=1}^{\infty} |D_{e_k} D_{e_j} (P_t g_{p,i})| (x) \leq O_s(C) + \sum_{t=1}^{\infty} O_s(\sqrt{Ct^{-\frac{d}{2}}}) \leq O_s(C).

On the other hand, to handle eq. (3.38)-b, we choose \(\varepsilon > 0\) and study at first

\[
|D_{e_k} D_{e_j} (P_t - \tilde{P}_t) g_{p,i}| (x) \leq \int_{|y| \geq t^{\frac{1}{2} + \varepsilon}} |D_{e_k} D_{e_j} (P_t - \tilde{P}_t)| (y) |g_{p,i}| (x-y) dy
\]

\[
+ 4 \int_{|y| \geq t^{\frac{1}{2} + \varepsilon}} \left( |P_t| + |\tilde{P}_t| \right) (y) |g_{p,i}| (x-y) dy
\]

\[
\leq 3 \int_{|y| \leq t^{\frac{1}{2} + \varepsilon}} O_s \left( \frac{C}{t^{\frac{d+1}{2}}} \right) dy + 4 \int_{|y| \geq t^{\frac{1}{2} + \varepsilon}} \left( |P_t| + |\tilde{P}_t| \right) (y) O_s(C) dy
\]

\[
eq \leq O_s \left( Ct^{-\left( \frac{d}{2} - \varepsilon \right)} \right) + \int_{|y| \geq t^{\frac{1}{2} + \varepsilon}} \left( |P_t| + |\tilde{P}_t| \right) (y) dy.
\]

We divide the estimation into two terms since Lemma 3.6 is uniform but not optimal for the tail probability, which is of type sub-Gaussian so that the mass outside \(|t|^{\frac{1}{2} + \varepsilon}\) should be very small. By direct calculation, we have that \(\int_{|y| \geq t^{\frac{1}{2} + \varepsilon}} \left( |P_t| \right) (y) dy \leq C(d) e^{-t^{2\varepsilon}}\) and by Hoeffding’s inequality for the lazy simple random walk \((S_t)_{t \geq 0}\)

\[
\int_{|y| \geq t^{\frac{1}{2} + \varepsilon}} \left( |P_t| \right) (y) dy = \mathbb{P} \left[ |S_t| \geq t^{\frac{1}{2} + \varepsilon} \right] \leq 2 \exp \left( -\frac{2t^{1+2\varepsilon}}{\text{Var}[S_t]} \right) \leq 2e^{-4t^{2\varepsilon}}.
\]

Combining these tail event estimates and by choosing \(\varepsilon = \frac{1}{4t}\), we obtain that

\[
|D_{e_k} D_{e_j} (P_t - \tilde{P}_t) g_{p,i}| (x) \leq O_s(Ct^{-\frac{d}{2}}),
\]

and this concludes that eq. (3.38)-b \(\leq O_s(C)\), so eq. (3.37) holds, eq. (3.30) holds and that \(D_{e_k} S_{p,ij}\) is well defined.

Remark. In the proof, we also obtained one quantitative estimate of the following type: There exist two constants \(s := s(d, p, \Lambda), C := C(d, p, \Lambda, s)\) such that for every random field \(X : \mathbb{Z}^d \rightarrow \mathbb{R}\) satisfying for every \(z \in \mathbb{Z}^d, |X(z)| \leq O_s(\theta)\), we have

\[
\forall 1 \leq i, j, k \leq d, x \in \mathbb{Z}^d, |D_{e_k} D_{e_j} (P_t - \tilde{P}_t) X| (x) \leq O_s(C \theta t^{-\frac{d}{2}}).
\]

By a similar approach with the classical local central limit theorem [49, Theorem 2.3.5], we can also prove that

\[
\forall x \in \mathbb{Z}^d, |P_t g_{p,i}| (x) \leq O_s(\sqrt{Ct^{-\frac{d}{2}}}).
\]
Step 2: Verification of eq. (3.29). The verification of eq. (3.29) is direct thanks to eq. (3.37). We will also use the semigroup property that

\[\tag{3.41} P_t(x) - P_{t-1}(x) = \frac{1}{4d} \Delta P_{t-1}(x).\]

\[
(D^* \cdot S_p)_i(x) = \frac{1}{4d} \sum_{t=0}^{\infty} \sum_{j=1}^{d} D^*_j D^*_e P_t g_{p,j}(x) - \sum_{t=0}^{\infty} \sum_{j=1}^{d} D^*_e P_t \left(D^* \cdot g_p\right)(x)
\]

\[= \sum_{t=0}^{\infty} (P_t - P_{t+1}) g_{p,i}(x) = g_{p,i}(x).
\]

In the last step, we use implicitly that \(\lim_{t \to \infty} P_t g_{p,i}(x) = 0\) almost surely. This is true by Borel-Cantelli lemma and the estimation \(|P_t g_{p,i}(x)| \leq O_s(Ct^{-\frac{s}{2}})\) (see eq. (3.40)):

\[
\sum_{t=1}^{\infty} \mathbb{P}[|P_t g_{p,i}|(x) \geq \varepsilon] \leq \sum_{t=1}^{\infty} \exp \left( - \left( C \varepsilon t^{\frac{1}{2}} \right)^s \right) < \infty.
\]

The second part of eq. (3.29), concerning \(-\Delta S_{p,ij}\), is easy to verify by a similar calculation

\[-\Delta S_{p,ij}(x) = \frac{1}{4d} \sum_{t=0}^{\infty} D^*_k D^*_e D^*_e (P_t g_{p,i}) - D^*_e D^*_e D^*_e (P_t g_{p,j})(x)
\]

\[= \frac{1}{4d} \sum_{t=0}^{\infty} \left(-\Delta P_t D^*_e g_{p,i}(x) - (-\Delta P_t D^*_e g_{p,j})(x)
\]

\[= (D^*_e g_{p,i})(x) - (D^*_e g_{p,j})(x).
\]

Finally, by the definition, we can define \(S_{p,ij}\) just by integration of \(DS_{p,ij}\) along a path. This construction does not depend on the choice of path since \(DS_{p,ij}\) is a potential field.

Step 3: Estimation of \(|\Phi_R \ast [D_{e_k} S_{p,ij}]](x)\). This is a result of the convolution. Thanks to the eq. (3.37), we can apply Fubini lemma to \(|\Phi_R \ast [D_{e_k} S_{p,ij}]](x)\) that

\[|\Phi_R \ast [D_{e_k} S_{p,ij}]](x) = \frac{1}{4d} \sum_{t=0}^{\infty} \Phi_R \ast [D_{e_k} D^*_e P_t] \ast [g_{p,i}] - \Phi_R \ast [D_{e_k} D^*_e P_t] \ast [g_{p,j}](x)
\]

\[= \frac{1}{4d} \sum_{t=0}^{\infty} [D_{e_k} D^*_e P_t] \ast (\Phi_R \ast [g_{p,i}]) - [D_{e_k} D^*_e P_t] \ast (\Phi_R \ast [g_{p,j}]) (x)\]
We apply this estimate and use eq. (2.4) to obtain that

\[
\frac{1}{4d} \sum_{t=0}^{\infty} \left[ D_{e_k} D_{e_j} P_t \right] (\Phi_R \ast [g_{p,i}]) \leq \frac{1}{4d} \sum_{t=0}^{\infty} \left[ D_{e_k} D_{e_j} P_t - D_{e_k} D_{e_j} \bar{P}_t \right] (\Phi_R \ast [g_{p,i}])
\]

(3.42) -a

\[
\leq \frac{1}{4d} \sum_{t=0}^{\infty} \left[ D_{e_k} D_{e_j} \bar{P}_t \right] (\Phi_R \ast [g_{p,i}])
\]

(3.42) -b

\[
\leq \frac{1}{4d} \sum_{t=0}^{\infty} D_{e_k} D_{e_j} \Phi \sqrt{\frac{t}{2}} (\Phi_R \ast [g_{p,i}])
\]

(3.42) -c

The main idea is that \( \Phi_R \ast [g_{p,i}] \leq O_s \left( CR^{-\frac{d}{2}} \right) \) by Proposition 1.2, then we repeat the main argument of stochastic integrability of \( D_{e_k} S_{p_{ij}} \) to get a better estimate. We focus on just one term:

We treat the three terms one by one. For eq. (3.42)-a, we apply eq. (3.39) with \( X := \Phi_R \ast [g_{p,i}] \) and we use also eq. (2.4)

\[
eq O_s \left( CR^{-\frac{d}{2}} \right) + \sum_{t=1}^{\infty} O_s \left( C t^{-\frac{d}{2}} R^{-\frac{d}{2}} \right)
\]

(3.43)

\[
\leq O_s \left( CR^{-\frac{d}{2}} \right).
\]

For the term eq. (3.42)-b, we observe that for every \( y \in \mathbb{R}^d \),

\[
\left| \left[ D_{e_k} D_{e_j} \bar{P}_t \right] (y) - D_{e_k} D_{e_j} \Phi \sqrt{\frac{t}{2}} (y) \right| \leq \left| \left[ D_{e_k} D_{e_j} \Phi \sqrt{\frac{t}{2}} \right] (y) \right| \leq \frac{C(d)}{t^{\frac{d}{2}}} \Phi \sqrt{t}(y).
\]

We apply this estimate and use eq. (2.4) to obtain that

\[
\leq \frac{1}{4d} \sum_{t=0}^{\infty} \left[ D_{e_k} D_{e_j} \bar{P}_t \right] (\Phi_R \ast [g_{p,i}]) (x)
\]

(3.44)

\[
\leq \frac{1}{4d} \sum_{t=0}^{\infty} \left[ D_{e_k} D_{e_j} \bar{P}_t \right] (\Phi_R \ast [g_{p,i}]) (x)
\]

\[
\leq \frac{1}{d} |\Phi_R \ast [g_{p,i}]| (x) + \frac{1}{4d} \sum_{t=1}^{\infty} C(d) \Phi \sqrt{t} (\Phi_R \ast [g_{p,i}]) (x)
\]

\[
\leq O_s \left( CR^{-\frac{d}{2}} \right) + \sum_{t=1}^{\infty} O_s \left( C t^{-\frac{d}{2}} R^{-\frac{d}{2}} \right)
\]

\[
\leq O_s \left( CR^{-\frac{d}{2}} \right).
\]
For the last term $\left| \frac{1}{4d} \sum_{t=0}^{\infty} D_{E_k} D_{E_j} \Phi \sqrt{\frac{t}{2}} \ast (\Phi_R \ast [g_{p,i}]) \right|(x)$, we use the property of semigroup, the linearity of the finite difference operator and we apply Proposition 1.2 to the kernel $D_{E_k} D_{E_j} \Phi \sqrt{\frac{t}{2}}$

$$\left| \frac{1}{4d} \sum_{t=0}^{\infty} D_{E_k} D_{E_j} \Phi \sqrt{\frac{t}{2}} \ast (\Phi_R \ast [g_{p,i}]) \right|(x) = \left| \frac{1}{4d} \sum_{t=0}^{\infty} D_{E_k} D_{E_j} \left( \Phi \sqrt{\frac{t}{2} + R^2} \ast [g_{p,i}] \right) \right|(x)$$

$$= \frac{1}{4d} \sum_{t=0}^{\infty} \left( D_{E_k} D_{E_j} \Phi \sqrt{\frac{t}{2} + R^2} \right) \ast [g_{p,i}] (x)$$

$$\leq \sum_{t=0}^{\infty} O_s \left( C \left( \frac{t}{2} + R^2 \right)^{-\frac{d}{2}} \right)$$

$$\leq O_s (CR^{-\frac{d}{2}}).$$

This concludes the proof as we put the three estimates eq. (3.43), eq. (3.44) and eq. (3.45) in eq. (3.42) and eq. (3.31).

3.2.3. $L^q, L^\infty$ estimate of $S_p$. Once we establish the spatial average estimate for $\mathcal{D}S_p$, we also have its $L^q$ and $L^\infty$ estimate.

**Proposition 3.5.** There exist three finite positive constants $s := s(d,p,\Lambda), k := k(d,p,\Lambda)$ and $C := C(d,p,\Lambda,s)$ such that for each $1 \leq i,j \leq d, p \in \mathbb{R}^d$ and $q \in [1,\infty)$,

$$\left( \int_{B_R} |S_{p,ij} - (S_{p,ij})_{B_R}|^q \right)^{\frac{1}{q}} \leq \begin{cases} O_s(C|p|^k \log^\frac{1}{2}(R)) & d = 2, \\ O_s(C|p|k) & d = 3, \end{cases}$$

and for each $x, y \in \mathbb{Z}^d$,

$$|S_{p,ij}(x) - S_{p,ij}(y)| \leq \begin{cases} O_s(C|p|^k \log^\frac{1}{2} |x - y|) & d = 2, \\ O_s(C|p|) & d = 3. \end{cases}$$

**Proof.** Similar to [19, Theorems 1 and 2], these estimates are the results of local estimate and spatial average estimates proved in eq. (3.1), eq. (1.19), eq. (3.30) and eq. (3.31) by applying a heat kernel type multi-scale Poincaré’s inequality. We refer to [19, Sections 4 and 5].

4. Two-scale expansion on the cluster

In this part, we prove Theorem 1.2 which is the heart of all the analysis of our algorithm as stated in Section 1.3. Here we prove a more detailed version of the theorem.

**Proposition 4.1** (Two-scale expansion on percolation). Under the same context of Theorem 1.2, there exist three random variables $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$ satisfying

$$\mathcal{X} \leq O_s(C(d,p,\Lambda)m), \quad \mathcal{Y}_1 \leq O_s \left( C(d,p,\Lambda,\ell(\lambda)m^2) \right), \quad \mathcal{Y}_2 \leq O_s \left( C(d,p,\Lambda,\ell(\lambda)m^2) \right),$$

and we have the estimate

$$\|\nabla (w - v) 1_{a \neq 0}\|_{L^2(\mathcal{E}_q(\square_m))} \leq C(d,\Lambda) \left( \|D\tilde{w}\|_{L^2(\square_m)} \left( 3^{-\frac{m}{2}} \ell^\frac{1}{2}(\lambda) \mathcal{X}^d + 3^{-\frac{m}{2}} \ell^{-\frac{1}{2}}(\lambda) \mathcal{Y}_1 \mathcal{X}^d + \mu \mathcal{Y}_1 + \mathcal{Y}_2 \right) \right)^\frac{1}{2}$$

$$+ \|D\tilde{w}\|^\frac{1}{2}_{L^2(\square_m)} \left( \|D^*D\tilde{w}\|^\frac{1}{2}_{L^2(\square_m)} \left( \ell^\frac{1}{2}(\lambda) \mathcal{X}^d + \ell^{-\frac{1}{2}}(\lambda) \mathcal{Y}_1 \mathcal{X}^d \right) \right)^\frac{1}{2}$$

$$+ \|D^*D\tilde{w}\|_{L^2(\square_m)} \mathcal{Y}_1 \mathcal{X}^d.$$

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4.1. Main part of the proof. The main idea of the proof is to use the quantities \( \{\phi_{\varepsilon_k}\}_{k=1,\ldots,d} \) and \( \{S_{\varepsilon_k,ij}\}_{i,j,k=1,\ldots,d} \) analyzed in previous work and in Section 3, under the condition \( \square_m \in \mathcal{P}_* \). We do some simple manipulations at first. Throughout the proof, we use the notation \( h := v - w \).

Proof. Step 1: Setting up. We define a modified coarsened function \( \tilde{h} \)

\[
\tilde{h}(x) = \begin{cases} 
    h(x) & x \in \mathcal{C}_s(\square_m), \\
    [h]_P(x) & x \in \square_m \setminus \mathcal{C}_s(\square_m), \text{dist}(\square_P(x), \partial \square_m) \geq 1, \\
    0 & x \in \square_m \setminus \mathcal{C}_s(\square_m), \text{dist}(\square_P(x), \partial \square_m) = 0.
\end{cases}
\]

We put it as a test function in eq. (1.15)

\[
\left( \tilde{h}, (\mu_{\varepsilon,m}^2 - \nabla \cdot a_{\varepsilon,m} \nabla)v \right)_{\text{int}(\square_m)} = \left( \tilde{h}, (\mu_{\varepsilon,m}^2 - \nabla \cdot \bar{a} \nabla \bar{v}) \right)_{\text{int}(\square_m)}.
\]

Since \( \tilde{h} \in C_0(\square_m) \), we can apply the formula eq. (2.9) and get

\[
\left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} v \right)_{\square_m} + \left( \nabla \tilde{h}, a_{\varepsilon,m} \nabla v \right)_{\square_m} = \left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} \bar{v} \right)_{\square_m} + \left( \nabla \tilde{h}, \bar{a} \nabla \bar{v} \right)_{\square_m}.
\]

We subtract a term of \( w \) on the two sides to get

\[
\left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} (v - w) \right)_{\square_m} + \left( \nabla \tilde{h}, a_{\varepsilon,m} \nabla (v - w) \right)_{\square_m} = \left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} (\bar{v} - w) \right)_{\square_m} + \left( \nabla \tilde{h}, \bar{a} \nabla \bar{v} - a_{\varepsilon,m} \nabla w \right)_{\square_m}.
\]

We put \( v - w = h \) into the identity and obtain that

\[
\left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} h \right)_{\square_m} + \left( \nabla \tilde{h}, a_{\varepsilon,m} \nabla h \right)_{\square_m} = \left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} (\bar{v} - w) \right)_{\square_m} + \left( \nabla \tilde{h}, \bar{a} \nabla \bar{v} - a_{\varepsilon,m} \nabla w \right)_{\square_m}.
\]

Step 2: Restriction tricks. There are three observations:

- **Observation 1.** The effect of \( \mu_{\varepsilon,m} \) restricts the inner product to \( \mathcal{C}_s(\square_m) \), and on \( \mathcal{C}_s(\square_m) \) we have \( \tilde{h} = h \) by eq. (4.1). Thus we have

  \[
  \left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} h \right)_{\square_m} = \mu^2 \langle h, h \rangle_{\mathcal{C}_s(\square_m)}, \quad \left( \mu_{\varepsilon,m} \tilde{h}, \mu_{\varepsilon,m} (\bar{v} - w) \right)_{\square_m} = \mu^2 \langle h, \bar{v} - w \rangle_{\mathcal{C}_s(\square_m)}.
  \]

- **Observation 2.** The definition of \( a_{\varepsilon,m} \) also restricts the inner product on \( E_d^a(\mathcal{C}_s(\square_m)) \) and we have

  \[
  \left( \nabla \tilde{h}, a_{\varepsilon,m} \nabla h \right)_{\square_m} = \left( \nabla h, a \nabla h \right)_{E_d^a(\mathcal{C}_s(\square_m))},
  \]

  as \( a_{\varepsilon,m} = 0 \) outside \( E_d^a(\mathcal{C}_s(\square_m)) \) by eq. (2.20).

- **Observation 3.** This step is the key where we gain much in the estimate and where we use the condition \( \square_m \in \mathcal{P}_* \). We apply the formula eq. (2.8) to \( \left( \nabla \tilde{h}, \bar{a} \nabla \bar{v} - a_{\varepsilon,m} \nabla w \right)_{\square_m} \) to obtain that

  \[
  \left( \nabla \tilde{h}, \bar{a} \nabla \bar{v} - a_{\varepsilon,m} \nabla w \right)_{\square_m} = \left( \tilde{h}, -\nabla \cdot (\bar{a} \nabla \bar{v} - a_{\varepsilon,m} \nabla w) \right)_{\text{int}(\square_m)}
  = \left( \tilde{h}, \mathcal{D}^* \cdot (\bar{a} \nabla \bar{v} - a_{\varepsilon,m} \nabla w) \right)_{\text{int}(\square_m)}
  = \left( \tilde{h}, \mathcal{D}^* \cdot (\bar{a} \nabla \bar{v} - a_{\varepsilon,m} \nabla w) \right)_{\text{int}(\square_m)}
  + \left( \tilde{h}, \mathcal{D}^* \cdot (a_{\varepsilon} - a_{\varepsilon,m}) \nabla w \right)_{\text{int}(\square_m)}.
  \]

We use the condition \( \square_m \in \mathcal{P}_* \), which implies that \( \mathcal{C}_s(\square_m) \subset \mathcal{C}_\infty \) and

\[
\text{supp} (\mathcal{D}^* \cdot (a_{\varepsilon} - a_{\varepsilon,m}) \nabla w) \subset (\mathcal{C}_\infty \cap \square_m) \setminus \mathcal{C}_s(\square_m).
\]
In Definition B.1 and Lemma B.2, we prove that \((C_\infty \cap \square_m) \setminus \mathcal{C}_s(\square_m)\) is the union of small clusters contained in the partition cubes \(\square_\mathcal{P}\) with distance 1 to \(\partial \square_m\), where \(\tilde{h}\) equals 0. Therefore, we obtain that
\[
\left\langle \tilde{h}, D^* \cdot (\tilde{a}D\tilde{v} - a_{\tilde{e}_m} Dw) \right\rangle_{\text{int}(\square_m)} = \left\langle \tilde{h}, D^* \cdot (\tilde{a}D\tilde{v} - a_{\tilde{e}_m} Dw) \right\rangle_{\text{int}(\square_m)}.
\]

Using an identity
\[
D^* \cdot (a_{\tilde{e}} Dw - \tilde{a}D\tilde{v}) = D^* \cdot F \text{ on } \mathbb{Z}^d,
\]
which will be proved later in Lemma 4.1 and \(F\) is a vector field \(F : \mathbb{Z}^d \to \mathbb{R}^d\), we conclude
\[
\left\langle \nabla \tilde{h}, a\nabla \tilde{v} - a_{\tilde{e}_m} \nabla w \right\rangle_{\square_m} 1\{\square_m \in \mathcal{P}_*\} = \left\langle \tilde{h}, -D^* \cdot F \right\rangle_{\text{int}(\square_m)} 1\{\square_m \in \mathcal{P}_*\}
= - \left\langle D\tilde{h}, F \right\rangle_{\square_m} 1\{\square_m \in \mathcal{P}_*\}.
\]

Combining all these observations, we transform eq. (4.3) to
\[
(\mu^2 \langle h, h \rangle_{\mathcal{C}_s(\square_m)} + \langle \nabla h, a\nabla h \rangle E^a(\mathcal{C}_s(\square_m))) 1\{\square_m \in \mathcal{P}_*\}
= (\mu^2 \langle h, \tilde{v} - w \rangle_{\mathcal{C}_s(\square_m)} - \langle D\tilde{h}, F \rangle_{\square_m}) 1\{\square_m \in \mathcal{P}_*\}.
\]

Using Hölder’s inequality and Young’s inequality, we obtain that
\[
\langle \nabla h, a\nabla h \rangle_{E^a(\mathcal{C}_s(\square_m)))} 1\{\square_m \in \mathcal{P}_*\} \leq \left( \frac{\mu^2}{4} \right) \| \tilde{v} - w \|_{L^2(\mathcal{C}_s(\square_m))}^2 + \| D\tilde{h} \|_{L^2(\square_m)} \| F \|_{L^2(\square_m)} 1\{\square_m \in \mathcal{P}_*\}.
\]

\textbf{Step 3: Study of } \| D\tilde{h} \|_{L^2(\square_m)}. \text{ The next step is to estimate the size of } \| D\tilde{h} \|_{L^2(\square_m)}. \text{ Since } \tilde{h} \in C^0(\square_m), \text{ we have that } \| D\tilde{h} \|_{L^2(\square_m)} = \| \nabla \tilde{h} \|_{L^2(\square_m)}. \text{ We use the function } [h]_{\mathcal{P}, \square_m} \text{ defined in eq. (2.23)}
\]
\[
[h]_{\mathcal{P}, \square_m}(x) = \left\{ \begin{array}{ll}
[h]_{\mathcal{P}}(x) & \text{dist}(\square_\mathcal{P}(x), \partial \square_m) \geq 1, \\
0 & \text{dist}(\square_\mathcal{P}(x), \partial \square_m) = 0.
\end{array} \right.
\]

as a function to do comparison and apply eq. (2.10) (see Figure 5 for the errors from the two terms)
\[
\| \nabla \tilde{h} \|_{L^2(\square_m)} = \| \nabla (\tilde{h} - [h]_{\mathcal{P}, \square_m}) \|_{L^2(\square_m)} + \| \nabla [h]_{\mathcal{P}, \square_m} \|_{L^2(\square_m)}
\leq 2d \| \tilde{h} - [h]_{\mathcal{P}, \square_m} \|_{L^2(\square_m)} + \| \nabla [h]_{\mathcal{P}, \square_m} \|_{L^2(\square_m)}
\leq 2d \| \tilde{h} - [h]_{\mathcal{P}, \square_m} \|_{L^2(\text{int}(\square_m) \cap \mathcal{C}_s(\square_m))} + \| \nabla [h]_{\mathcal{P}, \square_m} \|_{L^2(\square_m)}
\]
The last step is correct since \(\tilde{h}\) and \([h]_{\mathcal{P}, \square_m}\) coincide at the boundary and also on the part \text{int}(\square_m) \setminus \mathcal{C}_s(\square_m). \text{ We define}
\[
\mathcal{X} := \max_{x \in \square_m} \text{size}(\square_\mathcal{P}(x)),
\]
which can be estimated using eq. (2.6) and eq. (3.28) as
\[
\mathcal{X} \leq \mathcal{O}(C(d, \mathcal{P}, \Lambda)m),
\]
Figure 5. The figure shows the sources contributing to $\|\nabla \tilde{h}\|_{L^2(\Box_m)}$. The black segments represent the cluster $\mathcal{C}_*(\Box_m)$ while the blue segments represent the partition of good cubes. Using the coarsened function, we see that the quantity can be controlled by the sum of three terms: the difference between $\nabla \tilde{h}$ and $[h]_{\Box_m}$ near the cluster $\mathcal{C}_*(\Box_m)$, marked with red cross in the image; the gradient $[h]_{\Box_m}$ at the interface of different partition cubes $\Box_p$, marked with orange disk.

and apply Proposition 2.2

$$\|\nabla \tilde{h}\|_{L^2(\Box_m)} \leq 2d \left\| h - [h]_{\Box_m} \right\|_{L^2(\text{int}(\Box_m) \cap \mathcal{C}_*(\Box_m))} + \| \nabla [h]_{\Box_m} \|_{L^2(\Box_m)}$$

$$\leq C \left( \sum_{\{x,y\} \in E(a)_{\mathcal{C}_*(\Box_m)}} \text{size}(\Box_p(x))^2 \| \nabla h \|_2(x,y) \right)^{\frac{1}{2}}$$

$$\leq C \Lambda^d \| \nabla h \mathbf{1}_{\{a \neq 0\}} \|_{L^2(\mathcal{C}_*(\Box_m))}.$$
The two random variables used in the estimation are defined as

\[
\begin{align*}
Y_1 & := \max_{1 \leq i, j, k \leq d, \text{dist}(x, \square_m) \leq 1} \left| \phi_{e_k}^{(\lambda)}(x) \right| + \left| S_{e_k, i, j}^{(\lambda)}(x) \right|, \\
Y_2 & := \max_{1 \leq i, j, k \leq d, \text{dist}(x, \square_m) \leq 1} \left| \Phi_{\lambda^{-1}} \ast D_{e_l} \left[ \phi_{e_k} \right]_D^{\eta}(x) \right| + \left| \Phi_{\lambda^{-1}} \ast [D_{e_l}^* S_{e_k, i, j}](x) \right|,
\end{align*}
\]

where they involved the spatial average of corrector and flux corrector, the modified corrector defined in eq. (1.14) and modified flux corrector defined in eq. (4.18). They have estimates following eq. (2.6), eq. (2.27), eq. (3.31) and also Proposition 3.5, Proposition 2.4 that there exists 0 < s(d, p, \Lambda) < \infty and 0 < C(d, p, \Lambda) < \infty such that

\[
Y_1 \leq C_s \left( C(d, p, \Lambda, s) \ell(\lambda)m^\frac{1}{2} \right) \quad Y_2 \leq C_s \left( C(d, p, \Lambda, s)^\frac{d}{2} m^\frac{1}{2} \right).
\]

For \( \| \bar{v} - w \|_{L^2(\square_m)} 1_{\{\square_m \in \mathcal{P}_* \}} \), we have

\[
\| w - \bar{v} \|_{L^2(\square_m)}^2 1_{\{\square_m \in \mathcal{P}_* \}} \leq d \left( \max_{1 \leq j \leq d, x \in \square_m} \phi_{e_j}^{(\lambda)}(x) \right)^2 \sum_{j=1}^d 1_{\{\square_m \in \mathcal{P}_* \}} \leq Y_1^2 \leq dY_1^2 \| D\bar{v} \|_{L^2(\square_m)}^2.
\]

For \( \| F \|_{L^2(\square_m)} 1_{\{\square_m \in \mathcal{P}_* \}} \), we use the formula eq. (4.19)

\[
\| F \|_{L^2(\square_m)}^2 \leq C(d) \left( \sum_{i=1}^d \| (1 - Y) (a_{\varphi} - \bar{a}) (D_{e_i} \bar{v}) \|_{L^2(\square_m)}^2 + \sum_{i, k=1}^d \| \phi_{e_k}^{(\lambda)} (\cdot + e_k) a_{\varphi} D_{e_i} (YD_{e_k} \bar{v}) \|_{L^2(\square_m)}^2 \right) \quad \text{eq. (4.13)-a}
\]

\[
+ \sum_{i, j, k=1}^d \| S_{e_k, i, j}^{(\lambda)} (\cdot - e_j) D_{e_j}^* (YD_{e_k} \bar{v}) \|_{L^2(\square_m)}^2 + \sum_{i, j, k=1}^d \| D_{e_j}^* (S_{e_k, i, j} \Phi_{\lambda^{-1}}, e_j) (YD_{e_k} \bar{v}) \|_{L^2(\square_m)}^2 \quad \text{eq. (4.13)-b}
\]

\[
+ \sum_{i, k=1}^d \| a_{\varphi} D_{e_i} \left( \phi_{e_k}^{(\lambda)} P \Phi_{\lambda^{-1}} \right) (YD_{e_k} \bar{v}) \|_{L^2(\square_m)}^2 \quad \text{eq. (4.13)-c}
\]

\[
+ \sum_{i, k=1}^d \| a_{\varphi} D_{e_i} \left( \phi_{e_k}^{(\lambda)} P \Phi_{\lambda^{-1}} \right) (YD_{e_k} \bar{v}) \|_{L^2(\square_m)}^2 \quad \text{eq. (4.13)-d}
\]

\[
+ \sum_{i, k=1}^d \| a_{\varphi} D_{e_i} \left( \phi_{e_k}^{(\lambda)} P \Phi_{\lambda^{-1}} \right) (YD_{e_k} \bar{v}) \|_{L^2(\square_m)}^2 \quad \text{eq. (4.13)-e}
\]
We treat them term by term. For eq. (4.13)-a, noticing that \((1 - \Upsilon) \leq 1_{\{\text{dist}(\cdot, \partial) \leq 2\ell(\lambda)\}}\), we apply the trace formula eq. (2.14)

\[
eq (4.13)-a = \sum_{i=1}^{d} \| (1 - \Upsilon)(a_{g} - \bar{a})(D_{e_{i}} \bar{v}) \|_{L^{2}(\square_{m})}^{2} \\
\leq 2 \sum_{i=1}^{d} \| (D_{e_{i}} \bar{v}) 1_{\{\text{dist}(\cdot, \partial) \leq 2\ell(\lambda)\}} \|_{L^{2}(\square_{m})}^{2} \\
\leq C(d)\ell(\lambda) \left( 3^{-m} \| \mathcal{D}\bar{v} \|_{L^{2}(\square_{m})}^{2} + \| \mathcal{D}\bar{v} \|_{L^{2}(\square_{m})} \| \mathcal{D}^{*}\mathcal{D}\bar{v} \|_{L^{2}(\text{int}(\square_{m}))} \right).
\]

For the term eq. (4.13)-b, we notice that

\[
\mathcal{D}_{e_{i}} (\Upsilon \mathcal{D}_{e_{k}} \bar{v}) = (\mathcal{D}_{e_{i}} \Upsilon) (\mathcal{D}_{e_{k}} \bar{v}) + \Upsilon (\cdot + e_{i})(\mathcal{D}_{e_{i}} \mathcal{D}_{e_{k}} \bar{v}),
\]

and the support of \(\mathcal{D}_{e_{i}} \Upsilon\) is contained in the region of distance between \(\ell(\lambda)\) and \(2\ell(\lambda)\) from \(\partial \square_{m}\) i.e.

\[
\mathcal{D}_{e_{i}} \Upsilon \leq \frac{1}{\ell(\lambda)} 1_{\{\cdot \in \square_{m}, \frac{1}{2}\ell(\lambda) \leq \text{dist}(\cdot, \partial) \leq 3\ell(\lambda)\}},
\]

then we apply these in eq. (4.13)-b and also eq. (2.14) and obtain that

\[
eq (4.13)-b = \sum_{i,k=1}^{d} \| \phi_{e_{k}}^{(\lambda)} (\cdot + e_{k}) a_{g} \mathcal{D}_{e_{i}} (\Upsilon \mathcal{D}_{e_{k}} \bar{v}) \|_{L^{2}(\square_{m})}^{2} \\
\leq \sum_{i,k=1}^{d} \| \phi_{e_{k}}^{(\lambda)} (\cdot + e_{k}) (\mathcal{D}_{e_{i}} \Upsilon)(\mathcal{D}_{e_{k}} \bar{v}) \|_{L^{2}(\square_{m})}^{2} + \sum_{i,k=1}^{d} \frac{1}{\ell(\lambda)} 1_{\{\text{dist}(\cdot, \partial) \leq 3\ell(\lambda)\}} \| (\mathcal{D}_{e_{i}} \mathcal{D}_{e_{k}} \bar{v}) \|_{L^{2}(\square_{m})}^{2} \\
\leq \sum_{i,k=1}^{d} \| \mathcal{D}_{e_{k}} \bar{v} \|_{L^{2}(\square_{m})}^{2} + \sum_{i,k=1}^{d} \frac{1}{\ell(\lambda)} \left( 3^{-m} \| \mathcal{D}\bar{v} \|_{L^{2}(\square_{m})}^{2} + \| \mathcal{D}\bar{v} \|_{L^{2}(\square_{m})} \| \mathcal{D}^{*}\mathcal{D}\bar{v} \|_{L^{2}(\text{int}(\square_{m}))} \right).
\]

In the last step, we apply eq. (2.14) and use the interior \(H^{2}\) norm of \(\bar{v}\) since the function \(\Upsilon\) is supported just in the interior with distance \(\ell(\lambda)\) from \(\partial \square_{m}\). eq. (4.13)-c follows the similar estimate.

For the term eq. (4.13)-d, we use the quantity \(\mathcal{Y}_{2}\) to estimate it in

\[
eq (4.13)-d = \sum_{i,j,k=1}^{d} \| \mathcal{D}_{e_{j}}^{*} (S_{e_{k},ij} * \Phi_{\lambda-1}) (\Upsilon \mathcal{D}_{e_{k}} \bar{v}) \|_{L^{2}(\square_{m})}^{2} \\
\leq C(d)\mathcal{Y}_{2}^{2} \| \mathcal{D}\bar{v} \|_{L^{2}(\square_{m})}^{2}.
\]

The term eq. (4.13)-e follows the similar estimate.

We combine eq. (4.14), eq. (4.15) and eq. (4.16) together and obtain that

\[
\| \Phi \|_{L^{2}(\square_{m})} \leq C(d) \left( \| \mathcal{D}\bar{v} \|_{L^{2}(\square_{m})} \left( 3^{-\frac{m}{2}} \ell^{\frac{3}{2}}(\lambda) + 3^{-\frac{m}{2}} \ell^{-\frac{1}{2}}(\lambda) \mathcal{Y}_{1} + \mathcal{Y}_{2} \right) \\
\| \mathcal{D}\bar{v} \|_{L^{2}(\square_{m})} \| \mathcal{D}^{*}\mathcal{D}\bar{v} \|_{L^{2}(\text{int}(\square_{m}))}^{\frac{1}{2}} \left( \ell^{\frac{3}{2}}(\lambda) + \ell^{-\frac{1}{2}}(\lambda) \mathcal{Y}_{1} + \| \mathcal{D}^{*}\mathcal{D}\bar{v} \|_{L^{2}(\text{int}(\square_{m}))} \mathcal{Y}_{1} \right) \right).
\]

We put the two estimates eq. (4.12) and eq. (4.13) into eq. (4.9) and get the desired result.
4.2. Construction of a vector field. In this part we calculate the vector field $F$ used in the last paragraph. We define at first the modified flux corrector $\{S_{ek,ij}\}_{1 \leq i,j,k \leq d}$ similar to eq. (14) that

\[ S_{ek,ij}^{(\lambda)} := S_{ek,ij} - |S_{ek,ij}| \Phi_{\lambda^{-1}}. \]

Lemma 4.1. There exists a vector field $F : \mathbb{Z}^d \to \mathbb{R}^d$ such that $D^* \cdot (a_\phi Dw - \bar{a}D\bar{v}) = D^* \cdot F$, with the formula

\[ F_i = (1 - \Upsilon) (a_\phi - \bar{a}) (D_{ei} \bar{v}) + \sum_{k=1}^d \phi_{e_k}^{(\lambda)} (x + e_k) a_\phi D_{e_k} (\Upsilon D_{ek} \bar{v}) \]

(4.19)

\[ - \sum_{j,k=1}^d S_{ek,ij}^{(\lambda)} (\cdot - e_j) D_{e_j} (\Upsilon D_{ek} \bar{v}) + \sum_{j,k=1}^d D_{e_j}^* (|S_{ek,ij}| \Phi_{\lambda^{-1}}) (\Upsilon D_{ek} \bar{v}) \]

\[ - \sum_{k=1}^d a_\phi D_{e_k} \left[ (\phi_{e_k})_{\bar{v}}^\eta \Phi_{\lambda^{-1}} \right] (\Upsilon D_{ek} \bar{v}). \]

Proof. We write

(4.20)

\[ [a_\phi Dw - \bar{a}D\bar{v}]_i (x) = \left[ (a_\phi - \bar{a}) D\bar{v} + \sum_{k=1}^d a_\phi D \left( (\Upsilon D_{ek} \bar{v}) \phi_{e_k}^{(\lambda)} \right) \right] (x) \]

\[ = \left[ \left( (1 - \Upsilon)(a_\phi - \bar{a}) D\bar{v} \right)_i (x) + \sum_{k=1}^d \phi_{e_k}^{(\lambda)} (x + e_i) a_\phi (x, x + e_i) D_{e_i} (\Upsilon D_{ek} \bar{v}) \right] (x) \]

\[ + \sum_{k=1}^d \left[ (a_\phi D\phi_{e_k}^{(\lambda)} + (a_\phi - \bar{a}) Dl_{e_k}) (\Upsilon D_{ek} \bar{v}) \right]_i (x) \]

The terms eq. (4.20)-a and eq. (4.20)-b appear in the eq. (4.19) as the first and second term on the right hand side, so it suffices to treat the remaining terms in eq. (4.20), where we apply the definition of $S_{ek}$ eq. (3.29)

(4.21)

\[ \sum_{k=1}^d \left[ a_{\phi} D\phi_{e_k}^{(\lambda)} + (a_{\phi} - \bar{a}) Dl_{e_k} \right] (\Upsilon D_{ek} \bar{v}) \right]_i (x) \]

\[ = \sum_{k=1}^d \left[ a_{\phi} (D\phi_{e_k} + Dl_{e_k}) - \bar{a} Dl_{e_k} \right] (\Upsilon D_{ek} \bar{v}) \right]_i (x) - \sum_{k=1}^d a_{\phi} \left( D [\phi_{e_k}]^\eta_{\bar{v}} \Phi_{\lambda^{-1}} \right) (\Upsilon D_{ek} \bar{v}) \right]_i (x) \]

\[ = \sum_{k=1}^d \left[ D^* \cdot S_{ek}^{(\lambda)} (\Upsilon D_{ek} \bar{v}) \right]_i (x) + \sum_{k=1}^d \left[ D^* \cdot (|S_{ek}| \Phi_{\lambda^{-1}}) (\Upsilon D_{ek} \bar{v}) \right]_i (x) \]

\[ - \sum_{k=1}^d a_{\phi} \left( D [\phi_{e_k}]^\eta_{\bar{v}} \Phi_{\lambda^{-1}} \right) (\Upsilon D_{ek} \bar{v}) \right]_i (x). \]
The terms eq. (4.21)-b and eq. (4.21)-c also appear in the definition of $F$ with standard this gives the formula in eq. (4.19).

\[ D^* \cdot \text{eq. (4.21)-a} = \sum_{i,k=1}^{d} D^*_{e_i} \left[ D^* \cdot S^{(\lambda)}_{e_k} (\nabla D_{e_k} \bar{v}) \right] (x) \]

\[ = \sum_{i,j,k=1}^{d} D^*_{e_i} \left( \left( D^*_{e_j} S^{(\lambda)}_{e_k,ij} \right) (\nabla D_{e_k} \bar{v}) \right) (x) \]

\[ = \sum_{i,j,k=1}^{d} D^*_{e_i} \left( D^*_{e_j} D^*_{e_k} \left( S^{(\lambda)}_{e_k,ij} (\nabla D_{e_k} \bar{v}) \right) \right) (x) - \sum_{i,j,k=1}^{d} D^*_{e_i} \left( S^{(\lambda)}_{e_k,ij} (\nabla D_{e_k} \bar{v}) \right) (x) \]

\[ = \sum_{i=1}^{d} D^*_{e_i} \left( - \sum_{j,k=1}^{d} \left( S^{(\lambda)}_{e_k,ij} (\nabla D_{e_k} \bar{v}) \right) \right) (x). \]

This gives the formula in eq. (4.19).

\[ \square \]

5. Analysis of the Algorithm

We are now ready to complete the proof of Theorem 1.1, and we start by analyzing our algorithm with standard $H^1$ and $H^2$ estimates for $\bar{u}$ in eq. (1.5).

Lemma 5.1 ($H^1$ and $H^2$ estimates). In the iteration eq. (1.5) we have the following estimates

\[ \|\nabla \bar{u}\|_{L^2(\bar{\Omega}_m)} \leq |\bar{a}|^{-1} (1 + \Lambda) \|\nabla (u - u_0) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))}, \]

\[ \|D^* \nabla \bar{u}\|_{L^2(\nabla(\bar{\Omega}_m))} \leq C(d, \Lambda)|\bar{a}|^{-1} \lambda \|\nabla (u - u_0) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))}, \]

\[ \|\nabla (u - u_0) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))} \leq 2|\bar{a}|^{-1} (1 + \Lambda)^2 \|\nabla (u - u_0) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))}. \]

Proof. We start by testing eq. (1.9) and eq. (1.10) with the function $u_1$, and we also use the trick that $\lambda_{\mathcal{E},m}$ and $a_{\mathcal{E},m}$ restrict the problem on $(\mathcal{E}_s(\bar{\Omega}_m))$

\[ \langle \lambda_{\mathcal{E},m} u_1, \lambda_{\mathcal{E},m} u_1 \rangle_{\bar{\Omega}_m} + \langle \nabla u_1, a_{\mathcal{E},m} \nabla u_1 \rangle_{\bar{\Omega}_m} = \langle \nabla u_1, a_{\mathcal{E},m} (u - u_0) \rangle_{\bar{\Omega}_m}, \]

\[ \Rightarrow \lambda^2 \|u_1\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))} + \Lambda^{-1} \|\nabla u_1 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))} \leq \|\nabla (u - u_0) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))} \|\nabla u_1 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))}. \]

We obtain that

\[ \lambda \|u_1\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))} \leq \Lambda \|\nabla (u - u_0) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))}, \]

\[ \|\nabla u_1 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))} \leq \Lambda \|\nabla (u - u_0) 1_{\{a \neq 0\}}\|_{L^2(\mathcal{E}_s(\bar{\Omega}_m))}. \]

Combining the first equation and the second equation in eq. (1.9) and eq. (1.10), we obtain that

\[ -\nabla \cdot \bar{a} \nabla \bar{u} = -\nabla \cdot a_{\mathcal{E},m} \nabla (u - u_0 - u_1) \quad \text{in int}(\bar{\Omega}_m), \]

then we test it by the function $\bar{u}$ and use Cauchy-Schwarz inequality to obtain that

\[ \langle \nabla \bar{u}, \bar{a} \nabla \bar{u} \rangle_{\bar{\Omega}_m} = \langle \nabla \bar{u}, a_{\mathcal{E},m} \nabla (u - u_0 - u_1) \rangle_{L^2(\bar{\Omega}_m)} \]

\[ \leq \|\nabla \bar{u}\|_{L^2(\bar{\Omega}_m)} \|\nabla (u - u_0 - u_1) 1_{\{a \neq 0\}}\|_{\mathcal{E}_s(\bar{\Omega}_m)} \]

\[ \Rightarrow \|\nabla \bar{u}\|_{L^2(\bar{\Omega}_m)} \leq |\bar{a}|^{-1} \|\nabla (u - u_0 - u_1) 1_{\{a \neq 0\}}\|_{\mathcal{E}_s(\bar{\Omega}_m)}. \]
Using eq. (5.5) we obtain that

\[ \| \nabla \bar{u} \|_{L^2(\square_m)} \leq |\bar{a}|^{-1} \left( \| \nabla(u - u_0 - u_1) 1_{\{a \neq 0\}} \|_{\mathcal{E}_0(\square_m)} \right) \]

\[ \leq |\bar{a}|^{-1} \left( \| \nabla(u - u_0) 1_{\{a \neq 0\}} \|_{\mathcal{E}_0(\square_m)} + \| \nabla u_1 1_{\{a \neq 0\}} \|_{\mathcal{E}_0(\square_m)} \right) \]

\[ \leq |\bar{a}|^{-1} (1 + \Lambda) \| \nabla(u - u_0) 1_{\{a \neq 0\}} \|_{\mathcal{E}_0(\square_m)}. \]

This proves the formula eq. (5.1).

Concerning eq. (5.2), we use the estimation of $H^2$ regularity eq. (2.13) for $-\nabla \cdot \bar{a} \nabla \bar{u} = \lambda_{\varepsilon, m}^2 u_1$ since $\bar{a}$ is constant and obtain that

\[ \sum_{i,j=1}^{d} \left\| D_i^* D_j \bar{u} \right\|_{L^2(\text{int}(\square_m))}^2 \leq C(d) |\bar{a}|^{-2} \| \lambda_{\varepsilon, m}^2 u_1 \|_{L^2(\square_m)}. \]

We put the result from eq. (5.4) and eq. (5.1) and obtain that

\[ \| D^* D \bar{u} \|_{L^2(\text{int}(\square_m))} \leq C(d, \Lambda) |\bar{a}|^{-1} \lambda \| \nabla(u - u_0) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))}. \]

To prove eq. (5.3), we put eq. (1.10), the first equation and the second equation of eq. (1.9) into the right hand side of the third equation and obtain that

\[ (\lambda_{\varepsilon, m}^2 - \nabla \cdot a_{\varepsilon, m} \nabla) u_2 = \lambda_{\varepsilon, m}^2 \bar{u}^2 - \nabla \cdot a_{\varepsilon, m} \nabla(u - u_0 - u_1) \text{ in } \text{int}(\square_m). \]

We subtract $(\lambda_{\varepsilon, m}^2 - \nabla \cdot a_{\varepsilon, m} \nabla) \bar{u}$ on the two sides to obtain

\[ (\lambda_{\varepsilon, m}^2 - \nabla \cdot a_{\varepsilon, m} \nabla)(u_2 - \bar{u}) = -\nabla \cdot a_{\varepsilon, m} \nabla(u - u_0 - u_1 - \bar{u}) \text{ in } \text{int}(\square_m), \]

and then we test it by $(u_2 - \bar{u})$ to obtain that

\[ \| \nabla(u_2 - \bar{u}) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} \leq \Lambda \| \nabla(u - u_0 - u_1 - \bar{u}) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))}. \]

Therefore, combining eq. (5.6), eq. (5.4) and eq. (5.1) we can obtain a trivial bound for our algorithm

\[ \| \nabla(\bar{u} - u) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} \leq \| \nabla(u - u_0 - u_1 - \bar{u}) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} + \| \nabla(u_2 - \bar{u}) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} \]

\[ \leq 2|\bar{a}|^{-1} (1 + \Lambda)^2 \| \nabla(u - u_0) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))}. \]

\[ \square \]

The trivial bound eq. (5.3) is not optimal. In the typical case $\square_m \in \mathcal{P}_*$ in large scale, we can use Theorem 1.2 to help us get a better bound, and this help use conclude the performance of our algorithm.

**Proof of Theorem 1.1.** We analyze the algorithm in two cases: $\square_m \in \mathcal{P}_*$ and $\square_m \notin \mathcal{P}_*$. In the case $\square_m \notin \mathcal{P}_*$, we use eq. (5.3) that

\[ \| \nabla(\bar{u} - u) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} 1_{\{\square_m \notin \mathcal{P}_*\}} \leq 2|\bar{a}|^{-1} (1 + \Lambda)^2 \| \nabla(u - u_0) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} 1_{\{\square_m \notin \mathcal{P}_*\}}. \]

In the case $\square_m \in \mathcal{P}_*$, we combine the first equation and the second equation of eq. (1.9) and eq. (1.10), together with the third term they give

\[ -\nabla \cdot a_{\varepsilon, m} \nabla(u - u_0 - u_1) = -\nabla \cdot \bar{a} \nabla \bar{u} \text{ in } \text{int}(\square_m), \]

\[ (\lambda_{\varepsilon, m}^2 - \nabla \cdot a_{\varepsilon, m} \nabla) u_2 = (\lambda_{\varepsilon, m}^2 - \nabla \cdot \bar{a} \nabla) \bar{u} \text{ in } \text{int}(\square_m). \]

This gives us two equations of two-scale expansion. As in eq. (1.12), we define $w := \bar{u} + \sum_{k=1}^{d} (\lambda^D_k \phi_k) \phi_{k}^{(\lambda)}$ and apply Theorem 1.2

\[ \| \nabla(\bar{u} - u) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} 1_{\{\square_m \in \mathcal{P}_*\}} \]

\[ \leq \left( \| \nabla(w - (u - u_0 - u_1)) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} + \| \nabla(u_2 - w) 1_{\{a \neq 0\}} \|_{L^2(\mathcal{E}_0(\square_m))} \right) 1_{\{\square_m \in \mathcal{P}_*\}}. \]
The last equation gives a bound of type proposition 4.1. Together with the Lemma 5.1 and the estimate for case $\square_m \notin \mathcal{P}_s$, we obtain that
\[ \| \nabla (\hat{u} - u) 1_{\{\nabla \neq 0\}} \|_{L^2(\mathcal{V}_s(\square_m))} \leq Z \| \nabla (u_0 - u) 1_{\{\nabla \neq 0\}} \|_{L^2(\mathcal{V}_s(\square_m))}, \]
where $Z$ is given by
\begin{equation}
Z = C(d, \Lambda) \left( 3^{-\frac{m}{2}} \ell^{-\frac{1}{2}} (\lambda) \mathcal{Y}_1 \mathcal{X}_d + \mathcal{Y}_2 \mathcal{X}_d + \lambda^\frac{1}{2} \ell^{-\frac{1}{2}} (\lambda) \mathcal{Y}_1 \mathcal{X}_d + \lambda \mathcal{Y}_1 \mathcal{X}_d + 1_{\{\square_m \notin \mathcal{P}_s\}} \right).
\end{equation}
This gives the exact expression of the quantity $Z$. To conclude, we have to quantify $Z$ and we use eq. (4.8), eq. (4.11), eq. (2.18) and eq. (2.4) that there exist two positive constants $s(d, p, \Lambda)$ and $C(d, p, \Lambda, s)$ such that
\begin{equation}
Z \leq O_s \left( C(d, p, \Lambda, s) \left( 3^{-\frac{m}{2}} + \lambda + \lambda^\frac{1}{2} + \lambda^2 \right) \ell^{\frac{1}{2}} (\lambda) + \lambda m^\frac{1}{2} \ell (\lambda) + 3^{-m} \right).
\end{equation}
Observing that $3^{-m} \leq \lambda$, then the dominating order writes $Z \leq O_s \left( C \lambda^\frac{1}{2} \ell (\lambda) m^\frac{1}{2} \right)$ and this concludes the proof of Theorem 1.1.

6. Numerical experiments

We report on numerical experiments corresponding to our algorithm. In a cube $\square$ of size $L$, we try to solve a localized corrector problem, that is, we look for the function $\phi_{L,p} \in C_0(\square)$ such that
\begin{equation}
-\nabla \cdot a \nabla (\phi_{L,p} + l_p) = 0 \quad \text{in } \mathcal{V}_s(\square).
\end{equation}
The quantity $\phi_{L,p}$ is very similar to the corrector $\phi_p$ and has sublinear growth. This is a good example for illustrating the usefulness of our algorithm, since the homogenized approximation to this function is simply the null function, which is not very informative.

In our example, we take $d = 2$, $p = e_1$ and $L = 243$. We implement the algorithm to get a series of approximated solutions $\tilde{u}_n$ where $\tilde{u}_0 = 0$. Moreover, we use the residual error to see the convergence
\[ \text{res}(\tilde{u}_n) := \frac{1}{|\square|} \| -\nabla \cdot a \nabla (\tilde{u}_n + l_p) \|_{L^2(\mathcal{V}_s(\square))}^2 = \frac{1}{|\square|} \| -\nabla \cdot a \nabla (\tilde{u}_n - \phi_{L,p}) \|_{L^2(\mathcal{V}_s(\square))}^2. \]
See the Figure 6 for a simulation of the corrector $\phi_{L,p}$ with high resolution, and Figure 7 for its residual errors.

\section*{Appendix A. Proof of some discrete functional inequality}

\textbf{Lemma A.1} ($H^2$ interior estimate for elliptic equation). \textit{Given two functions $v, f \in C_0(\square_m)$ satisfying the discrete elliptic equation}
\begin{equation}
-\Delta v = f, \quad \text{in } \text{int}(\square_m),
\end{equation}
\textit{we have an interior estimate}
\begin{equation}
\| D^* D v \|_{L^2(\text{int}(\square_m))}^2 := \sum_{i,j=1}^d \| D_{e_i}^d D_{e_j} v \|_{L^2(\text{int}(\square_m))}^2 \leq d \| f \|_{L^2(\text{int}(\square_m))}^2.
\end{equation}
\textit{Proof.} We extend the elliptic equation to the whole space at first. The function $v, f$ have a natural null extension on $\mathbb{Z}^d$ satisfying
\[ D^* \cdot D v = f + (D^* \cdot D v) 1_{\{\psi_{\square_m} \}}, \quad \text{in } \mathbb{Z}^d. \]
To simplify the notation, we denote by $\tilde{f}$ the term on the right hand side. Then, by one step difference of direction $e_j$, we have
\[ D^* \cdot D (D_{e_j} v(x)) = D_{e_j} \tilde{f}(x). \]
We test this equation with a function $\phi$ of compact support, then by eq. (2.9) we obtain
\[ \langle D\phi, D(D_{e_j} v) \rangle_{\mathbb{Z}^d} = \langle D_{e_j}^* \phi, \tilde{f} \rangle_{\mathbb{Z}^d}. \]
Figure 6. A simulation for the corrector on the maximal cluster in a cube $243 \times 243$.

<table>
<thead>
<tr>
<th>round</th>
<th>errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.02597982969</td>
</tr>
<tr>
<td>2</td>
<td>0.0126490361046</td>
</tr>
<tr>
<td>3</td>
<td>0.0070540548365</td>
</tr>
<tr>
<td>4</td>
<td>0.0045201077274</td>
</tr>
<tr>
<td>5</td>
<td>0.00282913420116</td>
</tr>
<tr>
<td>6</td>
<td>0.00190945842802</td>
</tr>
<tr>
<td>7</td>
<td>0.00132483912845</td>
</tr>
<tr>
<td>8</td>
<td>0.000939101476657</td>
</tr>
</tbody>
</table>

Figure 7. A table of errors $\text{res}(\hat{u}_n)$.

Putting $\phi = (D_{e_j} v)$ in this formula, we obtain that

$$\langle D(D_{e_j} v), D(D_{e_j} v) \rangle_{\mathbb{Z}^d} = \langle D_{e_j}^* D_{e_j} v, f \rangle_{\mathbb{Z}^d}$$

$$= \langle D_{e_j}^* D_{e_j} v, f \rangle_{\mathbb{Z}^d} + \langle D_{e_j}^* D_{e_j} v, (D^* \cdot D v) 1_{\{\partial \square m\}} \rangle_{\mathbb{Z}^d}.$$  

We do the sum over the $d$ canonical directions and get

$$\sum_{i,j=1}^d \langle D_{e_i} D_{e_j} v, D_{e_i} D_{e_j} v \rangle_{\mathbb{Z}^d} = \sum_{j=1}^d \langle D_{e_j}^* D_{e_j} v, f \rangle_{\mathbb{Z}^d} + \sum_{j=1}^d \langle D_{e_j}^* D_{e_j} v, (D^* \cdot D v) 1_{\{\partial \square m\}} \rangle_{\mathbb{Z}^d}$$

$$= \sum_{j=1}^d \langle D_{e_j}^* D_{e_j} v, f \rangle_{\mathbb{Z}^d} + \langle D^* \cdot D v, (D^* \cdot D v) 1_{\{\partial \square m\}} \rangle_{\mathbb{Z}^d}.$$  

Since $D_{e_j}^* v(x) = -D_{e_j} v(x - e_j)$, we have

$$\sum_{i,j=1}^d \langle D_{e_i}^* D_{e_j} v, D_{e_i}^* D_{e_j} v \rangle_{\mathbb{Z}^d} = \sum_{j=1}^d \langle D_{e_j}^* D_{e_j} v, f \rangle_{\mathbb{Z}^d} + \langle D^* \cdot D v, (D^* \cdot D v) 1_{\{\partial \square m\}} \rangle_{\mathbb{Z}^d}.$$  

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There are three observations for this equation.

- supp($\mathcal{D}^*_i \mathcal{D}^*_j v$) $\subset \square_m$.
- For any $x \in \partial \square_m$, 
  \[ (\mathcal{D}^* \cdot \mathcal{D}v)^2 = \left( \sum_{j=1}^d \mathcal{D}^*_j \mathcal{D}^*_j v(x) \right)^2 = \sum_{j=1}^d \left( \mathcal{D}^*_j \mathcal{D}^*_j v(x) \right)^2, \]
  since $v = 0$ on the boundary and only one term of $\left\{ \mathcal{D}^*_j \mathcal{D}^*_j v(x) \right\}_{j=1}^d$ is not null.
- On the boundary $\partial \square_m$, $f = 0$ since $f \in C_0(\square_m)$.

Combining the three observations, we get
\[ \sum_{i,j=1}^d \langle \mathcal{D}^*_i \mathcal{D}^*_j v, \mathcal{D}^*_i \mathcal{D}^*_j v \rangle_{\text{int}(\square_m)} = \sum_{j=1}^d \langle \mathcal{D}^*_j \mathcal{D}^*_j v, f \rangle_{\text{int}(\square_m)} + \sum_{j=1}^d \langle \mathcal{D}^*_j \mathcal{D}^*_j v, \mathcal{D}^*_j \mathcal{D}^*_j v \rangle_{\partial \square_m}. \]

Thus, all the terms in the last sum on the right hand side can be found on the left hand side. We use Cauchy-Schwarz inequality and Young’s inequality
\[ \sum_{i,j=1}^d \langle \mathcal{D}^*_i \mathcal{D}^*_j v, \mathcal{D}^*_i \mathcal{D}^*_j v \rangle_{\text{int}(\square_m)} \leq \sum_{j=1}^d \langle \mathcal{D}^*_j \mathcal{D}^*_j v, f \rangle_{\text{int}(\square_m)} \]
\[ \leq \sum_{j=1}^d \left( \frac{1}{2} \langle \mathcal{D}^*_j \mathcal{D}^*_j v, \mathcal{D}^*_j \mathcal{D}^*_j v \rangle_{\text{int}(\square_m)} + \frac{1}{2} \langle f, f \rangle_{\text{int}(\square_m)} \right) \]
\[ \Rightarrow \sum_{i,j=1}^d \langle \mathcal{D}^*_i \mathcal{D}^*_j v, \mathcal{D}^*_i \mathcal{D}^*_j v \rangle_{\text{int}(\square_m)} \leq d \langle f, f \rangle_{\text{int}(\square_m)}, \]
which concludes the proof.

The same technique to do an integration along the path helps us to get an estimate of trace.

**Lemma A.2** (Trace inequality). For every $u : \square_m \to \mathbb{R}$ and $0 \leq K \leq \frac{3^m}{4}$, we have the following inequality
\[ \| u \|_{L^2(\square_m)}^2 \leq C(d)(K + 1) \left( 3^{-m} \| u \|_{L^2(\square_m)}^2 + \| \nabla u \|_{L^2(\square_m)} \right). \]

**Proof.** We use the notation $L_{m,t}$ to define the level set in $\square_m$ with distance $t$ to the boundary
\[ L_{m,t} := \{ x \in \square_m : \text{dist}(x, \partial \square_m) = t \}. \]

Then, we observe that $L_{m,0} = \partial \square_m$ and we have the partition
\[ \square_m = \bigsqcup_{t=0}^{\lfloor \frac{3^m}{4} \rfloor} L_{m,t}. \]

Using the pigeonhole principle, it is easy to prove that there exists a $t^* \in \left[ 0, \lfloor \frac{3^m}{4} \rfloor - 1 \right]$ such that
\[ \| u \|_{L^2(L_{m,t^*})}^2 \leq \frac{4}{3^m} \| u \|_{L^2(\square_m)}^2, \]
and we define $t^* := \arg \min_{t \in [0, \lfloor \frac{3^m}{4} \rfloor]} \| u \|_{L^2(L_{m,t})}^2$. We call $L_{m,t^*}$ the pivot level and it plays the same role as the null boundary in the proof of Poincaré’s inequality. In the following, we will apply the
trick of integration along the path to prove the eq. (2.14) for one lever \( L_{m,t} \). For every \( x \in L_{m,t} \), we denote by \( r(x, t^*) \) a root on the pivot level \( L_{m,t^*} \), and choose a path \( \gamma_{x,t^*} = \{ \gamma_{k} \}_{0 \leq k \leq n} \) such that
\[
\gamma_{0} = r(x, t^*), \quad \gamma_{k} \sim \gamma_{k+1}, \quad \gamma_{n} = x.
\]
Moreover, we use \( |\gamma_{x,t^*}| \) to represent the number of steps of the path, for example here \( |\gamma_{x,t^*}| = n \). We apply a discrete Newton-Leibniz formula to get
\[
u(x) - \nu(r(x, t^*)) = \sum_{k=0}^{|\gamma_{x,t^*}|} (\nu(\gamma_{k+1}^{x,t^*}) - \nu(\gamma_{k}^{x,t^*})) \leq \sum_{k=0}^{|\gamma_{x,t^*}|} \nabla u(\gamma_{k}^{x,t^*}, \gamma_{k+1}^{x,t^*}) (u(\gamma_{k+1}^{x,t^*}) + u(\gamma_{k}^{x,t^*})).
\]
We put this formula into the norm of \( \|u\|_{L^2(L_{m,t})} \), \( t \in [0, \frac{3m}{4}] \) and and apply Cauchy-Schwarz inequality to obtain that
\[
\|u\|_{L^2(L_{m,t})}^2 = \sum_{x \in L_{m,t}} \left( \|u(x) - u(r(x, t^*))\|^2 + \sum_{k=0}^{|\gamma_{x,t^*}|} \nabla u(\gamma_{k}^{x,t^*}, \gamma_{k+1}^{x,t^*}) (u(\gamma_{k+1}^{x,t^*}) + u(\gamma_{k}^{x,t^*})) \right)
\leq \sum_{x \in L_{m,t}} \left( \nu(x) - \nu(r(x, t^*)) \right)
+ 2 \left( \sum_{x \in L_{m,t}} \sum_{k=0}^{|\gamma_{x,t^*}|} \nabla u(\gamma_{k}^{x,t^*}, \gamma_{k+1}^{x,t^*})^2 \left( \sum_{x \in L_{m,t}} \sum_{k=0}^{|\gamma_{x,t^*}|} (u(\gamma_{k+1}^{x,t^*}) + u(\gamma_{k}^{x,t^*})) \right) \right)^{\frac{1}{2}}
+ 4 \left( \sum_{(y_1, y_2) \in E_d(\square_m)} (\nabla u(y_1, y_2))^2 \left( \sum_{x \in L_{m,t}} \sum_1^{|\gamma_{x,t^*}|} \nabla u(\gamma_{k}^{x,t^*}) \right) \right)^{\frac{1}{2}}
\times \left( \sum_{y \in \square_m} \nu(y) \left( \sum_{x \in L_{m,t}} \sum_1^{|\gamma_{x,t^*}|} \nabla u(\gamma_{k}^{x,t^*}) \right) \right)^{\frac{1}{2}}.
\]
The next step is to decide how to choose the root \( r(x, t^*) \) and the path. The main idea is to make every edge and every vertex as root is passed by \( \{ \gamma_{x,t^*}\} \) a finite number of times bounded by a constant \( C(d) \). One possible plan is to choose the root \( r(x, t^*) \) and the path \( \gamma_{x,t^*} \) a discrete path in \( (\mathbb{Z}^d, E_d) \) which is the closest to the vector \( \bar{O}x \), then it is a simple exercise to see that it gives us a bound \( C(d) \). See Figure 8 as a visualization. Then we get that
\[
\|u\|_{L^2(L_{m,t})} \leq C(d) \left( \|u\|_{L^2(L_{m,t^*})} + \|u\|_{L^2(\square_m)} \|\nabla u\|_{L^2(\square_m)} \right).
\]
Then we put the eq. (A.4) and get
\[
\|u\|_{L^2(L_{m,t})} \leq 4C(d) \left( 3^{-m} \|u\|_{L^2(\square_m)} + \|u\|_{L^2(\square_m)} \|\nabla u\|_{L^2(\square_m)} \right).
\]
eq (2.14) is just a result by summing all the levels of distance less than \( K \).

\textbf{Appendix B. Small clusters}

This part is devoted to studying the small clusters in the percolation. Many of the arguments presented here have appeared in the previous work [3]. We extract those results from [3] and expand
Figure 8. To construct the path $\gamma^{x,t^*}$ for every $x \in L_{m,t}$, one can find at first the pivot level $L_{m,t^*}$. Then we connect $O$ and $x$ and find one of its closest discrete path in $(\mathbb{Z}^d, E_d)$ and denote by $\gamma^{x,t^*}$ the segment from $L_{m,t^*}$ to $x$. By this construction, every edge and vertex is passed by the paths $\{\gamma^{x,t^*}\}_{x \in L_{m,t}}$ at most $C(d)$ times. In this picture, the arrows in blud indicate the vectors $Ox_1, Ox_2, Ox_3$ and the segments with arrow in red are the paths $\gamma^{x_1,t^*}, \gamma^{x_2,t^*}, \gamma^{x_3,t^*}$.

upon certain points that are useful for our purposes. The motivation to state these results comes from the technique of partition of good cubes:

**Question B.1.** In a cube $\square \in T$ and its enlarged domain $\text{cl}_P(\square)$, besides the maximal cluster $C^*(\square)$, what is the behavior of the other finite connected clusters?

**Question B.2.** When we apply Lemma 2.3, since $C^*(\square)$ and $\bigcup_{z \in \square} C^*(\text{cl}_P(z))$ are not necessarily equal, how can we describe the difference between the two?

**Question B.3.** What is the difference between $C_\infty \cap \square$ and $C^*(\square)$?

We start with a first very elementary lemma:

**Lemma B.1.** For any $\square \in T$ and $z \in (C_\infty \cap \text{cl}_P(\square)) \setminus C^*(\square)$, there exists a cluster $C'$ such that $z \in C'$ and $C' \xrightarrow{\mu} \partial \text{cl}_P(\square)$.

**Proof.** For a cube $\square \in T$ and its enlarged domain $\text{cl}_P(\square)$, there exist three types of clusters:

1. One unique maximal cluster $C^*(\square)$;
2. The isolated clusters which connect neither to $C^*(\square)$ nor to the boundary $\partial \text{cl}_P(\square)$;
3. The clusters which do not connect to $C^*(\square)$ but connect to the boundary.

Then it is clear the cluster $C'$ containing $z \in (C_\infty \cap \text{cl}_P(\square)) \setminus C^*(\square)$ can only be of the third type and this proves the lemma.

We define the third class above as small clusters (see Figure 10). For any $z \in \mathbb{Z}^d$, we denote by $C'(z)$ the clusters containing $z$.

**Definition B.1** (Small clusters). For any $\square \in T$, we define *small clusters* in $\square$ as the union of clusters, restricted to $\text{cl}_P(\square)$, different from $C^*(\square)$ but connecting to $\partial \text{cl}_P(\square)$, and we denote it by $C_s(\square)$, i.e.

$$C_s(\square) := \bigcup_{z \in \partial \text{cl}_P(\square) \setminus C^*(\square)} C'(z).$$
Intuitively, these small clusters should be of order size(□)^d−1 when the cube □ is large. This is indeed true, as we prove the following lemma:

**Lemma B.2.** For any □ ∈ T, the set Cs(□) has the following decomposition

\[(B.1)\quad Cs(□) \subset \bigcup_{z \in \partial clP(□)} □P(z),\]

and has the estimate

\[(B.2)\quad |Cs(□)|1_{□ \in P^*} \leq O_1(C size(□)^{d-1}).\]

*Proof.* We prove at first eq. (B.1). In the case that □ /∈ P^*, it is obvious since it has to enlarge to clP(□) which is a larger cube, and all the terms on the right hand side of eq. (B.1) refer to clP(□).

In the case that □ ∈ P^*, we consider one cluster C′ connecting to x ∈ ∂□. We suppose that it is not contained in the union of the elements of P lying on ∂□, then it has to cross into the interior. As illustration in Figure 9, it has several situations:

![Figure 9](image)

**Figure 9.** The image explain why C' is contained in the union of partition cubes lying at the ∂□. Without loss of generality, we suppose the big blue cube is □P(x) and the small one is its neighbor good cube. The cube in color of green is the part of size 3/4 of the good cube. The paths in red represent different typical situations that if a finite cluster connects to the boundary ∂□ and wants to cross \( \bigcup_{z \in \partial clP(□)} □P(z) \).

1. The first case is that C' cross at least one pair of (d−1)-dimensional opposite face of partition cube, as showed as γ1 or γ2. For the case γ1, we have \( |diam(C')| > size(□P(x)) \); for the case γ2, we have \( |diam(C')| > \frac{1}{3} size(□P(x)) \). Then by the definition of partition cube, we can find a cube □' of \( \frac{1}{2} size(□P(x)) \) to contain parts of γ1 and □' intersects \( \frac{3}{4} □P(x) \), so by the definition of good cube we have necessarily C' ⊆ Cs(□P(x)). Same discussion can be also applied to the case γ2. This gives a contradiction.

2. The second case is that C' does not cross any pair of (d−1)-dimensional opposite face of partition cube, but also enter the interior of □ by ∂□P(x) or the boundary of its neighbor, so \( |diam(C')| > \frac{1}{3} size(□P(x)) \). One can always find a cube □' of size \( \frac{1}{3} size(□P(x)) \) crossed by C'. If it is the case in γ3 that □' intersects \( \frac{3}{4} □P(x) \), then we apply the definition of good cubes and C' ⊆ Cs(□P(x)). Otherwise, in the case γ4, C' must cross a cube □'' of size \( \frac{1}{3} size(□P(x)) \) in its neighbor and we apply the same discussion, which also gives a contradiction.
To estimate the upper bound eq. (B.2), we use the decomposition above and calculate the volume of $\bigcup_{z \in \partial \Box} \Box p(z)$ by doing a contour integration along $\partial \Box$ of height function $\text{size}(\Box p(z))$ and then applying eq. (2.4),

$$|\mathcal{C}_s(\Box)|1_{\{\Box \in \mathcal{P}_*\}} \leq \left| \bigcup_{z \in \partial \Box} \Box p(z) \right| \leq \sum_{z \in \partial \Box} \text{size}(\Box p(z)) \leq O_1(C \text{ size}(\Box)^{d-1}).$$

![Figure 10](image)

**Figure 10.** The black cluster is $\mathcal{C}_s(\Box)$. The cubes in blue are the good cubes at the boundary, which contains the small cluster $\mathcal{C}_s(\Box)$ in color red. Its volume can be controlled by the integration along $\partial \Box$ of the size of the partition cubes.

Thus, Lemmas B.1 and B.2 answer Question B.1, and the notation of $\mathcal{C}_s(\Box)$ also helps us to solve Question B.2:

**Lemma B.3.** For $\Box \in \mathcal{T}$ such that $\text{size}(\Box) > n > 0$, we have the estimate

$$(B.3) \quad \left| \mathcal{C}_s(\Box) \setminus \left( \bigcup_{z \in 3^n \mathbb{Z}^d \cap \Box} \mathcal{C}_s(z + \Box_n) \right) \right| 1_{\{\Box \in \mathcal{P}_*\}} \leq O_1(C |\Box| 3^{-n}).$$

**Proof.** We decompose this difference in every cube of size $3^n$

$$\left| \mathcal{C}_s(\Box) \setminus \left( \bigcup_{z \in 3^n \mathbb{Z}^d \cap \Box} \mathcal{C}_s(z + \Box_n) \right) \right| 1_{\{\Box \in \mathcal{P}_*\}} \leq \sum_{z \in 3^n \mathbb{Z}^d \cap \Box} \left| (\mathcal{C}_s(\Box) \cap (z + \Box_n)) \setminus \left( \bigcup_{z \in 3^n \mathbb{Z}^d \cap \Box} \mathcal{C}_s(z + \Box_n) \right) \right| 1_{\{\Box \in \mathcal{P}_*\}}$$

$$\leq \sum_{z \in 3^n \mathbb{Z}^d \cap \Box} |(\mathcal{C}_s(\Box) \cap (z + \Box_n)) \setminus \mathcal{C}_s(z + \Box_n)| 1_{\{\Box \in \mathcal{P}_*\}}$$

$$\leq \sum_{z \in 3^n \mathbb{Z}^d \cap \Box} |(\mathcal{C}_s(\Box) \cap (z + \Box_n)) \setminus \mathcal{C}_s(z + \Box_n)| 1_{\{z + \Box_n \in \mathcal{P}_*\}}$$

$$+ \sum_{z \in 3^n \mathbb{Z}^d \cap \Box} |(\mathcal{C}_s(\Box) \cap (z + \Box_n)) \setminus \mathcal{C}_s(z + \Box_n)| 1_{\{z + \Box_n \notin \mathcal{P}_*\}}.$$
The two terms can be treated separately. For the case $\hat{\square} + n \in \mathcal{P}_* $, as we have mentioned, we have $d_{\mathcal{P}}(z + \hat{\square} + n) = z + \hat{\square} + n$ and $|((\mathcal{C}_s(\square) \setminus (z + \hat{\square} + n)) \setminus \mathcal{C}_s(z + \hat{\square} + n))|$ can be counted at the boundary $\partial (z + \hat{\square} + n)$.

We turn this argument into the estimate using eq. (2.15) and eq. (2.4)

\[
\sum_{z \in 3^n \mathbb{Z}^d \cap \square} |((\mathcal{C}_s(\square) \setminus (z + \hat{\square} + n)) \setminus \mathcal{C}_s(z + \hat{\square} + n))| 1_{\{z + \hat{\square} + n \in \mathcal{P}_* \}} \leq \sum_{z \in 3^n \mathbb{Z}^d \cap \square} |\mathcal{C}_s(z + \hat{\square} + n)| 1_{\{z + \hat{\square} + n \in \mathcal{P}_* \}} \leq \mathcal{O}_1(C|\square|^3 - n).
\]

For another part, we use eq. (2.15) and eq. (2.4) directly that

\[
\sum_{z \in 3^n \mathbb{Z}^d \cap \square} |((\mathcal{C}_s(\square) \setminus (z + \hat{\square} + n)) \setminus \mathcal{C}_s(z + \hat{\square} + n))| 1_{\{z + \hat{\square} + n \notin \mathcal{P}_* \}} \leq \sum_{z \in 3^n \mathbb{Z}^d \cap \square} |z + \hat{\square} + n| 1_{\{z + \hat{\square} + n \notin \mathcal{P}_* \}} \leq \mathcal{O}_1(C|\square|^3 - n).
\]

We combine all these estimates and conclude the result. \( \square \)

Finally, we study Question B.3 on $(\mathcal{C}_\infty(\square) \cap \square) \setminus \mathcal{C}_s(\square)$:

**Lemma B.4.** Under the condition $\square \in \mathcal{P}_*$, and we use $\tilde{\square}$ to represent its predecessor, then we have $(\mathcal{C}_\infty(\square) \cap \square) = (\mathcal{C}_s(\tilde{\square}) \cap \square)$, and we have the estimate that

\[
|((\mathcal{C}_\infty(\square) \cap \square) \setminus \mathcal{C}_s(\tilde{\square}))| 1_{\{\square \in \mathcal{P}_* \}} \leq \mathcal{O}_1(C|\square|^\frac{d-1}{3}).
\]

**Proof.** The lemma says when the cube $\square_m$ is even better than a good cube, $\mathcal{C}_s(\tilde{\square}) \cap \square$ can contain all the part of $\mathcal{C}_\infty \cap \square$. One direction $(\mathcal{C}_s(\tilde{\square}) \cap \square) \subset (\mathcal{C}_\infty \cap \square)$ is obvious. We prove the other direction $(\mathcal{C}_\infty \cap \square) \subset (\mathcal{C}_s(\tilde{\square}) \cap \square)$ by contradiction. We suppose that this direction is not correct so that there exists $z \in (\mathcal{C}_\infty \cap \square)$ but $z \notin (\mathcal{C}_s(\tilde{\square}) \cap \square)$. By Lemma B.1, there exists a cluster $\mathcal{C}'$ different from $\mathcal{C}_s(\tilde{\square})$ and $z \in \mathcal{C}'$ and $\mathcal{C}'$ connects to $\partial \tilde{\square} \in \mathcal{P}_*$ (then $\tilde{\square} \in \mathcal{P}_*$). Since $\mathcal{C}_s(\tilde{\square})$ is part of $\mathcal{C}_s(\tilde{\square})$, $\mathcal{C}'$ cannot connect to $\mathcal{C}_s(\tilde{\square})$. Thus, there exists an open path $\gamma$ such that $z \in \gamma \subset \mathcal{C}'$ intersecting $\partial \tilde{\square}$ and we have $|\gamma| > \frac{1}{3} \text{size}(\tilde{\square})$. This violates the second term in Proposition 2.1 that a large path should belong to part of $\mathcal{C}_s(\tilde{\square})$. We suppose that $\text{size}(\square) = 3^n$ and then apply Lemma B.3 to obtain that

\[
|((\mathcal{C}_\infty \cap \square) \setminus \mathcal{C}_s(\tilde{\square}))| 1_{\{\square \in \mathcal{P}_* \}} = \left| (\mathcal{C}_s(\tilde{\square}) \cap \square) \setminus \mathcal{C}_s(\tilde{\square}) \right| 1_{\{\square \in \mathcal{P}_* \}} \leq \left| \mathcal{C}_s(\tilde{\square}) \setminus \left( \bigcup_{z \in 3^n \mathbb{Z}^d \cap \tilde{\square}} \mathcal{C}_s(z + \hat{\square} + n) \right) \right| 1_{\{\square \in \mathcal{P}_* \}} \leq \mathcal{O}_1(C|\square|^3 - n).
\]

\( \square \)

**Remark.** The same argument can prove even a stronger result that $(\mathcal{C}_\infty \cap \frac{3}{4} \tilde{\square}) = (\mathcal{C}_s(\tilde{\square}) \cap \frac{3}{4} \tilde{\square})$.

**Appendix C. Characterization of the effective conductance**

In the literature, there are several approaches to define the effective conductance $\hat{a}$, and the object of this section is to give a proof of the equivalence of these definitions in the context of percolation.

Let us at first recall the definition and some useful propositions in the previous work [3, Definition 5.1]: we define the energy in the domain $U \subset \mathbb{Z}^d$ with $l_p(x) := p \cdot x$ boundary condition

\[
\nu(U, p) := \inf_{v \in l_p + C_0(\text{cl}_p(U))} \frac{1}{2\text{cl}_p(U)} \left( \nabla v \cdot a \nabla v \right)_{E^2_\nu(\mathcal{C}_s(U))},
\]

where $E^2_\nu(\mathcal{C}_s(U))$ is the quadratic form of the $\nu$-energy.
and we denote by \( v(\cdot, U, p) \) its minimiser. The effective conductance \( \bar{a} \) is a deterministic positive scalar defined by

\[
\frac{1}{2} p \cdot \bar{a} := \lim_{m \to \infty} \mathbb{E}[\nu(\square_m, p)],
\]
with the rate of convergence \( [3, \text{Lemma } 4.8] \): there exists \( s(d) > 0, \alpha(d, p, \Lambda) \in \left( 0, \frac{1}{4} \right] \) and \( C(d, p, \Lambda) < \infty \) such that for every \( \square \in \mathcal{T} \)

\[
\frac{1}{2} p \cdot \bar{a} - \nu(\square, p) \leq O_s(C|p|^2 \text{size}(\square)^{-\alpha}).
\]

We will also use the following trivial bound several times in the proof

\[
\nu(U, p) \leq \frac{1}{2|c_l^p(U)|} \langle \nabla v \cdot a \nabla v \rangle_{\mathcal{E}^a(U)} \leq d|p|^2.
\]

The main theorem in this part is to prove the following characterization.

**Theorem C.1** (Characterization of the effective conductance). *In the context of homogenization in supercritical percolation, the effective conductance \( \bar{a} \) is a positive scalar constant and the following definitions are equivalent:

1. \( p \cdot \bar{a} = \lim_{m \to \infty} \frac{1}{|c_l^p(\square_m)|} \langle \nabla v \cdot a \nabla v \rangle_{\mathcal{E}^a(\square_m)} \).
2. \( p \cdot \bar{a} = \lim_{m \to \infty} \inf_{v \in \mathcal{C}_s(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{\mathcal{E}^a(\square_m)} \).
3. \( p \cdot \bar{a} = \mathbb{E}[\mathcal{D}(\phi_p + l_p) \cdot a \mathcal{D}(\phi_p + l_p)] \).
4. \( \bar{a} = \mathbb{E}[a \mathcal{D}(\phi_p + l_p)] \).

Before starting the proof, we give some remarks on these definitions. Equation (C.5) is just a variant of eq. (C.2). Equation (C.6) differs from the first one in that just it does the minimization but does not enlarge the domain to \( c_l^p(\square_m) \) nor restricts the problem to \( \mathcal{C}_s(\square_m) \). Equation (C.7) uses the linear \( a \)-harmonic function in the whole space instead of that in \( \square_m \), so it is stationary. The last one is a little different from the previous three ones, but we need it in Proposition 1.2, thus we add it to the list of equivalent definitions.

**Proof.** Equation (C.5) is a direct consequence of eq. (C.2) and eq. (C.3), Markov’s inequality and the lemma of Borel-Cantelli to transform it to an “almost sure” version.

Equation (C.6) is a variant from the first one, especially when \( \square_m \in \mathcal{P}_s \) they are very close. So we do the decomposition

\[
\frac{1}{|\square_m|} \inf_{v \in l_p + C_0(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{\mathcal{E}^a(\square_m)} - p \cdot \bar{a} 
\]

\[
\leq \frac{1}{|\square_m|} \inf_{v \in l_p + C_0(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{\mathcal{E}^a(\square_m)} - p \cdot \bar{a} \mathbf{1}_{\{\square_m \in \mathcal{P}_s\}}
\]

\[
+ \frac{1}{|\square_m|} \inf_{v \in l_p + C_0(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{\mathcal{E}^a(\square_m)} - p \cdot \bar{a} \mathbf{1}_{\{\square_m \notin \mathcal{P}_s\}}
\]

and the second one can be handled easily by a trivial bound by comparing with \( l_p \) as in eq. (C.4)

\[
\frac{1}{|\square_m|} \inf_{v \in l_p + C_0(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{\mathcal{E}^a(\square_m)} - p \cdot \bar{a} \mathbf{1}_{\{\square_m \notin \mathcal{P}_s\}} \leq O_1(C(d, p, \Lambda)|p|^{2.3^{-m}}).
\]

By an argument of Borel-Cantelli, we prove this term converges almost surely to 0. Then we focus on the case \( \square_m \in \mathcal{P}_s \). In fact, in this case the minimiser on \( \mathcal{E}^a(\square_m) \) is the sum of the one on each clusters. Observing that the one on isolated cluster from \( \partial \square_m \) can be null since it has no boundary
condition, so we have to deal with the one on $\mathcal{C}_x(\square_m)$ and the one on the small clusters $\mathcal{C}_x(\square_m)$.

We apply eq. (B.2), the estimate eq. (2.15) and a trivial bound eq. (C.4) to get

$$
\begin{align*}
& \left| \inf_{\square_m \subset \mathcal{P}(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{E^a_d(\square_m)} - p \cdot \bar{a} \right| 1_{\square_m \in \mathcal{P}_*} \\
\leq & \left| \frac{1}{|\mathcal{P}(\square_m)|} \langle \nabla v(\cdot, \square_m, p) \cdot a \nabla v(\cdot, \square_m, p) \rangle_{E^a_d(\mathcal{C}_x(\square_m))} - p \cdot \bar{a} \right| 1_{\square_m \notin \mathcal{P}_*} \\
+ & \left| \frac{1}{|\mathcal{P}(\square_m)|} \langle \nabla v(\cdot, \square_m, p) \cdot a \nabla v(\cdot, \square_m, p) \rangle_{E^a_d(\mathcal{C}_x(\square_m))} - p \cdot \bar{a} \right| 1_{\square_m \in \mathcal{P}_*} \\
+ & \left| \frac{1}{|\mathcal{P}(\square_m)|} \inf_{v \in \mathcal{P}(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{E^a_d(\mathcal{C}_x(\square_m))} \right| 1_{\square_m \in \mathcal{P}_*} \\
\leq & \left| \frac{1}{|\mathcal{P}(\square_m)|} \langle \nabla v(\cdot, \square_m, p) \cdot a \nabla v(\cdot, \square_m, p) \rangle_{E^a_d(\mathcal{C}_x(\square_m))} - p \cdot \bar{a} \right| + O_1(C(d, p, \Lambda)|p|^{2\gamma - m}).
\end{align*}
$$

So its almost sure limit is the same as the first one when $m \to \infty$.

By a similar calculation, one can prove a variant of eq. (C.6) that reads

$$
(C.9) \quad p \cdot \bar{a} \overset{a.s}{=\lim_{m \to \infty}} \frac{1}{|\square_m|} \inf_{v \in \mathcal{P}(\square_m)} \langle \nabla v \cdot a \nabla v \rangle_{E^a_d(\square_m)},
$$

and we recall, see [38, Theorem 9.1], that this definition coincides with eq. (C.7). By a calculus of variation argument, we have

$$
\forall p, q \in \mathbb{R}^d, \quad q \cdot \bar{a} = \mathbb{E}[D(\phi_q + l_q) \cdot a \cdot D(\phi_p + l_p)].
$$

Moreover, observing that $1_{\{a \neq 0\}} D \phi_q + q$ and $a \cdot D(\phi_p + p)$ are stationary, and the former is a the gradient and the latter is divergence free, we can use the Div-Curl and Birkhoff theorems

$$
q \cdot \bar{a} = \mathbb{E}[1_{\{a \neq 0\}} D \phi_q + q] \mathbb{E}[a \cdot D(\phi_p + p)] = q \cdot \mathbb{E}[a \cdot D(\phi_p + p)].
$$

This concludes the equivalence with eq. (C.8). \qed

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