STABILITY OF MARTINGALE OPTIMAL TRANSPORT AND WEAK OPTIMAL TRANSPORT

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Abstract. Under mild regularity assumptions, the transport problem is stable in the following sense: if a sequence of optimal transport plans \( \pi_1, \pi_2, \ldots \) converges weakly to a transport plan \( \pi \), then \( \pi \) is also optimal (between its marginals).

Alfonsi, Corbetta and Jourdain [1] asked whether the same property is true for the martingale transport problem. This question seems particularly pressing since martingale transport is motivated by robust finance where data is naturally noisy. On a technical level, stability in the martingale case appears more intricate than for classical transport since martingale optimal transport plans are not characterized by a ‘monotonicity’-property of their supports.

In this paper we give a positive answer and establish stability of the martingale transport problem. As a particular case, this recovers the stability of the left curtain coupling established by Juillet [35]. An important auxiliary tool is an unconventional topology which takes the temporal structure of martingales into account. Our techniques also apply to the weak transport problem introduced by Gozlan, Roberto, Samson and Tetali.

Keywords: stability, martingale transport, weak transport, causal transport, weak adapted topology, robust finance.

1. Introduction and main results

Let \( X \) and \( Y \) be Polish spaces and consider a continuous function \( c : X \times Y \to [0, \infty) \). Given probability measures \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \), the classical transport problem is

\[
\inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) \pi(dx, dy),
\]

where \( \Pi(\mu, \nu) \) denotes the set of couplings with \( X \)-marginal \( \mu \) and \( Y \)-marginal \( \nu \). A classical result in optimal transport asserts that \( \pi \in \Pi(\mu, \nu) \) is optimal for \((OT)\) iff its support \( \text{supp} \pi \) is \( c \)-cyclically monotone [42, 43]. One useful consequence of this characterization of optimality is the stability of \((OT)\) with respect to the marginals \( \mu, \nu \) as well as the cost function \( c \). Indeed, the link between monotonicity and stability becomes apparent once one realizes that the notion of monotonicity is itself stable.

In this article we consider the martingale optimal transport problem from the point of view of monotonicity and stability. In fact, since this problem is an instance of a weak optimal transport problem (where the cost is singular, i.e., also takes the value \(+\infty\)), we will likewise study the latter class of problems for regular cost functions from this viewpoint.

1.1. Stability of martingale optimal transport. The martingale optimal transport problem is a variant of \((OT)\) stemming from robust mathematical finance (cf. [32, 13, 41, 23, 20, 17, 11, 7, 34, 35, 38, 19, 33, 24, 30] among many others). In order to define this problem, we take \( X = Y = \mathbb{R} \), suppose that \( \mu, \nu \) have finite first moments, and introduce the set \( \Pi_M(\mu, \nu) \) of martingale couplings with marginals \( \mu, \nu \). To be precise, a transport plan \( \pi \) is a martingale coupling iff

\[
\int_{\mathbb{R}} y \pi_x(dy) = x \quad \mu\text{-a.s.},
\]

where \( \{\pi_x\}_{x \in \mathbb{R}} \) denotes a regular disintegration of the second coordinate given the first one. By a famous result of Strassen, the set \( \Pi_M(\mu, \nu) \) is non-empty iff \( \mu \) is smaller than \( \nu \) in......
convex order. The martingale optimal transport problem\(^1\) is given by

\[ \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \pi(dx, dy), \tag{MOT} \]

under the convention that the infimum over the empty set is $+\infty$.

The main result of the article is the stability of (MOT). This gives a positive answer to the question posed by Alfonsi, Corbetta and Jourdain in [3, Section 5.3] in the case $d = 1$.

Let $r \geq 1$. We denote by $\mathcal{P}_r(\mathbb{R})$ the set of probability measures with finite $r$-th moments and by $\mathcal{W}_r$, the topology of $r$-Wasserstein convergence on $\mathcal{P}_r(\mathbb{R})$, cf. [42].

**Theorem 1.1 (MOT Stability).** Let $c, c_k : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$, $k \in \mathbb{N}$, be continuous cost functions such that $c_k$ converges uniformly to $c$. Let $\{\mu_k\}_{k \in \mathbb{N}}, \{\nu_k\}_{k \in \mathbb{N}}$ be sequences in $\mathcal{P}_1(\mathbb{R})$ converging in $\mathcal{W}_1$ to $\mu$ and $\nu$, respectively. For each $k \in \mathbb{N}$ let $\pi^k \in \Pi_M(\mu_k, \nu_k)$ be an optimizer of (MOT) with cost $c_k$ between the marginals $\mu_k$ and $\nu_k$. If $c(x, y) \leq a(x) + b(y)$ with $a \in L^1(\mu)$, $b \in L^1(\nu)$, and

\[ \lim_{k \to \infty} \sup \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y) \pi^k(dx, dy) < \infty, \]

then any weak accumulation point of $\{\pi^k\}_{k \in \mathbb{N}}$ is an optimizer of (MOT) for the cost function $c$. In particular if the latter has a unique optimizer $\pi$, then $\pi^k \rightharpoonup \pi$ weakly.

**Corollary 1.2.** Let $c, c_k : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$, $k \in \mathbb{N}$, be continuous cost functions such that $c_k$ converges uniformly to $c$. Let $\{\mu_k\}_{k \in \mathbb{N}}, \{\nu_k\}_{k \in \mathbb{N}}$ be sequences in $\mathcal{P}_1(\mathbb{R})$ converging in $\mathcal{W}_r$, to $\mu$ and $\nu$, respectively, and $\mu_k$ is smaller in convex order than $\nu_k$, $k \in \mathbb{N}$. Suppose that $c(x, y) \leq K (1 + |x|^r + |y|^r)$, for some $K > 0$.

Then we have

\[ \lim_{k \to \infty} \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y) \pi(dx, dy) = \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \pi(dx, dy). \]

We remark that Juillet has obtained in [35] the stability of the left-curtain coupling and hence stability for martingale transport for specific costs. These results are recovered as particular cases of our main result.

Guo and Oblój in [28] introduce and study the convergence of a computational method for martingale transport where the marginals are discretely approximated and the martingale constraint is allowed to fail with a vanishing error.\(^2\) In independent work Wiesel [44] proved stability of the value function of the martingale optimal transport problem in 1 dimension by estimating the distance between an arbitrary coupling to its projection w.r.t. the adapted Wasserstein distance. For further details on the adapted Wasserstein distance we refer to [5].

### 1.2. Stability of optimal weak transport

Gozlan, Roberto, Samson and Tetali [26] proposed the following non-linear generalization of (OT). Given a cost function $C : X \times \mathcal{P}(Y) \rightarrow \mathbb{R}$ the optimal weak transport problem is

\[ \inf_{\pi \in \Pi(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx). \tag{WOT} \]

\(^1\)The multidimensional version of the martingale transport problem is defined analogously, although the mathematical finance application is less clear.

\(^2\)Note that the updated version [29] (listed on arxiv.org on April 8th 2019) shows in Proposition 4.7 that the optimal value of (MOT) is continuous w.r.t. $(\mu, \nu) \in \mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$ provided that $\mathcal{P}_2(\mathbb{R}) \times \mathcal{P}_2(\mathbb{R})$ is equipped with $\mathcal{W}_2$-convergence and $c$ is assumed to be Lipschitz continuous.

The authors of the present article decided to post this article on arxiv.org concurrently to emphasize the independence of our work. We also note that the main focus of [23] [29] lies on the numerics of martingale transport. In contrast to Theorem 1.1 the proof of [29] Proposition 4.7 is based on the dual problem rather than stability of the optimal couplings.
Observe that one may consider cost functions of the form

\[
C_M(x, p) := \begin{cases} \int_0^\infty c(x, y) \, p(dy) & x, \\ +\infty & \text{else}, \end{cases}
\]

and in this way (MOT) is a special case of (WOT).

While the original motivation for (WOT) mainly stems from applications to geometric inequalities (cf. Marton [32, 36] and Talagrand [39, 40]), weak transport problems appear also in a number of further topics, including martingale transport [2, 4, 15, 7, 9], the causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5]. In fact, recently some works have considered non-linear martingale transport problems for cost functionals causal transport problem [8, 1], and stability in mathematical finance [5].

The second main contribution of the article is the stability of optimal weak transport. Throughout the article we fix a compatible metric on \( X \). We briefly review this idea since it points to the right notion of monotonicity which will be useful in proving the above stability results.

**Theorem 1.3 (WOT Stability).** Let \( C, C_k : X \times \mathcal{P}_1(Y) \to [0, \infty), \ k \in \mathbb{N}, \) be continuous cost functions such that \( C_k \) converges uniformly to \( C \), and either one of the following holds

(a) \( |C(x, \cdot): x \in X| \subseteq C(\mathcal{P}_1(Y)) \) is an equicontinuous family of convex functions,

(b) \( \mu \in \mathcal{P}_1(X) \), and there is a constant \( K > 0 \), \( x_0 \in X, y_0 \in Y \) such that

\[
C(x, p) \leq K \left( 1 + d_Y(x, x_0)' + \int d_Y(y, y_0)' \, p(dy) \right).
\]

Let \( \{\mu_k\} \subseteq \mathcal{P}(X) \) and \( \{\nu_k\} \subseteq \mathcal{P}(Y) \), which converge weakly to \( \mu \) and in \( \mathcal{W}_r \) to \( \nu \), respectively. For each \( k \in \mathbb{N} \) let \( \pi^k \in \Pi(\mu_k, \nu_k) \) be an optimizer of (WOT) with cost function \( C_k \) between the marginals \( \mu_k \) and \( \nu_k \). If

\[
\lim_{k \to \infty} \sup \int_X C_k(x, \pi^k_x) \mu_k(dx) < \infty,
\]

then any weak accumulation point of \( \{\pi^k\} \subseteq \mathcal{P}(Y) \) is an optimizer of (WOT) for the cost function \( C \). In particular if the latter has a unique optimizer \( \pi \), then \( \pi^k \to \pi \) weakly.

We now describe the main idea used in the proofs of Theorems 1.1 and 1.3.

### 1.3. Monotonicity and the correct topology on the set of couplings.

The article [9] investigates the optimal weak transport problem by essentially enlarging the original state space \( X \times Y \) to \( X \times \mathcal{P}(Y) \). We briefly review this idea since it points to the right notion of monotonicity which will be useful in proving the above stability results.

First we introduce the embedding map

\[
J : \mathcal{P}(X \times Y) \to \mathcal{P}(X \times \mathcal{P}(Y)), \quad \pi \mapsto \text{proj}_X(\pi)(dx) \delta_x(dp),
\]

in words, if \( (X, Y) \sim \pi \) then \( (X, \pi_x) \sim J(\pi) \), and its left-inverse, the \( X \times Y \)-intensity map \( \hat{I} \),

\[
\hat{I} : \mathcal{P}(X \times \mathcal{P}(Y)) \to \mathcal{P}(X \times Y),
\]

\[
P \mapsto \int_{\mathcal{P}(Y)} p(dy) \, P(dx, dp).
\]

Despite the fact that \( J \) is seldom continuous, it does enjoy a key property: it preserves the relative compactness of a set. As a consequence, one can easily obtain optimizers for the following suitable extension of (WOT) as soon as \( C \) is lower semicontinuous:

\[
\inf_{P \in \Lambda(\mu, \nu)} \int_{X \times \mathcal{P}(Y)} C(x, p) \, P(dx, dp), \quad \text{(WOT')}
\]
where $\Lambda(\mu, \nu)$ is the set of couplings $P \in \mathcal{P}(X \times \mathcal{P}(Y))$ with $\hat{H}(P) \in \Pi(\mu, \nu)$. When $C(x, \cdot)$ is furthermore convex, the extended problem $\text{(WOT)}$ is equivalent to the original one, and in addition, $\text{(WOT)}$ can be shown to admit an optimizer by means of the natural projection operator $\hat{H}$ from $\mathcal{P}(X \times \mathcal{P}(Y))$ onto $\mathcal{P}(X \times Y)$.

This idea of using an embedding (which preserves relative compactness) into a larger space can be appreciated in the following terms: On the original space $\mathcal{P}(X \times Y)$ we consider the initial topology of $J$ when the target space is given the weak topology. This initial topology has been studied in [6, 5] and given the name adapted weak topology. One immediate observation is that the cost functional of $\text{(WOT)}$ is lower semicontinuous. Since under the stated conditions it makes no difference to work with $\text{(WOT)}$ or $\text{(WOT')}$, it helps to consider the latter simpler linear problem in order to inform our intuition. By analogy with optimal transport we define

**Definition 1.4 (C-monotonicity).** A coupling $\pi \in \Pi(\mu, \nu)$ is C-monotone if there exists a $\mu$-full set $\Gamma \subseteq X$ such that for any finite number of points $x_1, \ldots, x_N \in \Gamma$ and $q_1, \ldots, q_N \in \mathcal{P}(Y)$ with $\sum_{i=1}^N \pi_{x_i} = \sum_{i=1}^N q_i$, we have

$$\sum_{i=1}^N C(x_i, \pi_{x_i}) \leq \sum_{i=1}^N C(x_i, q_i).$$

It was shown under mild assumptions in [9] that optimality of $\pi$ for $\text{(WOT)}$ implies $C$-monotonicity in the sense of Definition 1.4 above. The reverse implication(sufficiency) was shown to be true under the additional assumption that the cost function $C$ is uniformly $\mathcal{W}_1$-Lipschitz in the second argument. In the present article we will generalize this result (and largely simplify the arguments) in Theorem 2.2. Once we are equipped with this necessary and sufficient criterion for optimality, the stability result Theorem 1.3 becomes a consequence of the fact that the notion of $C$-monotonicity is itself stable.

Although martingale optimal transport is a particular case of optimal weak transport, in this work we treat the two problems separately. The reason for this lies in the fact that our approach to optimal weak transport requires regularity of the cost, which the ‘embedded cost’ $C_M$, see [11], does not provide any longer. To deal with the singular cost $C_M$, we refine the notion of $C$-monotonicity when it comes to martingale couplings and additionally employ techniques which currently only work in dimension one. We define

**Definition 1.5 (Martingale C-monotonicity).** A coupling $\Pi_M(\mu, \nu)$ is martingale C-monotone if there exists a $\mu$-full set $\Gamma \subseteq \mathbb{R}^d$ such that for any finite number of points $x_1, \ldots, x_N \in \Gamma$ and $q_1, \ldots, q_N \in \mathcal{P}_1(\mathbb{R}^d)$ with $\sum_{i=1}^N \pi_{x_i} = \sum_{i=1}^N q_i$ and $\int_{\mathbb{R}^d} y q_i(\mathrm{d}y) = x_i$, we have

$$\sum_{i=1}^N C(x_i, \pi_{x_i}) \leq \sum_{i=1}^N C(x_i, q_i).$$

Then the key to proving Theorem 1.1 boils down to two arguments: that martingale C-monotonicity is sufficient for optimality, and that this notion of monotonicity is itself stable.

The idea that a monotonicity viewpoint for martingale optimal transport has to be based on the disintegrations of a coupling, or put otherwise, that optimality cannot be read-off from the support of a martingale coupling, goes to Juillet [35]. This author in fact showed that for the left-curtain coupling (c.f. [14]) the support of an optimizer may contain suboptimal paths; see Figure 1 for an illustration. This is in stark contrast with classical optimal transport, where for say continuous bounded cost functions $c$ it is known that $c$-cyclical monotonicity of the support is equivalent to optimality. This failure ultimately underlies the need for our Definition 1.5.

### 1.4. Outline.

The article is written so that the sections concerning optimal weak transport and martingale optimal transport are largely independent. Section 2 deals with the stability of optimal weak transport, whereas Section 3 explores the stability of martingale optimal transport.
transport. Section 4 studies the relationship between classical optimal transport and optimal weak transport. Finally Section 5 wraps up the article with a number of remarks, followed by an appendix containing auxiliary results.

2. ON THE WEAK TRANSPORT PROBLEM

Let us complement Definition 1.4 with a more complete list of monotonicity properties: see Remark 2.4(a) for a comparison of these definitions.

**Definition 2.1.**

1. We call \( \Gamma \subseteq X \times \mathcal{P}(Y) \) C-monotone iff for any finite number of points

\[
(x_1, p_1), \ldots, (x_N, p_N) \in \Gamma \text{ and } q_1, \ldots, q_N \in \mathcal{P}(Y) \text{ with } \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i,
\]

we have

\[
\sum_{i=1}^{N} C(x_i, p_i) \leq \sum_{i=1}^{N} C(x_i, q_i).
\]

2. A probability measure \( P \in \mathcal{P}(X \times \mathcal{P}(Y)) \), which is concentrated on a C-monotone set, is called C-monotone.

The next theorem greatly extends the sufficiency of C-monotonicity first obtained in [9, Theorem 5.5]. We recall that \( \mathcal{P}_r(Y) \) denotes the space of probability measures on \( Y \) with finite \( r \)-th moment, \( r \geq 1 \), i.e., \( p \in \mathcal{P}_r(Y) \) iff \( p \in \mathcal{P}(Y) \) and

\[
\int_Y d_y(y, y_0)^r \, p(dy) < \infty
\]

for some \( y_0 \in Y \), and thus all \( y_0 \in Y \). It is well-known that \( \mathcal{P}_r(Y) \) equipped with the topology of the Wasserstein metric \( \mathcal{W}_r \) becomes a Polish space, where \( \mathcal{W}_r(p, q) \) for \( p, q \in \mathcal{P}_r(Y) \) is defined via

\[
\mathcal{W}_r(p, q)^r = \inf_{\pi \in \Pi(p, q)} \int_{Y \times Y} d_y(y_1, y_2)^r \, \pi(dy_1, dy_2).
\]

Furthermore, we fix on \( X \times \mathcal{P}_r(Y) \) the metric \( d((x, p), (x', p')) := d_x(x, x') + \mathcal{W}_r(p, p') \) for \((x, p), (x', p') \in X \times \mathcal{P}_r(Y)\), and denote by \( \mathcal{P}_r(X \times \mathcal{P}_r(Y)) \) the space of probability measures on \( X \times \mathcal{P}_r(Y) \), which finitely integrate \((x, p) \mapsto d((x, p), (x', p')) \) for given \((x', p') \in X \times \mathcal{P}_r(Y)\), equipped with the \( \mathcal{W}_r \)-metric associated with the metric \( d \). Similarly, we fix on \( \mathcal{P}_r(Y) \) the Wasserstein metric \( \mathcal{W}_r \), and denote by \( \mathcal{P}_r(\mathcal{P}_r(Y)) \), the space of probability measures on \( \mathcal{P}_r(Y) \), which finitely integrate \( p \mapsto \mathcal{W}_r(p, p') \) for given \( p' \in \mathcal{P}_r(Y) \), equipped with the \( r \)-th Wasserstein topology. Note that these definitions are independent of the choice of \((x', p') \in X \times \mathcal{P}_r(Y)\) and \( p' \in \mathcal{P}_r(Y) \), respectively.
Recall the definition of the set $\Lambda(\mu, \nu)$ given after \textbf{WOT}, i.e.,

$$\Lambda(\mu, \nu) := \left\{ P \in \mathcal{P}(X \times \mathcal{P}(Y)) : \hat{R}(P) \in \Pi(\mu, \nu) \right\}. \quad (2.1)$$

**Theorem 2.2.** Let $\mu \in \mathcal{P}(X), \nu \in \mathcal{P}_r(Y), C : X \times \mathcal{P}_r(Y) \to \mathbb{R}$ measurable. Assume either of the following conditions:

(a) \(\{C(x, \cdot) : x \in X\} \subseteq \mathcal{C}(\mathcal{P}_r(Y))\) is equicontinuous continuous, i.e.

$$\theta(\delta) := \sup \left\{ |C(x, p) - C(x, q)| : x \in X, (p, q) \in \mathcal{P}_r(Y)^2 \text{ s.t. } \mathcal{W}_r(p, q) \leq \delta \right\},$$

vanishes for $\delta \searrow 0$.

(b) $\mu \in \mathcal{P}_r(X), C$ is jointly continuous and there is $K \in \mathbb{R}, x_0 \in X, y_0 \in Y$ s.t. for all $x \in X$:

$$C(x, p) \leq K \left( 1 + d_2(x, x_0)^r + \int d_2(y, y_0) \cdot p(dy) \right).$$

Then $P \in \Lambda(\mu, \nu)$ is optimal for \textbf{WOT} iff $P$ is $C$-monotone. Similarly, if $p \mapsto C(x, p)$ is convex, then $\pi \in \Pi(\mu, \nu)$ is optimal for \textbf{WOT} iff $\pi$ is $C$-monotone.

**Remark 2.3.** Note that under the assumptions of Theorem \textbf{2.2} $C$-monotonicity of $\pi \in \Pi(\mu, \nu)$ yields $C$-monotonicity of $J(\pi) \in \Lambda(\mu, \nu)$, thus, $J(\pi)$ is optimal for \textbf{WOT} and particularly $\pi$ is optimal for \textbf{WOT}. We will complete this discussion in Remark \textbf{2.4}.

At the same time, we see in Section \textbf{5} Example \textbf{5.1} that convexity of $p \mapsto C(x, p)$ is necessary for optimal couplings being $C$-monotone. Example \textbf{5.2} shows that additional regularity (to lower semicontinuity and boundedness) of the cost $C$ is required for $C$-monotonicity being a sufficient optimality criterion.

**Proof.** Let $P$ be $C$-monotone concentrated on the $C$-monotone set $\Gamma$. Fix $P' \in \Lambda(\mu, \nu)$. We argue as in \textbf{[10]} for classical (linear) optimal transport. Take any iid sequences $(X_n)_{n \in \mathbb{N}}$ of $X$-valued random variables, and any iid sequences $(Y_n)_{n \in \mathbb{N}}, (Z_n)_{n \in \mathbb{N}}$ of $\mathcal{P}(Y)$-valued random variables, on some probability space $(\Omega, \mathcal{F})$, with

$$(X_n, Y_n) \sim P, \quad (X_n, Z_n) \sim P'. \quad (2.2)$$

In particular by the law of large numbers, we find $\mathbb{P}$-almost surely

$$\int_{X \times \mathcal{P}_r(Y)} C(x, p)P'(dx, dp) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N C(X_n, Z_n) - C(X_n, Y_n).$$

Note that for any function $g \in \mathcal{C}(Y)$, which is majorized by $y \mapsto 1 + d_2(y, y_0)^r$, we have

$$\mathbb{E}[Y_n(g)] = v(g),$$

where $p(g)$ for $p \in \mathcal{P}(Y)$ denotes the integral $\int_{Y} g(y) p(dy)$. Then the law of large numbers implies almost surely

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N Y_n(g) = v(g) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N Z_n(g).$$

By standard separability arguments, we find $\mathbb{P}$-almost surely

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N Y_n = v = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N Z_n \quad \mathbb{P}\text{-a.s.} \quad (2.3)$$

where convergence holds in $\mathcal{W}_r$. Let $\omega \in \Omega$ be in a $\mathbb{P}$-full set s.t.

$$\lim_{N \to \infty} \mathcal{W}_r \left( \frac{1}{N} \sum_{n=1}^N Y_n(\omega), \frac{1}{N} \sum_{n=1}^N Z_n(\omega) \right) = 0,$$
and \((X_n, Y_n)(\omega) \in \Gamma\) for all \(n \in \mathbb{N}\). From now on we omit the \(\omega\) argument. For each \(N \in \mathbb{N}\), we denote a \(\mathcal{W}_r\)-optimal coupling in \(\Pi\left(\frac{1}{N} \sum_{n=1}^{N} Z_n, \frac{1}{N} \sum_{n=1}^{N} Y_n\right)\) by \(\chi_N\), i.e.

\[
\mathcal{W}_r\left(\frac{1}{N} \sum_{n=1}^{N} Z_n, \frac{1}{N} \sum_{n=1}^{N} Y_n\right) = \int_{\chi_{XY}} d\gamma(z, y)^r \chi_N^r(dz, dy).
\]

We denote by \(\{\chi_N^r\}_{r \in \mathbb{R}}\) a regular disintegration of \(\chi_N^r\) given its projection in the first coordinate (marginal). Defining \(\chi_N^r Z_n(dy) := \int_{\mathbb{R}} z \chi_N^r(dy)\), we find

\[
\frac{1}{N} \sum_{n=1}^{N} \mathcal{W}_r(Z_n, \chi_N^r Z_n)^r \leq \frac{1}{N} \sum_{n=1}^{N} \int_{\mathbb{R}} d\gamma(z, y)^r \chi_N^r(dy) Z_n(dy) = \mathcal{W}_r\left(\frac{1}{N} \sum_{n=1}^{N} Z_n, \frac{1}{N} \sum_{n=1}^{N} Y_n\right)^r.
\]

Moreover, \(\sum_{n=1}^{N} \chi_N Z_n = \sum_{n=1}^{N} Y_n\), so by C-monotonicity

\[
\frac{1}{N} \sum_{n=1}^{N} C(X_n, \chi_N Z_n) - C(X_n, Y_n) \geq 0. \tag{2.5}
\]

In the case of (a), we apply Lemma [6.2] and get \(\tilde{\theta} : \mathbb{R}^+ \to \mathbb{R}^+\), which is continuous, concave, and vanishing at 0, and which we rename \(\theta\) for simplicity. Hence, by Jensen’s inequality,

\[
\frac{1}{N} \sum_{n=1}^{N} \theta\left(\mathcal{W}_r(Z_n, \chi_N^r Z_n)^r\right) \leq \theta\left(\frac{1}{N} \sum_{n=1}^{N} \mathcal{W}_r(Z_n, \chi_N^r Z_n)^r\right) \leq \theta\left(\mathcal{W}_r\left(\frac{1}{N} \sum_{n=1}^{N} Z_n, \frac{1}{N} \sum_{n=1}^{N} Y_n\right)^r\right),
\]

which vanishes as \(N \to +\infty\). Using C-monotonicity of \(P \) and uniform continuity, we obtain \(\mathbb{P}\)-almost surely

\[
\frac{1}{N} \sum_{n=1}^{N} C(X_n, Z_n) - C(X_n, Y_n)
= \frac{1}{N} \sum_{n=1}^{N} C(X_n, Z_n) - C(X_n, \chi_N^r Z_n) + \frac{1}{N} \sum_{n=1}^{N} C(X_n, \chi_N^r Z_n) - C(X_n, Y_n)
\geq -\frac{1}{N} \sum_{n=1}^{N} \theta(\mathcal{W}_r(Z_n, \chi_N^r Z_n))
\geq -\theta\left(\mathcal{W}_r\left(\frac{1}{N} \sum_{n=1}^{N} Y_n(\omega), \frac{1}{N} \sum_{n=1}^{N} Z_n(\omega)\right)^r\right) \to 0. \tag{2.6}
\]

In the case of (b), we define the random variable \(P'_N\) taking values in \(X \times \mathcal{P}(Y)\) by

\[
P'_N := \frac{1}{N} \sum_{n=1}^{N} \delta_{X_n, \chi_N^r Z_n}.
\]

By (2.4) we find \(\mathbb{P}\)-almost surely

\[
\mathcal{W}_r(P'_N, P') \leq \frac{1}{N} \sum_{n=1}^{N} \mathcal{W}_r(Z_n, \chi_N^r Z_n) \leq \mathcal{W}_r\left(\frac{1}{N} \sum_{n=1}^{N} Z_n, \frac{1}{N} \sum_{n=1}^{N} Y_n\right)^r,
\]

the right-hand side of which converges a.s. to zero as we have already seen in (2.3) \(^3\). Then, continuity and growth of \(C\) yields

\[
\int_{X \times \mathcal{P}(Y)} C(x, p) P'(dx, dp) - \int_{X \times \mathcal{P}(Y)} C(x, p) P'_N(dx, dp) \to 0. \tag{2.7}
\]

\(^3\)Here, the Wasserstein distance on the left-hand side of the equation is taken on the space \(X \times \mathcal{P}(Y)\) w.r.t. the metric \(d((x_1, p_1), (x_2, p_2))^r = d(x_1, x_2)^r + \mathcal{W}_r(p_1, p_2)^r\).
Hence, in both cases we have \( P \)-almost surely
\[
\int_{X \times \mathcal{P}(Y)} C(x, p) P'(dx, dp) - \int_{X \times \mathcal{P}(Y)} C(x, p) P(dx, dp) \geq \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} C(X_n, Y_n) - C(X_n, Y_n) \geq 0,
\]
where we used (2.2), (2.5), (2.6), and (2.7). \( \square \)

2.1. **Stability of C-Monotonicity.** Recall the embedding of (1.2)
\[
J: \mathcal{P}(X \times Y) \to \mathcal{P}(X \times \mathcal{P}(Y)),
\]
\[
\pi \mapsto \text{proj}_Y(\pi)(dx) \delta_x(dm).
\]
The intensity \( I(Q) \) of some measure \( Q \in \mathcal{P}(\mathcal{P}(Y)) \) is uniquely defined as the probability measure \( I(Q) \in \mathcal{P}(Y) \) with
\[
I(Q)(f) = \int_{\mathcal{P}(Y)} \int_{Y} f(y)p(dy) Q(dp) \quad \forall f \in C_b(Y),
\]
that is \( I(Q)(dy) = \int_{\mathcal{P}(Y)} p(dy) Q(dp) \).

**Remark 2.4.** In the light of this embedding it appears to be natural to consider C-monotonicity on the enhanced space \( X \times \mathcal{P}(Y) \).

(a) \( \pi \in \Pi(\mu, \nu) \) is C-monotone iff \( J(\pi) \) is C-monotone: On the one hand, if \( \pi \) is C-monotone it is possible to find a measurable set \( \Gamma \subseteq X \) such that \( \mu(\Gamma) = 1 \) which defines via
\[
\Gamma = \{(x, p) \in \Gamma \times \mathcal{P}(Y): p = \pi_x\},
\]
a C-monotone set. Therefore, equivalently to Definition 2.1, we can demand that there exists a C-monotone set \( \Gamma \subseteq X \times \mathcal{P}(Y) \) such that \( (x, \pi_x) \in \Gamma \) for \( \mu \)-almost every \( x \in X \).

On the other hand, if \( J(\pi) \) is C-monotone, then there exists a C-monotone set \( \Gamma \)
\[
\Gamma = \{(x, p) \in \Gamma \times \mathcal{P}(Y): p = \pi_x\} = \mu(\Gamma) = 1.
\]
Consider the \( \mu \)-full, analytically measurable set \( \{ x \in X: (x, \pi_x) \in \Gamma \} \). As analytically measurable sets are universally measurable, it admits a Borel measurable subset \( \bar{\Gamma} \) with \( \mu(\bar{\Gamma}) = 1 \). Thus, \( \pi \) is C-monotone (on \( \bar{\Gamma} \)) in the sense of Definition 1.3.

(b) If \( \Gamma \subseteq X \times \mathcal{P}(Y) \) is C-monotone, and \( C: \mathcal{P}(Y) \to \mathbb{R} \) is convex in the second argument, i.e., for all \( x \in X \), \( (p, q) \in \mathcal{P}(Y)^2 \), and \( a \in [0, 1] \)
\[
C(x, ap + (1-a)q) \leq aC(x, p) + (1-a)C(x, q),
\]
then the enlarged set
\[
\hat{\Gamma} := \left\{ \left(x, \frac{1}{k} \sum_{i=1}^{k} p_i \right): x \in X, (x, p_i) \in \Gamma, i = 1, \ldots, k \in \mathbb{N} \right\}
\]
is also C-monotone. Likewise, if \( C(x, \cdot) \) is further continuous for all \( x \in X \), then
\[
\hat{\Gamma} := \{(x, p) \in X \times \mathcal{P}(Y): x \in X, p \in \text{co}(\Gamma_x)\},
\]
is C-monotone, where \( \text{co} \) stands for the closed convex hull and \( \Gamma_x \) denotes the \( x \)-fibre of \( \Gamma \), that is \( \{ p \in \mathcal{P}(Y): (x, p) \in \Gamma \} \).

(c) We observe that the set \( \Lambda(\mu, \nu) \) (see (2.1)) can be characterized by a family of continuous functions \( F \subseteq C(X \times \mathcal{P}(Y)): P \in \Lambda(\mu, \nu) \) if and only if
\[
\int_{X \times \mathcal{P}(Y)} f(x) P(dx, dp) = \int_X f(x) \mu(dx) \quad \forall f \in C_b(X),
\]
\[
\int_{X \times \mathcal{P}(Y)} \int_Y g(y) p(dy) P(dx, dp) = \int_Y g(y) \nu(dy) \quad \forall g \in C_b(Y).
\]
As a further observation we have the equivalence of C-monotonicity as in Definition 2.1 and C-finite optimality under the linear constraints \( \mathcal{F} \)

\[
\mathcal{F} = \{ f \in C_b(X \times \mathcal{P}(Y)) : \exists g \in C_b(X), h \in C_b(Y) \text{ s.t. } f(x, p) \equiv g(x) \text{ or } f(x, p) = \int h(y) p(dy) \},
\]

which was introduced in [12, Definition 1.2].

**Theorem 2.5.** Let \( C : X \times \mathcal{P}_r(Y) \to [0, \infty] \) be measurable and \( P^* \in \mathcal{P}_r(X \times \mathcal{P}_r(Y)) \) optimal for \( \text{(WOT)} \) with finite value. Then \( P^* \) is C-monotone. In particular, if \( C \) satisfies for all \( x \in X \) and \( Q \in \mathcal{P}(\mathcal{P}(Y)) \)

\[
C(x, I(Q)) \leq \int_{\mathcal{P}(Y)} C(x, p) Q(dp), \tag{2.8}
\]

then any optimizer \( \pi^* \) of \( \text{(WOT)} \) with finite value is C-monotone.

**Proof.** The first assertion is a consequence of Remark 2.4(c) and [12, Theorem 1.4]. To show the second assertion, let \( P \in \Lambda(\mu, \nu) \). Then \( I(P_i) \mu(dx) \in \Pi(\mu, \nu) \) and by (2.8)

\[
\int_{X \times \mathcal{P}(Y)} C(x, p) P(dx, dp) \geq \int_X C(x, I(P_i)) \mu(dx).
\]

Hence, \( J(\pi^*) \) is optimal for \( \text{(WOT)} \). We deduce from the first part of the proof combined with Remark 2.4(a) C-monotonicity of \( \pi \). \( \square \)

The assumption that \( C \) is lower bounded is as a matter of fact not necessary to deduce C-monotonicity in the classical optimal weak transport setting, cf. [9, Theorem 5.2]. Note that (2.8) holds when \( C(x, \cdot) \) is lower semicontinuous and convex. Indeed, by similar approximation arguments as in Theorem 2.2 we find for any \( Q \in \mathcal{P}(\mathcal{P}(Y)) \) a sequence of measures \( \{p_i\}_{i \in \mathbb{N}} \subseteq \mathcal{P}(Y) \) such that

\[
\frac{1}{N} \sum_{i=1}^{N} p_i \to I(Q) \quad \text{in } \mathcal{W}_r, \quad \frac{1}{N} \sum_{i=1}^{N} C(x, p_i) \to \int_{X \times \mathcal{P}(Y)} C(x, p) Q(dp).
\]

Thus,

\[
\int_{X \times \mathcal{P}(Y)} C(x, p) Q(dp) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} C(x, p_i) \geq \liminf_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} p_i \geq C(x, I(Q)).
\]

**Lemma 2.6.** Let \( p_i, m_i \in \mathcal{P}_r(Y), i = 1, \ldots, N, \) with \( \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} m_i, \) and \( \{p_i^k, \ldots, p_N^k\}, k \in \mathbb{N}, \) be sequences in \( \mathcal{P}_r(Y) \) with \( p_i^k \to p_i \) in \( \mathcal{W}_r. \) Then there exist approximative sequences \( \{m_1^k, \ldots, m_N^k\}_{k \in \mathbb{N}} \) of competitors, i.e.,

\[
\sum_{i=1}^{N} p_i^k = \sum_{i=1}^{N} m_i^k \text{ and } m_i^k \to m_i \text{ in } \mathcal{W}_r.
\]

**Proof.** Since \( \sum_{i=1}^{N} m_i = \sum_{i=1}^{N} p_i, \) we find sub-probability measures \( m_{i,j}, \) with the property that \( m_{i,j}(A) \leq \min[p_i(A), m_j(A)] \) for all \( A \) measurable, and

\[
m_j = \sum_{i=1}^{N} m_{i,j}, \quad p_i = \sum_{j=1}^{N} m_{i,j}.
\]

Denote by \( (\chi_{z}^{k,j})_{z \in \mathcal{Z}} \) a regular disintegration of a \( \mathcal{W}_r \)-optimal transport plan \( \chi^{k,j} \in \Pi(p_i, p_i^k) \)

w.r.t. to its first marginal \( p_i. \) Let \( i, j \in \{1, \ldots, N\} \) and define

\[
m_{i,j}^k(dy) := \int_{\mathcal{Z}} \chi_{z}^{k,j}(dy) m_{i,j}(dz), \quad m_i^k := \sum_{j=1}^{N} m_{i,j}^k.
\]
Since
\[ \sum_{j=1}^{N} \mathcal{W}_{r}(m_{i,j}^k, m_{i,j}) \leq \sum_{j=1}^{N} \int_{Y \times Y} d_{Y}(z, y) \chi_{z}^{k,j}(dy) m_{i,j}(dz) \]
we deduce the convergence of \( m_{i,j}^k \) to \( m_{i,j} \), and in consequence, the convergence of \( m_{i}^k \) to \( m_{i} \) in \( \mathcal{W}_{r} \). Finally observe that
\[ \sum_{i=1}^{N} m_{i}^k = \sum_{j=1}^{N} \sum_{i=1}^{N} m_{i,j}^k = \sum_{j=1}^{N} \int_{Y \times Y} \chi_{z}^{k,j}(dy) p_{j}(dz) = \sum_{j=1}^{N} p_{j}^k \]
so indeed \( \{m_{1}^k, \ldots, m_{N}^k\} \) are feasible competitors of \( \{p_{1}^k, \ldots, p_{N}^k\} \) such that for \( i = 1, \ldots, N \), \( m_{i}^k \rightarrow m_{i} \) in \( \mathcal{W}_{r} \).
\[ \square \]

**Lemma 2.7.** Let \( C \in C(X \times \mathcal{P}_{r}(Y)) \), \( \varepsilon \geq 0 \), and \( N \in \mathbb{N} \). Then the set
\[ \Gamma_{N}^{\varepsilon} := \{ (x_{i}, p_{i})_{i=1}^{N} \in (X \times \mathcal{P}_{r}(Y))^{N} | \forall m_{1}, \ldots, m_{N} \in \mathcal{P}_{r}(Y) \text{ s.t. } \sum_{i=1}^{N} p_{i} = \sum_{i=1}^{N} m_{i}, \text{ we have } \sum_{i=1}^{N} C(x_{i}, p_{i}) \leq \sum_{i=1}^{N} C(x_{i}, m_{i}) + \varepsilon \} \]
(2.9)
is a closed subset of \( (X \times \mathcal{P}_{r}(Y))^{N} \).

**Proof.** Take any convergent sequence \( (x_{i}^k, p_{i}^k)_{i=1}^{N} \in \Gamma_{N}^{\varepsilon} \), \( k \in \mathbb{N} \), such that
\[ x_{i}^k \rightarrow x_{i} \text{ in } X, \quad p_{i}^k \rightarrow p_{i} \text{ in } \mathcal{W}_{r} \].
Assume that \( (x_{i}, m_{i})_{i=1}^{N} \) is a competitor, i.e., \( \sum_{i=1}^{N} m_{i} = \sum_{i=1}^{N} m_{i} \). Lemma 2.6 provides an approximative sequence of competitors, and by continuity of \( C \) we conclude. \[ \square \]

The key ingredient towards stability of \( \{\text{WOT}\} \) is the following result concerning stability of the notion of \( C \)-monotonicity.

**Theorem 2.8.** Let \( C, C_{k} \in C(X \times \mathcal{P}_{r}(Y)) \), \( k \in \mathbb{N} \), and \( C_{k} \) converges uniformly to \( C \). If \( P, P^{k} \in \mathcal{P}_{r}(X \times \mathcal{P}_{r}(Y)) \), \( k \in \mathbb{N} \), such that
\begin{enumerate}
  \item[(a)] for all \( k \in \mathbb{N} \) the measure \( P^{k} \) is \( C_{k} \)-monotone,
  \item[(b)] the sequence \( (P^{k})_{k=1}^{\infty} \) converges to \( P \),
\end{enumerate}
then \( P \) is \( C \)-monotone. Moreover, if \( \pi, \pi^{k} \in \mathcal{P}_{r}(X \times Y) \) and \( C_{k} \) is convex in the second argument, \( k \in \mathbb{N} \), such that
\begin{enumerate}
  \item[(a')] for all \( k \in \mathbb{N} \) the measure \( \pi^{k} \) is \( C_{k} \)-monotone,
  \item[(b')] the sequence \( (\pi^{k})_{k=1}^{\infty} \) converges to \( \pi \),
\end{enumerate}
then \( \pi \) is \( C \)-monotone.

**Proof.** The aim is to construct a \( C \)-monotone set \( \Gamma \) on which \( P \) is concentrated. So, we write \( P^{k,\otimes N} \) and \( P^{0,N} \) for the \( N \)-fold product measure of \( P^{k} \) and \( P \) where \( N \in \mathbb{N} \). By \( C_{k} \)-monotonicity and uniform convergence we find for any \( \varepsilon > 0 \) a natural number \( k_{0} \) such that \( P^{k,\otimes N}, k \geq k_{0} \), is concentrated on \( \Gamma_{N}^{\varepsilon} \), see (2.9). Lemma 2.7 combined with the Portmanteau theorem yield that \( P^{0,N} \) is concentrated on \( \Gamma_{N}^{\varepsilon} \):
\[ 1 = \limsup_{k} P^{k,\otimes N}(\Gamma_{N}^{\varepsilon}) \leq P^{0,N}(\Gamma_{N}^{\varepsilon}) = 1. \]
As a consequence, we find that \( P^{0,N} \) gives full measure to the closed set \( \Gamma_{N} := \Gamma_{N}^{0} \). Sets of the form \( \bigotimes_{i=1}^{N} O_{i} \), where \( O_{i} \) is open in \( X \times \mathcal{P}_{r}(Y) \) form a basis of the product topology on
concentrated on the determines a coupling, which is likewise an accumulation point of \( \{\Lambda\} \), respectively, where \( \Lambda \) is open in \( X \times \mathcal{P}_r(Y) \).

\[
\Gamma^c_N = \bigcup_{i=1}^N \bigotimes_{i=1}^N O_{i,k}.
\]

In particular, we deduce for any \( k \in \mathbb{N} \)

\[
0 = P^\otimes N \left( \bigotimes_{i=1}^N O_{i,k} \right) = \prod_{i=1}^N P(O_{i,k}).
\]

We find open sets \( A_N \) such that

\[
A_N := \bigcup_{i=1}^N \bigotimes_{i=1}^N O_{i,k}, \quad P(A_N) = 0,
\]

\[
\Gamma^c_N \subseteq \bigcup_{i=1}^N (X \times \mathcal{P}_r(Y))^{N-1} \times A_N \times (X \times \mathcal{P}_r(Y))^{N-i}.
\]

Since \( N \in \mathbb{N} \) was arbitrary we defined the closed and \( C \)-monotone set

\[
\Gamma := \left( \bigcup_{N \in \mathbb{N}} A_N \right)^c, \quad P(\Gamma) = 1, \quad \Gamma^c \subseteq \Gamma_N.
\]

With the taken precautions it poses no challenge to verify that \( P \) is \( C \)-monotone on \( \Gamma \).

To show the second assertion, we embed \( \pi^t \in \mathcal{P}(X \times Y) \) into \( \mathcal{P}(X \times \mathcal{P}(Y)) \) owing to the map \( J \). Note that \( \Lambda(\mu, \nu) \) is a closed subset of \( \mathcal{P}_r(X \times \mathcal{P}_r(Y)) \). By the same line of argument as in [9, Lemma 2.6] we find relative compactness, thus compactness of \( \Lambda(\mu, \nu) \).

Therefore, we find an accumulation point \( P \in \mathcal{P}(X \times \mathcal{P}(Y)) \) of \( (J(\pi^t))_{t \in \mathbb{N}} \). Note that

\[
\mu(dx) I(P_x) := \pi \in \Pi(\mu, \nu),
\]

determines a coupling, which is likewise an accumulation point of \( \{\pi^t\}_{t \in \mathbb{N}} \). Since \( P \) is concentrated on the \( C \)-monotone set \( \Gamma \), we find for any \( x \in X \) such that \( P_x(\Gamma_x) = 1 \) a sequence of measures \( \pi^t \in \Gamma_x \subseteq \mathcal{P}(Y) \), \( t \in \mathbb{N} \), with

\[
g^n_x := \frac{1}{n} \sum_{i=1}^n \pi^t_i \to I(P_x) = \pi_x, \quad n \to \infty, \quad \text{in} \ \mathcal{W}_r.
\]

By Remark 2.4[b] we know that \( (x, g^n_x) \) is contained in the \( C \)-monotone set \( \Gamma_x \). By closure of \( \Gamma \) we conclude \( (x, \pi_x) \in \Gamma \) for \( \mu \)-a.e. \( x \), and \( C \)-monotonicity of \( \pi_x \).

From Theorems 2.2 and 2.8 we easily deduce the following corollary, which has Theorem 1.3 in the introduction as a particular case:

**Corollary 2.9.** Let \( C, C_k \in \mathcal{C}(X \times \mathcal{P}_r(Y)), k \in \mathbb{N} \), be non-negative cost functions such that \( C_k \) converges uniformly to \( C \) and one of the following holds:

1. \( C(x, \cdot) : x \in X \) is equicontinuous,
2. \( \mu \in \mathcal{P}_r(X) \) and there is a constant \( K > 0 \), \( x_0 \in X \), \( y_0 \in Y \) such that for all \( (x, p) \in X \times \mathcal{P}(X) \)

\[
C(x, p) \leq K \left( 1 + d_k(x, x_0) + \int d_r(y, y_0) \, p(dy) \right).
\]

Let \( \mu_k \) and \( \nu_k \) be two sequences of probability measures on \( \mathcal{P}(X) \) and \( \mathcal{P}_r(Y) \), respectively, where \( \mu_k \) converges weakly to \( \mu \) and \( \nu_k \) converges in \( \mathcal{W}_r \) to \( \nu \). Let \( P^k \in \Lambda(\mu_k, \nu_k) \) be optimizers of \( \text{WOT} \) with cost function \( C_k \) between the marginals \( \mu_k \) and \( \nu_k \). If

\[
\limsup_k P^k(C_k) < \infty,
\]

then any weak accumulation point of \( \{P^k\}_{k \in \mathbb{N}} \) is an optimizer of \( \text{WOT} \) for the cost \( C \).

If moreover \( C_k(x, \cdot) \) and \( C(x, \cdot) \) are convex, then an analogous statement holds in the case of \( \text{WOT} \).
3. Stability of martingale optimal transport

In this section we consider the martingale optimal transport problem (MOT), and \( X = Y = \mathbb{R}^d \). A generalization of \( c \)-cyclical monotonicity under additional linear constraints were suggested by [12, 25], which also encompass (MOT). For (MOT), the set of linear constraints \( \mathcal{F}_M \subseteq C(\mathbb{R}^d \times \mathbb{R}^d) \) takes the shape

\[
\mathcal{F}_M := \{ f \in C(\mathbb{R}^d \times \mathbb{R}^d) : f(x, y) = g(x)(y - x) \text{ and } g \in C_b(\mathbb{R}^d) \}.
\]

**Definition 3.1.**

1. A measure \( \alpha' \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) is called a \( \mathcal{F}_M \)-competitor of \( \alpha \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) iff their marginals coincide and \( \alpha(f) = \alpha'(f) \) for all \( f \in \mathcal{F}_M \).
2. We call \( \Gamma \subseteq \mathbb{R}^d \times \mathbb{R}^d \) \( (c, \mathcal{F}_M) \)-monotone iff for any probability measure \( \alpha \), finitely supported on \( \Gamma \), and any competitor \( \alpha' \), we have \( \alpha(c) \leq \alpha'(c) \).
3. A martingale coupling \( \pi \in \Pi_M \), which is supported on a \( (c, \mathcal{F}_M) \)-monotone set, is called \( (c, \mathcal{F}_M) \)-monotone.

**Remark 3.2.** Definition [3.1], Point (2), implies for any \( (c, \mathcal{F}_M) \)-monotone set \( \Gamma \) that for all sequences \( (x_1, p_1), \ldots, (x_N, p_N) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \) with \( p_i \) finitely supported on the fibre \( \Gamma_{x_i} \) that

\[
\sum_{i=1}^N \int_{\mathbb{R}^d} c(x_i, y) p_i(dy) \leq \sum_{i=1}^N \int_{\mathbb{R}^d} c(x_i, y) q_i(dy), \tag{3.1}
\]

where \( q_1, \ldots, q_N \in \mathcal{P}_1(\mathbb{R}^d) \), \( \sum_{i=1}^N q_i = \sum_{i=1}^N p_i \) and \( \int_{\mathbb{R}^d} y q_i(dy) = \int_{\mathbb{R}^d} y q_i(dy) \).

On the other hand, suppose \( \Gamma \) is a set such that (3.1) holds for all \( N \in \mathbb{N} \), and collections \( (x_1, p_1), \ldots, (x_N, p_N) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \) with \( p_i \) finitely supported on \( \Gamma_{x_i} \). Given any measure \( \alpha \in \mathcal{P}(\Gamma) \) supported on the finite set \( \{(x_1, y_1), \ldots, (x_n, y_n)\} \), we have optimality of \( \{(x_1, \alpha_{x_1}), \ldots, (x_n, \alpha_{x_n})\} \) where \( \alpha_{x_i} = \frac{\alpha_{x_i}(x_i, y_i)}{\alpha_{x_i}(x_i, \mathbb{R}^d)} \) over all competing sequences as in (3.1). Therefore, \( \hat{\alpha}(dx, dy) := \sum_{i=1}^n \delta_{x_i}(dx) \alpha_{x_i}(dy) \) defines an optimal coupling between its marginals under all competitors, i.e., for all \( \gamma \in \Pi(\text{proj}_1(\hat{\alpha}), \text{proj}_2(\hat{\alpha})) \) with \( \int_{\mathbb{R}^d} y \gamma(dy) = \int_{\mathbb{R}^d} y \alpha(dy) \) we have

\[
\frac{1}{n} \sum_{i=1}^n c(x_i, y_i) = \hat{\alpha}(c) \leq \gamma(c). \tag{3.2}
\]

By the duality theorem of linear programming, we find dual optimizers of the linear program given by (3.2), i.e., maximizers of

\[
\max_{f(x+y\Delta(x)) \leq f(x)} \sum_{i=1}^N f(x_i) + \sum_{i=1}^N \alpha_{x_i}(g) + \sum_{i=1}^N \Delta(x_i) \cdot \int_{\mathbb{R}^d} (x_i - y) \alpha_{x_i}(dy).
\]

The complementary slackness condition, which reads here as

\[
(f(x_i) + g(y_j) + \Delta(x_i) \cdot (x_i - y_j) \hat{\alpha}(x_i, y_j) = 0 \quad \forall 1 \leq i, j \leq n,
\]
yields that optimality of the dual optimizer is independent of the definitive choice of the measure \( \alpha \) – as long as \( \text{supp} \alpha \subseteq \text{supp} \hat{\alpha} \). Hence, we deduce \( (c, \mathcal{F}_M) \)-monotonicity of \( \alpha \), and \( \Gamma \) is \( (c, \mathcal{F}_M) \)-monotone. In Definition 3.3 we introduce a notion of martingale \( C \)-monotonicity for weak transport costs which by above reasoning naturally extends \( (c, \mathcal{F}_M) \)-monotonicity, see Definition 3.1, to weak transport costs.

We will see in Lemma 3.7 that under given conditions \( (c, \mathcal{F}_M) \)-monotonicity of a coupling is equivalent to martingale \( C \)-monotonicity (cf. Definition 1.5).

By [12], optimizers of (MOT) are concentrated on \( (c, \mathcal{F}_M) \)-monotone sets. If \( c \) is continuous, then the reverse implication was shown in one dimension by Beiglböck and Juillet [14] and Griessler [27], but for arbitrary dimensions \( d \in \mathbb{N} \) it remains unanswered.
Let us regard two natural generalizations of (MOT):

$$\inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R}} C(x, \pi) \mu(dx), \quad \text{(MWOT)}$$

$$\inf_{P \in \Lambda_M(\mu, \nu)} \int_{\mathbb{R}^d} C(x, p) P(dx, dp), \quad \text{(MWOT')}$$

where $\Lambda_M(\mu, \nu)$ is the set of all $P \in \Lambda(\mu, \nu)$ giving full measure to

$$\left\{(x, p) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) : x = \int_{\mathbb{R}^d} y \rho(dy) \right\},$$

i.e., $P \in \Lambda_M(\mu, \nu)$ iff $P \in \Lambda(\mu, \nu)$ and

$$\int_{\mathbb{R}^d} f(x, p) P(dx, dp) = 0 \quad \forall f \in \tilde{\mathcal{F}}_M,$$

where the set of martingale constraints $\tilde{\mathcal{F}}_M$ is given by

$$\tilde{\mathcal{F}}_M := \left\{ f \in C_b(\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)) : \exists h \in C_b(\mathcal{P}(\mathbb{R}^d)), h \in C_b(\mathbb{R}^d) \right\},$$

s.t. $f(x, p) = g(p)h(x) \int_{\mathbb{R}^d} (x - y) \rho(dy)$.

**Definition 3.3.**

1. We call $\Gamma \subseteq \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ martingale $C$-monotone iff for any $N \in \mathbb{N}$, any collection $(x_1, p_1), \ldots, (x_N, p_N) \in \Gamma$, and $q_1, \ldots, q_N \in \mathcal{P}_1(\mathbb{R}^d)$ such that $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i$, and $\int_{\mathbb{R}^d} y \rho(dy) = \int_{\mathbb{R}^d} y q_i(dy)$, we have

$$\sum_{i=1}^N C(x_i, p_i) \leq \sum_{i=1}^N C(x_i, q_i).$$

2. A probability measure $P \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d))$, which is supported on a martingale $C$-monotone set, is then called martingale $C$-monotone.

3. A probability measure $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ is called martingale $C$-monotone if $I(\pi) \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d))$ is martingale $C$-monotone. (This is equivalent to Definition [1,3].)

Again, by [12] Theorem 1.4] we show in the following theorem that martingale $C$-

**Theorem 3.4.** Let $C : X \times \mathcal{P}_1(\mathbb{R}^d) \to [0, \infty]$ be measurable and $P^* \in \Lambda_M(\mu, \nu)$ optimal for (MWOT') with finite value. Then $P^*$ is martingale $C$-monotone. Moreover, if $C$ additionally satisfies for all $x \in X$ and $Q \in \mathcal{P}_1(\mathcal{P}(\mathbb{R}^d))$

$$C(x, I(Q)) \leq \int_{\mathbb{R}^d} C(x, p) Q(dp), \quad \text{(3.3)}$$

then any optimizer $\pi^*$ of (MWOT') with finite value is martingale $C$-monotone.

As before [3.3] holds when $C(x, \cdot)$ is lower semicontinuous and convex, and in particular for $C(x, p) = \int_{\mathbb{R}^d} c(x, y) \rho(dy)$ when $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous and lower bounded.

**Proof.** Since (MWOT') is an optimal transport problem under additional linear constraints, the first statement is a consequence of [12] Theorem 1.4]. To show the second assertion, we note that any martingale coupling $\pi \in \Pi_M(\mu, \nu)$ naturally induces an element in $\Lambda_M(\mu, \nu)$ by the embedding $I$, c.f. Section [1,3]. Let $P \in \Lambda_M(\mu, \nu)$, then $I(P_\pi) \mu(dx) \in \Pi_M(\mu, \nu)$ and by (3.3)

$$\int_{\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)} C(x, p) P(dx, dp) \geq \int_{\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)} C(x, I(P_\pi)) \mu(dx).$$

As before [3.3] holds when $C(x, \cdot)$ is lower semicontinuous and convex, and in particular for $C(x, p) = \int_{\mathbb{R}^d} c(x, y) \rho(dy)$ when $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous and lower bounded.
Hence, \( J(\pi^*) \) is optimal for \( \text{MWOT} \), and we deduce from the first part martingale C-monotonicity of \( J(\pi^*) \). Due to similar reasoning as in Remark 2.4(3) we conclude that \( \pi^* \) is also martingale C-montone. \( \square \)

From here on we assume that
\[ d = 1, \]
but we hope that in the future a similar approach can be developed for higher dimensions.

**Lemma 3.5.** Let \( N \in \mathbb{N} \) and \( p_i \in \mathcal{P}_r(\mathbb{R}) \) with competitor \( q_i \in \mathcal{P}_r(\mathbb{R}) \), \( i = 1, \ldots, N \), i.e.,
\[
\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i, \quad \int_{\mathbb{R}} y p_i(dy) = \int_{\mathbb{R}} y q_i(dy).
\]
Suppose there are sequences \( \{p^k_i\}_{k \in \mathbb{N}} \), \( k \in \mathbb{N} \), of measures in \( \mathcal{P}_r(\mathbb{R}) \) with \( p^k_i \to p_i \) in \( \mathcal{W}_r \). Then there exist approximative sequences \( \{q^k_i\}_{k \in \mathbb{N}} \) of competitors, i.e.,
\[
\sum_{i=1}^{N} p^k_i = \sum_{i=1}^{N} q^k_i, \quad \int_{\mathbb{R}} y p^k_i(dy) = \int_{\mathbb{R}} y q^k_i(dy), \quad \text{and} \quad q^k_i \to q_i \text{ in } \mathcal{W}_r.
\]
Since the proof of Lemma 3.5 is slightly demanding, we first give for convenience of the reader a more concrete version of the argument in the simpler setting of \( N = 2 \):

**Proof of Lemma 3.5 for \( N = 2 \).** W.l.o.g. \( q_1 \neq p_1 \). Applying Lemma 2.6 we find a sequence \( \{q^k_i\}_{k \in \mathbb{N}} \) which converges to \( q_1 \) in \( \mathcal{W}_r \) and such that \( \sum_{i=1}^{N} q^k_i = \sum_{i=1}^{N} p^k_i \). We may further assume \( \int_{\mathbb{R}} y q^k_1(dy) < \int_{\mathbb{R}} y p^k_1(dy) \). We can decompose the measures \( q_1, q_2, p_1, p_2 \) into sub-probability measures \( m_{i,j}, i, j \in \{1, 2\} \) such that
\[
p_i = m_{i,1} + m_{i,2}, \quad q_j = m_{1,j} + m_{2,j}.
\]
By equality of the mean values of \( q_1 \) and \( p_1 \), we find that
\[
\int_{\mathbb{R}} y m_{1,2}(dy) = \int_{\mathbb{R}} y m_{2,1}(dy).
\]
Thus, we find disjoint, open intervals \( I_1, I_2 \) with \( \min(m_{1,2}(I_1), m_{2,1}(I_2)) > 0 \) and \( \sup(I_2) < \inf(I_1) \). Similarly, for each \( k \in \mathbb{N} \) we can decompose \( q^k_1, q^k_2, p^k_1, p^k_2 \) in the same manner and obtain by the construction in Lemma 2.6 that \( m^k_{i,j} \) converges to \( m_{i,j} \) in \( \mathcal{W}_r \). When well-defined denote by \( \alpha_k > 0 \) the constant such that
\[
\int_{\mathbb{R}} y q^k_1(dy) + \alpha_k \left( \frac{1}{m^k_{1,2}(I_1)} \int_{I_1} y m^k_{1,2}(dy) - \frac{1}{m^k_{2,1}(I_2)} \int_{I_2} y m^k_{2,1}(dy) \right) = \int_{\mathbb{R}} y p^k_1(dy).
\]
By \( \mathcal{W}_r \)-convergence, we have on the one hand \( \lim_{k \to \infty} m^k_{1,2}(I_1) > 0 \) and \( \lim_{k \to \infty} m^k_{2,1}(I_2) > 0 \), and on the other,
\[
\lim_{k \to \infty} \int_{\mathbb{R}} y q^k_1(dy) - \int_{\mathbb{R}} y p^k_1(dy) = 0,
\]
implying that \( \alpha_k \) is well-defined for \( k \) sufficiently large, and \( \alpha_k \to 0 \). Therefore, there is an index \( k_0 \in \mathbb{N} \) such that
\[
q^k_1 = q^k_1 + \alpha_k \left( \frac{m^k_{1,2}(I_1)}{m^k_{1,2}(I_1)} - \frac{m^k_{2,1}(I_2)}{m^k_{2,1}(I_2)} \right), \quad q^k_2 = q^k_2 - \alpha_k \left( \frac{m^k_{1,2}(I_1)}{m^k_{1,2}(I_1)} - \frac{m^k_{2,1}(I_2)}{m^k_{2,1}(I_2)} \right),
\]
are both probability measures for \( k \geq k_0 \), and \( \{q^k_1, q^k_2\}_{k \geq k_0} \) has the desired properties. \( \square \)

**Proof of Lemma 3.5. General Case.** Let \( s \in \mathbb{R}, F_p: \mathbb{R} \to [0, 1] \) the cumulative distribution function of \( p \in \mathcal{P}(\mathbb{R}) \), and define
\[
I^1_s := \{ i \in \{1, \ldots, N\}: F_p(s) = 1 \}, \quad I^2_s := \{ i \in \{1, \ldots, N\}: F_q(s) = 1 \}.
\]
As a preparatory step, we show that
\[
j \in \{1, \ldots, N\} \setminus I^1_s \implies p_j((-\infty, s)) = 0, \quad (3.4)
\]
already implies that \( P^j = 1 \) and
\[
j \in \{1, \ldots, N\} \setminus I^j \implies q_j((-\infty, s)) = 0. \tag{3.5}
\]
This is achieved by observing the barycenters: assume (3.4) and note that
\[
0 = \sum_{i \in I^j} \int_{\mathbb{R}} y \, p_i(dy) - \int_{\mathbb{R}} y \, q_j(dy)
= \int_{\mathbb{R}} y \left( \sum_{i \in I^j} p_i - q_j \right)(dy) - \int_{\mathbb{R}} y \left( \sum_{i \in I^j} q_i - p_i \right)(dy), \tag{3.6}
\]
where \((\cdot)^+\) of a signed measure denotes its positive part. Moreover, as \( \sum_{i=1}^N p_i = \sum_{i=1}^N q_i \) we obtain
\[
\sum_{i \in I^j} q_i(-\infty, a) \leq \sum_{i \in I^j} p_i(-\infty, a) \leq \sum_{i \in I^j} p_i,
\]
and consequently \( (\sum_{i \in I^j} p_i - q_j)^+ \) is concentrated on \((-\infty, s]\), whereas \( (\sum_{i \in I^j} q_i - p_i)^+ \) is concentrated on \([s, +\infty)\). From (3.6) follows that \( (\sum_{i \in I^j} p_i - q_j)^+ = (\sum_{i \in I^j} q_i - p_i)^+ = 0 \), thus \( \sum_{i \in I^j} p_i = \sum_{i \in I^j} q_i \) and \( I^j \subseteq I^j \). Assume that there is \( j \in I^j \setminus I^j \). Then
\[
\sum_{i=1}^N q_i(-\infty, a) = \sum_{i=1}^N p_i(-\infty, a) = \sum_{i \in I^j} q_i(-\infty, a)\]
shows that \( q_j((-\infty, s)) = 0 \). Since \( j \in I^j \) we have that \( q_j((s, +\infty)) = 0 \), and consequently \( q_j = 0 \). The barycenter of \( p_j \) coincides with \( s \), but from \( q_j((s, +\infty)) = 0 \) and \( p_j((-\infty, s)) > 0 \), which violates (3.4). Therefore \( I^j = I^j \), and (3.5) is satisfied.

Clearly, if there exists \( s \in \mathbb{R} \) such that either (3.4) or (3.5) holds, then by the preparatory step we get \( I^j = I^j : I_s \) and \( \sum_{i \in I^j} p_i = \sum_{i \in I^j} q_i \), and
\[
j \in \{1, \ldots, N\} \setminus I_s \implies p_j((-\infty, s)) = 0, \quad j \in \{1, \ldots, N\} \setminus I_s \implies q_j((-\infty, s)) = 0.
\]
In this case we can split the problem into two parts: Finding sequences of competitors for the index sets \( I_s \) and \( \{1, \ldots, N\} \setminus I_s \). It is sufficient to show the existence of such a sequence for the sub problem \( I_s \), where we also assume that \( s \) is minimal.

Thus, assume without loss of generality that \( s \) is minimal and \( I^j_s = I^j_s = \{1, \ldots, N\} \), \( N > 1 \). Denote the convex hull of the support of \( q_i \) by
\[
S_i := \text{co}(\text{supp} \, q_i) \quad i = 1, \ldots, N.
\]
We can also assume without loss of generality that \((p_i, q_j)\) are pairwise different for \( i, j \in \{1, \ldots, N\} \).

Let \( t_{\min}, t_{\max} \in \{1, \ldots, N\} \) be such that \( \inf t_{\min} \) is minimal and \( \sup t_{\max} \) is maximal. Under these assumptions, we get that \( p_{\min}, q_{\max}, p_{\max}, \) and \( q_{\max} \) cannot be concentrated on a single point, thus,
\[
\lambda(S_{\min}) > 0 \quad \text{and} \lambda(S_{\max}) > 0, \tag{3.7}
\]
where \( \lambda \) is the Lebesgue measure.

Applying now Lemma 2.6 we find for every \( i \in \{1, \ldots, N\} \) a sequence \( \{q^k_i\}_{k \in \mathbb{N}} \) with \( \sum_{i=1}^N q^k_i = \sum_{i=1}^N p_i \) and which converges to \( q_i \) in \( \mathcal{W}^\gamma \), in particular,
\[
\lim_{k \to \infty} \left| \int_{\mathbb{R}} \gamma p^k_i(dy) - \int_{\mathbb{R}} \gamma q^k_i(dy) \right| = 0. \tag{3.8}
\]

Consider the following case. Let \( i, j \in \{1, \ldots, N\} \) such that one of the following is true: either
\[
\lambda(S_i \cap S_j) > 0 \quad \text{or} \quad S_j \subseteq \text{int}(S_i), \tag{3.9}
\]
where \( \text{int}(S_i) \) denotes the interior of \( S_i \). Then there are open intervals \( O^+_{ij}, O^-_{ij}, O^+_{ji}, O^-_{ji} \) such that
\[
q_j(O^+_{ij}) > 0, \quad q_j(O^-_{ij}) > 0, \quad t \in \{-, +\},
\sup O^+_{ij} < \inf O^-_{ji}, \quad \sup O^+_{ji} < \inf O^-_{ij},
\]

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\]
By weak convergence of \( q^k_i \) to \( q_i \), the Portmanteau theorem implies
\[
\liminf_{k \to \infty} q^k_i (O_i^j) > 0, \quad \liminf_{k \to \infty} q^k_j (O_j^i) > 0,
\]
\[
\liminf_{k \to \infty} q^k_i (O_i^j) > 0, \quad \liminf_{k \to \infty} q^k_j (O_j^i) > 0.
\]
(3.10)
In particular, when \( k \) is sufficiently large we can by (3.8) and (3.10) replace \( q^k_i \) and \( q^k_j \) with the probability measures
\[
\tilde{q}^k_i := q^k_i + \alpha^k_i \left( \frac{q^k_i \| \delta_{O^i_i} - q^k_j \| \delta_{O^j_j}}{q^k_j \| \delta_{O^j_j} - q^k_i \| \delta_{O^i_i}} \right) + \alpha^k \left( \frac{q^k_i \| \delta_{O^i_i} - q^k_j \| \delta_{O^j_j}}{q^k_j \| \delta_{O^j_j} - q^k_i \| \delta_{O^i_i}} \right),
\]
and \( \tilde{q}^k_j := q^k_i - \tilde{q}^k_i + q^k_j \), where \( \{\alpha^k_i\} \subseteq \mathbb{N} \) and \( \{\alpha^k_i\} \subseteq \mathbb{N} \) are non-negative sequences converging to zero, and \( q^k_i \) and \( p^k_i \) have the same barycenter (i.e. if the barycenter of \( q^k_i \) is smaller than that of \( p^k_j \) we set \( \alpha^k_i > 0 \) and \( \alpha^k_i = 0 \), etc.). Thus, \( \{\tilde{q}^k_i\} \subseteq \mathbb{N} \) and \( \{\tilde{q}^k_j\} \subseteq \mathbb{N} \) converge in \( \mathcal{W} \) to \( q_i \) and \( q_j \), respectively.

We will call \( i \in \{1, \ldots, N\} \) a pivot, if for all \( \tilde{i} \in \{1, \ldots, N\} \) with \( \sup S_{\tilde{i}} \leq \inf S_i \), the barycenters of \( \tilde{q}^k_i \) are correct (when \( k \) is sufficiently large), that means, there is \( k_0 \in \mathbb{N} \) such that
\[
k \geq k_0 \implies \int_{\mathbb{R}} y \tilde{q}^k_i (dy) = \int_{\mathbb{R}} y p^k_i (dy).
\]
(3.11)
For such a pivot \( i \), consider all \( j \neq i \) with (3.9). By the previous step in this proof, there exists \( k_1 \in \mathbb{N} \) such that we can change \( \tilde{q}^k_i \) and \( \tilde{q}^k_j \) for \( k \geq k_1 \in \mathbb{N} \) (and denote the probability measures \( \tilde{q}^k_i \) and \( \tilde{q}^k_j \) for notational convenience again by \( q^k_i \) and \( q^k_j \) respectively) such that
\[
k \geq k_1 \implies \int_{\mathbb{R}} y q^k_j (dy) = \int_{\mathbb{R}} y p^k_j (dy).
\]
(3.12)
There are two possible cases:

1. (3.9) holds true for all \( j \neq i \) with \( \inf S_j < \sup S_j \), which would imply that (given \( k \) is sufficiently large), we have not only corrected the barycenter of \( q^k_i \) but also the one of \( q^k_j \), since
\[
\int_{\mathbb{R}} y q^k_i (dy) = \sum_{j=1}^{N} \int_{\mathbb{R}} y q^k_j (dy) - \sum_{j=1}^{N} \int_{\mathbb{R}} y q^k_j (dy) = \int_{\mathbb{R}} y p^k_j (dy).
\]
Hence, we have found the desired sequences.

2. There are indices \( k \in \{1, \ldots, N\} \) with \( \sup S_i \leq \inf S_k \leq \sup S_k \). Due to minimality of \( s \), there is an index \( l \in \{1, \ldots, N\} \) with \( \inf S_l < \sup S_i < \sup S_l \). Hence (3.9) holds and we can use (3.11) to fix the barycenters of \( q^k_l \) for \( k \) sufficiently large. Moreover, \( l \) is a pivot: Let \( \tilde{l} \in \{1, \ldots, N\} \) with \( \sup S_{\tilde{l}} \leq \inf S_l \), then either \( \sup S_l \leq \inf S_{\tilde{l}} \) whereby (3.11) holds for \( \tilde{l} \), or if \( \inf S_l < \sup S_{\tilde{l}} \leq \inf S_l < \sup S_{\tilde{l}} \). In the latter case, we have that \( \tilde{l} \) satisfies (3.9) and therefore (3.12). Notice that
\[
\sup S_l > \sup S_i.
\]
(3.13)
Taking \( i = i_{\text{min}} \) as our initial pivot, we can iterate the above reasoning. Since there are only finitely many elements, the procedure terminates as ensured by (3.13).

The key ingredient of this part is the following stability result concerning the notion of martingale \( C \)-monotonicity:

**Theorem 3.6.** Let \( C, C_k \in C(\mathbb{R} \times \mathcal{P}_r(\mathbb{R})) \), \( k \in \mathbb{N} \), and \( C_k \) converges uniformly to \( C \). If \( P, P^k \in \mathcal{P}_r(\mathbb{R} \times \mathcal{P}_r(\mathbb{R})) \), \( k \in \mathbb{N} \), are such that

- (a) for all \( k \in \mathbb{N} \) the measure \( P^k \) is martingale \( C_k \)-monotone,
- (b) the sequence \( \{P^k\}_{k \in \mathbb{N}} \) converges to \( P \),

then \( P \) is martingale \( C \)-monotone. Moreover, if \( \pi, \pi^k \in \mathcal{P}_r(\mathbb{R} \times \mathbb{R}) \) and \( C_k \) is convex in the second argument and such that

- (a’) for all \( k \in \mathbb{N} \) the measure \( \pi^k \) is martingale \( C_k \)-monotone,
(b’) the sequence \( \{\pi^k\}_{k \in \mathbb{N}} \) converges to \( \pi \), then \( \pi \) is martingale \( C \)-monotone.

**Proof.** The proof runs parallel to the one of Theorem 2.8 Using Lemma 3.5 we can alter Lemma 2.7 such that for all \( \epsilon \geq 0 \) and \( N \in \mathbb{N} \) the set
\[
\tilde{\Gamma}^\epsilon_N := \left\{ (x, p_i)_{i=1}^N \in (\mathbb{R} \times \mathcal{P}_1(\mathbb{R}))^N \mid \int m_i \leq \sum_{i=1}^N p_i = \sum_{i=1}^N m_i + \epsilon \right\}
\]
is closed. The aim is to construct a martingale \( C \)-monotone set \( \Gamma \) on which \( P \) is concentrated. So, we write \( P^{k,\delta N} \) and \( P^{\delta N} \) for the \( N \)-fold product measure of \( P^k \) resp. \( P \) where \( N \in \mathbb{N} \). By martingale \( C \)-monotonicity and uniform convergence we find for any \( \epsilon > 0 \) a natural number \( k_0 \) such that \( P^{k,\delta N}, k \geq k_0 \), is concentrated on \( \tilde{\Gamma}^\epsilon_N \). As \( \tilde{\Gamma}^\epsilon_N \) is closed, the Portmanteau theorem yields that \( P^{\delta N} \) is concentrated on \( \tilde{\Gamma}^\epsilon_N \):
\[
1 = \limsup_k P^{k,\delta N}(\tilde{\Gamma}^\epsilon_N) \leq P^{\delta N}(\tilde{\Gamma}^\epsilon_N) = 1.
\]
As a consequence, we find that \( P^{\delta N} \) gives full measure to the closed set \( \tilde{\Gamma}^\epsilon_N \). With the same line of argument as in the proof of Theorem 2.8 we can find from here a closed, martingale \( C \)-monotone set \( \tilde{\Gamma} \) with \( P(\tilde{\Gamma}) = 1 \).

To show the second assertion, we embed \( \pi^k \in \mathcal{P}_1(\mathbb{R} \times \mathbb{R}) \) into \( \mathcal{P}_1(\mathbb{R} \times \mathcal{P}_1(\mathbb{R})) \) owing to the map \( J \). Then, by compactness of \( \Lambda_\mu(\mu, \nu) \), we find an accumulation point \( P \in \mathcal{P}_1(\mathbb{R} \times \mathcal{P}_1(\mathbb{R})) \) of \( (J(\pi^k))_{k \in \mathbb{N}} \). Note that \( P \) gives full measure to \( \{(x, p) \in \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \mid x = \mathbb{E}_p(dy)\} \), and
\[
\mu(dx) I(P_x) =: \pi \in \Pi_\mu(\mu, \nu)
\]
determines a martingale coupling, which is likewise a \( \mathcal{W}_r \)-accumulation point of \( \{\pi^k\}_{k \in \mathbb{N}} \).

Since \( P \in \mathcal{P}_1(\mathbb{R} \times \mathcal{P}_1(\mathbb{R})) \) we have \( \mu \)-almost surely that \( I(P_x) \in \mathcal{P}_1(\mathbb{R}) \). As \( P \) is concentrated on the martingale \( C \)-monotone set \( \Gamma \), we find for any \( x \in X \) such that \( P_x(\Gamma_x) = 1 \) and \( I(P_x) \in \mathcal{P}_1(\mathbb{R}) \) a sequence of measures \( \pi^k_i \in \Gamma, i \in \mathbb{N}, \) with
\[
q^k_n := \frac{1}{n} \sum_{i=1}^n \pi^k_i \rightarrow I(P_x) = \pi_x, \quad n \rightarrow \infty, \quad \text{in } \mathcal{W}_r.
\]
Since \( C \) is convex in the second argument, we can reason as in Remark 2.4(b) and find that for \( \mu \)-a.e. \( x \) \( (x, q^k_n) \) is already contained in the martingale \( C \)-monotone set \( \Gamma \). Then by closure of \( \Gamma \) we conclude \( (x, \pi_x) \in \Gamma \) for \( \mu \)-a.e. \( x \), and by the same reasoning as in Remark 2.4(a) martingale \( C \)-monotonicity of \( \pi \). \( \square \)

**Lemma 3.7.** Let \( \pi \in \Pi_\mu(\mu, \nu), b \in L^1(\nu), c: X \times Y \rightarrow \mathbb{R} \) be jointly measurable, and \( c(x, \cdot) \) be upper semicontinuous and upper bounded by a positive multiple of \( b \) for all \( x \in \mathbb{R} \). Then \( \pi \) is \( (c, \mathcal{F}_\mu) \)-monotone if and only if \( \pi \) is martingale \( C \)-monotone (with \( C(x, p) := \int_{\mathbb{R}} c(x, y)p(dy) \)).

**Proof.** Let \( \tilde{\Gamma} \subseteq \mathbb{R} \times \mathbb{R} \) be \( (c, \mathcal{F}_\mu) \)-monotone with \( \pi(\tilde{\Gamma}) = 1 \). Consider the Borel measurable set
\[
\Gamma := \left\{ (x, p) \in X \times \mathcal{P}_1(Y) : p(\tilde{\Gamma}_x) = 1, \int_{\mathbb{R}} y p(dy) = x, \int_{\mathbb{R}} |b(y)| p(dy) < \infty \right\},
\]
where \( (x, \pi_x) \in \Gamma \) for \( \mu \)-almost every \( x \in X \). Take any sequence \( (x_1, p_1), \ldots, (x_N, p_N) \in \Gamma \) with competitors \( q_1, \ldots, q_N \in \mathcal{P}_1(\mathbb{R}) \), i.e.,
\[
\sum_{i=1}^N p_i = \sum_{i=1}^N q_i, \quad \int_{\mathbb{R}} y p_i(dy) = \int_{\mathbb{R}} y q_i(dy).
\]
We find for any \((x, p) \in \Gamma\), a sequence of finitely supported measures \(\{p_k^\delta\}_{k \in \mathbb{N}}\) where for all \(k \in \mathbb{N}\) we have that \(p_k^\delta(\hat{\Gamma}_x) = 1\), \(\int \psi \, p_k^\delta(dy) = x_i\), and

\[
\lim_{k \to \infty} \int_{\mathbb{R}} c(x, y) \, p_k^\delta(dy) = \int_{\mathbb{R}} c(x, y) \, p(dy), \tag{3.14}
\]

\[
\lim_{k \to \infty} \int_{\mathbb{R}} |b(y)| \, p_k^\delta(dy) = \int_{\mathbb{R}} |b(y)| \, p(dy). \tag{3.15}
\]

Thus, Lemma 3.5 provides \(q_1^\delta, \ldots, q_N^\delta\), sequences of feasible and finitely supported competitors with corresponding limit points \(q_1, \ldots, q_N\). Then \((c, F_M)\)-monotonicity yields

\[
\sum_{i=1}^N \int_{\mathbb{R}} c(x, y) \, p_i(dy) = \lim_{k \to \infty} \sum_{i=1}^N \int_{\mathbb{R}} c(x, y) \, p_k^\delta(dy) \leq \lim sup_{k \to \infty} \sum_{i=1}^N \int_{\mathbb{R}} c(x, y) \, q_k^\delta(dy),
\]

where we used (3.14) for the first equality, and upper semicontinuity of \(c(x, \cdot)\), upper boundedness of \(c(x, \cdot) - a(x)h(\cdot)\) for some \(a(x) > 0\), and (3.15) in the last inequality. We have shown that \(\Gamma\) is martingale C-monotone. Since \((x, \pi_i) \in \Gamma\) for \(\mu\)-a.e. \(x \in \mathbb{R}\), we conclude that \(\pi\) is martingale C-monotone.

Now, let \(\pi\) be martingale C-monotone on a Borel measurable set \(\Gamma \subseteq \mathbb{R} \times \mathcal{P}(\mathbb{R})\), that is, \(J(\pi)(\Gamma) = 1\). Due to the variation of Lusin’s theorem, Lemma 6.3 there is an analytically measurable set \(\hat{\Gamma} \subseteq X \times Y\) satisfying:

(i) For any \((x, p) \in \Gamma\) we have that \(p\) is concentrated on the fibre \(\hat{\Gamma}_x = \{y \in Y: (x, y) \in \hat{\Gamma}\}\), i.e., \(p(\hat{\Gamma}_x) = 1\).

(ii) For any \((x, y) \in \hat{\Gamma}\) we find \((x, p) \in \Gamma\) and a Borel measurable set \(K \subseteq \hat{\Gamma}_x\) such that

1. \(c\) restricted to the fibre \([x] \times K\) is continuous,
2. \(y \in \text{supp}(p_K) \cap K\) and

\[
\int_{B(y) \cap K} \frac{c(x, z)}{p(B(y) \cap K)} \, p(dy) \to c(x, y) \quad \text{for } \delta \searrow 0. \tag{3.16}
\]

In particular, we have that \(\pi\) is concentrated on \(\hat{\Gamma}\) by property (i). To see that \(\hat{\Gamma}\) is \((c, F_M)\)-monotone, take a finite subset of \(\hat{\Gamma}\); i.e., \(G := \{(x_1, y_1), \ldots, (x_N, y_N)\} \subseteq \hat{\Gamma}\). Let \(\alpha\) be supported on \(G\) and \(\beta\) be a competitor, i.e.,

\[
\text{proj}_1(\alpha) = \text{proj}_1(\beta), \quad \text{proj}_2(\alpha) = \text{proj}_2(\beta), \quad \int_{\mathbb{R}} y \, \alpha_i(dy) = \int_{\mathbb{R}} y \, \beta_i(dy), \quad i = 1, \ldots, N.
\]

By property (ii) of \(\Gamma\) we find for each \((x_i, y_i), 1 \leq i \leq N, (x_i, p_i) \in \Gamma\) and sets \(K_i\) such that \(c\) is continuous on \([x_i] \times K_i, y_i \in K_i\), and (3.16) holds. Let

\[
a^\delta(dx, dy) := \sum_{i=1}^N \delta_{(x_i, y_i)}(dx) \frac{a^\delta(x_i, y_i)}{p(K_i \cap B_{y_i}(y_i))} p|_{K_i \cap B_{y_i}(y_i)}(dy), \quad k \in \mathbb{N}.
\]

Then \(a^\delta\) converges weakly to \(a\), and by Lemma 3.5 we find a sequence \(\{\beta^k\}_{k \in \mathbb{N}}\), where \(\beta^k\) is a competitor of \(a^\delta\), which converges weakly to \(\beta\). As \(c(x, \cdot)\) is upper semicontinuous and finitely valued, there exists \(k_0 \in \mathbb{N}\) with

\[
\sup \left\{ c(x, y): y \in \bigcup_{i=1}^N B_{y_i}(y_i) \right\} < \infty. \tag{3.17}
\]

Thus, we have

\[
\int_{\mathbb{R} \times \mathbb{R}} c(x, y) \, a(dx, dy) = \lim_{k \to \infty} \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \, a^k(dx, dy) \leq \lim inf_{k \to \infty} \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \beta^k(dx, dy) \leq \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \beta(dx, dy),
\]
where we obtain the first equality due to \((3.16)\), the first inequality due to martingale C-monotonicity, and the final inequality due to upper semicontinuity and \((3.17)\). □

**Proof of Theorem 7.7** Since \(\pi^k\) is optimizer of \((\text{MOT})\) for cost \(c_k\) with marginals \(\mu_k\) and \(\nu_k\), \(\pi^k\) is \((c, F_M)\)-monotone by \([12\), Theorem 1.4\]. By Lemma \([5.7]\) we find that \(\pi^k\) is martingale \(C_k\)-monotone. Then Theorem \([3.6]\) shows that martingale monotonicity is preserved for \(k \to \infty\). Hence, \(\pi\) is martingale C-monotone and thus \((c, F_M)\)-monotone by Lemma \([3.7]\).

Finally, \(\pi\) is optimal for \((\text{MOT})\) with cost \(c\) by \([27\), Theorem 1.3\]. □

**Proof of Corollary 7.2** Let \(\pi^k\) be optimal for \((\text{MOT})\) for the cost function \(c_k\) and marginal measures \(\mu_k\), \(\nu_k\). We may apply Theorem \([1.1]\) showing that every accumulation point (with respect to weak convergence) of \((\pi^k)^{k \in \mathbb{N}}\) is an optimizer for \((\text{MOT})\) for the cost function \(c\) and marginal measures \(\mu, \nu\). On \(\mathbb{R}^2\) we may choose the metric \(D((x, y), (\bar{x}, \bar{y})) = \sqrt{|x - \bar{x}|^2 + |y - \bar{y}|^2}\) in order to define the \(r\)-Wasserstein metric on \(\mathcal{P}(\mathbb{R}^2)\). Then

\[
\int_{\mathbb{R} \times \mathbb{R}} D((0, 0), (x, y))^r \pi^k(dx, dy) = \int_{\mathbb{R}} |x|^r \mu_k(dx) + \int_{\mathbb{R}} |y|^r \nu_k(dy)
\]

\[
\to \int_{\mathbb{R}} |x|^r \mu(dx) + \int_{\mathbb{R}} |y|^r \nu(dy) = \int_{\mathbb{R} \times \mathbb{R}} D((0, 0), (x, y)) \pi(dx, dy),
\]

as \(k\) tends to \(\infty\) for any coupling \(\pi\) with marginals \(\mu, \nu\). It follows by \([43\), Definition 6.8 (i)\] that the accumulation points of \((\pi^k)^{k \in \mathbb{N}}\) under the weak topology or under the \(W_r\)-topology coincide. The following inequality is immediate

\[
\liminf_{k \to \infty} \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y) \pi^k(dx, dy) \geq \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \pi(dx, dy).
\]

In order to prove

\[
\limsup_{k \to \infty} \int_{\mathbb{R} \times \mathbb{R}} c_k(x, y) \pi^k(dx, dy) \leq \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \pi(dx, dy),
\]

it suffices to observe that if (for some subsequence which we do not track) \(\pi^k \to \pi\) in \(\mathcal{W}_r\), then \(\int_{\mathbb{R} \times \mathbb{R}} c_k(x, y) \pi^k(dx, dy) \to \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \pi(dx, dy)\), since \(\pi\) must be optimal for the r.h.s. But this is clear since \(c_k \to c\) uniformly and \(c\) is dominated by a positive multiple of \((x, y) \mapsto 1 + D((0, 0), (x, y))^r\). □

4. The relation of OT and WOT

In this part we explore the relationship between (classical) c-cyclical monotonicity in optimal transport, and C-monotonicity in optimal weak transport, in the case when \(C(x, p) = \int_{\mathbb{R}} c(x, y) p(dy)\). We recall the notion of c-cyclical monotonicity, cf. \([43\), Definition 5.1]:

**Definition 4.1** (c-cyclical monotonicity). Let \(c : X \times Y \to \mathbb{R}\)

1. A set \(\Gamma \subseteq X \times Y\) is called c-cyclical monotone iff for any \(N \in \mathbb{N}\), and any collection \((x_1, y_1), \ldots, (x_N, y_N)\) ∈ Γ, we have

\[
\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}),
\]

with the convention \(y_{N+1} = y_1\).

2. A probability measure \(\pi \in \mathcal{P}(X \times Y)\), which is concentrated on a c-cyclical monotone set, is then called c-cyclical monotone.

Our main result of this section is:

**Theorem 4.2.** Let \(\pi \in \Pi(\mu, \nu), b \in L^1(\nu), c : X \times Y \to \mathbb{R}\) be jointly measurable, and \(c(x, \cdot)\) be upper semicontinuous and upper bounded by a positive multiple of \(b\) for all \(x \in X\). Then \(\pi\) is c-cyclical monotone if and only if \(\pi\) is C-monotone (with \(C(x, p) := \int_{\mathbb{R}} c(x, y) p(dy)\)).
Lemma 4.3. Let \( c : X \times Y \rightarrow \mathbb{R} \) be jointly measurable. Let \( \pi \in \Pi(\mu, \nu) \) be \( C \)-monotone with \( C(x, y) := \int_{Y} c(x, y) \, p(dy) \) and \( c(x, \cdot) \in L^1(\pi_x) \) for \( \mu \)-almost every \( x \in X \). Then for all \( A \subseteq \mathcal{B}(X \times Y) \) the restriction of \( \pi \) to \( A \), i.e., \( \pi|_A \), is also \( C \)-monotone.

Proof. Let \( x_1, \ldots, x_N \in \Gamma \cap \{ x \in X : \pi_x(A_x) > 0 \} \), \( N \in \mathbb{N} \), where \( \Gamma \) is a \( C \)-monotone subset and \( A_x := \{ y \in Y : (x, y) \in A \} \). To shorten notation we write
\[
p_i := \pi_{i|x_i}, \quad \tilde{p}_i := \frac{1}{\tilde{p}_i(A_{i|x_i})} p_i, \quad i = 1, \ldots, N.
\]
Without loss of generality we can assume that restricting and disintegrating commutes, \( \tilde{p}_i = (\pi_{i|x_i}) \) for all \( i = 1, \ldots, N \). Let \( n > 1 \) be a natural number with reciprocal value smaller than min, \( p_i(A_{i|x_i}) \). Define the probability measures
\[
r_i = \frac{n \pi_{i|x_i} - \tilde{p}_i}{n - 1} \in \mathcal{P}(Y), \quad i = 1, \ldots, N,
\]
which by linearity of \( p \mapsto C(x, p) \) satisfy
\[
n \sum_{i=1}^{N} C(x_i, \pi_{i|x_i}) = \sum_{i=1}^{N} C(x_i, \tilde{p}_i) + (n - 1) C(x_i, r_i). \tag{4.1}
\]
To show \( C \)-monotonicity of \( \pi|_A \), let \( \tilde{q}_1, \ldots, \tilde{q}_N \in \mathcal{P}(Y) \) with \( \sum_{i=1}^{N} \tilde{q}_i = \sum_{i=1}^{N} \tilde{p}_i \). Define
\[
q_{i,j} := \begin{cases} \tilde{q}_i & j = 1, \\ r_i & 2 \leq j \leq n, \end{cases} \quad i = 1, \ldots, N,
\]
whence, \( \tilde{q}_i := \frac{1}{n} \sum_{j=1}^{n} q_{i,j} \in \mathcal{P}(Y), i = 1, \ldots, N \), define competitors of \( (\pi_{i|x_i}) \), since
\[
\sum_{i=1}^{N} \tilde{q}_i = \frac{1}{n} \sum_{i=1}^{N} \sum_{j=1}^{n} q_{i,j} = \sum_{i=1}^{N} \pi_{i|x_i}.
\]
From \( C \)-monotonicity of \( \pi \) we derive that
\[
n \sum_{i=1}^{N} C(x_i, \pi_{i|x_i}) \leq n \sum_{i=1}^{N} C(x_i, q_i) = \sum_{i=1}^{N} \sum_{j=1}^{n} C(x_i, q_{i,j}),
\]
which is by (4.1) equivalent to
\[
\sum_{i=1}^{N} C(x_i, \tilde{p}_i) \leq \sum_{i=1}^{N} C(x_i, \tilde{q}_i).
\]

The next proposition has Theorem 4.2 as an immediate corollary:

Proposition 4.4. Let \( c : X \times Y \rightarrow \mathbb{R} \) be jointly measurable, \( b : Y \rightarrow \mathbb{R} \) be measurable, and \( c(x, \cdot) \) be upper semicontinuous and dominated by a positive multiple of \( b \) for each \( x \in X \). If \( \Gamma \subseteq X \times \mathcal{P}(Y) \) is a \( C \)-monotone analytic set (where \( C(x, p) := \int_{Y} c(x, y) \, p(dy) \)), then the set \( \hat{\Gamma} \subseteq X \times Y \) from Lemma 6.3 is \( c \)-cyclically monotone. Conversely, if a \( c \)-cyclic monotone set \( \Gamma \subseteq X \times Y \) is given, then
\[
\Gamma := \{ (x, p) \in X \times \mathcal{P}(Y) : p(\hat{\Gamma}_x) = 1, b \in L^1(p) \}
\]
is \( C \)-monotone.

Proof. Suppose that \( \Gamma \) is \( C \)-monotone. Let a finite number of points \( (x_1, y_1), \ldots, (x_N, y_N) \) in \( \hat{\Gamma} \) be given. We find \( (x_i, p_i) \in \Gamma \) with \( p_i(\hat{\Gamma}_x) = 1 \) as in Lemma 6.3. This means that for each \( y_i \) there is a Borel measurable set \( K_i \subseteq Y \) with \( y_i \in \text{supp}(p_i|_{K_i}) \), \( c|_{(x_i) \times K_i} \) is continuous, and
\[
p_i^c := \frac{1}{p_i(B(Y_i) \cap K_i)} p_i|_{B(Y_i) \cap K_i} \Rightarrow \frac{c(x_i, z)}{p(K_i \cap B(Y_i))} \, \mu(dz) \rightarrow c(x_i, y_i).
\]
The measure \(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i, p_i}\) is by construction \(C\)-monotone. By the restriction property of Lemma 4.3, the measure \(\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i, p_i}\) is likewise \(C\)-monotone. Then by upper semicontinuity of \(c(x, \cdot)\) and (3.17) (which is here applicable), we conclude (with the convention that \(N + 1 = 1\)):

\[
\sum_{i=1}^{N} c(x_i, y_i) = \lim_{\varepsilon \searrow 0} \sum_{i=1}^{N} C(x_i, p_i^\varepsilon) \leq \limsup_{\varepsilon \searrow 0} \sum_{i=1}^{N} C(x_i, p_i^\varepsilon) \leq \sum_{i=1}^{N} c(x_i, y_{i+1}).
\]

Now let \(\hat{\Gamma}\) be a \(c\)-cyclical monotone set. Recall that \(c(x, \cdot)\) is dominated by \(b\) and \(b \in L^1(p)\) for all \((x, p) \in \Gamma\). Therefore, by the law of large numbers, we have for an iid sequence \((Y_i)_{i \in \mathbb{N}}\) distributed according to \(p\) that almost surely

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} c(x, Y_i) = \int \nabla c(x, y) p(dy), \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |b(Y_i)| = \int_{Y} |b(y)| p(dy).
\]

Thus, we can approximate \(p\) by discrete measures concentrated on \(\hat{\Gamma}\) and thereby obtain \(C\)-monotonicity of \(\hat{\Gamma}\): Let \((x_1, p_1), \ldots, (x_n, p_n) \in \Gamma\) and \(q_1, \ldots, q_N \in \mathcal{P}(\mathcal{Y})\) with \(\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i\). We find by discretely approximating \(p_1, \ldots, p_n\) on \(\hat{\Gamma}\), and then using \(c\)-cyclical monotonicity, the sequence of competitors constructed in Lemma 2.6 and upper semicontinuity of \(c(x, \cdot)\) and upper boundedness of \(c(x, \cdot) - a(x)b(\cdot)\) for some \(a(x) > 0\) that

\[
\sum_{i=1}^{N} C(x_i, p_i) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} c(x_i, y_j) \leq \limsup_{n} \frac{1}{n} \sum_{i=1}^{n} \int_{Y} c(x_i, y) q_i^\varepsilon(dy) \leq \sum_{i=1}^{N} C(x_i, q_i).
\]

□

5. Epilogue

Convexity is a natural assumption in the setting of weak transport. It is known that convexity of \(C(x, \cdot)\) is necessary to obtain general existence of minimizers in the space of couplings, see [4, Example 3.2]. Similarly, convexity is required for \(C\)-monotonicity to be a necessary optimality criterion:

**Example 5.1.** Let \(X = [0, 1], Y = [0, 1], \mu = \delta_0, \nu = \frac{1}{2}(\delta_0 + \delta_1)\), and

\[
C(x, p) = \min(p([0]), p([1])).
\]

Then \(C\) is continuous and concave on \(X \times \mathcal{P}(Y)\), but the only (and therefore optimal) coupling \(\mu \otimes \nu \in \Pi(\mu, \nu)\) is not \(C\)-monotone. Indeed,

\[
2C(0, \nu) = 1 > 0 = C(0, \delta_0) + C(0, \delta_1).
\]

In the classical optimal transport setting \(c\)-cyclical monotonicity implies optimality already when the cost function \(c\) is bounded from below and real valued, see [10]. A similar conclusion cannot be drawn in optimal weak transport as \(C\)-monotonicity does not even imply optimality when \(C\) is lower continuous:

**Example 5.2.** Let \(X = [0, 1], Y = [0, 1] \) and \(C(x, p) := p_x\delta_0([0, 1]) = p([0, 1] \setminus \{x\})\), which is a measurable cost function (c.f. Proposition 6.5) and lower semicontinuous for fixed \(x \in [0, 1]\): Given a weakly convergent sequence \(p_k \rightharpoonup p \in \mathcal{P}(Y)\), the Portmanteau theorem yields

\[
\liminf_k C(x, p_k) = \liminf_k p_k([0, 1] \setminus \{x\}) \geq p([0, 1] \setminus \{x\}) = C(x, p).
\]

The product coupling \(\pi := \lambda \otimes \lambda \in \Pi(\lambda, \lambda)\) where \(\lambda\) denotes the uniform distribution on \([0, 1]\) is in fact \(C\)-monotone: Since \(\pi_x = \lambda\) (\(\lambda\)-almost surely), we have for any \(x_1, \ldots, x_N \in [0, 1]\) and \(q_1, \ldots, q_N \in \mathcal{P}(\mathcal{Y})\) with

\[
\sum_{i=1}^{N} \pi_{x_i} = N \lambda = \sum_{i=1}^{N} q_i
\]
that $q_i$ is absolutely continuous to $\lambda$, hence,

$$\sum_{i=1}^{N} C(x_i, \pi_{x_i}) = N = \sum_{i=1}^{N} C(x_i, q_i).$$

But the unique optimizer of (WOT) with cost $C$ is given by $\pi^*(dx, dy) := \lambda(dx)\delta_x(dy)$ and

$$\int_{[0,1]} C(x, \pi^*) \lambda(dx) = 0 < 1 = \int_{[0,1]} C(x, \pi_x) \lambda(dx).$$

Even when $C$ is given as the integral with respect to some $c: X \times Y \to \mathbb{R}$, we cannot hope that $C$-monotonicity implies optimality and/or $c$-cyclical monotonicity, as the next example shows.

**Example 5.3.** Let $X = [0, 1]$, $Y = [0, 1]$ and $C(x, p) = \int_{[0,1]} c(x, y)p(dy)$ with $c(x, y) := \mathbb{1}_{y \in (1/2, 1]}$. As in the previous example, the product coupling $\pi = \lambda \otimes \lambda$ is $C$-monotone, but not optimal, whereas $\pi^*(dx, dy) = \lambda(dx)\delta_x(dy)$ is optimal and in particular $c$-cyclical monotone.

The failure of $C$-monotonicity to provide optimality in these simple settings (the cost function $C$ is even bounded and lower semicontinuous) is caused by the manner it varies over $X \times Y$: The variation over $X$ is pointwise (similar to $c$-cyclical monotonicity), whereas over $Y$ variations are taken in a weak sense, i.e., we require that the $Y$-intensities of the two competing sequences to agree. Here, $C$-monotonicity is unable to detect the jump from 1 to 0 at $x$, and we could argue that $C$-monotonicity yields optimality of $\pi$ under all couplings $\nu$ in $\Pi(\lambda, \lambda)$ such that $\nu_{x} \ll \lambda$ for $\lambda$-almost all $x \in [0, 1]$. To be able to compare with all competing couplings, more regularity of $C$ in $Y$-direction is necessary, e.g., upper semicontinuity as in Theorem 4.2 or uniform equicontinuity as in Theorem 2.2.

Notably, Example 5.2 when taking $\tilde{C} := -C$ as cost, provides an upper semicontinuous cost function which is convex (in fact linear) in the following sense: Let $Q \in \mathcal{P}(\mathcal{P}([0, 1]))$, then

$$\int_{\mathcal{P}([0,1])} \tilde{C}(x, p)Q(dp) = \tilde{C}
\left(x, \int_{\mathcal{P}([0,1])} pQ(dp)\right). \quad (5.1)$$

Therefore, $\tilde{C}$ being lower semicontinuous and convex is a strictly stronger assumption than the convexity property stated in (5.1) together with measurability, as demanded in Theorem 2.5.

6. Appendix

The existence of a concave modulus of continuity was employed in the proof of Theorem 2.2. We split the argument into a general part, and a part optimized for the setting of the paper.

**Lemma 6.1.** Let $(Y, d)$ be a metric space, $r \geq 1$, and $f : X \times Y \to \mathbb{R}$ be $d$-uniformly continuous uniformly in $x \in X$, i.e. such that

$$\vartheta(\delta) := \sup \{|f(x, a) - f(x, b)| : x \in X, a, b \in Y \text{ s.t. } d(a, b) \leq \delta\},$$

vanishes for $\delta \searrow 0$. Assume additionally that

$$\forall \xi > 0 : \sup_{a, b \in X, \forall x, \forall d(a, b) \leq \xi} \left|\frac{f(x, a) - f(x, b)}{d(a, b)}\right| < \infty.$$ 

Then there is $\tilde{\vartheta} : [0, \infty) \to [0, \infty]$ concave such that $\lim_{\delta \searrow 0} \tilde{\vartheta}(\delta) = 0$ and

$$|f(x, a) - f(x, b)| \leq \tilde{\vartheta}(d(a, b)).$$
Proof. Fixing \( \varepsilon > 0 \), there is \( \delta_\varepsilon \) such that \( |f(x, a) - f(x, b)| \leq \varepsilon \) if \( d(a, b) \leq \delta_\varepsilon \). If on the other hand \( d(a, b) > \delta_\varepsilon \), then

\[
\frac{|f(x, a) - f(x, b)|}{d(a, b)} \leq K_\varepsilon.
\]

Hence overall \( |f(x, a) - f(x, b)| \leq \varepsilon + d(a, b)^2 K_\varepsilon \) and in particular

\[
\theta(\delta) \leq \varepsilon + \delta K_\varepsilon,
\]

and \( K_\varepsilon < \infty \). Denoting by \( \tilde{\theta} \) the concave envelope of \( \theta \), i.e. the infimum over all affine functions majorizing \( \theta \), we conclude \( \tilde{\theta}(\delta) \leq \varepsilon + \delta K_\varepsilon \), and so \( \lim_{\delta \to 0} \tilde{\theta}(\delta) = 0 \), as \( \varepsilon \) was arbitrary. Finally remark that \( |f(x, a) - f(x, b)| \leq \theta(d(a, b)) \leq \tilde{\theta}(d(a, b)) \).

\[ \square \]

Lemma 6.2. Let \( C : X \times \mathcal{P}_r(Y) \to \mathbb{R} \) and suppose that \( C(x, \cdot) \) is \( W_r \)-uniformly continuous uniformly in \( x \in X \), i.e. such that

\[
\theta(\delta) := \sup \left( C(x, p) - C(x, q) : x \in X, (p, q) \in \mathcal{P}_r(Y)^2 \text{ s.t. } W_r(p, q) \leq \delta \right),
\]

vanishes for \( \delta \searrow 0 \). Then there is \( \tilde{\theta} : [0, \infty) \to [0, \infty] \) concave such that \( \lim_{\delta \to 0} \tilde{\theta}(\delta) = 0 \) and

\[
\theta(\delta) \leq \tilde{\theta}(W_r(p, q)).
\]

Proof. We apply Lemma 6.1. It suffices to show that for any \( c > 0 \)

\[
\sup_{p, q \in \mathcal{P}_r(Y) : W_r(p, q) \geq c} \frac{|C(x, p) - C(x, q)|}{W_r(p, q)} < \infty. \tag{6.1}
\]

To this end choose \( 0 < \delta \leq \frac{c}{\varepsilon} \) such that \( \theta(\delta) < \infty \). For any \( p, q \in \mathcal{P}_r(Y) \) with \( W_r(p, q) \geq c \), there is \( N \geq 2 \) such that

\[
(N - 1)\delta \leq W_r(p, q) \leq N\delta.
\]

Denoting \( [p, q]_\alpha = (1 - \alpha)p + \alpha q \), by convexity of optimal transport we have\(^4\) for all \( \alpha, \beta \in [0, 1] \)

\[
W_r([p, q]_\alpha, [p, q]_\beta) \leq |\alpha - \beta|W_r(p, q).
\]

Hence,

\[
|C(x, p) - C(x, q)| \leq \sum_{k=1}^{N} |C(x, [p, q]_{(k-1)/N}) - C(x, [p, q]_{k/N})| \leq \sum_{k=1}^{N} N\delta N\delta \leq N\delta \leq \frac{\theta(\delta)W_r(p, q)}{\delta(N - 1)} \leq 2W_r(p, q)\frac{\theta(\delta)}{\delta},
\]

and we conclude that the left-hand side of (6.1) is bounded by \( \frac{2\theta(\delta)}{\delta} < \infty \).

\[ \square \]

The following lemma can be viewed as a measurable version of Lusin’s theorem. It is applied in Section 3 and 4 to connect martingale \( C \)-monotone respectively \( C \)-monotone sets \( \Gamma \) with \( (c, \hat{\mathcal{F}}_M) \)-monotone respectively \( c \)-cyclically monotone sets \( \hat{\Gamma} \). Namely this is done in Lemma 3.7 and Proposition 4.4.

Lemma 6.3. Let \( \Gamma \subseteq X \times \mathcal{P}(Y) \) be analytic, and \( c : X \times Y \to \mathbb{R} \cup \{\pm \infty\} \) be Borel measurable. Then there exists an analytic set \( \hat{\Gamma} \subseteq X \times Y \) with the following properties:

(i) For any \( (x, p) \in \Gamma \) we have that \( p \) is concentrated on the fibre \( \Gamma_x = \{ y \in Y : (x, y) \in \hat{\Gamma} \} \), i.e., \( p(\hat{\Gamma}_x) = 1 \).

(ii) For any \( (x, y) \in \hat{\Gamma} \) we find \( (x, p) \in \Gamma \) and a Borel measurable set \( K \subseteq \Gamma_x \) such that

(a) \( c \) restricted to the fibre \( \{x\} \times K \) is continuous,

\(^4\)Indeed, if \( \beta > \alpha \) we write \( [p, q]_\beta = \frac{1 - \beta}{\beta - \alpha}[p, q]_\alpha + \frac{\beta - \alpha}{\beta - \alpha}q \) and first deduce \( W_r([p, q]_\alpha, [p, q]_\beta) \leq \frac{\beta - \alpha}{\beta - \alpha}W_r(p, q) \leq |\beta - \alpha|W_r(p, q) \), by convexity of optimal transport with respect to marginals. Iterating the argument we find \( W_r([p, q]_\alpha, [p, q]_\beta) \leq (\beta - \alpha)W_r(p, q) \).
(b) \( y \in \text{supp}(p) \cap K \) and
\[
\int_{B_d(y) \cap K} \frac{c(x, z)}{p(B_d(y) \cap K)} p(dz) = c(x, y) \quad \text{for } \delta \searrow 0.
\]

**Proof.** Without loss of generality, we can assume that \( c \) is bounded. We follow similar ideas as \([22]\).

To illustrate the idea, let for a moment \( K \subseteq Y \) be any measurable set, and write \( c(Y) := c(x, y) \). Continuity of \( c_{\|\cdot\|,\alpha}(\cdot) \) means that for any open set \( O \subseteq \mathbb{R} \) the preimage \( c^{-1}_c(O \cap K) \) is open in the trace topology on \( Y \cap K \). For any radius \( \alpha > 0 \) and open set \( O \subseteq \mathbb{R} \), we can consider an \( \alpha \)-doughnut around \( c^{-1}_c(O) \), i.e.,
\[
\tilde{A}(\alpha) := \{ y \in Y : \exists z \in c^{-1}_c(O) \text{ s.t. } d_Y(y, z) < \alpha \} \setminus c^{-1}_c(O).
\]
Clearly, \( \tilde{A}(\alpha) \cup c^{-1}_c(O) \) is open and \( c^{-1}_c(O) = (\tilde{A}(\alpha) \cup c^{-1}_c(O)) \cap \tilde{A}(\alpha) \), thus, \( c^{-1}_c(O) \) is a relatively open subset of \( \tilde{A}(\alpha) \). Note that for any \( p \in \mathcal{P}(Y) \), we have by outer regularity that \( p(\tilde{A}(\alpha)) \) vanishes with \( \alpha \).

Next, we generalize the above reasoning adequately to the product space \( X \times Y \) and countably many open sets. Denote by \( \{U_n\}_{n \in \mathbb{N}} \) a countable basis of the topology on \( \mathbb{R} \). In analogy to \( A \), we define for any sequence of radii in \( Y \)-direction \( \alpha = \{\alpha_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^N, \alpha_n > 0 \) for all \( n \in \mathbb{N} \), the \( \mathcal{B}(X \times Y) \)-measurable set
\[
A(\alpha) := \bigcup_{n \in \mathbb{N}} \{ (x, y) \in X \times Y : \exists \alpha \in c^{-1}_c(U_n) \text{ s.t. } d_Y(y, z) < \alpha_n \} \setminus c^{-1}_c(U_n).
\]
We see likewise that \( c^{-1}(U_n) \cap A(\alpha) \cap \{x \times Y \} \) is relatively open in \( A(\alpha) \cap \{x \times Y \} \). As \( \{U_n\}_{n \in \mathbb{N}} \) forms a basis of the topology on \( \mathbb{R} \), we find for any open set \( O \subseteq \mathbb{R} \) a subset \( N \subseteq \mathbb{N} \) with \( \bigcup_{n \in N} U_n = O \), thus,
\[
c^{-1}(O) \cap A(\alpha) \cap \{x \times Y \} = \bigcup_{n \in N} c^{-1}(U_n) \cap A(\alpha) \cap \{x \times Y \},
\]
is also relatively open in \( A(\alpha) \cap \{x \times Y \} \), and consequently \( c_{\|\cdot\|,\alpha}(\cdot) \) is continuous for all \( x \in X \).

Let \( (x, p, \alpha) \in X \times \mathcal{P}(Y) \times \mathbb{R}^N \) and \( \alpha_n > 0 \) for all \( n \in \mathbb{N} \), \( K := \text{proj}_Y(A(\alpha) \cap \{x \times Y \}) \). We have shown for any \( y \in \text{supp}(p|_X) \) that \( \delta \searrow 0 \) implies
\[
\frac{1}{p(K \cap B_d(y))} p(K \cap B_d(y)) \rightarrow \delta_y \quad \text{in } \mathcal{P}(Y).
\]
Since \( c \) is bounded and continuous on \( \{x \times K \) we have for \( \delta \searrow 0 \)
\[
\int_{K \cap B_d(y)} \frac{c(x, z)}{p(K \cap B_d(y))} p(dz) = c(x, y) \quad \text{for } \delta \searrow 0.
\]
For any \( (x, p) \in X \times \mathcal{P}(Y) \) we have by outer regularity of \( p \) that for any \( n \in \mathbb{N} \) and \( \delta \searrow 0 \)
\[
p\left( \{ y \in Y : \exists \alpha \in c^{-1}_c(U_n) \text{ s.t. } d_Y(y, z) < \delta \} \right) \rightarrow p\left( c^{-1}_c(U_n) \right).
\]
Therefore, for any \( \varepsilon > 0 \) there is \( \alpha \in \mathbb{R}^N \) with \( \alpha_n > 0, n \in \mathbb{N} \) such that
\[
p\left( \{ y \in Y : \exists \alpha \in c^{-1}_c(U_n) \text{ s.t. } d_Y(y, z) < \alpha_n \} \right) < \frac{\varepsilon}{2^n},
\]
whence
\[
\delta_y \otimes p(A(\alpha)) \leq \sum_{n \in \mathbb{N}} p\left( \{ y \in Y : \exists \alpha \in c^{-1}_c(U_n) \text{ s.t. } d_Y(y, z) < \alpha_n \} \setminus c^{-1}_c(U_n) \right) < \varepsilon.
\]
The set
\[
M^\varepsilon := \{ (x, p, \alpha) \in X \times \mathcal{P}(Y) \times \mathbb{R}^N : \delta_y \otimes p(A(\alpha)) < \varepsilon \}
\]
satisfies that \( \text{proj}_{X \times \mathcal{P}(Y)}(M^\varepsilon) = X \times \mathcal{P}(Y). \)

We claim that \( (x, p, \alpha) \mapsto \delta_y \otimes p(A(\alpha)) \) and \( M^\varepsilon \) are both Borel measurable: Evidently, \( \alpha \mapsto A(\alpha) \) meets the requirements of Lemma \([6,4]\) below, and consequently the function
\[(x,y,a) \mapsto f_a(x,y,a) := \mathbb{I}_{A_{\alpha_0}}(x,y)\] is Borel measurable. From here, we deduce Borel measurability of
\[(x,p,a) \mapsto \delta_x \otimes p([x] \times B \setminus A(a)) = \int_B [1 - f_a(x,y,a)] \, p(dy),\]
for every \(B \in \mathcal{B}(Y)\), which yields our claim.

Finally, we are in a position to define \(\hat{\Gamma}\): Since \(\Gamma\) is analytic and \(M^\varepsilon\) is Borel measurable, the set \(M^\varepsilon = M^\varepsilon \cap \Gamma \times \mathbb{R}^N\) is yet again analytic. Consider the analytic set
\[
\Theta^\varepsilon := \{(x,p,a,y) \in X \times \mathcal{P}(Y) \times \mathbb{R}^N \times Y : \exists (x,p,a) \in M^\varepsilon \text{ s.t. } y \in \text{supp}(p), \mathbb{I}_{A_{\alpha_0}}(x,y) = 0\}
\]
and let \(\hat{\Theta} = \Theta^\varepsilon \cap \{(x,p,a,y) \in X \times \mathcal{P}(Y) \times \mathbb{R}^N \times Y : y \in \text{supp}(p), f_A(x,y,a) = 0\}\), where we used that \(Y\) admits a countable basis \((O_k)_{k \in \mathbb{N}}\) of its topology, whereby
\[y \in \text{supp}(p) \iff \forall k \in \mathbb{N}: \text{ either } p(O_k) > 0 \text{ or } y \notin O_k.\]
Since \(\Theta^\varepsilon\) is analytic, its \(X \times Y\)-projection is analytic too. Thus,
\[
\hat{\Gamma} := \bigcup_{k \in \mathbb{N}} \text{proj}_{X \times Y} \Theta^\varepsilon^k
\]
is analytic. Recall that for any \((x,p) \in \Gamma\) and \(k \in \mathbb{N}\) there is \(\alpha_k \in (0,\infty)^N\) with \(\delta_x \otimes p(A(x^k)) < \frac{1}{k}\). As for every \(k \in \mathbb{N}\) we have
\[
p(\hat{\Gamma}_x) = \delta_x \otimes p(\hat{\Gamma} \cap X \times \text{supp}(p)) \geq \delta_x \otimes p([x] \times \{y \in Y : f_A(x,y,a^k) = 0\})
\]
we derive that \(\hat{\Gamma}\) satisfies item \(\boxed{[\text{i}]\rangle}\).

On the other hand, if \((x,y) \in \hat{\Gamma}\), then there is \(k \in \mathbb{N}\) and \((x,p,a) \in M^\varepsilon\) with \((x,p,a,y) \in \Theta^\varepsilon\). Set \(K := \text{proj}_Y(A(x^k) \cap [x] \times Y) \cap \text{supp}(p)\) and note that \([x] \times \{p\} \times \{a\} \times K \subseteq \Theta^\varepsilon\), that means \(K \subseteq \Gamma_x\). Recall \(\boxed{[\text{6.2}]\rangle}\). By the paragraph preceding \(\boxed{[\text{6.2}]\rangle}\), we have continuity of \(c_{[x] \times K}\), and conclude that item \(\boxed{[\text{ii}]\rangle}\) is satisfied. \(\square\)

**Lemma 6.4.** Let \(A : \mathbb{R}^N_+ \to \mathcal{B}(X \times Y)\) be given s.t.

\(\begin{align*}
(a) & \text{ for all } [\alpha]_{n \in \mathbb{N}} \subseteq \mathbb{R}^N_+, \alpha \leq \beta \text{ we have } A(\alpha) \subseteq A(\beta),
(b) & \text{ for any } [\alpha^k]_{k \in \mathbb{N}} \subseteq \mathbb{R}^N_+ \text{ with } \alpha_k \nearrow \alpha, n \in \mathbb{N} \text{ and } \alpha = (\alpha_n)_{n \in \mathbb{N}} \in \mathbb{R}^N_+, \text{ we have } A(\alpha^k) \nearrow A(\alpha).
\end{align*}\)

Then the function
\[
f_A : X \times Y \times \mathbb{R}^N_+ \to \mathbb{R} : (x,y,a) \mapsto \mathbb{I}_{A_{\alpha_0}}(x,y)
\]
is Borel measurable.

**Proof.** The function \(f_A\) is the indicator function of the set
\[
\mathcal{A} := \{(x,y,a) \in X \times Y \times \mathbb{R}^N_+ : (x,y) \in A(\alpha)\} = \bigcup_{\alpha \in \mathcal{Q}} A(\alpha) \times \{\alpha\}.
\]
We show that \(\mathcal{A}\) coincides with the Borel measurable set \(\mathcal{A}'\),
\[
\mathcal{A}' := \bigcup_{q \in \mathcal{Q}} A(q) \times \bigotimes_{n \in \mathbb{N}} [a_n, \infty),
\]
where \(\mathcal{Q} := \{\alpha \in \mathcal{Q}^N_+ : \exists n \in \mathbb{N} \text{ s.t. } a_k = 0 \forall k \geq n\}\.\) From property \((a)\) of \(A\) we deduce \(\mathcal{A} \supseteq \mathcal{A}'\). Now let \((x,y,a) \in \mathcal{A}\), then we find a sequence \([q^k]_{k \in \mathbb{N}} \subseteq \mathcal{Q}\) with \(q^k \nearrow \alpha\) and by \((b)\), there exists an index \(k_0 \in \mathbb{N}\) such that \((x,y) \in A(q^k)\) for all \(k \geq k_0\). Thus,
\[
(x,y,a) \in A(q^{k_0}) \times \bigotimes_{n \in \mathbb{N}} [a_{k_0}^n, \infty) \subseteq \mathcal{A'},
\]
and \(\mathcal{A} \subseteq \mathcal{A}'\). Therefore \(\mathcal{A}\) is a Borel measurable set and \(f_A\) a Borel measurable function. \(\square\)

---

5In this part we write \(\alpha \leq \beta \iff \alpha_i \leq \beta_i\) for all \(i\).
We conclude this section by showing the following measurable variant of the Lebesgue decomposition theorem, quoted in Example 5.2.

**Proposition 6.5.** Let $\mathcal{M}_{\pi}(X)$ the space of all finite measures on $X$ be equipped with the topology of weak convergence of measures. Then the map

$$T : \mathcal{M}_{\pi}(X) \times \mathcal{M}_{\pi}(X) \rightarrow \mathcal{M}_{\pi}(X) \times \mathcal{M}_{\pi}(X),$$

$$(p, q) \mapsto (q_{ac.p}, \mathcal{O}_{p}).$$

where $T(p, q)$ is the unique Lebesgue decomposition of $q$ w.r.t. $p$, i.e.,

$$q_{ac.p} + \mathcal{O}_{p} = q, \quad q_{ac.p} \ll p, \quad \mathcal{O}_{p} \perp p,$$

is measurable.

**Proof.** Essentially we are defining a family of measurable functions: for any $\delta > 0$ and $A \in \mathcal{B}(X)$ let

$$F_{\delta,A} : \mathcal{M}_{\pi}(X) \times \mathcal{M}_{\pi}(X) \times \mathcal{B}(X) \rightarrow \mathbb{R},$$

$$(p, q, B) \mapsto \begin{cases} q(A \setminus B) & p(B) < \delta, \\ q(A) & \text{else}. \end{cases} \quad (6.3)$$

In turn, this allows us to define $T(p, q)(A)$ as a the limit of countable infima of measurable functions. Since $X$ is Polish there exists a countable family $O$ of open sets on $X$ such that $\sigma(O) = \mathcal{B}(X)$. Denote by $\mathcal{R}(O)$ the set-theoretic ring generated by $O$, which is again countable. Then, for any $\varepsilon > 0$, $A \in \mathcal{B}(X)$ and $p \in \mathcal{P}(X)$ there is a set $A_\varepsilon \in \mathcal{R}(O)$ such that

$$p(A \setminus A_\varepsilon \cup A_\varepsilon \setminus A) < \varepsilon. \quad (6.4)$$

Using the classical Lebesgue decomposition theorem, for any pair $(p, q) \in \mathcal{M}_{\pi}(X) \times \mathcal{M}_{\pi}(X)$ there is a set $B \subseteq X$ with $p(B) = 0$ and

$$q_{ac.p}(A) = q(A \setminus \tilde{B}) \quad \forall A \in \mathcal{B}(X).$$

By [16, Proposition 4.5.3] we have

$$\lim_{\delta \searrow 0} \sup_{B \in \mathcal{B}(X) : p(B) < \delta} q_{ac.p}(B) = 0,$$

whence,

$$q_{ac.p}(A) = \lim_{\delta \searrow 0} \inf_{B \in \mathcal{B}(X) : p(B) < \delta} q_{ac.p}(A \setminus B)$$

$$= \lim_{\delta \searrow 0} \inf_{B \in \mathcal{B}(X) : p(B) < \delta} q_{ac.p}(A \setminus (\tilde{B} \cup B)) + \mathcal{O}_{p}(A \setminus (\tilde{B} \cup B))$$

$$= \lim_{\delta \searrow 0} \inf_{B \in \mathcal{B}(X) : p(B) < \delta} q(A \setminus B)$$

$$= \lim_{\delta \searrow 0} \inf_{B \in \mathcal{R}(O) : p(B) < \delta} q(A \setminus B),$$

where we used for the last equality the approximation property (6.4).

Thus $F_A : \mathcal{M}_{\pi}(X) \times \mathcal{M}_{\pi}(X) \rightarrow \mathbb{R}$ defined by

$$F_A (p, q) := q_{ac.p}(A) = \inf_{\delta \searrow 0} \inf_{\{B \in \mathcal{B}(X) : p(B) < \delta\}} q(A \setminus B) = \lim_{k \rightarrow \infty} \inf_{\mathcal{R}(O)} F_{\delta,k}(p, q, B) \quad (6.5)$$

is Borel measurable, and as $A \in \mathcal{B}(X)$ was arbitrary we conclude that $T$ is measurable. $\square$

**References**


