DICTATOR FUNCTIONS MAXIMIZE MUTUAL INFORMATION

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Let \((X, Y)\) denote \(n\) independent, identically distributed copies of two arbitrarily correlated Rademacher random variables \((X, Y)\). We prove that the inequality \(I(f(X); g(Y)) \leq I(X; Y)\) holds for any two Boolean functions: \(f, g: \{-1, 1\}^n \rightarrow \{-1, 1\}\) (\(I(\cdot; \cdot)\) denotes mutual information). We further show that equality in general is achieved only by the dictator functions \(f(x) = \pm g(x) = \pm x_i, i \in \{1, 2, \ldots, n\}\).

1. Introduction and Main Results. Let \((X, Y)\) be two dependent Rademacher random variables on \(\{-1, 1\}\), with correlation coefficient \(\rho := \mathbb{E}[XY] \in [-1, 1]\). For given \(n \in \mathbb{N}\), let \((X, Y) = (X, Y)^n\) be \(n\) independent, identically distributed copies of \((X, Y)\). We will use the notation from [3] for information-theoretic quantities. In particular, \(\mathbb{E}[X], \mathbb{H}(X),\) and \(I(X; Y)\) denote expectation, entropy, and mutual information, respectively. Motivated by problems in computational biology [4], Kumar and Courtade formulated the following conjecture [5, Conjecture 1].

CONJECTURE 1. For any Boolean function \(f: \{-1, 1\}^n \rightarrow \{-1, 1\}\),

\[ I(f(X); Y) \leq I(X; Y). \]  

This claim – while seemingly innocent at first sight – has received significant interest and resisted several efforts to find a proof (see the discussion in [2, Section IV]). Note that \(f = \chi_i\) for any dictator function [6, Definition 2.3] \(\chi_i(x) := x_i, i \in \{1, 2, \ldots, n\}\) achieves equality in (1).

We next state the main result of this paper, which is a relaxed version of Conjecture 1, involving two Boolean functions.

THEOREM 1. For any two Boolean functions \(f, g: \{-1, 1\}^n \rightarrow \{-1, 1\}\),

\[ I(f(X); g(Y)) \leq I(X; Y). \]

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If (1) were true, this statement would readily follow from the data processing inequality [3, Theorem 2.8.1]. Theorem 1 was stated as an open problem in [2] and [5, Section IV], and separately investigated in [1]. A proof of (2) was previously available only under the additional restrictive assumptions that $f$ and $g$ are equally biased (i.e., $E[f(X)] = E[g(X)]$) and satisfy the condition

\begin{equation}
P\{f(X) = 1, g(X) = 1\} \geq P\{f(X) = 1\}P\{g(X) = 1\}.
\end{equation}

The reader is invited to see [2, Section IV] for further details. In this paper, we use Fourier-analytic tools to prove Theorem 1 without any additional restrictions on $f$ and $g$. We suitably bound the Fourier coefficients of $f$ and $g$, and thereby reduce (2) to an elementary inequality, which is subsequently established.

A careful inspection of the proof of Theorem 1 reveals that in general, up to sign changes, the dictator functions $\chi_i, i \in \{1, 2, \ldots, n\}$ are the unique maximizers of $I(f(X); g(Y))$.

**Proposition 1.** If $0 < |\rho| < 1$, equality in (2) is achieved if and only if $f = \pm g = \pm \chi_i$ for some $i \in \{1, 2, \ldots, n\}$.

**2. Proof of Theorem 1.** Define $[n] := \{1, 2, \ldots, n\}$ and let $f, g$ be two Boolean functions on the Boolean hypercube, i.e., $f, g: \{-1, 1\}^n \to \{-1, 1\}$. Denote their Fourier expansions (cf. [6, (1.6)])

\begin{equation}
f(x) = \sum_{S \subseteq [n]} \hat{f}_S \chi_S(x) \quad \text{and} \quad g(x) = \sum_{S \subseteq [n]} \hat{g}_S \chi_S(x),
\end{equation}

using the basis \(\chi_S(x) := \prod_{i \in S} x_i\) for $S \subseteq [n]$. Define $a := \frac{1 + \hat{f}_\emptyset}{2} = P\{f(X) = 1\}, b := \frac{1 + \hat{g}_\emptyset}{2} = P\{g(X) = 1\}$ and $\theta_\rho := \frac{1}{4} \sum_{|S| \geq 1} |\hat{f}_S \hat{g}_S \rho|^{|S|}$. Without loss of generality, we may assume $\frac{1}{2} \leq a \leq b \leq 1$ and $\rho \in [0, 1]$, as mutual information is symmetric and we have, with $Y^* := \text{sgn}(\rho)Y$,

\begin{equation}
I(f(X); g(Y)) = I(\text{sgn}(\hat{f}_\emptyset)f(X); \text{sgn}(\hat{g}_\emptyset)g(\text{sgn}(\rho)Y^*)).
\end{equation}

In analogy to [6, Proposition 1.9], the inner product satisfies

\begin{equation}
\langle f, T_\rho g \rangle = E[f(X)g(Y)] = \hat{f}_\emptyset \hat{g}_\emptyset + 4\theta_\rho = 1 - 2P\{f(X) \neq g(Y)\},
\end{equation}

where $T_\rho$ is the noise operator [6, Definition 2.46]. Defining $\bar{t} := 1 - t$ for a generic $t$, we can express the probabilities

\begin{align}
&\text{(6)} \quad P\{f(X) = 1, g(Y) = -1\} = ab - \theta_\rho, \quad P\{f(X) = g(Y) = 1\} = ab + \theta_\rho, \\
&\text{(7)} \quad P\{f(X) = -1, g(Y) = 1\} = \bar{a}b - \theta_\rho, \quad P\{f(X) = g(Y) = -1\} = \bar{a}b + \theta_\rho,
\end{align}
Using (6), (7) and fundamental properties of mutual information [3, Section 2.4], we obtain

\[ I(f(X); g(Y)) = \xi(\theta, a, b) \]

with

\[ \xi(\theta, a, b) := H(a) + H(b) - H(ab + \theta, \bar{a}b - \theta, \bar{a}b + \theta), \]

where, slightly abusing notation, we defined the binary entropy function

\[ H(p) := H(p, \bar{p}) \quad \text{and} \quad H((p_i)_{i \in I}) := -\sum_{i \in I} p_i \log_2 p_i \quad \text{for} \ |I| > 1. \]

By the non-negativity of probabilities (6) and (7), for any \( \rho \in [0, 1] \),

\[ -\bar{a}b \leq \theta \rho \leq \rho \bar{a}b. \]

With \( P := \{ S \subseteq [n] : \hat{f}_S \hat{g}_S > 0 \} \setminus \{ \emptyset \} \) and \( N := \{ S \subseteq [n] : \hat{f}_S \hat{g}_S < 0 \} \), we define

\[ \tau^+ := \frac{1}{4} \sum_{S \in P} \hat{f}_S \hat{g}_S, \quad \tau^- := \frac{1}{4} \sum_{S \in N} \hat{f}_S \hat{g}_S \]

and apply the Schwarz inequality to show

\[ \tau^+ - \tau^- = \frac{1}{4} \sum_{S:|S|\geq1} |\hat{f}_S||\hat{g}_S| \leq \frac{1}{4} \sqrt{(1 - \hat{f}_\emptyset^2)(1 - \hat{g}_\emptyset^2)} = \sqrt{a\bar{a}b\bar{b}}. \]

As \( \theta_1 = \tau^+ + \tau^- \), we combine (9) and (12) to obtain

\[ \tau^+ \leq \frac{ab + \sqrt{a\bar{a}b\bar{b}}}{2}, \quad \tau^- \geq -\bar{a}b + \sqrt{a\bar{a}b\bar{b}} \]

By definition, \( \rho \tau^- \leq \theta_\rho \leq \rho \tau^+ \) and hence, \( \theta_\rho \in [\theta^-_\rho, \theta^+_\rho] \), where

\[ \theta^-_\rho := \max \left\{ -\bar{a}b, -\rho \bar{a}b + \sqrt{a\bar{a}b\bar{b}} \right\}, \quad \theta^+_\rho := \min \left\{ \bar{a}b, \rho \bar{a}b + \sqrt{a\bar{a}b\bar{b}} \right\}. \]

The function \( \xi(\theta, \alpha, \beta) \) is convex in \( \theta \) by the concavity of entropy [3, Theorem 2.7.3] and consequently, \( I(f(X); g(Y)) \leq \max_{\theta \in \{\theta^+_\rho, \theta^-_\rho\}} \xi(\theta, a, b) \). Thus, Theorem 1 can be proved by establishing \( 1 - H\left(\frac{\rho + 1}{2}\right) - \xi(\theta, a, b) \geq 0 \) for \( \theta \in \{\theta^+_\rho, \theta^-_\rho\} \). Furthermore, it suffices to consider \( \frac{1}{2} < a < b < 1 \) by continuity of \( \xi \).

Define \( C_{a,b} := \frac{ab + \sqrt{a\bar{a}b\bar{b}}}{2} \), \( \rho^+ := \min \left\{ \rho, \frac{a\bar{b}}{C_{a,b}} \right\}, \rho^- := \min \left\{ \rho, \frac{\bar{a}b}{C_{a,b}} \right\} \), and

\[ \phi(\rho, a, b) := 1 - H\left(\frac{\rho + 1}{2}\right) - \xi(\rho C_{a,b}, a, b). \]
Note that
\begin{align}
\phi(\rho^+, a, b) &= 1 - H\left(\frac{\rho^+ + 1}{2}\right) - \xi(\theta^+, a, b) \\
&\leq 1 - H\left(\frac{\rho + 1}{2}\right) - \xi(\theta^+, a, b) \tag{16}
\end{align}
by the monotonicity of the binary entropy function and accordingly we also have \( \phi(\rho^-, \bar{a}, b) \leq 1 - H\left(\frac{\rho + 1}{2}\right) - \xi(\theta^-, a, b) \). Theorem 1 thus follows from the following lemma.

**Lemma 1.** For \( 0 < \alpha < \beta < 1 \) and \( \rho \in \left[0, \frac{\alpha \beta}{\alpha - \beta}\right] \), we have \( \phi(\rho, \alpha, \beta) \geq 0 \) with equality if and only if \( \rho = 0 \).

Before proving Lemma 1, we note the following facts.

**Lemma 2.** For \( x \in (0, 1) \), we have
\[ \frac{1}{x^2 - 1} + \log(1 - x) > 0. \tag{18} \]

**Proof.** Using Taylor series expansion, we immediately obtain
\[ -\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n} < \sum_{n=1}^{\infty} x^n = \frac{x}{1 - x}. \tag{19} \]

The following lemma collects elementary facts about convex/concave functions and follows from elementary properties of convex functions on the real line (see, e.g., [7, Chapter I]).

**Lemma 3.** Let \( f: U \to \mathbb{R} \) be a continuous function, defined on the compact interval \( U := [u_1, u_2] \subset \mathbb{R} \). Assuming that \( f \) is twice differentiable on \( V \), where \( (u_1, u_2) \subseteq V \subseteq U \), the following properties hold.

1. If \( f''(u) \geq 0 \) for all \( u \in (u_1, u_2) \) and \( f'(u^*) = 0 \) for some \( u^* \in V \), then \( f(u) \geq f(u^*) \) for all \( u \in U \). Furthermore, if additionally \( f''(u) > 0 \) for all \( u \in (u_1, u_2) \), then \( f(u) > f(u^*) \) for all \( u \in U \setminus \{u^*\} \).
2. If \( f''(u) \leq 0 \) for all \( u \in (u_1, u_2) \), then \( f(u) \geq \min\{f(u_1), f(u_2)\} \) for all \( u \in U \). Furthermore, if \( f''(u) < 0 \) for all \( u \in (u_1, u_2) \), then \( f(u) > \min\{f(u_1), f(u_2)\} \) for all \( u \in (u_1, u_2) \).
Proof of Lemma 1. Let $I := \{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < \beta < 1\}$, fix arbitrary $(\alpha, \beta) \in I$ and define

\begin{equation}
\rho_- := \max\{\alpha \beta, \bar{\alpha} \bar{\beta}\} / C_{\alpha \beta}, \quad \rho_0 := \min\{\alpha \bar{\beta}, \alpha \bar{\beta}\} / C_{\alpha \beta}, \quad \rho_+ := \alpha \bar{\beta} / C_{\alpha \beta}.
\end{equation}

We shall adopt the simplified notation $\phi(\rho) := \phi(\rho, \alpha, \beta)$, suppressing the fixed parameters $(\alpha, \beta)$. For $\rho \in [0, \rho_+]$, we have the derivatives

\begin{equation}
\phi'(\rho) = \frac{1}{2} \log_2 \left( \frac{1 + \rho}{1 - \rho} \right) + C_{\alpha \beta} \log_2 \left( \frac{(\bar{\alpha} \bar{\beta} - C_{\alpha \beta} \rho)(\alpha \bar{\beta} - C_{\alpha \beta} \rho)}{(\alpha \beta + C_{\alpha \beta} \rho)(\bar{\alpha} \beta + C_{\alpha \beta} \rho)} \right),
\end{equation}

\begin{equation}
\phi''(\rho) = \frac{C_{\alpha \beta}^2}{\log_2(1 - \rho^2)} - \frac{1}{\bar{\alpha} \beta - C_{\alpha \beta} \rho} - \frac{1}{\alpha \beta + C_{\alpha \beta} \rho} - \frac{1}{\alpha \beta + C_{\alpha \beta} \rho}.
\end{equation}

We write $\phi''(\rho) = \frac{p(\rho)}{q(\rho)}$, where both $p$ and $q$ are polynomials in $\rho$, and choose

\begin{equation}
q(\rho) = \log(2)(1 - \rho^2)(\bar{\alpha} \beta - C_{\alpha \beta} \rho)
\end{equation}

\begin{equation}
\times (\bar{\alpha} \beta - C_{\alpha \beta} \rho)(\alpha \bar{\beta} + C_{\alpha \beta} \rho)(\alpha \beta + C_{\alpha \beta} \rho),
\end{equation}

such that $q(\rho) > 0$ for $\rho \in (0, \rho_+)$. By (22), $p(\rho)$ is given by

\begin{equation}
p(\rho) = (\bar{\alpha} \beta - C_{\alpha \beta} \rho)(\alpha \bar{\beta} - C_{\alpha \beta} \rho)(\alpha \beta + C_{\alpha \beta} \rho)
\end{equation}

\begin{equation}
- C_{\alpha \beta}^2(1 - \rho^2)(\bar{\alpha} \beta - C_{\alpha \beta} \rho)(\bar{\alpha} \beta + C_{\alpha \beta} \rho)(\alpha \beta + C_{\alpha \beta} \rho)
\end{equation}

\begin{equation}
+ (\bar{\alpha} \beta - C_{\alpha \beta} \rho)(\alpha \bar{\beta} + C_{\alpha \beta} \rho)(\alpha \beta + C_{\alpha \beta} \rho)
\end{equation}

\begin{equation}
+ (\bar{\alpha} \beta - C_{\alpha \beta} \rho)(\alpha \beta - C_{\alpha \beta} \rho)(\alpha \beta + C_{\alpha \beta} \rho)
\end{equation}

\begin{equation}
+ (\bar{\alpha} \beta - C_{\alpha \beta} \rho)(\alpha \beta - C_{\alpha \beta} \rho)(\bar{\alpha} \beta + C_{\alpha \beta} \rho).
\end{equation}

This entails $\deg(p) \leq 5$ and a careful calculation of the coefficients reveals $\deg(p) \leq 3$.

We will now demonstrate that there is a unique point $\rho^* \in (0, \rho_+)$, such that $p(\rho^*) = 0$. To this end, reinterpret $\phi''(\rho)$ as a rational function of $\rho$ on $\mathbb{R}$. By evaluating (24), we obtain the two inequalities

\begin{equation}
p(0) = \alpha \bar{\alpha} \beta \bar{\beta}(\alpha \bar{\alpha} \beta \bar{\beta} - C_{\alpha \beta}^2) > 0,
\end{equation}

\begin{equation}
p(\rho_+) = -(C_{\alpha \beta}^2 - (\alpha \bar{\beta})^2)(\beta - \alpha) \bar{\beta} \alpha < 0,
\end{equation}

from $\alpha < \beta$. The number of roots of $p$ in $(0, \rho_+)$ is thus odd and at most equal to its degree, i.e., either one or three. If we have $\rho_0 \leq 1$, then evaluation
of \((24)\) readily yields \(p(-\rho_o) \leq 0\). If, on the other hand, \(\rho_o > 1\), we obtain \(p(-\rho_-) \leq 0\) from \((24)\). Thus, \(p\) has at least one negative root and a unique root \(\rho^* \in (0, \rho_+)\). Figure 1 qualitatively illustrate the behavior of \(p(\rho)\) and \(\phi''(\rho)\).

Consequently, \(\phi''(\rho) > 0\) for \(\rho \in (0, \rho^*)\). By part 1 of Lemma 3, \(\phi(\rho) > \phi(0) = 0\) for \(\rho \in (0, \rho^*)\) as \(\phi'(0) = 0\). Since \(\phi''(\rho) < 0\) for \(\rho \in (\rho^*, \rho_+)\), we have \(\phi(\rho) > \min\{\phi(\rho^*), \phi(\rho_+)\}\) for all \(\rho \in (\rho^*, \rho_+)\), by part 2 of Lemma 3. In total, \(\phi(\rho) > \min\{0, \phi(\rho_+)\}\) for \(\rho \in (0, \rho_+)\).

As \(\phi(0) = 0\), it remains to show that \(\phi(\rho_+, \alpha, \beta) > 0\) for \((\alpha, \beta) \in I\). To this end, we introduce the transformation

\[
(27) \quad (\alpha, \beta) \mapsto (c, x) := \left(\frac{\log \frac{\alpha}{\beta}}{\log \frac{\alpha x}{\beta x}}, \sqrt{\frac{\alpha x}{\beta x}}\right),
\]

a bijective mapping from \(I\) to \((0, 1)^2\) with the inverse

\[
(28) \quad (c, x) \mapsto (\alpha, \beta) = \left(\frac{x^2 - x^2}{1 - x^2}, \frac{1 - x^2 - 2c}{1 - x^2}\right).
\]

In terms of \(c\) and \(x\), we have \(\phi(\rho_+, \alpha, \beta) = \psi(c, x)\), where

\[
(29) \quad \psi(c, x) := 1 - H\left(\frac{1}{2} + \frac{x}{1 + x}\right) - H\left(\frac{x^2 - x^2}{1 - x^2}\right) + \frac{1 - x^2 - 2c}{1 - x^2}H(x^2)
\]
\[ (30) \quad 0 = 1 - \frac{1 + 3x}{2 + 2x} + \frac{H(x^2)}{1 - x^2} + \frac{x^{2c}H(x^{2-2c}) + x^{2-2c}H(x^{2c})}{x^2 - 1}. \]

We fix a particular \( x \in (0, 1) \) and use the simplified notation \( \psi(c) := \psi(c, x) \), obtaining the derivatives

\[ (31) \quad \psi'(c) = \frac{2 \log(x)}{(x^2 - 1) \log(2)} \left[ 2ax^{2c} \log(x) + x^{2(1-c)} \log(1 - x^{2c}) - x^{2c} \log(x^{2c} - x^2) \right], \]

\[ (32) \quad \psi''(c) = \frac{4 \log(x)^2 x^{2c}}{(1 - x^2) \log(2)} \left[ \frac{1}{x^{-2(1-c)} - 1} + \log(1 - x^{2(1-c)}) \right] + \frac{x^2}{x^{4c}} \left( \log(1 - x^{2c}) + \frac{1}{x^{2c} - 1} \right). \]

By applying Lemma 2 twice, we obtain \( \psi''(c) > 0 \). Thus, \( \psi(c) > \psi(\frac{1}{2}) \) by part 1 of Lemma 3 as \( \psi'(\frac{1}{2}) = 0 \). It remains to show that \( \gamma := \psi(\frac{1}{2}, x) > 0 \). Note that \( \gamma(0) = \gamma(1) = 0 \) and

\[ (33) \quad \gamma'(x) = \frac{1}{(1 + x)^2} \log_2 \left[ (1 + 3x)(1 - x) \right], \]

for \( x \in [0, 1) \). If \( \gamma(x) \leq 0 \) for any \( x \in (0, 1) \) then \( f \) necessarily attains its minimum in \((0, 1)\) and there exists \( x^* \in (0, 1) \) with \( \gamma(x^*) \leq 0 \) and \( \gamma'(x^*) = 0 \). As \( x^* = \frac{2}{3} \) is the only point in \((0, 1)\) with \( \gamma'(x^*) = 0 \) and \( \gamma(\frac{2}{3}) = \log_2(\frac{27}{25}) > 0 \), this concludes the proof. \( \square \)

3. Proof of Proposition 1. We may assume \( 0 < \rho < 1 \) and \( \frac{1}{2} \leq a \leq b \leq 1 \) by virtue of (4). Clearly, \( g = \pm f = \pm \chi_i \) for some \( i \in [n] \) is a sufficient condition to maximize \( I(f(X); g(Y)) \). A careful inspection of the proof of Theorem 1 shows that this condition is also necessary.

In the following, we will use the notation of Section 2. As \( b = 1 \) implies \( I(f(X); g(Y)) = 0 \), we assume \( \frac{1}{2} \leq a \leq b < 1 \). For equality in Theorem 1, we need either \( \phi(\rho^+, a, b) = 0 \) or \( \phi(\rho^-, a, b) = 0 \). By Lemma 1, \( \phi(\rho^-, a, b) > 0 \) unless \( \bar{a} = a = \frac{1}{2} \), which in turn implies \( \phi(\rho^-, a, b) = \phi(\rho^+, a, b) \). The equality \( \phi(\rho^+, a, b) = 0 \) can only occur for \( b = a \), implying \( \rho^+ = \rho \). We want to show that \( \phi(\rho, a, a) = 0 \) implies \( a = \frac{1}{2} \). For \( a \neq \frac{1}{2} \) we have

\[ (34) \quad \frac{\partial \phi}{\partial \rho}(\rho, a, a) = \frac{1}{2} \log_2 \left( \frac{1 + \rho}{1 - \rho} \right) - a\bar{a} \log_2 \left( \frac{\rho}{a\bar{a}\rho^2} + 1 \right), \]
\[
\frac{\partial^2 \phi}{\partial \rho^2}(\rho, a, a) = \frac{\rho(1 - 2a)^2}{\log(2)(a + \rho \bar{a})(1 - a \bar{a})(1 - \rho^2)} > 0.
\]

Part 1 of Lemma 3 now yields \(0 = \phi(0, a, a) < \phi(\rho, a, a)\) as \(\frac{\partial \phi}{\partial \rho}(0, a, a) = 0\). By the strict convexity of \(\xi(\theta, \frac{1}{2}, \frac{1}{2})\) in \(\theta\), necessarily \(\theta_\rho = \frac{\langle f, T_\rho g \rangle}{2} \in \{\theta_\rho^+, \theta_\rho^-\} = \pm \frac{\rho}{4}\). The Cauchy-Schwarz inequality together with [6, Proposition 2.50] yields \(\rho^2 = \langle f, T_\rho g \rangle^2 = \langle T_{\sqrt{\rho} f}, T_{\sqrt{\rho} g} \rangle^2 \leq \langle f, T_\rho f \rangle \langle g, T_\rho g \rangle \leq \rho^2\). Thus, necessarily \(g = \pm f = \pm \chi_i\) for some \(i \in [n]\) by [6, Proposition 2.50].

4. Discussion. The key idea underlying the proof of Theorem 1 is to split \(\theta_1 = \tau^+ + \tau^-\) into its positive and negative part (see Section 2). After reducing the problem to the inequality in Lemma 1, the remaining proof is routine analysis. However, Lemma 1 might turn out to be useful in the context of other converse proofs, in particular for the optimization of rate regions with binary random variables.

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References.