

# Tightness for the Interface of the One-Dimensional Contact Process

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## Abstract

We consider a symmetric, finite range contact process with two types of infection; both have the same (supercritical) infection rate and heal at rate 1, but sites infected by Infection 1 are immune to Infection 2. We take the initial configuration where sites in  $(-\infty, 0]$  have Infection 1 and sites in  $[1, \infty)$  have Infection 2, then consider the process  $\rho_t$  defined as the size of the interface area between the two infections at time  $t$ . We show that the distribution of  $\rho_t$  is tight, thus proving a conjecture posed by Cox and Durrett in [3].

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# 1 Introduction

This paper addresses a conjecture of Cox and Durrett [3] concerning interfaces naturally arising in supercritical contact processes on  $\mathbb{Z}^d$ .

The contact process on  $\mathbb{Z}^d$  is a spin system with operator

$$\Omega f(\eta) = \sum_x (f(\eta^x) - f(\eta))c(x, \eta); \quad \eta \in \{0, 1\}^{\mathbb{Z}^d},$$

where

$$\begin{cases} \eta^x(y) = \eta(y) & \text{if } y \neq x; \\ \eta^x(x) = 1 - \eta(x) \end{cases}$$

and flip rates  $c(x, \eta)$  are given by

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1; \\ \lambda \sum p(y - x)\eta(y) & \text{if } \eta(x) = 0, \end{cases}$$

for  $\lambda > 0$  and probability kernel  $p(\cdot)$ .

In the following we take  $p(\cdot)$  to have finite range (*i.e.*  $\exists M < \infty : p(x) = 0$  for  $|x| > M$ ) and to be symmetric, though this latter hypothesis can be dispensed with via the techniques and results of Bezuidenhout and Gray [2].

Often the contact process is used as a model of the spread of an infection and a configuration  $\eta \in \{0, 1\}^{\mathbb{Z}^d}$  represents the state where there is an infection at  $x \in \mathbb{Z}^d$  if and only if  $\eta(x) = 1$ . We will adopt this point of view and speak of a site  $x$  being *infected* at time  $t$  (for a process  $(\eta_t : t \geq 0)$ ) if  $\eta_t(x) = 1$ . Sometimes we will identify configurations in  $\{0, 1\}^{\mathbb{Z}^d}$  with their sets of infected sites (*i.e.*, we will write  $\xi$  instead of  $\{x : \xi(x) = 1\}$ ). As defined above the contact process is attractive (see [8] for fundamental results associated with this property). Thus for two configurations  $\xi_0$  and  $\zeta_0$  satisfying  $\xi_0 \leq \zeta_0$  under the natural partial order, it is possible to construct in a single probability space two processes  $(\xi_t : t \geq 0)$  starting at  $\xi_0$  and  $(\zeta_t : t \geq 0)$  starting at  $\zeta_0$  and satisfying, with probability one,  $\xi_t \leq \zeta_t$  for all  $t$ .

A consequence is that  $\exists \lambda_c^1$  so that for  $\lambda > \lambda_c^1$  the invariant limit  $\lim_{t \rightarrow \infty} \delta_{\mathbb{1}} S(t)$  is a non-trivial measure and for  $\lambda < \lambda_c^1$  this limit is  $\delta_0$ . There also exists  $\lambda_c^2$  so that for  $\lambda > \lambda_c^2$ ,  $P^{\{0\}}(\tau = \infty) > 0$  for  $\tau = \inf\{t : \eta_t \equiv 0\}$ ; for  $\lambda < \lambda_c^2$ ,  $P^{\{0\}}(\tau = \infty) = 0$ . In fact via duality (see e.g. [8] or [4]),  $\lambda_c^1 = \lambda_c^2$  and this critical value will henceforth be denoted by  $\lambda_c$ .

We now introduce some notation. Suppose given independent Poisson processes on  $[0, \infty)$ ,  $\{D_x\}_{x \in \mathbb{Z}^d}$  of rate 1 and  $\{N^{(x,y)}\}_{x,y \in \mathbb{Z}^d}$  of rate  $\lambda p(y - x)$ . Denote by  $H$  a realization of all these independent processes; we say that  $H$  is a *Harris construction*.  $H$  is thus a Poisson measure on  $(\mathbb{Z}^d \cup (\mathbb{Z}^d)^2) \times [0, \infty)$  such that, if  $y, z \in \mathbb{Z}^d$  and  $I$  is a Borelian subset of  $[0, \infty)$ , we have  $H(\{z\} \times I) = D_z(I)$  and  $H(\{(y, z)\} \times I) = N^{(y,z)}(I)$ . Given a Harris construction  $H$  and  $(x, t) \in \mathbb{Z}^d \times [0, \infty)$ , denote by  $H^{(x,t)}$  the Harris construction obtained by shifting  $H$  so that the space origin becomes  $x$  and the time origin becomes  $t$ . Formally, if  $y, z \in \mathbb{Z}^d$  and  $I$  is a Borelian subset of  $[0, \infty)$ ,  $H^{(x,t)}(\{z\} \times I) = H(\{z + x\} \times (I + t))$  and  $H^{(x,t)}(\{(y, z)\} \times I) = H(\{(y + x, z + x)\} \times (I + t))$ .

Given a Harris construction  $H = \{(D_x)_{x \in \mathbb{Z}^d}, (N^{(x,y)})_{x,y \in \mathbb{Z}^d}\}$  and  $(x, s), (y, t) \in \mathbb{Z}^d \times \mathbb{R}_+$  with  $s < t$ , we write  $(x, s) \leftrightarrow (y, t)$  (in  $H$ ) if there exists a piecewise constant  $\gamma : [s, t] \rightarrow \mathbb{Z}^d$  so that

- (i)  $\gamma(s) = x, \gamma(t) = y$ ;
- (ii)  $\gamma(r) \neq \gamma(r-)$  only if  $r \in N^{\gamma(r-), \gamma(r)}$ ;

(iii)  $\nexists s \leq r \leq t$  with  $r \in D_{\gamma(r)}$ .

Given  $A, B, C \subset \mathbb{Z}^d$  and  $s, t \in \mathbb{R}_+$ , we write  $A \times s \leftrightarrow B \times t$  if  $(x, s) \leftrightarrow (y, t)$  for some  $x \in A, y \in B$ . Additionally,  $A \times \{s\} \leftrightarrow B \times \{t\}$  inside  $C$  if there exists a path connecting  $A \times \{s\}$  and  $B \times \{t\}$  and with image contained in  $C$ .

Given  $\xi_0 \in \{0, 1\}^{\mathbb{Z}^d}$  and a Harris construction  $H$ , we construct a trajectory  $(\eta_t^{\xi_0}(H) : t \geq 0)$  by specifying  $\eta_0^{\xi_0}(H) = \xi_0$  and  $[\eta_t^{\xi_0}(H)](x) = 1$  if and only if  $\xi_0 \times \{0\} \leftrightarrow (x, t)$  in  $H$ .

A moment's reflection shows that, under the law of  $H$ ,  $(\eta_t^{\xi_0}(H))_{t \geq 0}$  is a contact process with initial condition  $\xi_0$  and, if  $\xi_0 \leq \zeta_0$ , then putting  $\xi_t = \eta_t^{\xi_0}(H)$  and  $\zeta_t = \eta_t^{\zeta_0}(H)$ , we obtain the claimed coupling of two processes one of which is always inferior to the other.

As noted, we will be concerned with one dimensional contact processes with  $\lambda > \lambda_c$ . Define  $r_t^{\xi_0}(H) = \sup\{x : [\eta_t^{\xi_0}(H)](x) = 1\}$ . We will usually omit the dependency on  $H$ , and when we omit the initial condition and simply write  $r_t$ , we take  $\xi_0 = I_{(-\infty, 0]}$ . If  $\xi_0$  is such that  $\sum_x \xi_0(x) = \infty$  and  $\sup\{x : \xi_0(x) = 1\} < \infty$ , then almost surely  $\eta_t^{\xi_0} \neq \underline{0}$  and  $r_t^{\xi_0} < \infty$  for all  $t$ . It is classical that  $\frac{r_t}{t} \xrightarrow{t \rightarrow \infty} \alpha = \alpha(\lambda) > 0$ ; see Theorems 2.19 and 2.27 in [8] (even though the process treated there is nearest neighbor, the proof works for the finite range case as well).

We consider the following question. Define

$$l_t = l_t(H) = \inf\{x : [\eta_t^{(-\infty, 0]}(H)](x) \neq [\eta_t^{\mathbb{1}}(H)](x)\},$$

$$\rho_t = r_t^{(-\infty, 0]} - l_t, \rho_t^+ = \max\{\rho_t, 0\}, \rho_t^- = \max\{-\rho_t, 0\}.$$

While it is easy to see that  $\{r_t < l_t\}$  and  $\{l_t < r_t\}$  are events of strictly positive probability, it is reasonable to believe the two quantities are close. Cox and Durrett conjectured that  $\{|\rho_t|\}_{t \geq 0}$  would be a tight collection of random variables. We answer the conjecture affirmatively.

**Theorem 1.1** *The law of  $\{\rho_t\}_{t \geq 0}$  is tight; that is, for any  $\delta > 0$ , there exists  $L > 0$  such that  $\mathbb{P}(|\rho_t| > L) < \delta$  for every  $t \geq 0$ .*

From the joint process  $((\eta_t^{\mathbb{1}}, \eta_t^{(-\infty, 0]}) : t \geq 0)$  we can define a process  $(\chi_t : t \geq 0)$  on  $\{0, 1, 2\}^{\mathbb{Z}}$  by

$$\chi_t(x) = \begin{cases} 0 & \text{if } \eta_t^{(-\infty, 0]}(x) = \eta_t^{\mathbb{1}}(x) = 0; \\ 1 & \text{if } \eta_t^{(-\infty, 0]}(x) = 1; \\ 2 & \text{if } \eta_t^{(-\infty, 0]}(x) = 0, \eta_t^{\mathbb{1}}(x) = 1. \end{cases}$$

It is not difficult to see that  $\chi_t$  is a realization of a process taking values in  $\{0, 1, 2\}^{\mathbb{Z}}$  with initial configuration equal to  $I_{(-\infty, 0]} + 2 \cdot I_{(0, \infty)}$  and rates

$$\begin{aligned} 0 &\rightarrow 1 && \text{at rate } \lambda \sum p(y-x) I_{\chi(y)=1}; \\ 0 &\rightarrow 2 && \text{at rate } \lambda \sum p(y-x) I_{\chi(y)=2}; \\ 2 &\rightarrow 0 && \text{at rate } 1; \\ 1 &\rightarrow 0 && \text{at rate } 1; \\ 2 &\rightarrow 1 && \text{at rate } \lambda \sum p(y-x) I_{\chi(y)=1}. \end{aligned}$$

The particle system with above transition rates is a model for hierarchical competition considered in [6] and [7]; the following interpretation is provided. Sites in state 0 are said to contain grass, in state 1 to contain trees, and in state 2 to contain bushes. When trees attempt to occupy new territory, they are able to displace bushes, but bushes cannot displace

trees. Since in our case we take the initial configuration  $I_{(-\infty,0]} + 2 \cdot I_{(0,\infty)}$ , we expect the area taken by trees to grow to the right towards the area originally taken by bushes. However, since we allow for non-nearest neighbour interactions, we may observe a mixed area where the two coexist. In the above notation, this area appears when  $\rho_t > 0$ . Alternatively, it may happen that there is no mixed area and a gap of grass appears between the two homogeneous zones (in case  $\rho_t < 0$ ). Theorem 1.1 states that with large probability, and uniformly in time, neither the mixed nor the intermediate grass area is too large.

The proof is divided in two parts. The first part, namely the proof of tightness of  $\{\rho_t^+\}$ , is done in the end of Section 2. The key ingredients are the celebrated result of Bezuidenhout and Grimmett, the renormalization arguments employed, among others, by Durrett (see [4]) and the construction carried out in [10]. These permit us to argue that from a single  $(x, t)$  with  $\eta_t^{(-\infty,0]}(x) = 1$  there will be positive probability that inside a cone  $C_{x,t} = \{(y, s) : |y - x| \leq \beta(s - t)\}$ ,  $\eta^1$  and  $\eta^{(-\infty,0]}$  are equal. In Section 3, a much simpler argument is employed to establish tightness of  $\{\rho_t^-\}$ .

## 2 Tightness of $\{\rho_t^+\}$

### 2.1 Right edge speed

Given  $\gamma > 0$ , we say that  $(0, 0) \in \mathbb{Z} \times [0, \infty)$  is  $\gamma$ -slow up to time  $T$  if  $r_t \leq \gamma t \forall t \leq T$ . If this is satisfied for all  $T$ , we say that  $(0, 0)$  is  $\gamma$ -slow.

**Lemma 2.1** (i) *For any  $\epsilon > 0$  there exists  $\gamma > 0$  such that  $\mathbb{P}((0, 0) \text{ is } \gamma\text{-slow}) > 1 - \epsilon$ .*

(ii) *For any  $\gamma > \alpha$ , we have*

$$\mathbb{P}((0, 0) \text{ is } \gamma\text{-slow}) > 0 \tag{2.1}$$

*and there exist  $c, C > 0$  such that*

$$\mathbb{P}((0, 0) \text{ is } \gamma\text{-slow up to time } T \text{ but not } \gamma\text{-slow}) \leq C e^{-cT}. \tag{2.2}$$

*Proof.* Almost surely,  $t \mapsto r_t$  is right-continuous with left limits, identically zero in a neighborhood of 0 and satisfies  $r_t/t \rightarrow \alpha$ . It follows that almost surely  $\{r_t/t : t \geq 0\}$  is bounded, hence we have (i). It also follows that, given  $\gamma > \alpha$ , we can obtain  $R > 0$  such that  $\mathbb{P}(r_t/t < R/t + \gamma \forall t) > 0$ . Now,

$$\mathbb{P}((0, 0) \text{ is } \gamma\text{-slow}) \geq \mathbb{P}(r_t \leq 0 \forall t \in [0, R/\gamma], r_s^{(-\infty,0]}(H^{(r_{R/\gamma}, R/\gamma)}) < R + \gamma s \forall s \geq 0).$$

The first event on the above probability depends only on the Harris construction  $H$  on  $[0, R/\gamma]$ , whereas the second depends only on  $H$  on  $[R/\gamma, +\infty)$ , so they are independent. Also noticing that  $\mathbb{P}(r_s^{(-\infty,0]}(H^{(r_{R/\gamma}, R/\gamma)}) < R + \gamma s \forall s \geq 0) = \mathbb{P}(r_s < R + \gamma s \forall s \geq 0)$  we get by translation invariance:

$$\mathbb{P}((0, 0) \text{ is } \gamma\text{-slow}) \geq \mathbb{P}(r_t \leq 0 \forall t \in [0, R/\gamma]) \cdot \mathbb{P}(r_s < R + \gamma s \forall s \geq 0).$$

The second probability above is positive by our choice of  $R$ . The first one is also positive because it contains the event  $\{(-\infty, 0] \times [0, R/\gamma] \leftrightarrow (0, +\infty) \times [0, R/\gamma]\}$ , which has positive probability since it corresponds to a finite number of Poisson processes having no arrivals in a finite time interval. We thus have (2.1).

To establish (2.2), fix  $\gamma' \in (\alpha, \gamma)$  and notice that

$$\begin{aligned} \mathbb{P}(r_t \leq \gamma t \text{ for all } t \in [0, T] \text{ but not for all } t \geq 0) &\leq \mathbb{P}(\exists t > T : r_t > \gamma t) \\ &\leq \mathbb{P}(\exists t > T : r_t > \gamma t, r_T \leq \gamma' T) + \mathbb{P}(r_T > \gamma' T) \end{aligned}$$

By Lemma 2 in [10] (a large deviations result for  $r_t$ ),  $\gamma' > \alpha$  implies that the second term in the sum decays exponentially fast in  $T$ , and by translation invariance the first term is less than  $\mathbb{P}(\exists s > 0 : r_s > (\gamma - \gamma')T + \gamma s)$ . So, it will suffice to prove that  $\mathbb{P}(\exists s > 0 : r_s > k + \gamma s)$  decays exponentially fast as  $k$  tends to infinity. Indeed, put  $\theta = \mathbb{P}(\exists t > 0 : r_t \geq M + \gamma t)$  (remember that  $M$  is the range of the process) and  $T_N = \inf\{t \geq 0 : r_t \geq 2MN + \gamma t\}$  for  $N \geq 1$ . We have  $\theta < 1$  by (2.1) and:

$$\begin{aligned} \mathbb{P}(T_{N+1} < \infty) &= \mathbb{P}(\exists t > 0 : r_t \geq 2M(N+1) + \gamma t) \leq \\ &\quad \mathbb{P}(T_N < \infty, \exists s > 0 : r_s^{(-\infty, 0]}(H^{(r_{T_N}, T_N)}) \geq M + \gamma s) = \\ &\quad \mathbb{P}(T_N < \infty) \cdot \mathbb{P}(\exists s > 0 : r_s > M + \gamma s) = \mathbb{P}(T_N < \infty) \cdot \theta. \end{aligned}$$

Thus,  $\mathbb{P}(T_N < \infty) \leq \theta^N$ . Now, if  $k \geq 1$ ,

$$\mathbb{P}(\exists s > 0 : r_s > k + \gamma s) \leq \mathbb{P}(\exists s > 0 : r_s \geq 2M\sigma + \gamma s) \leq \mathbb{P}(T_\sigma < \infty) \leq \theta^\sigma,$$

where  $\sigma$  denotes the largest integer strictly smaller than  $k/2M$ . ■

## 2.2 Descendancy barriers

In this section we define an event called the formation of a descendancy barrier. This will mean that, inside a certain area delimited by a vertical cone that grows upwards from the origin, all infected sites will be connected to the origin. Additionally, no infection from one side of the cone will be able to pass to the other side without being connected to the origin. These barriers, which appear with positive probability as we will show, are the essential structure in our proof of tightness of  $\{\rho_t^+\}$ .

We first make a brief exposition of oriented percolation and state a result that will be needed later. For a detailed treatment of the subject, see the survey [5].

Let  $\Lambda = \{(m, n) \in \mathbb{Z} \times \mathbb{Z}_+ : m + n \text{ is even}\}$ ,  $\Omega = \{0, 1\}^\Lambda$  and  $\mathcal{F}$  the  $\sigma$ -algebra generated by cylinder sets of  $\Omega$ . Points of  $\Omega$  will be denoted by  $\Psi$ , with  $\Psi(m, n) \in \{0, 1\}$  for  $(m, n) \in \Lambda$ .  $\mathbb{P}_p$  will denote the product measure  $(p\delta_1 + (1-p)\delta_0)^{\otimes \Lambda}$ . The vertical axis of  $\Lambda$  will be interpreted as time.

Given  $k \geq 1, \epsilon > 0$  and a probability  $\mathbb{P}$  on  $\mathcal{F}$ , we say that  $(\Omega, \mathcal{F}, \mathbb{P})$  is a  $k$ -dependent oriented percolation system with closure below  $\epsilon$  if

$$\mathbb{P}(\Psi(m_i, n) = 0, 1 \leq i \leq r \mid \{\Psi(m, s) : 1 \leq s < n, (m, s) \in \Lambda\}) < \epsilon^r \quad (2.3)$$

where  $r \geq 1, (m_i, n) \in \Lambda \forall i$  and  $|m_{i_1} - m_{i_2}| > 2k$  when  $i_1 \neq i_2$ .

Given  $\Psi \in \Omega$ , we say that two points  $(x, m), (y, n) \in \Lambda$  with  $m < n$  are connected by an open path if there exists a sequence  $x_0 = x, x_1, \dots, x_{n-m} = y$  in  $\mathbb{Z}$  such that  $|x_{i+1} - x_i| = 1 \forall i \in \{0, \dots, n-m-1\}$  and  $\Psi(x_i, m+i) = 1 \forall i \in \{0, \dots, n-m\}$ . We say that  $(x, m)$  percolates up to time  $n$  when it is connected by an open path to a point at height  $n$ . Finally, we say that  $(x, m)$  percolates when there is an infinite open path starting from it.

In [5], it is proven that if  $p$  is sufficiently large, then the origin percolates with positive probability in  $(\Omega, \mathcal{F}, \mathbb{P}_p)$ . Moreover, the rightmost particle connected to the origin at time  $n$ , denoted  $R_n$ , almost surely satisfies  $\lim R_n/n = \tilde{\alpha}(p) > 0$  as  $n \rightarrow \infty$ . To obtain similar results for  $k$ -dependent systems, we use the following particular case of Theorem 0.0 in [9]:

**Lemma 2.2** *Fix  $k \in \mathbb{N}$  and  $0 < p < 1$ . There exists  $\epsilon > 0$  such that, if  $(\Omega, \mathcal{F}, \mathbb{P})$  is a  $k$ -dependent oriented percolation system with closure below  $\epsilon$ , then  $\mathbb{P}$  stochastically dominates  $\mathbb{P}_p$ .*

Using these facts and an argument similar to the one used in Lemma 2.1, we can prove

**Lemma 2.3** *Fix an arbitrary  $0 < \beta < 1$  and define the events*

$$\begin{aligned} \Gamma(i) &= \left\{ \begin{array}{l} \text{There exist two open paths, one starting at } (-2, 0) \text{ and the other at } (2, 0) \\ \text{and both reaching time } i. \text{ None of them intersect } \{(m, n) : -\beta n \leq m \leq \beta n\} \end{array} \right\}, \\ \Gamma &= \left\{ \begin{array}{l} \text{There exist two infinite open paths, one starting at } (-2, 0) \\ \text{and the other at } (2, 0). \text{ None of them intersect } \{(m, n) : -\beta n \leq m \leq \beta n\} \end{array} \right\}, \end{aligned} \quad (2.4)$$

For any  $k$  and  $\bar{\delta} > 0$ , there exists  $\epsilon > 0$  such that, if  $(\Omega, \mathcal{F}, \mathbb{P})$  is a  $k$ -dependent percolation system with closure below  $\epsilon$ , then

- (i)  $\mathbb{P}(\Gamma) > 1 - \bar{\delta}$ .
- (ii)  $\mathbb{P}(\Gamma(i) \setminus \Gamma) \leq D e^{-di}$  for some  $d, D > 0$ .

We now construct a mapping  $H \mapsto \Psi_H$  of Harris constructions into points of  $\Omega$ ; this is essentially a repetition of the mapping developed in [10]. The construction will depend on large integers  $K$  and  $N$  (in particular, much larger than the range  $M$ ) whose choice will be described in Proposition 2.4. Given  $m \in \mathbb{Z}, n \in \mathbb{Z}_+$ , define

$$\begin{aligned} I_m &= \left( \frac{mN}{2} - \frac{N}{2}, \frac{mN}{2} + \frac{N}{2} \right] \cap \mathbb{Z}, \\ J_{(m,n)} &= \left[ \frac{mN}{2} - M, \frac{mN}{2} + M \right] \times [KNn, KN(n+1)] \cap \mathbb{Z} \times [0, +\infty) \end{aligned} \quad (2.5)$$

We start defining an auxiliary  $\Phi_H \in \{0, 1, 2\}^\Lambda$ . Given  $(m, 0) \in \Lambda$ , put  $\Phi_H(m, 0) = 1$  if  $H$  and the trajectory  $\eta^1(H)$  satisfy the conditions

$$\begin{aligned} &\text{There is no vacant interval at time } KN \\ &\text{of length } N^{1/2} \text{ inside } I_{m-1} \cup I_{m+1}; \end{aligned} \quad (2.6)$$

$$\begin{aligned} &\text{Every occupied site in } I_{m-1} \cup I_{m+1} \text{ at time } KN \\ &\text{is a descendent of } I_m \times \{0\}; \end{aligned} \quad (2.7)$$

$$\begin{aligned} &\text{There does not exist } (z, s) \in J_{(m,0)} \text{ such that} \\ &I_m \times \{0\} \leftrightarrow (z, s) \text{ and } (I_m^C \times [0, s]) \leftrightarrow (z, s); \end{aligned} \quad (2.8)$$

put  $\Phi_H(m, 0) = 0$  otherwise. Given  $(m, n) \in \Lambda$  with  $n \geq 1$ , put  $\Phi_H(m, n) = 1$  if

$$1 \in \{\Phi_H(m-1, n-1), \Phi_H(m+1, n-1)\}; \quad (2.9)$$

There is no vacant interval at time  $KN(n+1)$   
of length  $N^{1/2}$  inside  $I_{m-1} \cup I_{m+1}$ ; (2.10)

Every occupied site in  $I_{m-1} \cup I_{m+1}$  at time  $KN(n+1)$   
is a descendent of  $(I_m \cap \eta_{KNn}^{\mathbb{I}}) \times KNn$ ; (2.11)

There does not exist  $(z, s) \in J_{(m,n)}$  such that  
 $((I_m \cap \eta_{KNn}^{\mathbb{I}}) \times KNn) \leftrightarrow (z, s)$  and  $(I_m^C \times [KNn, s]) \leftrightarrow (z, s)$ . (2.12)

If (2.9) fails, put  $\Phi_H(m, n) = 2$  and in every other case put  $\Phi_H(m, n) = 0$ . Finally, set

$$\Psi_H(m, n) = \begin{cases} 0, & \text{if } \Phi_H(m, n) = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Notice that, with this construction, if there is an infinite open path  $\{(m_i, n_i)\}_{i \geq 0}$  leaving the origin in  $\Psi_H$ , we must have  $\Phi_H(m_i, n_i) = 1$  for every  $i$ .

We now have the

**Proposition 2.4** (Mountford and Sweet [10]) There exist  $k, K$  - depending only on the parameter  $\lambda$  of the contact process - with the following property. For any  $\epsilon > 0$ , there exists  $N$  such that  $\Psi_H$  defined from  $K$  and  $N$  is a  $k$ -dependent percolation system with closure below  $\epsilon$ .

**Remark 2.5** *Conditions (2.6) and (2.10) are only necessary to establish Proposition 2.4, and will not be used in the sequel. Also,  $N$  in Proposition 2.4 can be chosen as large as we want; in particular, as already mentioned, we take both  $K$  and  $N$  larger than the range  $M$ .*

In what follows, the oriented percolation dependency parameter  $k$ , the constant  $\beta$  and associated events  $\Gamma, \Gamma(i)$ , the renormalization constants  $N, K$  and the closure density  $\epsilon$  will be fixed in the following way:

- $K$  and  $k$  are functions of  $\lambda$ , as explained in the above Proposition;
- $\beta$  will be any fixed number in  $(0, 1)$ ;
- $\Gamma$  and  $\Gamma(i)$  will be defined from  $\beta$  as in (2.4);
- $\delta > 0$  will be given during the proof of Theorem 1.1;
- $\epsilon$  will be chosen corresponding to  $\bar{\delta} = \delta/6, k, \beta$  as in Lemma 2.3;
- $N$  will be chosen corresponding to  $\epsilon$  as in Proposition 2.4.

Introducing some more terminology, call the origin  $\beta$ -expanding when:

$$\text{If } x \in \mathbb{Z}, y \in I_{-2} \cup I_0 \cup I_2, x \neq y, t \leq 1 \text{ and } (x, 0) \leftrightarrow (y, t) \text{ then } (0, 0) \leftrightarrow (y, t); \quad (2.13)$$

$$D_0 \cap [0, 1] = \emptyset; \quad (2.14)$$

$$(0, 0) \leftrightarrow (z, 1) \forall z \in I_{-2} \cup I_0 \cup I_2; \quad (2.15)$$

$$\Psi_{H(0,1)} \in \Gamma. \quad (2.16)$$

Condition (2.13) means that whenever an infection is transmitted to a site in  $I_{-2} \cup I_0 \cup I_2$  before time 1, there must exist an earlier/simultaneous (possibly indirect) transmission from  $(0, 0)$  to the same site. Condition (2.14) means that there is no healing at  $\{0\} \times [0, 1]$ . Condition (2.15) means that, at time 1, every site in  $I_{-2} \cup I_0 \cup I_2$  carries an infection that descends from the origin. Condition (2.16) states that the percolation structure defined after placing the origin at  $(0, 1)$  has the properties defined in (2.4). The  $\beta$  dependency is in the third event, since  $\Gamma$  depends on  $\beta$ , and also in the choice of the parameters of the renormalization.

We say that  $(0, 0)$  is  $\beta$ -expanding up to a time  $T > 1$  when (2.13), (2.14), (2.15) are satisfied and  $\Psi_{H(0,1)} \in \Gamma(i)$ , where  $i$  satisfies  $T \in (1 + KN(i - 1), 1 + KNi]$ . We then have the

**Lemma 2.6** (i)  $\mathbb{P}((0, 0) \text{ is } \beta\text{-expanding}) > 0$ .

(ii)  $\mathbb{P}((0, 0) \text{ is } \beta\text{-expanding up to time } T \text{ but not } \beta\text{-expanding}) \leq \bar{D}e^{-\bar{d}T}$  for some  $\bar{d}, \bar{D} > 0$ .

*Proof.* It is clear that with positive probability (2.13), (2.14) and (2.15) happen simultaneously. Also, they are independent of (2.16), which in turn has positive probability by Lemma 2.3, since  $\Psi_H$  is supercritical. Hence, the origin has positive probability of being  $\beta$ -expanding, proving (i). Now notice that

$$\{(0, 0) \text{ is } \beta\text{-expanding up to time } T \text{ but not } \beta\text{-expanding}\} \subset \{\Psi_{H(0,1)} \in \Gamma(\lfloor (T-1)/KN \rfloor) \setminus \Gamma\},$$

where  $\lfloor x \rfloor$  denotes the integer part of  $x$ . The probability of the last event in the above expression is bounded by  $De^{-d(\lfloor (T-1)/KN \rfloor)}$  by Lemma 2.3, so we have (ii).  $\blacksquare$

Let us now present the properties that motivated this construction. We start defining for  $\rho > 0$

$$V(\rho) = \{(z, s) \in \mathbb{Z} \times [0, +\infty) : -\rho s \leq z \leq \rho s\}.$$

Then, we have:

**Proposition 2.7** *Suppose the origin is  $\beta$ -expanding. Then, there exists a (deterministic)  $0 < \bar{\beta} < 1$  with the following three properties:*

(i) *If  $x, z \in \mathbb{Z}$ ,  $(x, 0) \leftrightarrow (z, s)$  and  $(z, s) \in V(\bar{\beta})$ , then  $(0, 0) \leftrightarrow (z, s)$ .*

(ii)  $r_s^0 \geq \begin{cases} \bar{\beta}s & \text{if } s \geq 1 \\ 0 & \text{if } s < 1 \end{cases} \geq \max\{0, \bar{\beta}s - 1\} \forall s \geq 0$ .

(iii) *If  $x, z \in \mathbb{Z}$  have different signs and  $(x, 0) \leftrightarrow (z, s)$ , then  $(0, 0) \leftrightarrow (z, s)$ .*

*Proof.* If  $s \leq 1$  in parts (i), (ii) or (iii), then the statements hold for any  $\bar{\beta} < 1$  by (2.13) and (2.14). Hence, from now on we assume that  $s > 1$  in all three parts. Suppose the origin is  $\beta$ -expanding. Since  $\Psi_{H(0,1)} \in \Gamma$ , there exist sequences  $\{m_n^r\}_{n \geq 0}, \{m_n^l\}_{n \geq 0}$  in  $\mathbb{Z}$  such that

$$\begin{aligned} m_0^l &= -2, m_0^r = 2 \\ |m_{n+1}^l - m_n^l| &= |m_{n+1}^r - m_n^r| = 1, \\ \Psi_{H(0,1)}(m_n^l, n) &= \Psi_{H(0,1)}(m_n^r, n) = 1, \\ m_n^l &< -\beta n < \beta n < m_n^r, \quad n \geq 0. \end{aligned} \tag{2.17}$$

Define

$$B^l = \bigcup_{n=0}^{\infty} [(I_{m_n^l} \times KNn) \cup J_{(m_n^l, n)}], \quad B^r = \bigcup_{n=0}^{\infty} [(I_{m_n^r} \times KNn) \cup J_{(m_n^r, n)}],$$

$$B = B^l \cup B^r \cup (I_0 \times \{0\}).$$

$B^l$  is a union of horizontal lines (the “ $I_m \times KNn$ ”’s), one for each height level  $KNn$ , and rectangles of base  $2M$  and height  $KN$  (the “ $J_{(m,n)}$ ”’s); each rectangle connects a pair of horizontal lines.  $B^l$  is thus a connected subset of  $\mathbb{R} \times [0, +\infty)$ . The same can be said about  $B^r$ . So,  $B$  is also connected and its complement in  $\mathbb{R} \times [0, +\infty)$  has two connected components, which will be referred to as “above” and “below”  $B$ . Also notice that, since  $N > 2M, \forall (x, t) \in B$  we either have  $[x - M, x] \times \{t\} \subset B$  or  $[x, x + M] \times \{t\} \subset B$ . In other words, the three sets whose union defines  $B$  ( $B^l, B^r$  and  $I_0 \times \{0\}$ ) have width larger than  $M$  at any time level.

Putting together (2.7), (2.8), (2.11), (2.12) and the three first conditions in (2.17), we can conclude that in the trajectory  $\eta^{\mathbb{1}}(H^{(0,1)})$ , every infected site in  $(0, 1) + B := \{(z, 1 + s) : (z, s) \in B\}$  descends from  $(I_{-2} \cup I_0 \cup I_2) \times \{1\}$ . Then, because of (2.15), in the trajectory  $\eta^{\mathbb{1}}(H)$  every infected site in  $(0, 1) + B$  descends from  $(0, 0)$ .

It follows from the last condition of (2.17) that there exists  $0 < \bar{\beta} < 1$  such that  $V(\bar{\beta})$  is contained in the union of  $(I_{-2} \cup I_0 \cup I_2) \times [0, 1]$  and the area above  $(0, 1) + B$ .

Now take  $x$  and  $z$  as in (i). Since  $s > 1$  and  $(z, s) \in V(\bar{\beta})$ ,  $(z, s)$  must be above  $(0, 1) + B$ . So, any path starting from  $(x, 0)$  and reaching  $(z, s)$  must have a point  $(y, t) \in (0, 1) + B$  and thus, as we have seen, it must be the case that  $(0, 0) \leftrightarrow (y, t) \leftrightarrow (z, s)$ .

Part (ii) follows from the facts that, for any  $s > 1$ ,  $(\bar{\beta}s, s)$  is to the left of  $(0, 1) + B^r$ , and that  $\eta_s^0 \cap \{x : (x, s) \in B^r\} \neq \emptyset$ .

Finally, take  $x, z$  as in (iii) and let  $\zeta$  be the path linking  $(x, 0)$  and  $(z, s)$ . We consider separately the two cases: there exist  $y \neq x$  and  $t < 1$  such that  $(y, t) \in \zeta$  or not. In the first case, (iii) follows from (2.13). In the second case, noticing that  $x$  and  $z$  have different signs and  $\zeta$  has horizontal displacements of size at most  $M$ , and using our remarks about  $B$  being connected and its width being larger than  $M$  at any time level, we conclude that  $(\gamma(t), t) \in (0, 1) + B$  for some  $t \in [0, s]$ . (iii) then follows from the fact that any infection in  $(0, 1) + B$  descends from  $(0, 0)$ .  $\blacksquare$

### 2.3 Proof of tightness of $\{\rho_t^+\}$

Call the origin  $(\beta, \gamma)$ -good up to time  $T$  (resp.  $(\beta, \gamma)$ -good) when it is both  $\beta$ -expanding and  $\gamma$ -slow up to time  $T$  (resp.  $\beta$ -expanding and  $\gamma$ -slow). Additionally, call a point  $(x, t)$   $\beta$ -expanding,  $\gamma$ -slow or  $(\beta, \gamma)$ -good when  $(0, 0)$  has the corresponding property on  $H^{(x,t)}$ .

**Lemma 2.8** *For  $\gamma > 0$  sufficiently large, we have*

- (i)  $\mathbb{P}((0, 0) \text{ is } (\beta, \gamma)\text{-good}) > 0$ ;
- (ii)  $\mathbb{P}((0, 0) \text{ is } (\beta, \gamma)\text{-good up to time } T \text{ but not } (\beta, \gamma)\text{-good}) \leq Fe^{-fT}$  for some  $f, F > 0$ ;
- (iii) *Given  $0 \leq a < b, \mathbb{P}((r_t, t) \text{ is not } (\beta, \gamma)\text{-good for any } t \in [a, b]) \leq Ge^{-g\sqrt{b-a}}$  for some  $g, G > 0$  not depending on  $a, b$ .*

*Proof.* The only point that does not follow directly from Lemmas 2.1 and 2.6 is (iii). We start proving the result when  $a = 0$ . Given a Harris construction  $H$ , define  $\mu(H) = \sup\{t \geq 0 :$

$(0, 0)$  is  $(\beta, \gamma)$ -good up to time  $t$  in  $H$ ,

$$\sigma_1(H) = \begin{cases} 1, & \text{if } \mu(H) < 1 \\ KN(n+1) + 1, & \text{if } \mu(H) \in [KNn + 1, KN(n+1) + 1) \\ +\infty, & \text{if } \mu(H) = +\infty \end{cases},$$

$\sigma_0(H) = 0$  and  $\sigma_{i+1}(H) = \sigma_i(H) + \sigma(H^{(r_{\sigma_i(H)}, \sigma_i(H))})$  if  $i \geq 1$  and  $\sigma_i(H) < +\infty$ . (The  $r_{\sigma_i}$  that appears is defined with respect to the original trajectory  $\eta^{(-\infty, 0]}(H)$ , with no change of coordinates). Each  $\sigma_i$  is a stopping time for the process  $t \mapsto H_t$ . It follows from the strong Markov property and translation invariance of the law of  $H$  that the law of  $H^{(r_{\sigma_i}, \sigma_i)}$  conditioned to  $\{\sigma_i < +\infty\}$  is the same as that of  $H$ . In particular, conditioned on  $\{\sigma_i < +\infty\}$ ,  $\sigma_{i+1} - \sigma_i$  has the law of  $\sigma_1$ , which satisfies:

- $\mathbb{P}(\sigma_1 = +\infty) \equiv \theta > 0$ , by (i).
- $\mathbb{P}(T < \sigma_1 < +\infty) < \bar{F}e^{-\bar{f}T}$  for some  $\bar{f}, \bar{F} > 0$ , by (ii).

Let  $\tau = \inf\{s : (r_s, s) \text{ is } (\beta, \gamma)\text{-good}\}$ . Now, if  $i_0$  is the first  $i$  such that  $\sigma_{i+1} = +\infty$ , we have  $\tau \leq \sigma_{i_0}$  and:

$$\begin{aligned} \mathbb{P}(\tau > b) &\leq \mathbb{P}(\sigma_{i_0} > b) \leq \mathbb{P}(i_0 > \sqrt{b}) + \mathbb{P}(i_0 \leq \sqrt{b}, \sigma_{i_0} > b) \\ &\leq (1 - \theta)^{\sqrt{b}} + \mathbb{P}(i_0 \leq \sqrt{b}, \sigma_{j+1} - \sigma_j > \sqrt{b} \text{ for some } 1 \leq j \leq i_0) \\ &\leq (1 - \theta)^{\sqrt{b}} + \sqrt{b} \cdot \bar{F}e^{-\bar{f}\sqrt{b}} \leq Ge^{-g\sqrt{b}} \end{aligned}$$

for some  $g, G$  suitably chosen.

For  $a > 0$  repeat the proof starting from  $(r_a, a)$  instead of  $(0, 0)$  and note that the constants  $\bar{f}$  and  $\bar{F}$  do not depend on  $a$ . ■

*Proof of Theorem 1.1 (First part).* Fix  $\delta > 0$ . This is the  $\delta$  that takes part in our renormalization construction as mentioned in the paragraph after Proposition 2.4. We want to prove that, for any  $T, \rho_T < L$  with probability larger than  $1 - \delta$ . To this end, we will proceed in two steps. First we will define a ‘‘good event’’ depending on  $T, \mathcal{H}(T)$ , with  $\mathbb{P}(\mathcal{H}(T)) > 1 - \delta$ . Then, we will choose  $L > 0$  and see that, in  $\mathcal{H}(T)$ , every infection in  $\eta_T^{\sharp}$  that is to the left of  $r_T - L$  must descend from  $(-\infty, 0] \times 0$ .

A.) *Choice of the good event.* By Lemma 2.1 (i), we can choose  $\gamma > 0$  such that the event

$$\mathcal{H}_1 = \{(0, 0) \text{ is } \gamma\text{-slow}\}$$

has probability larger than  $1 - \delta/3$ . We can also assume that  $\gamma$  satisfies (iii) in Lemma 2.8.

We can choose  $S > 0$  such that

$$\{\exists x \in [-S, 0] \text{ such that } H^{(x, 0)} \text{ satisfies (2.13), (2.14) and (2.15)}\}$$

has probability larger than  $1 - \delta/6$ ; notice that this event depends only on the Harris construction on the time interval  $[0, 1]$ . Also, for any  $x$ , we have  $\mathbb{P}(\Psi_{H^{(x, 1)}} \in \Gamma) = \mathbb{P}(\Psi_H \in \Gamma) > 1 - \delta/6$  by our choice of  $\epsilon$  (see remark after Proposition 2.4); for any  $x$  this event depends only on the Harris construction on the time interval  $[1, +\infty)$  and is thus independent of the former event. Therefore, putting

$$\mathcal{H}_2 = \{\text{There exists } x \in [-S, 0] \text{ such that } (x, 0) \text{ is } \beta\text{-expanding}\},$$

we have  $\mathbb{P}(\mathcal{H}_2) > 1 - \delta/3$ .

Now choose  $R > 0$  such that  $\sum_{n=1}^{\infty} Ge^{-g\sqrt{R+n}} < \delta/3$ , where  $g, G$  are defined in Lemma 2.8 (iii). Given  $\bar{R} > 0$ , define the time intervals

$$I_0 = [0, \bar{R}], I_n = (\sup I_{n-1}, \sup I_{n-1} + R + n] \text{ for } n \geq 1,$$

so that  $I_n = (\bar{R} + (n-1)R + \frac{n(n-1)}{2}, \bar{R} + nR + \frac{n(n+1)}{2}]$ ,  $|I_n| = R + n$  for  $n \geq 1$ . We now choose  $\bar{R}$  large enough so that

$$\forall n \geq 2, \forall t \in I_{n-1}, \frac{2\bar{\beta}t - S}{\bar{\beta} + \gamma} > |I_{n-1} \cup I_n|. \quad (2.18)$$

Given  $T > 0$ , define  $\bar{n}(T) = \sup\{n \geq 1 : I_n \subset [0, T]\}$ ; if  $I_0 \cup I_1 \not\subset [0, T]$ , put  $\bar{n}(T) = -\infty$ . The idea is that, given the time interval  $[0, T]$ , we will place the intervals  $I_n$  from top to bottom, i.e.,  $T - I_0, T - I_1, \dots$ , until the last one that fits, which will be  $I_{\bar{n}(T)}$ . Now define the event

$\mathcal{H}_3 = \mathcal{H}_3(T) = \{\text{For each } n \in [1, \bar{n}(T)], \text{ there exists } t \in T - I_n \text{ such that } (r_t, t) \text{ is } (\beta, \gamma)\text{-good}\}$ ;

if  $\bar{n}(T) = -\infty$ , simply take  $\mathcal{H}_3 =$  the whole space. Now, as a consequence of Lemma 2.8 (iii), we obtain

$$\begin{aligned} \mathbb{P}(\mathcal{H}_3(T)) &\geq 1 - \sum_{n=1}^{\bar{n}(T)} \mathbb{P}((t, r_t) \text{ is never } (\beta, \gamma)\text{-good when } t \in T - I_n) \\ &\geq 1 - \sum_{n=1}^{\infty} Ge^{-g\sqrt{|I_n|}} = 1 - \sum_{n=1}^{\infty} Ge^{-g\sqrt{R+n}} > 1 - \delta/3. \end{aligned}$$

In conclusion, if  $\mathcal{H}(T) = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3(T)$ , then  $\mathbb{P}(\mathcal{H}(T)) > 1 - \delta$  for any  $T$ .

B.) *Choice of  $L$  and proof that the interface area is smaller than  $L$  in the good event.* Let  $L = \gamma(\bar{R} + R + 1) + S$ ; notice that  $L$  does not depend on  $T$ . We first treat the case  $T \leq \bar{R} + R + 1$ . We might omit it: since  $\sup_{t \leq T} |\rho_t| < \infty$  almost surely, it suffices to prove its tightness in  $[T, +\infty)$  for sufficiently large  $T$ . However, we find that this case illustrates the main idea of the proof without the technical complications that appear in the general picture.

Let  $V = V(\bar{\beta}) = \{(z, s) \in \mathbb{Z} \times [0, +\infty) : -\bar{\beta}s \leq z \leq \bar{\beta}s\}$  where  $\bar{\beta} > 0$  is such that the conclusion of part (i) of Proposition (2.7) holds. Given  $A \subset \mathbb{Z} \times [0, +\infty)$  and  $t \geq 0$ , define  $\Pi_t(A) = \{z : (z, t) \in A\}$ .

Fix  $H \in \mathcal{H}(T)$ . Since  $H \in \mathcal{H}_2$ , we can take  $x \in [-S, 0]$  such that  $(x, 0)$  is  $\beta$ -expanding. Also, since  $H \in \mathcal{H}_1$ ,  $(0, 0)$  is  $\gamma$ -slow and in particular,  $r_T < \gamma T$ . Thus,

$$\begin{aligned} r_T - L &< \gamma T - L \\ &\leq \gamma(\bar{R} + R + 1) - \gamma(\bar{R} + R + 1) - S = -S < x + \bar{\beta}T < \sup \Pi_T((x, 0) + V) + 1; \end{aligned}$$

the  $+1$  is required because  $x + \bar{\beta}T$  may not be an integer. Assume that for  $y > 0$  and  $w$  satisfying  $r_T - w > L$  we have  $(y, 0) \leftrightarrow (w, T)$ . Notice that  $w < r_T - L \leq \sup \Pi_T((x, 0) + V)$ . If  $w \in \Pi_T((x, 0) + V)$ , then it follows from Proposition 2.7 (i) and translation invariance that  $(x, 0) \leftrightarrow (w, T)$ . If  $w < \inf \Pi_T((x, 0) + V)$ , then  $w$  and  $y$  are in opposite sides of  $x$  and it follows from Proposition 2.7 (ii) and translation invariance that  $(x, 0) \leftrightarrow (w, T)$ . This shows that any infection in  $(-\infty, r_T - L] \times T$  that descends from  $[1, +\infty) \times 0$  must also descend from  $(-\infty, 0] \times 0$ , completing the proof of this case.

Before starting the other case, we make some trivial remarks. Suppose  $(a, s), (b, t) \in \mathbb{Z} \times [0, +\infty)$  are such that  $a \leq b$  and  $s < t$ . Let  $\zeta^*$  be the smallest value of  $\zeta$  at which  $\Pi_\zeta((a, s) + V) \cap \Pi_\zeta((b, t) + V) \neq \emptyset$ .  $\zeta^*$  is either  $t$  (in the case  $(b, t) \in (a, s) + V$ ) or the time of intersection of the lines  $\zeta \mapsto a + \bar{\beta}(\zeta - s)$  and  $\zeta \mapsto b - \bar{\beta}(\zeta - t)$  i.e.,

$$\zeta^*((a, s), (b, t)) = \max \left\{ t, \frac{b - a + \bar{\beta}(t + s)}{2\bar{\beta}} \right\}. \quad (2.19)$$

Also,

$$\zeta > \zeta^*((a, s), (b, t)) \implies \Pi_\zeta((a, s) + V) \cup \Pi_\zeta((b, t) + V) \text{ is an interval.} \quad (2.20)$$

Now take  $T > \bar{R} + R + 1$  and  $H \in \mathcal{H}(T)$ . Again,  $(0, 0)$  is  $\gamma$ -slow and there exists  $x \in [-S, 0]$  such that  $(x, 0)$  is  $\beta$ -expanding. Also, since  $H \in \mathcal{H}_3(T)$ , there exist  $t_1 \in T - I_{\bar{n}}, t_2 \in T - I_{\bar{n}-1}, \dots, t_{\bar{n}} \in T - I_1$  such that  $(r_{t_i}, t_i)$  is  $(\beta, \gamma)$ -good for  $i = 1, \dots, \bar{n}$ . Notice that, since  $(0, 0)$  and each  $(r_{t_i}, t_i)$  is  $\gamma$ -slow, we have

$$\begin{aligned} r_{t_1} &\leq \gamma t_1, \\ r_{t_{n+1}} &\leq r_{t_n} + \gamma(t_{n+1} - t_n), n = 1, \dots, \bar{n} - 1. \\ r_T &\leq r_{t_{\bar{n}}} + \gamma(T - t_{\bar{n}}), \end{aligned} \quad (2.21)$$

and by Proposition 2.7 (ii) and translation invariance, we have

$$\begin{aligned} r_{t_1} &\geq x, \\ r_{t_{n+1}} &\geq r_{t_n}, n = 1, \dots, \bar{n} - 1. \end{aligned} \quad (2.22)$$

We claim that the cones  $(r_{t_i}, t_i) + V$  each overlap with their neighbors before time  $T$ , i.e.,

$$\begin{aligned} \zeta^*((x, 0), (r_{t_1}, t_1)) &< T \\ \zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) &< T, i = 1, \dots, \bar{n} - 1 \end{aligned} \quad (2.23)$$

Let us prove the first expression in (2.23). If  $(r_{t_1}, t_1) \in (x, 0) + V$ , then  $\zeta^*((x, 0), (r_{t_1}, t_1)) = t_1 < T$ . Assume  $(r_{t_1}, t_1) \notin (x, 0) + V$ . Since  $-S \leq x < r_{t_1} \leq \gamma t_1$ , we have  $r_{t_1} - x \leq \gamma t_1 + S$ . Also,

$$0 \in T - I_{\bar{n}+1} \implies T \in I_{\bar{n}+1} \xrightarrow{(2.18)} \frac{2\bar{\beta}T - S}{\bar{\beta} + \gamma} > |I_{\bar{n}+1} \cup I_{\bar{n}+2}| > |I_{\bar{n}} \cup I_{\bar{n}+1}|,$$

and since we also have  $t_1 \in T - I_{\bar{n}}$ , we obtain  $t_1 = t_1 - 0 < |I_{\bar{n}} \cup I_{\bar{n}+1}| < \frac{2\bar{\beta}T - S}{\bar{\beta} + \gamma}$ . Putting these inequalities together and using (2.19), we get

$$\zeta^*((x, 0), (r_{t_1}, t_1)) = \frac{r_{t_1} - x + \bar{\beta}t_1}{2\bar{\beta}} \leq \frac{\gamma t_1 + S + \bar{\beta}t_1}{2\bar{\beta}} < \frac{S}{2\bar{\beta}} + \frac{2\bar{\beta}T - S}{\bar{\beta} + \gamma} \cdot \frac{\gamma + \bar{\beta}}{2\bar{\beta}} = T.$$

For the second expression in (2.23): if  $(r_{t_{i+1}}, t_{i+1}) \in (r_{t_i}, t_i) + V$ , then  $\zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) = t_{i+1} < T$ . Assume  $(r_{t_{i+1}}, t_{i+1}) \notin (r_{t_i}, t_i) + V$  and write:

$$\begin{aligned} \zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) &= \frac{r_{t_{i+1}} - r_{t_i} + \bar{\beta}(t_{i+1} + t_i)}{2\bar{\beta}} \leq \frac{\gamma(t_{i+1} - t_i) + \bar{\beta}(t_{i+1} + t_i)}{2\bar{\beta}} = \\ &= \frac{\gamma(t_{i+1} - t_i)}{2\bar{\beta}} + \frac{t_{i+1} + t_i}{2}. \end{aligned}$$

Since  $t_i \in T - I_{\bar{n}-i+1}$  and  $t_{i+1} \in T - I_{\bar{n}-i}$ , we have  $t_{i+1} - t_i \leq |I_{\bar{n}-i} \cup I_{\bar{n}-i+1}| \leq \frac{2\bar{\beta}t - S}{\bar{\beta} + \gamma}$ , for any  $t \in I_{\bar{n}-i}$  by (2.18). In particular this holds for  $t = T - t_{i+1}$ . Therefore,

$$\begin{aligned} \zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) &\leq \frac{\gamma}{2\bar{\beta}} \cdot \frac{2\bar{\beta}(T - t_{i+1}) - S}{\bar{\beta} + \gamma} \\ &\leq \frac{2\bar{\beta}(T - t_{i+1}) - S}{2\bar{\beta}} + \frac{t_{i+1} + t_i}{2} \leq T - t_{i+1} + \frac{t_{i+1} + t_i}{2} \leq T. \end{aligned}$$

Now define the union of cones  $U = [\cup_{n=1}^{\bar{n}}((r_{t_n}, t_n) + V)] \cup [(x, 0) + V]$ . Using (2.20) and (2.23), we conclude that  $\Pi_T(U)$  is an interval.

Since  $t_{\bar{n}} \in T - I_1$ , we have  $T - t_{\bar{n}} \leq \bar{R} + R + 1$ . Also using the last inequality in (2.21), we obtain

$$r_T < r_{t_{\bar{n}}} + \gamma(T - t_{\bar{n}}) < r_{t_{\bar{n}}} + \gamma(\bar{R} + R + 1),$$

so, using  $L = \gamma(\bar{R} + R + 1) + S$ ,

$$r_T - L < r_{t_{\bar{n}}} + \gamma(\bar{R} + R + 1) - \gamma(\bar{R} + R + 1) - S < r_{t_{\bar{n}}} < \sup \Pi_T(U). \quad (2.24)$$

As before, take  $y > 0$  and  $w$  satisfying  $r_T - w > L$  and  $(y, 0) \leftrightarrow (w, T)$ . Since  $w < r_T - L < \sup \Pi_T(U)$  and  $\Pi_T(U)$  is an interval, there are two possibilities:

(a)  $w \in \Pi_T(U)$

In this case, by the definition of  $U$ , we either have  $w \in \Pi_T((x, 0) + V)$  (hence  $(x, 0) \leftrightarrow (w, T)$  as we already saw) or  $w \in \Pi_T((r_{t_i}, t_i) + V)$  for some  $i$ . In this last case there exists  $z$  such that  $(z, t_i) \leftrightarrow (w, T)$  and hence by part (i) of Proposition (2.7)  $(r_{t_i}, t_i) \leftrightarrow (w, T)$ , which implies  $(-\infty, 0] \times \{0\} \leftrightarrow (w, T)$ .

(b)  $w < \inf \Pi_T(U)$

By the same argument that was used in the case  $T < \bar{R} + R + 1$ , we have  $(x, 0) \leftrightarrow (w, T)$ .

In conclusion, in any case we have  $(-\infty, 0] \times 0 \leftrightarrow (w, T)$ . We have thus shown that any point  $(w, T)$  that is connected to  $[1, +\infty)$  but not to  $(-\infty, 0]$  must be to the right of  $(r_T - L, T)$ , *i.e.* that  $\rho_T < L$  as required.

### 3 Tightness of $\{\rho_t^-\}$

In the following Lemma, we will reuse the renormalization structure built in the last section. We fix an arbitrary  $\beta \in (0, 1)$  and  $k, K$  as in Proposition 2.4, then choose a closure density  $\epsilon$  such that the event  $\Gamma$  of Lemma 2.3 has positive probability. Finally, choose  $N$  such that  $\Psi_H$  has closure density below  $\epsilon$  (again as in Proposition 2.4).

**Lemma 3.1** *For any  $\sigma > 0$ , there exists  $L > 0$  such that, for any  $T > 0$ ,*

$$\mathbb{P}(\text{There exists } t \leq T \text{ such that } r_t > r_T + L) < \sigma. \quad (3.1)$$

*Proof.* As in the proof of Theorem 1.1, we will define an event  $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3(T)$  such that  $\mathbb{P}(\mathcal{G}) > 1 - \sigma$  and choose an appropriate  $L > 0$ ; then we will show that, in  $\mathcal{G}$ , we have

$$r_t < r_T + L \quad \forall t \leq T. \quad (3.2)$$

The first event is the same as before:  $\mathcal{G}_1 = \{(0, 0) \text{ is } \gamma\text{-slow}\}$ , with  $\gamma$  chosen so that this has probability  $> 1 - \sigma/3$  (see Lemma 2.1). Put  $\mathcal{G}_2 = \{r_t > -S \forall t \geq 0\}$  with  $S > 0$  chosen so that this has probability  $> 1 - \sigma/3$ ; this is possible because  $\inf\{r_t : t \geq 0\} > -\infty$  almost surely.

Increasing  $\gamma$  so that the conclusions of Lemma 2.8 hold, we may choose  $R > 0$  such that  $\sum_{n=0}^{\infty} Ge^{-g\sqrt{R+n}} < \sigma/3$ , where  $g, G$  are as in part (iii) of that Lemma 2.8. Then put

$$I_0 = [0, R], I_n = [\sup I_{n-1}, \sup I_{n-1} + R + n) \text{ for } n \geq 1,$$

so that  $I_n = [nR + \frac{(n-1)n}{2}, (n+1)R + \frac{(n+1)n}{2})$ ,  $|I_n| = R + n$  when  $n \geq 0$ . Also put  $\bar{n}(T) = \sup\{n \geq 0 : I_n \subset [0, T]\}$ ; if  $I_0 \not\subset [0, T]$ , put  $\bar{n}(T) = -\infty$ . Then, define

$$\mathcal{G}_3(T) = \{\text{For each } n \in [0, \bar{n}(T)], \text{ there exists } t \in T - I_n \text{ such that } (r_t, t) \text{ is } (\beta, \gamma)\text{-good}\};$$

if  $\bar{n}(T) = -\infty$ , take  $\mathcal{G}_3 =$  the whole space. By the choice of  $R$  and Lemma 2.8 (iii),  $\mathbb{P}(\mathcal{G}_3(T)) > 1 - \sigma/3$ . Thus,  $\mathbb{P}(\mathcal{G}) > 1 - \sigma$  as required.

Let us recall that

$$(r_s, s) \text{ is } (\beta, \gamma)\text{-good, } s' > s \implies r_s + \bar{\beta}(s' - s) - 1 \leq r_{s'} \leq r_s + \gamma(s' - s); \quad (3.3)$$

where  $\bar{\beta}$  is defined in Proposition 2.7. Choose  $L$  such that

$$L \geq \gamma(2R + 1) + S \text{ and} \quad (3.4)$$

$$L \geq S + \gamma(2R + 2n + 1) - \bar{\beta} \left( nR + \frac{n(n-1)}{2} \right) + 1 \quad \forall n \geq 0. \quad (3.5)$$

We proceed to prove that (3.2) is satisfied in  $\mathcal{G}$ . Fix  $0 < t < T$ . We deal with three cases:

- $t < T \leq 2R + 1$ . Since the origin is  $\gamma$ -slow, we have  $r_t \leq \gamma t \leq \gamma(2R + 1)$ . Since we are in  $\mathcal{G}_2$ , we have  $r_T > -S$ . So,  $r_T + L > -S + L \stackrel{(3.4)}{\geq} \gamma(2R + 1) \geq r_t$ .
- $T > 2R + 1, t \in (T - I_{\bar{n}}) \cup (T - I_{\bar{n}+1})$  (the point being that  $t$  is close to zero, so there does not necessarily exist a  $(\beta, \gamma)$ -good point below  $(r_t, t)$ ). By the definition of  $\bar{n}$ , we have  $0 \in I_{\bar{n}+1}$ , so  $t < |I_{\bar{n}} \cup I_{\bar{n}+1}| = 2R + 2\bar{n} + 1$  and  $r_t < \gamma t < \gamma(2R + 2\bar{n} + 1)$ . Also, by the definition of  $\mathcal{G}_3$ , there exists  $t^* \in T - I_{\bar{n}}$  such that  $(r_{t^*}, t^*)$  is  $(\beta, \gamma)$ -good.  $t^* \in T - I_{\bar{n}}$  implies  $T - t^* \geq \inf I_{\bar{n}} = \bar{n}R + \frac{(\bar{n}-1)\bar{n}}{2}$ . We then have

$$r_T + L \stackrel{(3.3)}{\geq} r_{t^*} + \bar{\beta}(T - t^*) + L - 1 > -S + \bar{\beta} \left( \bar{n}R + \frac{(\bar{n}-1)\bar{n}}{2} \right) + L - 1 \stackrel{(3.5)}{\geq} \gamma(2R + 2\bar{n} + 1) \geq r_t.$$

- $T > 2R + 1, t \in T - I_n$  with  $n < \bar{n}$ . Here,  $n + 1 \leq \bar{n}$ , so there exists  $t^* \in T - I_{n+1}$  such that  $(r_{t^*}, t^*)$  is  $(\beta, \gamma)$ -good. Notice that  $t > t^*, t - t^* < |I_n \cup I_{n+1}| = 2R + 2n + 1$ , so (3.3) gives

$$r_t \leq r_{t^*} + \gamma(t - t^*) \leq r_{t^*} + \gamma(2R + 2n + 1). \quad (3.6)$$

On the other hand,  $T - t^* \geq |I_0 \cup \dots \cup I_n| = (n+1)R + \frac{(n+1)n}{2}$ , so

$$\begin{aligned} r_T + L &\stackrel{(3.3)}{\geq} r_{t^*} + \bar{\beta}(T - t^*) + L - 1 \geq r_{t^*} + \bar{\beta} \left( nR + \frac{(n+1)n}{2} \right) + L - 1 \\ &\stackrel{(3.5)}{\geq} r_{t^*} + \gamma(2R + 2n + 1) \stackrel{(3.6)}{\geq} r_t. \end{aligned}$$

■

**Lemma 3.2** For any  $\sigma > 0$ , there exists  $L > 0$  such that, for any  $T > 0$ ,

$$\mathbb{P}([0, +\infty) \times 0 \leftrightarrow [0, L] \times T \text{ inside } (0, +\infty)) > 1 - \sigma. \quad (3.7)$$

This follows from the fact that  $r_t$  has positive asymptotic speed and a simple duality argument; we omit the proof.

For  $T > 0$ , define  $q_T = \max\{r_t : 0 \leq t \leq T\}$ . We now proceed to the *Proof of Theorem 1.1 (Second part)*. Fix  $\delta > 0$ . By Lemmas 3.1 and 3.2, we can obtain  $L_1, L_2 > 0$  such that

$$\begin{aligned} \mathbb{P}(q_T \leq r_T + L_1) &> \sqrt{1 - \delta}; \\ \mathbb{P}([0, +\infty) \times 0 \leftrightarrow [0, L_2] \times T \text{ inside } \{(x, t) : x \geq 0\}) &> \sqrt{1 - \delta}. \end{aligned}$$

Put  $L = L_1 + L_2 + M$ . For any  $T > 0$ , we have:

$$\begin{aligned} \mathbb{P}(\rho_t \geq -L) &= \\ \mathbb{P}((0, +\infty) \times 0 \leftrightarrow (r_T, r_T + L) \times T) &\geq \\ \mathbb{P}\left(q_T \leq r_T + L_1, \begin{array}{c} [r_T + L_1 + M + 1, +\infty) \times 0 \leftrightarrow [r_T + L_1 + M + 1, r_T + L] \times T \\ \text{inside } [r_T + L_1 + M + 1, +\infty) \end{array}\right) &= \\ \sum_{x=-L_1}^{+\infty} \mathbb{P}\left(r_T = x, q_T \leq x + L_1, \begin{array}{c} [x + L_1 + M + 1, +\infty) \times 0 \leftrightarrow [x + L_1 + M + 1, x + L] \times T \\ \text{inside } [x + L_1 + M + 1, +\infty) \end{array}\right) & \end{aligned}$$

(The sum starts at  $-L_1$  because  $q_T \geq 0$ , so we can only have  $q_T \leq r_T + L_1$  when  $r_T \geq -L_1$ ). Now, in each of the above probabilities, the first two events depend on the Harris construction on the set  $(-\infty, x + L_1 + M] \times [0, +\infty)$ , whereas the third event depends on the Harris construction on  $[x + L_1 + M + 1, +\infty) \times [0, +\infty)$ . They are thus independent and the sum becomes

$$\begin{aligned} &\sum_{x=-L_1}^{+\infty} \mathbb{P}(r_T = x, q_T \leq x + L_1) \cdot \mathbb{P}\left(\begin{array}{c} [x + L_1 + M + 1, +\infty) \times 0 \\ \leftrightarrow [x + L_1 + M + 1, x + L] \times T \\ \text{inside } [x + L_1 + M + 1, +\infty) \end{array}\right) \\ &= \mathbb{P}([0, +\infty) \times 0 \leftrightarrow [0, L_2] \times T \text{ inside } [0, +\infty)) \cdot \sum_{x=-L_1}^{+\infty} \mathbb{P}(r_T = x, q_T \leq x + L_1) \\ &= \mathbb{P}([0, +\infty) \times 0 \leftrightarrow [0, L_2] \times T \text{ inside } [0, +\infty)) \cdot \mathbb{P}(q_T \leq r_T + L_1) > 1 - \delta, \end{aligned}$$

completing the proof.

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