

Approximating a Geometric fractional Brownian motion and related processes via discrete Wick calculus

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We approximate the solution of some linear systems of SDEs driven by a fractional Brownian motion B^H with Hurst parameter $H \in (\frac{1}{2}, 1)$ in the Wick-Itô sense, including a geometric fractional Brownian motion. To this end we apply a Donsker-type approximation of the fractional Brownian motion by disturbed binary random walks due to Sottinen. Moreover, we replace the rather complicated Wick products by their discrete counterpart, acting on the binary variables, in the corresponding systems of Wick difference equations. As the solutions of the SDEs admit series representations in terms of Wick powers, a key for the proof of our Euler scheme is an approximation of the Hermite recursion formula for the Wick powers of B^H .

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1. Introduction

A fractional Brownian motion B^H with Hurst parameter $H \in (0, 1)$ is a continuous zero mean Gaussian process in \mathbb{R} with stationary increments and covariance function

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}).$$

The process $B^{\frac{1}{2}}$ is a standard Brownian motion, but a fractional Brownian motion is not a semimartingale for $H \neq \frac{1}{2}$. In this paper we restrict ourselves to the case $H > 1/2$, in which the corresponding fractional Gaussian noise $(B_{n+1}^H - B_n^H)_{n \in \mathbb{N}}$ exhibits long range dependence.

In recent years a lively interest in integration theory with respect to fractional Brownian motion has been emerged (see e.g. the monographs by Mishura or Biagini et al. [15, 4]). One of the extensions of the Itô integral beyond semimartingales is the fractional Wick-Itô integral. It is based on the Wick product \diamond , which has its origin as a

renormalization operator in quantum physics. In probability theory the Wick product with ordinary differentiation rule imitates the situation of ordinary multiplication with Itô differentiation rule (cf. Holden et al. [11]). Actually, this makes it a natural tool to apply for extending the Itô integral.

We first consider the fractional Doléans-Dade SDE $dS_t = S_t d^\circ B_t^H$, $S_0 = 1$ in terms of the fractional Wick-Itô integral. The well-known solution, $\exp(B_t^H - \frac{1}{2}t^{2H})$, is the geometric fractional Brownian motion, also known as the Wick exponential of fractional Brownian motion. Notice that the Wick exponential has expectation equal to one, and therefore can be interpreted as a multiplicative noise. Moreover, the ordinary exponential can be obtained from the Wick exponential by a deterministic scaling. Both processes are not semimartingales for $H \neq \frac{1}{2}$. The name Wick exponential is justified by the fact that it exhibits a power series expansion with Wick powers $(B_t^H)^{\circ k}$ instead of ordinary powers.

More generally, we consider a linear system of SDEs,

$$\begin{aligned} dX_t &= (A_1 X_t + A_2 Y_t) d^\circ B_t^H, & X_0 &= x_0, \\ dY_t &= (B_1 X_t + B_2 Y_t) d^\circ B_t^H, & Y_0 &= y_0. \end{aligned} \tag{1}$$

One can obtain Wick power series expansions for the solution of this system, too. Our goal is to approximate these Wick analytic functionals of a fractional Brownian motion. To this end we require an approximation of a fractional Brownian motion and an approximation of the Wick product.

There are several ways to approximate a fractional Brownian motion. One of the first approximations was given by Taqqu [22] in terms of stationary Gaussian sequences. We refer to Mishura [15, Section 1.15.3] for further approaches on weak convergence to a fractional Brownian motion. Sottinen constructed a simple approximation of a fractional Brownian motion on an interval for $H > \frac{1}{2}$ by sums of square integrable random variables in [21]. He used the Wiener integral representation of a fractional Brownian motion on an interval, $B_t^H = \int_0^t z_H(t, s) dB_s$, for a suitable deterministic kernel $z_H(t, s)$, due to Molchan and Golosov and Norros et al. [16, 17, 18]. For this purpose, he combined a pointwise approximation of the kernel $z_H(t, s)$ with Donsker's theorem. This approach was extended by Nieminen [19] to weak convergence of perturbed martingale differences to fractional Brownian motion. We shall utilize Sottinen's approximation with binary random variables throughout this paper.

The main problem of applying the Wick product on random variables with continuous distributions is that it is not a pointwise operation. Thus an explicit computation of the Wick-Itô integral is only possible in rare special cases. But exactly here is the advantage of the binary random walks. In such a purely discrete setup we apply the discrete counterpart of Wick product as introduced in Holden et al. [10]. Starting from the binary random walk one can build up a discrete Wiener space, and the discrete Wick product depends on this discretization. This Wiener chaos gives the analogy to the continuous Wick products. For a survey on discrete Wiener chaos we refer to Gzyl [9]. However, we will introduce the discrete Wick product in a self-contained way in Section 3.

We can now formulate a weak Euler scheme of the linear system of SDEs (1) in the Wick-Itô sense,

$$\begin{aligned} X_l^n &= X_{l-1}^n + (A_1 X_{l-1}^n + A_2 Y_{l-1}^n) \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), & X_0^n &= x_0, & l &= 1, \dots, n, \\ Y_l^n &= Y_{l-1}^n + (B_1 X_{l-1}^n + B_2 Y_{l-1}^n) \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), & Y_0^n &= y_0, & l &= 1, \dots, n, \end{aligned} \quad (2)$$

where \diamond_n is the discrete Wick product and $\left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right)$ are the increments of the disturbed binary random walk. As a main result we show that the piecewise constant interpolation of the solution of (2) converges weakly in the Skorokhod space to the solution of (1). This is the first rigorous convergence result connecting discrete and continuous Wick calculus we are aware of. As a special case (2) contains the Wick difference equation

$$X_l^n = X_{l-1}^n + X_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), \quad X_0^n = 1, \quad l = 1, \dots, n. \quad (3)$$

As a consequence, the piecewise constant interpolation of (3) converges weakly to a geometric fractional Brownian motion, the solution of the fractional Doléans-Dade SDE. This was conjectured by Bender and Elliott [3] in their study of the Wick fractional Black-Scholes market.

In [21] Sottinen considered the corresponding difference equation in the pathwise sense, i.e. with the ordinary multiplication instead of the discrete Wick product,

$$\hat{X}_l^n = \hat{X}_{l-1}^n + \hat{X}_{l-1}^n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), \quad \hat{X}_0^n = 1, \quad l = 1, \dots, n. \quad (4)$$

The solution is explicitly given by the multiplicative expression

$$\hat{X}_l^n = \prod_{j=1}^l \left(1 + \left(B_{\frac{j}{n}}^{H,n} - B_{\frac{j-1}{n}}^{H,n} \right) \right). \quad (5)$$

By the logarithmic transform of ordinary products into sums and a Taylor expansion, one obtains an additive expression for $\ln(\hat{X}_l^n)$ which converges weakly to a fractional Brownian motion. In this way Sottinen proved the convergence of \hat{X} to the ordinary exponential of a fractional Brownian motion [21, Theorem 3]. This approach fails for the solution of (3), since in a product representation, analogous to (5), the discrete Wick product \diamond_n appears instead of the ordinary multiplication. There is, however, no straightforward way to transform discrete Wick products into sums by application of a continuous functional.

However the solution of (2) exhibits an expression which is closely related to a discrete Wick power series representation. Therefore the convergence can be initiated explicitly for the Wick powers of a fractional Brownian motion, which fulfill the Hermite recursion formula. We obtain a discrete analogue to this recursion formula for discrete Wick powers of the disturbed binary random walks. Actually, the weak convergence of these discrete Wick powers is the key for the proof for our Euler scheme.

The paper is organized as follows: In Section 2 we give some preliminaries on the Wick-Itô integral with respect to a fractional Brownian motion and introduce the Wick exponential and other Wick analytic functionals. Then we define the approximating sequences and state the main results in Section 3. Section 4 is devoted to some L^2 - estimates of the approximating sequences. We prove convergence in finite-dimensional distributions in Section 5 and tightness in Section 6.

2. Wick exponential and Wick analytic functionals

In this section we introduce the Wick product, the Wick-Itô integral and describe the Hermite recursion formula for Wick powers of a zero mean Gaussian random variable. Then we obtain the Wick power series expansions for the solutions of SDEs (1).

We consider a *geometric fractional Brownian motion* or the so called *Wick exponential* of a fractional Brownian motion $\exp(B_t^H - \frac{1}{2}t^{2H})$. For $H = \frac{1}{2}$ this is exactly a geometric Brownian motion, also known as the stochastic exponential of a standard Brownian motion. For all $H \in (0, 1)$ and $t \geq 0$, it holds that $t^{2H} = \mathbf{E}[(B_t^H)^2]$ and thus the Wick exponential generalizes the stochastic exponential. It is well-known that $\exp(B_t - \frac{1}{2}t)$ solves the Doléans-Dade equation

$$dS_t = S_t dB_t, \quad S_0 = 1,$$

where the integral is an ordinary Itô integral. Actually the Wick exponential of fractional Brownian motion solves the corresponding fractional Doléans-Dade equation

$$dS_t = S_t d^\diamond B_t^H, \quad S_0 = 1$$

in terms of fractional Wick-Itô integral (cf. Mishura [15, Theorem 3.3.2]). We want to approximate solutions of similar SDEs.

Let Φ and Ψ be two zero mean Gaussian random variables. Then the *Wick exponential* is defined as

$$\exp^\diamond(\Phi) := \exp\left(\Phi - \frac{1}{2}\mathbf{E}[|\Phi|^2]\right).$$

For a standard Brownian motion $(B_t)_{t \geq 0}$ and $s < t < u$ it holds that

$$\exp^\diamond(B_u - B_t) \exp^\diamond(B_t - B_s) = \exp^\diamond(B_u - B_s).$$

Forcing this renormalization property to hold for all, possibly correlated, Φ and Ψ , leads to the definition of the *Wick product* \diamond of two Wick exponentials

$$\exp^\diamond(\Phi) \diamond \exp^\diamond(\Psi) := \exp^\diamond(\Phi + \Psi).$$

The Wick product can be extended to larger classes of random variables by density arguments (cf. [6, 20, 2]). For a general introduction to Wick product we refer to the monographs by Kuo and Holden et al. [14, 11] and Hu and Yan [13]. Note that the Wick

product is not a pointwise operation. Suppose $\Phi \sim \mathcal{N}(0, \sigma)$, then we have by definition $\Phi^{\diamond 0} = 1$, $\Phi^{\diamond 1} = \Phi$ and the recursion

$$\Phi^{\diamond n+1} = \Phi^{\diamond n} \diamond \Phi.$$

Observe that it holds that

$$\begin{aligned} \left. \frac{d}{dx} \exp^{\diamond}(x\Phi) \right|_{x=0} &= \left. \frac{d}{dx} \exp \left(x\Phi - \frac{1}{2} \mathbf{E} [|x\Phi|^2] \right) \right|_{x=0} \\ &= (\Phi - x\sigma^2) \exp \left(x\Phi - \frac{1}{2} \mathbf{E} [|x\Phi|^2] \right) \Big|_{x=0} = \Phi. \end{aligned}$$

Suppose we have

$$\Phi^{\diamond k} = \left. \frac{d^k}{dw^k} \exp^{\diamond}(w\Phi) \right|_{w=0}$$

for all positive integers $k \leq n$. Then with $z = w + x$, $\frac{dz}{dw} = \frac{dz}{dx} = 1$, we get

$$\begin{aligned} \Phi^{\diamond(n+1)} &= \left. \frac{d^n}{dw^n} \exp^{\diamond}(w\Phi) \right|_{w=0} \diamond \left. \frac{d}{dx} \exp^{\diamond}(x\Phi) \right|_{x=0} \\ &= \left. \frac{d^n}{dw^n} \frac{d}{dx} \exp^{\diamond}((w+x)\Phi) \right|_{w=0, x=0} = \left. \frac{d^{n+1}}{dz^{n+1}} \exp^{\diamond}(z\Phi) \right|_{z=0}. \end{aligned}$$

We now obtain, by differentiation and the Leibniz rule, the following Wick recursion formula

$$\begin{aligned} \Phi^{\diamond n+1} &= \left. \frac{d^n}{dw^n} \left((\Phi - w\sigma^2) \exp \left(w\Phi - \frac{1}{2} \mathbf{E} [|w\Phi|^2] \right) \right) \right|_{w=0} \\ &= (\Phi - w\sigma^2) \left. \frac{d^n}{dw^n} \exp \left(w\Phi - \frac{1}{2} \mathbf{E} [|x\Phi|^2] \right) \right|_{w=0} \\ &\quad + n(-\sigma^2) \left. \frac{d^{n-1}}{dw^{n-1}} \exp \left(w\Phi - \frac{1}{2} \mathbf{E} [|w\Phi|^2] \right) \right|_{w=0} \\ &= \Phi \Phi^{\diamond n} - n\sigma^2 \Phi^{\diamond n-1}. \end{aligned} \tag{6}$$

Define the *Hermite polynomial of degree $n \in \mathbb{N}$ with parameter σ^2* as

$$h_{\sigma^2}^n(x) := (-\sigma^2)^n \exp \left(\frac{x^2}{2\sigma^2} \right) \frac{d^n}{dx^n} \exp \left(\frac{-x^2}{2\sigma^2} \right).$$

Then the series expansion

$$\exp \left(x - \frac{1}{2} \sigma^2 \right) = \sum_{n=0}^{\infty} \frac{1}{n!} h_{\sigma^2}^n(x) \tag{7}$$

holds true. The first Hermite polynomials are $h_{\sigma^2}^0(x) = 1$, $h_{\sigma^2}^1(x) = x$. By the Leibniz rule we obtain the Hermite recursion formula

$$h_{\sigma^2}^{n+1}(x) = x h_{\sigma^2}^n(x) - n\sigma^2 h_{\sigma^2}^{n-1}(x). \quad (8)$$

By the equivalent first terms and recursions (6) and (8) we can conclude that for any Gaussian random variable $\Phi \sim \mathcal{N}(0, \sigma)$ and all $n \in \mathbb{N}$,

$$\Phi^{\diamond n} = h_{\sigma^2}^n(\Phi). \quad (9)$$

By (7) we have additionally

$$\exp^{\diamond}(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\diamond n}. \quad (10)$$

The fractional Wick-Itô integral, introduced by Duncan et al. [6], is an extension of the Itô integral beyond the semimartingale framework. There are several approaches to the fractional Wick-Itô integral. Essentially these approaches are via white noise theory as in Elliott and von der Hoek [7] and Hu and Øksendal [12], by Malliavin calculus in Alòs et al. [1], or by an S-transform approach in Bender [2]. In contrast to the forward integral, the fractional Wick-Itô integral has zero mean in general. This is the crucial property for an additive noise. The Wick-Itô integral is based on the Wick product. For a sufficiently good process $(X_s)_{s \in [0, t]}$ the fractional Wick-Itô integral with respect to fractional Brownian motion $(B_s^H)_{[0, t]}$ can be defined easily by Wick-Riemann sums (cf. Duncan et al. [6] or Mishura [15, Theorem 2.3.10]). Suppose $\pi_n = \{0 = t_0 < t_1 < \dots < t_n = t\}$ with $\max_{t_i \in \pi_n} |t_i - t_{i-1}| \rightarrow 0$ for $n \rightarrow \infty$, then

$$\int_0^t X_s d^{\diamond} B_s^H := \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n} X_{t_{i-1}} \diamond (B_{t_i}^H - B_{t_{i-1}}^H),$$

if the Wick products and the $L^2(\Omega)$ -limit exist. For more information on Wick-Itô integral with respect to fractional Brownian motion we refer to Mishura [15, chapter 2].

By the fractional Itô formula (cf. [2, Theorem 5.3] or [4, Theorem 3.7.2]) we have

$$d(B_t^H)^{\diamond k} = k(B_t^H)^{\diamond k-1} d^{\diamond} B_t^H, \quad (B_0^H)^{\diamond k} = \mathbf{1}_{\{k=0\}}. \quad (11)$$

For the Wick exponential

$$\exp^{\diamond}(B_t^H) = \sum_{k=0}^{\infty} \frac{1}{k!} (B_t^H)^{\diamond k}, \quad (12)$$

we obtain, by summing up the identity (11), the fractional Doléans-Dade equation,

$$dS_t = S_t d^{\diamond} B_t^H, \quad S_0 = 1. \quad (13)$$

For any analytic function $F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ we define the *Wick version* as

$$F^\diamond(\Phi) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \Phi^{\diamond k}.$$

From the recursive system of SDEs (11) we obtain SDEs for other *Wick analytic functionals* of a fractional Brownian motion

$$F^\diamond(B_t^H) = \sum_{k=0}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k}.$$

Recall the linear system of SDEs (1),

$$\begin{aligned} dX_t &= (A_1 X_t + A_2 Y_t) d^\diamond B_t^H, & X_0 &= x_0, \\ dY_t &= (B_1 X_t + B_2 Y_t) d^\diamond B_t^H, & Y_0 &= y_0. \end{aligned}$$

The coefficients of the solution,

$$X_t = \sum_{k=0}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k}, \quad Y_t = \sum_{k=0}^{\infty} \frac{b_k}{k!} (B_t^H)^{\diamond k}, \quad (14)$$

can be obtained recursively via (11) to be

$$a_0 = x_0, \quad b_0 = y_0, \quad a_k = A_1 a_{k-1} + A_2 b_{k-1}, \quad b_k = B_1 a_{k-1} + B_2 b_{k-1}.$$

Note that it holds $|a_k|, |b_k| \leq C^k$ for a $C \in \mathbb{R}_+$. This is according to the recursive derivation of the coefficients and it ensures that the Wick analytic functionals X_t and Y_t are square integrable (cf. the proof of Proposition 6).

3. The approximation results

Here we present the approximating sequences and discuss the main results. Precisely, we introduce the Donsker-type approximation of a fractional Brownian motion and the discrete Wick product, and obtain Wick difference equations, which correspond to the SDEs. We shall work with a fractional Brownian motion on the interval $[0, 1]$, but all results extend to any compact interval $[0, T]$.

We first consider the following kernel representation of a fractional Brownian motion on the interval $[0, 1]$ based on works by Molchan and Golosov [16, 17],

$$B_t^H = \int_0^t z_H(t, s) dB_s. \quad (15)$$

For $H > \frac{1}{2}$ the deterministic kernel takes the form

$$z_H(t, s) = \mathbf{1}_{\{t \geq s\}} c_H \left(H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{1}{2}} (u - s)^{H - \frac{3}{2}} du \quad (16)$$

with the constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}},$$

where Γ is the Gamma function (Norros et al. [18] or Nualart [20, section 5.1.3]). In order to simplify the notation we think of $H \in (\frac{1}{2}, 1)$ to be fixed from now on and omit the subscript H in the notation of the kernel. For an introduction to some elementary properties of fractional Brownian motion we refer to Nualart [20, chapter 5], Mishura [15], or Biagini et al. [4].

We apply Sottinen's approximation of a fractional Brownian motion by *disturbed binary random walks*. Suppose (Ω, \mathcal{F}, P) is a probability space and, for all $n \in \mathbb{N}$ and $i = 1, \dots, n$, we have independent and identically distributed symmetric Bernoulli random variables $\xi_i^n : \Omega \rightarrow \{-1, 1\}$ with $P(\xi_i^n = 1) = P(\xi_i^n = -1)$. By Donsker's theorem the sequence of random walks $B_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i^{(n)}$ converges weakly to a standard Brownian motion $B = (B_t)_{t \in [0,1]}$ [5, Theorem 16.1]. The idea of Sottinen [21] is to combine these random walks with a pointwise approximation of the kernel in representation (15). Define the pointwise approximation of $z(t, s)$ as

$$z^{(n)}(t, s) := n \int_{s - \frac{1}{n}}^s z\left(\frac{\lfloor nt \rfloor}{n}, u\right) du.$$

Then the sequence of binary random walks

$$B_t^{H,n} := \int_0^t z^{(n)}(t, s) dB_s^{(n)} = \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z\left(\frac{\lfloor nt \rfloor}{n}, s\right) ds \frac{1}{\sqrt{n}} \xi_i^{(n)}$$

converges weakly to a fractional Brownian motion $(B_t^H)_{t \in [0,1]}$ in the Skorokhod space $D([0,1], \mathbb{R})$ [21, Theorem 1].

A main advantage of the binary random walks is that we can avoid the difficult Wick product for random variables with continuous distributions. We approximate this operator on the binary random walks by *discrete Wick products*:

For any $n \in \mathbb{N}$ let $(\xi_1^n, \xi_2^n, \dots, \xi_n^n)$ be the n -tuple of independent and identically distributed symmetric Bernoulli random variables for the binary random walk $B_t^{H,n}$. The *discrete Wick product* is defined as

$$\prod_{i \in A} \xi_i^n \diamond_n \prod_{i \in B} \xi_i^n := \begin{cases} \prod_{i \in A \cup B} \xi_i^n & \text{if } A \cap B = \emptyset \\ 0 & \text{otherwise} \end{cases},$$

where $A, B \subseteq \{1, \dots, n\}$. We denote by

$$\mathcal{F}_n := \sigma(\xi_1^n, \xi_2^n, \dots, \xi_n^n)$$

the σ -field generated by the Bernoulli variables. Denote

$$\Xi_A^n := \prod_{i \in A} \xi_i^n.$$

Clearly the family of functions $\{\Xi_A^n : A \subseteq \{1, \dots, n\}\}$ is an orthonormal set in $L^2(\Omega, \mathcal{F}_n, P)$. Since its cardinality is equal to the dimension of $L^2(\Omega, \mathcal{F}_n, P)$ it constitutes a basis. Thus every $X \in L^2(\Omega, \mathcal{F}_n, P)$ has a unique expansion, called the *Walsh decomposition*,

$$X = \sum_{A \subseteq \{1, \dots, n\}} x_A^n \Xi_A^n,$$

where $x_A^n \in \mathbb{R}$. The Walsh decomposition can be regarded as a discrete version of the chaos expansion. By algebraic rules one obtains for $X = \sum_{A \subseteq \{1, \dots, n\}} x_A^n \Xi_A^n$ and $Y = \sum_{B \subseteq \{1, \dots, n\}} y_B^n \Xi_B^n$,

$$X \diamond_n Y = \sum_{C \subseteq \{1, \dots, n\}} \left(\sum_{\substack{A \cup B = C \\ A \cap B = \emptyset}} x_A^n y_B^n \right) \Xi_C^n.$$

Furthermore, the L^2 -inner product can be computed in terms of the Walsh decomposition as

$$\mathbf{E}[XY] = \sum_{A \subseteq \{1, \dots, n\}} x_A^n y_A^n. \quad (17)$$

There exists an analogous formula for the Wick product on the white noise space via chaos expansions that justifies the analogy between the discrete and ordinary Wick calculus (cf. Kuo [14]). For more information on the discrete Wick product we refer to Holden et al. [10]. More generally, the introduction of a discrete Wiener chaos depends on the class of discrete random variables $(\xi_1^n, \xi_2^n, \dots, \xi_n^n)$. We refer to Gzyl [9] for a survey on other discrete Wiener chaos approaches.

The representation

$$B_t^{H,n} = \sum_{i=1}^{\lfloor nt \rfloor} b_{t,i}^n \xi_i^n \quad \text{with} \quad b_{t,i}^n := \sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} z\left(\frac{\lfloor nt \rfloor}{n}, s\right) ds$$

is the Walsh decomposition for the binary random walk approximating B^H in $L^2(\Omega, \mathcal{F}_n, P)$. Notice that $b_{t,i}^n = b_{\frac{\lfloor nt \rfloor}{n}, i}^n$. Thus we can consider $B_t^{H,n} = B_{\frac{\lfloor nt \rfloor}{n}}^{H,n}$ as a process in discrete time. We can state our first convergence result:

Theorem 1. *Suppose*

1. $\lim_{n \rightarrow \infty} a_{n,k} = a_k$ exists for all $k \in \mathbb{N}$.
2. There exists a $C \in \mathbb{R}_+$, so that $|a_{n,k}| \leq C^k$ for all $n, k \in \mathbb{N}$.

Then the sequence of processes $\sum_{k=0}^n \frac{a_{n,k}}{k!} (B^{H,n})^{\diamond_k}$ converges weakly to the Wick power series $\sum_{k=0}^{\infty} \frac{a_k}{k!} (B^H)^{\diamond_k}$ in the Skorokhod space $D([0, 1], \mathbb{R})$.

The proof is given in Sections 5 and 6.

Consider now the following recursive system of Wick difference equations,

$$U_l^{k,n} = U_{l-1}^{k,n} + kU_{l-1}^{k-1,n} \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), \quad U_l^{0,n} = 1, \quad U_0^{k,n} = 0 \quad (18)$$

for all $l = 1, \dots, n$ and $k \in \mathbb{N}$. This is the discrete counterpart to the recursive system of SDEs in (11). We observe that $U^{0,n} = 1 = (B^{H,n})^{\diamond_n 0}$ and $U^{1,n} = (B^{H,n})^{\diamond_n 1}$, but

$$U_2^{2,n} = 2B_{\frac{1}{n}}^{H,n} \diamond_n B_{\frac{2}{n}}^{H,n} \neq B_{\frac{2}{n}}^{H,n} \diamond_n B_{\frac{2}{n}}^{H,n} = \left(B_{\frac{2}{n}}^{H,n} \right)^{\diamond_n 2}.$$

Thus, in contrast to the continuous case in (11), the discrete Wick powers are not the solutions for (18) if $k \geq 2$.

However we can prove a variant of Theorem 1 based on the system of recursive Wick difference equations, whose proof will also be given in Sections 5 and 6.

Theorem 2. *Under the assumptions of Theorem 1 define $\tilde{U}_t^{k,n} := U_{\lfloor nt \rfloor}^{k,n}$ as the piecewise constant interpolation of (18).*

Then the sequence of processes $\sum_{k=0}^n \frac{a_{n,k}}{k!} \tilde{U}^{k,n}$ converges weakly to the Wick power series $\sum_{k=0}^{\infty} \frac{a_k}{k!} (B^H)^{\diamond_k}$ in the Skorokhod space $D([0,1], \mathbb{R})$.

Example 1 (Wick powers of a fractional Brownian motion). *For $a_{n,k} = l! \mathbf{1}_{\{k=l\}}$,*

$$\begin{aligned} (B^{H,n})^{\diamond_n l} &\xrightarrow{d} (B^H)^{\diamond l}, \\ \tilde{U}^{l,n} &\xrightarrow{d} (B^H)^{\diamond l}. \end{aligned}$$

Example 2 (Geometric fractional Brownian motion). *For $a_{n,k} = a_k = 1$ we have*

$$\begin{aligned} \exp^{\diamond_n} \left(B_t^{H,n} \right) &:= \sum_{k=0}^{\lfloor nt \rfloor} \frac{1}{k!} \left(B_t^{H,n} \right)^{\diamond_n k} \xrightarrow{d} \exp^{\diamond} (B^H), \\ \tilde{S}^n &:= \sum_{k=0}^n \frac{1}{k!} \tilde{U}^{k,n} \xrightarrow{d} \exp^{\diamond} (B^H). \end{aligned}$$

Observe that by summing up the recursive system of Wick difference equations (18) we obtain

$$S_l^n = S_{l-1}^n + S_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), \quad S_0^n = 1 \quad (19)$$

for $l = 1, \dots, n$, where $S_l^n = \tilde{S}_{\frac{l}{n}}^n$. Hence, the piecewise constant interpolation of (19) converges weakly to the solution of the fractional Doléans-Dade equation (13).

The reasoning of the previous example can be generalized as follows:

Theorem 3 (Linear SDE with drift). *Suppose $\mu, s_0 \in \mathbb{R}$, $\sigma > 0$. Then $\tilde{S}_t^n := S_{[nt]}^n$, where S^n is the solution of the Wick difference equation*

$$S_l^n = \left(1 + \frac{\mu}{n}\right) S_{l-1}^n + \sigma S_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n}\right), \quad S_0^n = s_0, \quad l = 1, \dots, n, \quad (20)$$

converges weakly to the solution of the linear SDE with drift

$$dS_t = \mu S_t dt + \sigma S_t d^\diamond B_t^H, \quad S_0 = s_0 \quad (21)$$

in the Skorokhod space $D([0, 1], \mathbb{R})$.

Proof. Observe at first that for $\sigma_n \rightarrow \sigma > 0$ and $\tilde{a}_{n,k} = a_{n,k} \sigma_n^k$ we obtain by Theorem 2 that

$$\tilde{V}^n := \sum_{k=0}^n \frac{a_{n,k}}{k!} \sigma_n^k \tilde{U}^{k,n} \xrightarrow{d} \sum_{k=0}^{\infty} \frac{a_k}{k!} (\sigma B^H)^{\diamond k}.$$

With the choice $a_{n,k} := 1$ and

$$\sigma_n := \frac{\sigma}{1 + \frac{\mu}{n}} \rightarrow \sigma \quad \text{as } n \rightarrow \infty,$$

we observe by (18) that $V_l^n := \tilde{V}_{\frac{l}{n}}^n$ satisfies

$$V_l^n = V_{l-1}^n + \left(\frac{\sigma}{1 + \frac{\mu}{n}}\right) V_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n}\right), \quad V_0^n = 1, \quad l = 1, \dots, n.$$

Consider now the piecewise constant function $(\tilde{W}_t^n)_{t \in [0,1]}$ determined by $\tilde{W}_t^n := W_{[nt]}^n$ and

$$W_l^n = \left(1 + \frac{\mu}{n}\right) W_{l-1}^n, \quad W_0^n = s_0, \quad l = 1, \dots, n.$$

By this well-known Euler scheme,

$$(\tilde{W}_t^n)_{t \in [0,1]} \longrightarrow s_0 (\exp(\mu t))_{t \in [0,1]} \quad (22)$$

in the sup-norm on $[0, 1]$. The product

$$\begin{aligned} V_l^n W_l^n &= \left(1 + \frac{\mu}{n}\right) V_{l-1}^n W_{l-1}^n + \left[\left(\frac{\sigma}{1 + \frac{\mu}{n}}\right) V_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n}\right)\right] \left(1 + \frac{\mu}{n}\right) W_{l-1}^n \\ &= \left(1 + \frac{\mu}{n}\right) V_{l-1}^n W_{l-1}^n + \sigma V_{l-1}^n W_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n}\right), \\ V_0^n W_0^n &= s_0, \quad l = 1, \dots, n, \end{aligned}$$

satisfies the Wick difference equation (20) for $S_l^n = V_l^n W_l^n$. The multiplication by the deterministic function $s_0 \exp(\mu t)$ is continuous on the Skorokhod space. Thus with (22) and Billingsley [5, Theorem 4.1] we obtain

$$(\tilde{S}_t^n)_{t \in [0,1]} = (\tilde{V}_t^n \tilde{W}_t^n)_{t \in [0,1]} \xrightarrow{d} s_0 (\exp(\mu t) \exp^\diamond(\sigma B_t^H))_{t \in [0,1]}$$

in the Skorokhod space $D([0, 1], \mathbb{R})$. As $s_0 \exp(\mu t) \exp^\diamond(\sigma B_t^H)$ solves the SDE (21) (cf. Mishura [15, Theorem 3.3.2]), the proof is complete. \square

Remark 1. *Theorem 3 holds with additional approximations $(\sigma_n, \mu_n) \rightarrow (\sigma, \mu)$, too.*

Remark 2. *Theorem 3 was conjectured by Bender and Elliott [3] in their study of the discrete Wick-fractional Black-Scholes market. They deduced an arbitrage in this model for sufficiently large n . Although the arbitrage or no-arbitrage property is not preserved by weak convergence, this model showed that it is even possible to obtain arbitrage in this simple discrete Wick fractional market models. In a recent work [23] Valkeila shows that an alternative approximation to the exponential of a fractional Brownian motion by a superposition of some independent renewal reward processes leads to an arbitrage-free and complete model. We refer to Gaigalas and Kaj [8] for a general limit discussion for these superposition processes.*

Theorem 4 (Linear system of SDEs). *The piecewise constant interpolation*

$$\left(\tilde{X}_t^n, \tilde{Y}_t^n\right)^T := \left(X_{[nt]}^n, Y_{[nt]}^n\right)^T$$

for the solution of the linear system of Wick difference equations

$$\begin{aligned} X_l^n &= X_{l-1}^n + (A_1 X_{l-1}^n + A_2 Y_{l-1}^n) \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n}\right), & X_0^n &= x_0, & l &= 1, \dots, n, \\ Y_l^n &= Y_{l-1}^n + (B_1 X_{l-1}^n + B_2 Y_{l-1}^n) \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n}\right), & Y_0^n &= y_0 & l &= 1, \dots, n, \end{aligned}$$

converges weakly to the solution $(X, Y)^T$ of the corresponding linear system of SDEs (1) in the Skorokhod space $D([0, 1], \mathbb{R})^2$.

Proof. Analogously to (14) we obtain by the recursive system of Wick difference equations for $U^{k,n}$ in (18) the coefficients for the solution of the systems of difference equations

$$X_l^n = \sum_{k=0}^{\infty} \frac{a_k}{k!} U_l^{k,n}, \quad Y_l^n = \sum_{k=0}^{\infty} \frac{b_k}{k!} U_l^{k,n},$$

recursively by

$$a_0 = x_0, \quad b_0 = y_0, \quad a_k = A_1 a_{k-1} + A_2 b_{k-1}, \quad b_k = B_1 a_{k-1} + B_2 b_{k-1}.$$

We denote the upper bound

$$M_{AB} := 2 \max\{|A_1|, |A_2|, |B_1|, |B_2|\}.$$

Suppose $r_1, r_2 \in \mathbb{R}$ to be arbitrary. By the linear system and (18) the sequence of processes

$$r_1 \tilde{X}^n + r_2 \tilde{Y}^n = \sum_{k=0}^n \left(\frac{r_1 a_k + r_2 b_k}{k!} \right) \tilde{U}^{k,n}$$

fulfils the conditions in Theorem 2 with

$$|r_1 a_k + r_2 b_k| \leq \max\{|x_0|, |y_0|\} (|r_1| + |r_2|) M_{AB}^k.$$

Thus we obtain the weak convergence

$$r_1 \tilde{X}^n + r_2 \tilde{Y}^n \xrightarrow{d} \sum_{k=0}^{\infty} \left(\frac{r_1 a_k + r_2 b_k}{k!} \right) (B^{H,n})^{\diamond k} = r_1 X + r_2 Y.$$

Now the Cramér-Wold device (Billingsley [5, Theorem 7.7]) concludes. \square

Remark 3. *Theorem 4 can be extended to higher dimensional linear cases. It holds also for an additional approximation of the coefficients $A_{n,i} \rightarrow A_i$ and $B_{n,i} \rightarrow B_i$ for $n \rightarrow \infty$.*

Example 3 (Wick-sine and Wick-cosine). *The piecewise constant interpolation of*

$$\begin{aligned} X_l^n &= X_{l-1}^n + Y_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), & X_0^n &= 0, & l &= 1, \dots, n, \\ Y_l^n &= Y_{l-1}^n - X_{l-1}^n \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), & Y_0^n &= 1, & l &= 1, \dots, n \end{aligned}$$

converges weakly to the solution of the linear system

$$\begin{aligned} dX_t &= Y_t d^\diamond B_t^H & X_0 &= 0, \\ dY_t &= -X_t d^\diamond B_t^H & Y_0 &= 1, \end{aligned}$$

the process $(\sin^\diamond(B_t^H), \cos^\diamond(B_t^H))^T$. By Theorem 1 it can be approximated by the discrete Wick version functional $(\sin^{\diamond n}(B_t^{H,n}), \cos^{\diamond n}(B_t^{H,n}))^T$ as well.

4. Walsh decompositions and L^2 -estimates

In this section we give the Walsh decompositions for the approximating sequences and obtain some L^2 -estimates. A key for the approximation results will be the convergence of the L^2 -norms of the discrete Wick powers of $B_t^{H,n}$ to the corresponding L^2 -norms of the Wick powers of B_t^H .

Recall the Walsh decomposition $B_t^{H,n} = \sum_{i=1}^{\lfloor nt \rfloor} b_{t,i}^n \xi_i^n$. Denote

$$b_{t,A}^n := \prod_{i \in A} b_{t,i}^n, \quad \Xi_A^n := \prod_{i \in A} \xi_i^n, \quad d_{l,i}^n := b_{\frac{l}{n},i}^n - b_{\frac{l-1}{n},i}^n$$

for $l = 1, \dots, n$. Notice that $d_{i,i}^n = b_{\frac{i}{n},i}^n$, $d_{l,i}^n = 0$ for $i > l$ and that the increment has the representation

$$B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} = \sum_{i=1}^l d_{l,i}^n \xi_i^n.$$

Recall the recursive system of Wick difference equations,

$$U_l^{k,n} = U_{l-1}^{k,n} + kU_{l-1}^{k-1,n} \diamond_n \left(B_{\frac{l}{n}}^{H,n} - B_{\frac{l-1}{n}}^{H,n} \right), \quad U_l^{0,n} = 1, \quad U_0^{k,n} = 0 \quad (23)$$

for $l = 1, \dots, n$ and $k \in \mathbb{N}$.

Proposition 1. *For all $n, k \in \mathbb{N}$ and $l = 0, \dots, n$, we have the Walsh decompositions*

$$\frac{1}{k!} U_l^{k,n} = \sum_{\substack{C \subseteq \{1, \dots, l\} \\ |C|=k}} \left(\sum_{\substack{m: C \rightarrow \{1, \dots, l\} \\ \text{injective}}} \prod_{p \in C} d_{m(p),p}^n \right) \Xi_C^n, \quad (24)$$

$$\frac{1}{k!} \left(B_{\frac{l}{n}}^{H,n} \right)^{\diamond_n k} = \sum_{\substack{C \subseteq \{1, \dots, l\} \\ |C|=k}} b_{\frac{l}{n}, C}^n \Xi_C^n, \quad (25)$$

$$\frac{1}{k!} \left(B_{\frac{l}{n}}^{H,n} \right)^{\diamond_n k} - \frac{1}{k!} U_l^{k,n} = \sum_{\substack{C \subseteq \{1, \dots, l\} \\ |C|=k}} \left(\sum_{\substack{m: C \rightarrow \{1, \dots, l\} \\ \text{not injective}}} \prod_{p \in C} d_{m(p),p}^n \right) \Xi_C^n. \quad (26)$$

Proof. We use the conventions that an empty sum is zero, an empty product is one, and that there exists exactly one map from the emptyset to an arbitrary set. For these reasons the formulas hold for $k = 0$ or $l = 0$. We prove (24) by induction: For all $l = 0, \dots, n$ and all $k \in \mathbb{N}$ is obviously $U_l^{0,n} = 1$ and $U_0^{k,n} = 0$ as in formula (24). Suppose the formula is proved for all positive integers less or equal to a certain k and all $l = 0, \dots, n$. Furthermore, for $k+1$, suppose the formula is proved for all positive integers less or equal to a certain l . For $k+1$ and $l+1$ we compute, by the difference equation (23) and the induction hypothesis,

$$\begin{aligned} U_{l+1}^{k+1,n} - U_l^{k+1,n} &= (k+1)k! \left(\sum_{\substack{C \subseteq \{1, \dots, l\} \\ |C|=k}} \sum_{\substack{m: C \rightarrow \{1, \dots, l\} \\ \text{injective}}} \prod_{p \in C} d_{m(p),p}^n \Xi_C^n \right) \diamond_n \sum_{i=1}^{l+1} d_{l+1,i}^n \xi_i^n \\ &= (k+1)! \sum_{\substack{C \subseteq \{1, \dots, l\} \\ i \in \{1, \dots, l+1\} \\ |C|=k, i \notin C}} \sum_{\substack{m: C \rightarrow \{1, \dots, l\} \\ \text{injective}}} \prod_{p \in C} d_{m(p),p}^n d_{l+1,i}^n \Xi_{C \cup \{i\}}^n \\ &= (k+1)! \sum_{\substack{C' \subseteq \{1, \dots, l+1\} \\ |C'|=k+1}} \sum_{\substack{m': C' \rightarrow \{1, \dots, l+1\} \\ \text{injective}, \exists q: m(q)=l+1}} \prod_{p \in C'} d_{m'(p),p}^n \Xi_{C'}^n. \end{aligned} \quad (27)$$

Note that $d_{m,p} = 0$ for all $p-1 \geq m$. Thus, by the induction hypothesis,

$$U_l^{k+1,n} = (k+1)! \sum_{\substack{C \subseteq \{1, \dots, l+1\} \\ |C|=k+1}} \sum_{\substack{m: C \rightarrow \{1, \dots, l+1\} \\ \text{injective}, \forall q: m(q) < l+1}} \prod_{p \in C} d_{m(p),p}^n \Xi_C^n. \quad (28)$$

Thanks to equations (27) and (28) we obtain (24). In particular, $U_l^{k,n} = 0$ for all $k > l$. We now compute the k -th Wick power of $(B_{\frac{l}{n}}^{H,n})$ as follows:

$$\begin{aligned} \left(\sum_{i=1}^n b_{\frac{l}{n},i}^n \xi_i^n \right)^{\diamond_n k} &= \sum_{\substack{i_1, i_2, \dots, i_k=1 \\ \text{pairwise distinct}}}^l \left(\prod_{j=1}^k b_{\frac{l}{n}, i_j}^n \prod_{j=1}^k \xi_{i_j}^n \right) \\ &= \sum_{\substack{C \subseteq \{1, \dots, l\} \\ |C|=k}} k! \left(\prod_{i \in C} b_{\frac{l}{n}, i}^n \prod_{i \in C} \xi_i^n \right) = \sum_{\substack{C \subseteq \{1, \dots, l\} \\ |C|=k}} k! b_{\frac{l}{n}, C}^n \Xi_C^n. \end{aligned}$$

In particular, $(B_{\frac{l}{n}}^{H,n})^{\diamond_n k} = 0$ for all $k > l$. This yields (25).

The telescoping sum yields

$$\sum_{m(p)=1}^l d_{m(p),p}^n = \sum_{m(p)=p}^l d_{m(p),p}^n = \sum_{m(p)=p}^l \left(b_{\frac{m(p)}{n}, p}^n - b_{\frac{m(p)-1}{n}, p}^n \right) = b_{\frac{l}{n}, p}^n,$$

and thus we get

$$\sum_{m: C \rightarrow \{1, \dots, l\}} \prod_{p \in C} d_{m(p), p}^n = \prod_{p \in C} \left(\sum_{m(p)=1}^l d_{m(p), p}^n \right) = \prod_{p \in C} b_{\frac{l}{n}, p}^n = b_{\frac{l}{n}, C}^n. \quad (29)$$

Equation (26) is, thus, implied by (24) and (25). \square

In the next Propositions we obtain some elementary estimates for the L^2 -norm of discrete Wick powers of $B_t^{H,n}$.

Proposition 2. For all $t \in [0, 1]$ and $i \in \{1, \dots, \lfloor nt \rfloor\}$,

$$b_{t,i}^n \leq 2c_H n^{-(1-H)}.$$

Proof. We estimate

$$\begin{aligned} b_{t,i}^n &= n^{\frac{1}{2}} c_H \left(H - \frac{1}{2} \right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{\frac{1}{2}-H} \int_s^{\frac{\lfloor nt \rfloor}{n}} u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du ds \\ &\leq n^{\frac{1}{2}} c_H \left(H - \frac{1}{2} \right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} s^{\frac{1}{2}-H} \left(\frac{\lfloor nt \rfloor}{n} \right)^{H-\frac{1}{2}} \frac{1}{H-\frac{1}{2}} \left(\frac{\lfloor nt \rfloor}{n} - s \right)^{H-\frac{1}{2}} ds \\ &\leq n^{\frac{1}{2}} c_H \frac{1}{\frac{3}{2}-H} \left(\left(\frac{i}{n} \right)^{\frac{3}{2}-H} - \left(\frac{i-1}{n} \right)^{\frac{3}{2}-H} \right) \left(\frac{\lfloor nt \rfloor}{n} \right)^{2(H-\frac{1}{2})}. \end{aligned}$$

Since $t \leq 1$, $\frac{1}{\frac{3}{2}-H} \leq 2$ and $|x|^{\frac{3}{2}-H} - |y|^{\frac{3}{2}-H} \leq |x-y|^{\frac{3}{2}-H}$, the assertion follows. \square

Remark 4. Observe that,

$$\mathbf{E} \left[B_t^{H,n} B_s^{H,n} \right] = \mathbf{E} \left[\sum_{i_1, i_2=1}^{\lfloor nt \rfloor} b_{t, i_1}^n b_{s, i_2}^n \xi_{i_1}^n \xi_{i_2}^n \right] = \sum_{i=1}^{\lfloor nt \rfloor} (b_{t, i}^n b_{s, i}^n). \quad (30)$$

By Nieminen [19] we thus get for any $s, t \in [0, 1]$ the following convergence

$$\begin{aligned} \mathbf{E} \left[B_t^{H,n} B_s^{H,n} \right] &= \sum_{i=1}^{\lfloor nt \rfloor} n \int_{\frac{i-1}{n}}^{\frac{i}{n}} z \left(\frac{\lfloor nt \rfloor}{n}, u \right) du \int_{\frac{i-1}{n}}^{\frac{i}{n}} z \left(\frac{\lfloor ns \rfloor}{n}, u \right) du \\ &\longrightarrow \int_0^1 z(t, u) z(s, u) du = \mathbf{E} [B_t^H B_s^H]. \end{aligned} \quad (31)$$

Moreover, we have by the Cauchy-Schwarz inequality the upper bound

$$\begin{aligned} \mathbf{E} \left[\left(B_t^{H,n} - B_s^{H,n} \right)^2 \right] &= \sum_{i=1}^{\lfloor nt \rfloor} \left(\sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(z \left(\frac{\lfloor nt \rfloor}{n}, u \right) du - z \left(\frac{\lfloor ns \rfloor}{n}, u \right) du \right) \right)^2 \\ &\leq \sum_{i=1}^{\lfloor nt \rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(z \left(\frac{\lfloor nt \rfloor}{n}, u \right) - z \left(\frac{\lfloor ns \rfloor}{n}, u \right) \right)^2 du = \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}. \end{aligned} \quad (32)$$

Proposition 3. For all $t \geq s$ in $[0, 1]$ and all $N \in \mathbb{N}$ such that $\lfloor ns \rfloor \geq N$, we have

$$\begin{aligned} 0 &\leq \mathbf{E} \left[(B_t^{H,n})^2 \right]^N + \mathbf{E} \left[(B_s^{H,n})^2 \right]^N - 2 \mathbf{E} \left[B_t^{H,n} B_s^{H,n} \right]^N \\ &\quad - \frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] \\ &\leq 2c_H^2 N^2 t^{2H(N-1)} n^{-(2-2H)} \end{aligned} \quad (33)$$

In particular,

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] = \mathbf{E} \left[\left((B_t^H)^{\diamond N} - (B_s^H)^{\diamond N} \right)^2 \right].$$

Proof. As $N \leq \lfloor ns \rfloor$, we get, making use of Proposition 1, (17) and (30) in Remark 4,

$$\begin{aligned} &\frac{1}{N!} \mathbf{E} \left[(B_t^{H,n})^{\diamond_n N} (B_s^{H,n})^{\diamond_n N} \right] \\ &= \frac{1}{N!} \mathbf{E} \left[\left(N! \sum_{\substack{C \subseteq \{1, \dots, \lfloor nt \rfloor\} \\ |C|=N}} b_{t, C}^n \Xi_C^n \right) \left(N! \sum_{\substack{C \subseteq \{1, \dots, \lfloor ns \rfloor\} \\ |C|=N}} b_{s, C}^n \Xi_C^n \right) \right] \end{aligned}$$

$$\begin{aligned}
&= N! \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=N}} b_{t,C}^n b_{s,C}^n = \sum_{i_1, \dots, i_N=1}^{[nt]} \prod_{j=1}^N (b_{t,i_j}^n b_{s,i_j}^n) - \sum_{\substack{i_1, \dots, i_N=1 \\ \exists k, l : i_k=i_l}}^{[nt]} \prod_{j=1}^N (b_{t,i_j}^n b_{s,i_j}^n) \\
&= \mathbf{E} \left[B_t^{H,n} B_s^{H,n} \right]^N - \sum_{\substack{i_1, \dots, i_N=1 \\ \exists k, l : i_k=i_l}}^{[nt]} \prod_{j=1}^N (b_{t,i_j}^n b_{s,i_j}^n). \tag{34}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathbf{E} \left[(B_t^{H,n})^2 \right]^N + \mathbf{E} \left[(B_s^{H,n})^2 \right]^N - 2\mathbf{E} \left[B_t^{H,n} B_s^{H,n} \right]^N \\
&\quad - \frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] \\
&= \sum_{\substack{i_1, \dots, i_N=1 \\ \exists k, l : i_k=i_l}}^{[nt]} \left(\prod_{j=1}^N (b_{t,i_j}^n)^2 + \prod_{j=1}^N (b_{s,i_j}^n)^2 - 2 \prod_{j=1}^N (b_{t,i_j}^n b_{s,i_j}^n) \right) \\
&= \sum_{\substack{i_1, \dots, i_N=1 \\ \exists k, l : i_k=i_l}}^{[nt]} \left(\prod_{j=1}^N (b_{t,i_j}^n) - \prod_{j=1}^N (b_{s,i_j}^n) \right)^2 \geq 0. \tag{35}
\end{aligned}$$

Hence, the left hand side of the inequality in (33) follows. By Proposition 2, (30) and (32) in Remark 4, and $\left| \frac{[nt]}{n} \right| \leq t$ we obtain

$$\begin{aligned}
&\sum_{\substack{i_1, \dots, i_N=1 \\ \exists k, l : i_k=i_l}}^{[nt]} \left(\prod_{j=1}^N (b_{t,i_j}^n) - \prod_{j=1}^N (b_{s,i_j}^n) \right)^2 \\
&\leq \sum_{\substack{i_1, \dots, i_N=1 \\ \exists k, l : i_k=i_l}}^{[nt]} \left(\prod_{j=1}^N (b_{t,i_j}^n) \right)^2 \leq \binom{N}{2} (\max_i (b_{t,i}^n)^2) \sum_{i_1, \dots, i_{N-1}=1}^{[nt]} \left(\prod_{j=1}^{N-1} (b_{t,i_j}^n) \right)^2 \\
&\leq 2c_H^2 N^2 \mathbf{E} \left[(B_t^{H,n})^2 \right]^{N-1} n^{-(2-2H)} \leq 2c_H^2 N^2 t^{2H(N-1)} n^{-(2-2H)} \rightarrow 0 \tag{36}
\end{aligned}$$

for $n \rightarrow \infty$. The representation of Wick powers of B_t^H by Hermite polynomials as in (9), their orthonormality (cf. Kuo [14, p. 355]) and the polarization identity yields $\mathbf{E} \left[(B_t^H)^{\diamond_n N} (B_s^H)^{\diamond_n N} \right] = N! \mathbf{E} \left[(B_t^H) (B_s^H) \right]^N$, (cf. also [20, Lemma 1.1.1]). Thus we have by (35)

$$\mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] - \mathbf{E} \left[\left((B_t^H)^{\diamond_n N} - (B_s^H)^{\diamond_n N} \right)^2 \right]$$

$$\begin{aligned}
&= N! \left(\mathbf{E} \left[(B_t^{H,n})^2 \right]^N - \mathbf{E} \left[(B_t^H)^2 \right]^N + \mathbf{E} \left[(B_s^{H,n})^2 \right]^N - \mathbf{E} \left[(B_s^H)^2 \right]^N \right. \\
&\quad \left. - 2\mathbf{E} \left[B_t^{H,n} B_s^{H,n} \right]^N + 2\mathbf{E} \left[B_t^H B_s^H \right]^N \right) \\
&\quad - N! \sum_{\substack{i_1, \dots, i_N=1 \\ \exists k, l : i_k = i_l}}^{\lfloor nt \rfloor} \left(\prod_{j=1}^N (b_{t, i_j}^n) - \prod_{j=1}^N (b_{s, i_j}^n) \right)^2.
\end{aligned}$$

Applying the convergences (36) and (31) yields

$$\mathbf{E} \left[\left((B_t^{H,n})^{\diamond n N} - (B_s^{H,n})^{\diamond n N} \right)^2 \right] - \mathbf{E} \left[\left((B_t^H)^{\diamond N} - (B_s^H)^{\diamond N} \right)^2 \right] \rightarrow 0.$$

□

Remark 5. In particular we obtain by (34), (32) and $\left| \frac{\lfloor nt \rfloor}{n} \right| \leq t$

$$\begin{aligned}
\sum_{\substack{C \subseteq \{1, \dots, \lfloor nt \rfloor\} \\ |C|=N}} (b_{t,C}^n)^2 &= \left(\frac{1}{N!} \right)^2 \mathbf{E} \left[\left((B_t^{H,n})^{\diamond n N} \right)^2 \right] \\
&\leq \frac{1}{N!} \mathbf{E} \left[(B_t^{H,n})^2 \right]^N \leq \frac{1}{N!} \left| \frac{\lfloor nt \rfloor}{n} \right|^{2HN} \leq \frac{1}{N!} t^{2HN}. \quad (37)
\end{aligned}$$

The next Proposition estimates the difference between the approximating sequences in Theorems 1 and 2.

Proposition 4. Under the assumptions of Theorem 1 there exists a constant $K > 0$ such that for all $t \in [0, 1]$, $n \geq 1$ and $k \in \mathbb{N}$,

$$\mathbf{E} \left[\left| \sum_{k=0}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond n k} - \sum_{k=0}^n \frac{a_{n,k}}{k!} \tilde{U}_t^{k,n} \right|^2 \right] \leq K n^{1-2H} \quad (38)$$

for the approximating processes in Theorems 1 and 2.

Proof. Recall that $d_{r,i}^n = b_{\frac{r}{n}, i}^n - b_{\frac{r-1}{n}, i}^n = \sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} (z(\frac{r}{n}, s) - z(\frac{r-1}{n}, s)) ds$. By (32) in

Remark 4 we obtain

$$(d_{r,i}^n)^2 \leq \sum_{i=1}^r (d_{r,i}^n)^2 \leq \left| \frac{r}{n} - \frac{r-1}{n} \right|^{2H} = n^{-2H}.$$

Thus we have $d_{r,i}^n \leq n^{-H}$ for all $i, n, r \geq 1$. Hence we obtain, as the sum in (29) telescopes, for $|C| \geq 2$,

$$\begin{aligned}
 & \sum_{\substack{m:C \rightarrow \{1, \dots, [nt]\} \\ \text{not injective}}} \prod_{l \in C} d_{m(l),l}^n = \sum_{\substack{m:C \rightarrow \{1, \dots, [nt]\} \\ \exists u, v \in C : m(u)=m(v)}} \prod_{l \in C} d_{m(l),l}^n \\
 &= \sum_{u \in C} \sum_{m:C \setminus \{u\} \rightarrow \{1, \dots, [nt]\}} \left(\prod_{l \in C \setminus \{u\}} d_{m(l),l}^n \right) \sum_{v \in C \setminus \{u\}} d_{m(v),u}^n \\
 &\leq \max_{i,r} d_{r,i}^n (|C| - 1) \sum_{\substack{C' \subset C \\ |C'|=|C|-1}} \sum_{m:C' \rightarrow \{1, \dots, [nt]\}} \prod_{l \in C'} d_{m(l),l}^n \\
 &\leq n^{-H} (|C| - 1) \sum_{\substack{C' \subset C \\ |C'|=|C|-1}} b_{t,C'}^n. \tag{39}
 \end{aligned}$$

By (26), (39), (37), and since $([nt] - (k-1)) \leq n$ we obtain, for $k \geq 1$,

$$\begin{aligned}
 & \mathbf{E} \left[\left| \frac{1}{k!} \left((B_t^{H,n})^{\diamond_n k} - \tilde{U}_t^{k,n} \right) \right|^2 \right] \\
 &\leq \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=k}} \left(n^{-H} (k-1) \sum_{\substack{C' \subset C \\ |C'|=|C|-1}} b_{t,C'}^n \right)^2 \\
 &\leq n^{-2H} (k-1)^2 \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=k}} (k-1) \sum_{\substack{C' \subset C \\ |C'|=|C|-1}} (b_{t,C'}^n)^2 \\
 &\leq n^{-2H} (k-1)^3 ([nt] - (k-1)) \sum_{\substack{C' \subseteq \{1, \dots, [nt]\} \\ |C'|=k-1}} (b_{t,C'}^n)^2 \leq \frac{(k-1)^3}{(k-1)!} t^{2H(k-1)} n^{1-2H}.
 \end{aligned}$$

Since $a_{n,k} \leq C^k$ in Theorems 1 and 2 and $\left((B_t^{H,n})^{\diamond_n k} - \tilde{U}_t^{k,n} \right)$ are zero for $k = 0, 1$ and orthogonal for different k by Proposition 1 and (17), we get

$$\begin{aligned}
 & \mathbf{E} \left[\left| \sum_{k=0}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^n \frac{a_{n,k}}{k!} \tilde{U}_t^{k,n} \right|^2 \right] \\
 &= \sum_{k=2}^{[nt]} \mathbf{E} \left[\left| \frac{a_{n,k}}{k!} \left((B_t^{H,n})^{\diamond_n k} - \tilde{U}_t^{k,n} \right) \right|^2 \right] \leq \left(\sum_{k=2}^{\infty} C^{2k} \frac{(k-1)^3}{(k-1)!} t^{2H(k-1)} \right) n^{1-2H}.
 \end{aligned}$$

As the series on the right hand side converges uniformly in $t \in [0, 1]$, the assertion follows. \square

5. Convergence of the finite-dimensional distributions

We first prove that Theorems 1 and 2 hold with weak convergence replaced by convergence of the finite-dimensional distributions. To this end we first approximate the Wick powers of B_t^H by induction. Then we combine these convergence results to approximate the Wick analytic functionals $F^\diamond(B_t^H) = \sum_{k=0}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k}$. Finally we conclude that convergence in finite dimensions holds in Theorem 2.

We observed in Section 2, that $(B_t^H)^{\diamond N} = h_{|t|^{2H}}^N(B_t^H)$, and that the Hermite recursion formula

$$(B_t^H)^{\diamond(N+1)} = (B_t^H)(B_t^H)^{\diamond N} - |t|^{2H} N (B_t^H)^{\diamond(N-1)} \quad (40)$$

holds. For the discrete Wick powers of the discrete variables we now obtain a discrete variant of (40).

Proposition 5 (Discrete Hermite recursion). *For all $N \geq 1$ and $t \in [0, 1]$,*

$$(B_t^{H,n})^{\diamond_n(N+1)} = B_t^{H,n} (B_t^{H,n})^{\diamond_n N} - N \mathbf{E} \left[(B_t^{H,n})^2 \right] (B_t^{H,n})^{\diamond_n(N-1)} + R(B_t^{H,n}, N), \quad (41)$$

with remainder

$$R(B_t^{H,n}, N) = N! \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=N-1}} b_{t,C}^n \Xi_C^n \sum_{i \in C} (b_{t,i}^n)^2, \quad (42)$$

and

$$\mathbf{E} \left[\left(R(B_t^{H,n}, N) \right)^2 \right] \leq 16c_H^4 N! N^3 n^{-(4-4H)}. \quad (43)$$

In particular we will use that the discrete Hermite recursion (41) converges weakly to Hermite recursion (40) for $n \rightarrow \infty$.

Proof. By Proposition 1 we get

$$\begin{aligned} (B_t^{H,n})^{\diamond_n(N+1)} &= B_t^{H,n} \diamond_n (B_t^{H,n})^{\diamond_n N} = \left(\sum_{i=1}^{[nt]} b_{t,i}^n \xi_i^n \right) \diamond_n \left(\sum_{\substack{A \subseteq \{1, \dots, [nt]\} \\ |A|=N}} N! b_{t,A}^n \Xi_A^n \right) \\ &= B_t^{H,n} (B_t^{H,n})^{\diamond_n N} - \sum_{\substack{A \subseteq \{1, \dots, [nt]\} \\ |A|=N}} \sum_{i \in A} N! b_{t,i}^n b_{t,A}^n \Xi_A^n \xi_i^n. \end{aligned} \quad (44)$$

For the second term in equation (44), by (30) in Remark 4 and Proposition 1, we obtain

$$\sum_{\substack{A \subseteq \{1, \dots, [nt]\} \\ |A|=N}} \sum_{i \in A} N! b_{t,i}^n b_{t,A}^n \Xi_A^n \xi_i^n$$

$$\begin{aligned}
&= N! \sum_{\substack{A \subseteq \{1, \dots, [nt]\} \\ |A|=N}} \sum_{i \in A} b_{t, A \setminus \{i\}}^n \Xi_{A \setminus \{i\}}^n (b_{t, i}^n \xi_i^n)^2 = N! \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=N-1}} b_{t, C}^n \Xi_C^n \sum_{i \notin C} (b_{t, i}^n)^2 \\
&= N(N-1)! \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=N-1}} b_{t, C}^n \Xi_C^n \left(\sum_{i=1}^{[nt]} (b_{t, i}^n)^2 - \sum_{i \in C} (b_{t, i}^n)^2 \right) \\
&= N(B_t^{H, n})^{\diamond_n(N-1)} \mathbf{E} \left[(B_t^{H, n})^2 \right] - N! \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=N-1}} b_{t, C}^n \Xi_C^n \sum_{i \in C} (b_{t, i}^n)^2,
\end{aligned}$$

which yields (41) and (42). Thus, thanks to Proposition 2, Remark 5 and $t \leq 1$, we obtain

$$\begin{aligned}
\mathbf{E} \left[\left(R(B_t^{H, n}, N) \right)^2 \right] &= (N!)^2 \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=N-1}} (b_{t, C}^n)^2 \left(\sum_{i \in C} (b_{t, i}^n)^2 \right)^2 \\
&\leq (N!)^2 \frac{1}{(N-1)!} t^{2H(N-1)} \left((N-1) 4c_H^2 n^{-(2-2H)} \right)^2 \leq 16c_H^4 N! N^3 n^{-(4-4H)}.
\end{aligned}$$

□

Theorem 5. For all $N \in \mathbb{N}$,

$$(1, B^{H, n}, \dots, (B^{H, n})^{\diamond_n N}) \xrightarrow{fd} (1, B^H, \dots, (B^H)^{\diamond N}). \quad (45)$$

Proof. The proof goes by induction on N . By Sottinen's approximation, $(1, B^{H, n}) \xrightarrow{fd} (1, B^H)$. Suppose equation (45) is proved for some $N \geq 1$. Assume $k \in \mathbb{N}$ and $r_j^i \in \mathbb{R}$ for $j = 0, \dots, N+1$, $i = 1, \dots, k$ and $t_1, t_2, \dots, t_k \in [0, 1]$ are chosen arbitrarily. By the pointwise convergence $\mathbf{E} \left[(B_t^{H, n})^2 \right] \rightarrow |t|^{2H}$ and the generalized continuous mapping theorem (Billingsley [5, Theorem 5.5]) the induction hypothesis implies

$$\begin{aligned}
&\sum_{l=0}^N \left(\sum_{j=1}^k r_j^l (B_{t_j}^{H, n})^{\diamond_n l} \right) + \sum_{j=1}^k r_j^{N+1} \left(B_{t_j}^{H, n} (B_{t_j}^{H, n})^{\diamond_n N} - N \mathbf{E} \left[(B_{t_j}^{H, n})^2 \right] (B_{t_j}^{H, n})^{\diamond_n(N-1)} \right) \\
&\xrightarrow{d} \sum_{l=0}^N \left(\sum_{j=1}^k r_j^l (B_{t_j}^H)^{\diamond l} \right) + \sum_{j=1}^k r_j^{N+1} \left(B_{t_j}^H (B_{t_j}^H)^{\diamond N} - N |t_j|^{2H} (B_{t_j}^H)^{\diamond(N-1)} \right).
\end{aligned}$$

Since $H > \frac{1}{2}$, (43) yields $R(B_t^{H, n}, N) \rightarrow 0$ in $L^2(\Omega, P)$. Thus by Slutsky's theorem [5, Theorem 4.1] and the Hermite recursions (40) and (41), we obtain

$$\sum_{l=0}^{N+1} \left(\sum_{j=1}^k r_j^l (B_{t_j}^{H, n})^{\diamond_n l} \right) \xrightarrow{d} \sum_{l=0}^{N+1} \left(\sum_{j=1}^k r_j^l (B_{t_j}^H)^{\diamond l} \right).$$

By the Cramér-Wold device (Billingsley [5, Theorem 7.7]) we have

$$(1, B^{H,n}, \dots, (B^{H,n})^{\diamond_n N+1}) \xrightarrow{fd} (1, B^H, \dots, (B^H)^{\diamond_{N+1}}),$$

and the induction is complete. \square

Proposition 6. *In the context of Theorem 1 convergence holds in finite dimensional distributions.*

Proof. By Billingsley [5, Theorem 4.2] it suffices to show, that the following three conditions hold

$$\forall m \in \mathbb{N} \quad \sum_{k=0}^m \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \xrightarrow{fd} \sum_{k=0}^m \frac{a_k}{k!} (B_t^H)^{\diamond k} \quad \text{as } n \rightarrow \infty, \quad (46)$$

$$\forall t \in [0, 1] \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E} \left[\left| \sum_{k=0}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^m \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \right| \wedge 1 \right] = 0, \quad (47)$$

$$\sum_{k=0}^m \frac{a_k}{k!} (B_t^H)^{\diamond k} \xrightarrow{fd} \sum_{k=0}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k} \quad \text{as } m \rightarrow \infty. \quad (48)$$

Condition (46) directly follows from Theorem 5 and the generalized continuous mapping theorem ([5, Theorem 5.5]). For the second condition we compute

$$\begin{aligned} \mathbf{E} \left[\left(\sum_{k=0}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^m \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \right)^2 \right] &= \mathbf{E} \left[\left(\sum_{k=m+1}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \right)^2 \right] \\ &= \sum_{k=m+1}^n \left(\frac{a_{n,k}}{k!} \right)^2 \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n k} \right)^2 \right] \leq \sum_{k=m+1}^n \left(\frac{C^k}{k!} \right)^2 k! t^{2Hk}, \end{aligned}$$

applying the estimate of Remark 5. Here we used that discrete Wick powers of different order are orthogonal. Thus we even obtain, $\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=m+1}^n \frac{C^{2k}}{k!} = 0$ and for all $t \in [0, 1]$ a stronger result than condition (47),

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E} \left[\left(\sum_{k=0}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^m \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} \right)^2 \right] = 0.$$

By the orthogonality of the different Wick powers, we have

$$\mathbf{E} \left[\left(\sum_{k=m+1}^{\infty} \frac{a_k}{k!} (B_t^H)^{\diamond k} \right)^2 \right] = \sum_{k=m+1}^{\infty} \left(\frac{a_k}{k!} \right)^2 \mathbf{E} \left[\left((B_t^H)^{\diamond k} \right)^2 \right] \leq \sum_{k=m+1}^{\infty} \left(\frac{C^{2k}}{k!} \right) t^{2Hk} \rightarrow 0.$$

for $m \rightarrow \infty$, which implies that condition (48) even holds in $L^2(\Omega, P)$. \square

In view of Proposition 4 and Slutsky's theorem, the previous proposition also implies:

Proposition 7. *In the context of Theorem 2, convergence holds in finite dimensional distributions.*

6. Tightness

We now show the tightness of the sequences in Theorems 1 and 2 by the following criterion, which is a variant of Theorem 15.6 in Billingsley [5].

Theorem 6. *Suppose for the random elements Y^n in the Skorokhod space $D([0, 1], \mathbb{R})$ and $\sum_{k=0}^{\infty} \frac{a_k}{k!} (B^H)^{\diamond k}$ in $C([0, 1], \mathbb{R})$,*

$$Y^n \xrightarrow{fd} \sum_{k=0}^{\infty} \frac{a_k}{k!} (B^H)^{\diamond k},$$

and for $s \leq t$ in $[0, 1]$

$$\mathbf{E} \left[(Y_t^n - Y_s^n)^2 \right] \leq L \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H},$$

where $L > 0$ is a constant. Then Y^n converges weakly to $\sum_{k=0}^{\infty} \frac{a_k}{k!} (B^H)^{\diamond k}$ in $D([0, 1], \mathbb{R})$.

Proof. Let $s < t < u$ in $[0, 1]$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} & \mathbf{E} [|Y_t^n - Y_s^n| |Y_u^n - Y_t^n|] \\ & \leq \left(\mathbf{E} [|Y_t^n - Y_s^n|^2] \right)^{\frac{1}{2}} \left(\mathbf{E} [|Y_u^n - Y_t^n|^2] \right)^{\frac{1}{2}} \\ & \leq L \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^H \left| \frac{\lfloor nu \rfloor}{n} - \frac{\lfloor nt \rfloor}{n} \right|^H \leq L \left| \frac{\lfloor nu \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}. \end{aligned}$$

If $u - s \geq \frac{1}{n}$, we have, since $\lfloor nu \rfloor \leq nu$ and $-\lfloor ns \rfloor \leq -ns + 1$,

$$\left| \frac{\lfloor nu \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H} \leq (2(u - s))^{2H},$$

and thus

$$\mathbf{E} [|Y_t^n - Y_s^n| |Y_u^n - Y_t^n|] \leq L 2^{2H} (u - s)^{2H}. \quad (49)$$

If $u - s < \frac{1}{n}$, we have either $\lfloor ns \rfloor = \lfloor nt \rfloor$ or $\lfloor nt \rfloor = \lfloor nu \rfloor$ and so the left hand side in (49) is zero. Thus, the inequality (49) holds for all $s < t < u$. By the convergence of the finite-dimensional distributions and [5, Theorem 15.6] we get the weak convergence of the processes. \square

For the application of this criterion to the discrete Wick powers we need two lemmas.

Lemma 1. *Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space and $\|x\| := \langle x, x \rangle$ the corresponding norm on X . Then for all $x, y \in X$ and $N \geq 1$,*

$$\|x\|^{2N} + \|y\|^{2N} - 2(\langle x, y \rangle)^N \leq 2^{N+1} (\|x\| + \|y\|)^{2(N-1)} \|x - y\|^2.$$

Proof. It holds that

$$\begin{aligned} 2(\langle x, y \rangle)^N &= 2 \left(\frac{1}{2} (\|x\|^2 + \|y\|^2 - \|x - y\|^2) \right)^N \\ &= \frac{1}{2^{N-1}} \left[(\|x\|^2 + \|y\|^2)^N + \sum_{k=0}^{N-1} \binom{N}{k} (\|x\|^2 + \|y\|^2)^k (-1)^{N-k} \|x - y\|^{2(N-k)} \right] \\ &= \frac{1}{2^{N-1}} (\|x\|^2 + \|y\|^2)^N \\ &\quad - \|x - y\|^2 \frac{1}{2^{N-1}} \sum_{k=0}^{N-1} \binom{N}{k} (\|x\|^2 + \|y\|^2)^k (-1)^{N-k-1} \|x - y\|^{2(N-k-1)}. \end{aligned}$$

Hence we get

$$\begin{aligned} \|x\|^{2N} + \|y\|^{2N} - 2(\langle x, y \rangle)^N &= \|x\|^{2N} + \|y\|^{2N} - \frac{1}{2^{N-1}} (\|x\|^2 + \|y\|^2)^N \\ &\quad + \|x - y\|^2 \frac{1}{2^{N-1}} \sum_{k=0}^{N-1} \binom{N}{k} (\|x\|^2 + \|y\|^2)^k (-1)^{N-k-1} \|x - y\|^{2(N-k-1)}. \end{aligned} \tag{50}$$

Since $(\frac{1}{2})^{N-1} \sum_{k=0}^N \binom{N}{k} = 2$ the first line on the right hand side of (50) can be treated as follows,

$$\begin{aligned} \|x\|^{2N} + \|y\|^{2N} - \frac{1}{2^{N-1}} (\|x\|^2 + \|y\|^2)^N \\ = \frac{1}{2^{N-1}} \sum_{k=1}^{N-1} \binom{N}{k} \left(\frac{\|x\|^{2N} + \|y\|^{2N}}{2} - \|x\|^{2k} \|y\|^{2(N-k)} \right). \end{aligned} \tag{51}$$

As $\binom{N}{k} = \binom{N}{N-k}$, we now collect the corresponding summands in sum (51) for $k \neq \frac{N}{2}$. We obtain by mean value theorem with

$$M_{x,y} := \max_{\lambda \in [0,1]} (\lambda \|x\| + (1-\lambda) \|y\|) = \max \{ \|x\|, \|y\| \}$$

and since $k(N-k) \leq \frac{N^2}{4}$,

$$\begin{aligned} & \|x\|^{2N} + \|y\|^{2N} - \|x\|^{2k} \|y\|^{2(N-k)} - \|x\|^{2(N-k)} \|y\|^{2k} \\ &= \left(\|x\|^{2k} - \|y\|^{2k} \right) \left(\|x\|^{2(N-k)} - \|y\|^{2(N-k)} \right) \\ &\leq 2k M_{x,y}^{2k-1} \|x - y\| 2(N-k) M_{x,y}^{2(N-k)-1} \|x - y\| \leq N^2 M_{x,y}^{2(N-1)} \|x - y\|^2. \end{aligned}$$

Analogously we obtain, for $k = \frac{N}{2}$,

$$\frac{\|x\|^{2N} + \|y\|^{2N}}{2} - \|x\|^N \|y\|^N = \frac{1}{2} \left(\|x\|^N - \|y\|^N \right)^2 \leq \frac{1}{2} M_{x,y}^{2(N-1)} N^2 \|x - y\|^2.$$

Plugging these estimates into (51) and since $\left(\frac{1}{2}\right)^{N-1} \sum_{k=1}^{N-1} \binom{N}{k} \frac{1}{2} = \left(1 - \frac{1}{2^{N-1}}\right)$, we obtain

$$\|x\|^{2N} + \|y\|^{2N} - \left(\frac{1}{2}\right)^{N-1} \left(\|x\|^2 + \|y\|^2\right)^N \leq \left(1 - \frac{1}{2^{N-1}}\right) N^2 M_{x,y}^{2(N-1)} \|x - y\|^2. \quad (52)$$

For the term in the second line on the right hand side of (50) we observe that, by the triangle inequality,

$$\begin{aligned} & \left(\|x\|^2 + \|y\|^2\right)^k (-1)^{N-k-1} \|x - y\|^{2(N-k-1)} \\ & \leq (\|x\| + \|y\|)^{2k} (\|x\| + \|y\|)^{2(N-k-1)} = (\|x\| + \|y\|)^{2(N-1)}. \end{aligned}$$

Applying

$$M_{x,y} \leq \|x\| + \|y\|, \quad \left(\frac{1}{2}\right)^{N-1} \sum_{k=0}^{N-1} \binom{N}{k} = 2 - \frac{1}{2^{N-1}},$$

and (52) to (50), we have

$$\begin{aligned} & \|x\|^{2N} + \|y\|^{2N} - 2 \langle x, y \rangle^N \\ & \leq \left[\left(1 - \frac{1}{2^{N-1}}\right) N^2 + \left(2 - \frac{1}{2^{N-1}}\right) \right] (\|x\| + \|y\|)^{2(N-1)} \|x - y\|^2. \end{aligned}$$

By a short calculation and induction we obtain

$$\left(1 - \frac{1}{2^{N-1}}\right) N^2 + \left(2 - \frac{1}{2^{N-1}}\right) \leq \mathbf{1}_{\{N \neq 3\}} 2^N + \mathbf{1}_{\{N=3\}} \frac{17}{2} < 2^{N+1}.$$

□

Lemma 2. For all $t > s$ in $[0, 1]$ we have

$$\frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] \leq 8^N \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}.$$

Proof. For $N = 1$ the inequality is fulfilled by (32) in Remark 4. For $N > 1$ we consider the cases $N > \lfloor ns \rfloor$ and $\lfloor ns \rfloor \geq N$ separately. For $\lfloor nt \rfloor \geq N > \lfloor ns \rfloor$ we have $(B_s^{H,n})^{\diamond_n N} = 0$. Hence, Proposition 3 and Remark 5 imply

$$\frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] = \frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} \right)^2 \right] \leq \left| \frac{\lfloor nt \rfloor}{n} \right|^{2HN}.$$

Since $N \geq 2$, $2H > 1$, and $\frac{\lfloor nt \rfloor}{n} \leq 1$, we obtain

$$\begin{aligned} \left| \frac{\lfloor nt \rfloor}{n} \right|^{2HN} &\leq \left| \frac{\lfloor nt \rfloor}{n} \right|^2 = \left(\frac{1}{n} \right)^2 \left((\lfloor nt \rfloor - \lfloor ns \rfloor) + \lfloor ns \rfloor \right)^2 \\ &\leq \left(\frac{1}{n} \right)^2 \left((\lfloor nt \rfloor - \lfloor ns \rfloor) + N \right)^2 \leq \left(\frac{1}{n} \right)^2 \left((\lfloor nt \rfloor - \lfloor ns \rfloor) (N+1) \right)^2 \\ &= (N+1)^2 \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^2 \end{aligned}$$

Since $(N+1)^2 \leq 3^N$ for $N \geq 2$ and $2H < 2$ we obtain

$$\frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] \leq 3^N \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}$$

for all $\lfloor nt \rfloor \geq N > \lfloor ns \rfloor$. Recall that, by Proposition 3, for $\lfloor nt \rfloor > \lfloor ns \rfloor \geq N$,

$$\begin{aligned} \frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] \\ \leq \mathbf{E} \left[(B_t^{H,n})^2 \right]^N + \mathbf{E} \left[(B_s^{H,n})^2 \right]^N - 2 \mathbf{E} \left[(B_t^{H,n})(B_s^{H,n}) \right]^N. \end{aligned}$$

For any $n \in \mathbb{N}$ we can rewrite

$$\mathbf{E} \left[(B_t^{H,n})(B_s^{H,n}) \right] = \sum_{i=1}^n b_{t,i}^n b_{s,i}^n$$

as an ordinary inner product on \mathbb{R}^n of the vectors $(b_{t,1}^n, \dots, b_{t,n}^n)^T$ and $(b_{s,1}^n, \dots, b_{s,n}^n)^T$. Thus the application of Lemma 1 with $t, s \in [0, 1]$ gives

$$\begin{aligned} &\frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] \\ &\leq 2^{N+1} \left(\mathbf{E} \left[(B_t^{H,n})^2 \right]^{\frac{1}{2}} + \mathbf{E} \left[(B_s^{H,n})^2 \right]^{\frac{1}{2}} \right)^{2(N-1)} \mathbf{E} \left[\left((B_t^{H,n}) - (B_s^{H,n}) \right)^2 \right] \\ &\leq 2^{N+1} 2^{2(N-1)} \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H} \leq 8^N \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}. \end{aligned}$$

If $N > \lfloor nt \rfloor$, then the left hand side of the assertion vanishes. \square

Remark 6. *The proofs for a fractional Brownian motion on some interval $[0, T] \subset \mathbb{R}$ follow by a straightforward modification. As $\mathbf{E} \left[\left(B_t^{H,n} \right)^2 \right] \leq T^{2H}$ for $t \in [0, T]$ and*

$$\left| \frac{\lfloor nt \rfloor}{n} \right|^{2HN} \leq T^{2H(N-1)} \left| \frac{\lfloor nt \rfloor}{n} \right|^{2H}$$

we obtain the previous Lemma for $t > s$ in $[0, T]$ as

$$\frac{1}{N!} \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n N} - (B_s^{H,n})^{\diamond_n N} \right)^2 \right] \leq (8T^{2H})^N \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}.$$

Now we are able to prove the weak convergence to the Wick analytic functionals of a fractional Brownian motion.

Proof of Theorem 1. We apply Theorem 6. The convergence of finite-dimensional distributions was shown in Proposition 6. Let be $s < t$ in $[0, 1]$. Recall $a_{n,k} \leq C^k$. Then, by the orthogonality of $\left((B_t^{H,n})^{\diamond_n k} - (B_s^{H,n})^{\diamond_n k} \right)$ for different k and Lemma 2, we have

$$\begin{aligned} & \mathbf{E} \left[\left(\sum_{k=0}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^n \frac{a_{n,k}}{k!} (B_s^{H,n})^{\diamond_n k} \right)^2 \right] \\ &= \sum_{k=0}^n \left(\frac{a_{n,k}}{k!} \right)^2 \mathbf{E} \left[\left((B_t^{H,n})^{\diamond_n k} - (B_s^{H,n})^{\diamond_n k} \right)^2 \right] \leq \sum_{k=0}^n \frac{C^{2k}}{k!} 8^k \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}. \end{aligned}$$

Since $0 < \sum_{k=0}^{\infty} \frac{8^k C^{2k}}{k!} = \exp(8C^2) =: L < \infty$, we have

$$\mathbf{E} \left[\left(\sum_{k=0}^n \frac{a_{n,k}}{k!} (B_t^{H,n})^{\diamond_n k} - \sum_{k=0}^n \frac{a_{n,k}}{k!} (B_s^{H,n})^{\diamond_n k} \right)^2 \right] \leq L \left| \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} \right|^{2H}. \quad (53)$$

□

The alternative approximation, stated in Theorem 2, follows similarly:

Proof of Theorem 2. Let be $s < t$ in $[0, 1]$. Recall that $d_{m,i}^n > 0$ only if $i \leq m$. Thus, by Proposition 1, we can write

$$\sum_{k=0}^n \frac{a_{n,k}}{k!} \tilde{U}_s^{k,n} = \sum_{k=0}^{\lfloor nt \rfloor} a_{n,k} \sum_{\substack{C \subseteq \{1, \dots, \lfloor nt \rfloor\} \\ |C|=k}} \sum_{\substack{m: C \rightarrow \{1, \dots, \lfloor ns \rfloor\} \\ \text{injective}}} \prod_{l \in C} d_{m(l), l}^n \Xi_C^n.$$

Observe that, by the telescoping sum in (29), we have

$$\begin{aligned}
& \sum_{\substack{m:C \rightarrow \{1, \dots, [nt]\} \\ \text{injective} \\ \exists u: m(u) > [ns]}} \prod_{l \in C} d_{m(l), l}^n \\
& \leq \sum_{\substack{m:C \rightarrow \{1, \dots, [nt]\} \\ \exists u: m(u) > [ns]}} \prod_{l \in C} d_{m(l), l}^n = \sum_{m:C \rightarrow \{1, \dots, [nt]\}} \prod_{l \in C} d_{m(l), l}^n - \sum_{m:C \rightarrow \{1, \dots, [ns]\}} \prod_{l \in C} d_{m(l), l}^n \\
& = b_{t, C}^n - b_{s, C}^n.
\end{aligned}$$

Thus, due to the orthogonality of $\tilde{U}_t^{k, n} - \tilde{U}_s^{k, n}$ for different values of k , Proposition 1 and estimate (53) we obtain

$$\begin{aligned}
& \mathbf{E} \left[\left(\sum_{k=0}^n \frac{a_{n, k}}{k!} \tilde{U}_t^{k, n} - \sum_{k=0}^n \frac{a_{n, k}}{k!} \tilde{U}_s^{k, n} \right)^2 \right] \\
& = \sum_{k=0}^{[nt]} a_{n, k}^2 \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=k}} \left(\sum_{\substack{m:C \rightarrow \{1, \dots, [nt]\} \\ \text{injective} \\ \exists u: m(u) > [ns]}} \prod_{l \in C} d_{m(l), l}^n \right)^2 \\
& \leq \sum_{k=0}^{[nt]} a_{n, k}^2 \sum_{\substack{C \subseteq \{1, \dots, [nt]\} \\ |C|=k}} (b_{t, C}^n - b_{s, C}^n)^2 \leq L \left| \frac{[nt]}{n} - \frac{[ns]}{n} \right|^{2H},
\end{aligned}$$

and the result follows from Proposition 7 and Theorem 6. \square

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