

Fractional Pure Birth Processes

Enzo Orsingher^(*), Federico Polito₂

*Dipartimento di Statistica, Probabilità e Stat. Appl., "Sapienza" Università di Roma
pl. A. Moro 5, 00185 Rome, Italy.*

(1) – Tel: +39 06 49910585, fax: +39 06 4959241

E-mail address: enzo.orsingher@uniroma1.it (corresponding author)

(2) – Tel: +39 0649910499, fax: +39 06 4959241

E-mail address: federico.polito@uniroma1.it

August 10, 2009

Abstract

We consider a fractional version of the classical non-linear birth process of which the Yule-Furry model is a particular case. Fractionality is obtained by replacing the first-order time derivative in the difference-differential equations which govern the probability law of the process, with the Dzherbashyan-Caputo fractional derivative. We derive the probability distribution of the number $\mathcal{N}_\nu(t)$ of individuals at an arbitrary time t . We also present an interesting representation for the number of individuals at time t , in the form of the subordination relationship $\mathcal{N}_\nu(t) = \mathcal{N}(T_{2\nu}(t))$ where $\mathcal{N}(t)$ is the classical generalised birth process and $T_{2\nu}(t)$ is a random time whose distribution is related to the fractional diffusion equation. The fractional linear birth process is examined in detail in section 3 and various forms of its distribution are given and discussed.

Keywords: Airy Functions, Branching Processes, Dzherbashyan-Caputo Fractional Derivative, Iterated Brownian Motion, Mittag-Leffler Functions, Non-Linear Birth Process, Stable Processes, Vandermonde Determinants, Yule-Furry Process.

1 Introduction

We consider a birth process and denote by $\mathcal{N}(t)$, $t > 0$ the number of components in a stochastically developing population at time t . Possible examples are the number of particles produced in a radioactive disintegration and the number of particles in a cosmic ray shower where death is not permitted. The probabilities $p_k(t) = \Pr\{\mathcal{N}(t) = k\}$ satisfy the difference-differential equations

$$\frac{dp_k}{dt} = -\lambda_k p_k + \lambda_{k-1} p_{k-1}, \quad k \geq 1 \quad (1.1)$$

where at time $t = 0$

$$p_k(0) = \begin{cases} 1 & k = 1 \\ 0 & k \geq 2 \end{cases} . \quad (1.2)$$

This means that we have initially one progenitor igniting the branching process. For information on this process consult Gikhman and Skorokhod [5] page 322.

We here examine a fractional version of the birth process where the probabilities are governed by

$$\frac{d^\nu p_k}{dt^\nu} = -\lambda_k p_k + \lambda_{k-1} p_{k-1}, \quad k \geq 1 \quad (1.3)$$

and where the fractional derivative is understood in the Dzherbashyan-Caputo sense, that is as

$$\frac{d^\nu p_k}{dt^\nu} = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{d}{ds} p_k(s) (t-s)^{\nu-1} ds, \quad \text{for } 0 < \nu < 1 \quad (1.4)$$

(see Podlubny [12]). The use of Dzherbashyan-Caputo derivative is preferred because in this case initial conditions can be expressed in terms of integer-order derivatives.

Extensions of continuous-time point processes like the homogeneous Poisson process to the fractional case have been considered in Jumarie [7], Cahoy [3], Laskin [9], Wang and Wen [17], Wang et al. [18], Wang et al. [19], Uchaikin and Sibatov [15], Repin and Saichev [13] and Beghin and Orsingher [2]. A recently published paper (Uchaikin et al. [16]) considers a fractional version of the Yule-Furry process where the mean value $\mathbb{E}N_\nu(t)$ is analysed.

By solving recursively the equation (1.3) (we write $p_k(t)$, $t > 0$ in equations (1.3) and $p_k^\nu(t)$ for the solutions) we obtain that

$$p_k^\nu(t) = \Pr\{N_\nu(t) = k\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \left\{ \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\}, & k > 1 \\ E_{\nu,1}(-\lambda_1 t^\nu) & k = 1 \end{cases}. \quad (1.5)$$

Result (1.5) generalises the classical distribution of the birth process (see Gikhman and Skorokhod [5] page 322, or Bartlett [1], page 59) where instead of the exponentials we have the Mittag-Leffler functions defined as

$$E_{\nu,1}(x) = \sum_{h=0}^{\infty} \frac{x^h}{\Gamma(\nu h + 1)}, \quad x \in \mathbb{R}, \quad \nu > 0. \quad (1.6)$$

The fractional pure birth process has some specific features entailed by the fractional derivative appearing in (1.4) which is a non-local operator. The process governed by fractional equations (and therefore the related probabilities $p_k^\nu(t) = \Pr\{N_\nu(t) = k\}$, $k \geq 1$) displays a slowly decreasing memory which seems a characteristic feature of all real systems (for example the hereditarity and the related aspects observed in phenomena such as the fatigue of metals, magnetic hysteresis and others). Fractional equations of various types have proved to be useful in representing different phenomena in optics (light propagation through random media), transport of charge carriers and also in economics (a survey of applications can be found in Podlubny [12]). We show below that for the linear birth process $N_\nu(t)$, $t > 0$ the mean values $\mathbb{E}N_\nu(t)$, $\mathbb{V}arN_\nu(t)$ are increasing functions as the order of fractionality ν decreases.

This shows that the fractional birth process is capable of representing explosively developing epidemics, accelerated cosmic showers and in general very rapidly expanding populations. This is a feature which the fractional pure birth process shares with its Poisson fractional counterpart whose practical applications are studied in recent works (see for example Laskin [9] and Cahoy [3]).

We are able to show that the fractional birth process $\mathcal{N}_\nu(t)$ can be represented as

$$\mathcal{N}_\nu(t) = \mathcal{N}(T_{2\nu}(t)), \quad t > 0, 0 < \nu \leq 1 \quad (1.7)$$

where $T_{2\nu}(t)$, $t > 0$ is the random time process whose distribution at time t is obtained from the fundamental solution to the fractional diffusion equation (the fractional derivative is defined in (1.4))

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial s^2}, \quad 0 < \nu \leq 1 \quad (1.8)$$

subject to the initial conditions $u(s, 0) = \delta(s)$ for $0 < \nu \leq 1$ and also $u_t(s, 0) = 0$ for $1/2 < \nu \leq 1$, as

$$\Pr\{T_{2\nu}(t) \in ds\} = \begin{cases} 2u_{2\nu}(s, t) ds & \text{for } s > 0 \\ 0 & \text{for } s < 0 \end{cases}. \quad (1.9)$$

This means that the fractional birth process is a classical birth process with a random time $T_{2\nu}(t)$ which is the sole component of (1.7) affected by the fractional derivative. In equation (1.8) and throughout the whole paper the fractional derivative must be understood in Dzherbashyan-Caputo sense (1.3). The representation (1.7) leads to

$$\Pr\{\mathcal{N}_\nu(t) = k\} = \int_0^\infty \Pr\{\mathcal{N}(s) = k\} \Pr\{T_{2\nu}(t) \in ds\} \quad (1.10)$$

where

$$\Pr\{\mathcal{N}(s) = k\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{e^{-\lambda_m s}}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)}, & k > 1, t > 0 \\ e^{-\lambda_1 t}, & k = 1, t > 0 \end{cases}. \quad (1.11)$$

Formula (1.10) immediately shows that $\sum_k \Pr\{\mathcal{N}_\nu(t) = k\} = 1$ iff $\sum_k \Pr\{\mathcal{N}(t) = k\} = 1$. It is well-known that the process $\mathcal{N}(t)$, $t > 0$ is such that $\Pr(\mathcal{N}(t) < \infty) = 1$ for all $t > 0$ (non-exploding) if $\sum_k \lambda_k^{-1} = \infty$ (consult Feller [4], page 452).

A special case of the above fractional birth process is the fractional linear birth process where $\lambda_k = \lambda k$. The distribution (1.5) reduces in this case to the simple form

$$p_k^\nu(t) = \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu), \quad k \geq 1, t > 0. \quad (1.12)$$

For $\nu = 1$, we retrieve from (1.12) the classical geometric structure of the linear birth process with a single progenitor, that is

$$p_k^1(t) = (1 - e^{-\lambda t})^{k-1} e^{-\lambda t}, \quad k \geq 1, t > 0. \quad (1.13)$$

An interesting qualitative feature of the fractional linear birth process can be extracted from (1.12); it permits us to highlight the dependence of the branching speed on the order of fractionality ν . We show in section 3 that

$$\Pr \{N_\nu(dt) = n_0 + 1 \mid N_\nu(0) = n_0\} \sim \frac{\lambda n_0 (dt)^\nu}{\Gamma(\nu + 1)} \quad (1.14)$$

and this proves that a decrease in the order of fractionality ν speeds up the reproduction of individuals. We are not able to generalise (1.14) to the case

$$\Pr \{N_\nu(t + dt) = n_0 + 1 \mid N_\nu(t) = n_0\} \quad (1.15)$$

because the process we are investigating is not time-homogeneous. For the fractional linear birth process the representation (1.7) reduces to the form

$$N_\nu(t) = N(T_{2\nu}(t)), \quad t > 0, 0 < \nu \leq 1 \quad (1.16)$$

and has an interesting special structure when $\nu = 1/2^n$. For example for $n = 2$ the random time appearing in (1.16) becomes a folded iterated Brownian motion. This means that

$$N_{\frac{1}{4}}(t) = N(|\mathcal{B}_1(|\mathcal{B}_2(t)|)|). \quad (1.17)$$

Clearly $|\mathcal{B}_2(t)|$ is a reflecting Brownian motion starting from zero and $|\mathcal{B}_1(|\mathcal{B}_2(t)|)|$ is a reflecting iterated Brownian motion. This permits us to write the distribution of (1.17) in the following form

$$\Pr \left\{ N_{\frac{1}{4}}(t) = k \mid N_{\frac{1}{4}}(0) = 1 \right\} = \int_0^\infty (1 - e^{-\lambda s})^{k-1} e^{-\lambda s} \left\{ 2^2 \int_0^\infty \frac{e^{-\frac{s^2}{4\omega}}}{\sqrt{2\pi 2\omega}} \frac{e^{-\frac{\omega^2}{4t}}}{\sqrt{2\pi 2t}} d\omega \right\} ds. \quad (1.18)$$

The case $\nu = 1/2^n$ involves the $(n - 1)$ -times iterated Brownian motion

$$\mathcal{I}_{n-1}(t) = \mathcal{B}_1(|\mathcal{B}_2(\dots|\mathcal{B}_n(t)|\dots)|) \quad (1.19)$$

with distribution

$$\begin{aligned} \Pr \{|\mathcal{B}_1(|\mathcal{B}_2(\dots|\mathcal{B}_n(t)|\dots)|) \in ds\} &= \\ &= 2^n \int_0^\infty \frac{e^{-\frac{s^2}{4\omega_1}}}{\sqrt{4\pi\omega_1}} d\omega_1 \int_0^\infty \frac{e^{-\frac{\omega_1^2}{4\omega_2}}}{\sqrt{4\pi\omega_2}} d\omega_2 \dots \int_0^\infty \frac{e^{-\frac{\omega_{n-1}^2}{4t}}}{\sqrt{4\pi t}} d\omega_{n-1}. \end{aligned} \quad (1.20)$$

For details on this point see Orsingher and Beghin [11].

2 The distribution function for the generalised fractional birth process

We present now the explicit distribution

$$\Pr \{N_\nu(t) \mid \mathcal{N}(0) = 1\} = p_k^\nu(t) \quad t > 0, k \geq 1, 0 < \nu \leq 1 \quad (2.1)$$

of the number of individuals in the population expanding according to (1.3). Our technique is based on successive applications of the Laplace transform. Our first result is the next theorem below.

Theorem 2.1. *The solution to the fractional equations*

$$\begin{cases} \frac{d^\nu p_k}{dt^\nu} = -\lambda_k p_k + \lambda_{k-1} p_{k-1} & k \geq 1, 0 < \nu \leq 1 \\ p_k(0) = \begin{cases} 1 & k = 1 \\ 0 & k \geq 2 \end{cases} \end{cases} \quad (2.2)$$

is given by

$$p_k^\nu(t) = Pr\{\mathcal{N}_\nu(t) = k\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \left\{ \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\}, & k > 1 \\ E_{\nu,1}(-\lambda_1 t^\nu), & k = 1 \end{cases}. \quad (2.3)$$

Proof. We prove result (2.3) by a recursive procedure.

For $k = 1$, the equation

$$\frac{d^\nu p_1}{dt^\nu} = -\lambda_1 p_1, \quad p_1(0) = 1 \quad (2.4)$$

is immediately solved by

$$p_1^\nu(t) = E_{\nu,1}(-\lambda_1 t^\nu). \quad (2.5)$$

For $k = 2$ equation (1.3) becomes

$$\begin{cases} \frac{d^\nu p_2}{dt^\nu} = -\lambda_2 p_2 + \lambda_1 E_{\nu,1}(-\lambda_1 t^\nu) \\ p_2(0) = 0 \end{cases}. \quad (2.6)$$

In view of the fact that

$$\int_0^\infty e^{-\mu t} E_{\nu,1}(-\lambda_1 t^\nu) dt = \frac{\mu^{\nu-1}}{\mu^\nu + \lambda_1} \quad (2.7)$$

the Laplace transform of (2.6) yields

$$L_2(\mu) = \frac{\lambda_1 \mu^{\nu-1}}{\lambda_2 - \lambda_1} \left[\frac{1}{\mu^\nu + \lambda_1} - \frac{1}{\mu^\nu + \lambda_2} \right]. \quad (2.8)$$

In light of (2.7), from (2.8) we can extract the probability $p_2^\nu(t)$

$$p_2^\nu(t) = [E_{\nu,1}(-\lambda_1 t^\nu) - E_{\nu,1}(-\lambda_2 t^\nu)] \frac{\lambda_1}{\lambda_2 - \lambda_1}. \quad (2.9)$$

Now the Laplace transform of

$$\frac{d^\nu p_3}{dt^\nu} = -\lambda_3 p_3 + \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} [E_{\nu,1}(-\lambda_1 t^\nu) - E_{\nu,1}(-\lambda_2 t^\nu)] \quad (2.10)$$

yields, after some computations

$$L_3(\mu) = \lambda_2 \lambda_1 \mu^{\nu-1} \left[\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \frac{1}{\mu^\nu + \lambda_1} + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \frac{1}{\mu^\nu + \lambda_2} + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \frac{1}{\mu^\nu + \lambda_3} \right]. \quad (2.11)$$

From this result it is clear that

$$p_3^\nu(t) = \lambda_2 \lambda_1 \left[\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} E_{\nu,1}(-\lambda_1 t^\nu) + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} E_{\nu,1}(-\lambda_2 t^\nu) + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} E_{\nu,1}(-\lambda_3 t^\nu) \right]. \quad (2.12)$$

The procedure for $k > 3$ becomes more complicated. However the special case $k = 4$ is instructive and we treat it first.

The Laplace transform of the equation

$$\frac{d^\nu p_4}{dt^\nu} = -\lambda_4 p_4 + \lambda_1 \lambda_2 \lambda_3 \left[\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} E_{\nu,1}(-\lambda_1 t^\nu) + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} E_{\nu,1}(-\lambda_2 t^\nu) + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} E_{\nu,1}(-\lambda_3 t^\nu) \right] \quad (2.13)$$

subject to the initial condition $p_4(0) = 0$ becomes

$$L_4(\mu) = \lambda_1 \lambda_2 \lambda_3 \mu^{\nu-1} \left[\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} \left\{ \frac{1}{\mu^\nu + \lambda_1} - \frac{1}{\mu^\nu + \lambda_4} \right\} + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \left\{ \frac{1}{\mu^\nu + \lambda_2} - \frac{1}{\mu^\nu + \lambda_4} \right\} + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} \left\{ \frac{1}{\mu^\nu + \lambda_3} - \frac{1}{\mu^\nu + \lambda_4} \right\} \right]. \quad (2.14)$$

The critical point of the proof is to show that

$$\begin{aligned} & - \frac{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3) - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_3) + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} \\ & = \frac{1}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)}. \end{aligned} \quad (2.15)$$

We note that

$$0 = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \end{pmatrix} + \quad (2.16)$$

$$\begin{aligned}
& -\det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_3^2 & \lambda_4^2 \end{pmatrix} + \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_4^2 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} = \\
& = (\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3) - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_3) + \\
& \quad + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2) - (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)
\end{aligned}$$

where in the last step the Vandermonde formula is applied.

By inserting (2.16) into (2.14) we now have that

$$\begin{aligned}
L_4(\mu) &= \lambda_1 \lambda_2 \lambda_3 \mu^{\nu-1} \left[\frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} \frac{1}{\mu^\nu + \lambda_1} + \right. \\
& \quad + \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \frac{1}{\mu^\nu + \lambda_2} + \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} \frac{1}{\mu^\nu + \lambda_3} + \\
& \quad \left. + \frac{1}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \frac{1}{\mu^\nu + \lambda_4} \right]
\end{aligned} \tag{2.17}$$

so that by inverting (2.17) we extract the following result

$$p_4^\nu(t) = \prod_{j=1}^3 \lambda_j \left\{ \sum_{m=1}^4 \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^4 (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\}. \tag{2.18}$$

We now tackle the problem of showing that (2.3) solves the Cauchy problem (2.2) for all $k > 1$, by induction. This means that we must solve

$$\begin{cases} \frac{d^\nu p_k}{dt^\nu} = -\lambda_k + \prod_{j=1}^{k-1} \lambda_j \left\{ \sum_{m=1}^{k-1} \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^{k-1} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right\} \\ p_k(0) = 0 \end{cases}, k > 4. \tag{2.19}$$

The Laplace transform of (2.19) reads

$$L_k(\mu) = \prod_{j=1}^{k-1} \lambda_j \left[\sum_{m=1}^{k-1} \frac{\mu^{\nu-1}}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} \frac{1}{\mu^\nu + \lambda_m} - \frac{\mu^{\nu-1}}{\mu^\nu + \lambda_k} \sum_{m=1}^{k-1} \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} \right]. \tag{2.20}$$

We must now prove that

$$-\sum_{m=1}^{k-1} \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} = \frac{1}{\prod_{\substack{l=1 \\ l \neq k}}^k (\lambda_l - \lambda_k)} \tag{2.21}$$

and this relationship is important also for the proof of (1.11).

In order to prove (2.21) we rewrite the left-hand side as

$$-\sum_{m=1}^{k-1} \frac{\prod_{h=1}^{k-1} \prod_{l>h}^k (\lambda_l - \lambda_h)}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} \cdot \frac{1}{\prod_{i=1}^{k-1} \prod_{l>i}^k (\lambda_l - \lambda_i)} \quad (2.22)$$

and concentrate our attention on the numerator of (2.22). In analogy with the calculations in (2.16) we have that

$$\begin{aligned} 0 &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m & \cdots & \lambda_k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \cdots & \lambda_m^{k-2} & \cdots & \lambda_k^{k-2} \end{pmatrix} = \\ &= \sum_{m=1}^k (-1)^{m-1} \det \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_{m-1} & \lambda_{m+1} & \cdots & \lambda_k \\ \lambda_1^{k-2} & \cdots & \lambda_{m-1}^{k-2} & \lambda_{m+1}^{k-2} & \cdots & \lambda_k^{k-2} \end{pmatrix} = \\ &= \sum_{m=1}^k \frac{\prod_{h=1}^{k-1} \prod_{l>h}^k (\lambda_l - \lambda_h)}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} = \sum_{m=1}^{k-1} \frac{\prod_{h=1}^{k-1} \prod_{l>h}^k (\lambda_l - \lambda_h)}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} + \frac{\prod_{h=1}^{k-1} \prod_{l>h}^k (\lambda_l - \lambda_h)}{\prod_{\substack{l=1 \\ l \neq k}}^k (\lambda_l - \lambda_k)}. \end{aligned} \quad (2.23)$$

In the third step of (2.23) we applied the Vandermonde formula and considered that the n -th column is missing. It must also be taken into account that

$$\begin{aligned} &\frac{\prod_{l>1}^k (\lambda_l - \lambda_1)}{(\lambda_m - \lambda_1)} \cdot \frac{\prod_{l>2}^k (\lambda_l - \lambda_2)}{(\lambda_m - \lambda_2)} \cdots \frac{\prod_{l>m-1}^k (\lambda_l - \lambda_{m-1})}{(\lambda_m - \lambda_{m-1})} \\ &\cdot \frac{\prod_{l>m}^k (\lambda_l - \lambda_m)}{\prod_{l>m}^k (\lambda_l - \lambda_m)} \cdot \prod_{l>m+1}^k (\lambda_l - \lambda_{m+1}) \cdots \prod_{l>k-1}^k (\lambda_l - \lambda_{k-1}) = \frac{\prod_{h=1}^{k-1} \prod_{l>h}^k (\lambda_l - \lambda_h)}{(-1)^{m-1} \prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)}. \end{aligned} \quad (2.24)$$

From (2.22) and (2.23) we have that

$$-\sum_{m=1}^{k-1} \frac{1}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} = -\sum_{m=1}^{k-1} \frac{\prod_{h=1}^{k-1} \prod_{l>h}^k (\lambda_l - \lambda_h)}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} \cdot \frac{1}{\prod_{i=1}^{k-1} \prod_{l>i}^k (\lambda_l - \lambda_i)} = \frac{1}{\prod_{\substack{l=1 \\ l \neq k}}^k (\lambda_l - \lambda_k)} \quad (2.25)$$

In view of (2.25) we can write that

$$L_k(\mu) = \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{\mu^{\nu-1}}{\prod_{\substack{l=1 \\ l \neq m}}^k (\lambda_l - \lambda_m)} \cdot \frac{1}{\mu^\nu + \lambda_m} \quad (2.26)$$

because the k -th term of (2.26) coincides with the last term of (2.20) and therefore by inversion of the Laplace transform we get (2.3). \square

Remark 2.1. We now prove that for the generalised fractional birth process the representation

$$\mathcal{N}_\nu(t) = \mathcal{N}(T_{2\nu}(t)), \quad t > 0, 0 < \nu \leq 1 \quad (2.27)$$

holds. This means that the process under investigation can be viewed as a generalised birth process at a random time $T_{2\nu}(t)$, $t > 0$ whose distribution is the folded solution to the fractional diffusion equation (1.8).

$$\begin{aligned} \int_0^\infty e^{-\mu t} \mathcal{G}_\nu(u, t) dt &\stackrel{\text{by (2.3)}}{=} \int_0^\infty \left\{ \sum_{k=2}^\infty u^k \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{E_{\nu,1}(-\lambda_m t^\nu)}{\prod_{j \neq m}^k (\lambda_j - \lambda_m)} + u E_{\nu,1}(-\lambda_1 t^\nu) \right\} e^{-\mu t} dt \quad (2.28) \\ &= \sum_{k=2}^\infty u^k \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{\mu^{\nu-1}}{\mu^\nu + \lambda_m} \frac{1}{\prod_{j \neq m}^k (\lambda_j - \lambda_m)} + \frac{u \mu^{\nu-1}}{\mu^\nu + \lambda_1} \\ &= \int_0^\infty \left\{ \sum_{k=1}^\infty u^k \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^k \frac{\mu^{\nu-1}}{\prod_{j \neq m}^k (\lambda_j - \lambda_m)} e^{-s(\mu^\nu + \lambda_m)} + u e^{-s(\mu^\nu + \lambda_1)} \right\} ds \\ &= \int_0^\infty \mathcal{G}(u, s) \mu^{\nu-1} e^{-s\mu^\nu} ds = \int_0^\infty \mathcal{G}(u, s) \int_0^\infty e^{-\mu t} f_{T_{2\nu}}(s, t) dt ds \\ &= \int_0^\infty e^{-\mu t} \left\{ \int_0^\infty \mathcal{G}(u, s) f_{T_{2\nu}}(s, t) ds \right\} dt \end{aligned}$$

where

$$\int_0^\infty e^{-\mu t} f_{T_{2\nu}}(s, t) dt = \mu^{\nu-1} e^{-s\mu^\nu}, \quad s > 0 \quad (2.29)$$

is the Laplace transform of the folded solution to (1.8). From (2.28) we infer that

$$\mathcal{G}_\nu(u, t) = \int_0^\infty \mathcal{G}(u, s) f_{T_{2\nu}}(s, t) ds \quad (2.30)$$

and from this the representation (2.27) follows.

Remark 2.2. The relationship (2.27) permits us to conclude that the functions (2.3) are non-negative because

$$\Pr\{\mathcal{N}_\nu(t) = k\} = \int_0^\infty \Pr\{\mathcal{N}(s) = k\} \Pr\{T_{2\nu}(t) \in ds\} \quad (2.31)$$

and $\Pr \{\mathcal{N}(s) = k\} > 0$ and $\sum_k \Pr \{\mathcal{N}(s) = k\} = 1$ as shown, for example in Feller [4], page 452. Furthermore the fractional birth process is non-exploding iff $\sum_k (1/\lambda_k) = \infty$, for all values of $0 < \nu \leq 1$.

3 The fractional linear birth process

In this section we examine in detail a special case of the previous fractional birth process, namely the fractional linear birth process which generalises the classical Yule-Furry model. The birth rates in this case have the form

$$\lambda_k = \lambda k, \quad \lambda > 0, k \geq 1 \quad (3.1)$$

and indicate that new births occur with a probability proportional to the size of the population. We denote by $N_\nu(t)$ the number of individuals in the population expanding according to the rates (3.1) and we have that the probabilities

$$p_k^\nu(t) = \Pr \{N_\nu(t) = k | N_\nu(0) = 1\}, \quad k \geq 1 \quad (3.2)$$

satisfy the difference-differential equations

$$\begin{cases} \frac{d^\nu p_k}{dt^\nu} = -\lambda k p_k + \lambda(k-1)p_{k-1}, & 0 < \nu \leq 1, k \geq 1 \\ p_k(0) = \begin{cases} 1 & k = 1 \\ 0 & k \geq 2 \end{cases} \end{cases} \quad (3.3)$$

The distribution (3.2) can be obtained as a particular case of (2.3) or directly by means of a completely different approach, as follows.

Theorem 3.1. *The distribution of the fractional linear birth process with a simple initial progenitor reads*

$$p_k^\nu(t) = \Pr \{N_\nu(t) = k | N_\nu(0) = 1\} = \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu), \quad k \geq 1, 0 < \nu \leq 1 \quad (3.4)$$

where $E_{\nu,1}(x)$ is the Mittag-Leffler function (1.6).

Proof. We can prove result (3.4) by solving equation (3.3) recursively. This means that $p_{k-1}^\nu(t)$ has the form (3.4) so $p_k^\nu(t)$ maintains the same structure. This is tantamount to solving the Cauchy problem

$$\begin{cases} \frac{d^\nu p_k(t)}{dt^\nu} = -\lambda k p_k(t) + \lambda(k-1) \sum_{j=1}^{k-1} \binom{k-2}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) \\ p_k(0) = 0, \quad k > 1 \end{cases} \quad (3.5)$$

By applying the Laplace transform $L_{k,\nu}(\mu) = \int_0^\infty e^{-\mu t} p_k(t) dt$ to (3.5) we have that

$$L_{k,\nu}(\mu) = \lambda(k-1) \left\{ \sum_{j=1}^{k-1} \binom{k-2}{j-1} (-1)^{j-1} \frac{\mu^{k-1}}{\mu^\nu + \lambda j} \right\} \frac{1}{\mu^\nu + \lambda k}. \quad (3.6)$$

The Laplace transform (3.6) can conveniently be written as

$$\begin{aligned} L_{k,\nu}(\mu) &= \mu^{\nu-1} \left\{ \left[\frac{1}{\mu^\nu + \lambda} - \frac{1}{\mu^\nu + \lambda k} \right] - (k-1) \left[\frac{1}{\mu^\nu + 2\lambda} - \frac{1}{\mu^\nu + \lambda k} \right] + \right. \\ &+ \frac{(k-1)(k-2)}{2} \left[\frac{1}{\mu^\nu + 3\lambda} - \frac{1}{\mu^\nu + \lambda k} \right] + \cdots + (k-1) (-1)^{k-2} \left[\frac{1}{\mu^\nu + (k-1)\lambda} - \frac{1}{\mu^\nu + \lambda k} \right] \left. \right\} \\ &= \mu^{\nu-1} \sum_{j=1}^{k-1} \binom{k-1}{j-1} (-1)^{j-1} \frac{1}{\mu^\nu + j\lambda} - \frac{\mu^{\nu-1}}{\mu^\nu + \lambda k} \sum_{j=1}^{k-1} \binom{k-1}{j-1} (-1)^{j-1}. \end{aligned} \quad (3.7)$$

This permits us to conclude that

$$L_{k,\nu}(\mu) = \mu^{\nu-1} \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} \frac{1}{\mu^\nu + j\lambda}. \quad (3.8)$$

By inverting (3.8) we arrive immediately at result (3.4). \square

For $\nu = 1$ (3.8) can be written as

$$\int_0^\infty e^{-\lambda t} p_k^1(t) dt = \int_0^\infty e^{-\lambda t} e^{-\mu t} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j e^{-\lambda j t} = \int_0^\infty e^{-\mu t} \left\{ e^{-\lambda t} (1 - e^{-\lambda t})^{k-1} \right\} dt \quad (3.9)$$

and this is an alternative derivation of the Yule-Furry linear birth process distribution.

Remark 3.1. An alternative form of the distribution (3.4) can be derived by writing explicitly the Mittag-Leffler function and manipulating conveniently the double sums obtained. We have therefore

$$\begin{aligned} p_k^\nu(t) &= \sum_{m=0}^{k-1} \frac{(-\lambda t^\nu)^m}{\Gamma(\nu m + 1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (j+1)^m + \sum_{m=k}^\infty \frac{(-\lambda t^\nu)^m}{\Gamma(\nu m + 1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (j+1)^m \\ &= \frac{(\lambda t^\nu)^{k-1} (k-1)!}{\Gamma(\nu(k-1) + 1)} + \sum_{m=k}^\infty \frac{(-\lambda t^\nu)^m}{\Gamma(\nu m + 1)} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (j+1)^m \end{aligned} \quad (3.10)$$

The last step of (3.10) is justified by the following formulas (0.154(6) and 0.154(5) Gradshteyn and Ryzhik [6], page 4)

$$\sum_{k=0}^N (-1)^k \binom{N}{k} (\alpha + k)^{n-1} = 0, \quad \text{valid for } N \geq n \geq 1 \quad (3.11)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (\alpha + k)^n = (-1)^n n! \quad . \quad (3.12)$$

What is remarkable about (3.12) is that the result is independent of α . This can be ascertained as follows

$$S_n^\alpha = \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{r=0}^n \binom{n}{r} \alpha^r k^{n-r} = \sum_{r=0}^n \binom{n}{r} \alpha^r \sum_{k=0}^n (-1)^k \binom{n}{k} k^{n-r+1-1} \quad (3.13)$$

By formula 0.154(3) of Gradshteyn and Ryzhik [6], page 4, the inner sum in the second line of (3.13) equals zero for $1 \leq n - r + 1 \leq n$, that is for $1 \leq r \leq n$. Therefore (see Gradshteyn and Ryzhik [6] formula 0.154(4), page 4)

$$S_n^\alpha = \binom{n}{0} \alpha^0 \sum_{k=0}^n (-1)^k \binom{n}{k} k^n = (-1)^n n!. \quad (3.14)$$

We now provide a direct proof that the distribution (3.4) sums to unity. This is based on combinatorial arguments and will subsequently be validated by resorting to the representation of $N_\nu(t)$ as a composition of the Yule-Furry model with the random time $T_{2\nu}(t)$.

Theorem 3.2. *The distribution (3.4) is such that*

$$\sum_{k=1}^{\infty} p_k^\nu(t) = \sum_{k=1}^{\infty} \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) = 1 \quad (3.15)$$

Proof. We start by evaluating the Laplace transform $L_\nu(\mu)$ of (3.15) as follows

$$L_\nu(\mu) = \sum_{k=1}^{\infty} \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} \frac{\mu^{\nu-1}}{\mu^\nu + \lambda j} = \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{1}{\frac{\mu^\nu}{\lambda} + 1 + j}. \quad (3.16)$$

A crucial role is here played by the well-known formula (see Kirschenhofer [8])

$$\sum_{k=0}^N \binom{N}{k} (-1)^k \frac{1}{x+k} = \frac{N!}{x(x+1)\cdots(x+N)}. \quad (3.17)$$

Therefore

$$\begin{aligned} L_\nu(\mu) &= \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^{\infty} \frac{(k-1)!}{\left(\frac{\mu^\nu}{\lambda} + 1\right) \left(\frac{\mu^\nu}{\lambda} + 2\right) \cdots \left(\frac{\mu^\nu}{\lambda} + k\right)} = \frac{\mu^{\nu-1}}{\lambda} \sum_{l=0}^{\infty} \frac{\Gamma(l+1) \Gamma\left(\frac{\mu^\nu}{\lambda} + 1\right)}{\Gamma\left(\frac{\mu^\nu}{\lambda} + 1 + (l+1)\right)} \\ &= \frac{\mu^{\nu-1}}{\lambda} \sum_{l=0}^{\infty} B\left(l+1, \frac{\mu^\nu}{\lambda} + 1\right) = \frac{\mu^{\nu-1}}{\lambda} \int_0^1 \sum_{l=0}^{\infty} x^l (1-x)^{\frac{\mu^\nu}{\lambda}} dx \\ &= \frac{\mu^{\nu-1}}{\lambda} \int_0^1 (1-x)^{\frac{\mu^\nu}{\lambda} - 1} dx = \int_0^\infty e^{-\mu t} dt \end{aligned} \quad (3.18)$$

where $B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$, for $p, q > 0$. This concludes the proof of (3.15). \square

The presence of alternating sums in (3.4) imposes the check that $p_k^\nu(t) \geq 0$ for all k . This is the task of the next remark.

Remark 3.2. In order to check the non-negativity of (3.4) we exploit the results of the proof of theorem 3.2 suitably adapted. The expression

$$\sum_{k=1}^{\infty} \int_0^{\infty} e^{-\mu t} p_k^\nu(t) dt = \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^{\infty} B\left(k, \frac{\mu^\nu}{\lambda} + 1\right) \quad (3.19)$$

which emerges from (3.18) permits us to write that

$$\begin{aligned} \int_0^{\infty} e^{-\mu t} p_k^\nu(t) dt &= \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} (1-x)^{\frac{\mu^\nu}{\lambda}} dx = \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} e^{\frac{\mu^\nu}{\lambda} \ln(1-x)} dx \\ &= \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} e^{-\frac{\mu^\nu}{\lambda} \sum_{r=1}^{\infty} \frac{x^r}{r}} dx = \int_0^1 x^{k-1} \frac{\mu^{\nu-1}}{\lambda} e^{-\frac{\mu^\nu x}{\lambda} \prod_{r=2}^{\infty} e^{-\frac{\mu^\nu x^r}{\lambda r}}} dx. \end{aligned} \quad (3.20)$$

The terms

$$e^{-\frac{\mu^\nu x^r}{\lambda r}} = \mathbb{E}e^{-\mu X_r} = \int_0^{\infty} e^{-\mu t} q_\nu^r(x, t) dt \quad (3.21)$$

are the Laplace transforms of stable r.v's $X_r = S(\sigma_r, 1, 0)$ where $\sigma_r = (\frac{x^r}{\lambda r} \cos \frac{\pi\nu}{2})^{\frac{1}{\nu}}$ (for details on this point see Samorodnitsky and Taqqu [14], page 15). The term $\frac{\mu^{\nu-1}}{2\lambda} \exp(-\frac{\mu^\nu x}{\lambda})$ is the Laplace transform of the solution of the fractional diffusion equation

$$\begin{cases} \frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \lambda^2 \frac{\partial^2 u}{\partial x^2}, & 0 < \nu \leq 1 \\ u(x, 0) = \delta(x) \end{cases} \quad (3.22)$$

with the additional condition $u_t(x, 0) = 0$ for $1/2 < \nu \leq 1$, and can be written as

$$u_{2\nu}(x, t) = \frac{1}{2\lambda\Gamma(1-\nu)} \int_0^t \frac{p_\nu(x, s)}{(t-s)^\nu} ds \quad (3.23)$$

(see formula (3.5) Orsingher and Beghin [10]), where $p_\nu(x, 1) = q_\nu^1(x, 1)$ is the stable law with $\sigma_1 = (\frac{x}{\lambda} \cos \frac{\pi\nu}{2})^{\frac{1}{\nu}}$. We can represent the product

$$\frac{\mu^{\nu-1}}{\lambda} e^{-\frac{x\mu^\nu}{\lambda}} \prod_{r=2}^{\infty} e^{-\frac{\mu^\nu x^r}{\lambda r}} = \int_0^{\infty} e^{-\mu t} \left\{ \int_0^t u_{2\nu}(x, s) q_\nu(x, t-s) ds \right\} dt \quad (3.24)$$

where

$$\int_0^{\infty} e^{-\mu t} q_\nu(x, t) dt = \prod_{r=2}^{\infty} e^{-\frac{\mu^\nu x^r}{\lambda r}}. \quad (3.25)$$

Thus $q_\nu(x, t)$ appears as an infinite convolution of stable laws whose parameters depend on r and x . In light of (3.24) we have therefore that

$$\int_0^{\infty} e^{-\mu t} p_k^\nu(t) dt = 2 \int_0^{\infty} e^{-\mu t} \int_0^1 x^{k-1} \int_0^t u_{2\nu}(x, s) q_\nu(x, t-s) ds dx dt \quad (3.26)$$

Since $p_k^\nu(t)$ appears as the result of the integral of probability densities, we can conclude that $p_k^\nu(t) \geq 0$ for all $k \geq 1$ and $t > 0$.

We provide an alternative proof of the non-negativity of $p_k^\nu(t)$, $t > 0$ and of $\sum_k p_k^\nu(t) = 1$ based on the representation of the fractional linear birth process $N_\nu(t)$ as

$$N_\nu(t) = N(T_{2\nu}(t)), \quad 0 < \nu \leq 1 \quad (3.27)$$

where $T_{2\nu}(t)$ possesses distribution coinciding with the folded solution of the fractional diffusion equation

$$\begin{cases} \frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial x^2}, & 0 < \nu \leq 1 \\ u(x, 0) = \delta(x) \end{cases} \quad (3.28)$$

with the further condition $u_t(x, 0) = 0$ for $1/2 < \nu \leq 1$.

Theorem 3.3. *The probability generating function $G_\nu(u, t) = \mathbb{E}u^{N_\nu(t)}$ of $N_\nu(t)$, $t > 0$ has the following Laplace transform*

$$\int_0^\infty e^{-\mu t} G_\nu(u, t) dt = \int_0^\infty \frac{ue^{-\lambda t}}{1-u(1-e^{-\lambda t})} \mu^{\nu-1} e^{-\mu^\nu t} dt. \quad (3.29)$$

Proof. We evaluate the Laplace transform (3.29) as follows

$$\begin{aligned} \int_0^\infty e^{-\mu t} G_\nu(u, t) dt &= \int_0^\infty e^{-\mu t} \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} E_{\nu,1}(-\lambda j t^\nu) dt \\ &= \sum_{k=1}^\infty u^k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} \frac{\mu^{\nu-1}}{\mu^\nu + \lambda j} = \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^\infty u^k \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{1}{\frac{\mu^\nu}{\lambda} + 1 + j} = (\text{by (3.17)}) \\ &= \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^\infty u^k \frac{(k-1)!}{\left(\frac{\mu^\nu}{\lambda} + 1\right) \left(\frac{\mu^\nu}{\lambda} + 2\right) \cdots \left(\frac{\mu^\nu}{\lambda} + k\right)} = \frac{u\mu^{\nu-1}}{\lambda} \sum_{l=0}^\infty u^l \frac{l!}{\left(\frac{\mu^\nu}{\lambda} + 1\right) \cdots \left(\frac{\mu^\nu}{\lambda} + 1 + l\right)} \\ &= \frac{u\mu^{\nu-1}}{\lambda} \sum_{l=0}^\infty u^l \mathbf{B}\left(l+1, \frac{\mu^\nu}{\lambda} + 1\right) = \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \sum_{l=0}^\infty u^l x^l (1-x)^{\frac{\mu^\nu}{\lambda}} dx, \quad \text{for } 0 < ux < 1 \\ &= \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{(1-ux)} dx = (1-x = e^{-t}) = \int_0^\infty \frac{ue^{-\lambda t}}{1-u(1-e^{-\lambda t})} e^{-t\mu^\nu} \mu^{\nu-1} dt. \end{aligned} \quad (3.30)$$

□

Remark 3.3. In order to extract from (3.29) the representation (3.27) we note that

$$\begin{aligned} \int_0^\infty e^{-\mu t} \left\{ \sum_{k=0}^\infty u^k \Pr\{N(T_{2\nu}(t)) = k\} \right\} dt &= \\ = \int_0^\infty e^{-\mu t} \left\{ \int_0^\infty \sum_{k=0}^\infty u^k \Pr\{N(s) = k\} f_{T_{2\nu}}(s, t) ds \right\} dt &= \int_0^\infty G(u, s) \mu^{\nu-1} e^{-\mu^\nu s} ds \end{aligned} \quad (3.31)$$

which coincides with (3.29). It can be shown that

$$\int_0^\infty e^{-\mu t} f_{T_{2\nu}}(s, t) dt = \mu^{\nu-1} e^{-s\mu^\nu}, \quad s > 0 \quad (3.32)$$

is the Laplace transform of the folded solution to

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < \nu \leq 1 \quad (3.33)$$

with the initial condition $u(s, 0) = \delta(s)$ for $0 < \nu \leq 1$ and also $u_t(x, 0) = 0$ for $1/2 < \nu \leq 1$.

In force of (3.27) the non-negativity of $p_k^\nu(t)$ is immediate because

$$\Pr\{N_\nu(t) = k\} = \int_0^\infty \Pr\{N(s) = k\} \Pr\{T_{2\nu}(t) \in ds\}. \quad (3.34)$$

The relationship (3.34) immediately leads to the conclusion that $\sum_{k=1}^\infty \Pr\{N_\nu(t) = k\} = 1$.

Some explicit expressions for (3.34) can be given when the $\Pr\{T_{2\nu}(t) \in ds\}$ can be worked out in detail.

We know that for $\nu = 1/2^n$ we have that

$$\begin{aligned} \Pr\left\{T_{\frac{1}{2^{n-1}}}(t) \in ds\right\} &= \Pr\{|\mathcal{B}_1(|\mathcal{B}_2(\cdots|\mathcal{B}_n(t)|\cdots)|) \in ds\} = \\ &= 2^n \int_0^\infty \frac{e^{-\frac{s^2}{4\omega_1}}}{\sqrt{4\pi\omega_1}} d\omega_1 \int_0^\infty \frac{e^{-\frac{\omega_1^2}{4\omega_2}}}{\sqrt{4\pi\omega_2}} d\omega_2 \cdots \int_0^\infty \frac{e^{-\frac{\omega_{n-1}^2}{4t}}}{\sqrt{4\pi t}} d\omega_{n-1}. \end{aligned} \quad (3.35)$$

For details about (3.35) see theorem 2.2 of Orsingher and Beghin [11], where the differences of the constants depend on the fact that the diffusion coefficient in equation (3.33) equals 1 instead of $2^{(1/2^n)-2}$. The distribution (3.35) represents the density of the folded $(n-1)$ -times iterated Brownian motion and therefore $\mathcal{B}_1, \dots, \mathcal{B}_n$ are independent Brownian motions with volatility equal to 2.

For $\nu = 1/3$ the process (3.27) has the form $N_{\frac{1}{3}}(t) = N(|\mathcal{A}(t)|)$ where $\mathcal{A}(t)$ is a process whose law is the solution of

$$\frac{\partial^{\frac{2}{3}} u}{\partial t^{\frac{2}{3}}} = \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = \delta(x). \quad (3.36)$$

In Orsingher and Beghin [11] it is shown that the solution to (3.36) is

$$u_{\frac{2}{3}}(x, t) = \frac{3}{2} \frac{1}{\sqrt[3]{3t}} \mathcal{A}_i\left(\frac{|x|}{\sqrt[3]{3t}}\right) \quad (3.37)$$

where

$$\mathcal{A}_i(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\alpha x + \frac{\alpha^3}{3}\right) d\alpha \quad (3.38)$$

is the Airy function. Therefore the distribution (3.34) in this case reads

$$p_k^{\frac{1}{3}}(t) = \int_0^\infty e^{-\lambda s} (1 - e^{-\lambda s})^{k-1} \frac{3}{\sqrt[3]{3t}} \mathcal{A}_i\left(\frac{s}{\sqrt[3]{3t}}\right) ds, \quad k \geq 1, t > 0. \quad (3.39)$$

Remark 3.4. From (3.3) it is straightforward to show that the probability generating function $G_\nu(u, t) = \mathbb{E}u^{N_\nu(t)}$ satisfies the partial differential equation

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} G(u, t) = \lambda u(u-1) \frac{\partial}{\partial u} G(u, t), & 0 < \nu \leq 1 \\ G(u, 0) = u \end{cases} \quad (3.40)$$

and thus $\mathbb{E}N_\nu(t) = \left. \frac{\partial G}{\partial u} \right|_{u=1}$ is the solution to

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} \mathbb{E}N_\nu = \lambda \mathbb{E}N_\nu, & 0 < \nu \leq 1 \\ \mathbb{E}N_\nu(0) = 1 \end{cases} \quad (3.41)$$

The solution of (3.41) is

$$\mathbb{E}N_\nu(t) = E_{\nu,1}(\lambda t^\nu), \quad t > 0. \quad (3.42)$$

Clearly, result (3.42) can be also derived by evaluating the following Laplace transform

$$\begin{aligned} \int_0^\infty e^{-\mu t} \mathbb{E}N_\nu(t) dt &= \int_0^\infty e^{-\mu t} \left\{ \sum_{k=1}^\infty k \int_0^\infty \Pr\{N(s) = k\} \Pr\{T_{2\nu}(t) \in ds\} \right\} dt \\ &= \int_0^\infty e^{-\mu t} \int_0^\infty e^{\lambda s} \Pr\{T_{2\nu}(t) \in ds\} dt = \int_0^\infty e^{\lambda s} \mu^{\nu-1} e^{-s\mu^\nu} ds = \frac{\mu^{\nu-1}}{\mu^\nu - \lambda} = \int_0^\infty e^{-\mu t} E_{\nu,1}(\lambda t^\nu) dt \end{aligned}$$

and this confirms (3.42). The mean-value (3.42) can be obtained in a third manner.

$$\begin{aligned} \int_0^\infty e^{-\mu t} \mathbb{E}N_\nu(t) dt &= \sum_{k=1}^\infty k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} \int_0^\infty E_{\nu,1}(-\lambda j t^\nu) e^{-\lambda t} dt \quad (3.43) \\ &= \sum_{k=1}^\infty k \sum_{j=1}^k \binom{k-1}{j-1} (-1)^{j-1} \frac{\mu^{\nu-1}}{\mu^\nu + \lambda j} = \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^\infty k \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{1}{\frac{\mu^\nu}{\lambda} + 1 + j} \\ &= \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^\infty k \frac{(k-1)!}{\left(\frac{\mu^\nu}{\lambda} + 1\right) \cdots \left(\frac{\mu^\nu}{\lambda} + k\right)} = \frac{\mu^{\nu-1}}{\lambda} \sum_{k=1}^\infty k \frac{\Gamma(k) \Gamma\left(\frac{\mu^\nu}{\lambda} + 1\right)}{\Gamma\left(\frac{\mu^\nu}{\lambda} + k + 1\right)} \\ &= \frac{\mu^{\nu-1}}{\lambda} \int_0^1 \sum_{k=1}^\infty k x^{k-1} (1-x)^{\frac{\mu^\nu}{\lambda}} = \frac{\mu^{\nu-1}}{\mu^\nu - \lambda} = \int_0^\infty e^{-\mu t} E_{\nu,1}(\lambda t^\nu) dt \end{aligned}$$

The result of remark 3.4, $\mathbb{E}N_\nu(t) = E_{\nu,1}(\lambda t^\nu)$ should be compared with the results of Uchaikin et al. [16].

An interesting representation of (3.42) following from (3.27) gives that

$$\mathbb{E}N_\nu(t) = \int_0^\infty e^{\lambda s} \Pr\{T_{2\nu}(t) \in ds\} = \int_0^\infty \mathbb{E}N(s) \Pr\{T_{2\nu}(t) \in ds\}. \quad (3.44)$$

The expansion of the population subject to the law of the fractional birth process is increasingly rapid as the order of fractionality ν decreases. This is shown in figure 1 and this

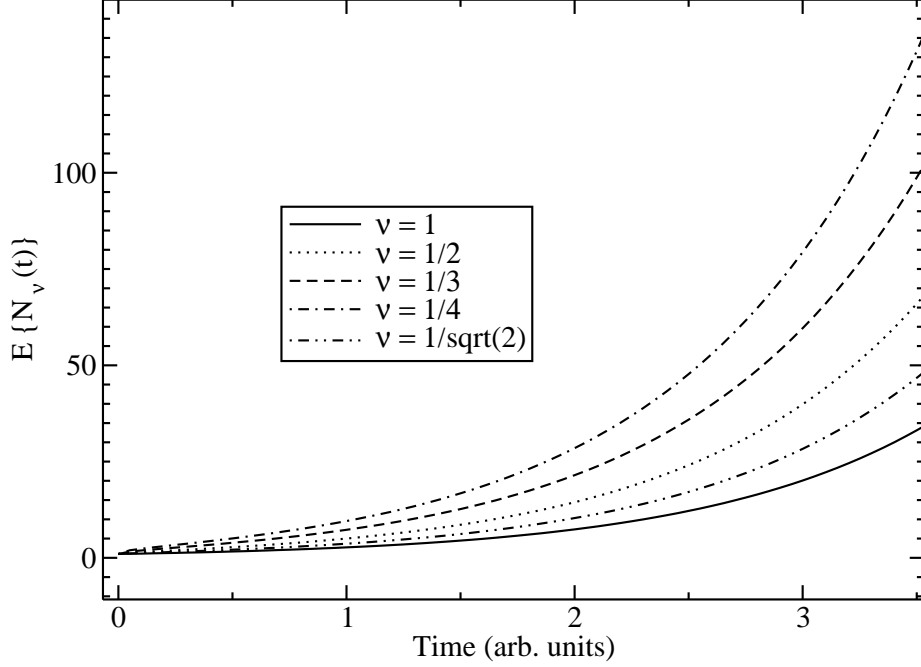


Figure 1: Mean number of individuals at time t for various values of ν .

behaviour is due to the increasing structure of the gamma function for $\nu > 0$ appearing in the Mittag-Leffler function $E_{\nu,1}$. This qualitative feature of the process investigated here shows that it conveniently applies to explosively expanding populations.

Remark 3.5. By twice deriving (3.40) w.r.t. u we obtain the fractional equation for the second-order factorial moment

$$\mathbb{E}\{N_\nu(t)(N_\nu(t) - 1)\} = g_\nu(t) \quad (3.45)$$

that is

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} g_\nu(t) = 2\lambda g_\nu(t) + 2\lambda \mathbb{E}N_\nu(t), & 0 < \nu \leq 1 \\ g_\nu(0) = 0 \end{cases} \quad (3.46)$$

The Laplace transform of the solution to (3.46) is

$$H_\nu(t) = \int_0^\infty e^{-\mu t} g_\nu(t) dt = \frac{2\lambda \mu^{\nu-1}}{(\mu^\nu - \lambda)(\mu^\nu - 2\lambda)} = 2\mu^{\nu-1} \left\{ \frac{1}{\mu^\nu - 2\lambda} - \frac{1}{\mu^\nu - \lambda} \right\}. \quad (3.47)$$

The inverse Laplace transform of (3.47) is

$$\mathbb{E}\{N_\nu(t)(N_\nu(t) - 1)\} = 2E_{\nu,1}(2\lambda t^\nu) - 2E_{\nu,1}(\lambda t^\nu). \quad (3.48)$$

It is now straightforward to obtain the variance from (3.48)

$$\text{Var}N_\nu(t) = 2E_{\nu,1}(2\lambda t^\nu) - E_{\nu,1}(\lambda t^\nu) - E_{\nu,1}^2(\lambda t^\nu). \quad (3.49)$$

For $\nu = 1$ we retrieve from (3.49) the well-known expression of the variance of the linear birth process

$$\text{Var}N_1(t) = e^{\lambda t} (e^{\lambda t} - 1). \quad (3.50)$$

Remark 3.6. If X_1, \dots, X_n are i.i.d. r.v's with common distribution $F(x) = \Pr(X < x)$ then we can write the following probability

$$\begin{aligned} \Pr \left\{ \max \left(X_1, \dots, X_{N_\nu(t)} \right) < x \right\} &= \sum_{k=1}^{\infty} (\Pr \{X < x\})^k \Pr \{N_\nu(t) = k\} = \quad \text{by (3.27)} \quad (3.51) \\ &= \int_0^\infty G(F(x), s) \Pr \{T_{2\nu}(t) \in ds\} = \int_0^\infty \frac{F(x) e^{-\lambda s}}{1 - F(x)(1 - e^{-\lambda s})} \Pr \{T_{2\nu}(t) \in ds\}. \end{aligned}$$

Analogously we have that

$$\Pr \left\{ \min \left(X_1, \dots, X_{N_\nu(t)} \right) > x \right\} = \int_0^\infty \frac{(1 - F(x)) e^{-\lambda s}}{1 - (1 - F(x))(1 - e^{-\lambda s})} \Pr \{T_{2\nu}(t) \in ds\}. \quad (3.52)$$

Remark 3.7. If the initial number of the components of the population is n_0 then the p.g.f. becomes

$$\mathbb{E} \left(u^{N_\nu(t)} \mid N_\nu(0) = n_0 \right) = \sum_{k=0}^{\infty} u^{k+n_0} \int_0^\infty e^{-\lambda z n_0} \binom{n_0 + k - 1}{k} (1 - e^{-\lambda z})^k \Pr \{T_{2\nu}(t) \in dz\}. \quad (3.53)$$

From (3.53) we can extract the distribution of the population size at time t as

$$\begin{aligned} \Pr \{N_\nu(t) = k + n_0 \mid N_\nu(0) = n_0\} &= \quad (3.54) \\ &= \binom{n_0 + k - 1}{k} \int_0^\infty e^{-\lambda z n_0} (1 - e^{-\lambda z})^k \Pr \{T_{2\nu}(t) \in dz\}, \quad k \geq 0. \end{aligned}$$

If we write $k + n_0 = k'$ we can rewrite (3.54) as

$$\begin{aligned} \Pr \{N_\nu(t) = k' \mid N_\nu(0) = n_0\} &= \quad (3.55) \\ &= \binom{k' - 1}{k' - n_0} \int_0^\infty e^{-\lambda z n_0} (1 - e^{-\lambda z})^{k' - n_0} \Pr \{T_{2\nu}(t) \in dz\}, \quad k' \geq n_0 \end{aligned}$$

where k' is the number of individuals in the population at time t . For $n_0 = 1$ formulas (3.54), (3.55) coincide with (3.4). The random time $T_{2\nu}(t)$, $t > 0$ appearing in (3.54) and (3.55) has a distribution which is related to the fractional equation

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \frac{\partial^2 u}{\partial z^2}, \quad 0 < \nu \leq 1. \quad (3.56)$$

It is possible to change a little bit the structure of formulas (3.54), (3.55) by means of the transformation $\lambda z = y$ so that the distribution of $T_{2\nu}(t)$ becomes related to equation

$$\frac{\partial^{2\nu} u}{\partial t^{2\nu}} = \lambda^2 \frac{\partial^2 u}{\partial y^2}, \quad 0 < \nu \leq 1 \quad (3.57)$$

where (3.1) shows the connection between the diffusion coefficient in (3.57) and the birth rate.

Remark 3.8. If we assume that the initial number of individuals in the population is $N_\nu(0) = n_0$ we can generalise the result (3.4) offering a representation of the distribution of $N_\nu(t)$ alternative to (3.55). If we take the Laplace transform of (3.55) we have that

$$\begin{aligned}
& \int_0^\infty e^{-\mu t} \Pr\{N_\nu(t) = k + n_0 \mid N_\nu(0) = n_0\} dt = & (3.58) \\
& = \int_0^\infty \binom{n_0 + k - 1}{k} \int_0^\infty e^{-\lambda z n_0} (1 - e^{-\lambda z})^k \Pr\{T_{2\nu}(t) \in dz\} dz = & \text{by (3.32)} \\
& = \int_0^\infty \binom{n_0 + k - 1}{k} e^{-\lambda z n_0} (1 - e^{-\lambda z})^k \mu^{\nu-1} e^{-\mu^\nu z} dz \\
& = \binom{n_0 + k - 1}{k} \mu^{\nu-1} \int_0^\infty e^{-z(\lambda n_0 + \mu^\nu)} (1 - e^{-\lambda z})^k dz \\
& = \binom{n_0 + k - 1}{k} \mu^{\nu-1} \sum_{r=0}^k \binom{k}{r} (-1)^r \int_0^\infty e^{-z(\lambda n_0 + \lambda r + \mu^\nu)} dz \\
& = \binom{n_0 + k - 1}{k} \mu^\nu \sum_{r=0}^k \binom{k}{r} (-1)^r \frac{1}{\lambda(n_0 + r) + \mu^\nu}.
\end{aligned}$$

By taking the inverse Laplace transform of (3.58) we have that

$$\Pr\{N_\nu(t) = k + n_0 \mid N_\nu(0) = n_0\} = \binom{n_0 + k - 1}{k} \sum_{r=0}^k \binom{k}{r} (-1)^r E_{\nu,1}(-(n_0 + r)\lambda t^\nu). \quad (3.59)$$

From (3.59) we can infer the following interesting information.

$$\begin{aligned}
\Pr\{N_\nu(dt) = n_0 + 1 \mid N_\nu(0) = n_0\} &= n_0 \sum_{r=0}^1 \binom{1}{r} (-1)^r E_{\nu,1}(-(n_0 + r)\lambda(dt)^\nu) & (3.60) \\
&= n_0 [E_{\nu,1}(-n_0\lambda(dt)^\nu) - E_{\nu,1}(-\lambda(n_0 + 1)(dt)^\nu)] \sim n_0 \frac{\lambda(dt)^\nu}{\Gamma(\nu + 1)}
\end{aligned}$$

by writing only the lower order terms. This shows that the probability of a new offspring at the beginning of the process is proportional to $(dt)^\nu$ and to the initial number of progenitors. From our point of view this is the most important qualitative feature of our results, since it makes explicit the dependence on the order ν of the fractional birth process.

Theorem 3.4. *The Laplace transform of the probability generating function $G_\nu(t, u)$ of the fractional linear birth process has the form*

$$H_\nu(\mu, u) = \int_0^\infty e^{-\mu t} G_\nu(t, u) dt = \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx, \quad 0 < u < 1, \mu > 0. \quad (3.61)$$

Proof. We have seen above that the function G solves the Cauchy problem

$$\begin{cases} \frac{\partial^\nu G_\nu}{\partial t^\nu} = \lambda u(u-1) \frac{\partial G_\nu}{\partial u}, & 0 < \nu \leq 1 \\ G_\nu(u, 0) = u \end{cases}. \quad (3.62)$$

By taking the Laplace transform of (3.62) we have that

$$\mu^\nu H_\nu - \mu^{\nu-1}u = \lambda u(u-1) \frac{\partial H_\nu}{\partial u}. \quad (3.63)$$

By inserting (3.61) into (3.63) and performing some integration by parts we have that

$$\begin{aligned} & \frac{u\mu^{2\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx - u\mu^{\nu-1} = \\ & = \lambda u(u-1) \left[\frac{\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx + \frac{u\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}} x}{(1-xu)^2} dx \right] \\ & = \lambda u(u-1) \left[\frac{\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} dx + \frac{\mu^{\nu-1}}{\lambda} \frac{x(1-x)^{\frac{\mu^\nu}{\lambda}}}{1-xu} \Big|_{x=0}^{x=1} \right. \\ & \quad \left. - \frac{\mu^{\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{(1-xu)} dx + \frac{\mu^{2\nu-1}}{\lambda^2} \int_0^1 \frac{x(1-x)^{\frac{\mu^\nu}{\lambda}-1}}{(1-xu)} dx \right] \\ & = \frac{u(u-1)\mu^{2\nu-1}}{\lambda} \int_0^1 \frac{x(1-x)^{\frac{\mu^\nu}{\lambda}-1}}{(1-xu)} dx = -u\mu^{\nu-1} + \frac{u\mu^{2\nu-1}}{\lambda} \int_0^1 \frac{(1-x)^{\frac{\mu^\nu}{\lambda}}}{(1-xu)} dx \end{aligned} \quad (3.64)$$

and this concludes the proof of theorem 3.4. \square

Remark 3.9. We note that $H_\nu(\mu, u)|_{u=1} = 1/\mu$ because $G_\nu(t, 1) = 1$. Furthermore

$$\frac{\partial H_\nu(\mu, u)}{\partial u} \Big|_{u=1} = \frac{\mu^{\nu-1}}{\mu^\nu - \lambda} = \int_0^\infty e^{-\mu t} E_{\nu,1}(\lambda t^\nu) dt \quad (3.65)$$

which accords well with (3.42).

Acknowledgement: The authors acknowledge with pleasure the remarks of an unknown referee which improved the quality of the paper.

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