Uniform error bounds for a continuous approximation of nonnegative random variables.

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Abstract

In this work we deal with approximations for distribution functions of nonnegative random variables. More specifically, we construct continuous approximants using an acceleration technique over a well-know inversion formula for Laplace transforms. We give uniform error bounds using a representation of these approximations in terms of gamma-type operators. We apply our results to certain mixtures of Erlang distributions which contain the class of continuous phase-type distributions.

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1 Introduction

Frequent operations in probability such as convolution or random summation of random variables, produce probability distributions which are difficult to evaluate in an explicit way. In these cases one needs to use numerical evaluation methods. For instance one can use numerical inversion of the Laplace or Fourier transfom of the distribution at hand (see [2] for the general use of Laplace-Stieltjes transforms in applied probability or [9, 11] for the method of Fast Fourier Transform in a context of risk theory). Another approach is the use of recursive evaluation methods, of special interest for random sums (see [11, 18], for instance). Some of the methods above mentioned require a previous discretization step of the initial random variables, when these ones are continuous. The usual way to do so, is by means of rounding methods. However, it is not always possible to evaluate the distribution of the rounded random variable in an explicit way, and it is not always clear by using these methods how the rounding error propagates when one takes successive convolutions. In these cases it seems interesting to consider alternative discretization methods. For instance, when dealing with nonnegative random variables, the following method ([10, p.233]) has been proposed in the literature. Let X be a random variable taking values on $[0,\infty)$ with distribution function F. Denote by $\phi_X(\cdot)$ the Laplace-Stieltjes (L-S) transform of X , that is

$$\phi_X(t) := Ee^{-tX} = \int_{[0,\infty)} e^{-tu} dF(u), \quad t > 0.$$

For each $t > 0$ we define a random variable $X^{\bullet t}$ taking values on $k/t, \quad k \in \mathbb{N}$, and such that

$$P(X^{\bullet t} = k/t) = \frac{(-t)^k}{k!} \phi_X^{(k)}(t), \quad k \in \mathbb{N}, \tag{1}$$

where $\phi_X^{(k)}$ denotes the k-th derivative ($\phi_X^{(0)} \equiv \phi_X$).

Thus, if we denote by L_t^*F the distribution function of $X^{\bullet t}$ we have that,

$$L_t^*F(x) := P(X^{\bullet t} \leq x) = \sum_{k=0}^{[tx]} \frac{(-t)^k}{k!} \phi_X^{(k)}(t), \quad x \geq 0, \tag{2}$$

where $[x]$ indicates the largest integer less than or equal to x. The use of this method allows to obtain the probability mass function in an explicit way in situations in which rounding methods maybe couldn't (see for instance [4] for gamma distributions). Moreover, this method allows an easy representation of L_t^*F in terms of F which makes it possible the study of rates of convergence in the approximation ([4, 5]). In [4] the problem was studied in a general setting, whereas in [5] a detailed analysis was carried out for the case of gamma distributions that is, whose density function is given by

$$f_{a,p}(x) := \frac{a^p x^{p-1} e^{-ax}}{\Gamma(p)}, \quad x > 0. \tag{3}$$

Also, in [16] error bounds for random sums of mixtures of gamma distributions were obtained, uniformly controlled on the parameters of the random summation index. In all these papers, the measure of distance considered was the Kolmogorov (or sup-norm) distance. More specifically, for a given real function f, defined on $[0, \infty)$ we denote by $\|f\|$ the sup-norm, that is

$$\|f\| := \sup_{x \geq 0} |f(x)|.$$

It was shown in [5] that for gamma distributions with shape parameter $p \geq 1$, we have that $\|L_t^*F - F\|$ is of order $1/t$, length of the discretization interval. Note that $\|L_t^*F - F\|$ is the Kolmogorov distance between X and $X^{\bullet t}$, as both are nonnegative random variables.

The aim of this paper is twofold. First of all, we will consider a continuous modification of (2) and give conditions under which this continuous modification has rate of convergence of $1/t^2$ instead of $1/t$ (see Sections 2 and 3). In Section 4 we will consider the case of gamma distributions to see that the error bounds are also uniform on the shape parameter. Finally, in Section 5 we will consider the application of the results in Section 4 to the class of mixtures of Erlang distributions, recently studied in [19]. This class contains many of the distributions used in applied probability (in particular phase-type distributions) and is closed under important operations such as mixtures, convolution or compounding.

2 The approximation procedure

The representation of $L_t^* F$ in (2) in terms of a Gamma process (cf. [4]) will play an important role in our proofs. We recall this representation. Let $(S(u)\ , u \geq 0)$ be a gamma process, in which $S(0) = 0$ and for $u > 0$, each $S(u)$ has a gamma density with parameters $a = 1$ and $p = u$, as given in (3) . Let g be a function defined on $[0, \infty)$. We consider the gamma-type operator L_t given by

$$L_t g(x) := Eg\left(\frac{S(tx)}{t}\right), \quad x \geq 0, \quad t > 0, \tag{4}$$

provided that this operator is well defined, that is, $L_t|g|(x) < \infty,\ x \geq 0,\ t > 0$. Then, for F continuous on $(0, \infty)$, $L_t^* F$ in (2), can be written as (cf. [4, p.228])

$$L_t^* F(x) = L_t F\left(\frac{[tx] + 1}{t}\right) = EF\left(\frac{S([tx] + 1)}{t}\right) \quad x \geq 0, \quad t > 0. \tag{5}$$

It can be seen that the rates of convergence of $L_t g$ to g are, at most, of order $1/t$ (observe (40) below). Our aim now is to get faster rates of convergence. To this end, we will consider the following operator, built using a classical acceleration technique (Richardson's extrapolation, see [9, 11], for instance)

$$L_t^{[2]} g(x) := 2L_{2t} g(x) - L_t g(x) = 2Eg\left(\frac{S(2tx)}{2t}\right) - Eg\left(\frac{S(tx)}{t}\right), \quad x \geq 0. \tag{6}$$

We will obtain a rate of uniform convergence from $L_t^{[2]} g$ to g of order $1/t^2$, on the following class of functions functions

$$\mathcal{D} := \{g \in C^4([0, \infty)) : \quad \|x^2 g^{iv}(x)\| < \infty\}. \tag{7}$$

The problem with $L_t^{[2]}g$ is that when tx is not a natural number, $L_t g(x)$ is given in terms of Weyl fractional derivatives of the Laplace transform (cf. [6, p. 92]) and, in general, we are not able to compute them in an explicit way. However, if we modify $L_t^{[2]}g$ using linear interpolation, that is

$$M_t^{[2]}g(x) := (tx-[tx])\left(L_t^{[2]}g\left(\frac{[tx]+1}{t}\right)\right)+([tx]+1-tx)\left(L_t^{[2]}g\left(\frac{[tx]}{t}\right)\right) \quad (8)$$

we observe that the order of convergence of $M_t^{[2]}g$ to g is also $1/t^2$, on the following class of functions

$$\mathcal{D}_1 := \{g \in C^4([0,\infty)) : \quad \|g''(x)\| \leq \infty \text{ and } \|x^2 g^{iv}(x)\| < \infty\}. \quad (9)$$

Moreover, the advantage of using $M_t^{[2]}g$ instead of $L_t^{[2]}g$ to approximate g is the computability. In the following result we note that the last approximation applied to a distribution function F, is related to $L_t^* F$, as defined in (2). From now on, $\mathbb{N}^*$ will denote the set $\mathbb{N} \setminus \{0\}$.

Proposition 2.1 *Let X be a nonnegative random variable with Laplace transform ϕ_X. Let $L_t^* F$, $t > 0$ be as defined in* (2)*, and let $M_t^{[2]}F$ be as defined in* (8)*. We have*

$$M_t^{[2]}F\left(\frac{k}{t}\right) = \begin{cases} F(0), & \text{if } k=0; \\ 2L_{2t}^*F\left(\frac{2k-1}{2t}\right) - L_t^*F\left(\frac{k-1}{t}\right), & \text{if } k \in \mathbb{N}^* \end{cases} \quad (10)$$

and

$$M_t^{[2]}F(x) = (tx-[tx])M_t^{[2]}F\left(\frac{[tx]+1}{t}\right) + ([tx]+1-tx)M_t^{[2]}F\left(\frac{[tx]}{t}\right). \quad (11)$$

Proof. Let $t > 0$ be fixed. First, observe that by (8), we can write

$$M_t^{[2]}F\left(\frac{k}{t}\right) = L_t^{[2]}F\left(\frac{k}{t}\right), \quad k \in \mathbb{N}. \quad (12)$$

Now, using (6) and (4), we have $M_t^{[2]}F(0) = L_t^{[2]}F(0) = F(0)$, which shows (10) for $k = 0$. Finally, using (6), (4) and (5), we have for $k \in \mathbb{N}^*$

$$L_t^{[2]}F\left(\frac{k}{t}\right) = 2EF\left(\frac{S(2k)}{2t}\right) - EF\left(\frac{S(k)}{t}\right) = 2L_{2t}^*F\left(\frac{2k-1}{2t}\right) - L_t^*F\left(\frac{k-1}{t}\right). \quad (13)$$

Thus, (12) and (13) show (10) for $k \in \mathbb{N}^*$. Note that (11) is obvious by (8) and (12). This completes the proof of Proposition 2.1. $\square$

In the following example we illustrate the use of the previous approximant in a context of random sums, defined in the following way. Let $(X_i)_{i\in\mathbb{N}^*}$ be a sequence of independent, identically distributed nonnegative random variables. Let M be a random variable concentrated on the nonnegative integers, independent of $(X_i)_{i\in\mathbb{N}^*}$. Consider the random variable

$$\sum_{i=1}^{M} X_i, \quad (14)$$

with the convention that the empty sum is 0.

Example 2.1 As pointed out in the Introduction, an explicit expression for the distribution of (14) is usually not possible. Our aim is to consider an example in which this distribution can be evaluated explicitely and to compare our approximation method with some others considered in the literature. To this end we consider that M follows a geometric distribution of parameter p, that is $P(M = k) = (1-p)^k p,\ k \in \mathbb{N}$ and $(X_i)_{i\in\mathbb{N}^*}$ are exponentially distributed (with mean 1, for the sake of simplicity). In this case, it is well-known (use L-S transforms, for instance) that (14) has the same distribution as mixture of the degenerate distribution at 0 (with probability p) and an exponential distribution, that is

$$F(x) := P(\sum_{i=1}^{M} X_i \le x) = p + (1-p)(1-e^{-px}) = 1-(1-p)e^{-px}, \quad x \ge 0. \quad (15)$$

When an explicit expression is not possible a usual approximate evaluation method is by discretizing the summands in (14) and then using recursive methods existing in the literature for discrete random sums. By considering (1) as a first discretization method, we have (cf. [5, p.391])

$$P\left(X_1^{\bullet t}=\frac{k}{t}\right)=\left(\frac{t}{t+1}\right)^k\frac{1}{t+1},\quad k=0,1,\ldots,\tag{16}$$

Thus, $t\sum_{i=1}^M X_i^{\bullet t}$ is a geometric sum of geometric distributions with parameter $r=(1+t)^{-1}$. It is easy to check (use L-S transforms for instance), that the distribution of such random variable is a mixture of the degenerate distribution at 0 (with probability p) and a geometric distribution with parameter $p^*=1-(1-r)(1-(1-p)r)^{-1}=1-t(t+p)^{-1}$, so that for each $k\in\mathbb{N}$,

$$L_t^*F\left(\frac{k}{t}\right)=P(\sum_{i=1}^M X_i^{\bullet t}\le\frac{k}{t})=p+(1-p)(1-(1-p^*)^{k+1})=1-(1-p)\left(\frac{t}{t+p}\right)^{k+1}.\tag{17}$$

Note that the first equality in (17) follows recalling (2) and noting that $\left(\sum_{i=1}^M X_i\right)^{\bullet t}$ has the same distribution as $\sum_{i=1}^M X_i^{\bullet t}$ (cf. [16, Prop.2.1.]). Actually, a more natural way (in this case) to compute (17) is to evaluate the L-S transform of $\left(\sum_{i=1}^M X_i\right)^{\bullet t}$ and then apply (1) and (2). But the previous computations ease comparisons with the following method. In fact, one of the most obvious (and used) method to discretize the summands in (14) is by a rounding method. For instance, a rounding down method (we round X_i to $[tX_i]t^{-1}$) yields

$$P\left(\frac{[tX_1]}{t}=\frac{k}{t}\right)=P\left(\frac{k}{t}\le X_1<\frac{k+1}{t}\right)=e^{-k/t}(1-e^{-1/t}),\ k\in\mathbb{N}.\tag{18}$$

In this case, $\sum_{i=1}^M[tX_i]$ is a geometric sum of geometric distributions with parameter $r'=1-e^{-1/t}$. We denote by R_tF the distribution function of $\sum_{i=1}^M\frac{[tX_i]}{t}$. Using the same arguments that in (17), we obtain for each $k\in\mathbb{N}$

$$R_tF\left(\frac{k}{t}\right)=P(\sum_{i=1}^M\frac{[tX_i]}{t}\le\frac{k}{t})=1-(1-p)\left(\frac{e^{-1/t}}{1-(1-p)(1-e^{-1/t})}\right)^{k+1}.\tag{19}$$

Finally, it would be interesting to compare the previous 'discretization methods' with a 'transform method'. To this end, we consider the Laplace transform of F in (15) (instead of its L-S transform), that is

$$w_F(\theta)=\int_0^\infty e^{-\theta u}F(u)du=\frac{1}{\theta}-\frac{1-p}{\theta+p},\quad\theta>0.$$

and apply Post-Widder inversion formula (see [10, p.233]), defined for $t \in \mathbb{N}^*$ as

$$W_t F(x) = \frac{(-1)^{t-1}}{(t-1)!} \left(\frac{t}{x}\right)^t w_F^{(t-1)} \left(\frac{t}{x}\right) = 1 - \frac{(1-p)t^t}{(px+t)^t}, \quad x \geq 0.$$

In Table 1 (computations with MATLAB) we consider a 'rough' discretization interval (t=5) , a small p ($p = 0.1$) and present, for different $x = k/5$, the exact values of F (column 2), the L_t^* approximation (column 3), the 'rounding down' discretization (column 4) and the Post-Widder inversion (column 5).

$x = \frac{k}{5}$	$F\left(\frac{k}{5}\right)$	$L_5^* F\left(\frac{k}{5}\right)$	$R_5 F\left(\frac{k}{5}\right)$	$W_5 F\left(\frac{k}{5}\right)$
$0=\frac{0}{5}$	0.1000	0.1176	0.1195	0.1000
$1=\frac{5}{5}$	0.1856	0.2008	0.2108	0.1848
$5=\frac{25}{5}$	0.4541	0.4622	0.4907	0.4412
$10=\frac{50}{5}$	0.6689	0.6722	0.7054	0.6383
$15=\frac{75}{5}$	0.7992	0.8002	0.8296	0.7576
$20=\frac{100}{5}$	0.8782	0.8782	0.9014	0.8327
$30=\frac{150}{5}$	0.9552	0.9548	0.9670	0.9142
$40=\frac{200}{5}$	0.9835	0.9832	0.9890	0.9524

Table 1: Comparison of different approximation methods for (15)

As we can see in Table 1, $L_5^* F$ provides better approximation than $R_5 F$. The intuitive explanation to this fact is that, when approximating $\sum_{i=1}^M X_i$ by $\sum_{i=1}^M X_i^{\bullet t}$, the error in the approximation can be controlled 'uniformly', no matter the distribution of M (see [16, Th. 4.3]). This effect is obvious when we choose M with a big expected value (our choice of a small p is for this reason, for bigger values of p checked, $L_5^* F$ is also better, but the difference is less appreciable). However, if we compare the approximations $L_5^* F$ and $W_5 F$ we see that the last one is better for small values, whereas the first one is better for big values. To explain this fact it is interesting to point out that $W_t F$, as $L_t^* F$, admits the following well-known representation. For a function g defined on $[0, \infty)$ we can write as in (5) (cf. [10, pp. 220,223])

$$W_t g(x) = Eg\left(x \frac{S(t)}{t}\right), \quad x > 0. \tag{20}$$

Note that the mean of the 'random points' defining W_t in (20) is $E(xt^{-1}S(t)) = x$, whereas for L_t^* in (5), we have $E(t^{-1}S([tx]+1)) = t^{-1}([tx]+1)$. This means that W_t is centered at x, whereas L_t^* is 'biased'. The benefits of this property for W_t are observed at small values in Table 1. However, we have $Var(xt^{-1}S(t)) = t^{-1}x^2$, whereas $Var(t^{-1}S([tx]+1)) = t^{-2}([tx]+1)$ this last one of order $t^{-1}x$, as $t \to \infty$. The 'more variability' of the random variables defining W_t for big values of x produces an undesired effect in the approximation.

Now, we show the improvement in the approximation with the use of $M_t^{[2]}$, as defined in (10), instead of L_t^*. In table 2 below (t=5) we make a comparison of $M_t^{[2]}F$ (column 5) with Richardson extrapolation for W_tF (or Stehfest enhancement of order two for the Post-Widder formula, see [1, p.40]), that is,

$$G_t^{[2]}F(x) := 2W_{2t}F(x) - W_tF(x), \quad x > 0. \tag{21}$$

As we can see, $M_5^{[2]}F$ provides us an exact value up to a 4 decimal places, whereas $G_5^{[2]}F$ doesn't get this accuracy.

$x = \frac{k}{5}$	$F\left(\frac{k}{5}\right)$	$L_5^*F\left(\frac{k-1}{5}\right)$	$L_{10}^*F\left(\frac{2k-1}{10}\right)$	$M_5^{[2]}F\left(\frac{k}{5}\right)$	$G_5^{[2]}F\left(\frac{k}{5}\right)$
$1=\frac{5}{5}$	0.1856	0.1848	0.1852	0.1856	0.1856
$5=\frac{25}{5}$	0.4541	0.4514	0.4528	0.4541	0.4538
$10=\frac{50}{5}$	0.6689	0.6656	0.6673	0.6689	0.6677
$15=\frac{75}{5}$	0.7992	0.7962	0.7977	0.7992	0.7975
$20=\frac{100}{5}$	0.8782	0.8758	0.8770	0.8782	0.8766
$30=\frac{150}{5}$	0.9552	0.9538	0.9545	0.9552	0.9553
$40=\frac{200}{5}$	0.9835	0.9829	0.9832	0.9835	0.9854

Table 2: Comparison of $M_5^{[2]}$ in (10) with $G_5^{[2]}$ in (21)

3 Error bounds for the approximation

Let $g \in \mathcal{D}$, as defined in (7). Our first aim is to give bounds of $\|L_t^{[2]}g - g\|$ in terms of $\|x^2 g^{iv}(x)\|$. To this end we will use as 'test function' the following one

$$\phi(x) = \begin{cases} 0, & \text{if } x = 0; \\ \dfrac{x^2}{2}\left(\dfrac{3}{2} - \log(x)\right), & \text{otherwise.} \end{cases} \tag{22}$$

Observe that $\phi \in \mathcal{D}$. In fact, by elementary calculus

$$\phi^{'}(x) = x(1 - \log x); \quad \phi^{''}(x) = -\log x; \quad \phi^{'''}(x) = -\frac{1}{x} \quad \text{and} \quad \phi^{iv}(x) = \frac{1}{x^2}. \tag{23}$$

In the next Lemma, we make an explicit computation of $L_t\phi(x)$, in terms of the Ψ (or digamma) function. This function is defined as (cf. [3, p.258])

$$\Psi(x) := \frac{d}{dx}\log(\Gamma(x)) = \frac{1}{\Gamma(x)}\int_0^\infty \log u \; e^{-u}u^{x-1}du, \quad x > 0 \tag{24}$$

and therefore, using the last equality we have the following probabilistic expression of the psi function in terms of the gamma process:

$$\Psi(x) = ElogS(x), \quad x > 0. \tag{25}$$

We will use the following property of this function (cf. [3, p.258]),

$$\Psi(x+1) = \frac{1}{x} + \Psi(x). \tag{26}$$

Lemma 3.1 *Let ϕ be as defined in* (22)*, and let $L_t, \; t > 0$ be as defined in* (4)*. We have that*

$$L_t\phi(x) = \frac{1}{2t^2}\left(\frac{3(tx)^2}{2} - \frac{tx}{2} - 1 + tx(tx+1)(-\Psi(tx) + \log(t))\right), \; x > 0. \tag{27}$$

Proof. Let $t > 0$ and $x > 0$ be fixed. First of all, using elementary calculus, (4) and (26), we can write

$$\begin{aligned} L_t\phi(x) &= E\frac{S(tx)^2}{2t^2}\left(\frac{3}{2} - \log\left(\frac{S(tx)}{t}\right)\right) \\ &= \frac{1}{2t^2}\frac{1}{\Gamma(tx)}\int_0^\infty u^2\left(\frac{3}{2} - \log\left(\frac{u}{t}\right)\right)e^{-u}u^{tx-1}du \\ &= \frac{(tx)(tx+1)}{2t^2}\frac{1}{\Gamma(tx+2)}\int_0^\infty \left(\frac{3}{2} - \log\left(\frac{u}{t}\right)\right)e^{-u}u^{tx+1}du \\ &= \frac{(tx)(tx+1)}{2t^2}\left(\frac{3}{2} - E\log\left(\frac{S(tx+2)}{t}\right)\right). \end{aligned} \tag{28}$$

Therefore, using (25), we can write

$$L_t\phi(x) = \frac{(tx)(tx+1)}{2t^2}\left(\frac{3}{2} - \Psi(tx+2) + \log(t)\right). \tag{29}$$

Now, using twice (26), we have

$$\Psi(tx+2) = \frac{2(tx)+1}{tx(tx+1)} + \Psi(tx). \tag{30}$$

By (29), (30) we obtain

$$L_t\phi(x) = \frac{(tx)(tx+1)}{2t^2}\left(\frac{3}{2} - \frac{2(tx)+1}{tx(tx+1)} - \Psi(tx) + \log(t)\right).$$

The result follows using elementary algebra in the expression above. $\square$

In the next Lemma we will study the approximation properties of $L_t\phi$ to ϕ. We will make use of the following inequalities for the psi function.

$$\frac{1}{2x} \leq \log(x) - \Psi(x) \leq \frac{1}{x}, \quad x > 0, \quad \text{and} \tag{31}$$

$$\log(x) - \Psi(x) - \frac{1}{2x} \leq \frac{1}{12x^2}, \qquad x > 0. \tag{32}$$

We can find (31) in [7, p. 374], whereas (32) is an immediate consequence of the fact that the function

$$\Psi(x) - \log(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is completely monotonic (cf. [15, p.304]) and thus, nonnegative.

Lemma 3.2 *Let* ϕ *be as defined in* (22), *and let* $L_t,\ t > 0$ *be as defined in* (4). *We have*

$$\|L_t\phi(x) - \phi(x) + \frac{x\log x}{2t} + \frac{1}{3t^2}\| \leq \frac{3}{8t^2}. \tag{33}$$

Proof. Let $x > 0$ and $t > 0$ be fixed. First of all, we can write

$$\phi(x) = \frac{1}{2t^2}\left(\frac{3(tx)^2}{2} - (tx)^2\log(tx) + (tx)^2\log(t)\right). \tag{34}$$

On the other hand,

$$\frac{x\log x}{2t}+\frac{1}{3t^2}=\frac{1}{2t^2}\left((tx)\log tx-(tx)\log t+\frac{2}{3}\right). \tag{35}$$

Therefore, using Lemma 3.1, (34) and (35) we can write

$$\begin{aligned}
&L_t\phi(x)-\phi(x)+\frac{x\log x}{2t}+\frac{1}{3t^2}\\
&=\frac{1}{2t^2}\left(-\frac{tx}{2}-1-(tx)^2\Psi(tx)-(tx)\Psi(tx)+(tx)^2\log(tx)+(tx)\log(tx)+\frac{2}{3}\right)\\
&=\frac{1}{2t^2}\left((tx)^2\left(\log(tx)-\Psi(tx)-\frac{1}{2(tx)}\right)+tx(\log(tx)-\Psi(tx))-\frac{1}{3}\right). \qquad (36)
\end{aligned}$$

By (31) we have that $1/2\le x(\log(x)-\Psi(x))\le 1,\ x>0$, and thus

$$\frac{1}{6}\le tx(\log(tx)-\Psi(tx))-\frac{1}{3}\le\frac{2}{3}. \tag{37}$$

Thus, using (36), (37) and (32), we obtain (33). $\square$

We are in a position to enunciate the following.

Theorem 3.1 *Let $g\in\mathcal{D}$, as defined in* (7) *and let $L_t^{[2]}$, $t>0$ be as defined in* (6). *We have*

$$|L_t^{[2]}g(x)-g(x)|\le\frac{1}{6t^2}\|xg'''(x)\|+\frac{9}{16t^2}\|x^2g^{iv}(x)\|.$$

Proof. We will see firstly that $g\in\mathcal{D}$ implies that

$$\|xg'''(x)\|\le\|x^2g^{iv}(x)\|<\infty. \tag{38}$$

To begin with, the fact that $\|x^2g^{iv}(x)\|<\infty$ implies that $\lim_{x\to\infty}x^{1+\alpha}g^{iv}(x)=0$, for all $0<\alpha<1$. By L'Hopital's rule, we have also that $\lim_{x\to\infty}x^{\alpha}g'''(x)=0$, thus concluding that $\lim_{x\to\infty}g'''(x)=0$. Using this fact, we can write

$$g'''(x)=-\int_x^\infty g^{iv}(u)du$$

which implies easily (38), as

$$|xg'''(x)| \leq x \int_x^\infty \frac{|u^2 g^{iv}(u)|}{u^2} du \leq \|x^2 g^{iv}(x)\|,$$

Now, let $t > 0$ and let L_t be as in (4). As a previous step, we will prove that

$$|L_t g(x) - g(x) - \frac{xg''(x)}{2t} - \frac{xg'''(x)}{3t^2}| \leq \frac{3}{8t^2} \|x^2 g^{iv}(x)\|, \quad x > 0. \tag{39}$$

To this end, let $x > 0$. Using a Taylor's series expansion of the random point $u = S(tx)/t$ around x, and taking into account that $E(S(x)-x) = 0$, $E(S(x)-x)^2 = x$ and $E(S(x)-x)^3 = 2x$, we can write

$$\begin{aligned} L_t g(x) - g(x) &= Eg\left(\frac{S(tx)}{t}\right) - g(x) \\ &= \frac{E(S(tx)-tx)^2}{2t^2} g''(x) + \frac{E(S(tx)-tx)^3}{6t^3} g'''(x) + \frac{1}{6} E \int_x^{\frac{S(tx)}{t}} g^{iv}(\theta)(\frac{S(tx)}{t} - \theta)^3 d\theta \\ &= \frac{xg''(x)}{2t} + \frac{xg'''(x)}{3t^2} + \frac{1}{6} E \int_x^{\frac{S(tx)}{t}} g^{iv}(\theta)(\frac{S(tx)}{t} - \theta)^3 d\theta. \end{aligned} \tag{40}$$

Then, using (40) we get the bound

$$\begin{aligned} &\left| L_t g(x) - g(x) - \frac{xg''(x)}{2t} - \frac{xg'''(x)}{3t^2} \right| = \frac{1}{6} \left| E \int_x^{\frac{S(tx)}{t}} g^{iv}(\theta) \left(\frac{S(tx)}{t} - \theta \right)^3 d\theta \right| \\ &\leq \frac{\|x^2 g^{iv}(x)\|}{6} E \int_{\min(x, \frac{S(tx)}{t})}^{\max(x, \frac{S(tx)}{t})} \left| \frac{S(tx)}{t} - \theta \right|^3 \frac{1}{\theta^2} d\theta \\ &= \frac{\|x^2 g^{iv}(x)\|}{6} E \int_x^{\frac{S(tx)}{t}} \left(\frac{S(tx)}{t} - \theta \right)^3 \frac{1}{\theta^2} d\theta. \end{aligned} \tag{41}$$

Let $\phi(\cdot)$ be as in (22). Using (40) and (23) we have

$$L_t \phi(x) - \phi(x) + \frac{x \log x}{2t} + \frac{1}{3t^2} = \frac{1}{6} E \int_x^{\frac{S(tx)}{t}} \left(\frac{S(tx)}{t} - \theta \right)^3 \frac{1}{\theta^2} d\theta. \tag{42}$$

Then, by (41) and (42) we can write

$$|L_t g(x) - g(x) - \frac{xg''(x)}{2t} - \frac{xg'''(x)}{3t^2}| \leq \|x^2 g^{iv}(x)\| \|L_t \phi(x) - \phi(x) + \frac{x \log x}{2t} + \frac{1}{3t^2}\|.$$

Thus, (39) follows applying Lemma 3.2.

Observe that in (39), the only term of order $1/t$ is the one accompanying to the second derivative. By means of the operator $L_t^{[2]}$, as defined in (6), this term is eliminated. In fact, using (39) we have

$$\begin{aligned} L_t^{[2]} g(x) - g(x) &= 2\left(L_{2t} g(x) - g(x)\right) - \left(L_t g(x) - g(x)\right) \\ &= 2\left(L_{2t} g(x) - g(x) - \frac{x}{4t} g''(x) - \frac{x}{12t^2} g'''(x)\right) \\ &- \left(L_t g(x) - g(x) - \frac{x}{2t} g''(x) - \frac{x}{3t^2} g'''(x)\right) - \frac{x}{6t^2} g'''(x) \\ &\leq \frac{1}{6t^2} \|x g'''(x)\| + \frac{9}{16t^2} \|x^2 g^{iv}(x)\|. \end{aligned} \tag{43}$$

This completes the proof of Theorem 3.1. $\square$

Finally, in the next result we study the approximation properties of $M_t^{[2]}$.

Theorem 3.2 *Let $g \in \mathcal{D}_1$, as defined in* (9) *and let $M_t^{[2]}$, $t > 0$ be as defined in* (8)*. We have*

$$\|M_t^{[2]} g - g\| \leq \frac{1}{8t^2} \|g''(x)\| + \frac{1}{6t^2} \|x g'''(x)\| + \frac{9}{16t^2} \|x^2 g^{iv}(x)\|.$$

Proof. Note firstly that $g \in \mathcal{D}_1$ implies that $\|x g'''(x)\| < \infty$, thanks to (38). Now let $t > 0$ and $x > 0$ be fixed. We write,

$$\begin{aligned} M_t^{[2]} g(x) - g(x) &= (tx - [tx]) \left(L_t^{[2]} g\left(\frac{[tx]+1}{t}\right) - g\left(\frac{[tx]+1}{t}\right)\right) \\ &+ ([tx] + 1 - tx) \left(L_t^{[2]} g\left(\frac{[tx]}{t}\right) - g\left(\frac{[tx]}{t}\right)\right) \\ &+ (tx - [tx]) \left(g\left(\frac{[tx]+1}{t}\right) - g(x)\right) + ([tx] + 1 - tx) \left(g\left(\frac{[tx]}{t}\right) - g(x)\right). \end{aligned} \tag{44}$$

Using the usual expansion

$$|g(y) - g(x) - (y - x) g'(x)| \leq \frac{(y-x)^2}{2} \|g''\| \tag{45}$$

and taking into account that

$$\begin{aligned}
&(tx-[tx])\left(g\left(\frac{[tx]+1}{t}\right)-g(x)\right)+([tx]+1-tx)\left(g\left(\frac{[tx]}{t}\right)-g(x)\right)\\
&\qquad=(tx-[tx])\left(g\left(\frac{[tx]+1}{t}\right)-g(x)-\frac{[tx]+1-tx}{t}g'(x)\right)\\
&\qquad+([tx]+1-tx)\left(g\left(\frac{[tx]}{t}\right)-g(x)-\frac{[tx]-tx}{t}g'(x)\right), \qquad (46)
\end{aligned}$$

we obtain from the above expression and (45)

$$\begin{aligned}
&\left|(tx-[tx])\left(g\left(\frac{[tx]+1}{t}\right)-g(x)\right)+([tx]+1-tx)\left(g\left(\frac{[tx]}{t}\right)-g(x)\right)\right|\\
&\leq\left((tx-[tx])\frac{([tx]+1-tx)^2}{2t^2}+([tx]+1-tx)\frac{([tx]-tx)^2}{2t^2}\right)\|g''\|\\
&=\frac{(tx-[tx])([tx]+1-tx)}{2t^2}\|g''\|\leq\frac{1}{8t^2}\|g''\|, \qquad (47)
\end{aligned}$$

the last inequality as as for each $k\in\mathbb{N}$, the supremum of $(u-k)(k+1-u),\ k\leq u\leq k+1$ is attained at $u=k+1/2$. On the other hand, taking into account Theorem 3.1 we have

$$\begin{aligned}
&\left|(tx-[tx])\left(L_t^{[2]}g\left(\frac{[tx]+1}{t}\right)-g\left(\frac{[tx]+1}{t}\right)\right)\right.\\
&\qquad\left.+([tx]+1-tx)\left(L_t^{[2]}g\left(\frac{[tx]}{t}\right)-g\left(\frac{[tx]}{t}\right)\right)\right|\\
&\leq\|L_t^{[2]}g-g\|\leq\frac{1}{6t^2}\|xg'''(x)\|+\frac{9}{16t^2}\|x^2g^{iv}(x)\|. \qquad (48)
\end{aligned}$$

The result follows by (44), (47) and (48). □

4 Application to gamma distributions

In this Section we will study the case of gamma distributions, that is, with density function as given in (3). It is not hard to see that these distributions are in the class $\mathcal{D}_1$, for a shape parameter $p=1$ or $p\geq 2$, and therefore, we are a position of apply Theorem 3.2. The aim of this Section is to show that in fact, the bounds in this Theorem can uniformly bounded on the shape parameter, which will be an advantage when dealing with mixtures of these distributions. From now on, we denote by

$$f_p(x) := \begin{cases} \dfrac{e^{-x}x^{p-1}}{\Gamma(p)}, & x > 0, \quad \text{if } p \in \mathbb{R} \setminus \{0, -1, -2, \dots\}; \\ 0, & x > 0, \quad \text{if } p \in \{0, -1, -2, \dots\}, \end{cases} \tag{49}$$

The 'odd' definition of f_p for $p \in \{0, -1, -2, \dots\}$ is for notational convenience in (51). For $p > 0$ the function above is the density of a gamma random variable as in (3) with scale parameter $a = 1$. Results for another scale parameter will follow by a change of scale (see Proposition 5.2 below). First of all we will consider the case $p = 1$, that is an exponential random variable. As the distribution function of this random variable has no computational problems, it makes no sense to approximate it. However, when we consider the problem of approximating a general mixture of Gamma distributions, the exponential distribution could be a component.

Lemma 4.3 *Let $F(x) = 1 - e^{-x}$, $x \geq 0$. For $t > 0$, let $M_t^{[2]}F$ be as defined in* (8)*. We have that*

$$\|M_t^{[2]}F - F\| \leq \left(\frac{1}{8} + \frac{1}{6e} + \frac{9}{4e^2}\right)\frac{1}{t^2}$$

Proof. First of all, note that $|F^{(k)}(x)| = e^{-x}$, and that $\sup_{x\geq 0} x^k e^{-x} = k^k e^{-k}$, $k = 1, 2, \dots$. Thus, we have

$$\|F''\| = 1, \quad \|xF'''(x)\| = e^{-1} \quad \text{and} \quad \|x^2 F^{iv}(x)\| = 2^2 e^{-2} \tag{50}$$

The conclusion follows taking into account Theorem 3.2. $\square$

Now we will deal with the case $p \geq 2$ in (49). The two following Lemmas will be useful in order to bound the derivatives of this density. For the sake of brevity, they are stated without proof (only elementary calculus is required). For the proofs, we refer the interested reader to [17], a preliminary version of this paper (available online).

Lemma 4.4 *Let $f_p(\cdot)$, $p > 0$ be as defined in (49). We have for all $n \in \mathbb{N}$*

$$\begin{aligned} \frac{d^n}{dx^n} f_p(x) &= \frac{e^{-x} x^{p-n-1}}{\Gamma(p)} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(\prod_{j=1}^{n-i} (p-j) \right) x^i \\ &= \sum_{i=0}^{n} \binom{n}{i} (-1)^i f_{p-n+i}(x), \quad x > 0, \end{aligned} \tag{51}$$

in which $\prod_{j=1}^{0} (p-j) = 1$.

Next, we formulate a technical lemma in which we define certain decreasing functions, which will be used to bound the weighted derivatives of f_p.

Lemma 4.5 *We have*

(i) The function

$$g_1(p) := \frac{1}{\Gamma(p)} e^{-(p-1)} (p-1)^{p-1}, \quad p > 1, \quad (g_1(1) = 1), \tag{52}$$

is decreasing in p.

(ii) The function

$$g_2(p) := \frac{1}{\Gamma(p)} e^{-\left(p - \frac{1}{2} + \frac{1}{2}\sqrt{4p-3}\right)} \left(p - \frac{1}{2} + \frac{1}{2}\sqrt{4p-3} \right)^{p-1/2}, \quad p \geq 1 \tag{53}$$

is decreasing in p.

(iii) The function

$$g_3(p) := \frac{1}{\Gamma(p)} e^{-(p-1-\sqrt{p-1})} (\sqrt{p-1} - 1)^{p-2} (\sqrt{p-1})^{p-1}, \quad p > 2, \tag{54}$$

$(g_3(2) = 1)$, is decreasing in p.

(iv) The function

$$g_4(p) := \frac{1}{\Gamma(p)} e^{-(p-\sqrt{3p-2})} (p - \sqrt{3p-2})^{p-2} (\sqrt{3p-2} - 1)^3, \quad p > 2 \tag{55}$$

$(g_4(2) = 1)$ is decreasing in p.

In the following result we get bounds of the quantities required in Theorem 3.2, depending on the shape parameter p, but also decreasing on this one.

Lemma 4.6 *Let f_p be as in* (49)*, and g_i, $i = 1, 2, 3, 4$ be as in Lemma 4.5. We have*

(i) $\sup_{x\geq 0} |f_p(x)| = g_1(p), \quad p \geq 1.$

(ii) $\sup_{x\geq 0} |x f_p'(x)| = g_2(p), \quad p \geq 1.$

(iii) $\sup_{x\geq 0} |f_p'(x)| = g_3(p), \quad p \geq 2.$

(iv) $\sup_{x\geq 0} |x f_p''(x)| \leq \max\{g_1(p-1), g_2(p-1)\}, \quad p \geq 2.$

(v) $\sup_{x\geq 0} |x^2 f_p'''(x)| \leq g_4(p) + 3g_2(p-1) + g_1(p-1), \quad p \geq 2.$

Proof. To show part (i), it is clear that, for $p \geq 1$,

$$\sup_{x\geq 0} f_p(x) = f_p(p-1) = \frac{e^{-(p-1)}(p-1)^{p-1}}{\Gamma(p)},$$

and (i) follows recalling (52). To show part (ii) we have (cf. [16] Remark 3.2. and Lemma 5.2)

$$\sup_{x\geq 0} |x f_p'(x)| = \frac{1}{\Gamma(p)} \left(p - \frac{1}{2} + \frac{1}{2}\sqrt{4p-3}\right)^{p-1/2} e^{-p-\frac{1}{2}+\frac{1}{2}\sqrt{4p-3}}, \quad p > 1, \quad (56)$$

and (ii) follows recalling (53). To show part (iii), by (51), we have for $p \geq 2$,

$$f_p'(x) = \frac{1}{\Gamma(p)} e^{-x} x^{p-2}(p-1-x), \quad x > 0, \quad (57)$$

$$f_p''(x) = \frac{1}{\Gamma(p)} e^{-x} x^{p-3}((p-1)(p-2) - 2(p-1)x + x^2), \quad x > 0, \quad (58)$$

and it can be checked easily that the zeroes of $f_p''(x)$ are $p_1 := p - 1 - \sqrt{p-1}$ and $p_2 := p - 1 + \sqrt{p-1}$. Therefore, $|f_p'(x)|$ must attain its maximum value either at p_1 or p_2. Actually p_1 corresponds to the maximum. To show that we will see that

$$\frac{f_p'(p_1)}{|f_p'(p_2)|} = e^{2\sqrt{p-1}} \left(\frac{\sqrt{p-1}-1}{\sqrt{p-1}+1}\right)^{p-2} \geq 1, \quad p \geq 2. \quad (59)$$

To show the last inequality in (59), taking logarithms we will prove that

$$r_1(p) := 2\sqrt{p-1} + (p-2)\Big(\log(\sqrt{p-1}-1) - \log(\sqrt{p-1}+1)\Big) \geq 0, \quad p > 2. \tag{60}$$

Call

$$\rho_1(b) := \frac{2b}{b^2-1} + \left(\log(b-1) - \log(b+1)\right), \quad b > 1.$$

Note that

$$r_1(p) = (p-2)\rho_1(\sqrt{p-1}), \quad p > 2. \tag{61}$$

We will firstly prove that

$$\rho_1(b) \geq 0, \quad b > 1. \tag{62}$$

To show (62), it is readily seen that $\rho_1'(b) = -4(b^2-1)^{-2},\ b > 1$, so that ρ_1 is decreasing. As $\lim_{b\to\infty} \rho_1(b) = 0$, we have (62). This implies also (60), recalling (61). Therefore, we conclude that

$$\sup_{x>0} |f_p'(x)| = f_p'(p_1) = \frac{1}{\Gamma(p)} e^{-(p-1-\sqrt{p-1})}(\sqrt{p-1}-1)^{p-2}(\sqrt{p-1})^{p-1}, \tag{63}$$

this, together with (54), shows (iii).

To show part (iv), note that using (51), we can write $f_p'(x) = f_{p-1}(x) - f_p(x)$ and therefore,

$$xf_p''(x) = xf_{p-1}'(x) - xf_p'(x), \quad x > 0, \quad p \geq 2. \tag{64}$$

On the other hand, we see in (58) that $f_{p-1}'(x)$ and $f_p'(x)$ have the same sign for $0 < x < p-2$ and $p-1 < x < \infty$ and therefore, using part (ii), and Lemma 4.5(i), we can write

$$\sup_{x \notin [p-2,p-1]} |xf_p''(x)| \leq \max(g_2(p-1), g_2(p)) = g_2(p-1). \tag{65}$$

On the other hand we have by (58)

$$xf_p''(x) = \frac{1}{\Gamma(p)} e^{-x} x^{p-2}((p-1)(p-2) - 2(p-1)x + x^2) \tag{66}$$

using the above expression and taking into account that for $p-2 \leq x \leq p-1$

$$e^{-x}x^{(p-2)} \leq e^{-p-2}(p-2)^{p-2} \quad \text{and} \quad |(p-1)(p-2)-2(p-1)x+x^2| = p-1, \tag{67}$$

the last inequality as $|(p-1)(p-2)-2(p-1)x+x^2|,\ p-2 \leq x \leq p-1$ attains its maximum value at $p-1$. From (66) and (67), we conclude that

$$\sup_{x \in [p-2,p-1]} |xf_p''(x)| \leq \frac{1}{\Gamma(p)} e^{-(p-2)}(p-2)^{p-2}(p-1) = g_1(p-1), \tag{68}$$

the last inequality recalling (52). Thus (65) and (68) conclude the proof of (iv).

To show (v), let $p \geq 2$. We have firstly, by (51)

$$\begin{aligned} f_p'''(x) &= f_{p-3}(x) - 3f_{p-2}(x) + 3f_{p-1}(x) - f_p(x) \\ &= \frac{e^{-x}x^{p-4}}{\Gamma(p)}((p-1)(p-2)(p-3) - 3(p-1)(p-2)x + 3(p-1)x^2 - x^3) \\ &= \frac{e^{-x}x^{p-4}}{\Gamma(p)}((p-1-x)^3 + 3(p-1)(x-(p-2)) - (p-1)),\ x > 0. \end{aligned} \tag{69}$$

Therefore, if we call

$$h_p(x) := \frac{e^{-x}x^{p-2}}{\Gamma(p)}(p-1-x)^3, \quad x > 0,$$

we have, recalling (57)

$$\begin{aligned} x^2 f_p'''(x) &= \frac{e^{-x}x^{p-2}}{\Gamma(p)}((p-1-x)^3 - 3(p-1)(x-(p-2)) - (p-1)) \\ &= h_p(x) + 3xf_{p-1}'(x) - f_{p-1}(x), \quad x \geq 0. \end{aligned} \tag{70}$$

We will firstly see that

$$\sup_{x \geq 0} |h_p(x)| = g_4(p), \tag{71}$$

with $g_4(\cdot)$ as defined in (55). Note that

$$h_p'(x) = \frac{e^{-x}x^{p-3}}{\Gamma(p)}(p-1-x)^2(x^2 - 2px + (p-1)(p-2)), \quad x > 0$$

The maximum value of $|h_p|$ will be attained at the roots of the last polynomials, being $p_1 := p + \sqrt{3p-2}$ and $p_2 := p - \sqrt{3p-2}$. To check which value attains the maximum, call $u := \sqrt{3p-2}$. Note that $p_1 = (u+1)(u+2)/3$ and $p_2 = (u-1)(u-2)/3$. Then, with this notation we will prove that

$$\frac{|h_p(p_2)|}{|h_p(p_1)|} = e^{2u}\left(\frac{(u-1)(u-2)}{(u+1)(u+2)}\right)^{\frac{u^2-4}{3}}\left(\frac{u-1}{u+1}\right)^3 \geq 1, \quad u > 2. \tag{72}$$

To show the last inequality in (72), taking logarithms, we will show that

$$\rho_2(u) := 2u + \frac{u^2-4}{3}\log\left(\frac{(u-1)(u-2)}{(u+1)(u+2)}\right) + 3\log\left(\frac{u-1}{u+1}\right) \geq 0 \quad u > 2. \tag{73}$$

Note that

$$\begin{aligned}\rho_2'(u) =& 2 + \frac{2u}{3}\log\left(\frac{(u-1)(u-2)}{(u+1)(u+2)}\right) + \frac{u^2-4}{3}\left(\frac{1}{u-1} + \frac{1}{u-2} - \frac{1}{u+1} - \frac{1}{u+2}\right)\\ &+ 3\left(\frac{1}{u-1} - \frac{1}{u+1}\right) = \frac{4u^2}{u^2-1} + \frac{2u}{3}\log\left(\frac{(u-1)(u-2)}{(u+1)(u+2)}\right) \quad u > 2.\end{aligned}$$

We will show that $\rho_2'(u) \leq 0, u > 2$. In fact,

$$\frac{d}{du}\frac{3}{2u}\rho_2'(u) = \frac{36}{(u+1)^2(u-1)^2(u^2-4)^2} \geq 0, \quad u > 2.$$

and then $3(2u)^{-1}\rho_2'(u)$ is increasing. As $\lim_{u\to\infty} 3(2u)^{-1}\rho_2'(u) = 0$, we conclude that $3(2u)^{-1}\rho_2'(u) \leq 0$, and thus that $\rho_2'(u) \leq 0$. Therefore, $\rho_2(u)$ is decreasing. This, together with the fact that $\lim_{u\to\infty}\rho_2(u) = 0$, proves (73), and therefore (72). Then $\|h_p\| = h_p(p_2) = g_4(p)$, thus proving (71). Now, the proof of part (iv) follows easily recalling (70) and using (71) and parts (i) and (ii). $\square$

As an immediate consequence of Theorem 3.2 and Lemma 4.6 we have the following

Corollary 4.1 *Let F_p be a gamma distribution of shape parameter $p \geq 2$, that is whose density function is given by* (49). *Let $M_t^{[2]}$, $t > 0$ be as defined in* (8). *We have*

$$\|M_t^{[2]}F_p - F_p\| \leq \left(\frac{17}{12} + \frac{27}{16e}\right)\frac{1}{t^2} \approx \frac{2.0375}{t^2}$$

Proof. Let $p \geq 2$ be fixed. The result is an immediate consequence of Theorem 3.2, as $F_p' = f_p$ as defined in (49). Therefore by Lemma 4.6(iii) and Lemma 4.5 (ii) we have that

$$\|F_p''\| = \|f_p'\| = g_3(p) \leq g_3(2) = 1. \qquad (74)$$

On the other hand, we see we have by Lemma 4.5 (i) that

$$g_1(p-1) \leq g_1(1) = 1 \quad \text{and} \quad g_2(p-1) \leq g_2(1) = e^{-1}, \quad p \geq 2 \qquad (75)$$

Thus, using the above inequalities and Lemma 4.6(iv), we have

$$\|xF_p'''(x)\| = \|xf_p''(x)\| \leq 1. \qquad (76)$$

Finally by Lemma 4.6(v), Lemma 4.5 (iv) and (75) we have

$$\|x^2 F_p^{iv}(x)\| = \|x^2 f_p'''(x)\| \leq g_4(2) + 3g_2(1) + g_1(1) = 2 + 3e^{-1}. \qquad (77)$$

Using (74), (76), (77), and Theorem 3.2, we obtain the result. This completes the proof of Corollary 4.1. $\square$

5 Applications to mixtures of Erlang distributions and phase-type distributions

In this Section we apply the results in the previous one to mixtures of Erlang distributions, and to random sums of them. In order to make the study for an arbitrary scale parameter, we see in the following result the behaviour of $M_t^{[2]}F$ under changes of scale.

Proposition 5.2 *Let X be a random variable with distribution function F. For a given $c > 0$ denote by F^c the distribution function of cX. Let $M_t^{[2]}F$ and $M_t^{[2]}F^c$, $t > 0$ be the respective approximations for F and F^c, as defined in* (8)*. We have that*

$$M_t^{[2]}F^c(x) = M_{ct}^{[2]}F(x/c), \quad x \geq 0. \qquad (78)$$

Therefore,

$$\|M_t^{[2]}F^c - F^c\| = \|M_{ct}^{[2]}F - F\|. \tag{79}$$

Proof. Let $t > 0$ and $c > 0$ be fixed. First of all, we will see that

$$M_t^{[2]}F^c\left(\frac{k}{t}\right) = M_{ct}^{[2]}F\left(\frac{k}{ct}\right), \quad k \in \mathbb{N}, \tag{80}$$

and therefore, (78) is satisfied for points in the set $k/t,\ k \in \mathbb{N}$. To this end, we use (12) and (6), and take into account that

$$F^c(x) = F(x/c), \quad x \geq 0, \tag{81}$$

to write, for all $k \in \mathbb{N}$,

$$\begin{aligned} M_t^{[2]}F^c\left(\frac{k}{t}\right) &= 2EF^c\left(\frac{S(2k)}{2t}\right) - EF^c\left(\frac{S(k)}{t}\right) \\ &= 2EF\left(\frac{S(2k)}{2ct}\right) - EF\left(\frac{S(k)}{ct}\right) = M_{ct}^{[2]}F\left(\frac{k}{ct}\right), \end{aligned} \tag{82}$$

thus proving (80). For a general $x > 0$, we use (8) and (80), to see that

$$\begin{aligned} &M_t^{[2]}F^c(x) = (tx - [tx])M_t^{[2]}F^c\left(\frac{[tx]+1}{t}\right) + ([tx]+1-tx)M_t^{[2]}F^c\left(\frac{[tx]}{t}\right) \\ &= (tx - [tx])M_{ct}^{[2]}F\left(\frac{[tx]+1}{ct}\right) + ([tx]+1-tx)M_{ct}^{[2]}F\left(\frac{[tx]}{ct}\right) = M_{ct}^{[2]}F\left(\frac{x}{c}\right), \end{aligned}$$

the last inequality being trivial, as $tx = (ct)\dot{(}x/c)$. This concludes the proof of (78). Finally, (79) follows easily from (78) and (81), as we have

$$\sup_{x>0}|M_t^{[2]}F^c(x) - F^c(x)| = \sup_{x>0}|M_{ct}^{[2]}F(x/c) - F(x/c)|$$

This concludes the proof of Proposition 5.2. $\square$

As an application of the results in the previous Section, we will consider the class of (possibly infinite) mixtures of Erlang distributions recently studied by Willmot and Woo (cf. [19]). More specifically let $F_{(a,j)},\ a > 0,\ j \in \mathbb{N}^*$, be the distribution function corresponding to the density $f_{(a,j)}$ given in (3) (an Erlang j distribution with scale parameter a). We will consider a finite number of scale parameters arranged in increasing order ($0 < a_1 < \cdots < a_n$), and a set of nonnegative numbers p_{ij}, $i = 1, \ldots n$, $j = 0, 1, 2, \ldots$, such that $\sum_{i=1}^{n} \sum_{j=1}^{\infty} p_{ij} = p \leq 1$, and define the class of distribution function $\mathcal{ME}(a_1, \ldots, a_n)$ given as

$$F(x) = (1-p) + \sum_{i=1}^{n} \sum_{j=1}^{\infty} p_{ij} F_{a_i,j}(x), \quad x \geq 0 \tag{83}$$

(we consider a slight modification of the class in [19, p.103], as we allow the point mass at 0 with probability $1-p$). Based on [19, p.103], we can alternative write (83) by using only the maximum of the scale parameters, that is

$$F(x) = (1-p) + \sum_{j=1}^{\infty} p_j F_{a_n,j}(x), \quad x \geq 0. \tag{84}$$

Moreover, the class (84) is a wide class containing many of the distributions considered in applied probability, such as (obviously) finite mixtures of Erlangs, but also the class of phase-type distributions (see Proposition 5.4 below). Every random variable having a representation as in (83) can be approximated by means of $M_t^{[2]}$, as it is shown in the following.

Proposition 5.3 *Let F be a distribution function of the form $\mathcal{ME}(a_1, \ldots, a_n)$, $0 < a_1 < \cdots < a_n$, as in* (83)*. Let $M_t^{[2]}$, $t > 0$ be as defined in* (8)*. We have*

$$\|M_t^{[2]} F - F\| \leq \left(\frac{17}{12} + \frac{27}{16e} \right) \frac{\sum_{i=1}^{n} (\sum_{j=1}^{\infty} p_{ij}) a_i^2}{t^2}. \tag{85}$$

Proof. Let $t > 0$ and $0 < a_1 < \cdots < a_n$ be fixed. The linearity of $M_t^{[2]}$ yields

$$M_t^{[2]} F(x) = (1-p) + \sum_{i=1}^{n} \sum_{j=1}^{\infty} p_{ij} M_t^{[2]} F_{a_i,j}(x), \quad x \geq 0. \tag{86}$$

By Corollary 4.1 we can write, for a scale parameter 1,

$$\|M_t^{[2]} F_{1,j} - F_{1,j}\| \leq \left(\frac{17}{12} + \frac{27}{16e} \right) \frac{1}{t^2}, \quad j = 2, 3, \ldots \tag{87}$$

Moreover, using Lemma 4.3 we have

$$\|M_t^{[2]}F_{1,1} - F_{1,1}\| \leq \left(\frac{1}{2} + \frac{1}{6e} + \frac{9}{4e^2}\right)\frac{1}{t^2} \leq \left(\frac{17}{12} + \frac{27}{16e}\right)\frac{1}{t^2} \tag{88}$$

Let now the general scale parameters $a_i,\ i = 1, \dots, n$. We use that given X a gamma random variable of scale parameter 1, then, X/a_i is a gamma random variable of scale parameter a_i, and therefore, using Proposition 5.2, (87) and (88), we have for each $a_i,\ i = 1, \dots, n$ and $j \in \mathbb{N}^*$

$$\|M_t^{[2]}F_{a_i,j} - F_{a_i,j}\| = \|M_{t/a_i}^{[2]}F_{1,j} - F_{1,j}\| \leq \left(\frac{17}{12} + \frac{27}{16e}\right)\frac{a_i^2}{t^2}. \tag{89}$$

Thus using (86) and (89) we have

$$\begin{aligned}\|M_t^{[2]}F - F\| &\leq \sum_{i=1}^{n}\sum_{j=1}^{\infty} p_{ij}\|M_t^{[2]}F_{a_i,j} - F_{a_i,j}\| \\ &\leq \left(\frac{17}{12} + \frac{27}{16e}\right)\frac{\sum_{i=1}^{n}(\sum_{j=1}^{\infty} p_{ij})a_i^2}{t^2}. \end{aligned} \tag{90}$$

This completes the proof of Proposition 5.3. □

As a consequence of the previous result, we we can provide error bounds for compound distributions (that is, distribution functions of random sums as in (14)), when the summands are mixtures of Erlangs, as stated in the following

Corollary 5.2 *Let G be the distribution function of a random sum as in* (14)*, in which the sequence of $(X_i)_{i\in\mathbb{N}^*}$ has a common distribution $\mathcal{ME}(a_1, \dots, a_n), 0 < a_1 < \cdots < a_n$, as defined in* (83)*. Let $M_t^{[2]}$ be as in* (8)*. We have that*

$$\|M_t^{[2]}G - G\| \leq \left(\frac{17}{12} + \frac{27}{16e}\right)\frac{(1 - G(0))a_n^2}{t^2},$$

Proof. The proof is immediate taking into account that a mixture of Erlangs $\mathcal{ME}(a_1, \dots, a_n), 0 < a_1 < \cdots < a_n$ can be expressed as in (84), and compound distributions of these random variables are also mixtures of Erlang (cf. [19, p.106], with a slight modification in the coefficients, as we allow a point mass at 0), that is, we can write

$$G(x) = q_0 + \sum_{j=1}^{\infty} q_j F_{a_n,j}(x), \quad x \geq 0,$$

in which $\{q_j,\ j = 0, 1, \dots\}$ form a probability mass function (obviously, $q_0 = G(0)$). The result follows using the above expression and Proposition 5.3. $\square$

The class of phase type distributions, of great importance in applied probability, can be expressed as mixtures of Erlangs. Phase-type distribution are defined as the time until absorption in a continuous-time Markov chain with one absorbent state (cf., for instance [12, Ch.II], or [8, Ch.VIII], and the references therein). A phase-type distribution can be expressed in terms of a matrix exponential as follows. Consider a vector $\alpha = (\alpha_1, \dots \alpha_n)$ of nonnegative numbers such that $\alpha_1 + \cdots + \alpha_n \leq 1$. Let A be a $n \times n$ matrix with negative diagonal entries, non-negative off-diagonal entries and non-positive row sums. A nonnegative random variable X is a phase type distribution $PH(\alpha, A)$ if its distribution function is written as

$$F(x) = 1 - \alpha e^{xA} \mathbf{1'}, \quad x \geq 0,$$

in which $\mathbf{1'}$ represent the transpose of the n th dimensional vector $\mathbf{1} = (1, \dots, 1)$. Note that phase-type distributions are absolutely continuous random variables when $\alpha_1 + \cdots + \alpha_n = 1$, having positive mass at 0 (of magnitude $1 - (\alpha_1 + \cdots + \alpha_n)$) when $\alpha_1 + \cdots + \alpha_n < 1$. Phase-type distributions have been extensively studied both from a theoretical and practical point of view. For instance, it is well known that phase-type distributions have a rational Laplace transform, thus allowing numerical computations using our approximation procedures. Also, in the next Proposition we will give an expression of a phase-type distributions in terms of mixtures of Erlangs. This, together with Proposition 5.3 provides our approximations with rates of convergence. The proof of the next result is based on the following property of phase-type distributions, due to Maier (cf.[13, p.591]) Let f be the density of an absolutely continuous phase-type distribution. There exists some $c > 0$ verifying

$$c_j := \frac{d^j}{dx^j} e^{cx} f(x)\bigg|_{x=0} > 0, \quad j \in \mathbb{N}. \tag{91}$$

We are in a position to enunciate the following.

Proposition 5.4 *Let F be a phase-type distribution $PH(\alpha, A)$, with $\alpha_1 + \cdots + \alpha_n > 0$. Let $c > 0$ be such that the absolutely continuous part of F satisfies property* (91)*. Then F can be expressed as a mixture of Erlangs, that is*

$$F(x) = p_0 + \sum_{j=1}^{\infty} p_j F_{c,j}(x), \quad x \geq 0, \tag{92}$$

in which $p_0 = 1 - (\alpha_1 + \cdots + \alpha_n)$.

Proof. To prove (a) assume firstly that F is absolutely continuous, that is, $\alpha_1 + \cdots + \alpha_n = 1$. Then, its density is given by $f(x) = -\alpha e^{xA} A\mathbf{1}', \quad x > 0$. We choose $c > 0$ verifying (91). Note that we can write

$$e^{cx} f(x) = -\alpha e^{x(cI-A)} A\mathbf{1}', \quad x \geq 0. \tag{93}$$

It can be easily checked that the function $-\alpha e^{x(cI-A)} A\mathbf{1}', \quad x \in \mathbb{R}$ is analytic in $\mathbb{R}$, so that if we consider the Taylor's series expansion of this function around 0, and take into account (91) and (93), we have

$$e^{cx} f(x) = \sum_{j=0}^{\infty} c_j \frac{x^j}{j!}, \quad x > 0,$$

from which we can write (recall (3))

$$f(x) = \sum_{j=0}^{\infty} \frac{c_j}{c^{j+1}} \frac{c^{j+1} x^j e^{-cx}}{j!} = \sum_{j=0}^{\infty} \frac{c_j}{c^{j+1}} f_{c,j+1}(x), \quad x > 0$$

and in this way we obtain the expression of f in terms of a mixture of Erlang densities with shape parameter c (by construction the coefficients are non-negative, and integrating both sides in the above expression we see that their sum is 1). As a consequence we can write

$$F(x) = \sum_{j=1}^{\infty} \frac{c_{j-1}}{c^j} F_{c,j}(x), \quad x \geq 0, \tag{94}$$

thus having F expressed as a mixture of Erlangs, as in (92). Now assume that $0 < \alpha_1 + \cdots + \alpha_n < 1$. This means that F has a point mass at 0 of magnitude $p_0 := 1 - (\alpha_1 + \cdots + \alpha_n)$. The absolutely continuous part of F (F^{ac}) is a phase type distribution ($PH(\bar{\alpha}, A)$), with $\bar{\alpha} = (\alpha_1 + \cdots + \alpha_n)^{-1}\alpha$. Let $c > 0$ be such that F^{ac} verifies property (93). We can write thanks to (94)

$$F(x) = p_0 + (1-p_0)F^{ac}(x) = p_0 + \sum_{j=1}^{\infty}(1-p_0)\frac{c_{j-1}}{c^j} F_{c,j}(x), \quad x \geq 0$$

This completes the proof of Proposition 5.4. $\square$

Remark 5.1 Similar expansions to that given in Proposition 5.4 can be found in [12, p. 58]. These expansions are obtained using a representation $PH(\alpha, A)$ of the distribution under consideration. Note that if we denote by $\|A\|$ the maximum absolute value of the entries of A, it is easy to check using (93) (cf. [14, p.751]), that $c = \|A\|$ verifies (91). However, as the representation of a phase type is not unique this value might not be the optimum one. Observe also that the error bound given in (85) indicates that we should take c as small as possible. This problem then, is closely connected with Conjecture 6 in [14], concerning the minimum c satisfying (91) and its relation with a phase-type representation having $\|A\|$ as small as possible. To the best of our knowledge, this conjecture remains, nowadays, unsolved.

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