Between the LIL and the LSL

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Abstract

In two earlier papers two of us (A.G. and U.S.) extended Lai's (1974) law of the single logarithm for delayed sums to a multiindex setting in which the edges of the $\mathbf{n}$th window grow like $|\mathbf{n}|^\alpha$, or with different α's, where the $\alpha's \in (0,1)$. In this paper the edge of the nth window typically grows like $n/\log n$, thus at a higher rate than any power less than one, but not quite at the LIL-rate.

1 Introduction

Let X, $\{X_k,\, k \geq 1\}$ be i.i.d. random variables with mean 0 and partial sums $\{S_n,\, n \geq 1\}$. The Hartman-Wintner *Law of the Iterated Logarithm* (LIL) states that

$$\limsup_{n\to\infty} \,(\liminf_{n\to\infty})\, \frac{S_n}{\sqrt{2n\log\log n}} = \sigma \quad \big(-\sigma\big) \quad \text{a.s.} \quad \Longleftrightarrow\ E\,X^2 < \infty, \quad \text{and} \quad E\,X = 0\,,\ E\,X^2 = \sigma^2\,.$$

The sufficiency was proved by Hartman and Wintner [8] and the necessity by Strassen [11].

The *Law of the Single Logarithm* (LSL) is due to Lai [9], and deals with *delayed sums* or *windows*, viz., with

$$T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j, \quad n \geq 0,\, k \geq 1,$$

and states that, for $0 < \alpha < 1$,

$$\limsup_{n\to\infty} \frac{T_{n,n+n^\alpha}}{\sqrt{2n^\alpha \log n}} = \sigma\sqrt{1-\alpha} \quad \text{a.s.} \iff E\Big(|X|^{2/\alpha}\big(\log^+|X|\big)^{-1/\alpha}\Big) < \infty\,,\ E\,X^2 = \sigma^2,\ E\,X = 0,$$

where throughout $\log^+ x = \max\{\log x, 1\}$.

The degenerate boundary case $\alpha = 0$ contains the trivial one in that the window reduces to a single random variable. More precisely, in that case,

$$\frac{T_{n,n+1}}{b_n} = \frac{X_{n+1}}{b_n} \overset{a.s.}{\to} 0 \quad \text{as} \quad n\to\infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} P(|X| > b_n) < \infty,$$

which, in turn, holds iff $Eb^{-1}(|X|) < \infty$, where $b^{-1}(\cdot)$ is a (suitably defined) inverse of $\{b_n\}$.

The next interesting case with $\alpha = 0$ is when the span $a_n = \log n$, that is, the window $T_{n,n+\log n}$, in which case the so-called Erdős-Rényi law ([3], Theorem 2, [2], Theorem 2.4.3) tells us that if $E\,X = 0$, and the moment generating function $\psi_X(t) = E\,\exp\{tX\}$ exists in a neigbourhood of 0, then, for any $c > 0$,

$$\lim_{n\to\infty} \max_{0\leq k\leq n-k} \frac{T_{k,k+c\log k}}{c\log k} = \rho(c) \quad \text{a.s.},$$

where

$$\rho(c) = \sup\{x : \inf_t e^{-tx}\psi_X(t) \geq e^{-1/c}\}.$$

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Note that here the limit actually depends on the distribution of the summands.

For a generalization to more general window widths a_n, such that $a_n/\log n \to \infty$ as $n \to \infty$, but still assuming that the moment generating function exists, see e.g. [2], Theorem 3.1.1, where the limit, in contrast to the just cited result, does not depend on the distribution. Results where the moment condition is somewhat weaker than existence of a moment generating function were discussed in [10]; here the limit depends on both the variance and the distribution. In case that the moment condition and the size of a_n is roughly of algebraic order larger than two see e.g. [2], Theorem 3.2.1.

For the boundary case at the other end with $\alpha = 1$ one has $a_n = n$ and $T_{n,2n} \stackrel{d}{=} S_n$ and the correct norming is as in the LIL.

One interesting remaining case is when the window size is larger than any power less than one, and at the same time not quite linear, and this is the starting point of the present paper. Technically we wish to examine windows of the form

$$T_{n,n+a_n} \quad \text{where} \quad a_n = \frac{n}{L(n)} \quad \text{with} \tag{1.1}$$

$$\text{a differentiable function} \quad L(\cdot) \nearrow \infty \in \mathcal{SV} \quad \text{and} \quad \frac{xL'(x)}{L(x)} \searrow \quad \text{as} \quad x \to \infty\,. \tag{1.2}$$

Notation: $L \in \mathcal{SV}$ means that L is slowly varying at infinity (see e.g. [1] or [5], Section A.7).

The typical case one should have in mind is $L(n) = \log n$, that is, the window $T_{n,n+n/\log n}$.

Remark 1.1 Strictly speaking we should write $a_n = [n/L(n)]$ and $a_n = [n/\log n]$, and so on. However, in order to avoid trivial and boring technicalities we shall treat such sequences as integer valued whenever convenient.

In Section 2 we present the setup, the main result and the implication for some typical slowly varying functions, namely $L(x) = (\log x)^p$ for $p > 0$ and iterated logarithms $L(x) = \log_m x$, where $\log_m(x)$ denotes the m times iterated logarithm. For the proof in Section 3 we first review the exponential inequalities. Section 3.2 then introduces a family of subsequences within which sufficiency of the moment condition is proved in Sections 3.3–3.6. Section 3.7 deals with the same issue for the full sequence, while the question of necessity is taken care of in Section 3.8. Proofs of the corollaries in Section 2 are provided in Section 4, while Section 5 furnishes further examples, including some with more complicated slowly varying parts.

It turns out that the proof of the main result has some common ingredients with that of the classical LIL, primarily in the sense that one needs two truncations, one to match the Kolmogorov exponential bounds and one to match the moment requirements. Typically (and somewhat frustratingly) it is the thin central part that causes the main trouble in the proof. A weaker result is obtained if only the first truncation is made. The cost is that too much integrability will be required. However, for the reader who is not so much concerned with optimality we include a proof of this weaker version in Section 6, after which we revisit two examples in order to illustrate the consequences.

2 Setup and main result

Recall that the window widths, a_n, are assumed to be of the form $n/L(n)$, with a function L satisfying (1.2). Define $\{d_n,\, n \geq 2\}$ by

$$d_n = \log \frac{n}{a_n} + \log\log n = \log L(n) + \log\log n.$$

Note that $\{d_n\}$ may be viewed as *the additional norming sequence* in Theorem 2.1, in the sense that it corresponds to $\{\log_2 n\}$ in the LIL and $\{\log n\}$ in the LSL.

Furthermore, let

$$f(n) = \min\{a_n \cdot d_n,\, n\},$$

with $f(\cdot)$ an increasing interpolating function, i.e., $f(x) = f_{[x]}$ for $x > 0$ and with $f^{-1}(x)$ the corresponding (suitably defined) inverse function being $\geq x$.

Here is now our main result.

Theorem 2.1 *Suppose that X, X_1, X_2, … are i.i.d. random variables with mean 0 and finite variance σ^2, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. If*

$$E\left(f^{-1}(X^2)\right) < \infty, \tag{2.1}$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+a_n}}{\sqrt{2a_n d_n}} = \sigma \qquad a.s. \tag{2.2}$$

Conversely, if

$$P\Big(\limsup_{n\to\infty} \frac{|T_{n,n+a_n}|}{\sqrt{a_n d_n}} < \infty\Big) > 0, \tag{2.3}$$

then (2.1) holds, $E\,X = 0$ and (2.2) holds with $\sigma^2 = \operatorname{Var} X$.

Remark 2.1 The "natural" necessary moment assumption is (2.1) with $f(n) = a_n d_n$. However, for very slowly increasing functions L, for example $L(x) = \log\log\log\log x$ it turns out that finite variance is needed, and since then $f(n) = n$ (2.1) is equivalent to finite variance.

Remark 2.2 The result holds also for any sequence $\{a_n\}$ being of regular variation of order $\alpha \in (0, 1)$. Here the sufficiency part can be obtained from strong invariance principles, as e.g. described in Theorem 3.2.2 in the book [2]. However, for our situation no strong invariance principle is available.

Remark 2.3 In addition to the lim sup results there exist, throughout, lim inf counterparts such that $\liminf \cdots = -\limsup \cdots$ a.s. Actually, the set of limit points is the whole interval $[-\sigma, \sigma]$.

The immediate slowly varying function that comes to mind is (of course) the logarithmic function. The second one would be $L(x) = \log_2 x = \log\log x$, and, possibly $L(x) = \log_m(x)$. We precede the proofs by stating the conclusions for these cases as separate corollaries. For simplicity we omit the converse parts.

Corollary 2.1 *Suppose that X, X_1, X_2, … are i.i.d. random variables with mean 0 and finite variance σ^2, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. If for some $p > 0$*

$$E\,X^2 \frac{(\log^+|X|)^p}{\log^+\log^+|X|} < \infty, \tag{2.4}$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+n/(\log n)^p}}{\sqrt{2(p+1)\frac{n}{(\log n)^p}\log\log n}} = \sigma \qquad a.s. \tag{2.5}$$

Corollary 2.2 *Suppose that X, X_1, X_2, … are i.i.d. random variables with mean 0, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. If $\sigma^2 = \operatorname{Var} X < \infty$, then for any $m \ge 2$*

$$\limsup_{n\to\infty} \frac{T_{n,n+n/\log_m(n)}}{\sqrt{2\frac{n}{\log_m(n)}\log\log n}} = \sigma \qquad a.s. \tag{2.6}$$

Note that in case $m = 2$ the normalization is just $\sqrt{2n}$. Proofs of the corollaries are deferred to Section 4 and further examples are given in Section 5.

3 Proof of Theorem 2.1

In spite of the fact that we are dealing with limit laws for delayed sums, the present topic is in fact too close to the LIL to warrant LSL techniques. In contrast to the proofs in [9] and [6, 7], where one uses exponential bounds and Borel-Cantelli lemmas for the single primed contribution along a suitably subsequence, and takes care of the double- and triple primed contributions for the full sequence and fills the gaps, we have to resort to the LIL-technique where one proves Borel-Cantelli lemmas, and thus also the theorem itself, first for subsequences and then for the entire sequence.

We thus begin by providing Borel-Cantelli sums along subsequences, after which an appeal to the Borel-Cantelli lemmas completes the proof for subsequences.

Section 3.7 is devoted to the problem of "filling the gaps" in order to include arbitrary windows.

3.1 Truncation and exponential bounds

The typical pattern in proving results of the LIL type requires two truncations; the first one to match the Kolmogorov exponential bounds (see e.g. [5], Section 8.2), and the second one to match the moment requirements.

Toward this end we introduce parameters $\delta > 0$ and $\varepsilon > 0$, and set

$$b_n = \frac{\sigma\delta}{\varepsilon}\sqrt{\frac{a_n}{d_n}}, \tag{3.1}$$

and

$$\begin{aligned} X_n' &= X_n I\{|X_n| \le b_n\}, \quad X_n'' = X_n I\{b_n < |X_n| < \delta\sqrt{f(n)}\}, \\ X_n''' &= X_n I\{|X_n| \ge \delta\sqrt{f(n)}\}. \end{aligned}$$

In the following all objects with primes or multiple primes refer to the respective truncated summands.

Since truncation destroys centering, we obtain, using standard procedures, together with the fact that $E\,X = 0$,

$$|E\,X_k'| = |-E\,X_k I\{|X_k| > b_k\}| \le E|X|I\{|X_k| > b_k\} \le \frac{E\,X^2 I\{|X| > b_k\}}{b_k},$$

so that

$$\begin{aligned} |E\,T'_{n,n+a_n}| &\le \sum_{n\le k\le n+a_n} \frac{E\,X^2 I\{|X| > b_k\}}{b_k} \le a_n \cdot \frac{E\,X^2 I\{|X| > b_n\}}{b_n} \\ &= \frac{\varepsilon}{\sigma\delta}\cdot\sqrt{a_n d_n}\cdot E\,X^2 I\{|X| > b_n\} = o(\sqrt{a_n d_n}), \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.2}$$

Upper bounds

Since

$$\operatorname{Var} X_k' \le (E\,X_k')^2 \le E\,X^2 = \sigma^2,$$

it follows that

$$\operatorname{Var}(T'_{n,n+a_n}) \le a_n\sigma^2. \tag{3.3}$$

An application of the Kolmogorov upper exponential bound (see e.g. [5], Lemma 8.2.1) with $x = \varepsilon(1-\delta)\sqrt{2d_n}$ and $c_n = 2\delta/x$, together with (3.2) and (3.3) now yield

$$\begin{aligned} P(T'_{n,n+a_n} > \varepsilon\sqrt{2a_n d_n}) &\le P(T'_{n,n+a_n} - ET'_{n,n+a_n} > \varepsilon(1-\delta)\sqrt{2a_n d_n}) \\ &\le P(T'_{n,n+a_n} - ET'_{n,n+a_n} > \frac{\varepsilon(1-\delta)}{\sigma}\sqrt{2\operatorname{Var}(T'_{n,n+a_n})d_n}) \\ &\le \exp\Big\{-\frac{2\varepsilon^2(1-\delta)^2}{2\sigma^2}\cdot d_n(1-\delta)\Big\} \\ &= \exp\Big\{-\frac{\varepsilon^2(1-\delta)^3}{\sigma^2}\cdot d_n\Big\}. \end{aligned} \tag{3.4}$$

Lower bounds

In order to apply the lower exponential bound (see e.g. [5], Lemma 8.2.2) we first need a lower bound for the truncated variances:

$$\begin{aligned} \operatorname{Var} X_k' &= E\,{X_k'}^2 - (E\,X_k')^2 = E\,X^2 - E\,X^2 I\{|X_k| \ge b_k\} - (E\,X_k')^2 \\ &\ge \sigma^2 - 2E\,X^2 I\{|X_k| \ge b_k\} \ge \sigma^2(1-\delta), \end{aligned}$$

for n large, so that

$$\operatorname{Var}(T'_{n,n+a_n}) \ge a_n\sigma^2(1-\delta), \quad \text{for } n \text{ large}. \tag{3.5}$$

It now follows that for any $\gamma > 0$,

$$\begin{aligned} P(T'_{n,n+a_n} > \varepsilon\sqrt{2a_n d_n}) &\geq P\big(T'_{n,n+a_n} - ET'_{n,n+a_n} > \varepsilon(1+\delta)\sqrt{2a_n d_n}\big) \\ &\geq P\Big(T'_{n,n+a_n} - ET'_{n,n+a_n} > \frac{\varepsilon(1+\delta)}{\sigma\sqrt{(1-\delta)}}\sqrt{2\mathrm{Var}\,(T'_{n,n+a_n})d_n}\Big) \\ &\geq \exp\Big\{-\frac{2\varepsilon^2(1+\delta)^2}{2\sigma^2(1-\delta)}\cdot d_n(1+\gamma)\Big\} \\ &= \exp\Big\{-\frac{\varepsilon^2(1+\delta)^2(1+\gamma)}{\sigma^2(1-\delta)}\cdot d_n\Big\}\,, \quad \text{for } n \text{ large.} \end{aligned} \tag{3.6}$$

3.2 A family of subsequences

In order to choose a suitable subsequence, consider the difference equation $n_{k+1} - n_k = c\,n_k/L(n_k)$ with a suitable constant $c > 0$ to be determined later, or in continuous variables

$$y' = c\,y/L(y)\,. \tag{3.7}$$

With $\varphi(y) = \int^y \frac{L(u)\,du}{u}$ being in the class Π (see [1] for the notation and Thm. 3.7.3) and $\psi(x) = \varphi^{-1}(x)$ being in the class Γ (see [1] for the notation and Thm. 3.10.4) the solution of the differential equation is given by $\psi(cx)$ and the subsequence of interest is $n_k = \psi(c\,k)$. Note that $\frac{n_{k+1}}{n_k} = 1 + \frac{c}{L(n_k)} \to 1$ and that $L(n_{k+1})/L(n_k) \to 1$ as $k \to \infty$.

An important relation in the following is

$$d_{n_k} = \log(L(\psi(ck))\,\log\psi(ck)) \sim \log(ck) \sim \log k \quad \text{as} \quad t\to\infty \quad \text{for any } c > 0, \tag{3.8}$$

which is an immediate consequence of the following result.

Lemma 3.1 *With a slowly varying function $L(\cdot)$ satisfying (1.2) we have*

$$\frac{\log(L(t)\,\log t)}{\log\varphi(t)} \to 1 \quad \text{as} \quad t\to\infty. \tag{3.9}$$

PROOF. With $\varphi^*(t) = L(t)\,\log t$ we have $\varphi(t) \leq \varphi^*(t)$ since $L(\cdot) \nearrow$. Next,

$$\begin{aligned} \varphi^*(t) &= \int_1^t \Big(L'(u)\log u + \frac{L(u)}{u}\Big)\,du = \int_1^t \frac{L'(u)\,u\,L(u)}{L(u)\,u}\int_1^u \frac{1}{v}\,dv\,du + \varphi(t) \\ &\leq \int_1^t \frac{L(u)}{u}\int_1^u \frac{L'(v)}{L(v)}dv\,du + \varphi(t) \leq \varphi(t)(1+\log(L(t)))\,, \end{aligned}$$

where we used condition (1.2). Hence

$$1 \geq \frac{\log\varphi(t)}{\log\varphi^*(t)} \geq 1 - \frac{\log(1+\log L(t))}{\log(L(t)\log t)} \to 1 \quad \text{as} \quad t\to\infty. \qquad \square$$

REMARK 3.1 In the classical proof of the LIL, the subsequence is of geometric increase—$\{\lambda^k,\ k\geq 1\}$ for some λ close to 1. For the LSL, the subsequence is of polynomial increase—$\{(k/\log k)^{1/(1-\alpha)},\ k\geq 1\}$. It is therefore natural, in the present intermediate context, to search for a subsequence with growth rate in between geometric and polynomial, that is, to search for something like $\{\lambda^{k^\beta},\ k\geq 1\}$ for some $\beta\in(0,1)$. For the canonical case, $L(n) = \log n$, it turns out that $n_k \sim e^{c\sqrt{2k}}$.

3.3 Sufficiency along subsequences: $T'_{n,n+a_n}$

The upper bound

Here we use $c > 0$ small. Let $\{n_k = \psi(c\,k),\ k\geq 1\}$, where $n_k \nearrow \infty$ as $k\to\infty$ satisfy

$$\sum_{k=1}^{\infty}\exp\Big\{-\frac{\varepsilon^2(1-\delta)^3}{\sigma^2}\cdot d_{n_k}\Big\} < \infty. \tag{3.10}$$

Applying (3.4) to $\{X'_k,\ k\ge 1\}$ then yields

$$\sum_{k=1}^{\infty} P(|T'_{n_k,n_k+a_{n_k}}|>\varepsilon\sqrt{2a_{n_k}d_{n_k}})<\infty \tag{3.11}$$

for any $\varepsilon>\sigma$. Note that (3.11) is independent of the special choice of $c>0$.

The lower bound

Now we choose the sparser subsequence $\{n_k=\psi(ck),\ k\ge 1\}$, where $c>1$ and $n_k\nearrow\infty$ as $k\to\infty$ satisfying

$$\sum_{k=1}^{\infty}\exp\Big\{-\frac{\varepsilon^2(1+\delta)^2(1+\gamma)}{\sigma^2(1-\delta)}\cdot d_{n_k}\Big\}=\infty \tag{3.12}$$

for any $\varepsilon<\sigma$. Observe that the windows are now nonoverlapping, since $c>1$ implies that $n_{k+1}>n_k+n_k/L(n_k)$ eventually. Applying (3.6) to this sequence similarly shows that

$$\sum_{k=1}^{\infty} P(T'_{n_k,n_k+a_{n_k}}>\varepsilon\sqrt{2a_{n_k}d_{n_k}})=\infty. \tag{3.13}$$

3.4 Sufficiency along subsequences: $T''_{n,n+a_n}$

The next step is to prove the analog of (3.11) for $T''_{n,n+a_n}$, i.e. that

$$\sum_{k=1}^{\infty} P(|T''_{n,n+a_n}|>\delta\sqrt{f(n)})<\infty. \tag{3.14}$$

The symmetric case

We first consider symmetric random variables, and begin by recalling the Kahane-Hoffmann-Jørgensen inequality (see e.g. [5], Theorem 3.7.5):

Lemma 3.2 *Suppose that $X_1,\ X_2,\ \dots,\ X_n$ are independent symmetric random variables with partial sums $S_n,\ n\ge 1$.*
(i) *For any $x,y>0$,*

$$\begin{aligned} P(|S_n|>2x+y) &\le P(\max_{1\le k\le n}|X_k|>y)+4\big(P(|S_n|>x)\big)^2\\ &\le \sum_{k=1}^{n}P(|X_k|>y)+4\big((P(|S_n|>x)\big)^2. \end{aligned}$$

(ii) *If $X_1,\ X_2,\ \dots,\ X_n$ are identically distributed (and $x=y$), then an iteration yields that there are constants $\kappa_i>0,\ i=1,2$, such that*

$$P(|S_n|>9x)\le\kappa_1\, nP(|X_1|>x)+\kappa_2\,\big((P(|S_n|>x)\big)^4. \qquad\square$$

Applying the lemma to $T''_{n,n+a_n}$ we thus obtain, with $\eta=\delta/9$, that

$$\begin{aligned} P(|T''_{n,n+a_n}|>\delta\sqrt{f(n)}) &\le \kappa_1\sum_{k=n+1}^{n+a_n}P(|X''_k|>\eta\sqrt{f(n)})+\kappa_2\,\big(P(|T''_{n,n+a_n}|>\eta\sqrt{f(n)})\big)^4\\ &\le \kappa_1\, a_nP(|X|>\eta\sqrt{f(n)})+\kappa_2\,\big(P(|T''_{n,n+a_n}|>\eta\sqrt{f(n)})\big)^4. \end{aligned} \tag{3.15}$$

Summing over our subsequence for k_0 large (remembering that $d_{n_k} \sim \log k$ as $k \to \infty$), we now have

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} a_{n_k} P(|X| > \eta\sqrt{f(n_k)}) \le \sum_{k=k_0}^{\infty} a_{n_k} P(f^{(-1)}(X^2/\eta^2) > n_k) \\
&= \sum_{k=k_0}^{\infty} \frac{n_k}{L(n_k)} P(f^{(-1)}(X^2/\eta^2) > n_k) \le \int_1^{\infty} \frac{\psi(x)}{L(\psi(x))} P(f^{(-1)}(X^2/\eta^2) > c\,\psi(x))\,dx \\
&\Big[\text{ use } \frac{\psi(x-1)}{\psi(x)} \ge c > 0\,, \quad \text{a change of variable} \quad y = \psi(x), \quad x = \varphi(y), \quad \frac{dx}{dy} = \varphi'(y) = \frac{L(y)}{y}\ \Big] \\
&= \int_C^{\infty} \frac{y}{L(y)} P(f^{(-1)}(X^2/\eta^2) > c\,y) \frac{L(y)}{y}\,dy = \int_C^{\infty} P(f^{(-1)}(X^2/\eta^2) > c\,y)\,dy \\
&\le C\,E\,f^{(-1)}(X^2) < \infty\,, \qquad (3.16)
\end{aligned}$$

which takes care of the first term in the RHS of (3.15).

As for the second one, Chebyshev's inquality tells us that

$$\begin{aligned}
P(|T''_{n,n+a_n}| > \eta\sqrt{f(n)}) &\le \frac{\operatorname{Var} T''_{n,n+a_n}}{\eta^2 f(n)} \le \frac{a_n E\,X^2 I\{b_n < |X| < \delta\sqrt{f(n+n/L(n))}\}}{\eta^2 f(n)} \\
&= \frac{\varepsilon}{\sigma\delta} \frac{E\,X^2 I\{b_n < |X| < \delta\sqrt{f(n+n/L(n))}\}}{\eta^2 d_n} \le \frac{\varepsilon}{\sigma\delta} \frac{E\,X^2}{\eta^2 d_n}
\end{aligned}$$

and, hence that,

$$\big(P(|T''_{n,n+a_n}| > \eta\sqrt{f(n)})\big)^4 \le \Big(\frac{\varepsilon}{\sigma\delta\eta^2 d_n}\Big)^4 (E\,X^2)^3 E\,X^2 I\{b_n < |X| < \delta\sqrt{f(n+n/L(n))}\}\,,$$

so that

$$\begin{aligned}
&\Big(\frac{\sigma\delta\eta^2}{\varepsilon}\Big)^4 \frac{1}{(E\,X^2)^3} \sum_{k=k_0}^{\infty} P(|T''_{n_k,n_k+n_k/L(n_k)}| > \eta\sqrt{f(n_k)}) \qquad (3.17) \\
&\le C \sum_{k=k_0}^{\infty} \frac{E\,X^2 I\{b_{n_k} < |X_k| < \delta\sqrt{f(n_k+n_k/L(n_k))}\}}{d_{n_k}^4} \\
&\le C \sum_{k=k_0}^{\infty} \frac{1}{(\log k)^4} \int_{b_{n_k}}^{\delta\sqrt{f(n_k+n_k/L(n_k))}} x^2\,dF(x) = \int_{k_*}^{\infty} \Big(\sum_{A(k,x)} \frac{1}{(\log k)^4}\Big) x^2\,dF(x), \qquad (3.18)
\end{aligned}$$

where k_* is some lower irrelevant limit, and

$$A(k,x) = \{k : b_{n_k} < |x| < \delta\sqrt{f(n_k+n_k/L(n_k))}\}.$$

In order to invert the double inequality we first observe that in case $f(n) = a_n d_n$ (the case $f(n) = n$ is simpler and the necessary changes are indicated only)

$$a_{n_k+n_k/L(n_k)} = \frac{n_k + n_k/L(n_k)}{L(n_k + n_k/L(n_k))} \le \frac{n_k}{L(n_k)}\big(1 + \frac{1}{L(n_k}\big)\,,$$

and that

$$d_{n_k+n_k/L(n_k)} = \log L(n_k + n_k/L(n_k)) + \log\log(n_k + n_k/L(n_k) \sim \log d_{n_k} \sim \log k \sim \log(\varphi(n_k)),$$

because of the slow variation of L, $\log L$, and $\log x$ and the fact that we have chosen our subsequence via the relation $n_k = \psi(c\,k)$, which implies that $\varphi(n_k) \sim c\,k$.

Exploiting this yields

$$\begin{aligned}
f(n_k + n_k/L(n_k)) &= a_{n_k+n_k/L(n_k)} \cdot d_{n_k+n_k/L(n_k)} \le (1+\delta/2)\frac{n_k}{L(n_k)}\big(1 + \frac{1}{L(n_k}\big) \cdot \log(\varphi(n_k)) \\
&\le \frac{n_k}{L(n_k)}\big(1+\delta\big) \cdot \log(\varphi(n_k)) \quad \text{as} \quad k \to \infty. \qquad (3.19)
\end{aligned}$$

Next, let, for a slowly varying function L, $L^{\#}$ be its de Bruijn conjugate (see e.g. [1], Sec 1.5) obeying

$$L(x\,L^{\#}(x))\,L^{\#}(x) \to 1 \quad \text{and} \quad L(x)\,L^{\#}(x\,L(x)) \to 1 \quad \text{as} \quad x \to \infty\,, \tag{3.20}$$

with its help we can solve $t = \xi\,L(\xi)$ asymptotically by $\xi \sim tL^{\#}(t)$. Now, for evaluating $A(k,x)$ we define

$$L_1(u) = \frac{1}{L(u)\log(\varphi(u))} \quad \text{and} \quad L_2(u) = \frac{\log(\varphi(u))}{L(u)}\,,$$

both of which are slowly varying (in the case $f(n) = n$ we may define $L_2 \equiv 1$), and their de Bruijn conjugates $L_1^{\#}(x)$ and $L_2^{\#}(x)$. Then, with suitable constants c_i, we have

$$\begin{aligned}
A(k,x) &\subset \Big\{k : \Big(\frac{\delta\sigma}{\varepsilon}\Big)^2 \frac{n_k}{L(n_k)\log(\varphi(n_k))} \le x^2 \le \delta^2 \frac{n_k}{L(n_k)}\big(1+\delta\big)\cdot\log(\varphi(n_k))\Big\} \\
&\subset \Big\{k : \Big(\frac{\delta\sigma}{\varepsilon}\Big)^2 \frac{\psi(c\,k)}{L(\psi(c\,k))\log(\varphi(\psi(c\,k)))} \le x^2 \le \delta^2 \frac{\psi(c\,k)}{L(\psi(c\,k))}(1+\delta)\cdot\log(\varphi(\psi(c\,k)))\Big\} \\
&\subset \Big\{k : c_1\,\psi(c\,k)\,L_1(\psi(c\,k)) \le x^2 \le c_2\,\psi(c\,k)\,L_2(\psi(c\,k))\Big\} \\
&\subset \Big\{k : c_3 x^2 L_2^{\#}(x^2) \le \psi(c\,k) = n_k \le c_4 x^2 L_1^{\#}(x^2)\Big\} \\
&\subset \Big\{k : c_5\varphi(x^2 L_2^{\#}(x^2)) \le k \le c_6\varphi(x^2 L_1^{\#}(x^2))\Big\}\,.
\end{aligned}$$

An application of the mean-value theorem and the fact that $\varphi'(x) = L(x)/x \searrow$ as $x \to \infty$ and (3.20) therefore imply that

$$\begin{aligned}
\operatorname{Card}(A(k,x)) &\le c_6\varphi(x^2 L_1^{\#}(x^2)) - c_5\varphi(x^2 L_2^{\#}(x^2)) \\
&\le C\big(\varphi(x^2 L_1^{\#}(x^2)) - \varphi(x^2 L_2^{\#}(x^2))\big) \\
&\le C\varphi'(x^2 L_2^{\#}(x^2))\big(x^2 L_1^{\#}(x^2) - x^2 L_2^{\#}(x^2)\big) \\
&= C\frac{L(x^2 L_2^{\#}(x^2))}{x^2 L_2^{\#}(x^2)}\big(x^2 L_1^{\#}(x^2) - x^2 L_2^{\#}(x^2)\big) \\
&\le C\frac{\log(\varphi(x^2 L_2^{\#}(x^2)))}{L_2(x^2 L_2^{\#}(x^2)))L_2^{\#}(x^2)}\big(L_1^{\#}(x^2) - L_2^{\#}(x^2)\big) \\
&\le C\log(\varphi(x^2 L_2^{\#}(x^2)))L_1^{\#}(x^2)\,.
\end{aligned}$$

Inserting this into the inner sum in (3.18) we now obtain

$$\sum_{A(k,x)} \frac{1}{(\log k)^4} \le C\,\frac{L_1^{\#}(x^2)}{(\log\varphi(x^2 L_1^{\#}(x^2)))^3} \le C\,L_2^{\#}(x^2)\,, \tag{3.21}$$

where the last inequality follows from the fact that $\frac{L_1^{\#}(x^2)}{(\log\varphi(x^2 L_1^{\#}(x^2)))^3} \le C\,L_2^{\#}(x^2)$ since, using (3.20),

$$\begin{aligned}
\frac{L_1^{\#}(x)}{L_2^{\#}(x)} &\sim \frac{L_2(x\,L_2^{\#}(x))}{L_1(x\,L_1^{\#}(x))} \\
&\le \frac{L(x\,L_1^{\#}(x))}{L(x\,L_2^{\#}(x))}\cdot\big(\log(\varphi(xL_1^{\#}(x)))\big)^2 \\
&\le C\,\big(\log(\varphi(xL_1^{\#}(x)))\big)^2\exp\Big(\int_{xL_2^{\#}(x)}^{xL_1^{\#}(x)} \varepsilon(t)/t\,dt\Big) \\
&\le C\,\Big(\log(\varphi(xL_1^{\#}(x)))\Big)^2\exp\Big(o(1)\log\Big(\frac{L_1^{\#}(x)}{L_2^{\#}(x)}\Big)\Big)\,,
\end{aligned}$$

by the representation theorem for slowly varying functions. (In case $f(n) = n$ the inequality (3.21) is trivial since $L_2^{\#} \equiv 1$ and $L_1^{\#}$ is decreasing.) Finally, using the fact that $f^{-1}(x) \sim xL_2^{\#}(x)$, we conclude that the sum in (3.18) converges, which takes care of the second sum in (3.15).

Combining this with (3.16) proves the validity of (3.14) in the symmetric case.

Desymmetrization

In order to prove (3.14) for the general case we first estimate the truncated means. Remembering that $E\,X_k = 0$ for all k we obtain, by stretching the bounds to the extreme,

$$\begin{aligned} |E\,T''_{n,n+a_n}| &= |\sum_{k=n+1}^{n+a_n} E\,X_k I\{b_k < |X_k| < \delta\sqrt{f_k}\}| \\ &\le \sum_{k=n+1}^{n+a_n} E|X_k| I\{b_n < |X_j| < \delta\sqrt{f(n+n/L(n))}\} \\ &\le a_n E|X| I\{|X| \ge b_n\} \le \frac{a_n}{b_n}\, E\,X^2 I\{|X| \ge b_n\} \\ &= \frac{\varepsilon}{\sigma\delta}\sqrt{a_n d_n}\, E\,X^2 I\{|X| \ge b_n\} = o(\sqrt{a_n d_n}) \quad \text{as} \quad n \to \infty, \end{aligned}$$

since this is the same estimate as for $E\,T'_{n,n+a_n}$, after which the desired conclusion follows with the aid of the symmetrization inequalities (cf. [5], Proposition 3.6.2).

3.5 Sufficiency along subsequences: $T'''_{n,n+a_n}$

In order for $|T'''_{n,n+a_n}|$ to surpass the level $\eta\sqrt{a_n d_n}$ it is necessary that at least one of the X''':s is nonzero. For every $\eta > 0$ (recall that $a_{n_k} = n_k/L(n_k)$, $d_{n_k} \sim \log k$) this means that

$$\begin{aligned} \sum_{k=1}^{\infty} P(|T'''_{n_k,n_k+n_k/L(n_k)}| > \eta\sqrt{a_{n_k} d_{n_k}}) &\le \sum_{k=1}^{\infty}\sum_{j=1}^{a_{n_k}} P(|X_{n_k+j}| > \frac{\eta}{2}\sqrt{f(n_k+j)}) \\ \le \sum_{k=1}^{\infty} a_{n_k} P(|X| > \frac{\eta}{2}\sqrt{f(n_k)}) < \infty, & \end{aligned} \tag{3.22}$$

by (3.16).

3.6 Sufficiency along subsequences: combining the contributions

Combining (3.11), (3.14) and (3.22) we conclude that

$$\sum_{k=1}^{\infty} P(|T_{n_k,n_k+n_k/L(n_k)}| > (\varepsilon + 2\eta)\sqrt{2a_{n_k} d_{n_k}}) < \infty \tag{3.23}$$

provided $\varepsilon > \sigma/(1-\delta)^{3/2}$, and since η and δ may be arbitrarily chosen, that

$$\sum_{k=1}^{\infty} P(|T_{n_k,n_k+n_k/L(n_k)}| > \varepsilon\sqrt{2a_{n_k} d_{n_k}}) < \infty \quad \text{for} \quad \varepsilon > \sigma, \tag{3.24}$$

so that, in view of the first Borel-Cantelli lemma,

$$\limsup_{k\to\infty} \frac{T_{n_k,n_k+n_k/L(n_k)}}{\sqrt{2a_{n_k} d_{n_k}}} \le \sigma \quad \text{a.s.} \tag{3.25}$$

A completely analogous argument, combining (3.13), (3.14) and 3.22), yields

$$\sum_{k=1}^{\infty} P(T_{n_k,n_k+n_k/L(n_k)} > \varepsilon\sqrt{2a_{n_k} d_{n_k}}) = \infty \quad \text{for} \quad \varepsilon < \sigma, \tag{3.26}$$

and since the windows with this, sparser subsequence, are disjoint we may apply the second Borel-Cantelli lemma to conclude that

$$\limsup_{k\to\infty} \frac{T_{n_k, n_k+n_k/L(n_k)}}{\sqrt{2a_{n_k} d_{n_k}}} \geq \sigma \quad \text{a.s.} \tag{3.27}$$

Finally, combining (3.25) and (3.27) yields

$$\limsup_{k\to\infty} \frac{T_{n_k, n_k+n_k/L(n_k)}}{\sqrt{2a_{n_k} d_{n_k}}} = \sigma \quad \text{a.s.}, \tag{3.28}$$

which, in addition proves the sufficiency of the following result which is Theorem 2.1 for subsequences of the form $n_k = \psi(ck)$ with $c > 1$. The necessity follows, of course, from the necessity for the full sequence, the proof of which is given in Section 3.8 below.

Theorem 3.1 *Suppose that* X, X_1, X_2, ... *are i.i.d. random variables with mean* 0 *and finite variance* σ^2*, set* $T_n = \sum_{j=n+1}^{n+k} X_j$*, and let, for* $c > 1$*,*

$$n_k = \varphi^{(-1)}(ck), \quad k \geq 1,$$

where $\varphi^{(-1)}$ *is the inverse of* $\varphi(y) = \int^y \frac{L(u)}{u}\,du$*. If (2.1) holds, then*

$$\limsup_{n\to\infty} \frac{T_{n_k+n_k/L(n_k)}}{\sqrt{2\frac{n_k}{L(n_k)}\log k}} = \sigma \quad a.s. \tag{3.29}$$

Conversely, if

$$P\Big(\limsup_{n\to\infty} \frac{|T_{n_k+n_k/L(n_k)}|}{\sqrt{\frac{n_k}{L(n_k)}\log k}} < \infty\Big) > 0, \tag{3.30}$$

then (2.1) holds, $E\,X = 0$*,* $E\,X^2 < \infty$*, and (3.29) holds with* $\sigma^2 = \operatorname{Var} X$*.*

3.7 Sufficiency for the entire sequence

We thus have to show that our process behaves accordingly for the entire sequence. Here the second Lévy inequality (see e.g. [5], Theorem 3.7.2) is instrumental. Let $\eta > 0$ be given. Then

$$\begin{aligned} P\left(\max_{n_k\leq n\leq n_{k+1}} \frac{S_{n+a_n} - S_n}{\sqrt{2a_n\, d_n}} > (1+6\,\eta)\sigma\right) &\leq P(\max_{n_k\leq n\leq n_{k+1}} (S_{n+a_n} - S_{n_k+a_{n_k}}) > 2\eta\,\sigma\sqrt{2a_{n_k}\, d_{n_k}}) \\ &+P(\max_{n_k\leq n\leq n_{k+1}} (-S_n + S_{n_k}) > 2\eta\,\sigma\sqrt{2a_{n_k}\, d_{n_k}}) \\ &+P(\max_{n_k\leq n\leq n_{k+1}} (S_{n_k+a_{n_k}} - S_{n_k}) > (1+2\eta)\,\sigma\sqrt{2a_{n_k} d_{n_k}})\,, \end{aligned} \tag{3.1}$$

where $n_k = \psi(c\,n_k)$ as before with some suitable constant $c > 0$ to be fixed in a moment.

Set $\tilde{n_k} = n_k + a_{n_k}$. Since $n_{k+1}/n_k \to 1$ and $L(\cdot) \in \mathcal{SV}$, the following relations hold eventually (i.e. for k sufficiently large):

$$\begin{aligned} n_{k+1} - n_k &\leq c\psi'(c(k+1)) = c\frac{n_{k+1}}{L(cn_k)} \leq 2c\,a_{n_k}\,, \\ n_k &\leq \tilde{n_k} = n_k(1 + (L(n_k))^{-1}) \leq n_k(1+\eta)\,, \\ a_{n_k} &\leq a_{\tilde{n_k}} \leq a_{n_k(1+\eta)} \leq (1+\eta)a_{n_k}\,, \\ \tilde{n_{k+1}} - \tilde{n_k} &\leq (n_{k+1} - n_k)(1 + (L(n_k))^{-1}) \leq 2c(1+\eta)\,a_{n_k} \leq 2c(1+\eta)\,a_{\tilde{n_k}}\,, \\ d_{n_k} &\leq d_{\tilde{n_k}} \leq (1+\eta)d_{n_k}\,. \end{aligned}$$

In the following we exploit these relations without specific mentioning each time.

As a first application we note that (3.1) can be bounded by

$$\begin{aligned}\leq P(&\max_{\tilde{n_k}\leq n\leq \tilde{n_k}+2c\,(1+\eta)\,a_{\tilde{n_k}}}(S_n - S_{\tilde{n_k}}) > 2\eta\,\sigma\sqrt{2a_{n_k}\,d_{n_k}})\\ &+P(\max_{n_k\leq n\leq n_k+2c\,a_{n_k}}(-S_n + S_{n_k}) > 2\eta\,\sigma\sqrt{2a_{n_k}\,d_{n_k}})\\ &+P(\max_{n_k\leq n\leq n_{k+1}}(S_{n_k+a_{n_k}} - S_{n_k}) > (1+2\eta)\,\sigma\sqrt{2a_{n_k}d_{n_k}})\,. \end{aligned}\tag{3.2}$$

Now,

$$\begin{aligned}\mathrm{Var}\,(S_{\tilde{n_k}+2c\,(1+\eta)\,a_{\tilde{n_k}}} - S_{\tilde{n_k}}) &= 2c\,(1+\eta)\,a_{\tilde{n_k}}\sigma^2 = o(a_{n_k}d_{n_k}) \quad \text{as} \quad k\to\infty,\\ \mathrm{Var}\,(S_{n_k+2c\,a_{n_k}} - S_{n_k}) &= 2c\,a_{n_k}\sigma^2 = o(a_{n_k}d_{n_k}) \quad \text{as} \quad k\to\infty,\\ \mathrm{Var}\,(S_{n_k+a_{n_k}} - S_{n_k}) &= a_{n_k}\sigma^2 = o(a_{n_k}d_{n_k}) \quad \text{as} \quad k\to\infty,\end{aligned}$$

that is, the variances are $\leq \eta^4\sigma^2 a_{n_k}d_{n_k}$ for k sufficiently large.

An application of the Lévy inequality to the first two probabilities in (3.2), leaving the third one as is, then shows that (3.2) can be bounded by

$$\begin{aligned}\leq\quad & 2\,P((S_{\tilde{n_k}+2c\,(1+\eta)\,a_{\tilde{n_k}}} - S_{\tilde{n_k}}) > 2\eta\sigma\sqrt{2a_{n_k}d_{n_k}} - \sqrt{2}\cdot\eta^2\sigma\sqrt{a_{n_k}d_{n_k}})\\ &+2\,P(-(S_{n_k+2c\,a_{n_k}} - S_{n_k}) > 2\eta\sigma\sqrt{2a_{n_k}d_{n_k}} - \sqrt{2}\cdot\eta^2\sigma\sqrt{a_{n_k}d_{n_k}})\\ &+2\,P((S_{n_k+a_{n_k}} - S_{n_k}) > (1+\eta)\sigma\,\sqrt{2a_{n_k}d_{n_k}})\\ \leq\quad & 2\,P((S_{\tilde{n_k}+2c\,(1+\eta)\,a_{\tilde{n_k}}} - S_{\tilde{n_k}}) > \eta\sigma\sqrt{2a_{n_k}d_{n_k}})\\ &+2\,P(-(S_{n_k+2c\,(1+\eta)\,a_{n_k}} - S_{n_k}) > \eta\sigma\sqrt{2a_{n_k}d_{n_k}})\\ &+2\,P((S_{n_k+a_{n_k}} - S_{n_k}) > (1+\eta)\sigma\,\sqrt{2a_{n_k}d_{n_k}})\\ \leq\quad & 2\,P(|S_{\tilde{n_k}+2c\,(1+\eta)\,a_{\tilde{n_k}}} - S_{\tilde{n_k}}| > \frac{\eta}{\sqrt{2c(1+\eta)^3}}\,\sigma\sqrt{2\cdot 2c\,(1+\eta)\,a_{\tilde{n_k}}d_{\tilde{n_k}}})\\ &+2\,P(|S_{n_k+2c\,a_{n_k}} - S_{n_k}| > \frac{\eta}{\sqrt{2c}},\sigma\sqrt{2\cdot 2c\,a_{n_k}d_{n_k}})\\ &+2\,P(|S_{n_k+a_{n_k}} - S_{n_k}| > (1+\eta)\,\sigma\sqrt{2a_{n_k}d_{n_k}})\,.\end{aligned}\tag{3.3}$$

Summing the three probabilities over k and recalling (3.24) tells us that the total sum converges whenever

$$\min\Big\{\frac{\eta}{\sqrt{2c(1+\eta)^3}}, \frac{\eta}{\sqrt{2c}}, 1+\eta\Big\} > 1\,.$$

Since we can choose $c>0$ arbitrarily small we finally conclude that for any $\eta>0$ we have

$$\sum_k P\Big(\max_{n_k\leq n\leq n_{k+1}}\frac{S_{n+a_n}-S_n}{\sqrt{2a_n\,d_n}} > (1+6\,\eta)\sigma\Big) < \infty\,,$$

implying the upper inequality for the entire sequence, that is,

$$\limsup_{n\to\infty}\frac{T_{n,n+n/L(n)}}{\sqrt{2a_nd_n}} \leq \sigma \qquad \text{a.s.}$$

3.8 Necessity

By the zero-one law, the probability that the lim sup is finite is 0 or 1, hence, being positive it equals 1. Consequently, (cf. [9], p. 438),

$$\limsup_{n\to\infty}\frac{|X_n|}{\sqrt{a_nd_n}} < \infty \qquad \text{a.s.},$$

from which, in case $a_n\,d_n \le f(n)$, it follows, via the second Borel-Cantelli lemma and the i.i.d. assumption, that

$$\sum_{n=1}^{\infty} P(|X_n| > \sqrt{f(n)}) = \sum_{n=1}^{\infty} P(X^2 > f(n)),$$

which verifies (2.1).

If (2.1) is weaker than finite variance, i.e. if $a_n\,d_n \ge f(n)$, then, by following Feller's proof [4] (see also e.g. [5], Section 8.4) of the necessity in the LIL with only obvious changes, amounting to replacing sums by windows, we may conclude that (2.1)—i.e. finite variance—holds also in this case.

An application of the sufficiency part then tells us that (2.1) holds with $\sigma^2 = \operatorname{Var} X$.

Finally, if $E\,X = \mu$, then, by the law of large numbers, $S_n/f(n) \sim \mu n/f(n) \to \mu\cdot\infty$ as $n\to\infty$, which forces μ to be equal to zero.

4 Proofs of the Corollaries in Section 2

4.1 Proof of Corollary 2.1

In this case $a_n = n/(\log n)^p$ (for $n \ge 9$), so that

$$d_n = \log\frac{n}{n/(\log n)^p} + \log\log n = (p+1)\log\log n,$$

that is, $f(n) = (p+1)n\log\log n/(\log n)^p$. It follows that $f^{-1}(n) \sim \frac{n\,(\log n)^p}{(p+1)\log\log n}$ as $n\to\infty$, so that (2.1) turns out as

$$E\,X^2\,\frac{(\log^+|X|)^p}{\log^+\log^+|X|} < \infty.$$

It remains to verify that $xL'(x)/L(x)$ is decreasing. Now, $L(x) = (\log x)^p$ and $L'(x) = x^{-1}p(\log x)^{p-1}$, so that $xL'(x)/L(x) = p(\log x)^{-1}$, which indeed decreases.

4.2 Proof of Corollary 2.2

Thus, $a_n = n/\log_m(n)$, for n sufficiently large, so that

$$d_n = \log\log_m(n) + \log\log n = \log_{m+1}(n) + \log\log n \sim \log\log n \quad \text{as} \quad n\to\infty,$$

Since $a_n d_n > n$, we have $f^{-1}(n) = n$ as $n\to\infty$, which implies that finite variance is the appropriate necessary assumption.

As for (1.2), this time $L(x) = \log_m x$ and $L'(x) = x^{-1}\prod_{i=1}^{m-1}(\log_i x)^{-1}$, so that $xL'(x)/L(x) = \prod_{i=1}^{m}(\log_i x)^{-1}$, which indeed decreases.

5 Further examples

In this section we provide some additional examples to illustrate Theorem 2.1. As in Section 2 we omit stating converse results.

The first example mixes powers of logarithms and iterated logarithms.

Example 5.1 Let, for $n \ge 9$, $a_n = n(\log\log n)^q/(\log n)^p$, $p,\,q > 0$, which means that the slowly varying function part is $L(n) = (\log n)^p/(\log\log n)^q$. Differentiation and some algebraic simplification yield that $xL'(x)/L(x) = p/\log x - q/(\log x\log_2 x)$, which is ultimately decreasing. Moreover,

$$d_n = \log\Big(\frac{n(\log\log n)^q}{n/(\log n)^p}\Big) + \log\log n = (p+1)\log\log n - q\log\log\log n \sim (p+1)\log\log n \quad \text{as} \quad n\to\infty,$$

so that, $f(n) = (p+1)n(\log\log n)^{q+1}/(\log n)^p$ which implies that $f^{-1}(n) \sim Cn(\log n)^p/(\log\log n)^{q+1}$ as $n\to\infty$. The following result emerges.

Corollary 5.1 *Suppose that $X, X_1, X_2, \dots$ are i.i.d. random variables with mean 0 and finite variance σ^2, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. If, for some $p,\ q > 0$,*

$$E\,X^2 \frac{(\log^+ |X|)^p}{(\log^+ \log^+ |X|)^{q+1}} < \infty, \tag{5.1}$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+n(\log\log n)^q/(\log n)^p}}{\sqrt{2(p+1)\frac{n}{(\log n)^p}(\log\log n)^{q+1}}} = \sigma \qquad a.s. \tag{5.2}$$

The previous conclusion holds, in fact, also for $q = 0$, in which case Corollary 5.1 reduces to Corollary 2.1, since the loglog-contribution is of a lower order of magnitude. However, the case $p = 0$ requires a separate treatment.

Example 5.2 Let, for $n \geq 9$, $a_n = n/(\log\log n)^q$, $q > 1$. Now, $L(x) = (\log_2 x)^q$ gives $xL'(x)/L(x) = q/(\log x \log_2 x)$, which is decreasing. Moreover,

$$d_n = \log\Big(\frac{n}{n/(\log\log n)^q}\Big) + \log\log n = q\log\log\log n + \log\log n \sim \log\log n \quad \text{as} \quad n \to \infty,$$

that is, $f(n) = n(\log\log n)^{1-q}$, and $f^{-1}(n) \sim n(\log\log n)^{q-1}$ as $n \to \infty$.

Corollary 5.2 *Suppose that $X, X_1, X_2, \dots$ are i.i.d. random variables with mean 0 and finite variance σ^2, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. If, for some $q > 1$,*

$$E\,X^2(\log^+ \log^+ |X|)^{q-1} < \infty, \tag{5.3}$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+n/(\log\log n)^q}}{\sqrt{2n(\log\log n)^{1-q}}} = \sigma \qquad a.s. \tag{5.4}$$

Example 5.3 Let $a_n = n/\exp\{\sqrt{\log n}\}$ $n \geq 1$. Since $L(x) = \exp\{\sqrt{\log x}\}$, we have $xL'(x)/L(x) = (\log x)^{-1/2}/2$, which is decreasing. Moreover,

$$d_n = \log\exp\{\sqrt{\log n}\} + \log\log n = \sqrt{\log n} + \log\log n \sim \sqrt{\log n} \quad \text{as} \quad n \to \infty,$$

which gives $f(n) \sim n\sqrt{\log n}/\exp\{\sqrt{\log n}\}$ as $n \to \infty$, so that

$$f^{-1}(n) \sim n\exp\{\sqrt{\log n + 1/2}\}/\sqrt{\log n} \quad \text{as} \quad n \to \infty.$$

The following conclusion therefore holds.

Corollary 5.3 *Suppose that $X, X_1, X_2, \dots$ are i.i.d. random variables with mean 0 and finite variance σ^2, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. If*

$$E\,X^2 \frac{\exp\{\sqrt{2\log^+ |X|}\}}{\sqrt{\log^+ |X|}} < \infty, \tag{5.5}$$

then

$$\limsup_{n\to\infty} \frac{T_{n,n+n/\exp\{\sqrt{\log n}\}}}{\sqrt{2\frac{n}{\exp\{\sqrt{\log n}\}}\sqrt{\log n}}} = \sigma \qquad a.s. \tag{5.6}$$

The following example is a more general version.

Example 5.4 Let, for $n \geq 1$, $a_n = n(\log n)^{\gamma}/\exp\{(\log n)^{\beta}\}$, where $0 < \beta < 1$ and $\gamma \in \mathbb{R}$. Thus, $L(x) = (\log x)^{-\gamma}\exp\{(\log x)^{\beta}\}$, and $xL'(x)/ = \beta(\log x)^{\beta-1} - \gamma/\log x$, which is ultimately decreasing. Furthermore,

$$d_n = \log\exp\{(\log n)^{\beta}\} - \log\log\big((\log n)^{\gamma}\big) + \log\log n = (\log n)^{\beta} - \log\gamma \sim (\log n)^{\beta} \quad \text{as} \quad n \to \infty,$$

which gives

$$f(n) \sim n(\log n)^{\beta+\gamma}/\exp\{(\log n)^{\beta}\} \quad \text{as} \quad n \to \infty.$$

It follows that, for $0 < \beta < 1/2$,

$$f^{-1}(n) \sim n\frac{\exp\{(\log n)^{\beta}\}}{(\log n)^{\beta+\gamma}} \quad \text{as} \quad n \to \infty,$$

and for $1/2 \leq \beta < 2/3$,

$$f^{-1}(n) \sim n\frac{\exp\{(\log n)^{\beta} + \beta(\log x)^{2\beta-1}\}}{(\log n)^{\beta+\gamma}} \quad \text{as} \quad n \to \infty,$$

and so on. The following conclusion holds.

Corollary 5.4 *Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables with mean* 0 *and finite variance σ^2, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. If $0 < \beta < 1/2$ and*

$$E\,X^2\frac{\exp\{(2\log^+|X|)^{\beta}\}}{(\log^+|X|)^{\beta+\gamma}} < \infty, \tag{5.7}$$

then

$$\limsup_{n\to\infty}\frac{T_{n,n+n(\log n)^{\gamma}/\exp\{(\log n)^{\beta}\}}}{\sqrt{2\frac{n(\log n)^{\gamma+\beta}}{\exp\{(\log n)^{\beta}\}}}} = \sigma \quad a.s. \tag{5.8}$$

6 A simplified Theorem 2.1

As mentioned in the introduction, this section concerns a weaker result, the proof of which is *much* easier in that only one truncation is made. However, minimal moment conditions are not obtained.

Theorem 6.1 *Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables with mean* 0 *and finite variance σ^2, and set $T_{n,n+k} = \sum_{j=n+1}^{n+k} X_j$. Define sequences $\{a_n\}$ and $\{d_n\}$ as in Section 2 and set*

$$b_n = \sqrt{\frac{a_n}{d_n}}.$$

If $E\,X = 0$ and

$$E\,b^{-1}(|X|) < \infty, \tag{6.1}$$

then

$$\limsup_{n\to\infty}\frac{T_{n,n+a_n}}{\sqrt{2a_nd_n}} = \sigma \quad a.s. \tag{6.2}$$

6.1 Proof of Theorem 6.1

Set

$$X_n'' = X_nI\{|X_n| > \frac{\sigma\delta}{\varepsilon}b_n\} = X_n - X_n'.$$

The contribution of $T'_{n,n+a_n}$

This part requires no change. In other words, we first let $\{n_k,\, k \geq 1\}$, where $n_k \nearrow \infty$ as $k \to \infty$ satisfy (3.10), apply (3.4) to $\{X_{n_k},\, k \geq 1\}$ to obtain

$$\sum_{k=1}^{\infty} P(|T'_{n_k,n_k+a_{n_k}}| > \varepsilon\sqrt{2a_{n_k}d_{n_k}}) < \infty\,,$$

for the convergence part

After this we choose a sparser subsequence $\{n_k,\, k \geq 1\}$, where $n_k \nearrow \infty$ as $k \to \infty$ satisfying (3.12) and apply (3.6) to obtain

$$\sum_{k=1}^{\infty} P(T'_{n_k,n_k+a_{n_k}} > \varepsilon\sqrt{2a_{n_k}d_{n_k}}) = \infty\,,$$

for the divergence part.

The contribution of $T''_{n,n+a_n}$

Since the X'':s have changed, we can use the stronger LSL-argument here (cf. [6, 7]). Namely, in order for the $|T''_{n,n+a_n}|$:s to surpass the level $\eta\sqrt{a_n d_n}$ infinitely often it is necessary that infinitely many of the X'':s are nonzero. However, the latter event has zero probability in view of the first Borel-Cantelli lemma, since

$$\sum_{n=1}^{\infty} P(|X_n| > \frac{\sigma\delta}{\varepsilon}b_n) = \sum_{n=1}^{\infty} P(|X| > \frac{\sigma\delta}{\varepsilon}b_n) < \infty$$

if and only if (6.1) holds.

Finishing the proof

Departing from this point the arguments are identical to those of the proof of Theorem 2.1.We therefore omit the details.

6.2 Revisiting some examples

Corollary 2.1 revisited for $p = 1$. With $a_n = n/\log n$ and $b_n = \sqrt{\frac{n/\log n}{\log\log n}}$, the conclusion is, once again,

$$\frac{T_{n,n+n/\log n}}{\sqrt{4\frac{n}{\log n}\log\log n}} = \sigma \qquad \text{a.s.,}$$

however, provided (6.1) holds, that is, provided

$$E\big(X^2\log^+|X|\log^+\log^+|X|\big) < \infty\,,$$

since $b^{-1}(n) \sim n^2\log n\log\log n$. This should be compared with (2.4),

$$E\,X^2\frac{\log^+|X|}{\log^+\log^+|X|} < \infty\,.$$

Corollary 2.2 revisited for $m = 2$. With $m = 2$ and $a_n = n/\log\log n$ and $b_n = \sqrt{\frac{n/\log\log n}{\log\log n}}$, the conclusion is, once again,

$$\frac{T_{n,n+n/\log\log n}}{\sqrt{2n}} = \sigma \qquad \text{a.s.,}$$

however, provided (6.1) holds, that is, provided

$$E\,X^2(\log^+\log^+|X|)^2 < \infty\,,$$

which should be compared with the optimal one which was finite variance.

Corollary 5.1 revisited. Here $a_n = n(\log\log n)^q/(\log n)^p$, p, $q > 0$. With

$$b_n = \sqrt{\frac{n(\log\log n)^q/(\log n)^p}{\log\log n}} = \sqrt{\frac{n(\log\log n)^{q-1}}{(\log n)^p}}$$

assumption (6.1) turns out as

$$E\,X^2 \frac{(\log^+|X|)^p}{(\log^+\log^+|X|)^{q-1}} < \infty,$$

to be compared with the weaker

$$E\,X^2 \frac{(\log^+|X|)^p}{(\log^+\log^+|X|)^{q+1}} < \infty.$$

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