

# Asymptotic distributions for a class of generalized L-statistics

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## Abstract

We adapt the techniques in Stigler [23] to obtain a new, general asymptotic result for trimmed  $U$ -statistics via the generalized  $L$ -statistic representation introduced by Serfling [18]. Unlike existing results we do not require continuity of an associated distribution at the truncation points. Our results are quite general and are expressed in terms of the quantile function associated with the distribution of the  $U$ -statistic summands. This approach leads to improved conditions for the asymptotic normality of these trimmed  $U$ -statistics.

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# 1 Introduction and statement of results

Stigler [23] developed an asymptotic result for the trimmed mean without requiring continuity of the underlying distribution function associated with the observations. This result was extended to non-degenerate  $U$ -statistics based on trimmed samples in Borovskikh and Weber [4]. An alternative method for developing robust versions of  $U$ -statistics is to consider the statistic formed by trimming the kernel values rather than the observations upon which the statistic is based. This idea is discussed in, for example, Serfling [18], Choudhury and Serfling [7] and Gijbels, Janssen and Veraverbeke [10]. In this paper we use the generalized  $L$ -statistic representation developed in Serfling [18] to obtain an asymptotic result for trimmed  $U$ -statistics under quite general conditions. We will not require continuity of the relevant, associated distribution at the truncation points.

Let  $X, X_1, \dots, X_n$  be independent identically distributed random variables, taking values in a measurable space  $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$  and having common distribution  $F$ . Let  $h$  be a symmetric function from  $\mathbf{X}^m$  to  $R$  and denote by  $H_F$  the right continuous distribution function of the random variable  $h(X_1, \dots, X_m)$ . Set  $N = \binom{n}{m}$  and let  $h_1, \dots, h_N$  be an enumeration of the values of  $h(X_{i_1}, \dots, X_{i_m})$  taken over the  $N$   $m$ -tuples in  $\sigma_{nm} = \{(i_1, \dots, i_m) : 1 \leq i_1 < \dots < i_m \leq n\}$ . Note that these random variables  $h_i$  are in general dependent. Let  $h_{n1} \leq \dots \leq h_{nN}$  denote the ordered values of  $h_1, \dots, h_N$ .

The original  $U$ -statistic is defined as an average taken over the  $N$  possible outcomes  $h(X_{i_1}, \dots, X_{i_m})$ ,  $1 \leq i_1 < \dots < i_m \leq n$ , that is

$$\begin{aligned} U &= \binom{n}{m}^{-1} \sum_{\sigma_{nm}} h(X_{i_1}, \dots, X_{i_m}) \\ &= N^{-1} \sum_{i=1}^N h_{ni} = \int_R x dH_n(x), \end{aligned} \tag{1}$$

where the empirical distribution function  $H_n(x)$  of  $U$ -statistical structure is defined by

$$H_n(y) = \binom{n}{m}^{-1} \sum_{\sigma_{nm}} I\{h(X_{i_1}, \dots, X_{i_m}) \leq y\}, \quad y \in R, \tag{2}$$

and  $I\{A\}$  denotes the indicator of the set  $A$ . For any  $0 < \gamma < 1$  let  $N_\gamma = [\gamma N]$ , where  $[a]$  denotes the largest integer less than or equal to  $a$ . If  $0 < \alpha < \beta < 1$  then put  $N_{\alpha\beta} = N_\beta - N_\alpha$ . The trimmed versions of  $U$  are based on trimming the second sum in (1)

$$U_{\alpha\beta} = N_{\alpha\beta}^{-1} \sum_{i=N_\alpha+1}^{N_\beta} h_{ni} \tag{3}$$

or on trimming of the range of integration in (1)

$$L_{\alpha\beta} = \int_{[h_\alpha, h_\beta)} x dH_n(x) \quad (4)$$

with  $h_\alpha = h_{n\bar{N}_\alpha}$  and  $h_\beta = h_{n\bar{N}_\beta}$ , where  $\bar{N}_\gamma = -[-\gamma N]$ ,  $\gamma = \alpha, \beta$ . For the results that follow it is important to note that the lower bound for the integral in (4) is included and the upper bound excluded. This is critical since  $H_n$  is a step function. With this constraint we are able to obtain the asymptotic distribution of  $L_{\alpha\beta}$  without imposing any conditions on the nature of  $H_F$ . In Lemma 2.3 below we show that  $L_{\alpha\beta} = N^{-1} \sum_{i=\bar{N}_\alpha}^{\bar{N}_\beta-1} h_{ni}$ . Thus  $U_{\alpha\beta}$  and  $L_{\alpha\beta}$  differ in terms of their divisors, and possible subtle differences in the number of summands.

A class generalized  $L$ -statistics, which includes (3) and (4), was introduced by Serfling [18]. The trimmed  $U$ -statistics (3) and (4) are directly connected with generalized Lorenz curves, which are important in financial mathematics (see, for example, Goldie [9], Helmers and Zitikis [13]).

Clearly,  $H_n(y)$  is an unbiased estimator of  $H_F(y)$ . In the case  $m = 1$  and  $h(x) = x$ ,  $H_n$  reduces to the usual empirical distribution function. Define the left-continuous quantile function  $H_F^{-1}(t) = \inf\{y \in R : H_F(y) \geq t\}$ ,  $0 < t \leq 1$ ,  $H_F^{-1}(0) = H_F^{-1}(0+)$ , for any distribution function  $H_F$ . The empirical quantile function  $H_n^{-1}(t)$  has the form

$$H_n^{-1}(t) = \sum_{i=1}^N h_{ni} I\left\{\frac{i-1}{N} < t \leq \frac{i}{N}\right\}, \quad H_n^{-1}(0) = h_{n1}.$$

A large number of authors have studied the weak convergence of such  $L$ -statistics in the case  $m = 1$ ,  $h(x) = x$ . A partial list consists of Chernoff *et al.* [6], Bickel [2], Shorack [19, 20], Stigler [23, 24], Csörgo *et al.* [8], Griffin and Pruitt [11], Cheng [5], Mason and Shorack [16]. For  $m \geq 2$  under various sets of regularity conditions, asymptotic normality of various types of generalized  $L$ -statistics has been investigated by Silverman [21], Serfling [18], Akritas [1], Janssen *et al.* [15], Helmers and Ruymgaart [12], Gijbels *et al.* [10], Hössjer [14].

In the above-mentioned papers for  $m \geq 2$  the results always assumed that  $H_F$  is continuous or smooth. However in modern statistical robust procedures and for bootstrap procedures results allowing for the discontinuity of the underlying distribution function  $H_F$  are needed. We study the asymptotic behaviour of  $U_{\alpha\beta}$  and  $L_{\alpha\beta}$  for any  $H_F$  without imposing the requirement of continuity.

The conditions of our theorem and the limit random variable are defined via the values of quantile function  $H_F^{-1}$  at the points  $\alpha$  and  $\beta$ . Existing results handle the cases where  $H_F^{-1}(\gamma+) = H_F^{-1}(\gamma)$ ,  $\gamma = \alpha, \beta$ . Our main result is

derived without this assumption of continuity. We represent the trimmed  $U$ -statistic as a sum of classical  $U$ -statistics with bounded, non-degenerate kernels plus some smaller terms and then we apply standard results to such statistics.

For convenience in what follows, for the distribution function  $H_F$ , we denote the smallest quantile  $H_F^{-1}(\gamma)$  and the largest quantile  $H_F^{-1}(\gamma+)$ , respectively, as

$$\xi_\gamma^- := \inf\{x \in R : H_F(x) \geq \gamma\}, \quad \xi_\gamma^+ := \sup\{x \in R : H_F(x) \leq \gamma\}$$

and  $\Delta\xi_\gamma = \xi_\gamma^+ - \xi_\gamma^-$  with  $\gamma = \alpha, \beta$ . Let

$$\dot{N}_\gamma^\pm = \sum_{i=1}^N I\{h_i < \xi_\gamma^\pm\}, \quad N_\gamma^\pm = \sum_{i=1}^N I\{h_i \leq \xi_\gamma^\pm\}.$$

Note that

$$H_n(\xi_\gamma^\pm-) = N^{-1}\dot{N}_\gamma^\pm, \quad H_n(\xi_\gamma^\pm) = N^{-1}N_\gamma^\pm, \quad (5)$$

and  $H_n^{-1}(\gamma) = h_{n\bar{N}_\gamma}$  are valid for all  $0 < \gamma < 1$  and the following events coincide

$$\{H_n^{-1}(\gamma) > x\} = \{\gamma > H_n(x)\}, \quad \{H_n^{-1}(\gamma) \leq x\} = \{\gamma \leq H_n(x)\}, \quad x \in R. \quad (6)$$

Introduce the functional  $\theta = \theta(H_F)$  where

$$\theta = \int_R [((x - \xi_\beta^-)I\{x \leq \xi_\beta^-\} + \beta\xi_\beta^-) - ((x - \xi_\alpha^+)I\{x < \xi_\alpha^+\} + \alpha\xi_\alpha^+)] dH_F(x)$$

and the following functions with  $x \in \mathbf{X}$

$$\begin{aligned} g(x) &= [EI\{h(x, X_2, \dots, X_m) \leq \xi_\beta^-\}(h(x, X_2, \dots, X_m) - \xi_\beta^-) + \beta\xi_\beta^-] \\ &\quad - [EI\{h(x, X_2, \dots, X_m) < \xi_\alpha^+\}(h(x, X_2, \dots, X_m) - \xi_\alpha^+) + \alpha\xi_\alpha^+] - \theta, \\ g_\alpha(x) &= EI\{h(x, X_2, \dots, X_m) < \xi_\alpha^+\} - \theta_\alpha, \quad \theta_\alpha = H_F(\xi_\alpha^+), \\ g_\beta(x) &= EI\{h(x, X_2, \dots, X_m) \leq \xi_\beta^-\} - \theta_\beta \\ &= 1 - \theta_\beta - EI\{h(x, X_2, \dots, X_m) > \xi_\beta^-\}, \quad \theta_\beta = H_F(\xi_\beta^-). \end{aligned} \quad (7)$$

Note that for all  $0 < \alpha < \beta < 1$  and  $x \in \mathbf{X}$  we have  $|g(x)| \leq 4(|\xi_\alpha^+| + |\xi_\beta^-|)$ .

Let  $\sigma_g^2 = Eg^2(X)$ ,  $\sigma_{g_\alpha}^2 = Eg_\alpha^2(X)$ ,  $\sigma_{g_\beta}^2 = Eg_\beta^2(X)$ ,  $c_{gg_\alpha} = Eg(X)g_\alpha(X)$ ,  $c_{gg_\beta} = Eg(X)g_\beta(X)$ , and  $c_{g_\alpha g_\beta} = Eg_\alpha(X)g_\beta(X)$ .

**Theorem 1.1** *If  $\sigma_g^2 > 0$ , then for any underlying distribution function  $H_F$  we have*

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g - \Delta\xi_\alpha I(\tau_\alpha > 0) \tau_\alpha - \Delta\xi_\beta I(\tau_\beta < 0) \tau_\beta,$$

where  $(\tau_\alpha, \tau_g, \tau_\beta)$  is a trivariate Gaussian random vector with mean vector zero and covariance matrix

$$\begin{pmatrix} \sigma_{g_\alpha}^2 & c_{gg_\alpha} & c_{g_\alpha g_\beta} \\ c_{gg_\alpha} & \sigma_g^2 & c_{gg_\beta} \\ c_{g_\alpha g_\beta} & c_{gg_\beta} & \sigma_{g_\beta}^2 \end{pmatrix}.$$

**Corollary 1.2** For any underlying distribution function  $H_F$ , we have when  $\sigma_g^2 > 0$

$$\frac{\sqrt{n}}{m}(L_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g - \Delta\xi_\alpha I(\tau_\alpha > 0) \tau_\alpha - \Delta\xi_\beta I(\tau_\beta < 0) \tau_\beta,$$

where  $(\tau_\alpha, \tau_g, \tau_\beta)$  is a trivariate Gaussian random vector defined as in Theorem 1.1.

**Corollary 1.3** Suppose that the quantile function  $H_F^{-1}(x)$  is continuous at the points  $\alpha$  and  $\beta$ . If  $\sigma_g^2 > 0$ , then

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g.$$

For the simple case  $m = 1$  the functions in (7) reduce to

$$\begin{aligned} g(x) &= I\{\xi_\alpha^+ \leq h(x) \leq \xi_\beta^-\} h(x) - EI\{\xi_\alpha^+ \leq h(X) \leq \xi_\beta^-\} h(X) \\ &\quad + \xi_\alpha^+ g_\alpha(x) - \xi_\beta^- g_\beta(x), \\ g_\alpha(x) &= I\{h(x) < \xi_\alpha^+\} - \theta_\alpha, \\ g_\beta(x) &= I\{h(x) \leq \xi_\beta^-\} - \theta_\beta = 1 - \theta_\beta - I\{h(x) > \xi_\beta^-\}. \end{aligned}$$

A useful application of the theorem for the  $m = 2$  case is for the kernel  $h(x, y) = \frac{1}{2}(x - y)^2$ . This provides the asymptotic behaviour of a natural, alternative robust version of the sample variance. We will now develop explicit expressions for the terms in a more interesting example.

**Example** Let  $h(x_1, \dots, x_m) = \max\{x_1, \dots, x_m\}$  with  $m \geq 2$ . Let  $F(t)$  be the distribution function of  $X_i$  and denote  $Y = \max\{X_2, \dots, X_m\}$ . Then  $H_F(t) = (F(t))^m$  and

$$\begin{aligned}
g(x) &= g_{\alpha\beta}(x) + \xi_{\alpha}^{+}g_{\alpha}(x) - \xi_{\beta}^{-}g_{\beta}(x), \\
g_{\alpha\beta}(x) &= EI\{\xi_{\alpha}^{+} \leq \max\{x, Y\} \leq \xi_{\beta}^{-}\} \max\{x, Y\} - \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y dH_F(y) \\
&= I\{\xi_{\alpha}^{+} \leq x \leq \xi_{\beta}^{-}\} x (F(x))^{m-1} - \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y (F(y))^{m-1} dF(y) \\
&\quad + \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} (I\{x < y\} - F(y-)) y d(F(y))^{m-1}, \\
g_{\alpha}(x) &= I\{x < \xi_{\alpha}^{+}\} (F(\xi_{\alpha}^{+}-))^{m-1} - (F(\xi_{\alpha}^{+}-))^m, \\
g_{\beta}(x) &= I\{x \leq \xi_{\beta}^{-}\} (F(\xi_{\beta}^{-}))^{m-1} - (F(\xi_{\beta}^{-}))^m.
\end{aligned}$$

In addition,

$$\sigma_g^2 = E g_{\alpha\beta}^2(X) + E[\xi_{\alpha}^{+}g_{\alpha}(X) - \xi_{\beta}^{-}g_{\beta}(X)]^2 + 2E g_{\alpha\beta}(X)[\xi_{\alpha}^{+}g_{\alpha}(X) - \xi_{\beta}^{-}g_{\beta}(X)],$$

$$\sigma_{g_{\alpha}}^2 = (F(\xi_{\alpha}^{+}-))^{2m-1}(1 - F(\xi_{\alpha}^{+}-)), \quad \sigma_{g_{\beta}}^2 = (F(\xi_{\beta}^{-}))^{2m-1}(1 - F(\xi_{\beta}^{-})),$$

$$E g_{\alpha}(X)g_{\beta}(X) = (F(\xi_{\alpha}^{+}-)F(\xi_{\beta}^{-}))^{m-1}F(\xi_{\alpha}^{+}-)(1 - F(\xi_{\beta}^{-})) \quad ,$$

$$\begin{aligned}
E g_{\alpha\beta}(X)g_{\alpha}(X) &= (F(\xi_{\alpha}^{+}-))^m \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} (1 - F(y-)) y d(F(y))^{m-1} \\
&\quad - (F(\xi_{\alpha}^{+}-))^m \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y (F(y))^{m-1} dF(y),
\end{aligned}$$

$$\begin{aligned}
E g_{\alpha\beta}(X)g_{\beta}(X) &= (F(\xi_{\beta}^{-}))^{m-1}(1 - F(\xi_{\beta}^{-})) \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y (F(y))^{m-1} dF(y) \\
&\quad + (F(\xi_{\beta}^{-}))^{m-1}(1 - F(\xi_{\beta}^{-})) \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} F(y-) y d(F(y))^{m-1}.
\end{aligned}$$

Consider the distribution function

$$\begin{aligned}
F(t) &= 2tI\{0 \leq t < \frac{1}{2}\alpha^{1/m}\} + \alpha^{1/m}I\{\frac{1}{2}\alpha^{1/m} \leq t < \alpha^{1/m}\} \\
&\quad + tI\{\alpha^{1/m} \leq t < \beta^{1/m}\} + \beta^{1/m}I\{\beta^{1/m} \leq t < 2\beta^{1/m}\} \\
&\quad + \frac{1}{2}tI\{2\beta^{1/m} \leq t < 2\} + I\{t \geq 2\}, \quad t \in R.
\end{aligned}$$

Then

$$\xi_{\alpha}^{-} = \frac{1}{2}\alpha^{1/m}, \quad \xi_{\alpha}^{+} = \alpha^{1/m}, \quad \xi_{\beta}^{-} = \beta^{1/m}, \quad \xi_{\beta}^{+} = 2\beta^{1/m},$$

$F(\xi_\alpha^+ -) = \alpha^{1/m}$ ,  $F(\xi_\beta^-) = \beta^{1/m}$ ,  $F(t) = t$ ,  $t \in [\alpha^{1/m}, \beta^{1/m}]$ ,  $\sigma_g^2 > 0$   
and the limiting behaviour is given by

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g - \frac{1}{2} \alpha^{1/m} I(\tau_\alpha > 0) \tau_\alpha - \beta^{1/m} I(\tau_\beta < 0) \tau_\beta.$$

However for the simpler distribution function

$$F(t) = tI\{0 \leq t < 1\} + I\{t \geq 1\}, \quad t \in R.$$

we have

$$\xi_\alpha^- = \xi_\alpha^+ = \alpha^{1/m}, \quad \xi_\beta^- = \xi_\beta^+ = \beta^{1/m},$$

$F(\xi_\alpha^+ -) = \alpha^{1/m}$ ,  $F(\xi_\beta^-) = \beta^{1/m}$ ,  $F(t) = t$ ,  $t \in [\alpha^{1/m}, \beta^{1/m}]$ ,  $\sigma_g^2 > 0$   
and we get the asymptotic behaviour covered by Janssen *et al.* [15],

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g.$$

## 2 Proofs

The following two lemmas are key results for the proof.

**Lemma 2.1** *The following representation holds:*

$$\begin{aligned} \sum_{i=N_\alpha+1}^{N_\beta} h_{ni} &= \sum_{i=1}^N I\{\xi_\alpha^+ \leq h_i \leq \xi_\beta^-\} h_i + \xi_\alpha^+ (\dot{N}_\alpha^+ - N_\alpha) - \xi_\beta^- (N_\beta^- - N_\beta) \\ &\quad - \Delta \xi_\alpha I\{N_\alpha < \dot{N}_\alpha^+\} (\dot{N}_\alpha^+ - N_\alpha) - \Delta \xi_\beta I\{N_\beta^- < N_\beta\} (N_\beta^- - N_\beta) \\ &\quad + \mathbb{L}_\alpha + \mathbb{L}_\beta, \end{aligned} \tag{8}$$

where  $\mathbb{L}_\alpha = J_\alpha - \bar{J}_\alpha$  with

$$J_\alpha = I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (h_{ni} - \xi_\alpha^-), \quad \bar{J}_\alpha = I\{\dot{N}_\alpha^+ \leq N_\alpha\} \sum_{i=\dot{N}_\alpha^++1}^{N_\alpha} (h_{ni} - \xi_\alpha^+),$$

and  $\mathbb{L}_\beta = \bar{J}_\beta - J_\beta$  with

$$J_\beta = I\{N_\beta < N_\beta^-\} \sum_{i=N_\beta+1}^{N_\beta^-} (h_{ni} - \xi_\beta^-), \quad \bar{J}_\beta = I\{N_\beta^- \leq N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (h_{ni} - \xi_\beta^+).$$

**Proof of Lemma 2.1.** For  $i = 1, \dots, N$  write

$$\dot{h}_{ni} = (h_{ni} + \Delta\xi_\alpha)I\{h_{ni} < \xi_\alpha^+\} + h_{ni}I\{\xi_\alpha^+ \leq h_{ni} \leq \xi_\beta^-\} + (h_{ni} - \Delta\xi_\beta)I\{h_{ni} > \xi_\beta^-\}.$$

Since  $\dot{h}_{ni} = h_{ni} + \Delta\xi_\alpha I\{h_{ni} < \xi_\alpha^+\} - \Delta\xi_\beta I\{h_{ni} > \xi_\beta^-\}$  and  $I\{h_{ni} < \xi_\alpha^+\} = I\{i \leq \dot{N}_\alpha^+\}$  and by (6)  $I\{h_{ni} > \xi_\beta^-\} = I\{i > N_\beta^-\}$ , then we can write

$$\begin{aligned} \sum_{i=N_\alpha+1}^{N_\beta} h_{ni} &= \sum_{i=N_\alpha+1}^{N_\beta} \dot{h}_{ni} - \Delta\xi_\alpha I\{N_\alpha < \dot{N}_\alpha^+\}(\dot{N}_\alpha^+ - N_\alpha) \\ &\quad - \Delta\xi_\beta I\{N_\beta^- < N_\beta\}(N_\beta^- - N_\beta). \end{aligned} \quad (9)$$

Note that  $h_{n\dot{N}_\alpha^+} < \xi_\alpha^+ \leq h_{n,\dot{N}_\alpha^++1}$  and  $h_{nN_\beta^-} \leq \xi_\beta^- < h_{n,N_\beta^-+1}$ . From (6) we have  $I\{\xi_\alpha^+ \leq h_{ni} \leq \xi_\beta^-\} = I\{\dot{N}_\alpha^+ < i \leq N_\beta^-\}$ . Hence in (9)

$$\begin{aligned} \sum_{i=N_\alpha+1}^{N_\beta} \dot{h}_{ni} &= \sum_{i=\dot{N}_\alpha^++1}^{N_\beta^-} h_{ni} - I\{\dot{N}_\alpha^+ \leq N_\alpha\} \sum_{i=\dot{N}_\alpha^++1}^{N_\alpha} h_{ni} \\ &\quad + I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (h_{ni} + \Delta\xi_\alpha) - I\{N_\beta^- < N_\beta\} \sum_{i=N_\beta+1}^{N_\beta^-} h_{ni} \\ &\quad + I\{N_\beta^- \leq N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (h_{ni} - \Delta\xi_\beta) \\ &= \sum_{i=1}^N I\{\xi_\alpha^+ \leq h_i \leq \xi_\beta^-\} h_i + \xi_\alpha^+(\dot{N}_\alpha^+ - N_\alpha) - \xi_\beta^-(N_\beta^- - N_\beta) \\ &\quad + \mathbb{L}_\alpha + \mathbb{L}_\beta. \end{aligned} \quad (10)$$

Equation (8) follows from (9) and (10). This proves Lemma 2.1 .

**Lemma 2.2** *Note*

$$\begin{aligned} N^{-1} \sum_{i=N_\alpha+1}^{N_\beta} h_{ni} &= N^{-1} \sum_{i=1}^N I\{\xi_\alpha^+ \leq h_i \leq \xi_\beta^-\} h_i + \xi_\alpha^+ (H_n(\xi_\alpha^+) - \alpha) \\ &\quad - \xi_\beta^- (H_n(\xi_\beta^-) - \beta) - \Delta\xi_\alpha I\{N_\alpha < \dot{N}_\alpha^+\} (H_n(\xi_\alpha^+) - \alpha) \\ &\quad - \Delta\xi_\beta I\{N_\beta^- < N_\beta\} (H_n(\xi_\beta^-) - \beta) + n^{-1/2} \varrho_n, \end{aligned}$$

where  $\varrho_n \rightarrow 0$  in probability as  $n \rightarrow \infty$ .

**Proof of Lemma 2.2.** We shall estimate  $\mathbb{L}_\alpha$  and  $\mathbb{L}_\beta$ , taking into account the values of the distribution function  $H_F(x)$  at  $x = \xi_\gamma^\pm$  with  $\gamma = \alpha, \beta$ . Figures 1 and 2 illustrate the different situations that need to be considered.

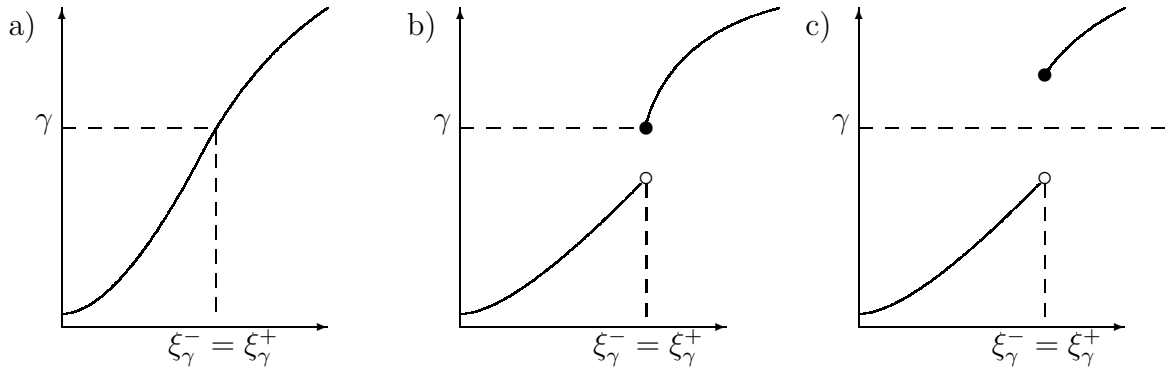


Figure 1: Plots of  $H(\cdot)$  with  $\xi_\gamma^- = \xi_\gamma^+$ .  
 (a)  $H(\xi_\gamma^{\pm-}) = \gamma = H(\xi_\gamma^{\pm})$ ; (b)  $H(\xi_\gamma^{\pm-}) < \gamma = H(\xi_\gamma^{\pm})$ ;  
 (c)  $H(\xi_\gamma^{\pm-}) < \gamma < H(\xi_\gamma^{\pm})$ .

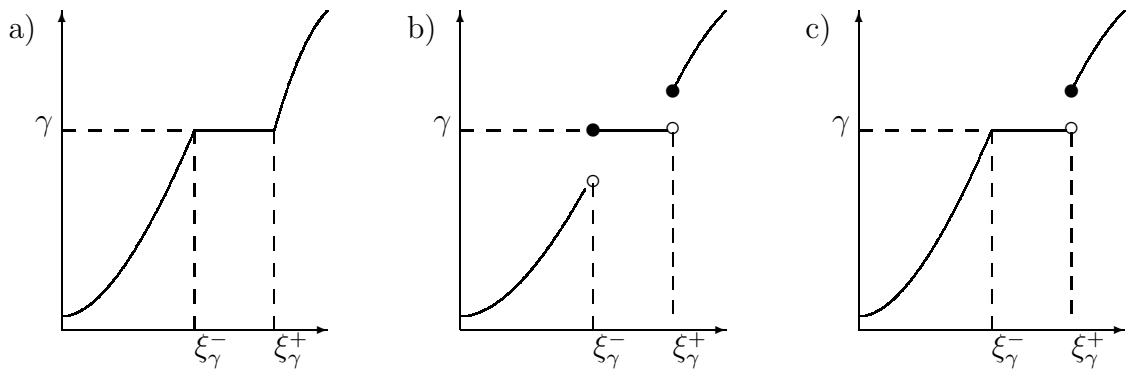


Figure 2: Plots of  $H(\cdot)$  with  $\xi_\gamma^- < \xi_\gamma^+$ .  
 (a)  $\gamma = H(\xi_\gamma^-) = H(\xi_\gamma^+)$ ; (b)  $H(\xi_\gamma^-) < \gamma = H(\xi_\gamma^-) = H(\xi_\gamma^+ -) < H(\xi_\gamma^+)$ ;  
 (c)  $H(\xi_\gamma^-) = \gamma = H(\xi_\gamma^+ -) < H(\xi_\gamma^+)$ .

Estimating  $\mathbb{L}_\alpha$ . Noting  $I\{\xi_\alpha^+ > h_{ni}\} = I\{i \leq \dot{N}_\alpha^+\}$  we write

$$\begin{aligned}
J_\alpha &= I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (h_{ni} - \xi_\alpha^-) I\{\xi_\alpha^+ > h_{ni}\} \\
&= I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (h_{ni} - \xi_\alpha^-) I\{\xi_\alpha^- < h_{ni} < \xi_\alpha^+\} I\{N_\alpha^- < i \leq \dot{N}_\alpha^+\} \\
&\quad - I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (\xi_\alpha^- - h_{ni}) I\{\xi_\alpha^- \geq h_{ni}\} I\{i \leq N_\alpha^-\} \\
&= J_\alpha^+ - J_\alpha^-.
\end{aligned}$$

It is clear that if  $\xi_\alpha^- = \xi_\alpha^+$  then  $J_\alpha^+ = 0$  *a.s.* Let  $\xi_\alpha^- \neq \xi_\alpha^+$  as is the case in Figure 2. In this case  $H_F(\xi_\alpha^-) = \alpha = H_F(\xi_\alpha^+ -)$  and we can write

$$\begin{aligned}
0 \leq J_\alpha^+ &\leq I\{N_\alpha^- < \dot{N}_\alpha^+\} (\xi_\alpha^+ - \xi_\alpha^-) \sum_{i=N_\alpha^-+1}^{\dot{N}_\alpha^+} I\{\xi_\alpha^- < h_{ni} < \xi_\alpha^+\} \\
&\leq \Delta\xi_\alpha \sum_{i=1}^N I\{\xi_\alpha^- < h_i < \xi_\alpha^+\} = 0 \text{ (a.s.)},
\end{aligned}$$

since  $EI\{\xi_\alpha^- < h_i < \xi_\alpha^+\} = H_F(\xi_\alpha^+ -) - H_F(\xi_\alpha^-) = 0$ . Hence we always have the relation  $J_\alpha^+ = 0$  *a.s.* To estimate  $J_\alpha^-$  we note first that if  $\dot{N}_\alpha^+ > N_\alpha^-$  then the indicator  $I\{i \leq N_\alpha^-\} = 0$  for all  $i = N_\alpha^- + 1, \dots, \dot{N}_\alpha^+$ . Therefore we have the inequalities

$$\begin{aligned}
0 \leq J_\alpha^- &\leq I\{N_\alpha < N_\alpha^-\} \sum_{i=N_\alpha+1}^{N_\alpha^-} (\xi_\alpha^- - h_{ni}) I\{\xi_\alpha^- \geq h_{ni}\} \\
&\leq I\{N_\alpha < N_\alpha^-\} (N_\alpha^- - N_\alpha) (\xi_\alpha^- - h_{nN_\alpha}) I\{\xi_\alpha^- \geq h_{nN_\alpha}\}. \quad (11)
\end{aligned}$$

Further we shall apply the technique used in Smirnov [22] with a probability inequality from Hoeffding [14] (or see, for example, Serfling [17], p.75 and p.201). Thus

$$\begin{aligned}
&P\{(\xi_\alpha^- - h_{nN_\alpha} > \varepsilon) \cap (\xi_\alpha^- \geq h_{nN_\alpha})\} \\
&\leq P\{\xi_\alpha^- - h_{nN_\alpha} \geq \varepsilon\} \\
&= P\{H_n(\xi_\alpha^- - \varepsilon) \geq N^{-1}N_\alpha\} \\
&= P\{H_n(\xi_\alpha^- - \varepsilon) - H(\xi_\alpha^- - \varepsilon) \geq N^{-1}N_\alpha - H(\xi_\alpha^- - \varepsilon)\} \\
&\leq c_1 \exp\{-c_2 n \theta_\alpha^2(\xi_\alpha^-, \varepsilon)\} \quad (12)
\end{aligned}$$

with some positive constants  $c_1$  and  $c_2$ , depending only on  $m$  and  $\theta_\alpha(\xi_\alpha^-, \varepsilon) = \alpha - H_F(\xi_\alpha^- - \varepsilon)$ . Further,  $\theta_\alpha(\xi_\alpha^-, \varepsilon) > 0$  for any small values of  $\varepsilon > 0$  by the

definition of the smallest  $\alpha$ -quantile  $\xi_\alpha^-$ . Under the conditions of the lemma  $\sqrt{n}N^{-1}(N_\alpha^- - N_\alpha) \xrightarrow{d} \tau_\alpha^-$  as  $n \rightarrow \infty$ . Hence  $\sqrt{n}N^{-1}J_\alpha \rightarrow 0$  in probability as  $n \rightarrow \infty$ .

Next we consider  $\bar{J}_\alpha$ . By definition  $\dot{N}_\alpha^+ \leq N_\alpha^+$  and since  $I\{h_{ni} < \xi_\alpha^+\} = I\{i \leq \dot{N}_\alpha^+\}$  and  $h_{nN_\alpha^+} \leq \xi_\alpha^+ < h_{n, N_\alpha^++1}$  then the indicator  $I\{h_{ni} = \xi_\alpha^+\} = 1$  for  $i = \dot{N}_\alpha^+ + 1, \dots, N_\alpha^+$  and we can write

$$\begin{aligned} 0 \leq \bar{J}_\alpha &= I\{\dot{N}_\alpha^+ \leq N_\alpha\} \sum_{i=\dot{N}_\alpha^++1}^{N_\alpha} (h_{ni} - \xi_\alpha^+) I\{h_{ni} \geq \xi_\alpha^+\} \\ &= I\{N_\alpha^+ \leq N_\alpha\} \sum_{i=N_\alpha^++1}^{N_\alpha} (h_{ni} - \xi_\alpha^+) I\{h_{ni} > \xi_\alpha^+\} \\ &\leq I\{N_\alpha^+ \leq N_\alpha\} (N_\alpha - N_\alpha^+) (h_{nN_\alpha} - \xi_\alpha^+) I\{h_{nN_\alpha} > \xi_\alpha^+\}. \end{aligned} \quad (13)$$

In (13) we need to consider two cases:  $H_F(\xi_\alpha^+) = \alpha$  and  $\alpha < H_F(\xi_\alpha^+)$ . In the first case  $H_F(\xi_\alpha^+) = \alpha$  and we have the weak convergence  $\sqrt{n}N^{-1}(N_\alpha^+ - N_\alpha) \xrightarrow{d} \tau_\alpha^+$  as  $n \rightarrow \infty$  and the following estimates which are similar to (12)

$$\begin{aligned} &P\{(h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon) \cap (h_{nN_\alpha} > \xi_\alpha^-)\} \\ &\leq P\{h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon\} \\ &= P\{N^{-1}N_\alpha > H_n(\xi_\alpha^+ + \varepsilon)\} \\ &= P\{H(\xi_\alpha^+ + \varepsilon) - H_n(\xi_\alpha^+ + \varepsilon) > H(\xi_\alpha^+ + \varepsilon) - N^{-1}N_\alpha\} \\ &\leq c_1 \exp\{-c_2 n \delta_\alpha^2(\xi_\alpha^+, \varepsilon)\}, \end{aligned} \quad (14)$$

where  $\delta_\alpha(\xi_\alpha^+, \varepsilon) = H_F(\xi_\alpha^+ + \varepsilon) - \alpha$ . In addition,  $\delta_\alpha(\xi_\alpha^+, \varepsilon) > 0$  for any small values of  $\varepsilon > 0$  because of the definition of the largest  $\alpha$ -quantile  $\xi_\alpha^+$ . Hence in the case  $H_F(\xi_\alpha^+) = \alpha$  we have  $\sqrt{n}N^{-1}\bar{J}_\alpha \rightarrow 0$  in probability as  $n \rightarrow \infty$ . In the second case in (13)  $\delta_\alpha(\xi_\alpha^+, 0) = H_F(\xi_\alpha^+) - \alpha > 0$ , and we have the following representation

$$\sqrt{n}N^{-1}(N_\alpha^+ - N_\alpha) = \sqrt{n}(H_n(\xi_\alpha^+) - H_F(\xi_\alpha^+)) + \sqrt{n}\delta_\alpha(\xi_\alpha^+, 0) + \omega_n(\alpha), \quad (15)$$

where  $\sqrt{n}(H_n(\xi_\alpha^+) - H_F(\xi_\alpha^+)) \xrightarrow{d} \tau_\alpha^+$  and  $\omega_n(\alpha) = \sqrt{n}N^{-1}(\alpha N - [\alpha N]) = O(n^{-1/2})$  as  $n \rightarrow \infty$  but the positive term  $\sqrt{n}\delta_\alpha(\xi_\alpha^+, 0)$  is unbounded. Therefore in this case we shall apply the estimate (14) with  $\varepsilon n^{-1}$  instead of  $\varepsilon$ , that is  $P\{(h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon n^{-1}) \cap (h_{nN_\alpha} > \xi_\alpha^-)\} \leq c_1 \exp\{-c_2 n \delta_\alpha^2(\xi_\alpha^+, \varepsilon n^{-1})\}$ . Since the distribution function  $H_F$  is continuous from the right at the point  $\xi_\alpha^+$  then  $\delta_\alpha(\xi_\alpha^+, 0) \leq \delta_\alpha(\xi_\alpha^+, \varepsilon n^{-1})$  for any small  $\varepsilon > 0$  and sufficiently large  $n$ . Hence in the second case  $\alpha < H_F(\xi_\alpha^+)$  and (14) is replaced by the inequality

$$P\{(h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon n^{-1}) \cap (h_{nN_\alpha} > \xi_\alpha^-)\} \leq c_1 \exp\{-c_2 n \delta_\alpha^2(\xi_\alpha^+, 0)\}, \quad (16)$$

which provides the desired convergence  $\sqrt{n}N^{-1}\bar{J}_\alpha \rightarrow 0$  in probability as  $n \rightarrow \infty$ . Thus we have proved that  $\sqrt{n}N^{-1}\mathbb{L}_\alpha \rightarrow 0$  in probability as  $n \rightarrow \infty$ .

Estimating  $\mathbb{L}_\beta$ . Noting  $I\{\xi_\beta^- \geq h_{ni}\} = I\{i \leq N_\beta^-\}$  we write

$$\begin{aligned} 0 \leq -J_\beta &= I\{N_\beta < N_\beta^-\} \sum_{i=N_\beta^-+1}^{N_\beta^-} (\xi_\beta^- - h_{ni}) I\{\xi_\beta^- \geq h_{ni}\} \\ &\leq I\{N_\beta < N_\beta^-\} (N_\beta^- - N_\beta) (\xi_\beta^- - h_{nN_\beta}) I\{\xi_\beta^- \geq h_{nN_\beta}\}. \end{aligned}$$

Here by analogy with (12) we have

$$\begin{aligned} &P\{(\xi_\beta^- - h_{nN_\beta} > \varepsilon) \cap (\xi_\beta^- \geq h_{nN_\beta})\} \\ &\leq P\{\xi_\beta^- - h_{nN_\beta} \geq \varepsilon\} \\ &= P\{H_n(\xi_\beta^- - \varepsilon) \geq N^{-1}N_\beta\} \\ &= P\{H_n(\xi_\beta^- - \varepsilon) - H(\xi_\beta^- - \varepsilon) \geq N^{-1}N_\beta - H(\xi_\beta^- - \varepsilon)\} \\ &\leq c_1 \exp\{-c_2 n \theta_\beta^2(\xi_\beta^-, \varepsilon)\} \end{aligned} \quad (17)$$

with  $\theta_\beta(\xi_\beta^-, \varepsilon) = \beta - H_F(\xi_\beta^- - \varepsilon)$ , and by analogy with (15)

$$\sqrt{n}N^{-1}(N_\beta^- - N_\beta) = \sqrt{n}(H_n(\xi_\beta^-) - H_F(\xi_\beta^-)) + \sqrt{n}\theta_\beta(\xi_\beta^-, 0) + \omega_n(\beta). \quad (18)$$

Here we need to consider two cases:  $\beta - H_F(\xi_\beta^-) = 0$  or  $\beta - H_F(\xi_\beta^-) > 0$ . In the first case we apply the inequality (17) with sufficiently small  $\varepsilon > 0$ . In the second case we use (17) again but with parameter  $\varepsilon n^{-1}$  as in (16) to get

$$P\{(\xi_\beta^- - h_{nN_\beta} > \varepsilon n^{-1}) \cap (\xi_\beta^- \geq h_{nN_\beta})\} \leq c_1 \exp\{-c_2 n \theta_\beta^2(\xi_\beta^-, 0)\}, \quad (19)$$

since the distribution function  $H_F$  has a limit from the left at the point  $\xi_\beta^-$ , and  $H_F(\xi_\beta^-) \geq H_F(\xi_\beta^- - \varepsilon n^{-1})$ . In this result we have  $\sqrt{n}N^{-1}J_\beta \rightarrow 0$  in probability as  $n \rightarrow \infty$ .

Finally, we consider  $\bar{J}_\beta$ . Since  $I\{h_{ni} > \xi_\beta^\pm\} = I\{i > N_\beta^\pm\}$  we write

$$\begin{aligned} \bar{J}_\beta &= I\{N_\beta^- < N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (h_{ni} - \xi_\beta^+) I\{h_{ni} > \xi_\beta^-\} \\ &= -I\{N_\beta^- < N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (\xi_\beta^+ - h_{ni}) I\{\xi_\beta^- < h_{ni} < \xi_\beta^+\} I\{N_\beta^- < i \leq N_\beta^+\} \\ &\quad + I\{N_\beta^- < N_\beta\} \sum_{i=N_\beta^++1}^{N_\beta} (h_{ni} - \xi_\beta^+) I\{h_{ni} > \xi_\beta^+\} I\{i > N_\beta^+\} \\ &= -\bar{J}_\beta^- + \bar{J}_\beta^+. \end{aligned}$$

If  $\xi_\beta^- = \xi_\beta^+$  then  $\bar{J}_\beta^- = 0$  a.s.. Now assume  $\xi_\beta^- \neq \xi_\beta^+$ . In this case  $H_F(\xi_\beta^-) = \beta = H_F(\xi_\beta^+ -)$  and we have

$$\begin{aligned} 0 \leq \bar{J}_\beta^- &\leq I\{N_\beta^- < \dot{N}_\beta^+\}(\xi_\beta^+ - \xi_\beta^-) \sum_{i=N_\beta^-+1}^{\dot{N}_\beta^+} I\{\xi_\beta^- < h_{ni} < \xi_\beta^+\} \\ &\leq \Delta\xi_\beta \sum_{i=1}^N I\{\xi_\beta^- < h_i < \xi_\beta^+\} = 0 \text{ (a.s.)}, \end{aligned}$$

since  $EI\{\xi_\beta^- < h_i < \xi_\beta^+\} = H_F(\xi_\beta^+ -) - H_F(\xi_\beta^-) = 0$ . Hence we always have  $\bar{J}_\beta^- = 0$  a.s.. To estimate  $\bar{J}_\beta^+$  we write

$$0 \leq \bar{J}_\beta^+ \leq I\{N_\beta^- < N_\beta\}(N_\beta^+ - N_\beta)(h_{ni} - \xi_\beta^+)I\{h_{ni} > \xi_\beta^+\}$$

and apply the estimates (13) - (16) with  $\beta$  instead of  $\alpha$ . We have  $\sqrt{n}N^{-1}\bar{J}_\beta \rightarrow 0$  in probability as  $n \rightarrow \infty$  and hence  $\sqrt{n}N^{-1}\mathbb{L}_\beta \rightarrow 0$  in probability as  $n \rightarrow \infty$ .

This proves Lemma 2.2.

**Proof of Theorem 1.1.** Let  $U(g)$  be a  $U$ -statistic of the form (1) with the kernel

$$\begin{aligned} g(x_1, \dots, x_m) &= [I\{h(x_1, \dots, x_m) \leq \xi_\beta^-\}(h(x_1, \dots, x_m) - \xi_\beta^-) + \beta\xi_\beta^-] \\ &\quad - [I\{h(x_1, \dots, x_m) < \xi_\alpha^+\}(h(x_1, \dots, x_m) - \xi_\alpha^+) + \alpha\xi_\alpha^+]. \end{aligned}$$

We see that

$$\begin{aligned} U(g) &= N^{-1} \sum_{i=1}^N I\{\xi_\alpha^+ \leq h_i \leq \xi_\beta^-\} h_i \\ &\quad + \xi_\alpha^+(H_n(\xi_\alpha^+ -) - \alpha) - \xi_\beta^-(H_n(\xi_\beta^-) - \beta). \end{aligned}$$

It is not difficult to verify for this function that  $Eg(X_1, \dots, X_m) = \theta$  and  $g(x) = Eg(x, X_2, \dots, X_m) - \theta$ ,  $x \in \mathcal{X}$ , in addition,  $Eg^2(X) > 0$  by the condition of the theorem. Hence the kernel  $g$  is non-degenerate and by the central limit theorem for  $U$ -statistics with such bounded kernels we have the weak convergence  $\tau_{ng} := m^{-1}\sqrt{n}(U(g) - \theta) \xrightarrow{d} \tau_g$  as  $n \rightarrow \infty$  (see, for example, Borovskikh [3]). By the same central limit theorem we have

$$\tau_{n\alpha} := m^{-1}\sqrt{n}(H_n(\xi_\alpha^+ -) - H(\xi_\alpha^+ -)) \xrightarrow{d} \tau_\alpha,$$

and

$$\tau_{n\beta} := m^{-1}\sqrt{n}(H_n(\xi_\beta^-) - H(\xi_\beta^-)) \xrightarrow{d} \tau_\beta,$$

as  $n \rightarrow \infty$ . Under the conditions of the theorem we have  
 $E|I\{\dot{N}_\alpha^+ - N_\alpha > 0\} - I\{\tau_\alpha > 0\}| \rightarrow 0$  if  $\Delta\xi_\alpha \neq 0$  (in this case  $H(\xi_\alpha^+ -) = \alpha$ )  
and  $E|I\{N_\beta^- - N_\beta < 0\} - I\{\tau_\beta < 0\}| \rightarrow 0$  if  $\Delta\xi_\beta \neq 0$  (in this case  $H(\xi_\beta^-) = \beta$ ).  
Further it is easy to prove that the covariances  $\text{Cov}(\tau_{n*}, \tau_{n\star}) \rightarrow \text{Cov}(\tau_*, \tau_\star)$  as  
 $n \rightarrow \infty$ , where  $*, \star = \alpha, g, \beta$ . Now apply Lemma 2.2 to complete the proof of  
Theorem 1.1.

**Lemma 2.3** *The following representation holds:*

$$L_{\alpha\beta} = N^{-1} \sum_{i=\bar{N}_\alpha}^{\bar{N}_\beta-1} h_{ni}.$$

**Proof of Lemma 2.3.** Write by definition

$$\begin{aligned} L_{\alpha\beta} &= \int_R I\{h_\alpha \leq x < h_\beta\} x dH_n(x) \\ &= \frac{1}{N} \sum_{i=1}^N I\{h_\alpha \leq h_{ni} < h_\beta\} h_{ni} \\ &= \frac{1}{N} \sum_{i=1}^N I\{h_{ni} < h_\beta\} h_{ni} - \frac{1}{N} \sum_{i=1}^N I\{h_{ni} < h_\alpha\} h_{ni} \\ &= \frac{1}{N} \sum_{i=1}^{\bar{N}_\beta-1} h_{ni} - \frac{1}{N} \sum_{i=1}^{\bar{N}_\alpha-1} h_{ni} \\ &= \frac{1}{N} \sum_{i=\bar{N}_\alpha}^{\bar{N}_\beta-1} h_{ni}. \end{aligned}$$

This proves Lemma 2.3.

The proof of Corollary 1.2 follows from Theorem 1.1 and Lemma 2.3.

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