Viscosity solutions for systems of parabolic variational inequalities

Lucian Maticiuc[a,*], Etienne Pardoux[b], Aurel Răşcanu[c,d,*], Adrian Zălinescu[d,*]

[a] Department of Mathematics, "Gh. Asachi" Technical University of Iaşi, Bd. Carol I, no. 11, 700506, Romania

[b] LATP, Université de Provence, CMI, 39, rue F. Joliot-Curie, 13453, Marseille Cedex 13, France,

[c] Faculty of Mathematics, "Al.I. Cuza" University of Iaşi, Bd. Carol I, no. 9, 700506, Romania,

[d] "O. Mayer"Mathematics Institute of the Romanian Academy, Iaşi, Bd. Carol I, no. 8, 700506, Romania.

E-mail addresses: lucianmaticiuc@yahoo.com (L. Maticiuc), pardoux@cmi.univ-mrs.fr (E. Pardoux), aurel.rascanu@uaic.ro (A. Răşcanu, corresponding author), adrian.zalinescu@gmail.com (A. Zălinescu)

[*] The work of this authors was partial done during the visits to the Université de Provence and it was supported by the Grants ID 395/2007 and CNCSIS 1156/2006.

April 11, 2009

Abstract

In this paper we first define the notion of viscosity solution for the following system of partial differential equations involving a subdifferential operator:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}_t u(t,x) + f\big(t,x,u(t,x)\big) \in \partial\varphi\big(u(t,x)\big), \\ u(T,x) = h(x),\ t\in[0,T],\ x\in\mathbb{R}^d, \end{cases}$$

where $\partial\varphi$ is the subdifferential operator of the proper convex lower semicontinuous function $\varphi:\mathbb{R}^k \to (-\infty,+\infty]$ and $\mathcal{L}_t$ is a second differential operator given by $\mathcal{L}_t v_i(x) = \frac{1}{2}\mathrm{Tr}\big[\sigma(t,x)\sigma^*(t,x)\mathrm{D}^2 v_i(x)\big] + \big\langle b(t,x), \nabla v_i(x)\big\rangle$, $i\in\overline{1,k}$.

We prove the uniqueness of the viscosity solution and then, by a stochastic approach, we prove the existence of a viscosity solution $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}^k$ of the above parabolic variational inequality.

AMS Classification subjects: 60H99, 60H30, 35K85, 49L25.

Keywords or phrases: Feynman-Kaç formula; Systems of variational inequalities; Viscosity solutions.

1 Introduction

The viscosity solutions theory was introduced by M.G. Crandall and P.L. Lions [3]. This theory allows to study nonlinear equations which admit as solutions continuous functions, without further smoothness constraint. The classical work in the field of viscosity solutions for second order partial differential equations is the survey paper of M.G. Crandall, H. Ishii and P.L. Lions [2], where the autors give several equivalent ways to define the notion of viscosity solution, and also some very general existence and uniqueness theorems. Starting with the study of backward stochastic differential equations (introduced by E. Pardoux and S. Peng [10]), generalised Feynman-Kaç representation formulas have been obtained for the viscosity solutions of semilinear partial differential equations. R.W.R. Darling and E. Pardoux in [4] studied elliptic equations with Dirichlet boundary conditions; furthermore, Y. Hu in [7] treated elliptic equations with homogeneous Neumann boundary conditions. E. Pardoux and S. Zhang in [13] extended these results for the case of parabolic systems with nonlinear Neumann boundary conditions.

N. El Karoui et al. [5] considered the case of reflected solutions of one-dimensional backward stochastic differential equations, related to an obstacle problem for a parabolic partial differential equation. The more general case of backward stochastic differential equations involving subdifferential operator and the connection with parabolic variational inequalities has been studied by E. Pardoux and A. Răşcanu in [11] and [12].

The aim of this paper is to consider the more general case of variational inequalities for systems of partial differential equations. The paper is organized as follows. In the first section we define the extended notion of viscosity solution for a system of parabolic variational inequalities, then we formulate the existence and uniqueness result and we prove the uniqueness. In the second section we use a stochastic approach in order to prove the existence result; by using a backward stochastic variational inequality, we obtain a probabilistic formula for the viscosity solution of our system.

2 Main results

The goal of this paper is to study the existence and uniqueness of the viscosity solution of the following system of parabolic variational inequalities:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}_t u\,(t,x) + f\big(t,x,u(t,x)\big) \in \partial\varphi\big(u\,(t,x)\,\big),\ t\in[0,T]\,,\ x\in\mathbb{R}^d, \\ u(T,x) = h(x),\ x\in\mathbb{R}^d, \end{cases} \tag{1}$$

where $\mathcal{L}_t v$, with $v \in C^2(\mathbb{R}^d;\mathbb{R}^k)$, is given by

$$
\begin{aligned}
(\mathcal{L}_t v)_i(x) &= \frac{1}{2}\operatorname{Tr}\left[\sigma(t,x)\sigma^*(t,x)\,\mathrm{D}^2 v_i(x)\right] + \left\langle b(t,x), \mathrm{D}\, v_i(x)\right\rangle \\
&= \frac{1}{2}\sum_{j,l=1}^{d}(\sigma\sigma^*)_{jl}(t,x)\frac{\partial^2 v_i(x)}{\partial x_j \partial x_l} + \sum_{j=1}^{d} b_j(t,x)\frac{\partial v_i(x)}{\partial x_j},\ i\in\overline{1,k},
\end{aligned}
$$

and $T > 0$ is the fixed finite horizon.

We make the following standard assumptions:

(A.1) the functions

$$
b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d \text{ and } \sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}
$$

are Lipschitz with constant L,

(A.2) the functions

$$
f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k \to \mathbb{R}^k \text{ and } h : \mathbb{R}^d \to \mathbb{R}^k
$$

are continuous and there exists $\gamma \in \mathbb{R}$ such that

$$
\left\langle y-\tilde{y}, f(t,x,y) - f(t,x,\tilde{y})\right\rangle \le \gamma |y-\tilde{y}|^2,
$$

for all $t\in[0,T]$, $x\in\mathbb{R}^d$, $y,\tilde{y}\in\mathbb{R}^k$,

(A.3) there exist some $M_1 > 0$, $p \ge 0$ such that, for all $t\in[0,T]$, $x\in\mathbb{R}^d$,

$$
\begin{aligned}
&(i) \quad \left|h(x)\right| \le M_1\left(1+|x|^p\right),\\
&(ii) \quad \int_0^T f_R^{\#}(t,x)dt < \infty,\ \forall R \ge 0,\\
&\qquad\qquad \text{where } f_R^{\#}(t,x) := \sup\left\{|f(t,x,y)| : |y| \le R\right\},
\end{aligned}
$$

(A.4) the function $\varphi : \mathbb{R}^k \to (-\infty,+\infty]$ is proper (*i.e.* $\varphi \not\equiv +\infty$), convex, lower semicontinuous (l.s.c) and there exist $M_2 > 0$ and $r \ge 0$ such that

$$
\left|\varphi\big(h(x)\big)\right| \le M_2\left(1+|x|^r\right),\ \forall x\in\mathbb{R}^d.
$$

We recall that the subdifferential $\partial\varphi$ is defined by

$$
\partial\varphi(u) = \left\{u^*\in\mathbb{R}^k : \langle u^*, v-u\rangle + \varphi(u) \le \varphi(v),\ \forall v\in\mathbb{R}^k\right\}.
$$

It is a common practice to regard sometimes $\partial\varphi$ as a subset of $\mathbb{R}^k\times\mathbb{R}^k$, by writing $(u,u^*)\in\partial\varphi$ instead of $u^*\in\partial\varphi(u)$. We denote by

$$
\begin{aligned}
\operatorname{Dom}(\varphi) &= \left\{u\in\mathbb{R}^k : \varphi(u) < +\infty\right\},\\
\operatorname{Dom}(\partial\varphi) &= \left\{u\in\mathbb{R}^k : \partial\varphi(u) \ne \emptyset\right\}.
\end{aligned}
$$

We give now a result which allows us to define the notion of directional derivative of a convex function (see [1] for more details).

Theorem 1 *Let $\varphi : \mathbb{R}^k \to (-\infty, +\infty]$ be a convex function. Then, for all $u \in \operatorname{Dom}(\varphi)$ and $z \in \mathbb{R}^k$, there exist*

$$\begin{aligned}\varphi'_-(u;z) &:= \lim_{t \nearrow 0} \frac{\varphi(u+tz)-\varphi(u)}{t} = \sup_{t<0} \frac{\varphi(u+tz)-\varphi(u)}{t}, \\ \varphi'_+(u;z) &:= \lim_{t \searrow 0} \frac{\varphi(u+tz)-\varphi(u)}{t} = \inf_{t>0} \frac{\varphi(u+tz)-\varphi(u)}{t}.\end{aligned} \tag{2}$$

Moreover, the following hold:
(a) $\varphi'_-(u;z) \le \varphi'_+(u;z)$, $\forall u \in \operatorname{Dom}(\varphi), \forall z \in \mathbb{R}^k$,
(b) $\varphi'_-(u;-z) = -\varphi'_+(u;z)$, $\forall u \in \operatorname{Dom}(\varphi), \forall z \in \mathbb{R}^k$,
(c) $\varphi'_-(u;\cdot)$ *is superlinear and* $\varphi'_+(u;\cdot)$ *is sublinear,*
(d) if u and z are such that there exists $\delta > 0$ such that $u + tz \in \operatorname{Dom}(\varphi), \forall t \in (-\delta, \delta)$, then $\varphi'_-(u;z), \varphi'_+(u;z) \in \mathbb{R}$.

If we take $k = 1$, then we know that, in every point $u \in \operatorname{Dom}(\varphi)$,

$$\partial\varphi(u) = \mathbb{R} \cap \left[\varphi'_-(u), \varphi'_+(u)\right], \tag{3}$$

where $\varphi'_-(u)$ and $\varphi'_+(u)$ are respectively, the left derivative and right derivative of φ at the point u.
The following proposition generalizes the above characterization to the case of $k \ge 1$:

Proposition 2 *Let $\varphi : \mathbb{R}^k \to (-\infty, +\infty]$ be a proper convex function and $u \in \operatorname{Dom}(\varphi)$. The following statements are equivalent:*

$$\begin{aligned} &(i) \quad u^* \in \partial\varphi(u), \\ &(ii) \quad \langle u^*, z\rangle \ge \varphi'_-(u;z), \ \forall z \in \mathbb{R}^k, \\ &(iii) \quad \langle u^*, z\rangle \le \varphi'_+(u;z), \ \forall z \in \mathbb{R}^k. \end{aligned}$$

Proof. First we prove that (i) implies assertion (ii). Let $u^* \in \partial\varphi(u)$. From the definition we have that

$$\langle u^*, y-u\rangle + \varphi(u) \le \varphi(y), \ \forall y \in \mathbb{R}^k. \tag{4}$$

For all $t < 0$ and any direction $z \in \mathbb{R}^k$, we obtain, using (4) with $y = u + tz$, that

$$\begin{aligned} &\langle u^*, tz\rangle + \varphi(u) \le \varphi(u+tz), \ \forall z \in \mathbb{R}^k, \ \forall t < 0 \\ \Leftrightarrow\ &\langle u^*, z\rangle \ge \frac{\varphi(u+tz)-\varphi(u)}{t}, \ \forall z \in \mathbb{R}^k, \ \forall t < 0 \end{aligned}$$

Passing to the limit as $t \to 0$, $t < 0$, we obtain that

$$\langle u^*, z\rangle \ge \varphi'_-(u;z), \ \forall z \in \mathbb{R}^k.$$

Suppose now that (ii) holds, which is

$$\langle u^*, z\rangle \geq \varphi'_-(u; z), \forall z \in \mathbb{R}^k.$$

From (2) we deduce that

$$\langle u^*, z\rangle \geq \frac{\varphi(u + tz) - \varphi(u)}{t}, \forall z \in \mathbb{R}^k, \ \forall t < 0.$$

Consequently

$$\langle u^*, tz\rangle \leq \varphi(u + tz) - \varphi(u) \ , \forall z \in \mathbb{R}^k, \ \forall t < 0.$$

Hence

$$\langle u^*, y - u\rangle \leq \varphi(y) - \varphi(u), \forall y \in \mathbb{R}^k$$

that is $u^* \in \partial\varphi(u)$.

The equivalence between (i) and (iii) follows in the same manner. ■

Let us define, for $u \in \overline{\mathrm{Dom}(\varphi)}$ and $z \in \mathbb{R}^k$,

$$\varphi'_*(u; z) = \liminf_{\substack{v \to u \\ v \in \mathrm{Dom}(\partial\varphi)}} \varphi'_-(v; z), \ \varphi'^{,*}(u; z) = \limsup_{\substack{v \to u \\ v \in \mathrm{Dom}(\partial\varphi)}} \varphi'_+(v; z).$$

For $u \in \mathbb{R}^k$, let (with the usual convention $\inf \emptyset = +\infty$)

$$|\partial\varphi|_0(u) = \inf |\partial\varphi(u)|.$$

If $u \in \mathrm{Dom}(\partial\varphi)$, then there is a unique $u^* \in \mathbb{R}^k$, denoted $(\partial\varphi)_0(u)$, such that $|\partial\varphi|_0(u) = |(\partial\varphi)_0(u)|$.

We can give now, using Proposition 2, the definition of a viscosity solution for the multidimensional parabolic variational inequality (1). First we denote the set of $d \times d$ symmetric real matrices by S^d, and we give the definition of the superjet of a function.

Definition 3 *Let $\bar{u} : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, a continuous function, and $(t, x) \in [0, T] \times \mathbb{R}^d$. We denote by $\mathcal{P}^{2,+}\bar{u}(t, x)$ (the parabolic superjet of $\bar{u}$ at (t, x)) the set of triples $(p, q, X) \in \mathbb{R} \times \mathbb{R}^d \times S^d$ which are such that for all $(s, y) \in [0, T] \times \mathbb{R}^d$ in a neighborhood of (t, x):*

$$\bar{u}(s, y) \leq \bar{u}(t, x) + p(s - t) + \langle q, y - x\rangle + \\ + \frac{1}{2}\big\langle X(y - x), y - x\big\rangle + o\big(|s - t| + |y - x|^2\big).$$

Similarly is defined $\mathcal{P}^{2,-}\bar{u}(t, x)$ (the parabolic subjet of $\bar{u}$ at (t, x)) as the set of triples $(p, q, X) \in \mathbb{R} \times \mathbb{R}^d \times S^d$ which are such that for all $(s, y) \in [0, T] \times \mathbb{R}^d$ in a neighborhood of (t, x):

$$\bar{u}(s, y) \geq \bar{u}(t, x) + p(s - t) + \langle q, y - x\rangle + \\ + \frac{1}{2}\big\langle X(y - x), y - x\big\rangle + o\big(|s - t| + |y - x|^2\big).$$

Here $r \mapsto o(r)$ denotes any continuous function such that $\lim_{r \to 0} \frac{o(r)}{r} = 0$.

Definition 4 *Let $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^k$ a continuous function satisfying $u(T,\cdot) = h(\cdot)$. The function u is a viscosity solution of (1) if:*

$$u(t,x) \in \mathrm{Dom}(\varphi), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d,$$

and, for all $z \in \mathbb{R}^k$, at any point $(t,x) \in [0,T] \times \mathbb{R}^d$, for any $(p,q,X) \in \mathcal{P}^{2,+} \langle u, z\rangle (t,x)$, we have that

$$p + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^*)(t,x)X\big) + \big\langle b(t,x), q\big\rangle + \langle f\big(t,x,u(t,x)\big), z\rangle \geq \varphi'_*\left(u\left(t,x\right); z\right)$$

Remark 5 *If $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^k$ is a continuous function, which satisfies $u(T,\cdot) = h(\cdot)$, then u is a viscosity solution of (1) if and only if*

$$u(t,x) \in \mathrm{Dom}(\varphi), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d,$$

and, for all $z \in \mathbb{R}^k$, at any point $(t,x) \in [0,T] \times \mathbb{R}^d$, for any $(p,q,X) \in \mathcal{P}^{2,-} \langle u, z\rangle (t,x)$, we have that

$$p + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^*)(t,x)X\big) + \langle b(t,x), q\rangle + \big\langle f\big(t,x,u(t,x)\big), z\big\rangle \leq \varphi'^{,*}\left(u\left(t,x\right); z\right)$$

We now present the main results.

Theorem 6 (Existence) *Let the assumptions $(A.1)$-$(A.4)$ be satisfied. Then the multidimensional parabolic variational inequality (1) has at least a viscosity solution.*

For the proof of the existence theorem we use a stochastic approach; we study a backward stochastic variational inequality which allows us to give a probabilistic formula for the solutions of (1). We present this approach in the last section.

Concerning the uniqueness for equation (1), we need to impose more restrictive assumptions.

$(A.5)$ For all $u \in \mathrm{Dom}(\varphi)$, there exists a neighborhood V of u such that $(\partial\varphi)_0$ is bounded on $\mathrm{Dom}(\partial\varphi) \cap V$.

$(A.6)$ If $u \in \mathrm{Dom}(\varphi)$ and $z \in \mathbb{R}^k$ such that $u + z \in \mathrm{Dom}(\varphi)$, then there exists a neighbourhood V of u such that

$$\forall v \in V \cap \mathrm{Dom}(\partial\varphi), \; \exists t > 0 : v + tz \in \mathrm{Dom}(\partial\varphi).$$

Theorem 7 (Uniqueness) *Let the assumptions $(A.1)$, $(A.2)$, $(A.5)$ and $(A.6)$ be satisfied. Then the multidimensional parabolic variational inequality (1) has at most one viscosity solution in the class of continuous functions with polynomial growth.*

3 Proof of the uniqueness theorem

First we prove the following

Lemma 8 *If* $\varphi : \mathbb{R}^k \to (-\infty, +\infty]$ *is a convex function, proper and l.s.c., then we have:*

(a) *for every* $u, v \in \mathrm{Dom}\,(\varphi)$,

$$\varphi'_-(u, u-v) \geq \varphi'_+(v, u-v), \tag{5}$$

(b) *for every* $u, v \in \mathrm{Dom}\,(\partial\varphi)$ *and* $z \in \mathbb{R}^k$, $\varphi'_*(u;z) \leq \varphi'_-(u;z)$ *and* $\varphi'^{,*}(u;z) \geq \varphi'_+(u;z)$,

(c) *for every* $z \in \mathbb{R}^k$, $\varphi'_*(\cdot;z)$ *is l.s.c on* $\mathrm{Dom}\,(\varphi)$ *and* $\varphi'^{,*}(\cdot;z)$ *is u.s.c. on* $\mathrm{Dom}\,(\varphi)$,

(d) *for every* $u, v \in \overline{\mathrm{Dom}\,(\varphi)}$, *if*

$$\limsup_{\substack{(u',v')\to(u,v)\\ u',v'\in\mathrm{Dom}(\partial\varphi)}} \left(\inf_{t>0}\left[|\partial\varphi|_0(u'-t(u-v)) + |\partial\varphi|_0(v'+t(u-v))\right]\right) < +\infty, \tag{6}$$

then

$$\varphi'_*(u, u-v) \geq \varphi'^{,*}(v, u-v).$$

Proof. (a) By the convexity of φ, for $t \in (0,1)$ we have

$$\varphi((1-t)u + tv) + \varphi(tu + (1-t)v) \leq \varphi(u) + \varphi(v).$$

Therefore,

$$-\frac{\varphi(u - t(u-v)) - \varphi(u)}{t} \geq \frac{\varphi(v + t(u-v)) - \varphi(v)}{t},$$

passing to the limit as $t \searrow 0$, we obtain $\varphi'_-(u, u-v) \geq \varphi'_+(v, u-v)$.

Results $b)$ and $c)$ follow in a standard way from the definition of φ'_* and $\varphi'^{,*}$, so we skip their proofs.

(d) For simplicity, let us denote $z = u - v$. From (6), we can find $M > 0$ and $r > 0$ such that, for $u' \in B(u,r) \cap \mathrm{Dom}\,(\partial\varphi)$, $v' \in B(v,r) \cap \mathrm{Dom}\,(\partial\varphi)$, there is a $t_{u',v'} < 0$ such that, for $t \in (t_{u',v'}, 0)$, we have

$$|\partial\varphi|_0(u'+tz) + |\partial\varphi|_0(v'-tz) \leq M.$$

Let u', v', and t be as above. Then

$$\begin{aligned}\frac{\varphi(u'+tz)-\varphi(u')}{t} &= \frac{\varphi(u'+t(u'-v'))-\varphi(u')}{t} + \frac{\varphi(u'+tz)-\varphi(u'+t(u'-v'))}{t}\\ &\geq \frac{\varphi(u'+t(u'-v'))-\varphi(u')}{t} + \langle(\partial\varphi)_0(u'+tz), z-(u'-v')\rangle.\end{aligned}$$

It follows that

$$\frac{\varphi(u'+tz)-\varphi(u')}{t} \geq \frac{\varphi(u'+t(u'-v'))-\varphi(u')}{t} - M|z-(u'-v')|.$$

Passing to the limit as $t \nearrow 0$,

$$\varphi'_-(u';z) \geq \varphi'_-(u';u'-v') - M|z-(u'-v')|.$$

In the similar manner we obtain that

$$\varphi'_+(v';u'-v') \geq \varphi'_+(v';z) - M|z-(u'-v')|.$$

Combining the last two inequalities with (5) we deduce that

$$\varphi'_-(u';z) \geq \varphi'_+(v';z) - 2M|z-(u'-v')|.$$

Passing to the limit as $u' \to u$ and $v' \to v$, we obtain $\varphi'_*(u;z) \geq \varphi'^{,*}(v;z)$. ■

Remark 9 *From the above Lemma, point (c) we can consider in Definition 4,* $\bar{\mathcal{P}}^{2,+}\langle u,z\rangle$ *and* $\bar{\mathcal{P}}^{2,-}\langle v,z\rangle$ *instead of* $\mathcal{P}^{2,+}\langle u,z\rangle$*, respectively* $\mathcal{P}^{2,-}\langle v,z\rangle$*.*

Proof of Theorem 7.

Let $u,v \in C\left([0,T]\times\mathbb{R}^d;\mathbb{R}^k\right)$ be viscosity solutions for (1). We must prove that $u=v$ on $[0,T]\times\mathbb{R}^d$.

For some $\lambda>0$, to be precised later, we make the following transformations:

$$\bar{u}(t,x) = e^{\lambda t}\xi^{-1}(x)\,u(t,x)\,,\ \bar{v}(t,x) = e^{\lambda t}\xi^{-1}(x)\,v(t,x)\,,$$

where $\xi(x) = \left(1+|x|^2\right)^{\mu/2}$, with μ larger than the order of growth of u and v.

Then $\bar{u}$ and $\bar{v}$ are bounded and they are solutions, in the viscosity sense, of

$$\begin{cases} \dfrac{\partial\bar{u}(t,x)}{\partial t} + \bar{\mathcal{L}}_t\bar{u}(t,x) + \bar{f}\big(t,x,\bar{u}(t,x)\big) \in e^{\lambda t}\xi^{-1}(x)\,\partial\varphi\big(e^{-\lambda t}\xi(x)\,\bar{u}(t,x)\big), \\ \bar{u}(T,x) = e^{\lambda T}\xi^{-1}(x)\,h(x),\ t\in[0,T]\,,\ x\in\mathbb{R}^d, \end{cases} \tag{7}$$

where, for $i=\overline{1,k}$,

$$(\bar{\mathcal{L}}_t\bar{u})_i(t,x) = (\mathcal{L}_t\bar{u})_i(t,x) + \big\langle(\sigma\sigma^*)(t,x)\,\xi^{-1}(x)\,\nabla\xi(x)\,,\nabla\bar{u}_i(t,x)\big\rangle$$

and

$$\begin{aligned}\bar{f}(t,x,\bar{u}) &= e^{\lambda t}\xi^{-1}(x)\,f(t,x,e^{-\lambda t}\xi(x)\,\bar{u}) - \lambda\bar{u} \\ &+\frac{1}{2}\operatorname{Tr}\left[(\sigma\sigma^*)(t,x)\,\xi^{-1}(x)\,\mathrm{D}^2\xi(x)\right]\bar{u} + \big\langle b(t,x)\,,\xi^{-1}(x)\,\nabla\xi(x)\,\big\rangle\bar{u}.\end{aligned}$$

Since $\lim\limits_{|x|\to+\infty}\bar{u}(t,x) = \lim\limits_{|x|\to+\infty}\bar{v}(t,x) = 0$ and $\bar{u}(T,x)=\bar{v}(T,x)$, there exists $(t_0,x_0)\in[0,T)\times\mathbb{R}^d$ such that

$$\theta := |\bar{u}(t_0,x_0)-\bar{v}(t_0,x_0)| \geq |\bar{u}(t,x)-\bar{v}(t,x)|\,,\ \forall\,(t,x)\in[0,T]\times\mathbb{R}^d.$$

Let us suppose that $u \neq v$. This implies $\theta > 0$. We set

$$z_0 = \frac{1}{\theta} \left(\bar{u}\left(t_0, x_0\right) - \bar{v}\left(t_0, x_0\right) \right).$$

Then

$$\theta := \left\langle \bar{u}\left(t_0, x_0\right) - \bar{v}\left(t_0, x_0\right), z_0 \right\rangle \geq \left\langle \bar{u}\left(t, x\right) - \bar{v}\left(t, x\right), z_0 \right\rangle, \ \forall \left(t, x\right) \in [0, T] \times \mathbb{R}^d.$$

We let, for $\alpha > 0$,

$$\Phi_\alpha\left(t, x, s, y\right) := \left\langle \bar{u}\left(t, x\right) - \bar{v}\left(s, y\right), z_0 \right\rangle - \frac{\alpha}{2}\left(\left|t - s\right|^2 + \left|x - y\right|^2 \right).$$

Since $\limsup_{|x| \vee |y| \to +\infty} \Phi_\alpha\left(t, x, s, y\right) \leq 0$ and $\Phi_\alpha\left(t_0, x_0, t_0, x_0\right) = \theta > 0$, there exists $\left(t_\alpha, x_\alpha, s_\alpha, y_\alpha\right) \in \left([0, T] \times \mathbb{R}^d\right)^2$ such that

$$M_\alpha := \Phi_\alpha\left(t_\alpha, x_\alpha, s_\alpha, y_\alpha\right) = \sup \Phi_\alpha.$$

By Lemma 3.1 from [2], we have:

$$\begin{aligned} &(a) \quad \lim_{\alpha \to +\infty} \alpha \left(\left|x_\alpha - y_\alpha\right|^2 + \left|t_\alpha - s_\alpha\right|^2 \right) = 0, \\ &(b) \quad \text{whenever } (\hat{t}, \hat{x}) \text{ is an accumulation point for } \left(t_\alpha, x_\alpha\right) \text{ as } \alpha \to +\infty, \\ &\qquad \text{we have that } \lim_{\alpha \to +\infty} M_\alpha = \left\langle \bar{u}(\hat{t}, \hat{x}) - \bar{v}(\hat{t}, \hat{x}), z_0 \right\rangle = \theta. \end{aligned} \tag{8}$$

Since, for large α, $\left(t_\alpha, x_\alpha\right)$ remains in a compact subset of $[0, T] \times \mathbb{R}^d$, there is at least one accumulation point $(\hat{t}, \hat{x})$ for $\left(t_\alpha, x_\alpha\right)$ as $\alpha \to +\infty$. We can suppose, without restricting the generality, that $\left(t_\alpha, x_\alpha\right) \to (\hat{t}, \hat{x})$. Of course, from (8), $\hat{t} < T$, and we also can assume that $t_\alpha, s_\alpha \in [0, T)$ for every α. Another consequence of (8) is that $\bar{u}(\hat{t}, \hat{x}) - \bar{v}(\hat{t}, \hat{x}) = \theta z_0$. Indeed,

$$\left|\bar{u}(\hat{t}, \hat{x}) - \bar{v}(\hat{t}, \hat{x})\right| \leq \theta = \left\langle \bar{u}(\hat{t}, \hat{x}) - \bar{v}(\hat{t}, \hat{x}), z_0 \right\rangle \leq \left|\bar{u}(\hat{t}, \hat{x}) - \bar{v}(\hat{t}, \hat{x})\right|,$$

from which we conclude $\bar{u}(\hat{t}, \hat{x}) - \bar{v}(\hat{t}, \hat{x}) = \left|\bar{u}(\hat{t}, \hat{x}) - \bar{v}(\hat{t}, \hat{x})\right| z_0 = \theta z_0$. Let us apply now Theorem 3.2 from [2], which asserts that, for every $\alpha > 0$, there exist $X, Y \in S^d$ such that:

$$\begin{aligned} &(a) \quad \left(\alpha\left(t_\alpha - s_\alpha\right), \alpha\left(x_\alpha - y_\alpha\right), X_\alpha\right) \in \bar{\mathcal{P}}^{2,+} \left\langle \bar{u}, z_0 \right\rangle \left(t_\alpha, x_\alpha\right) \\ &(b) \quad \left(\alpha\left(t_\alpha - s_\alpha\right), \alpha\left(x_\alpha - y_\alpha\right), Y_\alpha\right) \in \bar{\mathcal{P}}^{2,-} \left\langle \bar{v}, z_0 \right\rangle \left(s_\alpha, y_\alpha\right) \\ &(c) \quad \begin{pmatrix} X_a & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \end{aligned}$$

From the definition of the viscosity solution, and Remark 5, we obtain:

$$\begin{aligned} &\alpha\left(t_\alpha - s_\alpha\right) + \bar{\mathcal{L}}_{\alpha(x_\alpha - y_\alpha), X_\alpha}\left(t_\alpha, x_\alpha\right) + \left\langle \bar{f}\left(t_\alpha, x_\alpha, \bar{u}\left(t_\alpha, x_\alpha\right)\right), z_0 \right\rangle \\ &\qquad \geq e^{\lambda t_\alpha} \xi^{-1}\left(x_\alpha\right) \varphi'_*\left(e^{-\lambda t_\alpha} \xi\left(x_\alpha\right) \bar{u}\left(t_\alpha, x_\alpha\right); z_0\right), \\ &\alpha\left(t_\alpha - s_\alpha\right) + \bar{\mathcal{L}}_{\alpha(x_\alpha - y_\alpha), Y_\alpha}\left(s_\alpha, y_\alpha\right) + \left\langle \bar{f}\left(s_\alpha, y_\alpha, \bar{v}\left(s_\alpha, y_\alpha\right)\right), z_0 \right\rangle \\ &\qquad \leq e^{\lambda s_\alpha} \xi^{-1}\left(y_\alpha\right) \varphi'^{,*}\left(e^{-\lambda s_\alpha} \xi\left(y_\alpha\right) \bar{v}\left(s_\alpha, y_\alpha\right); z_0\right), \end{aligned}$$

where, for $q \in \mathbb{R}^d$, $X \in S^d$,

$$\bar{\mathcal{L}}_{q,X}(t,x) = \frac{1}{2}\mathrm{Tr}\left[(\sigma\sigma^*)(t,x)X\right] + \langle b(t,x),q\rangle + \left\langle (\sigma\sigma^*)(t,x)\xi^{-1}(x)\nabla\xi(x),q\right\rangle.$$

Substracting the two inequalities, we have:

$$\begin{aligned}
&\bar{\mathcal{L}}_{\alpha(x_\alpha-y_\alpha),X_\alpha}(t_\alpha,x_\alpha) - \bar{\mathcal{L}}_{\alpha(x_\alpha-y_\alpha),Y_\alpha}(s_\alpha,y_\alpha)\\
&\qquad + \left\langle \bar{f}(t_\alpha,x_\alpha,\bar{u}(t_\alpha,x_\alpha)) - \bar{f}(s_\alpha,y_\alpha,\bar{v}(s_\alpha,y_\alpha)),z_0\right\rangle\\
&\geq e^{\lambda t_\alpha}\xi^{-1}(x_\alpha)\varphi'_*\left(e^{-\lambda t_\alpha}\xi(x_\alpha)\bar{u}(t_\alpha,x_\alpha);z_0\right)\\
&\qquad - e^{\lambda s_\alpha}\xi^{-1}(y_\alpha)\varphi'^{,*}\left(e^{-\lambda s_\alpha}\xi(y_\alpha)\bar{v}(s_\alpha,y_\alpha);z_0\right).
\end{aligned} \tag{9}$$

From Lemma 8, point (c), we get

$$\begin{aligned}
&\liminf_{\alpha\to+\infty} e^{\lambda t_\alpha}\xi^{-1}(x_\alpha)\varphi'_*\left(e^{-\lambda t_\alpha}\xi(x_\alpha)\bar{u}(t_\alpha,x_\alpha);z_0\right)\\
&\qquad \geq e^{\lambda\hat{t}}\xi^{-1}(\hat{x})\varphi'_*\left(e^{-\lambda\hat{t}}\xi(\hat{x})\bar{u}(\hat{t},\hat{x});z_0\right)\\
&\limsup_{\alpha\to+\infty} e^{\lambda s_\alpha}\xi^{-1}(y_\alpha)\varphi'^{,*}\left(e^{-\lambda s_\alpha}\xi(y_\alpha)\bar{v}(s_\alpha,y_\alpha);z_0\right)\\
&\qquad \leq e^{\lambda\hat{t}}\xi^{-1}(\hat{x})\varphi'^{,*}\left(e^{-\lambda\hat{t}}\xi(\hat{x})\bar{v}(\hat{t},\hat{x});z_0\right).
\end{aligned}$$

Now, by Lemma 8, point d), and conditions $(A.5)$ and $(A.6)$,

$$\varphi'_*\left(e^{-\lambda\hat{t}}\xi(\hat{x})\bar{u}(\hat{t},\hat{x});z_0\right) \geq \varphi'^{,*}\left(e^{-\lambda\hat{t}}\xi(\hat{x})\bar{u}(\hat{t},\hat{x});z_0\right),$$

since $\bar{u}(\hat{t},\hat{x}) - \bar{v}(\hat{t},\hat{x}) = \theta z_0$. It follows that

$$\begin{aligned}
&\liminf_{\alpha\to+\infty}\Big[e^{\lambda t_\alpha}\xi^{-1}(x_\alpha)\varphi'_*\left(e^{-\lambda t_\alpha}\xi(x_\alpha)\bar{u}(t_\alpha,x_\alpha);z_0\right)\\
&\qquad - e^{\lambda s_\alpha}\xi^{-1}(y_\alpha)\varphi'^{,*}\left(e^{-\lambda s_\alpha}\xi(y_\alpha)\bar{v}(s_\alpha,y_\alpha);z_0\right)\Big]\\
&\geq e^{\lambda\hat{t}}\xi^{-1}(\hat{x})\varphi'_*\left(e^{-\lambda\hat{t}}\xi(\hat{x})\bar{u}(\hat{t},\hat{x});z_0\right)\\
&\qquad - e^{\lambda\hat{t}}\xi^{-1}(\hat{x})\varphi'^{,*}\left(e^{-\lambda\hat{t}}\xi(\hat{x})\bar{v}(\hat{t},\hat{x});z_0\right) \geq 0.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
&\bar{\mathcal{L}}_{\alpha(x_\alpha-y_\alpha),X_\alpha}(t_\alpha,x_\alpha) - \bar{\mathcal{L}}_{\alpha(x_\alpha-y_\alpha),Y_\alpha}(s_\alpha,y_\alpha)\\
&\qquad = \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(t_\alpha,x_\alpha)X_\alpha - \sigma\sigma^*(s_\alpha,y_\alpha)Y_\alpha\right] + \alpha\Big\langle b(t_\alpha,x_\alpha) - b(s_\alpha,y_\alpha)\\
&\qquad + \sigma\sigma^*(t_\alpha,x_\alpha)\xi^{-1}(x_\alpha)\nabla\xi(x_\alpha) - \sigma\sigma^*(s_\alpha,y_\alpha)\xi^{-1}(y_\alpha)\nabla\xi(y_\alpha),x_\alpha-y_\alpha\Big\rangle\\
&\leq \frac{3}{2}\alpha\left|\sigma(t_\alpha,x_\alpha)-\sigma(s_\alpha,y_\alpha)\right|^2 + \alpha\left|x_\alpha-y_\alpha\right|\Big(\left|b(t_\alpha,x_\alpha)-b(s_\alpha,y_\alpha)\right|\\
&\qquad + \left|\sigma\sigma^*(t_\alpha,x_\alpha)\xi^{-1}(x_\alpha)\nabla\xi(x_\alpha) - \sigma\sigma^*(s_\alpha,y_\alpha)\xi^{-1}(y_\alpha)\nabla\xi(y_\alpha)\right|\Big)\\
&\leq 3\alpha\left(L_1^2+L_1\right)\left(\left|x_\alpha-y_\alpha\right|^2 + \left|t_\alpha-s_\alpha\right|^2\right),
\end{aligned}$$

where L_1 is the Lipschitz constant of $(\sigma, b, \sigma\sigma^*\xi^{-1}\nabla\xi)$. Hence

$$\limsup_{\alpha\to+\infty}\left[\bar{\mathcal{L}}_{\alpha(x_\alpha-y_\alpha),X_\alpha}(t_\alpha,x_\alpha)-\bar{\mathcal{L}}_{\alpha(x_\alpha-y_\alpha),Y_\alpha}(s_\alpha,y_\alpha)\right]\leq 0.$$

Finally, from condition $(A.2)$,

$$\begin{aligned}
&\lim_{\alpha\to+\infty}\left\langle \bar{f}\left(t_\alpha,x_\alpha,\bar{u}\left(t_\alpha,x_\alpha\right)\right)-\bar{f}\left(s_\alpha,y_\alpha,\bar{v}\left(s_\alpha,y_\alpha\right)\right),z_0\right\rangle\\
&=\left\langle \bar{f}\left(\hat{t},\hat{x},\bar{u}(\hat{t},\hat{x})\right)-\bar{f}(\hat{t},\hat{x},\bar{v}(\hat{t},\hat{x})),z_0\right\rangle\\
&=\left(-\lambda+\frac{1}{2}\operatorname{Tr}\left[\sigma\sigma^*(\hat{t},\hat{x})\xi^{-1}(\hat{x})\operatorname{D}^2\xi(\hat{x})\right]+\left\langle b(\hat{t},\hat{x}),\xi^{-1}(\hat{x})\nabla\xi(\hat{x})\right\rangle\right)\theta\\
&\qquad+e^{\lambda\hat{t}}\xi^{-1}(\hat{x})\left\langle f(\hat{t},\hat{x},e^{-\lambda\hat{t}}\xi(\hat{x})\bar{u}(\hat{t},\hat{x}))-f(\hat{t},\hat{x},e^{-\lambda\hat{t}}\xi(\hat{x})\bar{v}(\hat{t},\hat{x})),z_0\right\rangle\\
&\leq\left(-\lambda+\tfrac{1}{2}\operatorname{Tr}\left[\sigma\sigma^*(\hat{t},\hat{x})\xi^{-1}(\hat{x})\operatorname{D}^2\xi(\hat{x})\right]+\left\langle b(\hat{t},\hat{x}),\xi^{-1}(\hat{x})\nabla\xi(\hat{x})\right\rangle+\gamma\right)\theta.
\end{aligned}$$

By taking

$$\lambda>\gamma+\sup\left[\frac{1}{2}\operatorname{Tr}\left[\sigma\sigma^*\xi^{-1}\operatorname{D}^2\xi\right]+\left\langle b,\xi^{-1}\nabla\xi\right\rangle\right],$$

we get

$$\lim_{\alpha\to+\infty}\left\langle \bar{f}\left(t_\alpha,x_\alpha,\bar{u}\left(t_\alpha,x_\alpha\right)\right)-\bar{f}\left(s_\alpha,y_\alpha,\bar{v}\left(s_\alpha,y_\alpha\right)\right),z_0\right\rangle<0.$$

Passing to the limit as $\alpha\to+\infty$ in (9) we obtain a contradiction. Hence $\bar{u}=\bar{v}$, and so $u=v$. ■

4 Proof of the existence theorem

In this section we prove Theorem 6. This is obtained by a stochastic approach. Using a certain backward stochastic variational inequality, we will obtain a generalized Feynman-Kaç representation formula for the viscosity solution of (1).

Let $\{W_t : t\geq 0\}$ be a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. We denote by $\{\mathcal{F}_t : t\geq 0\}$ the natural filtration generated by $\{W_t : t\geq 0\}$ and augmented by $\mathcal{N}$, the set of $\mathbb{P}$- null events of $\mathcal{F}$:

$$\mathcal{F}_t=\sigma\{W_r : 0\leq r\leq t\}\vee\mathcal{N}.$$

4.1 Backward stochastic variational inequality

We consider the following backward stochastic variational inequality

$$\begin{cases} dY_s+F(s,Y_s,Z_s)ds\in\partial\varphi\left(Y_s\right)ds+Z_sdW_s,\ s\in[0,T]\\ Y_T=\xi,\end{cases}\tag{10}$$

where:

(I) $F : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ satisfies that, for some $\alpha \in \mathbb{R}$, $\beta \geq 0$,

$$\left|\begin{array}{ll} (i) & F(\cdot,\cdot,y,z) \text{ is progressively measurable stochastic process (p.m.s.p.)}, \\ (ii) & y \longmapsto F(\omega,t,y,z) \text{ is continuous, a.s.}, \\ (iii) & \big\langle y-y', F(t,y,z)-F(t,y',z)\big\rangle \leq \alpha\,|y-y'|^2, \text{ a.s.}, \\ (iv) & \big|F(t,y,z)-F(t,y,z')\big| \leq L\,\|z-z'\|, \text{ a.s.}, \\ (v) & \mathbb{E}\Big(\int_0^T F_R^{\#}(t)dt\Big)^2 < \infty,\ \forall R \geq 0, \\ & \qquad\qquad \text{where } F_R^{\#}(t) := \sup\{|F(t,y,0)| : |y| \leq R\}, \end{array}\right. \tag{11}$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}^k$, $z, z' \in \mathbb{R}^{k\times d}$,

(II) $\varphi : \mathbb{R}^k \to (-\infty, +\infty]$ is proper, convex and l.s.c.

(III) The terminal date $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^k)$ such that

$$\mathbb{E}\big(\varphi(\xi)\big) < \infty. \tag{12}$$

For the proof of the next theorem see [9].

Theorem 10 *Let the assumptions (I)-(III) be satisfied. Then there exists a unique triple $\{(Y_t, Z_t, U_t) : t \in [0,T]\}$ of p.m.s.p. such that*

$$(a)\quad \mathbb{E}\left[\sup_{s\in[0,T]} |Y_s|^2 + \int_0^T \left(|Z_s|^2 + |U_s|^2\right) ds\right] < +\infty,$$

$$(b)\quad \mathbb{E}\int_0^T \varphi(Y_s)\,ds < +\infty,$$

$$(c)\quad (Y_t, U_t) \in \partial\varphi,\ \mathbb{P}\otimes dt \text{ a.e. on } \Omega\times[0,T],$$

$$(d)\quad Y_t + \int_t^T U_s ds = \xi + \int_t^T F(s, Y_s, Z_s)\,ds - \int_t^T Z_s dW_s, \text{ for all } t \geq 0, \text{ a.s.}$$

From assumption $(A.1)$ of the first section, it follows (see for example [6], [8]), that for each $(t,x) \in [0,T]\times\mathbb{R}^d$ there exists a unique continuous $\{\mathcal{F}_s^t\}$-p.m.s.p. $\{X_s^{t,x} : s \in [0,T]\}$, solution of the stochastic differential equation:

$$X_s^{t,x} = x + \int_t^{s\vee t} b(r, X_r^{t,x})dr + \int_t^{s\vee t} \sigma(r, X_r^{t,x})dW_r, \tag{13}$$

where

$$\mathcal{F}_s^t = \sigma\{W_r - W_t : t \leq r \leq s\} \vee \mathcal{N}.$$

The following proposition summarize some already well known properties of the solutions of the equations (13).

Proposition 11 *Under the assumption (A.1), the solution* $\{X_s^{t,x} : s \in [0,T]\}$ *of the equation (13) satisfies that for all* $p \geq 2$*, there exists some constant* $C = C(p,T,L) > 0$*, such that, for all* $t, \tilde{t} \in [0,T]$*,* $x, \tilde{x} \in \mathbb{R}^d$

$$(i) \quad \mathbb{E} \sup_{s\in[0,T]} |X_s^{t,x}|^p \leq C(1+|x|^p),$$

$$(ii) \quad \mathbb{E} \sup_{s\in[0,T]} |X_s^{t,x} - X_s^{\tilde{t},\tilde{x}}|^p \leq C\big(1+|x|^p+|\tilde{x}|^p\big)\big(|x-\tilde{x}|^p + |t-\tilde{t}|^{p/2}\big).$$

From the Theorem 10, with $\xi = h(X_T^{t,x})$, $F(\omega,s,y,z) = f(s, X_s^{t,x}(\omega), y)$, it follows, under assumptions (A.1)-(A.4) that, for $(t,x) \in [0,T]\times\mathbb{R}^d$, there exists a unique triple of $\{\mathcal{F}_s^t\}$-p.m.s.p. $\{(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x}) : s \in [0,T]\}$ solution of the backward stochastic variational inequality

$$Y_s^{t,x} + \int_s^T U_r^{t,x} dr = h\left(X_T^{t,x}\right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}\right) dr - \int_s^T Z_r^{t,x} dW_r, \qquad \text{for all } s \in [t,T] \text{ a.s.} \tag{14}$$

with $(Y_s^{t,x}, U_s^{t,x}) \in \partial\varphi$, $\mathbb{P}\times ds$ a.e. on $\Omega\times[t,T]$, and $Y_s^{t,x} = Y_t^{t,x}$, $Z_s^{t,x} = U_s^{t,x} = 0$, $\forall s \in [0,t]$.

Moreover we have the following properties of the solution of (14):

Proposition 12 *Under the assumptions (A.1)-(A.4) we have that there exists some constant* $C > 0$*, such that, for all* $t, \tilde{t} \in [0,T]$*,* $x, \tilde{x} \in \mathbb{R}^d$ *:*

$$(i) \quad \mathbb{E} \sup_{s\in[0,T]} |Y_s^{t,x}|^2 \leq C\left(1+|x|^2\right)$$

$$(ii) \quad \mathbb{E} \sup_{s\in[0,T]} |Y_s^{t,x} - Y_s^{\tilde{t},\tilde{x}}|^2 \leq \mathbb{E}\Big[\big|h(X_T^{t,x}) - h(X_T^{\tilde{t},\tilde{x}})\big|^2 + \int_0^T \big|1_{[t,T]}(r) f(r, X_r^{t,x}, Y_r^{t,x}) - 1_{[\tilde{t},T]}(r) f(r, X_r^{\tilde{t},\tilde{x}}, Y_r^{t,x})\big|^2 dr\Big]$$

We define

$$u(t,x) = Y_t^{t,x}, \quad (t,x) \in [0,T]\times\mathbb{R}^d, \tag{15}$$

which is a determinist quantity since $Y_t^{t,x}$ is $\mathcal{F}_t^t \equiv \sigma\{\mathcal{N}\}$–measurable.

From the Markov property we have that

$$u(s, X_s^{t,x}) = Y_s^{t,x} \tag{16}$$

For the proof of the next proposition see [11].

Proposition 13 *Under the assumptions (A.1)-(A.4), the function u defined by (15) satisfies:*

$$\begin{aligned} &(i) \quad u(t,x) \in \mathrm{Dom}(\varphi), \ \forall (t,x) \in [0,T]\times\mathbb{R}^d, \\ &(ii) \quad |u(t,x)| \leq C(1+|x|^p), \ \forall (t,x) \in [0,T]\times\mathbb{R}^d, \\ &(iii) \quad u \in C\big([0,T]\times\mathbb{R}^d; \mathbb{R}^k\big). \end{aligned} \tag{17}$$

4.2 Connection with variational inequalities for systems of partial differential equations

We prove that the function u defined above is a viscosity solution, in the sense of Definition 4, for the multidimensional parabolic variational inequality (1).

Theorem 6 is a consequence of the following:

Theorem 14 *Let the assumptions (A.1)-(A.4) be satisfied. Then the function u defined by (15) is a viscosity solution for the multidimensional parabolic variational inequality (1).*

Proof. Let $z \in \mathbb{R}^k$, $(t,x) \in [0,T] \times \mathbb{R}^d$ and $(p,q,X) \in \mathcal{P}^{2,+} \langle u, z \rangle (t,x)$. Let us denote for simplicity

$$V(t,x,u,p,q,X) = p + \frac{1}{2}\mathrm{Tr}\big((\sigma\sigma^*)(t,x)X\big) + \langle b(t,x), q\rangle + \langle f(t,x,u), z\rangle$$

Suppose, contrary to our claim, that

$$V(t,x,u(t,x),p,q,X) < \varphi'_*(u(t,x);z),$$

and we will find a contradiction.

It follows by continuity of f, u, b, σ that there exist $\varepsilon > 0$, $\delta > 0$ such that for all $|s-t| \le \delta$, $|y-x| \le \delta$,

$$V(s,y,u(s,y),p+\varepsilon, q + (X+\varepsilon I)(y-x), X + \varepsilon I) < \varphi'_*(u(s,y);z). \tag{18}$$

Now since $(p,q,X) \in \mathcal{P}^{2,+}\langle u, z\rangle (t,x)$, there exists $0 < \delta' \le \delta$ such that

$$\langle u(s,y), z\rangle < \Psi(s,y), \tag{19}$$

for all $s \in [0,T]$, $s \ne t$, $y \in \mathbb{R}^d$, $y \ne x$ with $0 < s - t \le \delta'$, $|y-x| \le \delta'$, where

$$\Psi(s,y) = \langle u(t,x), z\rangle + (p+\varepsilon)(s-t) + \langle q, y-x\rangle + \frac{1}{2}\big\langle (X+\varepsilon I)(y-x), y-x\big\rangle$$

Let

$$\tau = \inf\big\{s > t : |X_s^{t,x} - x| \ge \delta'\big\}$$

If we denote by

$$(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x}) = \big(\langle Y_s^{t,x}, z\rangle, \langle Z_s^{t,x}, z\rangle\big),\ t \le s,$$

then

$$\begin{cases} \bar{Y}_s^{t,x} = \langle u(\tau, X_\tau^{t,x}), z\rangle + \displaystyle\int_s^\tau \big(\langle f(r, X_r^{t,x}, u(r,X_r^{t,x})), z\rangle - \langle U_r^{t,x}, z\rangle\big) dr \\ \qquad\qquad - \displaystyle\int_s^\tau \bar{Z}_r^{t,x} dW_r, \\ \langle U_s^{t,x}, z\rangle \in \big[\varphi'_-(Y_s^{t,x}; z), \varphi'_+(Y_s^{t,x};z)\big]. \end{cases}$$

From Itô's formula it follows that

$$\left(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x}\right) := \big(\Psi(s, X_s^{t,x}), (\nabla\Psi\sigma)(s, X_s^{t,x})\big),\ t \le s \le t+\delta'$$

satisfies

$$\hat{Y}_s^{t,x} = \Psi(\tau, X_\tau^{t,x}) - \int_s^\tau \Big[\frac{\partial\Psi}{\partial r}(r, X_r^{t,x}) + \mathcal{L}_r\Psi(r, X_r^{t,x})\Big] dr - \int_s^\tau \hat{Z}_r^{t,x} dW_r$$

Let $(\tilde{Y}_s^{t,x}, \tilde{Z}_s^{t,x}) = (\hat{Y}_s^{t,x} - \bar{Y}_s^{t,x}, \hat{Z}_s^{t,x} - \bar{Z}_s^{t,x})$.

We have

$$\begin{aligned}\tilde{Y}_s^{t,x} &= \big[\Psi(\tau, X_\tau^{t,x}) - \langle u(\tau, X_\tau^{t,x}), z\rangle\big] + \int_s^\tau \Big[- \frac{\partial\Psi}{\partial r}(r, X_r^{t,x}) - \mathcal{L}_r\Psi(r, X_r^{t,x}) \\ &- \big\langle f(r, X_r^{t,x}, u(r, X_r^{t,x})), z\big\rangle + \langle U_r^{t,x}, z\rangle\Big] dr - \int_s^\tau \tilde{Z}_r^{t,x} dW_r\end{aligned} \tag{20}$$

We note that, from Lemma 8, point (c),

$$\big\langle U_s^{t,x}, z\big\rangle \ge \varphi'_-\big(u(s, X_s^{t,x}); z\big) \ge \varphi'_*\big(u(s, X_s^{t,x}); z\big),\ \mathbb{P}\otimes dt \text{ a.e.}$$

Moreover, the choice of δ' and τ implies that

$$\Psi(\tau, X_\tau^{t,x}) > \big\langle u(\tau, X_\tau^{t,x}), z\big\rangle .$$

From (18) it follows that

$$\begin{aligned}\varphi_*'\left(u(s, X_s^{t,x}); z\right) &> V\left(s, X_s^{t,x}, u(s, X_s^{t,x}), p+\varepsilon, q + (X+\varepsilon I)\left(X_s^{t,x} - x\right), X+\varepsilon I\right) \\ &= \frac{\partial\Psi}{\partial s}(s, X_s^{t,x}) + \mathcal{L}_s\Psi(s, X_s^{t,x}) + \left\langle f(s, X_s^{t,x}, u\left(s, X_s^{t,x}\right)), z\right\rangle .\end{aligned}$$

These inequalities and equation (20) imply that $\tilde{Y}_t^{t,x} > 0$, or equivalently

$$\Psi(t,x) > \langle u(t,x), z\rangle ,$$

which contradicts the definition of Ψ. Hence we have

$$V(t, x, u(t,x), p, q, X) \ge \varphi_*'(u(t,x); z).$$

This proves that u is a viscosity solution of (1). ■

References

[1] Borwein, J., Lewis, A. (2000). Convex analysis and nonlinear optimization: Theory and Examples *Springer-Verlag*, New-York.

[2] Crandall, M.G., Ishii, H., Lions, P.L. (1992). User's guide to the viscosity solutions of second order partial differential equations *Bull. Amer. Math. Soc.* 27, 1-67.

[3] Crandall, M.G., Lions, P.L. (1983). Viscosity solutions of Hamilton-Jacobi equations *Trans. Amer. Math. Soc.* 277, 1-42.

[4] Darling, R.W.R., Pardoux, E. (1997). Backward SDE with random terminal time and applications to semilinear elliptic PDE *Ann. Probab.* 25, 1135-1159.

[5] El-Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S., Quenez, M.C. (1997). Reflected solutions of backward SDE's and related obstacle problems for PDE's *Ann. Probab.* 25, 702-737.

[6] Friedman, A. (1975). Stochastic differential equations and applications, Vol. I, *Academic Press*, New-York.

[7] Hu, Y. (1993). Probabilistic interpretation for a system of quasilinear elliptic partial differential equations with Neumann boundary conditions *Stochastic Process. Appl.* 48, 107-121.

[8] Karatzas, I., Shreve, S.E. (1988). Brownian motion and stochastic calculus, *Springer-Verlag*, New-York.

[9] Maticiuc, L., Răşcanu, A., Zălinescu, A. (2008). Backward stochastic variational inequalities under weak assumptions on the data, available at http://arxiv.org/find/all/1/all:+maticiuc/0/1/0/all/0/1

[10] Pardoux, E., Peng, S. (1992). Backward SDE's and quasilinear parabolic PDE's *Stochastic PDE and Their Applications*, LNCIS 176, Springer, 200-217.

[11] Pardoux, E., Răşcanu, A. (1998). Backward stochastic differential equations with subdifferential operator and related variational inequalities *Stochastic Process. Appl.* 76, 191-215.

[12] Pardoux, E., Răşcanu, A. (1999). Backward stochastic variational inequalities *Stochastics* 67 (3-4), 159–167.

[13] Pardoux, E., Zhang, S., Generalized BSDE and nonlinear Neumann boundary value problems *Probab. Theory Related Fields* 110 (1998), 535-558.