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Dirichlet Mean Identities and Laws of a Class of Subordinators

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An interesting line of research is the investigation of the laws of random variables known as Dirichlet means. However there is not much information on inter-relationships between different Dirichlet means. Here we introduce two distributional operations, which consist of multiplying a mean functional by an independent beta random variable and an operation involving an exponential change of measure. These operations identify relationships between different means and their densities. This allows one to use the often considerable analytic work to obtain results for one Dirichlet mean to obtain results for an entire family of otherwise seemingly unrelated Dirichlet means. Additionally,it allows one to obtain explicit densities for the related class of random variables that have generalized gamma convolution distributions, and the finite-dimensional distribution of their associated Lévy processes. The importance of this latter statement is that Lévy processes now commonly appear in variety of applications in probability and statistics, but there are relatively few cases where one has described the relevant densities explicitly. We demonstrate how the technique allows one to obtain the finite-dimensional distribution of several interesting subordinators recently appearing in the literature

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1. Introduction

In this work we present two distributional operations which identify relationships between seemingly different classes of random variables which are representable as linear functionals of a Dirichlet process, otherwise known as Dirichlet means. Specifically the first operation consists of multiplication of a Dirichlet mean by an independent beta random variable and the second operation involves an exponential change of measure to the density of a related infinitely divisible random variable having a generalized gamma convolution distribution (GGC). This latter operation is often referred to in the statistical literature as *exponential tilting* or in mathematical finance as an *Esscher transform*. We believe our results add a significant component to the foundational work of Cifarelli

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and Regazzini (6; 7). In particular, our results allow one to use the often considerable analytic work to obtain results for one Dirichlet mean to obtain results for an entire family of otherwise seemingly unrelated mean functionals. It also allows one to obtain explicit densities for the related class of infinitely divisible random variables which are generalized gamma convolutions, and an explicit description of the finite-dimensional distribution of their associated Lévy processes,(see Bertoin (1) for the formalities of general Lévy processes). The importance of this latter statement is that Lévy processes now commonly appear in variety of applications in probability and statistics, but there are relatively few cases where one has described the relevant densities explicitly. A detailed summary and outline of our results may be found in section 1.2. Some background information, and notation, on Dirichlet processes and Dirichlet means, their connection with GGC random variables, recent references and some motivation for our work are given in the next section.

1.1. Background and motivation

Let X be a non-negative random variable with cumulative distribution function F_X. Note furthermore for a measurable set C, we use the notation $F_X(C)$ to mean the probability that X is in C. One may define a Dirichlet process random probability measure, see Freedman (18) and Ferguson (16; 17), say P_θ, on $[0,\infty)$ with total mass parameter θ and *prior* parameter F_X, via its finite dimensional distribution as follows; for any disjoint partition on $[0,\infty)$, say $(C_1,\ldots,C_k)$, the distribution of the random vector $(P_\theta(C_1),\ldots,P_\theta(C_k))$ is a k-variate Dirichet distribution with parameters $(\theta F_X(C_1),\ldots,\theta F_X(C_k))$. Hence for each C,

$$P_\theta(C) = \int_0^\infty \mathbb{I}(x\in C)P_\theta(dx)$$

has a beta distribution with parameters $(\theta F_X(C), \theta(1-F_X(C)))$. Equivalently setting $\theta F_X(C_i)=\theta_i$ for $i=1,\ldots,k$,

$$(P_\theta(C_1),\ldots,P_\theta(C_k)) \stackrel{d}{=} \left(\frac{G_{\theta_i}}{G_\theta}; i=1,\ldots,k\right)$$

where (G_{θ_i}) are independent random variables with gamma$(\theta_i,1)$ distributions and $G_\theta = G_{\theta_1}+\cdots+G_{\theta_k}$ has a gamma$(\theta,1)$ distribution. This means that one can define the Dirichlet process via the normalization of an independent increment gamma process on $[0,\infty)$, say $\gamma_\theta(\cdot)$, as

$$P_\theta(\cdot) = \frac{\gamma_\theta(\cdot)}{\gamma_\theta([0,\infty))}$$

where $\gamma_\theta(C_i) \stackrel{d}{=} G_{\theta_i}$ and whose almost surely finite total random mass is $\gamma_\theta([0,\infty)) \stackrel{d}{=} G_\theta$. A very important aspect of this construction is the fact that G_θ is independent of P_θ, and hence any functional of P_θ. This is a natural generalization of Lukacs'(35) characterization of beta and gamma random variables, whose work is fundamental to what is now referred to as the beta-gamma algebra, (for more on this, see Chaumont and Yor ((5), section 4.2)). See also Emery and Yor (12) for some interesting relationships between gamma processes, Dirichlet processes and Brownian bridges. Hereafter, for a random probability measure P on $[0, \infty)$, we write

$$P \sim \Pi_{\theta, F_X},$$

to indicate that P is a Dirichlet process with parameters (θ, F_X).

These simple representations and other nice features of the Dirichlet process have, since the important work of Ferguson (16; 17), contributed greatly to the relevance and practical utility of the field of Bayesian non and semi-parametric statistics. Naturally, owing to the ubiquity of the gamma and beta random variables, the Dirichlet process also arises in other areas. One of the more interesting, and we believe quite important, topics related to the Dirichlet process is the study of the laws of random variables called Dirichlet mean functionals, or simply Dirichlet means, which we denote as

$$M_\theta(F_X) \stackrel{d}{=} \int_0^\infty x P_\theta(dx),$$

initiated in the works of Cifarelli and Regazzini (6; 7). In (7) the authors obtained an important identity for the Cauchy-Stieltjes transform of order θ. This identity is often referred to as the Markov-Krein identity as can be seen in for example, Diaconis and Kemperman (10), Kerov (28) and Vershik, Yor and Tsilevich (40), where these authors highlight its importance to, for instance, the study of the Markov moment problem, continued fraction theory and exponential representation of analytic functions. This identity is later called the Cifarelli-Regazzini identity in (21). Cifarelli and Regazzini (7), owing to their primary interest, used this identity to then obtain explicit density and cdf formulae for $M_\theta(F_X)$. The density formulae may be seen as Abel type transforms and hence do not always have simple forms, although we stress that they are still useful for some analytic calculations. The general exception is the case of $\theta = 1$ which has a nice form. Some examples of works that have proceeded along these lines are Cifarelli and Melilli (8), Regazzini, Guglielmi and di Nunno (38), Regazzini, Lijoi and Prünster(39), Hjort and Ongaro (20), Lijoi and Regazzini (32), and Epifani, Guglielmi and Melilli (13; 14)). Moreover, the recent work of Bertoin, Fujita, Roynette and Yor (2) and James, Lijoi and Prünster (25) (see also (23) which is a preliminary version of this work) show that the study of mean functionals is relevant to the analysis of phenomena related to Bessel and Brownian processes. In fact the work of James, Lijoi and Prünster (25) identifies many new explicit examples of Dirichlet means which have interesting interpretations.

Related to these last points, Lijoi and Regazzini (32) have highlighted a close connection to the theory of generalized gamma convolutions (see (3)). Specifically, it is known that a rich sub-class of random variables having generalized gamma convolutions (GGC) distributions may be represented as

$$T_\theta \stackrel{d}{=} G_\theta M_\theta(F_X) \stackrel{d}{=} \int_0^\infty x \gamma_\theta(dx). \tag{1.1}$$

We call these random variables GGC(θ, F_X). In additon we see from (1.1) that T_θ is a random variable derived from a weighted gamma process, and hence the calculus discussed in Lo (33) and Lo and Weng (34) applies. In general GGC random variables are an important class of infinitely divisible random variables whose properties have been extensively studied by (3) and others. We note further that although we have written a GGC(θ, F_X) random variable as $G_\theta M_\theta(F_X)$ this representation is not unique and in fact it is quite rare to see T_θ represented in this way. We will show that one can in fact exploit this non-uniqueness to obtain explicit densities for T_θ even when it is not so easy to do so for $M_\theta(F_X)$. While the representation $G_\theta M_\theta(F_X)$ is not unique it helps one to understand the relationship between the Laplace transform of T_θ and the Cauchy-Stieltjes transform of order θ of $M_\theta(F_X)$, which indeed characterizes respectively the law of T_θ and $M_\theta(F_X)$. Specifically, using the independence property of G_θ and $M_\theta(F_X)$, leads to, for $\lambda \geq 0$,

$$\mathbb{E}[\mathrm{e}^{-\lambda T_\theta}] = \mathbb{E}[(1 + \lambda M_\theta(F_X))^{-\theta}] = \mathrm{e}^{-\theta \psi_{F_X}(\lambda)} \tag{1.2}$$

where

$$\psi_{F_X}(\lambda) = \int_0^\infty \log(1 + \lambda x) F_X(dx) = \mathbb{E}[\log(1 + \lambda X)]. \tag{1.3}$$

is the *Lévy exponent* of T_θ. We note that T_θ and $M_\theta(F_X)$ exist if and only if $\psi_{F_X}(\lambda) < \infty$ for $\lambda > 0$, (see for instance (9) and (3)). The expressions in (1.2) equates with the the identity obtained by Cifarelli and Regazzini (7), mentioned previously.

Despite these interesting results, there is very little work on the relationship between different mean functionals. Suppose, for instance, that for each fixed value of $\theta > 0$, $M_\theta(F_X)$ denotes a Dirichlet mean and $(M_\theta(F_{Z_c}); c > 0)$ denotes a collection of Dirichlet mean random variables indexed by a family of distributions $(F_{Z_c}; c > 0)$. Then one can ask the question, for what choices of X and Z_c are these mean functionals related, and in what sense? In particular, one may wish to know how their densities are related. The rationale here is that if such a relationship is established, then the effort that one puts forth to obtain results such as the explicit density of $M_\theta(F_X)$, can be applied to an entire family of Dirichlet means $(M_\theta(F_{Z_c}); c > 0)$. Furthermore since Dirichlet means are associated with GGC random variables this would establish relationships between a GGC(θ, F_X) random variable and a family of GGC(θ, F_{Z_c}) random variables. Simple examples are of course the choices $Z_c = X + c$ and $Z_c = cX$, which, due to the linearity properties of mean functionals, results easily in the identities in law

$$M_\theta(F_{X+c}) = c + M_\theta(F_X) \text{ and } M_\theta(F_{cX}) = cM_\theta(F_X)$$

Naturally, we are going to discuss more complex relationships, but with the same goal. That is, we will identify non-trivial relationships so that the often considerable efforts that one makes in the study of one mean functional $M_\theta(F_X)$ can be then used to obtain more easily results for other mean functionals, their corresponding GGC random variables and Lévy processes. In this paper we will describe two such operations which we elaborate on in the next subsection.

1.2. Outline and summary of results

Section 1.3 reviews some of the existing formulae for the density and cdf of Dirichlet means. In Section 2, we will describe the operation of multiplying a mean functional $M_{\theta\sigma}(F_X)$ by an independent beta random variable with parameters $(\theta\sigma, \theta(1-\sigma))$, say, $\beta_{\theta\sigma,\theta(1-\sigma)}$ where $0 < \sigma < 1$. We call this operation *beta scaling*. Theorem 2.1 shows that the resulting random variable $\beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_X)$ is again a mean functional but now of order θ. In addition, the GGC$(\theta\sigma, F_X)$ random variable $G_{\theta\sigma}M_{\theta\sigma}(F_X)$ is equivalently a GGC random variable of order θ. Now keeping in mind that tractable densities of mean functionals of order $\theta = 1$ are the easiest to obtain, Theorem 2.1 shows that by setting $\theta = 1$, the densities of the uncountable collection of random variables $(\beta_{\sigma,1-\sigma}M_\sigma(F_X); 0 < \sigma \leq 1)$, are all mean functionals of order $\theta = 1$. Theorem 2.2 then shows that efforts used to calculate the explicit density of any one of these random variables, via the formulae of (7), lead to the explicit calculation of the densities of all of them. Additionally, Theorem 2.2 shows that the corresponding GGC random variables may all be expressed as GGC random variables of order $\theta = 1$, representable in distribution as $G_1\beta_{\sigma,1-\sigma}M_\sigma(F_X)$. A key point here is that Theorem 2.2 gives a tractable density for $\beta_{\sigma,1-\sigma}M_\sigma(F_X)$ without requiring knowledge of the density of $M_\sigma(F_X)$, which is usually expressed in a complicated manner. These results also will yield some non-obvious integral identities. Furthermore, noting that a GGC(θ, F_X) random variable, T_θ, is infinitely divisible, we associate it with an independent increment process $(\zeta_\theta(t); t \geq 0)$ known as a subordinator ,(a non-decreasing non-negative Lévy process), where for each fixed t,

$$\mathbb{E}[\mathrm{e}^{-\lambda\zeta_\theta(t)}] = \mathbb{E}[\mathrm{e}^{-\lambda T_{\theta t}}] = \mathrm{e}^{-t\theta\psi_{F_X}(\lambda)}.$$

That is, marginally $\zeta_\theta(1) \overset{d}{=} T_\theta$ and $\zeta_\theta(t) \overset{d}{=} \zeta_{\theta t}(1) \overset{d}{=} T_{\theta t}$. In addition, for $s < t$, $\zeta_\theta(t) - \zeta_\theta(s) \overset{d}{=} \zeta_\theta(t-s)$ is independent of $\zeta_\theta(s)$. We say that the process $(\zeta_\theta(t); t \geq 0)$ is a GGC(θ, F_X) subordinator. Proposition 2.1 shows how Theorems 2.1 and 2.2, can be used to address the usually difficult problem of describing explicitly the densities of the finite-dimensional distribution of a subordinator (see (29)). This has implications in, for instance, the explicit description of densities of Bayesian nonparametric prior and posterior models. But clearly is of wider interest in terms of the distribution theory of infinitely divisible random variables and associated processes.

In Section 3, we describe how the operation of exponentially titling the density of a GGC(θ, F_X) random variable leads to a relationship between the densities of the mean functional $M_\theta(F_X)$ and yet another family of mean functionals. This is summarized in Theorem 3.1. Section 3.1 then discusses a combination of the two operations. Proposition 3.1 describes the density of beta scaled and tilted mean functionals of order 1. Using this, Proposition 3.2 describes a method to calculate a key quantity in the explicit description of the density and cdf of mean functionals. In section 4 we show how the results in sections 2 and 3 are used to derive the finite dimensional distribution and related quantities of a classes of subordinators suggested by the recent work of James, Lijoi and Prünster (25) and Bertoin, Fujita, Roynette and Yor (2).

1.3. Preliminaries

Suppose that X is a positive random variable with distribution F_X, and define the function

$$\Phi_{F_X}(t) = \int_0^\infty \log(|t-x|)\mathbb{I}(t \neq x)F_X(dx) = \mathbb{E}[\log(|t-X|)\mathbb{I}(t \neq X)].$$

Furthermore, define

$$\Delta_\theta(t|F_X) = \frac{1}{\pi}\sin(\pi\theta F_X(t))\mathrm{e}^{-\theta\Phi_{F_X}(t)},$$

where using a Lebesque-Stieltjes integral, $F_X(t) = \int_0^t F_X(dx)$. Cifarelli and Regazzini (7) (see also (8)), apply inversion formula to obtain the distributional formula for $M_\theta(F_X)$ as follows. For all $\theta > 0$, the cdf can be expressed as

$$\int_0^x (x-t)^{\theta-1}\Delta_\theta(t|F_X)dt \tag{1.4}$$

provided that θF_X possesses no jumps of size greater than or equal to one. If we let $\xi_{\theta F_X}(\cdot)$ denote the density of $M_\theta(F_X)$, it takes its simplest form for $\theta = 1$, which is

$$\xi_{F_X}(x) = \Delta_1(x|F_X) = \frac{1}{\pi}\sin(\pi F_X(x))\mathrm{e}^{-\Phi(x)}. \tag{1.5}$$

Density formulae for $\theta > 1$ are described as

$$\xi_{\theta F_X}(x) = (\theta-1)\int_0^x (x-t)^{\theta-2}\Delta_\theta(t|F_X)dt. \tag{1.6}$$

An expression for the density, which holds for all $\theta > 0$, was recently obtained by James, Lijoi and Prünster (25) as follows,

$$\xi_{\theta F_X}(x) = \frac{1}{\pi}\int_0^x (x-t)^{\theta-1}d_\theta(t|F_X)dt \tag{1.7}$$

where

$$d_\theta(t|F_X) = \frac{d}{dt}\sin(\pi\theta F_X(t))\mathrm{e}^{-\theta\Phi(t)}.$$

For additional formula, see (7; 38; 32).

Remark 1.1. *Throughout for random variables R and X, when we write the product RX we will assume unless otherwise mentioned that R and X are independent. This convention will also apply to the multiplication of the special random variables that are expressed as mean functionals. That is the product $M_\theta(F_X)M_\theta(F_Z)$ is understood to be a product of independent Dirichlet means.*

Remark 1.2. *Throughout we will be using the fact that if R is a gamma random variable, then the independent random variables R, X, Z satisfying $RX \overset{d}{=} RZ$ imply that $X \overset{d}{=} Z$. This is true because gamma random variables are simplifiable. For precise meaning of this term and conditions, Chaumont and Yor (5, sec. 1.12 and 1.13). This fact also applies to the case where R is a positive stable random variable.*

2. Beta Scaling

In this section we investigate the simple operation of multiplying a Dirichlet mean functional $M_\theta(F_X)$ by certain beta random variables. Note first that if M denotes an arbitrary positive random variable with density f_M, then by elementary arguments it follows that the random variable $W \stackrel{d}{=} \beta_{a,b}M$, where $\beta_{a,b}$ is beta(a,b) independent of M, has density expressible as

$$f_W(w) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 f_M(w/u) u^{a-2} (1-u)^{b-1} du.$$

However it is only in special cases where the density f_W can be expressed in even simpler terms. That is to say, it is not obvious how to carry out the integration. In the next results we show how remarkable simplifications can be achieved when $M = M_\theta(F_X)$, in particular for the range $0 < \theta \leq 1$, and $\beta_{a,b}$ is a symmetric beta random variable. First we will need to introduce some additional notation. Let Y_σ denote a Bernoulli random variable with success probability $0 < \sigma \leq 1$. Then if X is a random variable with distribution F_X, independent of Y_σ, it follows that the random variable XY_σ has distribution denoted as

$$F_{XY_\sigma}(dx) = \sigma F_X(dx) + (1-\sigma)\delta_0(dx), \tag{2.1}$$

and cdf

$$F_{XY_\sigma}(x) = \sigma F_X(x) + (1-\sigma)\mathbb{I}(x \geq 0). \tag{2.2}$$

Hence, there exists the mean functional

$$M_\theta(F_{XY_\sigma}) \stackrel{d}{=} \int_0^\infty y \tilde{P}_\theta(dy)$$

where $\tilde{P}_\theta(dy)$ denotes a Dirichlet process with parameters (θ, F_{XY_σ}). In addition we have for $x > 0$,

$$\Phi_{F_{XY_\sigma}}(x) = \mathbb{E}[\log(|x - XY_\sigma|)\mathbb{I}(XY_\sigma \neq x)] = \sigma \Phi_{F_X}(x) + (1-\sigma)\log(x). \tag{2.3}$$

When $\sigma = 1$, $Y_\sigma = 1$ and hence $F_{XY_1}(x) = F_X(x)$. Let E_σ denote a set such that $\mathbb{E}[P_\theta(E_\sigma)] = \sigma$. Now notice that every beta random variable, $\beta_{a,b}$, where a, b are arbitrary positive constants, can be represented as the simple mean functional,

$$P_\theta(E_\sigma) \stackrel{d}{=} \beta_{\theta\sigma,\theta(1-\sigma)} \stackrel{d}{=} M_\theta(F_{Y_\sigma}),$$

by choosing

$$\sigma = \frac{a}{a+b} \text{ and } \theta = a + b.$$

We note however that there are other choices of F_X that will also yield beta random variables as mean functionals. Throughout we will use the convention that $\beta_{\theta,0} := 1$, that is the case when $\sigma = 1$. We now present our first result.

Theorem 2.1. *For $\theta > 0$ and $0 < \sigma \leq 1$, let $\beta_{\theta\sigma,\theta(1-\sigma)}$ denote a beta random variable with parameters $(\theta\sigma, \theta(1-\sigma))$, independent of the mean functional $M_{\theta\sigma}(F_X)$. Then*

(i) $\beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X) \stackrel{d}{=} M_\theta(F_{XY_\sigma})$.
(ii) Equivalently, $M_\theta(F_{Y_\sigma}) M_{\theta\sigma}(F_X) \stackrel{d}{=} M_\theta(F_{XY_\sigma})$.
(iii) $G_{\theta\sigma} M_{\theta\sigma}(F_X) \stackrel{d}{=} G_\theta M_\theta(F_{XY_\sigma})$
(iv) That is, $GGC(\theta\sigma, F_X) = GGC(\theta, F_{XY_\sigma})$.

Proof. Since $M_\theta(F_{Y_\sigma}) \stackrel{d}{=} \beta_{\theta\sigma,\theta(1-\sigma)}$ statements (i) and (ii) are equivalent. We proceed by first establishing (iii) and (iv). Note that using (1.3),

$$\mathbb{E}[\log(1 + \lambda X Y_\sigma)] = \sigma \mathbb{E}[\log(1 + \lambda X)] = \sigma \int_0^\infty \log(1 + \lambda x) F_X(dx).$$

Hence

$$\mathbb{E}[\mathrm{e}^{-\lambda G_\theta M_\theta(F_{XY_\sigma})}] = \mathrm{e}^{-\theta\sigma \int_0^\infty \log(1+\lambda x) F_X(dx)} = \mathbb{E}[\mathrm{e}^{-\lambda G_{\theta\sigma} M_{\theta\sigma}(F_X)}],$$

which means that $G_\theta M_\theta(F_{XY_\sigma}) \stackrel{d}{=} G_{\theta\sigma} M_{\theta\sigma}(F_X)$, establishing statements (iii) and (iv). Now writing $G_{\theta\sigma} = G_\theta \beta_{\theta\sigma,\theta(1-\sigma)}$. It follows that

$$G_\theta M_\theta(F_{XY_\sigma}) \stackrel{d}{=} G_\theta \beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X).$$

Hence $\beta_{\theta\sigma,\theta(1-\sigma)} M_{\theta\sigma}(F_X) \stackrel{d}{=} M_\theta(F_{XY_\sigma})$, by the fact that gamma random variables are simplifiable. □

Remark 2.1. *We note that parts [(i)] and [(ii)] of Theorem 2.1 also follow as consequences of Ethier and Griffiths (15, Lemma 1). We state their interesting result for clarity;*

Lemma 2.1. *(Ethier and Griffiths (15)). Let ν_1 and ν_2 denote two probability measures. Now for $\theta_1, \theta_2 > 0$, define the probability measure*

$$\nu_{(\theta_1,\theta_2)}(dx) = \frac{\theta_1}{\theta_1 + \theta_2} \nu_1(dx) + \frac{\theta_2}{\theta_1 + \theta_2} \nu_2(dx).$$

Then for independent Dirichlet processes, $\mu_1 \sim \Pi_{\theta_1,\nu_1}$ and $\mu_2 \sim \Pi_{\theta_1,\nu_1}$

$$\mu_{1,2}(\cdot) \stackrel{d}{=} \beta_{\theta_1,\theta_2} \mu_1(\cdot) + (1 - \beta_{\theta_1,\theta_2}) \mu_2(\cdot)$$

where $\mu_{1,2}$ is a Dirichlet process with parameters $(\theta_1 + \theta_2, \nu_{(\theta_1,\theta_2)})$.

Hence, as a general consequence,

$$M_{\theta_1+\theta_2}(\nu_{(\theta_1,\theta_2)}) \stackrel{d}{=} \beta_{\theta_1,\theta_2} M_{\theta_1}(\nu_{\theta_1}) + (1 - \beta_{\theta_1,\theta_2}) M_{\theta_2}(\nu_{\theta_2})$$

Now from (2.1), we see that setting $\theta_1 = F_X, \nu_2 = \delta_0, \theta_1 = \theta\sigma$ and $\theta_2 = \theta(1-\sigma)$ yields statements [(i)] and [(ii)]. That is since $M_{\theta(1-\sigma)}(\delta_0) = 0$.

When $\theta = 1$, we obtain results for random variables $\beta_{\sigma,1-\sigma} M_\sigma(F_X)$. The symmetric beta random variables $\beta_{\sigma,1-\sigma}$ arise in a variety of important contexts, and are often referred to as generalized arcsine laws with density expressible as

$$\frac{\sin(\pi\sigma)}{\pi} u^{\sigma-1}(1-u)^{-\sigma} \text{ for } 0 < u < 1.$$

Now using (2.1) and (2.2), let $\mathcal{C}(F_X) = \{x : F_X(x) > 0\}$, then for $x > 0$,

$$\sin(\pi F_{XY_\sigma}(x)) = \begin{cases} \sin(\pi\sigma[1 - F_X(x)]), & \text{if } x \in \mathcal{C}(F_X), \\ \sin(\pi(1-\sigma)), & \text{if } x \notin \mathcal{C}(F_X). \end{cases} \tag{2.4}$$

Note also that $\sin(\pi[1-F_X(x)]) = \sin(\pi F_X(x))$. The next result yields another surprising property of these random variables.

Theorem 2.2. *Consider the setting in the Theorem 2.1. Then when $\theta = 1$, it follows that for each fixed $0 < \sigma \leq 1$, the random variable $M_1(F_{XY_\sigma}) \stackrel{d}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_X)$ has density*

$$\xi_{F_{XY_\sigma}}(x) = \frac{x^{\sigma-1}}{\pi} \sin(\pi F_{XY_\sigma}(x)) e^{-\sigma \Phi_{F_X}(x)} \text{ for } x > 0, \tag{2.5}$$

specified by (2.4). Since $GGC(\sigma, F_X) = GGC(1, F_{XY_\sigma})$, this implies that the random variable $G_\sigma M_\sigma(F_X) \stackrel{d}{=} G_1 M_1(F_{XY_\sigma})$ has density

$$g_{\sigma,F_X}(x) = \frac{1}{\pi} \int_0^\infty e^{-\frac{x}{y}} y^{\sigma-2} \sin(\pi F_{XY_\sigma}(y)) e^{-\sigma \Phi_{F_X}(y)} dy \tag{2.6}$$

Proof. Since $M_1(F_{XY_\sigma}) \stackrel{d}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_X)$, the density is of the form (1.5), for each fixed $\sigma \in (0, 1]$. Furthermore we use the identity in (2.3). □

Remark 2.2. *It is worthwhile to mention that transforming to the random variable $1/\beta_{\sigma,1-\sigma}$, (2.5) is equivalent to the otherwise not obvious integral identity,*

$$\frac{\sin(\pi\sigma)}{\pi} \int_1^\infty \frac{\xi_{\sigma F_X}(xy)}{(y-1)^\sigma} dy = \frac{x^{\sigma-1}}{\pi} \sin(\pi F_{XY_\sigma}(x)) e^{-\sigma\Phi(x)}.$$

This leads to interesting results when the density $\xi_{\sigma F_X}(x)$ has a known form. On the other hand, we see that one does not need the explicit density of $M_\sigma(F_X)$ to obtain the density of $M_1(F_{XY_\sigma}) \stackrel{d}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_X)$. In fact, owing to our goal of yielding simple densities for many Dirichlet means from one mean, we see that the effort to calculate the density of $M_1(F_{XY_\sigma})$, for each $0 < \sigma \leq 1$, is no more than what is needed to calculate the density of $M_1(F_X)$.

We now see how this translates into the usually difficult problem of describing explicitly the density of the finite-dimensional distribution of a subordinator. In the next result we write, for some set C,

$$\zeta_\theta(C) := \int_0^\infty \mathbb{I}(s \in C)\, \zeta_\theta(ds).$$

Proposition 2.1. *Let $(\zeta_\theta(t); t \le 1/\theta)$ denote a $GGC(\theta, F_X)$ subordinator on $[0, 1/\theta]$. Furthermore let $(C_1, \ldots, C_k)$ denote an arbitrary disjoint partition of the interval $[0, 1/\theta]$. Then the finite-dimensional distribution $(\zeta_\theta(C_1), \ldots, \zeta_\theta(C_k))$ has a joint density*

$$\prod_{i=1}^{k} g_{\sigma_i, F_X}(x_i), \tag{2.7}$$

where each $\sigma_i = \theta|C_i| > 0$ and $\sum_{i=1}^{k} \sigma_i = 1$. The density g_{σ_i, F_X} is given by (2.6). That is, $\zeta_\theta(C_i) \stackrel{d}{=} G_1 M_1(F_{XY_{\sigma_i}})$ and are independent for $i = 1, \ldots, k$, where $M_1(F_{XY_{\sigma_i}}) \stackrel{d}{=} \beta_{\sigma_i, 1-\sigma_i} M_{\sigma_i}(F_X)$ has density

$$\frac{1}{\pi} x^{\sigma_i - 1} \sin(\pi F_{XY_{\sigma_i}}(x)) e^{-\sigma_i \Phi_{F_X}(x)}.$$

Proof. First, since $(C_1, \ldots, C_k)$ partitions the interval $[0, 1/\theta]$, it follows that their sizes satisfy $0 < |C_k| \le 1/\theta$ and $\sum_{i=1}^{k} |C_k| = 1/\theta$. Since ζ_θ is a subordinator the independence of the $\zeta_\theta(C_i)$ is a consequence of its independent increment property. In fact these are essentially equivalent statements. Hence, we can isolate each $\zeta_\theta(C_i)$. It follows that for each i the Laplace transform is given by

$$\mathbb{E}[\mathrm{e}^{-\lambda \zeta_\theta(C_i)}] = \mathrm{e}^{-\theta |C_i| \psi_{F_X}(\lambda)} = \mathrm{e}^{-\sigma_i \psi_{F_X}(\lambda)},$$

which shows that each $\zeta_\theta(C_i)$ is $\mathrm{GGC}(\sigma_i, F_X)$ for $0 < \sigma_i \le 1$. Hence the result follows from Theorem 2.2. □

3. Exponential Tilting/Esscher Transform

In this section we describe how the operation of *exponential tilting* of the density of a $\mathrm{GGC}(\theta, F_X)$ random variable leads to a non-trivial relationship between a mean functional determined by F_X and θ, and an entire family of mean functionals indexed by an arbitrary constant $c > 0$. Additionally this will identify a non-obvious relationship between two classes of mean functionals. Exponential tilting is merely a catchy phrase for the operation of applying an exponential change of measure to a density or more general measure. In mathematical finance and other applications it is known as an *Esscher Transform* which is a key tool for option pricing. We mention that there is much known about exponential tilting of infinitely divisible random variables and in fact Bondesson (3, example 3.2.5) discusses explicitly the case of GGC random variables, albeit not in the way we shall describe it. In addition, examining the gamma representation in (1.1) one can see a relationship to Lo and Weng (34, Proposition 3.1) (see also Küchler and Sorensen (30) and James (22, Proposition 2.1) for results on exponential tilting of Lévy processes). However, here our focus is on the properties of related mean functionals which leads to genuinely new insights.

Before we elaborate on this, we describe generically what we mean by exponential tilting. Suppose that T denotes an arbitrary positive random variable with density, say

f_T. It follows that for each positive c, the random variable cT is well-defined and has density

$$\frac{1}{c} f_T(t/c).$$

Exponential tilting refers to the exponential change of measure resulting in a random variable, say $\tilde{T}_c$, defined by the density

$$f_{\tilde{T}_c}(t) = \frac{\mathrm{e}^{-t}(1/c) f_T(t/c)}{\mathbb{E}[\mathrm{e}^{-cT}]}.$$

Thus from the random variable T one gets a family of random variables $(\tilde{T}_c; c > 0)$. Obviously the density for each $\tilde{T}_c$ does not differ much. However something interesting happens when T is a scale mixture of a gamma random variables, i.e., $T = G_\theta M$, for some random positive random variable M independent of G_θ. In that case one can show, see (23), that $T_c = G_\theta \tilde{M}_c$ where $\tilde{M}_c$ is sufficiently distinct for each value of c. We demonstrate this for the case where $M = M_\theta(F_X)$.

First note that obviously, $cM_\theta(F_X) = M_\theta(F_{cX})$, for each $c > 0$, which in itself is not a very interesting transformation. Now setting $T_\theta = G_\theta M_\theta(F_X)$ with density denoted as g_{θ,F_X}, the corresponding random variable $\tilde{T}_{\theta,c}$ resulting from exponential tilting has density

$$\mathrm{e}^{-t}(1/c) g_{\theta,F_X}(t/c) \mathrm{e}^{\theta \psi_{F_X}(c)} \tag{3.1}$$

and Laplace transform

$$\frac{\mathbb{E}[\mathrm{e}^{-c(1+\lambda) G_\theta M_\theta(F_X)}]}{\mathbb{E}[\mathrm{e}^{-c G_\theta M_\theta(F_X)}]} = \mathrm{e}^{-\theta[\psi_{F_X}(c(1+\lambda)) - \psi_{F_X}(c)]}. \tag{3.2}$$

Now for each $c > 0$, define the random variable

$$A_c \stackrel{d}{=} \frac{cX}{(cX+1)}.$$

That is, the cdf of the random variable A_c, can be expressed as,

$$F_{A_c}(y) = F_X\left(\frac{y}{c(1-y)}\right) \text{ for } 0 < y < 1.$$

In the next theorem we will show that $M_\theta(F_X)$ relates to the family of mean functionals $(M_\theta(F_{A_c}); c > 0)$, by the tilting operation described above. Moreover, we will describe the relationship between their densities.

Theorem 3.1. *Suppose that X has distribution F_X and for each $c > 0$, $A_c \stackrel{d}{=} cX/(cX+1)$ is a random variable with distribution F_{A_c}. For each $\theta > 0$, let $T_\theta = G_\theta M_\theta(F_X)$ denote a GGC (θ, F_X) random variable having density g_{θ,F_X}. Let $\tilde{T}_{\theta,c}$ denote a random variable with density and Laplace transform described by (3.1) and (3.2) respectively. Then $\tilde{T}_{\theta,c}$ is a GGC(θ, F_{A_c}) random variable and hence representable as $G_\theta M_\theta(F_{A_c})$. Furthermore, the following relationships exists between the densities of the mean functionals $M_\theta(F_X)$ and $M_\theta(F_{A_c})$.*

(i) *Suppose that the density of $M_\theta(F_X)$, say, $\xi_{\theta F_X}$ is known. Then the density of $M_\theta(F_{A_c})$ is expressible as*

$$\xi_{\theta F_{A_c}}(y) = \frac{1}{c} e^{\theta\psi_{F_X}(c)}(1-y)^{\theta-2}\xi_{\theta F_X}\left(\frac{y}{c(1-y)}\right),$$

for $0 < y < 1$.

(ii) *Conversely, if the density of $M_\theta(F_{A_c})$, $\xi_{\theta F_{A_c}}(y)$, is known then the density of $M_\theta(F_X)$ is given by*

$$\xi_{\theta F_X}(x) = (1+x)^{\theta-2}\xi_{\theta F_{A_1}}\left(\frac{x}{1+x}\right)e^{-\theta\psi_{F_X}(1)}.$$

Proof. We proceed by first examining the Lévy exponent $[\psi_{F_X}(c(1+\lambda)) - \psi_{F_X}(c)]$ associated with $\tilde{T}_{\theta,c}$ as described in (3.2). Notice that

$$\psi_{F_X}(c(1+\lambda)) = \int_0^\infty \log(1 + c(1+\lambda)x)F_X(dx)$$

and $\psi_{F_X}(c)$ is of the same form with $\lambda = 0$. Hence isolating the logarithmic terms we can focus on the difference

$$\log(1 + c(1+\lambda)x) - \log(1 + cx).$$

This is equivalent to

$$\log\left(1 + \frac{cx}{1+cx}\lambda\right) = \log\left(\frac{1}{1+cx} + \frac{cx}{1+cx}(1+\lambda)\right),$$

showing that $\tilde{T}_{\theta,c}$ is $\mathrm{GGC}(\theta, F_{A_c})$. This fact can also be deduced from Proposition 3.1 in Lo and Weng (34). The next step is to identify the density of $M_\theta(F_{A_c})$, in terms of the density of $M_\theta(F_X)$. Using the fact that $T_\theta = G_\theta M_\theta(F_X)$, one may write the density of T_θ in terms of a gamma mixture as

$$g_{\theta,F_X}(t) = \frac{t^{\theta-1}}{\Gamma(\theta)}\int_0^\infty \mathrm{e}^{-t/m}m^{-\theta}\xi_{\theta F_X}(m)dm.$$

Hence, rearranging terms in (3.1), it follows that the density of $\tilde{T}_{\theta,c}$ can be written as

$$\mathrm{e}^{\theta\psi_{F_X}(c)}\frac{t^{\theta-1}}{\Gamma(\theta)}\int_0^\infty \mathrm{e}^{-t\frac{cm+1}{cm}}(cm)^{-\theta}\xi_{\theta F_X}(m)dm.$$

Now further algebraic manipulation makes this look like a mixture of a gamma$(\theta, 1)$ random variable, as follows,

$$\frac{t^{\theta-1}}{\Gamma(\theta)}\int_0^\infty \mathrm{e}^{-t\frac{cm+1}{cm}}\left[\frac{cm+1}{cm}\right]^\theta \frac{\mathrm{e}^{\theta\psi_{F_X}(c)}\xi_{\theta F_X}(m)}{(1+cm)^\theta}dm.$$

Hence it is evident that $M_\theta(F_{A_c})$ has the same distribution as a random variable $cM/(cM+1)$ where M has density

$$\mathrm{e}^{\theta\psi_{F_X}(c)}(1+cm)^{-\theta}\xi_{\theta F_X}(m).$$

Thus statements (i) and (ii) follow. □

3.1. Tilting and Beta Scaling

This section describes what happens when one applies the exponentially tilting operation relative to a mean functional resulting from beta scaling. Recall that the tilting operation applied to $G_\theta M_\theta(F_X)$ described in the previous section sets up a relationship between $M_\theta(F_X)$ and $M_\theta(F_{Ac})$. Consider the random variable $\beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_X) \overset{d}{=} M_\theta(F_{XY_\sigma})$. Then tilting $G_\theta M_\theta(F_{XY_\sigma})$ as in the previous section leads to the random variable $G_\theta M_\theta(F_{cXY_\sigma/(cXY_\sigma+1)})$ and hence relates

$$\beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_X) \overset{d}{=} M_\theta(F_{XY_\sigma})$$

to the Dirichlet mean of order θ,

$$M_\theta(F_{cXY_\sigma/(cXY_\sigma+1)}).$$

Now letting $F_{A_cY_\sigma}$ denote the distribution of A_cY_σ, one has

$$A_cY_\sigma \overset{d}{=} \frac{cXY_\sigma}{(cXY_\sigma+1)}$$

and hence

$$M_\theta(F_{cXY_\sigma/(cXY_\sigma+1)}) \overset{d}{=} M_\theta(F_{A_cY_\sigma}) \overset{d}{=} \beta_{\theta\sigma,\theta(1-\sigma)}M_{\theta\sigma}(F_{A_c}). \tag{3.3}$$

In a way this shows that the order of beta scaling and tilting can be interchanged. We now derive a result for the cases of $M_1(F_{XY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_X)$ and $M_1(F_{A_cY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_{A_c})$, related by the tilting operation described above. Combining Theorem 2.2 with Theorem 3.1 leads to the following result.

Proposition 3.1. *For each $0<\sigma\le 1$, the random variables $M_1(F_{XY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_X)$ and $M_1(F_{A_cY_\sigma}) = \beta_{\sigma,1-\sigma}M_\sigma(F_{A_c})$ satisfy the following;*

(i) The density of $M_1(F_{A_cY_\sigma})$ is expressible as

$$\xi_{F_{A_cY_\sigma}}(y) = \frac{e^{\sigma\psi_{F_X}(c)}y^{\sigma-1}}{\pi c^\sigma(1-y)^\sigma}\sin\left[\pi F_{XY_\sigma}\left(\frac{y}{c(1-y)}\right)\right]e^{-\sigma\Phi_{F_X}\left(\frac{y}{c(1-y)}\right)}$$

for $0<y<1$.

(ii) Conversely, the density of $M_1(F_{XY_\sigma})$ is given by

$$\xi_{F_{XY_\sigma}}(x) = \frac{e^{-\sigma\psi_{F_X}(1)}x^{\sigma-1}}{\pi(1+x)}\sin\left[\pi F_{A_1Y_\sigma}\left(\frac{x}{1+x}\right)\right]e^{-\sigma\Phi_{F_{A_1}}\left(\frac{x}{1+x}\right)}.$$

Proof. For clarity statement [(i)] is obtained by first using Theorem 3.1. Which gives,

$$\xi_{F_{A_c Y_\sigma}}(y) = \frac{1}{c}\mathrm{e}^{\psi_{F_{XY_\sigma}}(c)}(1-y)^{-1}\xi_{F_{XY_\sigma}}\left(\frac{y}{c(1-y)}\right),$$

for $0 < y < 1$. It then remains to substitute the form of the density (2.5) given in Theorem 2.2. Statement [(ii)] proceeds in the same way using (2.6). □

Note that even if one can calculate $\Phi_{F_{A_c}}$ for some fixed value of c, it may not be so obvious how to calculate it for another value. The previous results allow us to relate their calculation to that of Φ_{F_X} as described next.

Proposition 3.2. *Set $A_c = cX/(cX+1)$ and define $\Phi_{F_{A_c}}(y) = \mathbb{E}[\log(|y-A_c|)\mathbb{I}(A_c \neq y)]$. Then for $0 < y < 1$,*

$$\Phi_{F_{A_c}}(y) = \Phi_{F_X}\left(\frac{y}{c(1-y)}\right) - \psi_{F_X}(c) + \log(c(1-y)).$$

Proof. The result can be deduced by using Proposition 3.1 in the case of $\sigma = 1$. First notice that $\sin(\pi F_X(\frac{y}{c(1-y)})) = \sin(\pi F_{A_c}(y))$. Now equating the form of the density of $M_1(F_{A_c})$ given by (1.5) with the expression given in Proposition 3.1. It follows that

$$\mathrm{e}^{-\Phi_{F_{A_c}}(y)} = \frac{\mathrm{e}^{\psi_{F_X}(c)}}{c(1-y)}\mathrm{e}^{-\Phi_{F_X}\left(\frac{y}{c(1-y)}\right)},$$

which yields the result. □

Remark 3.1. *We point out that if G_κ represents a gamma random variable for $\kappa \neq \theta$, independent of $M_\theta(F_X)$, it is not necessarily true that $G_\kappa M_\theta(F_X)$ is a GGC random variable. For this to be true $M_\theta(F_X)$ would need to be equivalent in distribution to some $M_\kappa(F_R)$. In that case, our results above would be applied for a GGC(κ, F_R) model.*

4. Examples

In this section we will demonstrate how our results in section 2 and 3 can be applied to extend results for two random processes recently studied in the literature. The first involves a class of GGC subordinators that can be derived from a random mean of a two parameter Poisson Dirichlet process with a uniform base measure, which was studied as a special case in James, Lijoi and Prünster (25). The second involves a class of processes recently studied in Bertoin, Fujita, Roynette and Yor (2)[see also Maejima (36) for some discussion of this process]. A key component will be the ability to obtain an explicit expression for the respective Φ_{F_X}. In the first example we do not have much explicit information on the relevant density, $\xi_{\theta F_X}$, however we can rely on a general theorem of James, Lijoi and Prünster (25) to obtain Φ_{F_X}. In the second case of the models discussed in Bertoin, Fujita, Roynette and Yor (2), this theorem apparently does not apply. However, we will be able to use an explicit form of the density, obtained for a particular value of θ by Bertoin, Fujita, Roynette and Yor (2), to obtain Φ_{F_X}.

As we shall show, both these processes are connected to a random variable Z_α whose properties we now describe. For $0 < \alpha < 1$, let S_α denote a positive α-stable random variable specified by its Laplace transform

$$\mathbb{E}[\mathrm{e}^{-\lambda S_\alpha}] = \mathrm{e}^{-\lambda^\alpha}.$$

In addition define

$$Z_\alpha = (\frac{S_\alpha}{S'_\alpha})^\alpha$$

where S'_α is independent of S_α and has the same distribution. The density of this random variable was obtained by Lamperti (31) (see also Chaumont and Yor (5, exercise 4.2.1)))and has the remarkably simple form,

$$f_{Z_\alpha}(y) = \frac{\sin(\pi\alpha)}{\pi\alpha} \frac{1}{y^2 + 2y\cos(\pi\alpha) + 1} \text{ for } y > 0.$$

Furthermore, (see Fujita and Yor (19) and (James (24, Proposition 2.1)), it follows that cdf of Z_α satisfies for $z > 0$,

$$\begin{aligned} F_{Z_\alpha}(1/z) &= 1 - \frac{1}{\pi\alpha}\cot^{-1}\left(\frac{\cos(\pi\alpha) + 1/z}{\sin(\pi\alpha)}\right) \\ &= \frac{1}{\pi\alpha}\cot^{-1}\left(\frac{\cos(\pi\alpha) + z}{\sin(\pi\alpha)}\right) \\ &= 1 - F_{Z_\alpha}(z). \end{aligned}$$

and

$$\sin(\pi\alpha F_{Z_\alpha}(z)) = z\sin(\pi\alpha(1 - F_{Z_\alpha}(z))) = \frac{z\sin(\pi\alpha)}{[z^2 + 2z\cos(\pi\alpha) + 1]^{1/2}} \tag{4.1}$$

and

$$\begin{aligned} \sin(2\pi\alpha[1 - F_{Z_\alpha}(z)]) &= \frac{\sin(2\pi\alpha) + 2z\sin(\pi\alpha)}{1 + 2z\cos(\pi\alpha) + z^2} \\ &= \frac{2\sin(\pi\alpha)[\cos(\pi\alpha) + z]}{1 + 2z\cos(\pi\alpha) + z^2} \end{aligned} \tag{4.2}$$

When $\alpha = 1/2$,

$$\sin(\pi[1 - F_{Z_{1/2}}(z)]) = \frac{z}{z^2 + 1}.$$

4.1. Subordinators derived from an example in James, Lijoi and Prünster

For $0 < \alpha < 1$ and $\theta > -\alpha$, define a special case of of two parameter Poisson Dirichlet random probability measures as,

$$\tilde{P}_{\alpha,\theta}(\cdot) = \sum_{k=1}^{\infty} V_k \prod_{i=1}^{k-1}(1 - V_i)\delta_{U_k}(\cdot),$$

where U_k are iid Uniform[0,1] random variables and the V_k are a sequence of independent $\beta_{\alpha,\theta+k\alpha}$ random variables independent of (U_k). So in particular these random variables satisfy $\mathbb{E}[\tilde{P}_{\alpha,\theta}(\cdot)] = F_U(\cdot)$, where U denotes a Uniform$[0,1]$ random variable. In addition $\tilde{P}_{0,\theta}$ is a Dirichlet process. Then, consider the random means given as,

$$\tilde{M}_{\alpha,\theta}(F_U) := \mathbb{U}_{\alpha,\theta} = \sum_{k=1}^{\infty} U_k V_k \prod_{i=1}^{k-1}(1 - V_i) = \int_0^1 u\tilde{P}_{\alpha,\theta}(du).$$

The $\mathbb{U}_{\alpha,\theta}$ represent a special case of random variables representable as mean functionals of the class of two parameter (α, θ) Poisson Dirichlet random probability measures. That is to say random variables $\tilde{M}_{\alpha,\theta}(F_X)$ where F_X is a general distribution. An extensive study of this larger class was conducted by James, Lijoi and Prünster (25). In regards to $\mathbb{U}_{\alpha,\theta}$ they show that $\mathbb{U}_{\alpha,0}$ has an explicit density

$$\frac{\sin(\pi\alpha)}{\alpha\pi} \frac{(\alpha + 1)t^{\alpha}(1 - t)^{\alpha}}{[t^{2\alpha+2} + 2t^{\alpha+1}(1 - t)^{\alpha+1}\cos(\pi\alpha) + (1 - t)^{2\alpha+2}]}.$$

Furthermore, from James, Lijoi and Prünster (25, Theorem 2.1), for $\theta > 0$,

$$\mathbb{U}_{\alpha,\theta} \overset{d}{=} M_\theta(F_{\mathbb{U}_{\alpha,0}}).$$

This implies that

$$G_\theta \mathbb{U}_{\alpha,\theta} \overset{d}{=} G_\theta M_\theta(F_{\mathbb{U}_{\alpha,0}})$$

are GGC$(\theta, F_{\mathbb{U}_{\alpha,0}})$. Now from Vershik, Yor and Tslevich (40)[see also James, Lijoi and Prünster (25, eq (16))], it follows that

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\lambda G_\theta \mathbb{U}_{\alpha,\theta}}] &= \left(\frac{\lambda(\alpha + 1)}{(\lambda + 1)^{\alpha+1} - 1}\right)^{\frac{\theta}{\alpha}} \\
&= \exp(-\theta\mathbb{E}[\log(1 + \lambda\mathbb{U}_{\alpha,0})])
\end{aligned}$$

where this expression follows from the generalized Stieltjes transform of order $-\alpha$ of a Uniform[0,1] random variable,

$$\mathbb{E}[(1 + \lambda U)^{\alpha}] = \int_0^1 (1 + \lambda x)^{\alpha} dx = \frac{(\lambda + 1)^{\alpha+1} - 1}{\lambda(\alpha + 1)}.$$

A description of the densities of $\mathbb{U}_{\alpha,\theta}$ for $\theta > -\alpha$ is available from the results of (25). However, with the exceptions of $\mathbb{U}_{\alpha,1}$ and $U_{\alpha,1-\alpha}$, their densities are generally expressed in terms of integrals with respect to functions that possibly take on negative values. Here by focusing instead on random variables $\beta_{\theta,1-\theta}\mathbb{U}_{\alpha,\theta}$, for $0 < \theta < 1$, we can utilize the results in James, Lijoi and Prünster (25) to obtain explicit expressions for their densities and the corresponding $\mathrm{GGC}(\theta, F_{\mathbb{U}_{\alpha,0}})$ random variables.

In particular, we will obtain explicit descriptions for the finite dimensional distribution of a $\mathrm{GGC}(\alpha, F_{\mathbb{U}_{\alpha,0}})$ say $(\Upsilon_\alpha(t), t > 0)$ subordinator, where $\Upsilon_\alpha(1) \stackrel{d}{=} G_\alpha \mathbb{U}_{\alpha,\alpha}$, and hence

$$\mathbb{E}[\mathrm{e}^{-\lambda\Upsilon_\alpha(1)}] = \frac{\lambda(\alpha+1)}{(\lambda+1)^{\alpha+1} - 1}.$$

Although not immediately obvious one can show that

$$\mathbb{U}_{\alpha,0} \stackrel{d}{=} \frac{Z_\alpha^{1/(\alpha+1)}}{Z_\alpha^{1/(\alpha+1)} + 1}, \text{ and hence } F_{\mathbb{U}_{\alpha,0}}(t) = F_{Z_\alpha}((\frac{t}{1-t})^{\alpha+1}).$$

From this, due to the tilting relationship discussed in section 3, we see that we can also obtain results for the $\mathrm{GGC}(\alpha, F_{Z_\alpha^{1/(\alpha+1)}})$ subordinator say $(\Upsilon^\ddagger_\alpha(t), t > 0)$. To our knowledge this process and its mean functionals $M_\theta(F_{Z_\alpha^{1/(\alpha+1}})$ have not been studied. Now from James, Lijoi and Prünster (25, Theorem 5.2, (iii)) it follows that,

$$\mathrm{e}^{-\Phi_{F_{\mathbb{U}_{\alpha,0}}}(t)} = \frac{(\alpha+1)^{1/\alpha}}{[t^{2\alpha+2} + 2t^{\alpha+1}(1-t)^{\alpha+1}\cos(\pi\alpha) + (1-t)^{2\alpha+2}]^{\frac{1}{2\alpha}}} \tag{4.3}$$

This, combined with our results, leads to an explicit description of the finite dimensional distribution of the relevant subordinators.

Theorem 4.1. *Consider the GGC*$(\alpha, F_{\mathbb{U}_{\alpha,0}})$ *subordinator* $(\Upsilon_\alpha(t), t \leq 1/\alpha)$, *and the GGC*$(\alpha, F_{Z_\alpha^{1/(\alpha+1)}})$ *subordinator* $(\Upsilon^\ddagger_\alpha(t), t \leq 1/\alpha)$. *Let* $(C_1, \ldots, C_k)$, *denote an arbitrary disjoint partition of the interval* $(0, 1/\alpha]$, *with lengths* $|C_i|$ *and set* $\sigma_i = \alpha|C_i|$ *for* $i = 1, \ldots, k$. *Then the following results hold.*

(i) The finite dimensional distribution of $(\Upsilon_\alpha(C_1), \ldots, \Upsilon_\alpha(C_k))$ *is such that each* $\Upsilon_\alpha(C_i)$ *is independent and has distribution*

$$\Upsilon_\alpha(C_i) \stackrel{d}{=} G_1 M_1(F_{Y_{\sigma_i}\mathbb{U}_{\alpha,0}}),$$

where $M_1(F_{Y_{\sigma_i}\mathbb{U}_{\alpha,0}}) \stackrel{d}{=} \beta_{\sigma_i,1-\sigma_i}\mathbb{U}_{\alpha,\sigma_i}$. *Furthermore for any fixed* $0 < \sigma \leq 1$, *the density of* $M_1(F_{Y_\sigma\mathbb{U}_{\alpha,0}})$ *is given by , for* $0 < y < 1$,

$$\frac{(\alpha+1)^{\sigma/\alpha} y^{\sigma-1} \sin(\pi\sigma[1 - F_{\mathbb{U}_{\alpha,0}}(y)])}{[y^{2\alpha+2} + 2y^{\alpha+1}(1-y)^{\alpha+1}\cos(\pi\alpha) + (1-y)^{2\alpha+2}]^{\frac{\sigma}{2\alpha}}}$$

(ii) *The finite dimensional distribution of* $(\Upsilon^{\ddagger}_{\alpha}(C_1),\ldots,\Upsilon^{\ddagger}_{\alpha}(C_k))$ *is such that each* $\Upsilon^{\ddagger}_{\alpha}(C_i)$ *is independent and has distribution*

$$\Upsilon^{\ddagger}_{\alpha}(C_i) \overset{d}{=} G_1 M_1(F_{Y_{\sigma_i} Z_{\alpha}^{1/(\alpha+1)}}).$$

where $M_1(F_{Y_{\sigma_i} Z_{\alpha}^{1/(\alpha+1)}}) \overset{d}{=} \beta_{\sigma_i,1-\sigma_i} M_{\sigma_i}(F_{Z_{\alpha}^{1/(\alpha+1)}})$. *Furthermore for any fixed* $0 < \sigma \leq 1$, *the density of* $M_1(F_{Y_{\sigma} Z_{\alpha}^{1/(\alpha+1)}})$ *is given by , for* $x > 0$,

$$\frac{x^{\sigma-1}(x+1)^{\frac{\sigma(1+\alpha)}{\alpha}-1}\sin(\pi\sigma[1-F_{Z_\alpha}(x^{\alpha+1})])}{[x^{2\alpha+2}+2x^{\alpha+1}\cos(\pi\alpha)+1]^{\frac{\sigma}{2\alpha}}}$$

Proof. Statement [(i)] follows from Theorem 2.2. and Proposition 2.1 in combination with (4.3). Noting the relationship between $Z_{\alpha}^{1/(\alpha+1)}$ and $\mathbb{U}_{\alpha,0}$, statement [(ii)] follows from (Theorem 3.1,[ii]). □

From this, combined with an application of (4.1), we obtain a description for the densities of $\Upsilon^{\ddagger}_{\alpha}(1)$ and $\Upsilon_{\alpha}(1)$. In addition for $\alpha \leq 1/2$, we obtain a description of the density of $\Upsilon_{\alpha}(2)$ using (4.3).

Proposition 4.1. *Let* $\Upsilon_{\alpha}(1)$ *and* $\Upsilon^{\ddagger}_{\alpha}(1)$ *denote GGC random variables with parameters* $(\alpha, F_{\mathbb{U}_{\alpha,0}})$ *and* $(\alpha, F_{Z_{\alpha}^{1/(\alpha+1)}})$ *respectively. Then*

(i) $\Upsilon_{\alpha}(1) \overset{d}{=} G_1 M_1(F_{Y_\alpha \mathbb{U}_{\alpha,0}})$, *where* $M_1(F_{Y_\alpha \mathbb{U}_{\alpha,0}}) \overset{d}{=} \beta_{\alpha,1-\alpha}\mathbb{U}_{\alpha,\alpha}$ *has density for* $0 < y < 1$,

$$\frac{\sin(\pi\alpha)}{\pi}\frac{(\alpha+1)y^{\alpha-1}(1-y)^{\alpha+1}}{[y^{2\alpha+2}+2y^{\alpha+1}(1-y)^{\alpha+1}\cos(\pi\alpha)+(1-y)^{2\alpha+2}]}$$

(ii) $\Upsilon^{\ddagger}_{\alpha}(1) \overset{d}{=} G_1 M_1(F_{Y_\alpha Z_{\alpha}^{1/(\alpha+1)}})$, *where* $M_1(F_{Y_\alpha Z_{\alpha}^{1/(\alpha+1)}})$ *has density,*

$$\frac{\sin(\pi\alpha)}{\pi}\frac{x^{\alpha-1}(1+x)^{\alpha}}{[x^{2\alpha+2}+2x^{\alpha+1}\cos(\pi\alpha)+1]} \text{ for } x > 0.$$

(iii) *Suppose that* $\alpha \leq 1/2$, *then the GGC*$(2\alpha, F_{Z_{\alpha}^{1/(\alpha+1)}})$ *random variable,* $\Upsilon^{\ddagger}_{2\alpha}(1) \overset{d}{=} \Upsilon^{\ddagger}_{\alpha}(2)$, *is equivalent in distribution to* $G_1 M_1(F_{Y_{2\alpha} Z_{\alpha}^{1/(\alpha+1)}})$, *where* $M_1(F_{Y_{2\alpha} Z_{\alpha}^{1/(\alpha+1)}})$ *has density*

$$\frac{2x^{2\alpha-1}(x+1)^{2\alpha+1}\sin(\pi\alpha)[\cos(\pi\alpha)+x^{\alpha+1}]}{[x^{2\alpha+2}+2x^{\alpha+1}\cos(\pi\alpha)+1]^2} \text{ for } x > 0.$$

4.2. An example connected to Diaconis and Kemperman

Notice that we have the following convergence in distribution results, as $\alpha \to 0$,

$$\tilde{M}_{\alpha,\theta}(F_U) = M_\theta(F_{U_{\alpha,0}}) \overset{d}{\to} M_\theta(F_U), \text{ for } \theta > 0$$

and

$$\mathbb{U}_{\alpha,0} \overset{d}{\to} 1 - U.$$

Furthermore, setting $W = (1-U)/U = G_1/G_1'$,

$$M_\theta(F_{Z_\alpha^{1/(\alpha+1)}}) \overset{d}{\to} M_\theta(F_W) \text{ and } Z_\alpha^{1/(\alpha+1)} \overset{d}{\to} W.$$

Where the last statement can be read from Chaumont and Yor (5, p.155 and p.169). It is then natural to investigate the laws of the random processes connected with the GGC(θ, F_U) and GGC(θ, F_W) random variables. It is known from Diaconis and Kemperman (10) that the density of $M_1(F_U)$ is

$$\frac{e}{\pi}\sin(\pi y) y^{-y}(1-y)^{-(1-y)} \text{ for } 0 < y < 1. \tag{4.4}$$

Note furthermore that $\tilde{T}_1 \overset{d}{=} G_1 M_1(F_U)$ is GGC$(1, F_U)$ and has Laplace transform,

$$\mathbb{E}[\mathrm{e}^{-\lambda G_1 M_1(F_U)}] = \mathrm{e}^{-\psi_{F_U}(\lambda)} = \mathrm{e}(1+\lambda)^{-(\frac{\lambda+1}{\lambda})}.$$

Now $G_1 M_1(F_W)$, is a GGC$(1, F_W)$, with $\psi_{F_W}(\lambda) = \frac{\lambda}{\lambda-1}\log(\lambda)$. Theorem 3.1 shows that $M_1(F_U)$ arises from tilting the density of $G_1 M_1(F_W)$. The density of $M_1(F_W)$ is obtained by applying statement [(ii)] of Theorem 3.1. to (4.4), or by Proposition 3.1 statement [(ii)], and is given by

$$\xi_{F_W}(x) = \frac{1}{\pi}\sin(\frac{\pi x}{1+x})x^{-\frac{x}{(1+x)}} \text{ for } x > 0.$$

We now apply Theorem 2.2 and Proposition 2.1 to give a description of the finite-dimensional distribution of the subordinators associated with the two random variables above.

Proposition 4.2. *Let U denote a uniform [0,1] random variable and let $W = G_1/G_1'$ denote a ratio of independent exponential(1) random variables.*

(i) Suppose that $(\tilde{\zeta}_1(t); 0 < t < 1)$ is a GGC$(1, F_U)$ subordinator, then for $(C_1, \ldots, C_k)$ a disjoint partition of $(0,1)$, the finite dimensional distribution has joint density as in (2.7), with,

$$g_{\sigma_i,F_U}(x_i) = \int_0^1 e^{-\frac{x_i}{y}}\frac{e^{\sigma_i}}{\pi}\sin(\pi\sigma_i(1-y))y^{\sigma_i(1-y)-2}(1-y)^{-\sigma_i(1-y)}dy.$$

for $i = 1, \ldots, k$.

(ii) *That is* $\tilde{\zeta}_1(C_i) \overset{d}{=} G_1 M_1(F_{UY_{\sigma_i}})$ *and are independent for* $i = 1, \ldots, k$. *Furthermore, the density of* $M_1(F_{UY_{\sigma_i}}) \overset{d}{=} \beta_{\sigma_i, 1-\sigma_i} M_{\sigma_i}(F_U)$ *is*

$$\frac{e^{\sigma_i}}{\pi} \sin(\pi\sigma_i(1-y)) y^{\sigma_i(1-y)-1}(1-y)^{-\sigma_i(1-y)}$$

for $0 < y < 1$.

(iii) *If* $(\zeta_1(t); 0 < t < 1)$ *is a* $GGC(1, F_W)$ *subordinator then the finite dimensional distribution* $(\zeta_1(C_1), \ldots, \zeta_1(C_k))$ *is described now with,*

$$g_{\sigma_i, F_W}(x_i) = \int_0^\infty e^{-\frac{x_i}{w}} \frac{1}{\pi} \sin(\frac{\pi\sigma_i}{1+w}) w^{\frac{\sigma_i}{(1+w)}-2} dw.$$

(iv) *That is* $\zeta_1(C_i) \overset{d}{=} G_1 M_1(F_{WY_{\sigma_i}})$ *and are independent for* $i = 1, \ldots, k$. *Furthermore, the density of* $M_1(F_{WY_{\sigma_i}}) \overset{d}{=} \beta_{\sigma_i, 1-\sigma_i} M_{\sigma_i}(F_W)$ *is*

$$\frac{1}{\pi} \sin(\frac{\pi\sigma_i}{1+x}) x^{\frac{\sigma_i}{(1+x)}-1}$$

for $x > 0$.

Proof. This now follows from Theorem 2.2, Proposition 2.1 and (4.4). Specifically, note that $\mathcal{C}(F_U) = (0, \infty)$, hence for any $0 < \sigma < 1$,

$$\sin(\pi F_{UY_\sigma}(u)) = \sin(\pi\sigma(1-u))$$

for $0 < u < 1$ and 0 otherwise. Furthermore from (4.4), or by direct argument, it is easy to see that,

$$\Phi_{F_U}(y) = -\log\left(y^{-y}(1-y)^{-(1-y)}\right) - 1.$$

This fact also is evident from Diaconis and Kemperman (10). It follows that $M_1(F_{UY_\sigma})$ has density

$$\frac{e^\sigma}{\pi} \sin(\pi\sigma(1-y)) y^{\sigma(1-y)-1}(1-y)^{-\sigma(1-y)} \text{ for } 0 < y < 1.$$

The density for $M_1(F_{WY_\sigma})$ is obtained in a similar fashion by Proposition 3.1. □

Remark 4.1. *Setting*

$$A_c \overset{d}{=} \frac{cG_1}{cG_1 + G_1'}$$

One can easily obtain the density of the random variable $M_1(F_{A_c})$ *for each* $c > 0$ *by using Theorem 3.1, statement[(ii)]. Note also that one can deduce from the density of* $M_1(F_W)$ *that* $\Phi_{F_W}(x) = [x/(1+x)]\log(x)$. *Hence in this case an application of Proposition 3.2 shows that,*

$$\Phi_{F_{A_c}}(y) = \frac{y}{c(1-y)+y} \log(\frac{y}{c(1-y)}) - \frac{c\log(c)}{c-1} + \log(c(1-y))$$

We note that otherwise it is not easy to calculate Φ_{A_c}, *in this case, by direct arguments.*

4.3. The finite dimensional distribution of subordinators of BFRY

Our final example shows how one can apply the results in sections 2 and 3 to obtain new results for subordinators recently studied by Bertoin, Fujita, Roynette and Yor (2). In particular they investigate properties of the random variables corresponding to the lengths of excursions of Bessel processes straddling an independent exponential time, which can be expressed as

$$d_{\mathbf{e}}^{(\alpha)} - g_{\mathbf{e}}^{(\alpha)}$$

where for any $t > 0$

$$g_t^{(\alpha)} := \sup\{s \le t\ ;\ R_s = 0\}, \quad d_t^{(\alpha)} := \inf\{s \ge t,\ \ R_s = 0\} \tag{4.5}$$

for $(R_t,\ t \ge 0)$ a Bessel process starting from 0, with dimension $d = 2(1-\alpha)$, with $0 < d < 2$, or equivalently $0 < \alpha < 1$. Additionally, $\mathbf{e} \stackrel{d}{=} G_1$, an exponentially distributed random variable with mean 1.

In order to avoid confusion we will now denote relevant random variables appearing originally in Bertoin, Fujita, Roynette and Yor (2) as Δ_α and G_α, as Σ_α and $\mathbb{G}_\alpha$ respectively. From Bertoin, Fujita, Roynette and Yor (2), let $(\Sigma_\alpha(t); t > 0)$ denote a subordinator such that

$$\begin{aligned} \mathbb{E}[\mathrm{e}^{-\lambda\Sigma_\alpha(t)}] &= \left((\lambda+1)^\alpha - \lambda^\alpha\right)^t \\ &= \exp(-t(1-\alpha)\mathbb{E}[\log(1+\lambda/\mathbb{G}_\alpha)]) \end{aligned}$$

where from Bertoin, Fujita, Roynette and Yor (2, Theorems 1.1 and 1.3), $\mathbb{G}_\alpha$ denotes a random variable such that

$$\mathbb{G}_\alpha \stackrel{d}{=} \frac{Z_{1-\alpha}^{1/\alpha}}{1 + Z_{1-\alpha}^{1/\alpha}}$$

and has density on $(0,1)$ given by

$$f_{\mathbb{G}_\alpha}(u) = \frac{\alpha\sin(\pi\alpha)}{(1-\alpha)\pi}\,\frac{u^{\alpha-1}(1-u)^{\alpha-1}}{u^{2\alpha} - 2(1-u)^\alpha u^\alpha\cos(\pi\alpha) + (1-u)^{2\alpha}}.$$

Hence it follows that the random variable $1/\mathbb{G}_\alpha$ takes its values on $(1,\infty)$ with probability one and has cdf satisfying,

$$1 - F_{1/\mathbb{G}_\alpha}(x) = F_{\mathbb{G}_\alpha}(1/x) = F_{Z_{1-\alpha}}((x-1)^{-\alpha}).$$

As noted by Bertoin, Fujita, Roynette and Yor (2), $(\Sigma_\alpha(t); t > 0)$ is a GGC$(1-\alpha, F_{1/\mathbb{G}_\alpha})$ subordinator. Where the GGC$(1-\alpha, F_{1/\mathbb{G}_\alpha})$ random variable $\Sigma_\alpha \stackrel{d}{=} \Sigma_\alpha(1)$ satisfies,

$$\Sigma_\alpha \stackrel{d}{=} d_{\mathbf{e}}^{(\alpha)} - g_{\mathbf{e}}^{(\alpha)} \stackrel{d}{=} \frac{G_{1-\alpha}}{\beta_{\alpha,1}} \stackrel{d}{=} \frac{G_{1-\alpha}}{U^{1/\alpha}},$$

where U denotes a uniform$[0,1]$ random variable and for clarity $G_{1-\alpha}$ is a gamma$(1-\alpha,1)$ random variable. This means that the density of Σ_α is

$$\frac{\alpha}{\Gamma(1-\alpha)}x^{-\alpha-1}(1-\mathrm{e}^{-x}) \text{ for } x>0.$$

It is evident, as investigated in Fujita and Yor (19), that

$$M_{1-\alpha}(F_{1/\mathbb{G}_\alpha}) \overset{d}{=} \frac{1}{\beta_{\alpha,1}} \overset{d}{=} U^{-1/\alpha}$$

Remark 4.2. *Note that when $\alpha=1/2$, $\mathbb{G}_{1/2} \overset{d}{=} \beta_{1/2,1/2}$. It is known that for each fixed t,*

$$\Sigma_{1/2}(t) \overset{d}{=} \frac{G_{t/2}}{\beta_{1/2,(1+t)/2}},$$

which implies that

$$M_{t/2}(F_{1/\mathbb{G}_{1/2}}) = M_{t/2}(F_{1/\beta_{1/2,1/2}}) \overset{d}{=} \frac{1}{\beta_{1/2,(1+t)/2}} \tag{4.6}$$

This result may be found in James and Yor (27). Related to this fact, Cifarelli and Melilli (8) have shown that $M_{t/2}(F_{\beta_{1/2,1/2}}) \overset{d}{=} \beta_{(t+1)/2,(t+1)/2}$, for $t>0$.

In regards to exponentially tilting GGC$(1-\alpha, F_{1/\mathbb{G}_\alpha})$, notice that for $c>0$,

$$\frac{c/\mathbb{G}_\alpha}{c/\mathbb{G}_\alpha+1} = \frac{c}{\mathbb{G}_\alpha+c}.$$

Thus a GGC$(1-\alpha, F_{c/(\mathbb{G}_\alpha+c)})$ subordinator, say $(\Sigma^\dagger_{\alpha,c}(t), t \le 1/(1-\alpha))$, arises from exponential tilting. Naturally, the density of $\Sigma^\dagger_{\alpha,c}(1)/c$, is given by

$$\frac{\alpha x^{-\alpha-1}\mathrm{e}^{-cx}(1-\mathrm{e}^{-x})}{[(c+1)^\alpha - c^\alpha]\Gamma(1-\alpha)} \text{ for } x>0.$$

Equivalently, $\Sigma^\dagger_{\alpha,c}(1) \overset{d}{=} G_1 M_{1-\alpha}(F_{c/(\mathbb{G}_\alpha+c)})$ where $M_{1-\alpha}(F_{c/(\mathbb{G}_\alpha+c)})$ has density,

$$\frac{\alpha c^\alpha}{(c+1)^\alpha - c^\alpha}u^{-\alpha-1} \text{ for } \frac{c}{c+1} < u < 1.$$

Now using the facts discussed above, we will show how to use the results in section 2 to explicitly describe the finite dimensional distribution of the subordinators $(\Sigma_\alpha(t), t>0)$ and $(\Sigma^\dagger_{\alpha,c}(t), t>0)$ over the range $0<t\le 1/(1-\alpha)$, hence by infinite divisibility for all t. Additionally the analysis will also yield expressions for mean functionals based on $F_{1/\mathbb{G}_\alpha}$. First notice that, using (2.4), one has

$$\sin(\pi F_{Y_{1-\alpha}/\mathbb{G}_\alpha}(x)) = \begin{cases} \sin(\pi(1-\alpha)F_{\mathbb{G}_\alpha}(1/x)), & \text{if } x>1, \\ \sin(\pi(1-\alpha)), & \text{if } 0<x\le 1. \end{cases} \tag{4.7}$$

where again using the properties of $F_{Z_{1-\alpha}}$, as deduced from James (24, Proposition 2.1,(iii)),

$$\sin(\pi(1-\alpha)F_{\mathbb{G}_\alpha}(1/x)) = \frac{\sin(\pi(1-\alpha))}{\left[(x-1)^{2\alpha} - 2(x-1)^\alpha \cos(\pi\alpha) + 1\right]^{1/2}}. \tag{4.8}$$

We now use this to calculate,

$$\Phi_{F_{1/\mathbb{G}_\alpha}}(x) = \mathbb{E}[\log(|x - 1/\mathbb{G}_\alpha|)\mathbb{I}(x \neq 1/\mathbb{G}_\alpha)]. \tag{4.9}$$

Proposition 4.3. *For $0 < \alpha < 1$, consider $\Phi_{F_{1/\mathbb{G}_\alpha}}(x)$ as defined in (4.9). Then,*

$$\Phi_{F_{1/\mathbb{G}_\alpha}}(x) = \begin{cases} \frac{1}{2(1-\alpha)}[\log(\frac{x^2}{[(x-1)^{2\alpha}-2(x-1)^\alpha\cos(\pi\alpha)+1]})], & \textit{if } x > 1, \\ \frac{1}{1-\alpha}\log(x/[1-(1-x)^\alpha]), & \textit{if } 0 < x \leq 1. \end{cases} \tag{4.10}$$

Proof. Using simple beta gamma algebra one has

$$\Sigma_\alpha \overset{d}{=} \frac{G_{1-\alpha}}{\beta_{\alpha,1}} \overset{d}{=} G_1 \frac{\beta_{1-\alpha,\alpha}}{U^{1/\alpha}}$$

Hence applying Theorem 2.1, with $\theta = 1$, and $\sigma = 1-\alpha$, it follows that Σ_α is also GGC$(1, F_{Y_{1-\alpha}/\mathbb{G}_\alpha})$ and

$$B_\alpha := \frac{\beta_{1-\alpha,\alpha}}{\beta_{\alpha,1}} \overset{d}{=} \frac{\beta_{1-\alpha,\alpha}}{U^{1/\alpha}} \overset{d}{=} M_1(F_{Y_{1-\alpha}/\mathbb{G}_\alpha}). \tag{4.11}$$

By standard calculations the density of $B_\alpha = \beta_{1-\alpha,\alpha}/\beta_{\alpha,1}$ is given by

$$f_{B_\alpha}(x) = \frac{\sin(\pi(1-\alpha))}{\pi} x^{-\alpha-1}[1-(1-x)^\alpha \mathbb{I}(x \leq 1)]$$

However we see from (4.11) that $B_\alpha \overset{d}{=} M_1(F_{Y_{1-\alpha}/\mathbb{G}_\alpha})$. Hence Theorem 2.2 applies and the density of B_α can be written as

$$f_{B_\alpha}(x) = \frac{x^{-\alpha}}{\pi}\sin(\pi F_{Y_{1-\alpha}/\mathbb{G}_\alpha}(x))\mathrm{e}^{-(1-\alpha)\Phi_{F_{1/\mathbb{G}_\alpha}}(x)}$$

Now equating the two forms of the density of B_α and using (4.7) and (4.8), one then obtains the expression for $\Phi_{F_{1/\mathbb{G}_\alpha}}$. □

Now for $z > 0$, define the function

$$\mathcal{S}_{\alpha,\sigma}(z) = \sin(\pi\sigma F_{Z_{1-\alpha}}(z^{-\alpha}))[z^{2\alpha} - 2z^\alpha\cos(\pi\alpha) + 1]^{\frac{\sigma}{2(1-\alpha)}}$$

and define,

$$\mathcal{D}_{\alpha,\sigma}(x) = \begin{cases} \sin(\pi\sigma)[1-(1-x)^\alpha]^{\frac{\sigma}{1-\alpha}}, & \text{if } x \leq 1, \\ \mathcal{S}_{\alpha,\sigma}(x-1), & \text{if } x > 1. \end{cases}$$

Hereafter, $(C_1, \ldots, C_k)$ will denote an arbitrary disjoint partition of the interval $(0, 1/(1-\alpha)]$ with lengths $|C_i|$, and $\sigma_i = (1-\alpha)|C_i|$ for $i = 1, \ldots, k$.

Theorem 4.2. *Consider the GGC*$(1-\alpha, F_{1/\mathbb{G}_\alpha})$ *subordinator* $(\Sigma_\alpha(t), t \le 1/(1-\alpha))$, *and for each fixed* $c > 0$, *the GGC*$(1-\alpha, F_{c/(\mathbb{G}_\alpha+c)})$ *subordinator* $(\Sigma^\dagger_{\alpha,c}(t), t \le 1/(1-\alpha))$. *Then the following results hold.*

(i) The finite dimensional distribution of $(\Sigma_\alpha(C_1), \ldots, \Sigma_\alpha(C_k))$ *is such that each* $\Sigma_\alpha(C_i)$ *is independent and has distribution*

$$\Sigma_\alpha(C_i) \overset{d}{=} G_1 M_1(F_{Y_{\sigma_i}/\mathbb{G}_\alpha}),$$

where $M_1(F_{Y_{\sigma_i}/\mathbb{G}_\alpha}) \overset{d}{=} \beta_{\sigma_i,1-\sigma_i} M_{\sigma_i}(F_{1/\mathbb{G}_\alpha})$. *Furthermore for any fixed* $0 < \sigma \le 1$, *the density of* $M_1(F_{Y_\sigma/\mathbb{G}_\alpha})$ *is given by*

$$\frac{1}{\pi} x^{-(\frac{\sigma\alpha}{1-\alpha}+1)} \mathcal{D}_{\alpha,\sigma}(x) \textit{ for } x > 0.$$

(ii) For the GGC$(1-\alpha, F_{c/(\mathbb{G}_\alpha+c)})$ *process,* $\Sigma^\dagger_{\alpha,c}$, *it follows that each*

$$\Sigma^\dagger_{\alpha,c}(C_i) \overset{d}{=} G_1 M_1(F_{Y_{\sigma_i} c/(\mathbb{G}_{\alpha+c})}).$$

Where for each $0 < \sigma \le 1$, $M_1(F_{Y_\sigma c/(\mathbb{G}_{\alpha+c})})$ *has density,*

$$\frac{[c(1-y)]^{\frac{\sigma\alpha}{1-\alpha}} \mathcal{D}_\alpha(\frac{y}{c(1-y)})}{\pi[(c+1)^\alpha - c^\alpha]^\sigma y^{\frac{\sigma\alpha}{1-\alpha}+1}} \textit{ for } 0 < y < 1.$$

Proof. From Theorem 2.2 we have the general form of the density of $M_1(F_{Y_\sigma/\mathbb{G}_\alpha})$ is given by

$$\frac{x^{\sigma-1}}{\pi} \sin(\pi F_{Y_\sigma/\mathbb{G}_\alpha}(x)) \mathrm{e}^{-\sigma\Phi_{F_{1/\mathbb{G}_\alpha}}(x)}$$

The result is then concluded by applying Proposition 4.3 and (4.7) and (4.8). □

Remark 4.3. *The process* $\Sigma_{\alpha,c}(t)/c$ *is well defined for* $c \ge 0$ *and* $0 \le \alpha < 1$, *and presents itself as an interesting class worthy of further investigation. Letting* $c \to 0$ *it is evident that* $\Sigma^\dagger_{\alpha,c}(1)/c$, *converges to* $\Sigma_\alpha(1)$. *As shown by Bertoin, Fujita, Roynette and Yor (2, sec. 3.6.3),* $\Sigma_{0,c}(1)/c$, *for* $c > 0$, *has a similar interpretation as* $\Sigma_\alpha(1)$, *but where now the Bessel process* $(R_t, t > 0)$ *is replaced by a diffusion process whose inverse local time at* 0 *is distributed as a gamma subordinator* $(\gamma_l/c; l > 0)$. *Furthermore, albeit not explicitly addressed in Bertoin, Fujita, Roynette and Yor (2), the random variable* $\Sigma_{\alpha,c}(1)/c \overset{d}{=} d^{(\alpha,c)}_{\mathbf{e}} - g^{(\alpha,c)}_{\mathbf{e}}$, *has a similar interpretation where now* $(R_t, t > 0)$ *is replaced by a process* $(R^{(\alpha,c)}_t, t > 0)$ *whose inverse local time is distributed as a generalized gamma subordinator. That is to say a subordinator whose Lévy density is specified by* $Cy^{-\alpha-1}e^{-cy}$ *for* $y > 0$. *This interpretation may be deduced from Donati-Martin and Yor (11, see 1.c) where* $R^{(\alpha,c)}$ *equates with a downwards Bessel process with drift c.*

Bertoin, Fujita, Roynette and Yor (2) also show that a GGC $(1-\alpha, F_{\mathbb{G}_\alpha})$ random variable satisfies

$$G_{1-\alpha}M_{1-\alpha}(F_{\mathbb{G}_\alpha}) = G_{1-\alpha}U$$

Hence the Laplace transform of the GGC $(1-\alpha, F_{\mathbb{G}_\alpha})$ subordinator, say $(\mathcal{Z}^{\dagger}_{\alpha,1}(t), t > 0)$, is given by

$$(\frac{1}{\alpha\lambda}[(\lambda+1)^{\alpha}-1])^{t}$$

Additionally using the fact that,

$$\frac{1}{\mathbb{G}_\alpha} \overset{d}{=} \frac{1}{Z^{1/\alpha}_{1-\alpha}} + 1 \overset{d}{=} Z^{1/\alpha}_{1-\alpha} + 1 \tag{4.12}$$

leads to,

$$M_{1-\alpha}(F_{Z^{1/\alpha}_{1-\alpha}}) \overset{d}{=} M_{1-\alpha}(F_{1/\mathbb{G}_\alpha}) - 1 \overset{d}{=} \frac{G_1}{G_\alpha},$$

which leads to a description of a GGC$(1-\alpha, F_{Z^{1/\alpha}_{1-\alpha}})$ subordinator. The above points may also be found in the survey paper of James, Roynette, and Yor (26).

Theorem 4.3. *Consider the GGC*$(1-\alpha, F_{Z^{1/\alpha}_{1-\alpha}})$ *subordinator* $(\mathcal{Z}_\alpha(t), t \le 1/(1-\alpha))$ *and the GGC*$(1-\alpha, F_{\mathbb{G}_\alpha})$ *subordinator* $(\mathcal{Z}^{\dagger}_{\alpha,1}(t), t \le 1/(1-\alpha))$. *Then the following results hold.*

(i) The finite dimensional distribution of $(\mathcal{Z}_\alpha(C_1), \ldots, \mathcal{Z}_\alpha(C_k))$ *is such that each* $\mathcal{Z}_\alpha(C_i)$ *is independent and is equivalent in distribution to*

$$\mathcal{Z}_\alpha(C_i) \overset{d}{=} G_1 M_1(F_{Y_{\sigma_i} Z^{1/\alpha}_{1-\alpha}}).$$

Furthermore for any fixed $0 < \sigma \le 1$, *the density of* $M_1(F_{Y_\sigma Z^{1/\alpha}_{1-\alpha}}) \overset{d}{=} \beta_{\sigma,1-\sigma} M_\sigma(F_{Z^{1/\alpha}_{1-\alpha}})$ *is given by, for* $z > 0$,

$$\frac{z^{\sigma-1}}{\pi(1+z)^{\frac{\sigma}{1-\alpha}}}\mathcal{S}_{\alpha,\sigma}(z)$$

(ii) Similarly each $\mathcal{Z}^{\dagger}_{\alpha}(C_i) \overset{d}{=} G_1 M_1(F_{Y_{\sigma_i}\mathbb{G}_\alpha})$, *and for each fixed* $0 < \sigma \le 1$, $M_1(F_{Y_\sigma \mathbb{G}_\alpha})$ *has density*

$$\frac{\alpha^{\frac{\sigma}{1-\alpha}}}{\pi} y^{\sigma-1}(1-y)^{\frac{\sigma\alpha}{1-\alpha}} \mathcal{S}_{\alpha,\sigma}(\frac{y}{1-y})$$

Proof. Apply Theorem 2.2. and Theorem 3.1 where, from (4.12),

$$\Phi_{F_{Z^{1/\alpha}_{1-\alpha}}}(z) = \Phi_{F_{1/\mathbb{G}_\alpha}}(z+1).$$

□

Remark 4.4. *Note that as* $\alpha \to 1$,

$$M_\theta(F_{\mathbb{G}_\alpha}) \stackrel{d}{\to} M_\theta(F_U) \text{ and } M_\theta(F_{Z_{1-\alpha}^{1/\alpha}}) \stackrel{d}{\to} M_\theta(F_W)$$

Hence they have the same limiting behavior, described in section 4.2, as the random variables in section 4.1.

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