Closeness of convolutions of probability measures

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Abstract

We derive new explicit bounds for the total variation distance between two convolution products of $n \in \mathbb{N}$ probability distributions, one of which having identical convolution factors. Approximations by finite signed measures of arbitrary order are considered as well. We are interested in bounds with magic factors, i.e. roughly speaking n also appears in the denominator. Special emphasis is given to the approximation by the n-fold convolution of the arithmetic mean of the distributions under consideration. As an application, we consider the multinomial approximation of the generalized multinomial distribution. It turns out that here the order of some bounds given in Roos (2001) and Loh (1992) can significantly be improved. In particular, it follows that a dimension factor can be dropped. Moreover, better accuracy is achieved in the context of symmetric distributions with finite support. In the course of proof, we use a basic Banach algebra technique for measures on a measurable Abelian group. Though this method was already used by Le Cam (1960), our central arguments seem to be new. We also derive new smoothness bounds for convolutions of probability distributions, which might be of independent interest.

Keywords: Convolutions, explicit constants, generalized multinomial distribution, multivariate Krawtchouk polynomials, magic factor, multinomial approximation, signed measures, total variation distance.

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1 Introduction

1.1 Aim of the paper

Approximations of distributions of sums of independent random variables are needed in nearly all branches of probability theory and statistics. Many results for normal and compound Poisson approximations are nowadays available. However, if the distributions of the summands are similar to each other, much better accuracy can be achieved using identical convolutions of a certain distribution. In the present paper, we give total variation bounds for the accuracy of such approximations in a general framework, i.e. for probability distributions on a measurable Abelian group. We also consider higher order approximations by finite signed measures. All bounds contain magic factors, i.e. roughly speaking n appears in the denominator. As a consequence, this enables us to derive multidimensional results, some of which improve the order of bounds obtained in Roos (2001) and Loh (1992). It should be mentioned that Loh used Stein's method in a more general situation of dependent random variables. However, it seems to be unclear, whether Stein's method can be used to reproduce the results of the present paper. Furthermore, it turns out that our bounds have a better order in the case of symmetric probability distributions with finite support. Our proofs are based on a combination of some Banach algebra related techniques, which in principle were used by Le Cam (1960). On the other hand, the core arguments given in Sections 4.1 and 4.2 seem to be new. Further, the smoothness estimates for convolutions of probability distributions in Section 4.1 might be of independent interest; for instance, see (35) and (38).

We note that, at the beginning of our investigation, we tried to improve one of the central results of Roos (2001), see (5) and discussion thereafter. But unfortunately we were not able to use the multidimensional expansion of that paper for any substantial improvement. Surprisingly it turned out that it is better to forget the dimension, so to speak, and to use the properties of measures on a measurable Abelian group. This should explain, why we use this somewhat abstract approach.

The paper is structured as follows: The following two subsections are devoted to the notation and a review of known results. In Section 2, we present and discuss our main results. To get a first impression of the results of this paper, the reader may consult (15), (16), and (18). In Section 3, we give some numerical examples. The proofs are contained in Section 4.

1.2 Notation

Let $(\mathfrak{X}, +, \mathcal{A})$ be a measurable Abelian group, that is, $(\mathfrak{X}, +)$ is a commutative group with identity element 0 and $\mathcal{A}$ is a σ-algebra of subsets of $\mathfrak{X}$ such that the mapping $(x, y) \mapsto x - y$ from $(\mathfrak{X} \times \mathfrak{X}, \mathcal{A} \otimes \mathcal{A})$ to $(\mathfrak{X}, \mathcal{A})$ is measurable. We note that it is more convenient to formulate our results in terms of distributions or signed measures rather than in terms of random variables. Let $\mathcal{F}$ (resp. $\mathcal{M}$) be the set of all probability distributions (resp. finite signed measures) on $(\mathfrak{X}, \mathcal{A})$. Products and powers of finite signed measures in $\mathcal{M}$ are defined in the convolution sense, that is, for $V, W \in \mathcal{M}$ and $A \in \mathcal{A}$, we write $VW(A) = \int_{\mathfrak{X}} V(A - x)\, \mathrm{d}W(x)$. Empty products and powers of signed measures in $\mathcal{M}$ are understood to be $I := I_0$, where I_x is the Dirac measure at point $x \in \mathfrak{X}$. Let $V = V^+ - V^-$ denote the Hahn-Jordan decomposition of $V \in \mathcal{M}$ and let $|V| = V^+ + V^-$ be its total variation measure. The total variation norm of V is defined by $\|V\| = |V|(\mathfrak{X})$. We note that, in the literature, often the total variation distance $\sup_{A \in \mathcal{A}} |F(A) - G(A)| = \frac{1}{2}\|F - G\|$ between $F, G \in \mathcal{F}$ is used. In this paper, however, all distances will be given only in the total variation norm. With the usual operations of real scalar multiplication, addition, together with convolution and the total variation norm, $\mathcal{M}$ is a real commutative Banach algebra with unity I. For $V \in \mathcal{M}$ and a power series $g(z) = \sum_{m=0}^{\infty} a_m z^m$, $(a_m \in \mathbb{R})$ converging absolutely for each complex $z \in \mathbb{C}$

with $|z| \leqslant \|V\|$, we define $g(V) = \sum_{m=0}^{\infty} a_m V^m$. The above assumptions imply that the limit exists and is an element of the Banach algebra $\mathcal{M}$. On the other hand, the definition of $g(V)$ can also be understood in the pointwise sense. The exponential of $V \in \mathcal{M}$ is defined by the finite signed measure

$$\mathrm{e}^V = \exp(V) = \sum_{m=0}^{\infty} \frac{1}{m!} V^m \in \mathcal{M}.$$

We note that e^V is not necessarily a non-negative measure. Further, $\exp(t(F-I))$ is the compound Poisson distribution with parameters $t \in [0,\infty)$ and $F \in \mathcal{F}$. If F and G are non-negative measures on $(\mathfrak{X}, \mathcal{A})$ and F is absolutely continuous with respect to G, we write $F \ll G$. For $F \in \mathcal{F}$ and $A \in \mathcal{A}$, $F|_A$ is the restriction of F to the set A. The complement of $A \in \mathcal{A}$ is denoted by A^c. Set $\underline{0} = \emptyset$ and $\underline{n} = \{1, \dots, n\}$ for $n \in \mathbb{N} = \{1, 2, \dots\}$; further, for $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, set $\underline{n}_0 = \{0, \dots, n\}$. For a set J, let $|J|$ be the number of its elements. For $x \in \mathbb{R}$, let $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} \,|\, n \leqslant x\}$ and $\lceil x \rceil = \inf\{n \in \mathbb{Z} \,|\, n \geqslant x\}$. Always, let $0^0 = 1$, $1/0 = \infty$, and, for $k \in \mathbb{Z}$, $\sum_{m=k}^{k-1} = 0$ be the empty sum and $\prod_{m=k}^{k-1} = 1$ the empty product. For $a \in \mathbb{C}$ and $b \in \mathbb{Z}_+$, let $\binom{a}{b} = \prod_{m=1}^{b} (a-m+1)/m$. For $a, b \in \mathbb{R}$, set $a \wedge b = \min\{a, b\}$.

1.3 Known results

We first discuss some important results for discrete distributions on $\mathfrak{X} = \mathbb{R}^d$, $(d \in \mathbb{N})$. Let

$$H_0 = I, \quad H_r = I_{e_r}, \quad (r \in \underline{d}), \quad F_j = \sum_{r=0}^{d} p_{j,r} H_r, \quad (j \in \underline{n},\ n \in \mathbb{N}), \quad \overline{F} = \sum_{r=0}^{d} \overline{p}_r H_r, \tag{1}$$

where, for $r \in \underline{d}_0$, $p_{j,r} \in [0,1]$ with $\sum_{r=0}^{d} p_{j,r} = 1$, $\overline{p}_r = n^{-1} \sum_{j=1}^{n} p_{j,r} > 0$, and $e_r \in \mathbb{R}^d$, $(r \neq 0)$ is the vector with 1 at position r and 0 otherwise.

In the case $d = 1$, Ehm (1991, Theorem 1 and Lemma 2) proved with the help of Stein's method that the total variation distance between the Bernoulli convolution $\prod_{j=1}^{n} F_j$ and the binomial law $\overline{F}^n$ can be estimated by

$$\frac{\gamma_2}{62} \min\Big\{1, \frac{1}{n\overline{p}_1\overline{p}_0}\Big\} \leqslant \Big\| \prod_{j=1}^{n} F_j - \overline{F}^n \Big\| \leqslant 2\gamma_2 \min\Big\{1, \frac{1}{n\overline{p}_1\overline{p}_0}\Big\}, \tag{2}$$

where $\gamma_k = \sum_{j=1}^{n} (\overline{p}_1 - p_{j,1})^k$, $(k \in \mathbb{N})$. Here, the estimates depend on the behavior of the so-called magic factor $(n\overline{p}_1\overline{p}_0)^{-1}$ (cf. Introduction in Barbour *et al.* (1992)), and on the closeness of all $p_{j,1}$, $(j \in \underline{n})$, which is reflected by γ_2. In Theorem 3 of Roos (2000), a Krawtchouk expansion was used to show that an absolute constant $C > 0$ exists such that, if $\gamma_2 > 0$, then

$$\Big| \Big\| \prod_{j=1}^{n} F_j - \overline{F}^n \Big\| - \theta \sqrt{\frac{2}{\pi \mathrm{e}}} \Big| \leqslant C\,\theta \min\Big\{1, \frac{|\gamma_3|}{\gamma_2 \sqrt{n\overline{p}_1\overline{p}_0}} + \frac{1}{n\overline{p}_1\overline{p}_0} + \theta\Big\}, \qquad \Big(\theta = \frac{\gamma_2}{n\overline{p}_1\overline{p}_0}\Big).$$

For example, it easily follows that $\| \prod_{j=1}^{n} F_j - \overline{F}^n \| \sim \sqrt{2/(\pi \mathrm{e})}\,\theta$ as $\theta \to 0$ and $n\overline{p}_1\overline{p}_0 \to \infty$. Here, $\sim$ means that the quotient of both sides tends to one. Further results in this and a more general context can be found in Čekanavičius and Roos (2006) and the papers cited there.

The multivariate case $d \in \mathbb{N}$ was investigated by Loh (1992) using Stein's method. He gave an estimate for the closeness between the generalized multinomial distribution $\prod_{j=1}^n F_j$ and the multinomial distribution $\overline{F}^n$. This bound contains certain functions $C_1, C_2 > 0$ of $\overline{p}_r$, $(r \in \underline{d}_0)$, which can be estimated from above by absolute constants, if all $\overline{p}_r$'s are uniformly bounded away from 0 and 1. In his Theorem 5, he showed that, if $n \geqslant 2$ and $\max\{C_1 n^{-1/2},\, C_2[2(n-1)]^{-1}\} \leqslant 1$, then

$$\Big\| \prod_{j=1}^n F_j - \overline{F}^n \Big\| \leqslant 2 \sum_{j=1}^n \sum_{0 \leqslant r_1 < r_2 \leqslant d} |p_{j,r_1}\overline{p}_{r_2} - p_{j,r_2}\overline{p}_{r_1}|\, \varepsilon_{r_1,r_2}, \tag{3}$$

where

$$\varepsilon_{r_1,r_2} = \frac{C_2}{n-1} \ln\Big(\frac{2(n-1)}{C_2}\Big) + \Big(\frac{C_2}{2(n-1)}\Big)^2 + \frac{2C_1}{\sqrt{n}} \min\Big\{ \prod_{i=1}^n (1-p_{i,r_1}),\, \prod_{i=1}^n (1-p_{i,r_2}) \Big\}.$$

The quantities C_1, C_2 can be given explicitly as

$$C_1 = \sup_{0 \leqslant r_1 < r_2 \leqslant d} [\widetilde{C}_1(r_1,r_2) \wedge \widetilde{C}_1(r_2,r_1)], \quad C_2 = \sup_{\substack{0 \leqslant r_1, r_2, r_3 \leqslant d:\\ r_2 \neq r_1,\, r_3 \neq r_1}} \widetilde{C}_2(r_1,r_2,r_3), \tag{4}$$

where, for $r_1, r_2, r_3 \in \underline{d}_0$,

$$\begin{aligned} \widetilde{C}_1(r_1,r_2) &= \Big(\frac{2}{\overline{p}_{r_1}} + \frac{3}{\overline{p}_{r_2}} + \frac{1}{\mathrm{e}\overline{p}_{r_2}(1-\overline{p}_{r_2})}\Big)^{1/2} + \Big(\frac{1}{2\mathrm{e}\overline{p}_{r_2}(1-\overline{p}_{r_2})^2}\Big)^{1/2}, \\ \widetilde{C}_2(r_1,r_2,r_3) &= \begin{cases} \frac{2}{\overline{p}_{r_1}} + \frac{2}{\overline{p}_{r_2}}, & \text{if } r_2 = r_3, \\ \Big[\frac{1}{\overline{p}_{r_1}} + \frac{2}{\mathrm{e}\overline{p}_{r_2}(1-\overline{p}_{r_2})} + \frac{2}{\mathrm{e}\overline{p}_{r_3}(1-\overline{p}_{r_3})} & \\ \quad + \Big(\frac{2}{\overline{p}_{r_1}} + \frac{3}{\overline{p}_{r_2}} + \frac{1}{\mathrm{e}\overline{p}_{r_2}(1-\overline{p}_{r_2})}\Big)^{1/2} \Big(\frac{2}{\overline{p}_{r_1}} + \frac{3}{\overline{p}_{r_3}} + \frac{1}{\mathrm{e}\overline{p}_{r_3}(1-\overline{p}_{r_3})}\Big)^{1/2}\Big], & \text{if } r_2 \neq r_3. \end{cases} \end{aligned}$$

If $d = 1$, then it follows from Ehm's result and the equality $\sum_{0 \leqslant r_1 < r_2 \leqslant d} |p_{j,r_1}\overline{p}_{r_2} - p_{j,r_2}\overline{p}_{r_1}| = |\overline{p}_1 - p_{j,1}|$, that Loh's bound is not of the best possible order, because of the exponent of $|\overline{p}_1 - p_{j,1}|$ and the logarithmic term. It turned out that a bound better than (3) can be given using a multivariate Krawtchouk expansion, see Roos (2001, Theorem 2, Corollary 1). Indeed,

$$\Big\| \prod_{j=1}^n F_j - \overline{F}^n \Big\| \leqslant C_3 \Big(\sum_{r=1}^d \sqrt{\delta(r)} \Big)^2, \tag{5}$$

where $C_3 = \frac{\mathrm{e}}{2-\sqrt{3}} \leqslant 10.15$ and

$$\delta(r) = \sum_{j=1}^n (\overline{p}_r - p_{j,r})^2 \min\Big\{ \frac{4}{\mathrm{e}},\, \frac{1}{n\overline{p}_r\overline{p}_0} \Big\}, \quad (r \in \underline{d}).$$

A sometimes more precise bound is

$$\Big\| \prod_{j=1}^n F_j - \overline{F}^n \Big\| \leqslant \frac{(\sum_{r=1}^d \sqrt{\mathrm{e}\,\delta(r)})^2}{1 - \sum_{r=1}^d \sqrt{\mathrm{e}\,\delta(r)}}, \quad \text{if} \quad \sum_{r=1}^d \sqrt{\mathrm{e}\,\delta(r)} < 1. \tag{6}$$

In contrast to (3), for $d = 1$, the bounds in (5) and (2) have the same order. In the general case, from (5) and Cauchy's inequality, it follows that $\| \prod_{j=1}^n F_j - \overline{F}^n \| \leqslant C_3 d \sum_{r=1}^d \delta(r)$. We note that this estimate is of the same accuracy as (5) when the $\delta(r)$, $(r \in \underline{d})$ are of similar magnitude. In view of this bound, one might wonder, whether the dimension factor d can be dropped. However, as shown in Roos (2001, Remark 2 after Proposition 2), this is not generally possible. But if we concentrate on the estimate with the magic factors, i.e.

$$\Big\|\prod_{j=1}^{n} F_j - \overline{F}^n\Big\| \leqslant C_3\, d \sum_{r=1}^{d}\sum_{j=1}^{n} \frac{(\overline{p}_r - p_{j,r})^2}{n\overline{p}_r\overline{p}_0}, \tag{7}$$

the more general results of this paper imply that $C_3\, d$ can indeed be replaced by the constant 21.88, see Example 2.1 below. It should be mentioned that here the H_r, $(r \in \underline{d}_0)$ need not just be the Dirac measures as in (1).

2 Main results

In what follows, we present bounds which are small when the $F_j \in \mathcal{F}$, $(j \in \underline{n})$ are close or when n is large and the F_j are not too different. Our first result is the following.

Theorem 2.1 *Let $n \in \mathbb{N}$, $F_1, \dots, F_n, G \in \mathcal{F}$, $F_0 = \overline{F} = \frac{1}{n}\sum_{j=1}^{n} F_j$,*

$$V_k = \sum_{J \subseteq \underline{n}:\, |J|=k}\ \prod_{j\in J}(F_j - G), \quad (k \in \underline{n}_0), \qquad W_\ell = \sum_{k=0}^{\ell} V_k\, G^{n-k}, \quad (\ell \in \underline{n}_0).$$

For $j,k \in \underline{n}$ and $m \in \mathbb{N}$, set $M_{j,k} = (F_j - G)G^{\lfloor (n-k)/k \rfloor}$, $\nu_{k,m} = \sum_{j=1}^{n} \|M_{j,k}\|^m$, and $\widetilde{\nu}_k = \|\sum_{j=1}^{n} M_{j,k}\|$. Set

$$\eta_{\ell,\alpha} = \max_{k\,\in\,\underline{n}\setminus\underline{\ell}} \Big[\frac{1}{k^{1+\alpha}}\Big(\frac{\widetilde{\nu}_k^2}{4c_1} + \nu_{k,2}\Big)\Big], \quad (\ell \in \underline{n}_0,\ \alpha \in [0,\infty)), \qquad \eta_\ell = \eta_{\ell,0},$$

$$c_1 = \sup_{x\in(0,\infty)} \Big[\frac{\ln(2-(1-x)\mathrm{e}^x)}{x^2}\Big] = 0.694025\ldots.$$

(a) *Let $\alpha \in [0,\infty)$, $\ell \in \underline{n}_0$, and $\beta = \lceil \alpha(\ell+1)/2\rceil$. If $\eta_\ell < (2\mathrm{e}\, c_1)^{-1}$, then*

$$\Big\|\prod_{j=1}^{n} F_j - W_\ell\Big\| \leqslant (\ell+1)^\beta\, \beta!\, \frac{(2\mathrm{e}\, c_1\, \eta_{\ell,\alpha})^{(\ell+1)/2}}{(1-\sqrt{2\mathrm{e}\, c_1\, \eta_\ell})^{\beta+1}}. \tag{8}$$

In particular, for $\alpha = 0$, we have

$$\Big\|\prod_{j=1}^{n} F_j - W_\ell\Big\| \leqslant \frac{(2\mathrm{e}\, c_1\, \eta_\ell)^{(\ell+1)/2}}{1-\sqrt{2\mathrm{e}\, c_1\, \eta_\ell}}. \tag{9}$$

(b) *Assume that, for each $j \in \underline{n}_0$, $B_j \in \mathcal{A}$ exists such that $F_j|_{B_j^c} \ll G$ and let f_j denote a Radon-Nikodym density of $F_j|_{B_j^c}$ with respect to G. For $\ell \in \underline{n}_0$, we then have*

$$\begin{aligned}\eta_\ell \leqslant\ & \frac{1}{4c_1}\Big[\frac{n}{\sqrt{\ell+1}}\Big(\|(\overline{F}-G)|_{B_0}\| + |(\overline{F}-G)(B_0)|\Big) + \sqrt{2n}\Big(\int_{B_0^c}(f_0-1)^2\,\mathrm{d}G\Big)^{1/2}\Big]^2 \\ &+ \sum_{j=1}^{n}\Big[\frac{1}{\sqrt{\ell+1}}\Big(\|(F_j-G)|_{B_j}\| + |(F_j-G)(B_j)|\Big) + \sqrt{\frac{2}{n}}\Big(\int_{B_j^c}(f_j-1)^2\,\mathrm{d}G\Big)^{1/2}\Big]^2. \end{aligned} \tag{10}$$

We note that, if $G=\overline{F}$, then $\widetilde{\nu}_k=0$ and $\eta_{\ell,\alpha}$ simplifies to $\eta_{\ell,\alpha}=\max_{k\in\underline{n}\setminus\underline{\ell}}\frac{\nu_{k,2}}{k^{1+\alpha}}$. One might ask why we gave the complicated estimate (8). However, it turns out that, in special situations, the order of $\eta_{\ell,\alpha}$ for $\alpha>0$ can be much better than that of η_ℓ. See Proposition 2.1 below involving a bound for $\eta_{\ell,1}$ instead of just the estimate (10). Further, the reason why we formulated Theorem 2.1 in its present general form without the assumption that $G=\overline{F}$ is given with Lemma 4.3 and Example 4.2 below.

Let us first discuss the simple case when $\alpha=0$.

Remark 2.1 Let the assumptions of Theorem 2.1 hold. In what follows, whenever we consider V_k or W_k for a specified number $k\in\mathbb{Z}_+$, we assume that $k\leqslant n$.

(a) For $k\in\mathbb{N}$, let $\Gamma_k=\sum_{j=1}^n(G-F_j)^k$. We have $V_0=I$, $V_1=n(\overline{F}-G)$, $V_2=\frac{1}{2}(n^2(\overline{F}-G)^2-\Gamma_2)$, and, similarly as in Roos (2000, formula (10)), it can be shown that

$$V_k=-\frac{1}{k}\sum_{j=0}^{k-1}V_j\Gamma_{k-j},\quad (k\in\underline{n}).$$

This formula can easily be used to evaluate the signed measures W_ℓ for a given ℓ. In particular, we have $W_0=G^n$ and

$$W_1=G^n+n(\overline{F}-G)G^{n-1},\quad W_2=G^n+n(\overline{F}-G)G^{n-1}+\frac{1}{2}(n^2(\overline{F}-G)^2-\Gamma_2)G^{n-2}.$$

(b) In the important case $G=\overline{F}$, the formulas above become somewhat simpler. Here, we derive

$$V_1 = 0,\quad V_2=-\frac{1}{2}\Gamma_2,\quad V_3=-\frac{1}{3}\Gamma_3,\quad V_4=\frac{1}{8}\Gamma_2^2-\frac{1}{4}\Gamma_4, \tag{11}$$

$$V_5 = \frac{1}{6}\Gamma_2\Gamma_3-\frac{1}{5}\Gamma_5,\quad V_6=-\frac{1}{48}\Gamma_2^3+\frac{1}{8}\Gamma_2\Gamma_4+\frac{1}{18}\Gamma_3^2-\frac{1}{6}\Gamma_6, \tag{12}$$

$$V_7 = -\frac{1}{24}\Gamma_2^2\Gamma_3+\frac{1}{10}\Gamma_2\Gamma_5+\frac{1}{12}\Gamma_3\Gamma_4-\frac{1}{7}\Gamma_7, \tag{13}$$

$$V_8 = \frac{1}{384}\Gamma_2^4-\frac{1}{32}\Gamma_2^2\Gamma_4-\frac{1}{36}\Gamma_2\Gamma_3^2+\frac{1}{12}\Gamma_2\Gamma_6+\frac{1}{15}\Gamma_3\Gamma_5+\frac{1}{32}\Gamma_4^2-\frac{1}{8}\Gamma_8, \tag{14}$$

which, in particular, leads to $W_0=W_1=\overline{F}^n$,

$$W_2=\overline{F}^n-\frac{1}{2}\Gamma_2\overline{F}^{n-2},\qquad W_3=\overline{F}^n-\frac{1}{2}\Gamma_2\overline{F}^{n-2}-\frac{1}{3}\Gamma_3\overline{F}^{n-3}.$$

Letting $\ell=1$ and $\alpha=0$, we obtain under the present assumption that

$$\Big\|\prod_{j=1}^n F_j-\overline{F}^n\Big\|\leqslant\frac{2\mathrm{e}\,c_1\,\eta_1}{1-\sqrt{2\mathrm{e}\,c_1\,\eta_1}},\quad \text{if } \eta_1<(2\mathrm{e}\,c_1)^{-1}, \tag{15}$$

where

$$\eta_1=\max_{k\in\underline{n}\setminus\underline{1}}\frac{\nu_{k,2}}{k} \tag{16}$$

(see comment after Theorem 2.1) can be estimated with (18) below.

(c) Let us assume that, for each $j \in \underline{n}$, $F_j \ll G$ and let f_j be a G-density of F_j. Set $\overline{f} = \frac{1}{n}\sum_{j=1}^n f_j$. If in Theorem 2.1(b) we choose suitable $B_0, B_1, \dots, B_n \in \{\emptyset, \mathfrak{X}\}$, it then follows that, for $\ell \in \underline{n}_0$,

$$\begin{aligned}\eta_\ell \quad &\leqslant \quad \frac{1}{4c_1}\min\Big\{2n\int_{\mathfrak{X}}(\overline{f}-1)^2\,\mathrm{d}G,\ \frac{n^2}{\ell+1}\big\|\overline{F}-G\big\|^2\Big\}\\ &\quad +\sum_{j=1}^n \min\Big\{\frac{2}{n}\int_{\mathfrak{X}}(f_j-1)^2\,\mathrm{d}G,\ \frac{1}{\ell+1}\big\|F_j-G\big\|^2\Big\}.\end{aligned} \tag{17}$$

From the definition of η_ℓ it is clear that, if $G = F_1 = \dots = F_n$, then $\eta_\ell = 0$ for each $\ell \in \underline{n}_0$. The inequalities (17) and (10) reflect this fact. Moreover, in view of these bounds, if $G \approx \overline{F}$ in some sense and if the $F_1, \dots, F_n$ are not too different, then a large n leads to a small bound. Speaking in terms of Barbour *et al.* (1992, Introduction), our bound contains a magic factor (cf. Section 1.3 above).

(d) If $G = \overline{F}$, then, for each $j \in \underline{n}$, we clearly have $F_j \ll G$ and therefore a G-density f_j of F_j exists. In this case, (17) reduces to

$$\eta_\ell \leqslant \sum_{j=1}^n \min\Big\{\frac{2}{n}\int_{\mathfrak{X}}(f_j-1)^2\,\mathrm{d}\overline{F},\ \frac{1}{\ell+1}\big\|F_j-\overline{F}\big\|^2\Big\}, \qquad (\ell \in \underline{n}). \tag{18}$$

We note that, in (18), $\int_{\mathfrak{X}}(f_j-1)^2\,\mathrm{d}\overline{F}$ is finite for all $j \in \underline{n}$, which follows from

$$\int_{\mathfrak{X}} f_j^2\,\mathrm{d}\overline{F} = \int_{\mathfrak{X}} f_j\,\mathrm{d}F_j \leqslant n\int_{\mathfrak{X}} f_j\,\mathrm{d}\overline{F} = n.$$

One might ask whether the singularity in the right-hand side of (9) can be removed. The following theorem shows, that this is possible, if we enlarge the leading absolute constant and replace η_ℓ with η_0 (or with η_1 in the case $G = \overline{F}$).

Theorem 2.2 *Let the notation of Theorem 2.1 be valid.*

(a) *Let $\ell \in \underline{n}_0$ and let $u_\ell \in (0,\infty)$ be the smallest possible constant such that, without any restriction on η_0,*

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant u_\ell\,\eta_0^{(\ell+1)/2}. \tag{19}$$

We have

$$u_\ell \leqslant \frac{(2\mathrm{e}\,c_1)^{(\ell+1)/2}}{1-x_\ell}, \tag{20}$$

where $x_\ell \in (0,1)$ is the unique positive solution of the equation $x^{\ell+1} + x/2 = 1$. By (20), we get $u_0 \leqslant 5.9$, $u_1 \leqslant 17.3$, $u_2 \leqslant 44.5$, and $u_3 \leqslant 107.5$.

(b) *Let $\ell \in \underline{n}$ and let $\widetilde{u}_\ell \in (0,\infty)$ be the smallest possible constant such that, under the assumption $G = \overline{F}$ and without any restriction on η_1,*

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant \widetilde{u}_\ell\,\eta_1^{(\ell+1)/2}. \tag{21}$$

Then we get

$$\widetilde{u}_1 \leqslant 10.94, \quad \widetilde{u}_2 \leqslant 31.5, \quad \widetilde{u}_3 \leqslant 82.2, \quad \widetilde{u}_\ell \leqslant \frac{(2\mathrm{e}\, c_1)^{(\ell+1)/2}}{1-\widetilde{x}_\ell}, \quad (\ell \in \underline{n} \setminus \underline{3}), \tag{22}$$

where $\widetilde{x}_\ell \in (0,1)$ *is the unique positive solution of the equation* $\widetilde{x}^{\ell+1} - \widetilde{x}^2/2 + \widetilde{x} = 1$.

Remark 2.2 (a) If η_0, resp. η_1, is sufficiently small, the bounds given in Theorem 2.2 can be further improved as follows from Theorem 2.1 and Lemma 4.5 below. In particular, in the case $G = \overline{F}$, we have (cf. proof of Theorem 2.2)

$$\begin{aligned} \Big\| \prod_{j=1}^n F_j - \overline{F}^n \Big\| &\leqslant \frac{1}{2}\nu_{2,2} + \Big\| \prod_{j=1}^n F_j - W_2 \Big\| \leqslant (1 + \widetilde{u}_2\sqrt{\eta_1})\eta_1, \\ \Big\| \prod_{j=1}^n F_j - W_2 \Big\| &\leqslant \frac{1}{3}\nu_{3,3} + \Big\| \prod_{j=1}^n F_j - W_3 \Big\| \leqslant (\sqrt{3} + \widetilde{u}_3\sqrt{\eta_1})\eta_1^{3/2}, \\ \Big\| \prod_{j=1}^n F_j - W_3 \Big\| &\leqslant \frac{1}{8}\nu_{4,2}^2 + \Big\| \prod_{j=1}^n F_j - W_4 \Big\| \leqslant (2 + \widetilde{u}_4\sqrt{\eta_1})\eta_1^2. \end{aligned} \tag{23}$$

In view of (23), one may conjecture that $\widetilde{u}_1 \geqslant 1$. Indeed, this is correct and follows from the simple observation that, for $\mathfrak{X} = \mathbb{Z}$, $n \in 2\mathbb{N}$, $F_1 = \cdots = F_{n/2} = I_0$, $F_{n/2+1} = \cdots = F_n = I_1$, we have

$$\Big\| \prod_{j=1}^n F_j - \overline{F}^n \Big\| = 2\Big(1 - \binom{n}{n/2}\frac{1}{2^n}\Big) \longrightarrow 2, \quad (n \to \infty)$$

and, by (21) and (18), $\| \prod_{j=1}^n F_j - \overline{F}^n \| \leqslant \widetilde{u}_1\, \eta_1 \leqslant 2\widetilde{u}_1$.

(b) From (21) and (18), it follows that

$$\Big\| \prod_{j=1}^n F_j - \overline{F}^n \Big\| \leqslant 2\widetilde{u}_1 \max_{j\in\underline{n}} \int_{\mathfrak{X}} (f_j - 1)^2 \,\mathrm{d}\overline{F}.$$

(c) It is unclear, whether it is possible to remove the singularity in (8) for any $\alpha > 0$. Indeed, since the denominator of the right-hand side of (8) contains η_l and not $\eta_{\ell,\alpha}$, we cannot argue as in the proof of Theorem 2.2.

Example 2.1 In the situation of Theorem 2.1, let us assume that $F_j = \sum_{r=0}^d p_{j,r} H_r$, $(j \in \underline{n}$, $d \in \mathbb{N})$ and $G = \overline{F} = \sum_{r=0}^d \overline{p}_r H_r$, where $H_0, \ldots, H_d \in \mathcal{F}$, and for $r \in \underline{d}_0$, $p_{j,r} \in [0,1]$ with $\sum_{r=0}^d p_{j,r} = 1$ and $\overline{p}_r = \frac{1}{n}\sum_{j=1}^n p_{j,r} > 0$. Then, for each $r \in \underline{d}_0$, H_r has a $\overline{F}$-density h_r and we may assume that $\sum_{r=0}^d \overline{p}_r h_r = 1$. Consequently, F_j has the $\overline{F}$-density $f_j := \sum_{r=0}^d p_{j,r} h_r$, $(j \in \underline{n})$. Using the simple inequality

$$\frac{(\sum_{r=0}^d a_r)^2}{\sum_{r=0}^d a'_r} \leqslant \sum_{r=0}^d \frac{a_r^2}{a'_r}, \quad (a_r \in [0,\infty), a'_r \in (0,\infty) \text{ for } r \in \underline{d}_0), \tag{24}$$

we obtain, for $j \in \underline{n}$,

$$\begin{aligned}\int_{\mathfrak{X}} (f_j-1)^2\,\mathrm{d}\overline{F} &= \int_{\mathfrak{X}} f_j^2\,\mathrm{d}\overline{F}-1 = \int_{\mathfrak{X}} \frac{(\sum_{r=0}^d p_{j,r}h_r)^2}{\sum_{r=0}^d \overline{p}_r h_r}\,\mathrm{d}\overline{F}-1\\ &\leqslant \sum_{r=0}^d \frac{p_{j,r}^2}{\overline{p}_r}\int_{\{h_r>0\}}\frac{h_r^2}{h_r}\,\mathrm{d}\overline{F}-1 = \sum_{r=0}^d \frac{p_{j,r}^2}{\overline{p}_r}-1 = \sum_{r=0}^d \frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r}.\end{aligned}$$

Further, we have

$$\big\|F_j-\overline{F}\big\| \leqslant \sum_{r=0}^d |\overline{p}_r-p_{j,r}|.$$

Therefore, in this context, (18) implies that, for $\ell \in \underline{n}$,

$$\eta_\ell \leqslant \sum_{j=1}^n \min\Big\{2\sum_{r=0}^d \frac{(\overline{p}_r-p_{j,r})^2}{n\overline{p}_r},\ \frac{1}{\ell+1}\Big(\sum_{r=0}^d|\overline{p}_r-p_{j,r}|\Big)^2\Big\}. \tag{25}$$

Using (24), we get

$$\frac{(\overline{p}_0-p_{j,0})^2}{\overline{p}_0} = \frac{(1-\overline{p}_0)(\sum_{r=1}^d(\overline{p}_r-p_{j,r}))^2}{\overline{p}_0\sum_{r=1}^d\overline{p}_r} \leqslant \Big(\frac{1}{\overline{p}_0}-1\Big)\sum_{r=1}^d\frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r}$$

and hence

$$\sum_{r=0}^d \frac{(\overline{p}_r-p_{j,r})^2}{n\overline{p}_r} \leqslant \sum_{r=1}^d \frac{(\overline{p}_r-p_{j,r})^2}{n\overline{p}_r\overline{p}_0}. \tag{26}$$

We note that (26) is non-trivial in the sense that the sum on the right-hand side does not contain the summand for $r=0$. In view of (21), (22), and (25) with $\ell=1$, and (26), we see that, in (7), the factor $C_3 d$ can be replaced with $2\widetilde{u}_1$, which in turn is bounded by 21.88. We note that, if the H_r are given as in (1) then (25) and (18) coincide. But if $H_0 \approx \cdots \approx H_d$ in some sense, then (25) can be much worse than (18) and should therefore not be used in general.

The next proposition shows that, as claimed above, sometimes $\eta_{\ell,\alpha}$, $(\alpha>0)$ has a better order than η_ℓ. Here, we consider the case of symmetric distributions $F_1,\dots,F_n \in \mathcal{F}$ with finite support. For simplicity, we assume that $G=\overline{F}$.

Proposition 2.1 *Let the notation from Theorem 2.1 hold. Further, let* $b\in\mathbb{N}$, $x_1,\dots,x_b \in \mathfrak{X}\setminus\{0\}$, $F_j = p_{j,0}I+\sum_{r=1}^b p_{j,r}(I_{-x_r}+I_{x_r}) \in \mathcal{F}$, $(j\in\underline{n})$, *and* $G=\overline{F}=\overline{p}_0 I+\sum_{r=1}^b\overline{p}_r(I_{-x_r}+I_{x_r})$, *where* $p_{j,r}\in[0,1]$ *with* $p_{j,0}+2\sum_{r=1}^b p_{j,r}=1$ *and* $\overline{p}_r=\frac{1}{n}\sum_{j=1}^n p_{j,r}>0$, $(r\in\underline{b}_0)$. *For* $\ell\in\underline{n}$, *we then have*

$$\begin{aligned}\eta_{\ell,1} \leqslant \sum_{j=1}^n \min\Big\{&\frac{4}{n^2}\Big(\frac{(\overline{p}_0-p_{j,0})^2}{2\overline{p}_0^2}+2\sum_{r=1}^b\frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r\overline{p}_0}+\sum_{r=1}^b\frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r^2}\Big),\\ &\frac{1}{(\ell+1)^2}\Big(|\overline{p}_0-p_{j,0}|+2\sum_{r=1}^b|\overline{p}_r-p_{j,r}|\Big)^2\Big\}.\end{aligned} \tag{27}$$

We note that, in contrast to (18), the bound in (27) has the better magic factor n^{-2}. Hence, in the situation of Proposition 2.1, estimate (8) with $\alpha=1$ should be preferred over (9).

3 Numerical examples

In what follows, we compare the available bounds in the multinomial approximation of the generalized multinomial distribution. We assume the notation given in (1) with $d = 10$. Further let $\ell = 1$. The following two examples show that the results of the present paper can be considerably sharper than the bounds from the literature discussed in Section 1.3.

Example 3.1 For $j \in \underline{n}$, let $p_{j,r} = \binom{d}{r} q_j^r (1-q_j)^{d-r}$, $(r \in \underline{d}_0)$ be the binomial counting density with number of trials d and success probability $q_j = 0.4 + \frac{1}{(j+9)^a}$, where $a \geqslant 1$. Clearly we have $q_j \in (0.4, 0.5]$ for all $j \in \underline{n}$. We emphasize that, with this definition, F_j is not a binomial distribution. Further, if a or n is large, then $p_{j,r}$ should be close to $\overline{p}_r$ for a sufficient number of $j \in \underline{d}$ and $r \in \underline{d}_0$, so that we expect a small distance $\| \prod_{j=1}^n F_j - \overline{F}^n \|$ here. This is reflected in the bounds, given in Table 1.

Table 1: Numerical bounds for the distance in Example 3.1

n	a	C_1	C_2	(3)	(5)	(6)	(23) & (25)	(15) & (25)
100	1	111.4	15590.9	n.a.	$\geqslant 2$	n.a.	0.197438	0.173503
1000	1	145.7	26444.8	n.a.	$\geqslant 2$	n.a.	0.026902	0.032981
100	2	154.6	29809.2	n.a.	0.107737	0.034777	0.000366	0.000954
1000	2	156.3	30455.0	n.a.	0.110925	0.035914	0.000037	0.000120

Note that the bounds for the distance are always rounded up. Further, as the distance is always bounded by 2, larger bounds are omitted. The entry "n.a." means "not available" and describes a situation, where the bound cannot be used, since the respective condition does not hold. In all cases, the quantities C_1 and C_2 (see (4) for the definition) are quite large, which explains that the condition for (3) is not valid here. This is due to the fact that, in each case, some of the $\overline{p}_r$, $(r \in \underline{d}_0)$ are quite small. E.g. see Table 2 for the case $n = 100$ and $a = 1$.

Table 2: Point probabilities of $\overline{F}$ in Example 3.1 when $n = 100$, $a = 1$

r	0	1	2	3	4	5
$\overline{p}_r$	0.00416	0.03012	0.09851	0.19175	0.24611	0.21781
r	6	7	8	9	10	
$\overline{p}_r$	0.13473	0.05757	0.01628	0.00276	0.00021	

In the next example, we discuss a situation, where (3) gives non-trivial bounds.

Example 3.2 For $j \in \underline{n}$ and $r \in \underline{d}_0$, let

$$p_{j,r} = \frac{1 + (j+r)/(b(n+d))}{\sum_{r_1=0}^{d}(1 + (j+r_1)/(b(n+d)))},$$

where $b \geqslant 1$. Similarly as in Example 3.1, for large n or b, we expect good approximation, which indeed is reflected in the bounds for $\| \prod_{j=1}^n F_j - \overline{F}^n \|$ given in Table 3.

Table 3: Numerical bounds for the distance in Example 3.2

n	b	(3)	(5)	(6)	(23) & (25)	(15) & (25)
100	1	0.325253	0.008310	0.002337	0.000030	0.000098
1000	1	0.118021	0.000119	0.000033	3.9×10^{-7}	1.5×10^{-6}
100	2	0.112763	0.000978	0.000267	3.3×10^{-6}	1.2×10^{-5}
1000	2	0.040581	0.000014	3.8×10^{-6}	4.4×10^{-8}	1.7×10^{-7}

In contrast to Example 3.1, in each case the values $\overline{p}_r$, $(r \in \underline{d}_0)$ are quite similar, which implies that the condition for (3) is valid. E.g. see Table 4 for the case $n = 100$ and $b = 1$.

Table 4: Point probabilities of $\overline{F}$ in Example 3.2 when $n = 100$, $b = 1$

r	0	1	2	3	4	5
$\overline{p}_r$	0.08807	0.08864	0.08921	0.08978	0.09034	0.09091
r	6	7	8	9	10	
$\overline{p}_r$	0.09148	0.09204	0.09261	0.09318	0.09374	

In what follows, we discuss an example, where the distance can actually be evaluated.

Example 3.3 Suppose now that, in Example 3.1, we change the measures H_r to $H_r = I_r$ on $\mathbb{R}$ for $r \in \underline{d}_0$, i.e. all distributions $F_1, \dots, F_n, \overline{F}$ are one-dimensional. Then, using a computer, it is not difficult to get the exact numerical value for the distance, see Table 5.

Table 5: Exact numerical values for the distance (cf. with Table 1)

n	100	1000	100	1000
a	1	1	2	2
$\lVert \prod_{j=1}^n F_j - \overline{F}^n \rVert$	0.007152	0.001653	5.9×10^{-5}	7.6×10^{-6}

A basic property of the total variation distance tells us that, for distributions $\widetilde{H}_r \in \mathcal{F}$, $(r \in \underline{d}_0)$ in the case of a general measurable Abelian group, we have

$$\Big\| \prod_{j=1}^{n} \Big(\sum_{r=0}^{d} p_{j,r} \widetilde{H}_r \Big) - \Big(\sum_{r=0}^{d} \overline{p}_r \widetilde{H}_r \Big)^n \Big\| \leqslant \Big\| \prod_{j=1}^{n} \Big(p_{j,0} I + \sum_{r=1}^{d} p_{j,r} I_{e_r} \Big) - \Big(\overline{p}_0 I + \sum_{r=1}^{d} \overline{p}_r I_{e_r} \Big)^n \Big\|. \quad (28)$$

This can easily be seen by writing the difference of the measures on the left-hand side as a polynomial in $\widetilde{H}_r$, $(r \in \underline{d}_0)$ and then applying the triangle inequality. As a consequence of (28), each bound from Table 1 is valid here as well. A comparison shows that the bounds are getting closer to the actual distance as n or a is becoming large. For example, the bounds from (23) & (25) are about 27.6, 16.3, 6.2, and 4.9 times higher, respectively, than the values from Table 5.

We can apply this idea to Example 3.2 as well: if we again change the measures H_r to $H_r = I_r$ for $r \in \underline{d}_0$, we get the exact values of Table 6. A comparison with Table 3 shows that the bounds from (23) & (25) are about 4.8 to 4.0 times higher than these values.

Table 6: Exact numerical values for the distance (cf. with Table 3)

n	100	1000	100	1000
b	1	1	2	2
$\Vert \prod_{j=1}^n F_j - \overline{F}^n \Vert$	6.3×10^{-6}	9.1×10^{-8}	7.4×10^{-7}	1.1×10^{-8}

4 Proofs

4.1 Smoothness estimates for convolutions

In what follows, we use the standard multi-index notation: For $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, $(d \in \mathbb{N})$ and $w = (w_1, \dots, w_d) \in \mathbb{Z}_+^d$, we set $z^w = \prod_{r=1}^d z_r^{w_r}$, $|w| = \sum_{r=1}^d w_r$, and $w! = \prod_{r=1}^d w_r!$. Similarly, for $V = (V_1, \dots, V_d) \in \mathcal{M}^d$, set $V^w = \prod_{r=1}^d V_r^{w_r}$. For $v, w \in \mathbb{Z}_+^d$, we write $v \leqslant w$ in the case that $v_r \leqslant w_r$ for all $r \in \underline{d}$; let $v \wedge w = (v_1 \wedge w_1, \dots, v_d \wedge w_d)$. Sums over v, $\widetilde{v}$, and w are taken over subsets of $\mathbb{Z}_+^d$ as indicated. The following lemma is a counterpart of Lemma 5 in Roos (2001).

Lemma 4.1 *Let $k, n \in \mathbb{Z}_+$, $d \in \mathbb{N}$, and $a_v \in \mathbb{R}$ for $v \in \mathbb{Z}_+^d$ with $|v| = k$. Let $X = (X_r)_{r \in \underline{d}}$ be a random vector in $\mathbb{R}^d$ with $\mathrm{E}[(\sum_{r=1}^d |X_r|)^k] < \infty$ and put $X_0 = \sum_{r=1}^d X_r$. Let $p = (p_r)_{r \in \underline{d}} \in (0,1)^d$ such that $p_0 = 1 - \sum_{r=1}^d p_r \in (0,1)$. Further, let $H = (H_r)_{r \in \underline{d}} \in \mathcal{F}^d$, $H_0 \in \mathcal{F}$,*

$$G = \sum_{r=0}^d p_r H_r \in \mathcal{F}, \quad U_1 = \sum_{|v|=k} \frac{a_v}{v!} \prod_{r=1}^d (H_r - H_0)^{v_r}, \quad U_2 = \mathrm{E}\Big(\sum_{r=1}^d X_r (H_r - H_0)\Big)^k,$$

where, in the definition of U_2, the expectation is defined in the pointwise sense. Then we have

$$\Vert U_1\, G^n \Vert \leqslant \frac{\sqrt{n!}}{\sqrt{(n+k)!}} \Big(\sum_{|w| \leqslant k} \frac{w!(k-|w|)!}{p^w\, p_0^{k-|w|}} \Big[\sum_{|v|=k} \frac{a_v}{v!} \prod_{r=1}^d \binom{v_r}{w_r} \Big]^2 \Big)^{1/2}, \tag{29}$$

$$\Vert U_2\, G^n \Vert \leqslant \binom{n+k}{k}^{-1/2} \Big(\mathrm{E}\Big(\sum_{r=0}^d \frac{X_r Y_r}{p_r} \Big)^k \Big)^{1/2}, \tag{30}$$

where the random vector $Y = (Y_r)_{r \in \underline{d}}$ is an independent copy of X and $Y_0 = \sum_{r=1}^d Y_r$.

Proof. Let

$$\mathrm{Mult}(w, n, p) = \begin{cases} \frac{n!}{w!\,(n-|w|)!}\, p^w\, p_0^{n-|w|}, & \text{if } w \in \mathbb{Z}_+^d,\ |w| \leqslant n, \\ 0, & \text{otherwise} \end{cases}$$

denote the multinomial counting density with parameters n and p. For $f : \mathbb{Z}^d \longrightarrow \mathbb{R}$ and $r \in \underline{d}$, let $\Delta_r f : \mathbb{Z}^d \longrightarrow \mathbb{R}$ with $(\Delta_r f)(w) = f(w - e_r) - f(w)$ for $w \in \mathbb{Z}^d$. Products and powers of Δ-operators are understood in the sense of composition. Further, let $\Delta_r^0 f = f$. Clearly, $\Delta_{r_1}\Delta_{r_2} f = \Delta_{r_2}\Delta_{r_1} f$ for $r_1, r_2 \in \underline{d}$. For $v \in \mathbb{Z}_+^d$, let $\Delta^v f = \Delta_1^{v_1} \cdots \Delta_d^{v_d} f$. We set $\Delta^v \mathrm{Mult}(w, n, p) = (\Delta^v \mathrm{Mult}(\cdot, n, p))(w)$ for $w \in \mathbb{Z}_+^d$. We use the following properties of the multinomial distribution (see Roos (2001, formulas (20), (21), and (4))): For $v \in \mathbb{Z}_+^d$,

$$\sum_{|w| \leqslant n+|v|} \Delta^v \mathrm{Mult}(w, n, p)\, H^w\, H_0^{n+|v|-|w|} = G^n \prod_{r=1}^d (H_r - H_0)^{v_r} \tag{31}$$

and, for $v, w \in \mathbb{Z}_+^d$,

$$\Delta^v \mathrm{Mult}(w,n,p) = \mathrm{Kraw}(v;w,n+|v|,p)\mathrm{Mult}(w,n+|v|,p)\frac{v!\,n!}{(n+|v|)!\,p^v\,p_0^{|v|}}, \tag{32}$$

where

$$\mathrm{Kraw}(v;w,n,p) = \sum_{\widetilde{v}\leqslant v}\binom{n-|w|}{|v-\widetilde{v}|}\frac{|v-\widetilde{v}|!\,(-p)^{v-\widetilde{v}}\,p_0^{|\widetilde{v}|}}{(v-\widetilde{v})!}\prod_{r=1}^d\binom{w_r}{\widetilde{v}_r}, \tag{33}$$

is a Krawtchouk polynomial of degree v. Note that there is another set of Krawtchouk polynomials, which forms, together with the one from (33), a bi-orthogonal system of polynomials with respect to the multinomial distribution (see also Tratnik (1989)). From the more general Lemma 2 in Roos (2001), it follows that, for $v, \widetilde{v} \in \mathbb{Z}_+^d$ with $|v| = |\widetilde{v}|$, we have

$$\begin{aligned}\sum_{|w|\leqslant n+|v|} &\mathrm{Mult}(w,n+|v|,p)\,\mathrm{Kraw}(v;w,n+|v|,p)\,\mathrm{Kraw}(\widetilde{v};w,n+|v|,p)\\ &= \sum_{w\leqslant v\wedge\widetilde{v}}\frac{(n+|v|)!\,|v-w|!\,p^{v+\widetilde{v}-w}\,p_0^{|w+v|}}{w!\,n!\,(v-w)!\,(\widetilde{v}-w)!}.\end{aligned} \tag{34}$$

We note that the right-hand side of (34) is always positive, which shows that, if $d \geqslant 2$, then the Krawtchouk polynomials given above are not orthogonal with respect to the multinomial distribution. However, we do not need such a property. Using (31), (32), Cauchy's inequality, we now obtain

$$\begin{aligned}\big\|U_1\,G^n\big\| &= \Big\|\sum_{|v|=k}\frac{a_v}{v!}\sum_{|w|\leqslant n+|v|}\Delta^v\mathrm{Mult}(w,n,p)\,H^w\,H_0^{n+|v|-|w|}\Big\|\\ &\leqslant \frac{n!}{(n+k)!}\sum_{w\in\mathbb{Z}_+^d}\mathrm{Mult}(w,n+k,p)\Big|\sum_{|v|=k}\frac{a_v}{p^v\,p_0^k}\mathrm{Kraw}(v;w,n+k,p)\Big|\\ &\leqslant \frac{n!}{(n+k)!}\Big(\sum_{w\in\mathbb{Z}_+^d}\mathrm{Mult}(w,n+k,p)\Big[\sum_{|v|=k}\frac{a_v}{p^v\,p_0^k}\mathrm{Kraw}(v;w,n+k,p)\Big]^2\Big)^{1/2} =: T.\end{aligned}$$

Using (34), we get

$$\begin{aligned}T &= \frac{n!}{(n+k)!}\Big(\sum_{|v|=k}\sum_{|\widetilde{v}|=k}\frac{a_v\,a_{\widetilde{v}}}{p^{v+\widetilde{v}}\,p_0^{2k}}\sum_{w\leqslant v\wedge\widetilde{v}}\frac{(n+k)!\,(k-|w|)!\,p^{v+\widetilde{v}-w}\,p_0^{|w|+k}}{w!\,n!\,(v-w)!\,(\widetilde{v}-w)!}\Big)^{1/2}\\ &= \frac{\sqrt{n!}}{\sqrt{(n+k)!}}\Big(\sum_{|w|\leqslant k}\frac{w!(k-|w|)!}{p^w\,p_0^{k-|w|}}\Big[\sum_{|v|=k}\frac{a_v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\sum_{|\widetilde{v}|=k}\frac{a_{\widetilde{v}}}{\widetilde{v}!}\prod_{r=1}^d\binom{\widetilde{v}_r}{w_r}\Big]\Big)^{1/2}\\ &= \frac{\sqrt{n!}}{\sqrt{(n+k)!}}\Big(\sum_{|w|\leqslant k}\frac{w!(k-|w|)!}{p^w\,p_0^{k-|w|}}\Big[\sum_{|v|=k}\frac{a_v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]^2\Big)^{1/2}.\end{aligned}$$

Inequality (29) is shown. Since $U_2=\sum_{|v|=k}\frac{k!}{v!}\,\mathrm{E}\big[X^v\big]\prod_{r=1}^d(H_r-H_0)^{v_r}$, (29) gives

$$\begin{aligned}\|U_2\,G^n\| &\leqslant \frac{\sqrt{n!}\,k!}{\sqrt{(n+k)!}}\Big(\sum_{|w|\leqslant k}\frac{w!(k-|w|)!}{p^w\,p_0^{k-|w|}}\Big[\sum_{|v|=k}\frac{\mathrm{E}X^v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]^2\Big)^{1/2}\\ &= \frac{\sqrt{n!}\,k!}{\sqrt{(n+k)!}}\Big(\sum_{|w|\leqslant k}\frac{w!(k-|w|)!}{p^w\,p_0^{k-|w|}}\mathrm{E}\Big[\sum_{|v|=k}\frac{X^v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\sum_{|\widetilde{v}|=k}\frac{Y^{\widetilde{v}}}{\widetilde{v}!}\prod_{r=1}^d\binom{\widetilde{v}_r}{w_r}\Big]\Big)^{1/2}.\end{aligned}$$

For $|w|\leqslant k$, we have

$$\sum_{|v|=k}\frac{X^v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}=\frac{X^w\,X_0^{k-|w|}}{w!\,(k-|w|)!}.$$

Indeed, this follows from the identity theorem for power series taking into account the following equality of the corresponding generating functions

$$\sum_{k=0}^\infty\Big[\sum_{|v|=k}\frac{X^v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]z^k=\frac{z^{|w|}X^w}{w!}\mathrm{e}^{zX_0}=\sum_{k=|w|}^\infty\Big[\frac{X^w\,X_0^{k-|w|}}{w!\,(k-|w|)!}\Big]z^k,\qquad (z\in\mathbb{C}).$$

From the above, we get

$$\begin{aligned}\|U_2\,G^n\| &\leqslant \binom{n+k}{k}^{-1/2}\Big(\mathrm{E}\Big[\sum_{|w|\leqslant k}\frac{k!}{w!\,(k-|w|)!}\Big(\frac{X_0Y_0}{p_0}\Big)^{k-|w|}\prod_{r=1}^d\Big(\frac{X_rY_r}{p_r}\Big)^{w_r}\Big]\Big)^{1/2}\\ &= \binom{n+k}{k}^{-1/2}\Big(\mathrm{E}\Big(\sum_{r=0}^d\frac{X_rY_r}{p_r}\Big)^k\Big)^{1/2},\end{aligned}$$

which completes the proof of (30). □

The following lemma is an important application of Lemma 4.1 and generalizes formula (37) in Roos (2000). Another application is given in the proof of Proposition 2.1, see Section 4.3 below.

Lemma 4.2 *Let $k\in\mathbb{N}$, $n\in\mathbb{Z}_+$, $G\in\mathcal{F}$, and $U\in\mathcal{M}$, where we assume that $|U|\ll G$ and that $U(\mathfrak{X})=0$; let $f^\pm$ denote any Radon-Nikodym densities of $U^\pm$ with respect to G and put $f=f^+-f^-$. Then*

$$\|U^k\,G^n\|\leqslant\binom{n+k}{k}^{-1/2}\Big(\int f^2\,\mathrm{d}G\Big)^{k/2}. \tag{35}$$

Proof. If $\int f^2\,\mathrm{d}G=\infty$, then (35) is trivial. In what follows, we assume that $\int f^2\,\mathrm{d}G<\infty$. Let $\varepsilon\in(0,1)$ be fixed. Then $a_{j,\varepsilon}^\pm\in[0,\infty)$, $(j\in\mathbb{Z}_+)$ and pairwise disjoint $B_{j,\varepsilon}\in\mathcal{A}$, $(j\in\mathbb{Z}_+)$ exist such that

$$\bigcup_{j=0}^\infty B_{j,\varepsilon}=\mathfrak{X},\quad f_\varepsilon^\pm:=\sum_{j=0}^\infty a_{j,\varepsilon}^\pm\mathbf{1}(B_{j,\varepsilon}),\quad\text{and}\quad 0\leqslant f^\pm-f_\varepsilon^\pm\leqslant\varepsilon.$$

Here $\mathbf{1}(A)$ is the indicator function of a set A. Let $U_\varepsilon^\pm$ be the measures on $(\mathfrak{X},\mathcal{A})$ with G-densities $f_\varepsilon^\pm$. This implies that $U_\varepsilon^\pm=\sum_{j=0}^\infty q_{j,\varepsilon}^\pm\,H_{j,\varepsilon}$, where, for $j\in\mathbb{Z}_+$,

$$q_{j,\varepsilon}^\pm=a_{j,\varepsilon}^\pm\,G(B_{j,\varepsilon})\quad\text{and}\quad H_{j,\varepsilon}=\begin{cases}G(B_{j,\varepsilon}\cap\,\cdot\,)/G(B_{j,\varepsilon}), & \text{if } G(B_{j,\varepsilon})>0,\\ I, & \text{otherwise.}\end{cases}$$

Set $q_{j,\varepsilon} = q^+_{j,\varepsilon} - q^-_{j,\varepsilon}$, $f_\varepsilon = f^+_\varepsilon - f^-_\varepsilon$, and $U_\varepsilon = U^+_\varepsilon - U^-_\varepsilon$. We note that the latter equality indeed indicates the Hahn-Jordan decomposition of U_ε. Then $\|U_\varepsilon\| \leqslant \|U\|$ and $\|U - U_\varepsilon\| = \int \big((f^+ - f^+_\varepsilon) + (f^- - f^-_\varepsilon)\big)\,\mathrm{d}G \leqslant \varepsilon$, giving

$$\|U^k - U^k_\varepsilon\| \leqslant k\,\|U\|^{k-1}\|U - U_\varepsilon\| \leqslant k\,\|U\|^{k-1}\varepsilon.$$

Hence

$$\|U^k\,G^n\| \leqslant \|(U^k - U^k_\varepsilon)G^n\| + \|U^k_\varepsilon\,G^n\| \leqslant k\,\|U\|^{k-1}\varepsilon + \|U^k_\varepsilon G^n\|.$$

Since $\int f\,\mathrm{d}G = U(\mathfrak{X}) = 0$, we have $\big|\sum_{j=0}^\infty q_{j,\varepsilon}\big| = \big|\int (f^+_\varepsilon - f^-_\varepsilon)\,\mathrm{d}G\big| \leqslant \|U - U_\varepsilon\| \leqslant \varepsilon$, and therefore, for each $m \in \mathbb{N}$,

$$\Big\|\Big(\sum_{j=0}^\infty q_{j,\varepsilon}H_{j,\varepsilon}\Big)^k - \Big(\sum_{j=0}^m q_{j,\varepsilon}(H_{j,\varepsilon} - H_{0,\varepsilon})\Big)^k\Big\| \leqslant k\Big(\varepsilon + 2\sum_{j=m+1}^\infty |q_{j,\varepsilon}|\Big)\Big(2\sum_{j=0}^\infty |q_{j,\varepsilon}|\Big)^{k-1}.$$

Hence, we obtain

$$\begin{aligned}\|U^k_\varepsilon\,G^n\| &= \Big\|\Big(\sum_{j=0}^\infty q_{j,\varepsilon}H_{j,\varepsilon}\Big)^k\Big(\sum_{j=0}^\infty G(B_{j,\varepsilon})H_{j,\varepsilon}\Big)^n\Big\| \\ &\leqslant k\Big(\varepsilon + 2\sum_{j=m+1}^\infty |q_{j,\varepsilon}|\Big)\Big(2\sum_{j=0}^\infty |q_{j,\varepsilon}|\Big)^{k-1} + \Big\|\Big(\sum_{j=1}^m q_{j,\varepsilon}(H_{j,\varepsilon} - H_{0,\varepsilon})\Big)^k\Big(\sum_{j=0}^\infty G(B_{j,\varepsilon})H_{j,\varepsilon}\Big)^n\Big\|.\end{aligned}$$

From (30), it follows that the norm term on the right-hand side is bounded from above by

$$\begin{aligned}\binom{n+k}{k}^{-1/2}\Big(\sum_{j\in\underline{m}_0:\,G(B_{j,\varepsilon})>0}\frac{q^2_{j,\varepsilon}}{G(B_{j,\varepsilon})}\Big)^{k/2} &= \binom{n+k}{k}^{-1/2}\Big(\sum_{j=0}^m (a^+_{j,\varepsilon} - a^-_{j,\varepsilon})^2 G(B_{j,\varepsilon})\Big)^{k/2} \\ &\leqslant \binom{n+k}{k}^{-1/2}\Big(\int f^2_\varepsilon\,\mathrm{d}G\Big)^{k/2}.\end{aligned}$$

Letting $m \to \infty$, we obtain

$$\|U^k_\varepsilon\,G^n\| \leqslant k\varepsilon\Big(2\sum_{j=0}^\infty |q_{j,\varepsilon}|\Big)^{k-1} + \binom{n+k}{k}^{-1/2}\Big(\int f^2_\varepsilon\,\mathrm{d}G\Big)^{k/2}.$$

Since

$$\Big|\int (f^2 - f^2_\varepsilon)\,\mathrm{d}G\Big| \leqslant \int |f - f_\varepsilon|(|f| + |f_\varepsilon|)\,\mathrm{d}G \leqslant \varepsilon(\|U\| + \|U_\varepsilon\|) \leqslant 2\varepsilon\|U\|,$$

we obtain (35) by letting $\varepsilon \to 0$. This completes the proof. □

It may happen that the assumption in Lemma 4.2 does not hold directly. However, this can sometimes be overcome by shifting U. The following corollary is needed in the proof of Theorem 2.1.

Corollary 4.1 *Let $n \in \mathbb{Z}_+$, $G \in \mathcal{F}$, $U_1, U_2 \in \mathcal{M}$, and $U = U_1 + U_2$. We assume that $|U_2| \ll G$ and that both $U^\pm_2 \neq 0$. Put $\widetilde{U}^\pm_2 = U^\pm_2/\|U^\pm_2\|$. Let $f^\pm$ denote any Radon-Nikodym densities of $\widetilde{U}^\pm_2$ with respect to G and set $f = f^+ - f^-$. Then*

$$\|U\,G^n\| \leqslant \|U_1\| + |U_2(\mathfrak{X})| + \frac{\|U^+_2\| \wedge \|U^-_2\|}{\sqrt{n+1}}\Big(\int f^2\,\mathrm{d}G\Big)^{1/2}. \tag{36}$$

Proof. The assertion easily follows from the triangle inequality, Lemma 4.2, and the simple fact that $U_2 = U_2(\mathfrak{X})\widetilde{U}_2^{\tau} + (\|U_2^+\| \wedge \|U_2^-\|)(\widetilde{U}_2^+ - \widetilde{U}_2^-)$, where τ denotes $+$ or $-$ according to whether $\|U_2^+\| > \|U_2^-\|$ or not. □

Remark 4.1 (a) Let the assumptions of Corollary 4.1 hold. If μ is a σ-finite measure on $\mathfrak{X}$ and if $G \ll \mu$, then G and $\widetilde{U}_2^{\pm}$ have μ-densities v and $g^{\pm}$, say, and, letting $g = g^+ - g^-$, we can write $\int f^2 \,\mathrm{d}G = \int_{\{v>0\}} g^2 v^{-1} \,\mathrm{d}\mu$.

(b) Sometimes it is useful to simplify further the bound (36) by using the following inequality: $(\|U_2^+\| \wedge \|U_2^-\|)^2 \int f^2 \,\mathrm{d}G \leqslant \int h^2 \,\mathrm{d}G$, where $h = h^+ - h^-$ and $h^{\pm}$ denote any G-densities of $U_2^{\pm}$. Indeed, this follows from the representation

$$\int f^2 \,\mathrm{d}G = \int_A \frac{(h^+)^2}{\|U_2^+\|^2} \,\mathrm{d}G + \int_{A^c} \frac{(h^-)^2}{\|U_2^-\|^2} \,\mathrm{d}G,$$

whenever $A \in \mathcal{A}$ with $U_2^-(A) = U_2^+(A^c) = 0$.

The next corollary is an extension of Lemma 4.2 to compound distributions and may be particularly useful in the compound Poisson approximation.

Corollary 4.2 *Let $k \in \mathbb{N}$, $G \in \mathcal{F}$, and $U \in \mathcal{M}$, where we assume that $|U| \ll G$ and that $U(\mathfrak{X}) = 0$; let $f^{\pm}$ denote any Radon-Nikodym densities of $U^{\pm}$ with respect to G and put $f = f^+ - f^-$. Let N be a random variable in $\mathbb{Z}_+$ and $\varphi(z) = \mathrm{E}[z^N]$, $(z \in \mathbb{C}, |z| \leqslant 1)$ be its generating function. Set $\varphi(G) = \mathrm{E}[G^N] \in \mathcal{F}$, where the expectation is defined in the pointwise sense. Then we have*

$$\|U^k \varphi(G)\| \leqslant \Big(k \int_0^1 x^{k-1} \varphi(1-x) \,\mathrm{d}x\Big)^{1/2} \Big(\int f^2 \,\mathrm{d}G\Big)^{k/2}. \tag{37}$$

If N has Poisson distribution $\exp(t(I_1 - I))$ with $t \in (0, \infty)$, then

$$\|U^k \varphi(G)\| \leqslant \frac{1}{t^{k/2}} \sqrt{k!\, \mathrm{P}(N \geqslant k)} \Big(\int f^2 \,\mathrm{d}G\Big)^{k/2}. \tag{38}$$

Proof. Using the triangle inequality, Lemma 4.2, and Jensen's inequality, we obtain

$$\|U^k \varphi(G)\| \leqslant \mathrm{E}\|U^k G^N\| \leqslant \Big(\mathrm{E}\binom{N+k}{k}^{-1}\Big)^{1/2} \Big(\int f^2 \,\mathrm{d}G\Big)^{k/2}.$$

The integral representation of the beta function implies that $\mathrm{E}\binom{N+k}{k}^{-1} = k \int_0^1 x^{k-1} \varphi(1-x) \,\mathrm{d}x$, which, in turn, leads to (37). Inequality (38) easily follows from (37) and the series representation of the lower incomplete gamma function. □

We note that (38) is comparable to previous results of Roos (2003, Lemma 2) but is however much better because of the more general assumptions used in Corollary 4.2.

4.2 A general lemma

The results of Section 2 are based on the following general lemma. Here, a distribution $G \in \mathcal{F}$ is called infinitely divisible if, for each $n \in \mathbb{N}$, there exists a $G_n \in \mathcal{F}$ such that $G_n^n = G$. We note that, in general, such a n-th root G_n need not be unique (see Heyer (1977, proof of Theorem 3.5.15, pp. 222–223)); let $G^{1/n}$ denote any fixed n-th root of G.

Lemma 4.3 *Let* $n \in \mathbb{N}$, $F_1, \dots, F_n, G \in \mathcal{F}$, $L_1, \dots, L_n \in \mathcal{M}$. *Set* $\overline{L} = \frac{1}{n}\sum_{j=1}^n L_j$, $K_j = F_j \mathrm{e}^{-L_j}$, $(j \in \underline{n})$, $K_0 = G\mathrm{e}^{-\overline{L}}$,

$$V_k = \sum_{J \subseteq \underline{n}:\, |J|=k} \prod_{j\in J}(K_j - K_0), \quad (k \in \underline{n}_0), \qquad W_\ell = \sum_{k=0}^{\ell} V_k\, G^{n-k}\mathrm{e}^{k\overline{L}}, \quad (\ell \in \underline{n}_0),$$

$$M_{j,k} = \begin{cases} (K_j - K_0)(G^{n-k})^{1/k}\mathrm{e}^{\overline{L}}, & \text{if } G \text{ is infinitely divisible,} \\ (K_j - K_0)G^{\lfloor (n-k)/k\rfloor}\mathrm{e}^{\overline{L}}, & \text{otherwise,} \end{cases} \qquad (j,k \in \underline{n}),$$

$$\nu_{k,m} = \sum_{j=1}^n \|M_{j,k}\|^m, \quad \widetilde{\nu}_k = \Big\|\sum_{j=1}^n M_{j,k}\Big\|, \quad (k \in \underline{n}, m \in \mathbb{N}).$$

Let c_1 *be defined as in Theorem 2.1. Then, for* $\ell \in \underline{n}_0$,

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant \sum_{k=\ell+1}^{n} \Big[\Big(\frac{2\mathrm{e}\,c_1}{k}\Big(\frac{\widetilde{\nu}_k^2}{4c_1} + \nu_{k,2}\Big)\Big)^{k/2} \wedge \frac{\nu_{k,1}^k}{k!}\Big].$$

The following two examples show possible applications of Lemma 4.3. As a byproduct, results in the compound Poisson approximations can be derived.

Example 4.1 If we consider the case $L_1 = \dots = L_n = 0$, we see that Theorem 2.1(a) is a direct consequence of Lemma 4.3, (cf. proof of Theorem 2.1).

Example 4.2 Suppose that, for $j \in \underline{n}$, $H_j \in \mathcal{F}$, $p_j \in [0,1]$, $L_j = p_j(H_j - I)$, and $F_j = I + L_j$. Put $\overline{L} = n^{-1}\sum_{j=1}^n L_j$ and $G = \mathrm{e}^{\overline{L}}$. Then Lemma 4.3 implies that

$$\Big\|\prod_{j=1}^n F_j - G^n\Big\| \leqslant \sum_{k=1}^n \frac{1}{k!}\Big(\sum_{j=1}^n \|M_{j,k}\|\Big)^k, \tag{39}$$

where, for $j, k \in \underline{n}$,

$$\begin{aligned} K_j &= (I + p_j(H_j - I))\mathrm{e}^{-p_j(H_j - I)}, \qquad K_0 = I, \\ M_{j,k} &= (K_j - K_0)(G^{n-k})^{1/k}\mathrm{e}^{\overline{L}} = ((I + L_j)\mathrm{e}^{-L_j} - I)\exp\Big(\frac{n}{k}\overline{L}\Big). \end{aligned}$$

In principle, (39) is the same as estimate (26) in Roos (2003). The approach used there is based on a slight modification of an expansion due to Kerstan (1964). It is however not sufficient to get the results of the present paper.

For the proof of Lemma 4.3, we use formal power series over $\mathcal{M}$. In the following lemma, some basic properties in connection with the norm on $\mathcal{M}$ are summarized. The proof is simple and therefore omitted.

Lemma 4.4 *For $n \in \mathbb{N}$ and $k \in \underline{n}$, let $\psi_k^{(0)}(z) = \sum_{j=0}^{\infty} W_{j,k}\, z^j$, ($W_{j,k} \in \mathcal{M}$) be a formal power series over $\mathcal{M}$ with variable z and let $\operatorname{Coeff}(z^j, \psi_k^{(0)}(z))$ be its jth coefficient $W_{j,k}$. Further, consider the formal power series $\psi_k^{(1)}(z) = \sum_{j=0}^{\infty} \|W_{j,k}\|\, z^j$ and $\psi_k^{(2)}(z) = \sum_{j=0}^{\infty} a_{j,k}\, z^j$ for $a_{j,k} \in [\|W_{j,k}\|, \infty)$ and $k \in \underline{n}$. Then, for $j \in \mathbb{Z}_+$,*

$$\big\|\operatorname{Coeff}(z^j, \psi_1^{(0)}(z))\big\| = \operatorname{Coeff}(z^j, \psi_1^{(1)}(z)),$$
$$\Big\|\operatorname{Coeff}\Big(z^j, \prod_{k=1}^n \psi_k^{(0)}(z)\Big)\Big\| \leqslant \operatorname{Coeff}\Big(z^j, \prod_{k=1}^n \psi_k^{(1)}(z)\Big) \leqslant \operatorname{Coeff}\Big(z^j, \prod_{k=1}^n \psi_k^{(2)}(z)\Big).$$

Proof of Lemma 4.3. We first note that

$$\prod_{j=1}^n F_j = \Big(\prod_{j=1}^n (K_j - K_0 + K_0)\Big)\mathrm{e}^{n\overline{L}} = \sum_{k=0}^n V_k K_0^{n-k} \mathrm{e}^{n\overline{L}} = \sum_{k=0}^n V_k\, G^{n-k} \mathrm{e}^{k\overline{L}} = W_n.$$

For $k \in \underline{n}$, let $\lambda(n,k) = 0$ or $\lambda(n,k) = n - k - k\lfloor (n-k)/k \rfloor$ according to whether G is infinitely divisible or not. For $\ell \in \underline{n}_0$, we obtain

$$\begin{aligned}\prod_{j=1}^n F_j - W_\ell &= \sum_{k=\ell+1}^n V_k G^{n-k} \mathrm{e}^{k\overline{L}} = \sum_{k=\ell+1}^n \sum_{J\subseteq \underline{n}:\, |J|=k} \Big(\prod_{j\in J} M_{j,k}\Big) G^{\lambda(n,k)} \\ &= \sum_{k=\ell+1}^n \operatorname{Coeff}(z^k, \psi_k(z)) G^{\lambda(n,k)}, \end{aligned} \tag{40}$$

where $\psi_k(z) = \prod_{j=1}^n (I + M_{j,k} z)$ is regarded as a formal power series for $k \in \underline{n}$. It should be mentioned that it is essential here to extract the kth coefficient of a formal power series which itself depends on k. By Lemma 4.4, for $k \in \underline{n}$, we get

$$\big\|\operatorname{Coeff}(z^k, \psi_k(z))\big\| \leqslant \operatorname{Coeff}\Big(z^k, \prod_{j=1}^n (1 + \|M_{j,k}\| z)\Big) \leqslant \operatorname{Coeff}(z^k, \mathrm{e}^{\nu_{k,1} z}) = \frac{\nu_{k,1}^k}{k!}. \tag{41}$$

On the other hand, using

$$\begin{aligned}\psi_k(z) &= \exp\Big(\sum_{j=1}^n M_{j,k} z\Big) \prod_{j=1}^n \big(\mathrm{e}^{-M_{j,k} z}(I + M_{j,k} z)\big) \\ &= \exp\Big(\sum_{j=1}^n M_{j,k} z\Big) \prod_{j=1}^n \Big[\sum_{m=0}^{\infty} \frac{1-m}{m!} (-M_{j,k})^m z^m\Big],\end{aligned}$$

we derive

$$\big\|\operatorname{Coeff}(z^k, \psi_k(z))\big\| \leqslant \operatorname{Coeff}\Big(z^k, \mathrm{e}^{\widetilde{\nu}_k z} \prod_{j=1}^n g(\|M_{j,k}\| z)\Big), \tag{42}$$

where, for $y \in \mathbb{C}$,

$$g(y) = \sum_{m=0}^{\infty} \frac{|1-m|}{m!} y^m = 2 - (1-y)\mathrm{e}^y = 1 + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{8} + \dots.$$

From the definition of c_1, we obtain that

$$|g(y)| \leqslant g(|y|) \leqslant \mathrm{e}^{c_1 |y|^2}. \tag{43}$$

Here, we note that $h(x) := \ln(2-(1-x)\mathrm{e}^x)/x^2$ for $x \in (0,\infty)$ attains its maximum $c_1 = 0.694025\ldots$ at point $x_0 = 0.936219\ldots$. This can easily be shown using the representation

$$h(x) = \frac{1}{x^2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\ln(2-(1-tx)\mathrm{e}^{tx})\,\mathrm{d}t = \int_0^1 \frac{t}{2\mathrm{e}^{-tx}-1+tx}\,\mathrm{d}t,$$

which, after differentiation of the integrand, leads to a useful integral formula of the derivative

$$h'(x) = \frac{1}{x^3}\int_0^x \frac{t^2(2\mathrm{e}^{-t}-1)}{(2\mathrm{e}^{-t}-1+t)^2}\,\mathrm{d}t.$$

As a consequence, we learn that $h'(x) = 0$ has exactly one positive solution $x = x_0$, which can easily be calculated numerically. Let

$$\mathrm{Bessel}(0;y) = \sum_{m=0}^{\infty}\frac{(y^2/4)^m}{(m!)^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(y\cos(t))\,\mathrm{d}t, \qquad (y \in \mathbb{C})$$

be the modified Bessel function of first kind and order 0. Using (42), Cauchy's integral formula, and (43), we derive, for $k \in \underline{n}$ and arbitrary $R_k \in (0,\infty)$,

$$\begin{aligned}
\big\|\mathrm{Coeff}(z^k,\psi_k(z))\big\| &\leqslant \frac{1}{2\pi\,R_k^k}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}kt}\Big(\prod_{j=1}^n g(\|M_{j,k}\|R_k\mathrm{e}^{\mathrm{i}t})\Big)\exp(\widetilde{\nu}_kR_k\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t\\
&\leqslant \frac{1}{2\pi\,R_k^k}\int_{-\pi}^{\pi}\exp(\widetilde{\nu}_kR_k\cos(t))\,\mathrm{d}t\prod_{j=1}^n g(\|M_{j,k}\|R_k)\\
&\leqslant \frac{1}{R_k^k}\mathrm{Bessel}(0;\widetilde{\nu}_kR_k)\exp(c_1\,\nu_{k,2}\,R_k^2)\\
&= \frac{\varphi(\widetilde{\nu}_kR_k)}{R_k^k}\exp\Big(\Big(\frac{\widetilde{\nu}_k^2}{4}+c_1\,\nu_{k,2}\Big)R_k^2\Big),
\end{aligned}$$

where $\varphi(x) = \mathrm{Bessel}(0;x)\,\mathrm{e}^{-x^2/4} \leqslant 1$, $(x \in \mathbb{R})$. Choosing

$$R_k = \Big(\frac{k}{2(4^{-1}\widetilde{\nu}_k^2+c_1\,\nu_{k,2})}\Big)^{1/2},$$

we get

$$\big\|\mathrm{Coeff}(z^k,\psi_k(z))\big\| \leqslant \Big(\frac{2\mathrm{e}\,c_1}{k}\Big(\frac{\widetilde{\nu}_k^2}{4c_1}+\nu_{k,2}\Big)\Big)^{k/2}. \tag{44}$$

Taking into account (40), the fact that $\lambda(n,k) \in \underline{n}_0$, as well as (41) and (44), we obtain

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant \sum_{k=\ell+1}^n \big\|\mathrm{Coeff}(z^k,\psi_k(z))\big\| \leqslant \sum_{k=\ell+1}^n \Big[\Big(\frac{2\mathrm{e}\,c_1}{k}\Big(\frac{\widetilde{\nu}_k^2}{4c_1}+\nu_{k,2}\Big)\Big)^{k/2} \wedge \frac{\nu_{k,1}^k}{k!}\Big].$$

The proof is completed. □

4.3 Remaining proofs

Proof of Theorem 2.1. Part (a) follows from Lemma 4.3. Indeed, for $\eta_\ell < (2\mathrm{e}\, c_1)^{-1}$, we have

$$\begin{aligned}\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| &\leqslant \sum_{k=\ell+1}^n \Big(\frac{2\mathrm{e}\, c_1}{k^{1+\alpha}}\Big(\frac{\widetilde{\nu}_k^2}{4c_1}+\nu_{k,2}\Big)\Big)^{(\ell+1)/2} k^{\alpha(\ell+1)/2}\Big(\frac{2\mathrm{e}\, c_1}{k}\Big(\frac{\widetilde{\nu}_k^2}{4c_1}+\nu_{k,2}\Big)\Big)^{(k-\ell-1)/2}\\ &\leqslant (2\mathrm{e}\, c_1\,\eta_{\ell,\alpha})^{(\ell+1)/2}\sum_{k=\ell+1}^n k^\beta (2\mathrm{e}\, c_1\,\eta_\ell)^{(k-\ell-1)/2}\\ &\leqslant (\ell+1)^\beta\,\beta!\,\frac{(2\mathrm{e}\, c_1\,\eta_{\ell,\alpha})^{(\ell+1)/2}}{(1-\sqrt{2\mathrm{e}\, c_1\,\eta_\ell})^{\beta+1}}.\end{aligned}$$

Here we used that, for $x\in[0,1)$,

$$\sum_{k=\ell+1}^n k^\beta x^{k-\ell-1} \leqslant (\ell+1)^\beta \sum_{k=0}^\infty \frac{(k+\beta)!}{k!}x^k = (\ell+1)^\beta \frac{\mathrm{d}}{\mathrm{d}x^\beta}\frac{1}{1-x} = (\ell+1)^\beta \frac{\beta!}{(1-x)^{\beta+1}}.$$

Part (b) is shown by using Corollary 4.1 together with Remark 4.1. In fact, for $j,k\in\underline{n}$, we obtain

$$\|M_{j,k}\|^2 \leqslant \Big(\|(F_j-G)|_{B_j}\| + |(F_j-G)(B_j)| + \sqrt{\frac{2k}{n}}\Big(\int_{B_j^c}(f_j-1)^2\,\mathrm{d}G\Big)^{1/2}\Big)^2,$$

since, for $k\in\underline{n}$,

$$\Big\lfloor\frac{n-k}{k}\Big\rfloor + 1 \geqslant \frac{\max\{n-k,k\}}{k} \geqslant \frac{n}{2k}.$$

Similarly, we have

$$\widetilde{\nu}_k^2 = \Big\|\sum_{j=1}^n M_{j,k}\Big\|^2 \leqslant \Big(n\|(\overline{F}-G)|_{B_0}\| + n|(\overline{F}-G)(B_0)| + \sqrt{2kn}\Big(\int_{B_0^c}(f_0-1)^2\,\mathrm{d}G\Big)^{1/2}\Big)^2.$$

This yields (10) and completes the proof. □

For the proof of Theorem 2.2, we need the following lemma.

Lemma 4.5 *Let $n\in\mathbb{N}$, $L_1,\dots,L_n\in\mathcal{M}$ with $\sum_{j=1}^n L_j = 0$, and, for $k\in\underline{n}_0$ and $m\in\mathbb{N}$,*

$$\widetilde{V}_k = \sum_{J\subseteq \underline{n}:\,|J|=k}\,\prod_{j\in J} L_j, \qquad \vartheta_m = \sum_{j=1}^n \|L_j\|^m.$$

Then we have

$$\|\widetilde{V}_2\| \leqslant \frac{1}{2}\vartheta_2, \qquad \|\widetilde{V}_3\| \leqslant \frac{1}{3}\vartheta_3, \qquad \|\widetilde{V}_4\| \leqslant \frac{1}{8}\vartheta_2^2,$$
$$\|\widetilde{V}_5\| \leqslant \frac{1}{6}\vartheta_2\vartheta_3, \qquad \|\widetilde{V}_6\| \leqslant \frac{5}{144}\vartheta_2^3, \qquad \|\widetilde{V}_7\| \leqslant \frac{1}{24}\vartheta_2^2\vartheta_3.$$

Proof. The first two inequalities are easy. Taking into account (11)–(14), it is not difficult to show that, letting $\widetilde{\Gamma}_m=\sum_{j=1}^n(-L_j)^m$, $(m\in\mathbb{N})$,

$$\begin{aligned}
\big\|\widetilde{V}_4\big\| &= \frac{1}{8}\big\|[\widetilde{\Gamma}_2^2-\widetilde{\Gamma}_4]-\widetilde{\Gamma}_4\big\|\leqslant\frac{1}{8}\big(\big\|\widetilde{\Gamma}_2^2-\widetilde{\Gamma}_4\big\|+\big\|\widetilde{\Gamma}_4\big\|\big)\leqslant\frac{1}{8}\,\vartheta_2^2,\\
\big\|\widetilde{V}_5\big\| &= \frac{1}{6}\Big\|[\widetilde{\Gamma}_2\widetilde{\Gamma}_3-\widetilde{\Gamma}_5]-\frac{1}{5}\widetilde{\Gamma}_5\Big\|\leqslant\frac{1}{6}\vartheta_2\vartheta_3,\\
\big\|\widetilde{V}_6\big\| &= \frac{1}{144}\big\|3[\widetilde{\Gamma}_2^3-3\widetilde{\Gamma}_2\widetilde{\Gamma}_4+2\widetilde{\Gamma}_6]-9[\widetilde{\Gamma}_2\widetilde{\Gamma}_4-\widetilde{\Gamma}_6]-8[\widetilde{\Gamma}_3^2-\widetilde{\Gamma}_6]+\widetilde{\Gamma}_6\big\|\\
&\leqslant \frac{1}{144}\big(3[\vartheta_2^3-3\vartheta_2\vartheta_4+2\vartheta_6]+9[\vartheta_2\vartheta_4-\vartheta_6]+8[\vartheta_3^2-\vartheta_6]+\vartheta_6\big)\\
&= \frac{1}{144}\big(3\vartheta_2^3+8[\vartheta_3^2-\vartheta_6]-2\vartheta_6\big)\leqslant\frac{5}{144}\,\vartheta_2^3,\\
\big\|\widetilde{V}_7\big\| &= \frac{1}{840}\big\|35[\widetilde{\Gamma}_2^2\widetilde{\Gamma}_3-2\widetilde{\Gamma}_2\widetilde{\Gamma}_5-\widetilde{\Gamma}_3\widetilde{\Gamma}_4+2\widetilde{\Gamma}_7]-14[\widetilde{\Gamma}_2\widetilde{\Gamma}_5-\widetilde{\Gamma}_7]-35[\widetilde{\Gamma}_3\widetilde{\Gamma}_4-\widetilde{\Gamma}_7]+\widetilde{\Gamma}_7\big\|\\
&\leqslant \frac{1}{840}\big(35[\vartheta_2^2\vartheta_3-2\vartheta_2\vartheta_5-\vartheta_3\vartheta_4+2\vartheta_7]+14[\vartheta_2\vartheta_5-\vartheta_7]+35[\vartheta_3\vartheta_4-\vartheta_7]+\vartheta_7\big)\\
&= \frac{1}{840}\big(28[\vartheta_2^2\vartheta_3-2\vartheta_2\vartheta_5+\vartheta_7]+7\vartheta_2^2\vartheta_3-6\vartheta_7\big)\leqslant\frac{1}{24}\vartheta_2^2\vartheta_3.
\end{aligned}$$

Observe that, in order to obtain good constants, a convenient grouping of terms is essential. Further, for the bound of $\big\|\widetilde{V}_6\big\|$, we used the inequality $(\vartheta_3^2-\vartheta_6)/\vartheta_2^3\leqslant 4^{-1}$, which can be proved by using

$$\frac{\vartheta_3^2-\vartheta_6}{\vartheta_2^3}=\Big(\sum_{j=1}^n x_j^{3/2}\Big)^2-\sum_{j=1}^n x_j^3\leqslant\sum_{j=1}^n x_j^2(1-x_j)=:g_n((x_1,\dots,x_n)),$$

where $x_j=\big\|L_j\big\|^2(\sum_{i=1}^n\big\|L_i\big\|^2)^{-1}$, and the fact that the functions $g_n((x_1,\dots,x_n))$ for $n\in\{3,4,\dots\}$ and $(x_1,\dots,x_n)\in[0,\,1]^n$ with $\sum_{j=1}^n x_j=1$ satisfy

$$g_n((x_1,\dots,x_n))\leqslant g_{n-1}((x_1+x_2,x_3,\dots,x_n)),$$

whenever $0\leqslant x_1\leqslant\ldots\leqslant x_n\leqslant 1$. This completes the proof of the lemma. □

Proof of Theorem 2.2. In order to prove the assertions, we need a further bound. In fact, similarly as in the proof of Lemma 4.3, we get that, for $\ell\in\underline{n}_0$,

$$\begin{aligned}
\Big\|\prod_{j=1}^n F_j-W_\ell\Big\| &\leqslant 1+\big\|W_\ell\big\|\leqslant 2+\Big\|\sum_{k=1}^{\ell}\mathrm{Coeff}\Big(z^k,\prod_{j=1}^n\big(I+M_{j,k}z\big)\Big)G^{\lambda(n,k)}\Big\|\\
&\leqslant 2+\sum_{k=1}^{\ell}t^k=\frac{2-t-t^{\ell+1}}{1-t},
\end{aligned}$$

where $t=\sqrt{2\mathrm{e}\,c_1\,\eta_0}$. Similarly, if $G=\overline{F}$, then, for $\ell\in\underline{n}$,

$$\Big\|\prod_{j=1}^n F_j-W_\ell\Big\|\leqslant 2+\sum_{k=2}^{\ell}\widetilde{t}^k=\frac{2-2\widetilde{t}+\widetilde{t}^2-\widetilde{t}^{\ell+1}}{1-\widetilde{t}},$$

where $\widetilde{t}=\sqrt{2\mathrm{e}\,c_1\,\eta_1}$. We now prove (a). Let $\ell\in\underline{n}_0$. If $t\in[0,x_\ell]$, then (9) gives

$$\Big\|\prod_{j=1}^n F_j-W_\ell\Big\|\leqslant\frac{t^{\ell+1}}{1-t}\leqslant\frac{t^{\ell+1}}{1-x_\ell}.$$

On the other hand, if $t \in (x_\ell, \infty)$, then

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant \frac{2-t-t^{\ell+1}}{1-t} = \frac{t^{\ell+1}}{1-x_\ell}\Big(\frac{2-t-t^{\ell+1}}{t^{\ell+1}(1-t)}\Big)\Big(\frac{2-x_\ell-x_\ell^{\ell+1}}{x_\ell^{\ell+1}(1-x_\ell)}\Big)^{-1} \leqslant \frac{t^{\ell+1}}{1-x_\ell},$$

since $\frac{2-t-t^{\ell+1}}{t^{\ell+1}(1-t)} = t^{-\ell-1} + \sum_{j=1}^{\ell+1} t^{-j}$ is decreasing on $(0,\infty)$. This yields (19) and (20). The proof of (a) is easily completed. Let us now show (b). Set $G = \overline{F}$. Similarly to the above, one can show that, for $\ell \in \underline{n}$,

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant \frac{\widetilde{t}^{\ell+1}}{1-\widetilde{x}_\ell}.$$

This proves one part of (22). Using the norm inequalities in Lemma 4.5 and (9), we derive, for $\ell \in \underline{3}$,

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant \Big\|\prod_{j=1}^n F_j - W_7\Big\| + \Big\|W_7 - W_\ell\Big\| \leqslant \Big\|\prod_{j=1}^n F_j - W_7\Big\| + \sum_{k=\ell+1}^{7} \|V_k \overline{F}^{n-k}\| \leqslant \zeta_\ell(\eta_1),$$

where, for $x \in [0, (2\mathrm{e}\, c_1)^{-1})$,

$$\begin{aligned}
\zeta_1(x) &= x + \sqrt{3}\, x^{3/2} + 2\,x^2 + \frac{5^{5/2}}{6}\, x^{5/2} + \frac{15}{2}\, x^3 + \frac{7^{7/2}}{24} x^{7/2} + \frac{(2\mathrm{e}\, c_1\, x)^4}{1-\sqrt{2\mathrm{e}\, c_1\, x}},\\
\zeta_2(x) &= \sqrt{3}\, x^{3/2} + 2\,x^2 + \frac{5^{5/2}}{6}\, x^{5/2} + \frac{15}{2}\, x^3 + \frac{7^{7/2}}{24} x^{7/2} + \frac{(2\mathrm{e}\, c_1\, x)^4}{1-\sqrt{2\mathrm{e}\, c_1\, x}},\\
\zeta_3(x) &= 2\,x^2 + \frac{5^{5/2}}{6}\, x^{5/2} + \frac{15}{2}\, x^3 + \frac{7^{7/2}}{24} x^{7/2} + \frac{(2\mathrm{e}\, c_1\, x)^4}{1-\sqrt{2\mathrm{e}\, c_1\, x}}.
\end{aligned}$$

Note that, for $\ell \in \underline{3}$, we have $\zeta_\ell(\eta_1) \leqslant \frac{2-2\widetilde{t}+\widetilde{t}^2-\widetilde{t}^{\ell+1}}{1-\widetilde{t}}$, if and only if $\eta_1 \in [0, s_\ell]$, where $s_1 = 0.182839\ldots$, $s_2 = 0.196439\ldots$, and $s_3 = 0.205094\ldots$. If $\eta_1 \in [0, s_\ell]$, then $\|\prod_{j=1}^n F_j - W_\ell\| \leqslant \zeta_\ell(\eta_1) \leqslant \zeta_\ell(s_\ell)\eta_1^{(\ell+1)/2}/s_\ell^{(\ell+1)/2}$. If $\eta_1 \in (s_\ell, \infty)$, then, letting $\widetilde{t}_\ell = \sqrt{2\mathrm{e}\, c_1 s_\ell}$,

$$\Big\|\prod_{j=1}^n F_j - W_\ell\Big\| \leqslant \widetilde{t}^{\ell+1}\frac{2-2\widetilde{t}+\widetilde{t}^2-\widetilde{t}^{\ell+1}}{\widetilde{t}^{\ell+1}(1-\widetilde{t})} \leqslant \widetilde{t}^{\ell+1}\frac{2-2\widetilde{t}_\ell+\widetilde{t}_\ell^2-\widetilde{t}_\ell^{\ell+1}}{\widetilde{t}_\ell^{\ell+1}(1-\widetilde{t}_\ell)} = \eta_1^{(\ell+1)/2}\frac{\zeta_\ell(s_\ell)}{s_\ell^{(\ell+1)/2}}.$$

Numerical calculations give the bounds for $\widetilde{u}_\ell$, ($\ell \in \underline{3}$) as claimed in (22). This completes the proof. □

Proof of Proposition 2.1. Consider fixed $j, k \in \underline{n}$. Let $d = 2b$, $\overline{p} = (\overline{p}_1, \ldots, \overline{p}_b, \overline{p}_1, \ldots, \overline{p}_b) \in \mathbb{Z}_+^d$, and $\rho = \lfloor (n-k)/k \rfloor$. Further, for $v \in \mathbb{Z}_+^d$ with $|v| = 2$, let $a_v = \overline{p}_r - p_{j,r}$ if $v_r = v_{b+r} = 1$ and $a_v = 0$ otherwise. Let

$$H_0 = I, \qquad H_r = \begin{cases} I_{-x_r}, & r \in \underline{b},\\ I_{x_{r-b}}, & r \in \underline{d} \setminus \underline{b}. \end{cases}$$

Then we have $\overline{F}=\sum_{r=0}^d \overline{p}_r H_r$ and

$$\begin{aligned} F_j-\overline{F} &= (p_{j,0}-\overline{p}_0)I+\sum_{r=1}^b (p_{j,r}-\overline{p}_r)(I_{-x_r}+I_{x_r}) \\ &= \sum_{r=1}^b (p_{j,r}-\overline{p}_r)(I_{-x_r}+I_{x_r}-2I)=\sum_{r=1}^b(\overline{p}_r-p_{j,r})(I_{-x_r}-I)(I_{x_r}-I) \\ &= \sum_{|v|=2}\frac{a_v}{v!}\prod_{r=1}^d (H_r-H_0)^{v_r}. \end{aligned}$$

Here and henceforth, sums over v and w are taken over subsets of $\mathbb{Z}_+^d$ as indicated. In particular, we obtain

$$\|M_{j,k}\|^2=\|(F_j-\overline{F})\overline{F}^\rho\|^2\leqslant\|F_j-\overline{F}\|^2\leqslant\Big(|p_{j,0}-\overline{p}_0|+2\sum_{r=1}^b|p_{j,r}-\overline{p}_r|\Big)^2.$$

On the other hand, in view of

$$\|M_{j,k}\|^2=\|(F_j-\overline{F})\overline{F}^\rho\|^2=\Big\|\Big(\sum_{|v|=2}\frac{a_v}{v!}\prod_{r=1}^d(H_r-H_0)^{v_r}\Big)\Big(\sum_{r=0}^d\overline{p}_rH_r\Big)^\rho\Big\|^2,$$

we see that (29) can be applied, which together with the simple fact that $\frac{\rho!}{(\rho+2)!}\leqslant\frac{1}{(\rho+1)^2}\leqslant\frac{4k^2}{n^2}$ gives

$$\begin{aligned} \|M_{j,k}\|^2 &\leqslant \frac{\rho!}{(\rho+2)!}\sum_{|w|\leqslant 2}\frac{w!(2-|w|)!}{\overline{p}^w\,\overline{p}_0^{2-|w|}}\Big[\sum_{|v|=2}\frac{a_v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]^2 \\ &\leqslant \frac{4k^2}{n^2}\Big(\frac{2}{\overline{p}_0^2}\Big[\sum_{|v|=2}\frac{a_v}{v!}\Big]^2+\sum_{|w|=1}\frac{1}{\overline{p}^w\,\overline{p}_0}\Big[\sum_{|v|=2}\frac{a_v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]^2 \\ &\qquad\qquad+\sum_{|w|=2}\frac{w!}{\overline{p}^w}\Big[\sum_{|v|=2}\frac{a_v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]^2\Big). \end{aligned}$$

The special definition of a_v, ($v\in\mathbb{Z}_+^d, |v|=2$) implies that $a_{e_{r(1)}+e_{r(2)}}=0$ for $r(1),r(2)\in\underline{d}$ with $|r(1)-r(2)|\neq b$ and therefore the terms on the right-hand side can be evaluated as follows:

$$\begin{aligned} \frac{2}{\overline{p}_0^2}\Big[\sum_{|v|=2}\frac{a_v}{v!}\Big]^2 &= \frac{2}{\overline{p}_0^2}\Big[\sum_{r=1}^b(\overline{p}_r-p_{j,r})\Big]^2=\frac{(p_{j,0}-\overline{p}_0)^2}{2\overline{p}_0^2}, \\ \sum_{|w|=1}\frac{1}{\overline{p}^w\,\overline{p}_0}\Big[\sum_{|v|=2}\frac{a_v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]^2 &= \sum_{r=1}^d\frac{1}{\overline{p}_r\,\overline{p}_0}\Big[\sum_{|v|=2}\frac{a_v}{v!}\binom{v_r}{1}\Big]^2=2\sum_{r=1}^b\frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r\,\overline{p}_0}, \\ \sum_{|w|=2}\frac{w!}{\overline{p}^w}\Big[\sum_{|v|=2}\frac{a_v}{v!}\prod_{r=1}^d\binom{v_r}{w_r}\Big]^2 &= \sum_{|w|=2}\frac{w!}{\overline{p}^w}\Big[\frac{a_w}{w!}\Big]^2=\sum_{r=1}^b\frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r^2}. \end{aligned}$$

We note that some of the binomial coefficients above are equal to zero. This implies that

$$\|M_{j,k}\|^2 \leqslant \frac{4k^2}{n^2}\Big(\frac{(\overline{p}_0-p_{j,0})^2}{2\overline{p}_0^2}+2\sum_{r=1}^b\frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r\overline{p}_0}+\sum_{r=1}^b\frac{(\overline{p}_r-p_{j,r})^2}{\overline{p}_r^2}\Big).$$

Using this together with

$$\eta_{\ell,1} = \max_{k \in \underline{n} \setminus \underline{\ell}} \frac{\nu_{k,2}}{k^2} = \max_{k \in \underline{n} \setminus \underline{\ell}} \Big(\frac{1}{k^2} \sum_{j=1}^{n} \|M_{j,k}\|^2 \Big),$$

(see the comment after Theorem 2.1) the proof is easily completed. □

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