

Reflected BSDE with a Constraint and its applications in incomplete market

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Abstract. In this paper, we study a type of reflected BSDE with a constraint, and prove the existence of the smallest g -supersolution for this equation. Then we give its application on the pricing of American options in incomplete market.

Keywords: Reflected backward stochastic differential equation, backward stochastic differential equation with a constraint, American option in incomplete market.

1 Introduction

A backward stochastic differential equation (BSDE) driven by a d -dimensional Brownian motion $(B_t)_{t \geq 0}$ defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is of the following form

$$dy_t + g(t, y_t, z_t)dt - z_t dB_t = 0, \quad t \in [0, T],$$

where g is a given function called the generator of the BSDE. Here all processes are assumed to be square-integrable and progressively measurable with respect to the $(B_t)_{t \geq 0}$ -filtration. For a given terminal condition $y_T = \xi$, a solution $(y_t, z_t)_{t \in [0, T]}$ is a pair of processes satisfying the above relation. We often call it a g -solution to specify the generator g . In the case where the generator g is a Lipschitz function of (y, z) the existence and uniqueness of such BSDE was given by [18]. In this paper we consider 1-dimensional BSDE, i.e., g and y are assumed

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to be real valued. We are interested in a new type of BSDEs with the following type of singular generator:

$$g_{\Gamma}(t, \omega, y, z) = \begin{cases} g(t, y, z), & (y, z) \in \Gamma(t, \omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where, for each (t, ω) , $\Gamma(t, \omega)$ is a given closed subset of $\mathbb{R} \times \mathbb{R}^d$. This type of g_{Γ} -solution $(y_t)_{t \in [0, T]}$ is formulated as the smallest g -supersolution constrained in Γ with a given terminal condition ξ . Such type of BSDEs and its application to the problem of option pricing with constrained portfolios was studied in Cvitanic & Karatzas [1], [2] and Cvitanic, Karatzas & Soner [4] for a convex constraint and by Peng [20], Peng & Yang [24], for more general situations. The framework of the present paper is based on [20].

In this paper we mainly study g_{Γ} -reflected BSDE, i.e., a BSDE reflected by a lower obstacle $(L_t)_{t \in [0, T]}$ or an upper obstacle $(U_t)_{t \in [0, T]}$ with the above singular generator g_{Γ} . Our results have non-trivially generalized the original paper of El Karoui et al [6] as well as Hamadene [8], Hamadene & Lepeltier [13], Lepeltier & Xu [17] in which the generators are assumed to be a Lipschitz function. Since the obstacles L and U can be a very general L^2 -processes, our results is also a generalization of Peng & Xu [22].

Recently the study of reflected BSDE is very active since it can be applied to optimal stopping, optimal switching, American option pricing and the related dynamic risk measures, stochastic differential controls and games with mixed strategies (e.g. Dynkin games). We refer to [6] as well as Cvitanic & Karatzas [3], [8], [13], Karatzas & Kou [14], Lepeltier & San Martín [16], [17], Peng & Xu [23] as well as [22] for various situations of reflected BSDE and its applications with non-singular generators. In this paper we also discuss how to apply our results of g_{Γ} -reflected BSDE to American call and put options in an incomplete market with portfolio constraints.

Observe that a g_{Γ} -solution of a BSDE reflected by a lower obstacle $(L_t)_{t \in [0, T]}$ can be also considered as a BSDE with constraint $\Gamma_t \times \{y \in \mathbb{R} : y \geq L_t\}$. But it is theoretically and practically important to separate the reflecting process \bar{A} from the total increasing process $A + \bar{A}$, since the related (generalized) Skorokhod reflecting condition plays an important role (see Proposition 3.1). This type of separations is an important feature of our results.

This paper is organized as follows. In the next section we list the main notations and conditions used throughout this paper. In section 3, we present results and proofs of the existence and uniqueness of reflected BSDE with the singular generator g_{Γ} . Then we discuss some applications of our main results to the problem of pricing of American options in a market with portfolio constraints in section 4. Some of results needed in the proofs of this paper are will be given in Appendix.

2 g_{Γ} -solution: the smallest g -supersolution of BSDE with constraint Γ

Let (Ω, \mathcal{F}, P) be a probability space, and $B = (B^1, B^2, \dots, B^d)^T$ be a d -dimensional Brownian motion defined on $[0, \infty)$. The natural filtration generated by this Brownian motion is denoted by

$$\mathcal{F}_t = \sigma\{\{B_s; 0 \leq s \leq t\} \cup \mathcal{N}\},$$

where \mathcal{N} is the collection of all P -null sets of \mathcal{F} . The Euclidean norm of an element $x \in \mathbb{R}^m$ is denoted by $|x|$. We also need the following notations, for $p \in [1, \infty)$:

- $\mathbf{L}^p(\mathcal{F}_t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued } \mathcal{F}_t\text{-measurable random variables } X \text{ s.t. } E[|X|^p] < \infty\}$;
- $\mathbf{L}_{\mathcal{F}}^p(0, t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued and } \mathcal{F}_t\text{-progressively measurable processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E \int_0^t |\varphi_s|^p ds < \infty\}$;
- $\mathbf{D}_{\mathcal{F}}^p(0, t; \mathbb{R}^m) := \{\mathbb{R}^m\text{-valued and RCLL } \mathcal{F}_t\text{-progressively measurable processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E[\sup_{0 \leq s \leq t} |\varphi_s|^p] < \infty\}$;
- $\mathbf{A}_{\mathcal{F}}^p(0, t) := \{\text{increasing processes } A \text{ in } \mathbf{D}_{\mathcal{F}}^p(0, t; \mathbb{R}) \text{ with } A(0) = 0\}$.

When $m = 1$ we simply use $\mathbf{L}^p(\mathcal{F}_t)$, $\mathbf{L}_{\mathcal{F}}^p(0, t)$ and $\mathbf{D}_{\mathcal{F}}^p(0, t)$. In this section, we consider BSDE on the interval $[0, T]$, with a fixed $T > 0$.

We consider a function

$$g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

which always plays the role of the generator of our BSDE. It satisfies the following classical assumptions: there exists a constant $\mu > 0$, such that, for each y, y' in \mathbb{R} and z, z' in \mathbb{R}^d , we have

- (i) $g(\cdot, y, z) \in \mathbf{L}_{\mathcal{F}}^2(0, T)$;
 - (ii) $|g(t, \omega, y, z) - g(t, \omega, y', z')| \leq \mu(|y - y'| + |z - z'|)$, $dP \times dt$ a.s.
- (1)

The constraint Γ of our BSDE is a mapping $\Gamma(t, \omega) : \Omega \times [0, T] \rightarrow \mathcal{C}(\mathbb{R} \times \mathbb{R}^d)$, where $\mathcal{C}(\mathbb{R} \times \mathbb{R}^d)$ is the collection of all closed subsets of $\mathbb{R} \times \mathbb{R}^d$. Γ is assumed to be \mathcal{F}_t -adapted, namely,

- (i) $(y, z) \in \Gamma(t, \omega)$ iff $d_{\Gamma(t, \omega)}(y, z) = 0$, $t \in [0, T]$, a.s.;
 - (ii) $d_{\Gamma}(y, z)$ is an \mathcal{F}_t -adapted process, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,
- (2)

where $d_{\Gamma_t}(\cdot, \cdot)$ is a distance function from (y, z) to Γ : for $t \in [0, T]$,

$$d_{\Gamma_t}(y, z) := \inf_{(y', z') \in \Gamma_t} (|y - y'|^2 + |z - z'|^2)^{1/2} \wedge 1.$$

$d_{\Gamma_t}(y, z)$ is a Lipschitz function: for each y, y' in \mathbb{R} and z, z' in \mathbb{R}^d , we always have

$$|d_{\Gamma_t}(y, z) - d_{\Gamma_t}(y', z')| \leq (|y - y'|^2 + |z - z'|^2)^{1/2}.$$

Remark 2.1. *The above type of constraint Γ was firstly considered in Peng [20]. In fact Peng's constraint is formulated as*

$$\Gamma_t(\omega) = \{(y, z) \in \mathbb{R}^{1+d} : \Phi(\omega, t, y, z) = 0\}, \tag{3}$$

where $\Phi(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, \infty)$ is a given nonnegative function satisfying similar conditions as (1). In this paper we always consider the case

$$\Phi(t, y, z) = d_{\Gamma_t}(y, z).$$

In fact these two definitions are equivalent. In [4], the constraint is assumed to be convex.

We are then within the framework of supersolution and subsolution of BSDE of the following type:

Definition 2.1. (*g-supersolution, g-subsolution*, cf. El Karoui, Peng and Quenez (1997) [7] and Peng (1999) [20]) A process $y \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ is called a *g-supersolution* (resp. *g-subsolution*) if there exist a process $z \in \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$ and an increasing RCLL process $A \in \mathbf{A}_{\mathcal{F}}^2(0, T)$ (resp. $K \in \mathbf{A}_{\mathcal{F}}^2(0, T)$), such that $t \in [0, T]$,

$$\begin{aligned} y_t &= y_T + \int_t^T g(s, y_s, z_s) ds + A_T - A_t - \int_t^T z_s dB_s, \\ (\text{resp. } y_t &= y_T + \int_t^T g(s, y_s, z_s) ds - (K_T - K_t) - \int_t^T z_s dB_s.) \end{aligned} \quad (4)$$

Here z and A (resp. K) are called the martingale representing part and increasing part of y , respectively. y is called a *g-solution* if $A_t = K_t = 0$, $t \in [0, T]$. y is called a Γ -constrained *g-supersolution* if y and its corresponding martingale representing part z satisfy

$$(y_t, z_t) \in \Gamma_t, \quad (\text{or } d_{\Gamma_t}(y_t, z_t) = 0), \quad dP \times dt \text{ a.s. in } \Omega \times [0, T], \quad (5)$$

Remark 2.2. We observe that, if $y \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ is a *g-supersolution* or *g-subsolution*, then the pair (z, A) in (4) are uniquely determined since the martingale representing part z is uniquely determined. Occasionally, we also call the triple (y, z, A) a *g-supersolution* or *g-subsolution*.

A Γ -constraint *g-supersolution* can also be regarded as a supersolution of the BSDE with a singular generator g_{Γ} defined by

$$g_{\Gamma}(t, y, z) = g(t, y, z)1_{\Gamma_t}(y, z) + (+\infty) \cdot 1_{\Gamma_t^c}(y, z).$$

So we define the smallest Γ -constrained *g-supersolution* as g_{Γ} -solution.

Definition 2.2. (*g_{Γ} -solution*) A *g-supersolution* $(y_t, z_t, A_t)_{0 \leq t \leq T}$ is called g_{Γ} -supersolution on $[0, T]$ with a given terminal condition X if $d_{\Gamma_t}(y_t, z_t) = 0$, $dP \times dt$ almost surely. The smallest g_{Γ} -supersolution $(y_t, z_t, A_t)_{0 \leq t \leq T}$ with a given terminal condition $y_T = X$ is called g_{Γ} -solution. Here the “smallest” means that $y_t \geq y'_t$, $t \in [0, T]$, for any g_{Γ} -supersolution $(y'_t, z'_t, A'_t)_{0 \leq t \leq T}$ with $y'_T = X$.

Remark 2.3. The above definition is meaningful since, by [20] (see Appendix Theorem 5.2), if there exists at least one g_{Γ} -supersolution, then the smallest Γ -constrained *g-supersolution* also exists.

Remark 2.4. By the above definition if $(y_t, z_t, A_t)_{0 \leq t \leq T}$ is a g_{Γ} -solution on $[0, T]$ with terminal condition y_T , then for each $T_1 \leq T$, $(y_t, z_t, A_t)_{0 \leq t \leq T_1}$ is also a g_{Γ} -solution on $[0, T_1]$ with terminal condition y_{T_1} . The above definition does not imply that the increasing process A is also the smallest one. In fact the following example shows that there may exist a different g_{Γ} -supersolution $(\bar{y}, \bar{z}, \bar{A})$ on $[0, T]$ with the same terminal condition such that $A_t > \bar{A}_t$ for some t .

Example 2.1. Consider the case when $[0, T] = [0, 2]$, $X = 0$, $g = 0$ and $\Gamma_t = \{(y, z) : y \geq 1_{[0,1]}(t)\}$. So the g_{Γ} -solution of this equation is the solution of reflected BSDE with the lower obstacle $1_{[0,1]}(t)$. This g_{Γ} -solution smallest solution is expressed as $y_t = 1_{[0,1]}(t)$, $z_t = 0$, $A_t = 1_{[1,2]}(t)$ on $[0, T]$. One can also check that $\bar{y}_t = 1_{[0,2]}(t)$ with $\bar{z}_t = 0$, $\bar{A}_t = 1_{\{t=2\}}(t)$ is also a g_{Γ} -supersolution with the same terminal condition $\bar{y}_T = 0$. However we have $A_t > \bar{A}_t$ for $t \in [1, 2)$.

3 g_Γ -reflected BSDEs

3.1 Existence of g_Γ -reflected BSDEs: Definitions and results

In this section we consider the smallest g -supersolution with constraint Γ and a lower (resp. upper) reflecting obstacle L (resp. U). We assume that the two reflected obstacles L and U satisfy:

$$L, U \in \mathbf{L}_{\mathcal{F}}^2(0, T) \quad \text{and} \quad \text{ess sup}_{0 \leq t \leq T} L_t^+, \text{ess sup}_{0 \leq t \leq T} U_t^- \in \mathbf{L}^2(\mathcal{F}_T). \quad (6)$$

We only study the case of the constraint Γ not depending on y , only depending on z . In such case $\Gamma(t, \omega) = \mathbb{R} \times \Gamma_z(t, \omega)$. where $\Gamma_z(t, \omega)$ is a closed subset of \mathbb{R}^d . For more general situation see Remark xx and xx.

First let us introduce the definition of g_Γ -reflected solutions with a lower obstacle:

Definition 3.1. A g_Γ -reflected solution with a lower obstacle L is a quadruple of processes (y, z, A, \bar{A}) satisfying

(i) $(y, z, A, \bar{A}) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times (\mathbf{A}_{\mathcal{F}}^2(0, T))^2$ verifies

$$\begin{aligned} y_t &= X + \int_t^T g(s, y_s, z_s) ds + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T z_s dB_s, \\ d_{\Gamma_t}(z_t) &= 0, \quad dP \times dt \text{ a.s.} \end{aligned} \quad (7)$$

(ii) $y_t \geq L_t$, $dP \times dt$ -a.s., and the following generalized Skorokhod reflecting condition is satisfied: for each $L^* \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ such that $y_t \geq L_t^* \geq L_t$, $dP \times dt$ -a.s., we have

$$\int_0^T (y_{s-} - L_{s-}^*) d\bar{A}_s = 0, \quad \text{a.s.}, \quad (8)$$

(iii) y is the smallest one, i.e., for any quadruple $(y^*, z^*, A^*, \bar{A}^*)$ satisfying (i) and (ii), we have

$$y_t \leq y_t^*, \quad \forall t \in [0, T], \quad \text{a. s.}$$

In the above formulation we need to find two increasing processes A and \bar{A} in order to keep the solution in the constraints $y_t \geq L_t$ and $z_t \in \Gamma$. In fact these two increasing processes play different roles. A is used to keep the process z staying in the constraint Γ , while \bar{A} is the reflecting force to keep y to be above the obstacle L . Actually each of them has different meaning in finance.

Our first main result in this paper is:

Theorem 3.1. Suppose (1), (2) and (6) hold. For a given terminal condition $X \in \mathbf{L}^2(\mathcal{F}_T)$, we assume that there exists a triple $(y^*, z^*, A^*) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T) \times \mathbf{A}_{\mathcal{F}}^2(0, T)$, such that $dA^* \geq 0$ and the following holds

$$\begin{aligned} y_t^* &= X + \int_t^T g(s, y_s^*, z_s^*) ds + (A_T^* - A_t^*) - \int_t^T z_s^* dB_s, \\ (y_t^*, z_t^*) &\in [L_t, \infty) \times \Gamma_t, \quad dP \times dt\text{-a.s.} \end{aligned} \quad (9)$$

Then there exists the g_Γ -reflected solution (y, z, A, \bar{A}) with the barrier L of Definition 3.1.

Remark 3.1. This theorem can be generalized to the case when Γ also depends on y . In fact the basic idea of the proof of this theorem is based on a penalization method which still works for y -dependence situation. (cf. the proof of Theorem 3.1).

Under certain assumption, condition (9) can be verified easily.

Example 3.1. Assume that there exists C_0 a large enough constant, s.t. for $\forall y \geq C_0$,

$$g(t, y, 0) \leq C_0 + \mu |y|, \quad 0 \in \Gamma_t. \quad (10)$$

and there exists a deterministic process $a(t)$, s.t. $L_t \leq a(t)$ on $[0, T]$. For X is a r.v. in $\mathbf{L}_{+, \infty}^2(\mathcal{F}_T)$, i.e. $X \in \mathbf{L}^2(\mathcal{F}_t)$, $X^+ \in \mathbf{L}^\infty(\mathcal{F}_t)$, the triple of processes

$$(y_t^*, z_t^*, A_t^*) := (y_t^0, 0, \int_0^t [C_0 + \mu |y_s^0| - g(s, y_s^0, 0)] ds + A_t^0 + A_t^1)$$

is a solution of (9). Here (y_t^0, A_t^0) is the solution of ODE associated with coefficient $g(y) = C_0 + \mu |y|$, barrier $a(t)$ and terminal value $(\|X^+\|_\infty \vee C_0)e^{\mu(T-t)} + C_0(T-t)$, and $A_t^1 = (X - \|X^+\|_\infty \vee C_0)1_{\{t=T\}}$.

The smallest g_Γ -reflected solution with an upper obstacle U is relatively more complicated than the case of the lower obstacle.

Definition 3.2. A g_Γ -reflected solution with an upper obstacle U is a quadruple of processes

$$(y, z, A, K) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times (\mathbf{A}_{\mathcal{F}}^2(0, T))^2$$

satisfying

(i)

$$\begin{aligned} y_t &= X + \int_t^T g(s, y_s, z_s) ds + A_T - A_t - (K_T - K_t) - \int_t^T z_s dB_s, \\ d_{\Gamma_t}(z_t) &= 0, \quad dP \times dt\text{-a.s.} \quad \mathcal{V}_{[0, T]}[A - K] = \mathcal{V}_{[0, T]}[A + K], \end{aligned} \quad (11)$$

where $\mathcal{V}_{[0, T]}(\varphi)$ denotes the total variation of a process φ on $[0, T]$.

(ii) $y_t \leq U_t$, $dP \times dt\text{-a.s.}$, and the generalized Skorokhod reflecting condition is satisfied:

$$\int_0^T (U_{t-}^* - y_{t-}) dK_t = 0, \quad \text{a.s., for any } U^* \in \mathbf{D}_{\mathcal{F}}^2(0, T), \text{ s.t. } y_t \leq U_t^* \leq U_t, \text{ } dP \times dt\text{-a.s..}$$

(iii) For any other quadruple (y^*, z^*, A^*, K^*) satisfying (i) and (ii), we have

$$y_t \leq y_t^*, \quad 0 \leq t \leq T, \quad \text{a.s.}$$

Remark 3.2. The relation $\mathcal{V}_{[0, T]}[A - K] = \mathcal{V}_{[0, T]}[A + K]$ in (11) implies that A and K never increase at same time. This relation can help us to characterize the solution. Indeed it is easy to check that the quadruple $(y, z, A + K, 2K)$ satisfies all relations of the above relations except this one.

We also have the existence of a g_Γ -reflected solution with an upper obstacle U :

Theorem 3.2. *Assume that (1) and (2) hold for g and the constraint Γ respectively, U is a \mathcal{F}_t -adapted RCLL process satisfying (6). Then for each given terminal condition $X \in \mathbf{L}^2(\mathcal{F}_T)$, there exists a g_Γ -reflected solution (y, z, A, K) with upper obstacle U of Definition 3.2.*

Remark 3.3. *For the case when Γ depends on y , satisfying (2), Theorem 3.2 still holds under the following additional assumption: there exists a quadruple*

$$(y^*, z^*, A^*, K^*) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times (\mathbf{A}_{\mathcal{F}}^2(0, T))^2$$

such that

$$y_t^* = X + \int_t^T g(s, y_s^*, z_s^*) ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T z_s^* dB_s, \quad (12)$$

$$d_{\Gamma_t}(y_t^*, z_t^*) = 0, \quad y_t^* \leq U_t, \quad \text{a.s. a.e.}$$

$$\int_0^T (y_{t-}^* - U_{t-}^*) dK_t^* = 0 \quad \text{a.s., for any } U^* \in \mathbf{D}_{\mathcal{F}}^2(0, T), \text{ s.t. } y_t \leq U_t^* \leq U_t, \text{ } dP \times dt \text{ -a.s..}$$

In general this assumption is not easy to verify. One typical example is $\Gamma_t = [L_t, +\infty) \times \mathbb{R}^d$. Indeed the problem turns out to be a reflected BSDE with two barriers L and U . By [22] we know that if there exists a semimartingale X , such that $L \leq X \leq U$, $dP \times dt$ -a.s., then condition (12) can be satisfied.

3.2 Existence of g_Γ -reflected BSDE with a lower barrier: Proof of Theorem 3.1

The main idea of the proof is a penalization method. We prove theorem 3.1 by an approximation procedure. For given $m, n \in \mathbb{N}$, we consider the penalization equations,

$$y_t^{m,n} = X + \int_t^T g(s, y_s^{m,n}, z_s^{m,n}) ds + m \int_t^T d_{\Gamma_s}(z_s^{m,n}) ds \quad (13)$$

$$+ n \int_t^T (L_s - y_s^{m,n})^+ ds - \int_t^T z_s^{m,n} dB_s.$$

It is a classical BSDE with a generator

$$g^{m,n}(t, y, z) := g(t, y, z) + m d_{\Gamma_t}(z) + n(L_t - y)^-$$

which is a Lipschitz function. From [PP1990] this equation admits a unique solution $(y^{m,n}, z^{m,n})$. We denote $A_t^{m,n} := m \int_0^t d_{\Gamma_s}(y_s^{m,n}, z_s^{m,n}) ds$ and $\bar{A}_t^{m,n} := n \int_0^t (L_s - y_s^{m,n})^+ ds$. We have the following estimate.

Lemma 3.1. *Under the same assumptions as in Theorem 3.1, there exists a constant $C \in \mathbb{R}$ independent of m and n , such that*

$$E[\sup_{0 \leq t \leq T} (y_t^{m,n})^2] + E \int_0^T |z_s^{m,n}|^2 ds + E[(A_T^{m,n} + \bar{A}_T^{m,n})^2] \leq C. \quad (14)$$

Proof. Set $m = n = 0$, then we get a classical BSDE

$$y_t^{0,0} = X + \int_t^T g(s, y_s^{0,0}, z_s^{0,0}) ds - \int_t^T z_s^{0,0} dB_s.$$

For (y^*, z^*, A^*) given in (9), we have $d_{\Gamma_s}(z_s^*) \equiv 0$ and $(L_s - y_s^*)^+ \equiv 0$, thus

$$\begin{aligned} y_t^* &= X + \int_t^T g(s, y_s^*, z_s^*) ds + m \int_t^T d_{\Gamma_s}(z_s^*) ds + n \int_t^T (L_s - y_s^*)^+ ds \\ &\quad + (A_T^* - A_t^*) - \int_t^T z_s^* dB_s, \end{aligned}$$

By comparison theorem, it follows that $y_t^* \geq y_t^{m,n} \geq y_t^{0,0}$ on $[0, T]$. Thus $y^{m,n}$ satisfies the estimate:

$$E[\sup_{0 \leq t \leq T} (y_t^{m,n})^2] \leq C_1 = \max\{E[\sup_{0 \leq t \leq T} (y_t^*)^2], E[\sup_{0 \leq t \leq T} (y_t^{0,0})^2]\}. \quad (15)$$

The rest of the proof can be obtained by applying the following lemma.

We have

Lemma 3.2. *Let $(y^\alpha, z^\alpha, A^\alpha)_{\alpha \in \mathcal{A}}$ be a family g -supersolution of the form*

$$y_t^\alpha = y_T^\alpha + \int_t^T g(s, y_s^\alpha, z_s^\alpha) ds + (A_T^\alpha - A_t^\alpha) - \int_t^T z_s^\alpha dB_s \quad (16)$$

such that, for each α , y_t^α is continuous and such that $E[\sup_{0 \leq t \leq T} (y_t^\alpha)^2] \leq C_1$, where the constant C_1 is independent of α , then there exists a constant C , independent of α , such that

$$E \int_0^T |z_s^\alpha|^2 ds + E[(A_T^\alpha)^2] \leq C.$$

Proof. The method is borrow from [20]. By applying Itô formula to $|y_t^\alpha|^2$ on $[0, T]$ and taking expectation, with Lipschitz property of g , we get

$$\begin{aligned} & E[|y_t^\alpha|^2] + E\left[\int_t^T |z_s^\alpha|^2 ds\right] \\ & \leq E[(y_T^\alpha)^2] + E \int_t^T g^2(s, 0, 0) ds + (2\mu + \mu^2) \int_t^T |y_s^\alpha|^2 ds + \frac{1}{2} E\left[\int_t^T |z_s^\alpha|^2 ds\right] \\ & \quad + \frac{1}{\beta} E\left[\sup_{0 \leq t \leq T} (y_t^\alpha)^2\right] + \beta E[(A_T^\alpha - A_t^\alpha)^2], \end{aligned}$$

in view of $2ab \leq \frac{1}{\beta} a^2 + \beta b^2$, where β is a real number to be decided later. From integrability assumptions of $g(\cdot, 0, 0)$ and X , we get

$$E \int_0^T |z_s^\alpha|^2 ds \leq C_2 + 2\beta E[(A_T^\alpha)^2]. \quad (17)$$

Then we reformulate (16) as

$$A_T^\alpha = y_0^\alpha - y_T^\alpha - \int_0^T g(s, y_s^\alpha, z_s^\alpha) ds + \int_0^T z_s^\alpha dB_s,$$

and take square and expectation on both sides. With the Lipschitz condition of g we get

$$\begin{aligned} E[(A_T^\alpha)^2] &\leq 4E[(y_0^\alpha)^2] + 4E[(y_T^\alpha)^2] + 16TE \int_0^T g^2(s, 0, 0) ds \\ &\quad + 16\mu^2TE \int_0^T |y_s^\alpha|^2 ds + (16\mu^2T + 4)E \int_0^T |z_s^\alpha|^2 ds. \end{aligned}$$

It follows from (15), $X \in \mathbf{L}^2(\mathcal{F}_T)$ and $g(\cdot, 0, 0) \in \mathbf{L}^2_{\mathcal{F}}(0, T)$ that

$$E[(A_T^\alpha)^2] \leq C + (16\mu^2T + 4)E \int_0^T |z_s^\alpha|^2 ds. \quad (18)$$

Set $\beta = \frac{1}{32\mu^2T+8}$ in (17) and substitute (18) into it, we deduce that $E \int_0^T |z_s^\alpha|^2 ds \leq C$. Then in view of (18) $E[(A_T^\alpha)^2] \leq C$, the proof is complete. \square

We now give the proof of Theorem 3.1.

Proof of Theorem 3.1. In (13), we fix $m \in \mathbb{N}$, and set

$$g^m(t, y, z) := g(t, y, z) + md_{\Gamma_t}(z).$$

Since g^m is a Lipschitz function and the condition (14) is satisfied, it follows from theorem 4.1 in [22] that, as $n \rightarrow \infty$, the triple $(y^{m,n}, z^{m,n}, \bar{A}^{m,n})$ converges to $(y^m, z^m, \bar{A}^m) \in \mathbf{D}^2_{\mathcal{F}}(0, T) \times \mathbf{L}^2_{\mathcal{F}}(0, T) \times \mathbf{A}^2_{\mathcal{F}}(0, T)$, which is the solution of the following reflected BSDE whose coefficient is $g^m = g + md_{\Gamma}$,

$$\begin{aligned} y_t^m &= X + \int_t^T g^m(s, y_s^m, z_s^m) ds + \bar{A}_T^m - \bar{A}_t^m - \int_t^T z_s^m dB_s, \\ y_t^m &\geq L_t, \quad \text{a.s.a.e.}, \quad \int_0^T (y_{t-}^m - L_{t-}^*) d\bar{A}_t^m = 0, \\ &\text{for each } L^* \in \mathbf{D}^2_{\mathcal{F}}(0, T), \text{ such that } y^m \geq L^* \geq L, dP \times dt \text{ a.s.} \end{aligned} \quad (19)$$

We denote $A_t^m = m \int_0^t d_{\Gamma_s}(z_s^m) ds$. By (14) we have the following estimate:

$$E[\sup_{0 \leq t \leq T} (y_t^m)^2] + E \int_0^T |z_s^m|^2 ds + E[(A_T^m + \bar{A}_T^m)^2] \leq C.$$

Then by comparison theorem 5.5 for reflected BSDEs, we have $y_t^m \leq y_t^{m+1}$, $\bar{A}_t^m \geq \bar{A}_t^{m+1}$ and $d\bar{A}_t^m \geq d\bar{A}_t^{m+1}$ on $[0, T]$. Thus, when $m \rightarrow \infty$, $y_t^m \nearrow y_t$, $\bar{A}_t^m \searrow \bar{A}_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, for each $t \in [0, T]$, and $y_t \leq y_t^*$. Thanks to Fatou's lemma, we get $E[\sup_{0 \leq t \leq T} |y_t|^2] < \infty$, and thus $y^m \rightarrow y$ in $\mathbf{L}^2_{\mathcal{F}}(0, T)$. Since \bar{A}^m is an RCLL process, we can not directly apply the monotonic limit theorem, i.e. Theorem 2.1 in [20]. However following similar techniques of the proof of Theorem 2.1 in [20], we know that the limit y can be written in the following form

$$y_t = y_0 - \int_0^t g_s^0 ds - A_t - \bar{A}_t + \int_0^t z_s dB_s,$$

where z . and g^0 (resp. A_t) are the weak limits of z_s^m and $g_s^m = g(s, y_s^m, z_s^m)$ (resp. A_t^m) in $\mathbf{L}^2_{\mathcal{F}}(0, T)$ (resp. $\mathbf{L}^2(\mathcal{F}_t)$). Since $A^m + \bar{A}^m$ is an increasing process, by Lemma 2.2 in [20], we

know that y is RCLL. Then apply Itô formulae to $|y_t^m - y_t|^2$ on $[\sigma, \tau]$, with stopping times $0 \leq \sigma \leq \tau \leq T$, it follows that

$$\begin{aligned} & E|y_\sigma^m - y_\sigma|^2 + E \int_\sigma^\tau |z_s^m - z_s|^2 ds \\ = & E|y_\tau^m - y_\tau|^2 + E \sum_{t \in (\sigma, \tau]} [(\Delta A_t)^2 - (\bar{A}_t^m - \bar{A}_t)^2] - 2E \int_\sigma^\tau (y_s^m - y_s)(g_s^m - g_s^0) ds \\ & + 2E \int_{(\sigma, \tau]} (y_s^m - y_s) dA_s^m - 2E \int_{(\sigma, \tau]} (y_s^m - y_s) dA_s + 2E \int_{(\sigma, \tau]} (y_{s-}^m - y_{s-}) d(\bar{A}_s^m - \bar{A}_s). \end{aligned}$$

Since $E \int_{(\sigma, \tau]} (y_s^m - y_s) dA_s^m \leq 0$ and $E \int_{(\sigma, \tau]} (y_{s-}^m - y_{s-}) d(\bar{A}_s^m - \bar{A}_s) \leq 0$, we get

$$\begin{aligned} E \int_\sigma^\tau |z_s^m - z_s|^2 ds & \leq E|y_\tau^m - y_\tau|^2 + E \sum_{t \in (\sigma, \tau]} (\Delta A_t)^2 + 2E \int_\sigma^\tau |y_s^m - y_s| |g_s^m - g_s^0| ds \\ & \quad + 2E \int_{(\sigma, \tau]} |y_s^m - y_s| dA_s. \end{aligned}$$

Now we are in the same situation as in the proof of the monotonic limit theorem (cf. [20], Proof of Theorem 2.1). We then can follow the same approach to get the strong convergence of $z^m \rightarrow z$ in $\mathbf{L}_{\mathcal{F}}^p(0, T)$, for $p < 2$.

We pass to the limit on both sides of (19), using the above convergence results of $(y^m, z^m, A^m, \bar{A}^m)$. The limit (y, z, A, \bar{A}) satisfies

$$y_t = X + \int_t^T g(s, y_s, z_s) ds + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T z_s dB_s.$$

The estimate $E[(A_T^m)^2] \leq C$ implies $E[(\int_0^T d_{\Gamma_s}(z_s^m) ds)^2] \leq \frac{C}{m^2}$. When $m \rightarrow +\infty$, we get

$$E[\int_0^T d_{\Gamma_s}(z_s) ds] = 0, \quad \text{thus } d_{\Gamma_t}(z_t) \equiv 0, \quad dP \times dt - a.s..$$

It remains to prove that (y, A) satisfies condition **(ii)** in Definition 3.1, i.e., $y \geq L$ and

$$\int_0^T (y_{t-} - L_{t-}^*) d\bar{A}_t = 0. \quad \text{a.s., for any } L^* \in \mathbf{D}_{\mathcal{F}}^2(0, T), \quad \text{s.t. } y_t \geq L_t^* \geq L_t, \quad dP \times dt - \text{a.s..} \quad (20)$$

From $y^m \geq L$, $m \in \mathbb{N}$, we have $y \geq L$. Thus for each $L^* \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ such that $y \geq L^* \geq L$, we have

$$\begin{aligned} \int_0^T (y_{t-} - y_{t-}^m \wedge L_{t-}^*) d\bar{A}_t & = \int_0^T (y_{t-} - y_{t-}^m) d\bar{A}_t + \int_0^T (y_{t-}^m - y_{t-}^m \wedge L_{t-}^*) d\bar{A}_t \\ & \quad + \int_0^T (y_{t-}^m - y_{t-}^m \wedge L_{t-}^*) d(A_t - \bar{A}_t^m). \end{aligned}$$

As $m \rightarrow \infty$, the first term on the right side tends to zero due to Lebesgue domination theorem. The second term is null because of (19) and the fact $y^m \geq y^m \wedge L^* \geq L$. For the

third term we have

$$\begin{aligned} E\left|\int_0^T (y_{t-}^m - y_{t-}^m \wedge L_{t-}^*) d(A_t - \bar{A}_t^m)\right| &\leq E\left[\sup_{t \in [0, T]} |y_t^m - y_{t-}^m \wedge L_{t-}^*| (A_T^m - A_T)\right] \\ &\leq E\left[\sup_{t \in [0, T]} |y_t^m - y_{t-}^m \wedge L_{t-}^*|^2\right]^{1/2} E[(A_T^m - A_T)^2]^{1/2} \end{aligned}$$

which converges also to zero since $E[(A_T^m - A_T)^2]^{1/2} \searrow 0$ and the boundedness of $E[\sup_{t \in [0, T]} |y_t^m|^2]$. This with $y_t^m \wedge L^* \nearrow L^*$ yields (20).

For (iii) of Definition 3.1, we consider a quadruple $(y^*, z^*, A^*, \bar{A}^*)$ which satisfies (i) and (ii) of Definition 3.1. Since $d_{\Gamma_s}(y_s^*, z_s^*) \equiv 0$, we have for any $m \in \mathbb{N}$,

$$y_t^* = X + \int_t^T g(s, y_s^*, z_s^*) ds + m \int_t^T d_{\Gamma_s}(y_s^*, z_s^*) ds + A_T^* - A_t^* + \bar{A}_T^* - \bar{A}_t^* - \int_t^T z_s^* dB_s.$$

Since $dA^* \geq 0$, by comparison theorem 5.5, it follows that $y^* \geq y^m$, for all m . Thus (iii) holds. \square

Remark 3.4. *If L is continuous or only has positive jumps ($L_{t-} \leq L_t$), then \bar{A} is a continuous process. In this case \bar{A}^n in (19) are continuous, and $\bar{A}_t^n \geq \bar{A}_t^{n+1}$, $d\bar{A}_t^n \geq d\bar{A}_t^{n+1}$, $0 \leq t \leq T$, with $E[(\bar{A}_T^n)^2] \leq C$. Thus $\bar{A}_t^n \searrow \bar{A}_t$, $0 \leq t \leq T$. Moreover*

$$0 \leq \bar{A}_t^n - \bar{A}_t \leq \bar{A}_T^n - \bar{A}_T.$$

From

$$E\left[\sup_{0 \leq t \leq T} (\bar{A}_t^n - \bar{A}_t)^2\right] \leq E[(\bar{A}_T^n - \bar{A}_T)^2] \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

it follows that $\bar{A}_t^n \searrow \bar{A}_t$ uniformly. We then can pass to limit on both sides of (19) to obtain the g_Γ -reflected BSDE with the lower obstacle L .

3.3 Comparison of different limits of $y^{m,n}$ to the g_Γ -reflected solution

The g_Γ -reflected BSDE with a lower barrier is a special type of constrained BSDE, in which y and z are constrained in $[L_t, +\infty)$ and Γ respectively. Let us put the two constraints together and set $\widehat{\Gamma}_t = [L_t, +\infty) \times \Gamma_t \subset \mathbb{R} \times \mathbb{R}^d$. In this case the penalization equation becomes:

$$\begin{aligned} y_t^{n,n} &= X + \int_t^T g(s, y_s^{n,n}, z_s^{n,n}) ds + n \int_t^T d_{\widehat{\Gamma}_s}(y_s^{n,n}, z_s^{n,n}) ds - \int_t^T z_s^{n,n} dB_s \quad (21) \\ &= X + \int_t^T g(s, y_s^{n,n}, z_s^{n,n}) ds + n \int_t^T d_{\Gamma_s}(z_s^{n,n}) ds + n \int_t^T (L_s - y_s^{n,n})^+ ds \\ &\quad - \int_t^T z_s^{n,n} dB_s. \end{aligned}$$

Let $\widehat{A}_t^{n,n} = n \int_0^t d_{\widehat{\Gamma}_s}(z_s^{n,n})ds$. Again from the monotonic limit theorem in [20], we know that $(y^{n,n}, z^{n,n}, \widehat{A}^{n,n})$ converges to $(\widehat{y}, \widehat{z}, \widehat{A}) \in \mathbf{L}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times \mathbf{A}_{\mathcal{F}}^2(0, T)$ as $n \rightarrow \infty$, and that the limit is the $g_{\widehat{\Gamma}}$ -solution, i.e., the smallest g -supersolution constrained in $\widehat{\Gamma}$:

$$\widehat{y}_t = X + \int_t^T g(s, \widehat{y}_s, \widehat{z}_s)ds + \widehat{A}_T - \widehat{A}_t - \int_t^T \widehat{z}_s dB_s.$$

With $d_{\widehat{\Gamma}_t}(\widehat{y}_t, \widehat{z}_t) = 0$, a.e.a.s. on $[0, T]$.

Comparing this result to that of Theorem 3.1 for g_{Γ} -reflected BSDE, we have

Proposition 3.1. *The above $g_{\widehat{\Gamma}}$ -solution of BSDE $(\widehat{y}_t, \widehat{z}_t, \widehat{A}_t)_{t \in [0, T]}$ coincides with the g_{Γ} -reflected solution obtained in Theorem 3.1: $(\widehat{y}_t, \widehat{z}_t, \widehat{A}_t) \equiv (y_t, z_t, A_t + \overline{A}_t)$.*

Proof. For $m \leq n$, by comparison theorem for (13) and (21), we have

$$y_t^{m,m} \leq y_t^{m,n} \leq y_t^{n,n}$$

Letting $n \rightarrow \infty$ yields

$$y_t^{m,m} \leq y_t^m \leq \widehat{y}_t$$

then $m \rightarrow \infty$ yields

$$\widehat{y}_t \leq y_t \leq \widehat{y}_t.$$

Thus the two g -supersolution coincide with each others. \square

Let us consider another limit of $y^{m,n}$ by first letting $m \rightarrow \infty$. We have

$$y_t^{m,m} \geq y_t^{m,n} \geq y_t^{n,n}$$

Once again from the monotonic limit theorem in [20], when $m \rightarrow \infty$ the triple $(y^{m,n}, z^{m,n}, A^{m,n})$ converges to $(\widetilde{y}^n, \widetilde{z}^n, \widetilde{A}^n) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times \mathbf{A}_{\mathcal{F}}^2(0, T)$, which is the solution of the following Γ -constrained g^n -solution of BSDE with $g^n = g + n(L_t - y)^+$, or

$$\begin{aligned} \widetilde{y}_t^n &= X + \int_t^T g(s, \widetilde{y}_s^n, \widetilde{z}_s^n)ds + \widetilde{A}_T^n - \widetilde{A}_t^n + n \int_t^T (L_s - \widetilde{y}_s^n)^+ ds - \int_{ts}^T \widetilde{z}_s^n dB_s, \quad (22) \\ (\widetilde{z}_t^n) &\in \Gamma_t, \quad dP \times dt\text{-a.s.}, \quad dA^n \geq 0, \end{aligned}$$

and we have

$$\widehat{y}_t \geq \widetilde{y}_t^n \geq y_t^{n,n}.$$

By letting $n \rightarrow \infty$ we see that $\widetilde{y}_t^n \uparrow \widehat{y}_t = y_t$.

3.4 Existence of g_{Γ} -reflected solution with an upper barrier: Proof of Theorem 3.2

The main idea is still based on penalization method with more technicalities.

For each $n \in \mathbb{N}$, we consider the solution $(y^n, z^n, K^n) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times \mathbf{A}_{\mathcal{F}}^2(0, T)$ of the following reflected BSDE with the coefficient $g^n(t, y, z) = g(t, y, z) + nd_{\Gamma_t}(z)$ and the upper reflecting obstacle U :

$$\begin{aligned} y_t^n &= X + \int_t^T g^n(s, y_s^n, z_s^n) ds - (K_T^n - K_t^n) - \int_t^T z_s^n dB_s, \\ y^n &\leq U, \quad dP \times dt\text{-a.s.} \quad dK \geq 0, \quad \text{and} \quad \int_0^T (U_{t-}^* - y_{t-}^n) dK_t^n = 0, \\ \forall U^* &\in \mathbf{D}_{\mathcal{F}}^2(0, T), \quad \text{such that} \quad y^n \leq U^* \leq U \quad dP \times dt\text{-a.s.} \end{aligned} \quad (23)$$

Since $g^n(t, y, z)$ is Lipschitz with respect to (y, z) , from the existence theorem of [22] for reflected BSDEs with L^2 -obstacle, this equation has a unique solution. We denote $A_t^n = n \int_0^t d\Gamma_s(z_s^n) ds$.

In order to get an a priori estimate for (y^n, z^n, A^n, K^n) , we need the following lemma.

Lemma 3.3. *For any $X \in \mathbf{L}^2(\mathcal{F}_T)$, there exists a quadruple of processes $(y^*, z^*, A^*, K^*) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times (\mathbf{A}_{\mathcal{F}}^2(0, T))^2$ satisfying*

$$\begin{aligned} y_t^* &= X + \int_t^T g(s, y_s^*, z_s^*) ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T z_s^* dB_s, \\ d_{\Gamma_t}(z_t^*) &= 0 \text{ and } y_t^* \leq U_t, \quad dP \times dt\text{-a.s.}, \text{ with } \int_0^T (y_{t-}^* - U_{t-}^*) dK_t^* = 0, \text{ a.s.} \\ \forall U^* &\in \mathbf{D}_{\mathcal{F}}^2(0, T), \quad \text{such that} \quad y^* \leq U^* \leq U \quad dP \times dt\text{-a.s.} \end{aligned} \quad (24)$$

Proof. Fix a process $\sigma_t \in \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$ satisfying $\sigma_t \in \Gamma_t$, $t \in [0, T]$. We consider a forward SDE with the upper obstacle U_t . For $0 \leq t \leq T$

$$\begin{aligned} dx_t &= -g(t, x_t, \sigma_t) dt - d\bar{A}_t + \sigma_t dB_t, \\ x_0 &= 1 \wedge U_0, \quad \text{with } x_t \leq U_t, \text{ a.s.a.e.} \end{aligned}$$

Since $g(t, x, \sigma_t)$ is a Lipschitz function and $\text{ess sup}_{0 \leq t \leq T} U_t^- \in \mathbf{L}^2(\mathcal{F}_T)$, this equation admits a solution (x_t, \bar{A}_t) in $\mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{A}_{\mathcal{F}}^2(0, T)$. Set

$$\begin{aligned} y_t^* &= x_t, \quad z_t^* = \sigma_t, \\ A_t^* &= \bar{A}_t + (x_T - X)^+ 1_{\{t=T\}}, \quad K_t^* = (x_T - X)^+ 1_{\{t=T\}}. \end{aligned}$$

Then this quadruple is just what we need. \square

We have the following estimate:

Lemma 3.4. *There exists a constant $C > 0$, independent of n , such that*

$$E\left[\sup_{0 \leq t \leq T} (y_t^n)^2\right] + E \int_0^T |z_s^n|^2 ds + E[(A_T^n)^2] + E[(K_T^n)^2] \leq C. \quad (25)$$

Proof. Consider the following reflected BSDE

$$\begin{aligned} y_t^0 &= X + \int_t^T g(s, y_s^0, z_s^0) ds - (K_T^0 - K_t^0) - \int_t^T z_s^0 dB_s, \quad t \in [0, T], \\ y_t^0 &\leq U_t, \quad dK_t \geq 0, \quad \int_0^T (y_{t-}^0 - U_{t-}^*) dK_t^0 = 0. \\ \forall U^* &\in \mathbf{D}_{\mathcal{F}}^2(0, T), \quad \text{such that } y^0 \leq U^* \leq U \quad dP \times dt\text{-a.s.} \end{aligned}$$

This equation has a unique solution $(y^0, z^0, K^0) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times \mathbf{A}_{\mathcal{F}}^2(0, T)$. By the comparison theorem of reflected BSDEs, we have $y_t^n \geq y_t^0$ on $[0, T]$.

On the other hand, the quadruple (y^*, z^*, A^*, K^*) that we get from Lemma 3.3 satisfies

$$\begin{aligned} y_t^* &= X + \int_t^T (g + nd_{\Gamma_s})(s, y_s^*, z_s^*) ds + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T z_s^* dB_s, \\ y_t^* &\leq U_t, \quad \text{a.e.a.s.} \quad \int_0^T (y_{t-}^* - U_{t-}) dK_t^* = 0, \quad \text{a.s.} \end{aligned}$$

It follows from the comparison theorem 5.5 for reflected BSDEs that $y_t^n \leq y_t^*$, $K_t^n \leq K_t^*$ and $dK_t^n \leq dK_t^*$, for each $n \in \mathbb{N}$, $t \in [0, T]$. Thus there exists a constant $C > 0$, independent of n , such that

$$E[\sup_{0 \leq t \leq T} (y_t^n)^2] \leq E[\sup_{0 \leq t \leq T} \{(y_t^0)^2 + (y_t^*)^2\}] \leq C. \quad (26)$$

and

$$E[(K_T^n)^2] \leq E[(K_T^*)^2] \leq C. \quad (27)$$

To estimate (z^n, A^n) , we just need to rewrite (23)

$$y_t^n - K_t^n = X - K_T^n + \int_t^T g(s, y_s^n, z_s^n) ds + A_T^n - A_t^n - \int_t^T z_s^n dB_s.$$

and to check that the estimate in Lemma 3.2 can be applied for the triple $(y_t^n - K_t^n, z_t^n, A_t^n)$. \square

We are ready to prove Theorem 3.2.

Proof of Theorem 3.2. In (23), since $g^n(t, y, z) \leq g^{n+1}(t, y, z)$, by Comparison Theorem 5.5 for reflected BSDEs, $y_t^0 \leq y_t^n \leq y_t^{n+1} \leq y_t^*$. Thus $\{y_t^n\}_{n=1}^\infty$ increasingly converges to y_t as $n \rightarrow \infty$, and

$$E[\sup_{0 \leq t \leq T} (y_t)^2] \leq E[\sup_{0 \leq t \leq T} \{(y_t^0)^2 + (y_t^*)^2\}] \leq C.$$

It follows from the domination convergence theorem that

$$\lim_{n \rightarrow \infty} E[\int_0^T |y_t^n - y_t|^2 dt] = 0.$$

We can also get from Theorem 5.5 that $K_t^n \leq K_t^{n+1} \leq K_t^*$ and $dK_t^n \leq dK_t^{n+1} \leq dK_t^*$, $0 \leq t \leq T$. It follows that $\{K_t^n\}_{n=1}^\infty$ increasingly converges to an increasing process $K \in \mathbf{A}_{\mathcal{F}}^2(0, T)$ with $E[(K_T)^2] \leq C$. Meanwhile A^n are continuous increasing processes satisfying

$E[(A_T^n)^2 + \int_0^T |z_s^n|^2 ds] \leq C$ and there exists a process $z \in \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$, such that $z^n \rightarrow z$ weakly in $\mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$.

Now the conditions of the generalized monotonic limit theorem, Theorem 3.1 in [22], are satisfied. So we have $z^n \rightarrow z$ strongly in $\mathbf{L}_{\mathcal{F}}^p(0, T; \mathbb{R}^d)$, for $p < 2$. With the Lipschitz condition of g , the limit $y \in \mathbf{D}^2(0, T)$ can be written as

$$y_t = X + \int_t^T g(s, y_s, z_s) ds + (A_T - A_t) - (K_T - K_t) - \int_t^T z_s dB_s,$$

where, for each t , $A_t^n \rightarrow A_t$ weakly in $\mathbf{L}^2(\mathcal{F}_t)$, $K_t^n \rightarrow K_t$ strongly in $\mathbf{L}^2(\mathcal{F}_t)$ and $A, K \in \mathbf{A}_{\mathcal{F}}^2(0, T)$ are increasing processes.

From $E[(A_T^n)^2] = E[(n \int_0^T d_{\Gamma_s}(z_s^n) ds)^2] \leq C$, it follows that

$$E[(\int_0^T d_{\Gamma_s}(z_s^n) ds)^2] \leq \frac{C}{n^2},$$

while $d_{\Gamma_s}(z_s^n) \geq 0$, we get that $\int_0^T d_{\Gamma_s}(z_s^n) ds \rightarrow 0$, as $n \rightarrow \infty$. With the Lipschitz property of $d_{\Gamma_t}(z)$ and the convergence of z^n , we deduce that

$$d_{\Gamma_t}(z_t) = 0 \quad dP \times dt\text{-a.s.}$$

We now prove that the quadruple (y, z, A, K) satisfies (ii) of Definition 3.2. We have $y \leq U$, from $y^n \leq U$. Now for each $U^* \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ such that $U \geq U^* \geq y \geq y^n$, since $\int_0^T (y_{t-}^n - U_{t-}^*) dK_t^n = 0$, thus $\int_0^T (y_{t-} - U_{t-}^*) dK_t = 0$. Moreover we have $dK_t^n \leq dK_t$ and $K_T^n \nearrow K_T$ in $\mathbf{L}^2(\mathcal{F}_T)$. It then follows that

$$0 \leq \int_0^T (U_{t-}^* - y_{t-}) d(K_t - K_t^n) \leq \sup_{t \in [0, T]} (U_t^* - y_t) \cdot [K_T - K_T^n].$$

This with (6) and the estimate of y , it follows that (ii) of Definition 3.2 holds.

We now prove that (iii) in Definition 3.2 holds true for the quadruple. In fact, for any other quadruple $(\bar{y}, \bar{z}, \bar{A}, \bar{K}) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times (\mathbf{A}^2(0, T))^2$ satisfying

$$\begin{aligned} \bar{y}_t &= X + \int_t^T g(s, \bar{y}_s, \bar{z}_s) ds + \bar{A}_T - \bar{A}_t - (\bar{K}_T - \bar{K}_t) - \int_t^T \bar{z}_s dB_s, \\ d_{\Gamma_t}(\bar{z}_t) &= 0, \quad dP \times dt\text{-a.s.}, \quad d\bar{A} \geq 0, \quad d\bar{K} \geq 0, \\ \bar{y}_t &\leq U_t, \quad dP \times dt\text{-a.s.}, \quad \int_0^T (U_{t-}^* - \bar{y}_{t-}) d\bar{K}_t = 0, \quad \text{a.s.}, \end{aligned} \quad (28)$$

for any $U^* \in \mathbf{D}_{\mathcal{F}}^2(0, T)$, such that $\bar{y} \leq U^* \leq U$ $dP \times dt\text{-a.s.}$. Then it also satisfies for $\forall n \in \mathbb{N}$,

$$\bar{y}_t = X + \int_t^T g(s, \bar{y}_s, \bar{z}_s) ds + n \int_t^T d_{\Gamma_s}(\bar{z}_s) ds + \bar{A}_T - \bar{A}_t - (\bar{K}_T - \bar{K}_t) - \int_t^T \bar{z}_s dB_s.$$

Compare this to (23), since $d\bar{A}_t \geq 0$, we have $\bar{y} \geq y^n$, and $\bar{K} \geq K^n$. Let $n \rightarrow \infty$, it follows

$$\bar{y}_t \geq y_t, \quad \bar{K}_t \geq K_t, \quad \forall t \in [0, T], \quad \text{a.s.} \quad (29)$$

So y is the smallest process satisfying Definition 3.2 (i) and (ii).

It remains to prove the relation $\mathcal{V}_{[0,T]}(A + K) = \mathcal{V}_{[0,T]}(A + K)$ in (11), namely A and K is the Jordan decomposition of $A - K$. For this we set $\tilde{V}_t = \mathcal{V}_{[0,t]}(A - K)$ and define the Jordan decomposition of $A - K$ by

$$\tilde{A}_t = \frac{1}{2}(\tilde{V}_t + A_t - K_t), \quad \tilde{K}_t = \frac{1}{2}(\tilde{V}_t - A_t + K_t).$$

We have $d\tilde{K}_t = \frac{1}{2}d(\tilde{V}_t - A_t + K_t) \leq dK_t$ and thus, for each $U^* \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ with $U \geq U^* \geq y$, $dP \times dt$ -a.s.,

$$0 \leq \int_0^T (y_{t-} - U_{t-}^*)d\tilde{K}_t \leq \int_0^T (y_{t-} - U_{t-}^*)dK_t = 0.$$

So the quadruple $(y, z, \tilde{A}, \tilde{K})$ also satisfies (28). It then follows from the second inequality of (29) that $\tilde{K} \geq K$. This with $\tilde{K} \leq K$ yields $\tilde{K} \equiv K$ and thus A and K are indeed the Jordan decomposition of $A - K$. \square

Remark 3.5. *Since $y - K$ is the smallest process satisfying BSDE associated to X , g^K and constraint Γ , thus it is the $(g^K)_{\Gamma}$ -solution with terminal condition $X - K_T$, where*

$$g^K(t, y, z) = g(t, y + K_t, z).$$

Remark 3.6. *If U is continuous (or satisfies $U_{t-} \geq U_t$), then K is a continuous process. In fact, by [6], the solution y^n of (23) as well as the reflecting process K^n are continuous. This with $K^n \leq K^{n+1}$ and $dK^n \leq dK$ yields*

$$0 \leq K_t - K_t^n \leq K_T - K_T^n$$

and thus

$$E\left[\sup_{0 \leq t \leq T} (K_t - K_t^n)^2\right] \leq E[(K_T - K_T^n)^2] \rightarrow 0.$$

The continuity of K then follows from the uniform convergence of K^n to K .

4 Applications of g_{Γ} -reflected BSDEs: American option pricing in incomplete market

We follow the idea of El Karoui et al. (1997, [7]). In a financial market we consider the wealth strategy and portfolio (Y_t, π_t) of an investor which is a pair of adapted processes in $\mathbf{L}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d)$. This pair solves the following BSDE

$$-dY_t = g(t, Y_t, \pi_t^{\top} \sigma_t)dt - \pi_t^{\top} \sigma_t dB_t,$$

where g is a convex function of (y, π) satisfying the same Lipschitz condition given in (1). We suppose that the volatility matrix σ_t is invertible and $\sigma_t, (\sigma_t)^{-1}$ are bounded. We are concerned with the problem of pricing an American contingent claim.

Let S be a continuous process satisfying $E[\sup_t (S_t^+)^2] < \infty$, which is a given continuous time payoff during $[t, T)$ and ξ be a given terminal payoff at T . For a given $t \geq 0$, let \mathcal{T}_t

be the set of stopping times valued in $[t, T]$. Then the corresponding total payoff at time $s \in [t, T]$ is

$$\tilde{S}_s = \xi 1_{\{s=T\}} + S_s 1_{\{s<T\}}.$$

According to [7], in a complete market, i.e., a market without constraints on (Y, π) , the price of the American contingent claim $(\tilde{S}_s)_{0 \leq s \leq T}$ at time t is given by

$$Y_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} Y_t(\tau, \tilde{S}_\tau).$$

Here $Y_t(\tau, \tilde{S}_\tau)$ is the solution of BSDE with the terminal time τ and terminal condition \tilde{S}_τ . In fact the price $(Y_t)_{0 \leq t \leq T}$ is the unique solution of the reflected BSDE associated with the terminal condition ξ and the obstacle S : there exists $(\pi_t) \in \mathbf{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$ and an increasing continuous process (A_t) with $A_0 = 0$ such that

$$\begin{aligned} -dY_t &= g(s, Y_t, \pi_t^\top \sigma_t) ds + dA_t - \pi_t^\top \sigma_t dB_t, \quad Y_T = \xi, \\ Y_t &\geq S_t, \quad 0 \leq t \leq T, \quad \int_0^T (Y_t - S_t) dA_t = 0. \end{aligned}$$

Furthermore, the stopping time $D_t = \inf(t \leq s \leq T \mid dA_s > 0) \wedge T$ is the biggest optimal time after t , and

$$Y_t = Y_t(D_t, \tilde{S}_{D_t}).$$

Our problem is to price the American contingent claim $(\tilde{S}_s, 0 \leq s \leq T)$ for an incomplete market where the portfolios π_t are constrained in Γ_t which is a closed subset of \mathbb{R}^d . This problem can be solved as follows: We set $\Gamma_t^1 = \{z \in \mathbb{R}^d : z^\top \sigma_t^{-1} \in \Gamma_t\}$.

Theorem 4.1. *We assume that ξ is attainable, i.e., there exists a g -supersolution (Y', z', A') on $[t, T]$ with $z'_t \in \Gamma_t^1$, t -a.e. and with the terminal condition ξ . Then the solution (Y, z, A, \bar{A}) of the g_{Γ^1} -reflected BSDE with lower obstacle S exists and Y is the price process of the American option in the incomplete market. The quadruple (Y, z, A, \bar{A}) solves*

$$\begin{aligned} Y_t &= \xi + \int_t^T g(s, Y_s, \pi_s^\top \sigma_s) ds + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T \pi_s^\top \sigma_s dB_s, \quad (30) \\ Y_t &\geq S_t, \quad 0 \leq t \leq T, \quad z^\top \sigma_t^{-1} \in \Gamma_t^1, \quad \int_0^T (Y_t - S_t) d\bar{A}_t = 0. \end{aligned}$$

Furthermore \bar{A} is continuous and $D_0 = \inf(0 \leq s \leq T \mid d\bar{A}_s > 0) \wedge T$ is the corresponding optimal stopping time.

Proof. Let $\tau \in \mathcal{T}_t$ be any given stopping time and let $(\bar{Y}, \bar{z}, \bar{A})$ be a g_{Γ^1} -solution on $[0, \tau]$ with terminal condition \tilde{S}_τ . By comparison theorem we know that $Y_t^n \leq \bar{Y}_t$ on $[0, \tau]$, where Y^n is the solution of the reflected BSDE on $[0, T]$ associated to $(\xi, g + nd_{\Gamma^1}, S)$. Since Y^n upwardly converges to Y . Thus $Y_t \leq \bar{Y}_t$ on $[0, \tau]$. It follows that Y is the smallest g -supersolution constrained in Γ^1 among all g_{Γ^1} -solution \bar{Y} defined on $[t, \tau]$ with terminal condition $\bar{Y}_\tau = \tilde{S}_\tau$. Moreover Y is the g_{Γ^1} -solution defined on $[0, D_0]$. Thus D_0 is the optimal stopping time. \square

4.1 Some examples of American call option

We study the American call option, set $S_t = (X_t - k)^+$, $\xi = (X_T - k)^+$, where X is the price of underlying stock and k is the strike price. More precisely, X is the solution of

$$X_t = x_0 + \int_0^t \mu_s X_s ds + \int_0^t \sigma_s X_s dB_s. \quad (31)$$

Correspondingly, in (30) g is a linear function

$$g(t, y, \pi) = -r_t y - (\mu_t - r_t) \pi^\tau \sigma_t.$$

Proposition 4.1. *If ξ is attainable, then the maturity time of American call option in incomplete market is still T .*

Proof. Consider the price process Y^0 of American call option, without constraint, which is a solution of reflected BSDE

$$\begin{aligned} Y_t^0 &= \xi + \int_t^T g(s, Y_s^0, \pi_s^0) ds + \bar{A}_T^0 - \bar{A}_t^0 - \int_t^T (\pi_s^0)^\tau \sigma_s dB_s, \\ Y_t^0 &\geq S_t, \quad \int_0^T (Y_t^0 - S_t) d\bar{A}_t^0 = 0. \end{aligned}$$

Comparing it with (30), we have that $Y_t \geq Y_t^0$, $\bar{A}_t \leq \bar{A}_t^0$, $t \in [0, T]$.

Since in a complete market, an American call option always exercises at the terminal time T , which implies $D_t^0 = T$, where $D_t^0 = \inf(t \leq s \leq T \mid dA_s^0 > 0) \wedge T$. So we have $\bar{A}_t^0 = 0$ on $[0, T]$. It follows that $\bar{A}_t \leq \bar{A}_t^0 = 0$, $t \in [0, T]$. Then by the definition, $D_t = T$. \square

From this proposition, we know that maybe the seller's price process Y in incomplete market is bigger than in complete market. But their exercise times are the same, i.e. at T . So the seller's price is the same as the seller's price for the corresponding European contingent claim.

Now we consider an interesting example.

Example 4.1. *No short-selling: In this case $\Gamma_t = [0, \infty)$, for $t \in [0, T]$. We set $d = 1$. By Proposition 4.1 and Example 7.1 in [2], the price process of the American call option takes same value as European call option. This means that the constraint $K = [0, \infty)$ does not make any difference.*

From this example, we know that the constraint $\Gamma_t = [0, \infty)$ does not influence the price processes of the American contingent claim. In fact, we have a more general result.

Proposition 4.2. *Consider the constraint $\Gamma_t = [0, \infty)$, for $t \in [0, T]$. If $\xi = \Phi(X_T)$, $S_t = l(X_t)$, where $\Phi, l: \mathbb{R} \rightarrow \mathbb{R}$ are both increasing in x , and σ satisfies the uniformly elliptic condition, then the price process Y takes same value as in complete market, i.e. the constraint Γ does not influence the price.*

Proof. It is sufficient to prove that $\bar{\pi}_t \geq 0$ for) the solution $(\bar{Y}, \bar{\pi}, \bar{A})$ of following reflected BSDE

$$\begin{aligned}\bar{Y}_t &= \Phi(X_T) + \int_t^T g(s, \bar{Y}_s, \bar{\pi}_s) ds + \bar{A}_T - \bar{A}_t - \int_t^T \bar{\pi}_s^\tau \sigma_s dB_s, \\ \bar{Y}_t &\geq l(X_t), \quad \int_0^T (\bar{Y}_t - l(X_t)) d\bar{A}_t = 0.\end{aligned}\tag{32}$$

We put $(X_s^{t,x}, \bar{Y}_s^{t,x}, \bar{\pi}_s^{t,x}, \bar{A}_s^{t,x})_{t \leq s \leq T}$ under Markovian framework with (31). Define

$$u(t, x) = \bar{Y}_t^{t,x},$$

then by [6], we know that u is the viscosity solution of the PDE with an obstacle l ,

$$\begin{aligned}\min\{u(t, x) - l(x), -\frac{\partial u}{\partial t} - \mathcal{L}u - g(t, x, u, \nabla u \sigma)\} &= 0, \\ u(T, x) &= \Phi(x),\end{aligned}$$

where $\mathcal{L} = \frac{1}{2}(\sigma_s)^2 \frac{\partial^2}{\partial x \partial x} + \mu \frac{\partial}{\partial x}$. Since $(\bar{\pi}_r^{t,x})^\tau \sigma_r = \nabla u(r, X_r^{t,x}) \sigma_r$, and σ_r is uniformly elliptic, we only need to prove that $\nabla u(t, x)$ is non-negative. Indeed, it is easy to obtain by comparison theorem. For $x_1, x_2 \in \mathbb{R}$, with $x_1 \geq x_2$, so $X_s^{t,x_1} \geq X_s^{t,x_2}$. It follows that $\Phi(X_T^{t,x_1}) \geq \Phi(X_T^{t,x_2})$ and $l(X_s^{t,x_1}) \geq l(X_s^{t,x_2})$ in view of assumptions. By comparison theorem of BSDE, we get $\bar{Y}_t^{t,x_1} \geq \bar{Y}_t^{t,x_2}$, which implies $u(t, x_1) \geq u(t, x_2)$. So u is increase in x i.e. $\nabla u(t, x) \geq 0$, it follows that $\bar{\pi}_t^{t,x} \geq 0$. \square

4.2 Some examples of American put option

In this case, we set $S_t = (k - X_t)^+$, $\xi = (k - X_T)^+$, where X is the price of underlying stock given in (31) and k is the strike price. Parallel to Proposition 4.2, we have

Proposition 4.3. *Consider the constraint $\Gamma_t = (-\infty, 0]$, for $t \in [0, T]$. If $\xi = \Phi(X_T)$, $S_t = l(X_t)$, where $\Phi, l : \mathbb{R} \rightarrow \mathbb{R}$ are both decreasing functions, and σ satisfies uniformly elliptic condition, then the price process Y takes same value as in complete market, i.e. the constraint Γ has no influence on price process.*

Example 4.2. *No borrowing: $\Gamma_t = (-\infty, Y_t]$. Obviously, $Y_t \geq 0$, in view of $Y_t \geq S_t \geq 0$. So $\Gamma_t \supset (-\infty, 0]$, by Proposition 4.3, we know that the price process Y takes the same value as in a complete market.*

Under "No short-selling" constraint, we will get a totally different result.

Example 4.3. *No short-selling: $\Gamma_t = [0, \infty)$, for $t \in [0, T]$. The pricing process Y with hedging π satisfies*

$$\begin{aligned}Y_t &= \xi + \int_t^T g(s, Y_s, \pi_s) ds + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T \pi_s^* \sigma_s dB_s, \\ Y_t &\geq S_t, 0 \leq t \leq T, \quad \int_0^T (Y_t - S_t) d\bar{A}_t = 0, \quad \pi_t \geq 0, t\text{-a.e.}\end{aligned}$$

Notice that $S_t = (k - X_t)^+ < k$. So the g_Γ -reflected solution of the above equation is

$$\begin{aligned} Y_t &= \begin{cases} k, & t \in [0, T) \\ (k - X_T)^+, & t = T \end{cases}; \\ \pi_t &= 0, \\ A_t &= \begin{cases} k \int_0^t r_s ds, & t \in [0, T) \\ k \int_0^T r_s ds + k - (k - X_T)^+, & t = T \end{cases}; \\ \bar{A}_t &= 0. \end{aligned}$$

In particular the price of American put option under 'no short-selling' constraint is $Y_0 = k$.

5 Appendix

In this appendix, we present monotonic limit theorem introduced in [20] and a generalized version in [22]. We consider the following sequence of Itô processes

$$y_t^i = y_0^i + \int_0^t g_s^i ds - A_t^i + K_t^i + \int_0^t z_s^i dB_s, \quad t \in [0, T], \quad i = 1, 2, \dots \quad (33)$$

Here $g^i \in \mathbf{L}_{\mathcal{F}}^2(0, T)$ and $A^i, K^i \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ are given increasing processes. We assume

- (i) (y_t^i) increasingly converges to $y \in \mathbf{L}_{\mathcal{F}}^2(0, T)$ with $E[\sup_{0 \leq t \leq T} |y_t|^2] < \infty$;
 - (ii) (g_t^i, z_t^i) weakly converges to (g^0, z) in $\mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R} \times \mathbb{R}^d)$;
 - (iii) A^i is continuous and increasing with $A_0^i = 0$ and $E[(A_T^i)^2] < \infty$;
- (34)

Furthermore for K^i , we assume

- (iv) $K_t^j - K_s^j \geq K_t^i - K_s^i, \forall 0 \leq s \leq t \leq T$, a.s., $\forall i \leq j$;
 - (v) For each $t \in [0, T]$, $K_t^j \nearrow K_t$ in j , with $E[K_T^2] < \infty$.
- (35)

An easy consequence is

- (i) $E[\sup_{0 \leq t \leq T} |y_t^i|^2] \leq C$;
 - (ii) $E \int_0^T |y_t^i - y_t|^2 ds \rightarrow 0$,
- (36)

The generalized monotonic limit theorem given in [22] is as follows.

Theorem 5.1. *We assume (34) and (35) hold. Then the limit y of the sequence $\{y^i\}_{i=1}^\infty$ is of the form*

$$y_t = y_0 + \int_0^t g_s^0 ds - A_t + K_t + \int_0^t z_s dB_s, \quad (37)$$

where $A, K \in \mathbf{A}_{\mathcal{F}}^2(0, T)$ are increasing processes. Here, for each $t \in [0, T]$, A_t (resp. K_t) is the weak (resp. strong) limit of $\{A_t^i\}_{i=1}^\infty$ (resp. $\{K_t^i\}_{i=1}^\infty$) in $\mathbf{L}^2(\mathcal{F}_t)$. Furthermore, for any $p \in [0, 2)$, $\{z^i\}_{i=1}^\infty$ strongly converges to z in $\mathbf{L}_{\mathcal{F}}^p(0, T; \mathbb{R}^d)$, i.e.,

$$\lim_{i \rightarrow \infty} E \int_0^T |z_s^i - z_s|^p ds = 0. \quad (38)$$

If moreover A is a continuous process, then we have

$$\lim_{i \rightarrow \infty} E \int_0^T |z_s^i - z_s|^2 ds = 0. \quad (39)$$

The monotonic limit theorem was originally obtained in [20]

$$y_t^i = y_0^i + \int_0^t g_s^i ds - A_t^i + \int_0^t z_s^i dB_s, \quad t \in [0, T], \quad i = 1, 2, \dots. \quad (40)$$

Since this result is used in this paper, we state it as following:

Theorem 5.2. *We assume that assumption (34) holds. Then the limit y of the sequence $\{y^i\}_{i=1}^\infty$ given in (40) has a form*

$$y_t = y_0 + \int_0^t g_s^0 ds - A_t + \int_0^t z_s dB_s, \quad 0 \leq t \leq T,$$

where $A \in \mathbf{A}_{\mathcal{F}}^2(0, T)$ is an increasing process. Here, for each $t \in [0, T]$, A_t is the weak limit of $\{A_t^i\}_{i=1}^\infty$ in $\mathbf{L}^2(\mathcal{F}_t)$. Furthermore, $\{z^i\}_{i=1}^\infty$ strongly converges to z in $\mathbf{L}_{\mathcal{F}}^p(0, T, \mathbb{R}^d)$, i.e.

$$\lim_{i \rightarrow \infty} E \int_0^T |z_s^i - z_s|^p ds = 0, \quad p \in [0, 2). \quad (41)$$

If furthermore $(A_t)_{t \in [0, T]}$ is continuous, then we have

$$\lim_{i \rightarrow \infty} E \int_0^T |z_s^i - z_s|^2 ds = 0. \quad (42)$$

The smallest g -supersolution with constraint Γ was firstly considered in [20], when Γ is defined as

$$\Gamma_t(\omega) = \{(y, z) \in \mathbb{R}^{1+d} : \Phi(\omega, t, y, z) = 0\}.$$

Here Φ is a nonnegative, measurable Lipschitz function and $\Phi(\cdot, y, z) \in \mathbf{L}_{\mathcal{F}}^2(0, T)$, for $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. Under the following assumption, the result of the existence of the smallest solution, obtained in [20], can be stated as follows.

Theorem 5.3. *Suppose that the function g satisfies (1) and the constraint Γ satisfies (2). We assume that there is at least one Γ -constrained g -supersolution $y' \in \mathbf{D}_{\mathcal{F}}^2(0, T)$:*

$$\begin{aligned} y'_t &= X' + \int_t^T g(s, y'_s, z'_s) ds + A'_T - A'_t - \int_t^T z'_s dB_s, \\ A' &\in \mathbf{A}_{\mathcal{F}}^2(0, T), \quad (y'_t, z'_t) \in \Gamma_t, \quad t \in [0, T], \quad \text{a.s. a.e.} \end{aligned} \quad (43)$$

Then, for each $X \in \mathbf{L}^2(\mathcal{F}_T)$ with $X \leq X'$, a.s., there exists the g_{Γ} -solution $y \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ with the terminal condition $y_T = X$ (defined in Definition 2.2). Moreover, this g_{Γ} -solution is the limit of a sequence of g^n -solutions with $g^n = g + nd_{\Gamma}$, i.e.,

$$y_t^n = X + \int_t^T (g + nd_{\Gamma})(s, y_s^n, z_s^n) ds - \int_t^T z_s^n dB_s, \quad (44)$$

where the convergence is in the following sense:

$$y_t^n \nearrow y_t, \text{ with } \lim_{n \rightarrow \infty} E[|y_t^n - y_t|^2] = 0, \lim_{n \rightarrow \infty} E \int_0^T |z_t - z_t^n|^p dt = 0, p \in (0, 2) \quad (45)$$

$$A_t^n := n \int_0^t d_{\Gamma_s}(s, y_s^n, z_s^n) ds \rightarrow A_t \text{ weakly in } \mathbf{L}^2(\mathcal{F}_t), \quad (46)$$

where z and A are the corresponding martingale representing part and increasing part of y , respectively.

Proof. By the comparison theorem of BSDE, $y_t^n \leq y_t^{n+1} \leq y_t'$. It follows that there exists a $y \leq y'$ such that, for each $t \in [0, T]$,

$$y_t^1 \leq y_t^n \nearrow y_t \leq y_t'.$$

Consequently, there exists a constant $C > 0$, independent of n , such that

$$E[\sup_{0 \leq t \leq T} (y_t^n)^2] \leq C, \text{ so } E[\sup_{0 \leq t \leq T} (y_t)^2] \leq C.$$

Thanks to the monotonic limit Theorem 5.2, we can take the limit on both sides of BSDE (44) and obtain

$$y_t = X + \int_t^T g(s, y_s, z_s) ds + A_T - A_t - \int_t^T z_s dB_s.$$

On the other hand, by $E[(A_T^n)^2] = n^2 E[(\int_0^t d_{\Gamma_s}(y_s^n, z_s^n) ds)^2] \leq C$ we have

$$d_{\Gamma_t}(y_t, z_t) \equiv 0, \quad dP \times dt - a.s..$$

□

Remark 5.1. If the constraint Γ is of the following form $\Gamma_t = (-\infty, U_t] \times \mathbb{R}^d$, where $U_t \in \mathbf{L}^2(\mathcal{F}_t)$, then the g_Γ -solution with terminal condition $y_T = X$ exists, if and only if $d_{\Gamma_t}(Y_t, Z_t) \equiv 0$, a.s. a.e., where (Y, Z) is the solution of the BSDE

$$-dY_t = g(t, Y_t, Z_t) dt - Z_t dB_t, \quad t \in [0, T], \quad Y_T = X.$$

This follows easily by comparison theorem.

We also have

Theorem 5.4 (Comparison Theorem of g_Γ -solution). *We assume that g^1, g^2 satisfy (1) and Γ^1, Γ^2 satisfy (2). And suppose that, for each $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, we have*

$$X^1 \leq X^2, \quad g^1(t, y, z) \leq g^2(t, y, z), \quad \Gamma_t^1 \supseteq \Gamma_t^2, \quad (47)$$

For $i = 1, 2$, Let $Y^i \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ be the $g_{\Gamma^i}^i$ -solution with terminal condition $Y_T^i = X^i$. Then we have

$$Y_t^1 \leq Y_t^2, \text{ for } t \in [0, T], \text{ a.s.}$$

Proof. Consider the penalization equations for the two constrained BSDE: for $n \in \mathbf{N}$

$$\begin{aligned} y_t^{1,n} &= X^1 + \int_t^T g^{1,n}(s, y_s^{1,n}, z_s^{1,n}) ds - \int_t^T z_s^{1,n} dB_s, \\ y_t^{2,n} &= X^2 + \int_t^T g^{2,n}(s, y_s^{2,n}, z_s^{2,n}) ds - \int_t^T z_s^{2,n} dB_s, \end{aligned} \quad (48)$$

where

$$\begin{aligned} g^{1,n}(t, y, z) &= g^1(t, y, z) + nd_{\Gamma_t^1}(y, z), \\ g^{2,n}(t, y, z) &= g^2(t, y, z) + nd_{\Gamma_t^2}(y, z). \end{aligned}$$

From (47) we have $g^{1,n}(t, y, z) \leq g^{2,n}(t, y, z)$. It follows from the classical comparison theorem of BSDE that $y_t^{1,n} \leq y_t^{2,n}$. While as $n \rightarrow \infty$, $y_t^{1,n} \nearrow y_t^1$ and $y_t^{2,n} \nearrow y_t^2$, where y^1, y^2 are the g_Γ -solutions of the BSDEs respectively. It follows that $y_t^1 \leq y_t^2$, $0 \leq t \leq T$. \square

The comparison theorem is a powerful tool and useful concept in BSDE Theorem (cf. [7]). Here let us recall the main theorem of reflected BSDE and related comparison theorem for the case of lower obstacle L . We do not repeat the case for the upper obstacle since it is essentially the same. This result, obtained in [22], is a generalized version of [6], [8] and [17] for the part of existence, and [11] for the part of comparison theorem.

Theorem 5.5 (Reflected BSDE and related Comparison Theory). *We assume that the coefficient g satisfies Lipschitz condition (1) and the lower obstacle L satisfies (6). Then, for each $X \in \mathbf{L}^2(\mathcal{F}_T)$ with $X \geq L_T$ there exists a unique triple $(y, z, A) \in \mathbf{D}_{\mathcal{F}}^2(0, T) \times \mathbf{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^d) \times \mathbf{A}_{\mathcal{F}}^2(0, T)$, such that*

$$y_t = X + \int_t^T g(s, y_s, z_s) ds + A_T - A_t - \int_t^T z_s dB_s$$

and the generalized Skorokhod reflecting condition is satisfied: for each $L^* \in \mathbf{D}_{\mathcal{F}}^2(0, T)$ such that $y_t \geq L_t^* \geq L_t$, $dP \times dt$ -a.s., we have

$$\int_0^T (y_{s-} - L_{s-}^*) dA_s = 0, \quad a.s.,$$

Moreover, if a coefficient g' an obstacle L' and terminal condition X' satisfy the same condition as g , L and X , respectively, with for $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$X' \leq X, g'(t, y, z) \leq g(t, y, z), L'_t \leq L_t, \quad dP \times dt - a.s..$$

If the triple (y', z', A') is the corresponding reflected solution, then we have

$$Y'_t \leq Y_t, \quad \forall t \in [0, T], \quad a.s..$$

And if $L = L'$, then for each $0 \leq s \leq t \leq T$,

$$A'_t \geq A_t, \quad A'_t - A'_s \geq A_t - A_s.$$

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