Compound Poisson and signed compound Poisson approximations to the Markov binomial law

V. Čekanavičius* and P. Vellaisamy

Department of Mathematics and Informatics, Vilnius University,

Naugarduko 24, Vilnius 03225, Lithuania.

E-mail: vydas.cekanavicius@mif.vu.lt and

and

Department of Mathematics, Indian Institute of Technology Bombay,

Powai, Mumbai- 400076, India.

E-mail: pv@math.iitb.ac.in

Abstract

Compound Poisson distributions and signed compound Poisson measures are used for approximation of the Markov binomial distribution. The upper and lower bound estimates are obtained for the total variation, local and Wasserstein norms. In a special case, asymptotically sharp constants are calculated. For the upper bounds, the smoothing properties of compound Poisson distribution are applied. For the lower bound estimates the characteristic function method is used.

Key words: Markov binomial distribution, compound Poisson approximation, signed compound Poisson measure, geometric distribution, total variation norm, local norm, Wasserstein norm.

MSC 2000 Subject Classification: Primary 60J10; Secondary 60F05.

1 Introduction

The closeness of a compound Poisson (CP) distribution to the Markov binomial (MB) one was investigated in numerous papers, see, for example, [5], [6], [12], [14], [20], [31], [34], [37], and the references therein. Related problems were considered in [4], [7], [8], [13], [18], [19], [30], [33] and [38]. One can expect the MB-CP case to be comprehensively studied. As it turns out, it is not. Many papers deal with the convergence facts only. Only a few of the papers dealing with the estimates of accuracy of approximation have no assumption about the stationarity of the Markov chain.

The aim of this paper is to discuss some compound approximations for non-stationary Markov chain. We show that, for our version of the MB distribution, the natural approximation is a convolution of CP and compound binomial distributions, both having the same compounding geometric law. We outline some principles of construction of asymptotic expansions and consider second-order approximations. Part of the paper is devoted to signed compound Poisson approximations which can be viewed as the second-order expansions in the exponent. We obtain upper and lower bound estimates and show that, under certain conditions, they are of the same order of accuracy. All estimates are proved for the total variation, local and Wasserstein norms. For the upper bound estimates, we employ convolution technique which can be dated back to [23]. For the lower bound estimates, we use the characteristic function method. The methods of proof do not allow for reasonably small absolute constants. However, in special cases, asymptotically sharp constants are calculated.

We now introduce the following notations. Let I_k denote the distribution concentrated at an integer $k \in \mathbb{Z}$, the set of integers, and set $I = I_0$. In what follows, let V and M be two finite signed measures on $\mathbb{Z}$. Products and powers of V and M are understood in the convolution sense, i.e, $VM\{A\} = \sum_{k=-\infty}^{\infty} V\{A-k\}\, M\{k\}$ for a set $A \subseteq \mathbb{Z}$; further $M^0 = I$. The total variation norm, the local norm, and the Wasserstein norm of M are denoted by

$$\|M\| = \sum_{k=-\infty}^{\infty} |M\{k\}|, \qquad \|M\|_\infty = \sup_{k\in\mathbb{Z}} |M\{k\}|, \qquad \|M\|_{\mathrm{W}} = \sum_{k=-\infty}^{\infty} |M\{(-\infty,k]\}|,$$

respectively. Note that $\|(I_1 - I)M\|_{\mathrm{W}} = \|M\|$. The logarithm and exponential of M are given by

$$\ln M = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\,(M-I)^k \quad (\text{if } \|M-I\|<1), \qquad \mathrm{e}^M = \exp\{M\} = \sum_{k=0}^{\infty} \frac{1}{k!}\, M^k.$$

Note that

$$\|VM\|_\infty \leqslant \|V\|\|M\|_\infty, \qquad \|VM\| \leqslant \|V\|\|M\|, \qquad \|\mathrm{e}^M\| \leqslant \mathrm{e}^{\|M\|}.$$

Let $\widehat{M}(t)$ $(t \in \mathbb{R})$ be the Fourier transform of M. We denote by C positive absolute constants. The letter Θ stands for any finite signed measure on $\mathbb{Z}$ satisfying $\|\Theta\| \leqslant 1$. The values of C and Θ can vary from line to line, or even within the same line. Sometimes to avoid possible ambiguity, the C are supplied with indices. For $x \in \mathbb{R}$ and $k \in \mathbb{N} = \{1, 2, 3, \dots\}$, we set

$$\binom{x}{k} = \frac{1}{k!}\, x(x-1)\dots(x-k+1), \qquad \binom{x}{0} = 1.$$

Let $\xi_0, \xi_1, \dots, \xi_n, \dots$ be a Markov chain with the initial distribution

$$\mathrm{P}(\xi_0 = 1) = p_0, \qquad \mathrm{P}(\xi_0 = 0) = 1 - p_0, \qquad p_0 \in [0, 1]$$

and transition probabilities

$$\begin{gathered}
\mathrm{P}(\xi_i = 1 \,|\, \xi_{i-1} = 1) = p, \qquad \mathrm{P}(\xi_i = 0 \,|\, \xi_{i-1} = 1) = q,\\
\mathrm{P}(\xi_i = 1 \,|\, \xi_{i-1} = 0) = \overline{q}, \qquad \mathrm{P}(\xi_i = 0 \,|\, \xi_{i-1} = 0) = \overline{p},\\
p + q = \overline{q} + \overline{p} = 1, \qquad p, \overline{q} \in (0, 1), \qquad i \in \mathbb{N}.
\end{gathered}$$

The distribution of $S_n = \xi_1 + \dots + \xi_n$ $(n \in \mathbb{N})$ is called the Markov binomial distribution. We denote it by F_n. We should note that the definition of the Markov binomial distribution slightly varies from paper to paper, see [12], [30], and [36]. We choose the definition which, on the one hand, contains the binomial distribution as a special case and, on the other hand, allows comparison to the Dobrushin's results. Dobrushin [12] assumed that $p_0 = 1$ and considered $S_{n-1} + 1$.

Further on, we need various characteristics of S_n. Let

$$\begin{aligned}
\gamma_1 &= \frac{q\overline{q}}{q + \overline{q}}, \qquad \gamma_2 = -\frac{q\overline{q}^2}{(q + \overline{q})^2}\left(p + \frac{q}{q + \overline{q}}\right) - \frac{\gamma_1^2}{2},\\
\gamma_3 &= \gamma_1^2 \tilde{\gamma}_3, \quad \tilde{\gamma}_3 = \frac{\gamma_1}{3} + \frac{1}{q(q + \overline{q})}\left\{p^2\overline{q} + \frac{pq(2\overline{q} - q)}{q + \overline{q}} + \frac{2\overline{q}q^2}{(q + \overline{q})^2}\right\} + \frac{\overline{q}}{q + \overline{q}}\left(p + \frac{q}{q + \overline{q}}\right),\\
\lambda &= n - p_0, \quad \varkappa_1 = \gamma_1\left(\frac{\overline{q} - p}{q + \overline{q}} - p_0\right), \quad \varkappa_2 = p_0 \frac{pq}{q + \overline{q}}, \quad C_1 = \ln\frac{30}{19} = 0.4567\dots
\end{aligned}$$

We use the following measures also:

$$\begin{aligned}
G &= qI_1 \sum_{j=0}^{\infty} p^j I_j \quad \left(\widehat{G}(t) = \frac{q\mathrm{e}^{\mathrm{i}t}}{1 - p\mathrm{e}^{\mathrm{i}t}}\right), \quad H = I + \varkappa_2(G - I),\\
H_1 &= (1 - \gamma_1)I + \gamma_1 G, \quad H_1^\lambda = \exp\Big\{\lambda \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \gamma_1^j (G - I)^j\Big\}, \quad \Big(\widehat{H}_1^\lambda(t) = (\widehat{H}_1(t))^\lambda\Big),\\
D_j &= \exp\{\sum_{i=1}^{j} \gamma_i (G - I)^i\}, \quad 1 \le j \le 3, \quad D_1^\lambda = \exp\{\lambda\gamma_1(G - I)\}.
\end{aligned}$$

2 Known results

In this section, we discuss some of the known results on the compound Poisson approximations to the MB distribution. Many papers deal with the convergence facts only, see, for example, [14], [19], [20], [36]. Usually, the chain is assumed to be stationary. Typical example is Theorem 4.1 in [31, p. 1397] which states that if $p_0 = \overline{q}/(q+\overline{q})$ and $\tilde{\alpha} > 0$, then

$$\|F_n - \exp\{\tilde{\alpha}(G-I)\}\| \leqslant 2|n\overline{q} - \tilde{\alpha}| + \frac{2\overline{q}(1+p+n\overline{q}(2-p))}{q+\overline{q}}. \tag{1}$$

Even if we choose $\tilde{\alpha} = n\overline{q}$, the order of accuracy in (1) is not better than $n\overline{q}^2$. Similar estimate was obtained in Theorem 5 [37, p. 865]. If we use terminology of the book [2], we can say that estimate (1) contains no 'magic' factor. If we turn to the papers with 'magic' factors, then we have the following results. In [5] it was proved that, if $0 \leqslant p \leqslant C_0 < 1$, then

$$\|F_n - D_1^n\| \leqslant C\max(p_0, \overline{q})\min\Big(1, \frac{1}{\sqrt{n\overline{q}}}\Big) + C\min(\overline{q}, n\overline{q}^2) + C\mathrm{e}^{-Cn}. \tag{2}$$

The accuracy can be improved by some asymptotic expansions to

$$\begin{aligned}\Big\|F_n - D_1^n\Big(I + p_0\frac{q^2(p-\overline{q})}{(q+\overline{q})^2}I_1\Big)\Big\| \;&\leqslant\; C\overline{q}(p+\overline{q})\min\Big(1, \frac{1}{\sqrt{n\overline{q}}}\Big)\\ &\quad + C\min(\overline{q}, n\overline{q}^2) + C\mathrm{e}^{-Cn}.\end{aligned} \tag{3}$$

Note that in [5] formulas (4.5), (4.12) and (4.23) contain misprints. The parameter p_0 is misplaced and should be in the brackets. If $p \to \tilde{p} = Const$, $\overline{q} \to 0$ and $n\overline{q} \to \infty$, then the order of accuracy in (2) is $\max(\overline{q}, (n\overline{q})^{-1/2})$. Meanwhile, the order of accuracy in (3) is $\overline{q}$. We can hardly call (3) the second-order expansion, since the improvement of the accuracy was achieved due to the more precise approximation of the initial distribution of ξ_0 only.

The main idea of signed CP approximations is to leave more than one factorial cumulant in the exponent. In short, the signed CP measure is of the same structure as the CP one, but can have negative Poisson parameters. Such approximations are commonly used in insurance models and in limit theorems, see [1], [10], [17], [21], [22], [25], [28], and the references therein. For the MB distribution in [5] the following result is proved. If

$$p \leqslant \tilde{C} < 1, \quad \frac{\overline{q}}{q+\overline{q}} \leqslant \frac{1-\tilde{C}}{30}, \tag{4}$$

then

$$\|F_n - D_2^n\| \leqslant C(p+\overline{q})\Big\{\min\Big(\sqrt{\frac{\overline{q}}{n}}, n\overline{q}^2\Big) + \max(p_0, \overline{q})\min\Big(1, \frac{1}{\sqrt{n\overline{q}}}\Big) + \mathrm{e}^{-Cn}\Big\}. \tag{5}$$

Note that $\gamma_2 < 0$ and, therefore, D_2 is a signed measure rather than a distribution. As a rule, the signed CP approximations are more accurate than CP ones. Indeed, if $n \to \infty$, then (5) gives the estimate converging to zero, even if p and $\overline{q}$ are positive constants. Meanwhile, (2) and (3) are nontrivial if $\overline{q} = o(1)$ only.

We are unaware about any lower bound estimate for the compound Poisson approximation to the Markov binomial distribution.

3 The Main Results

3.1 Geometric expansions

Before formulating our results, it is necessary to explain the choice of approximating measures. Dobrushin [12] proved that, if $p \to \tilde{p}$, $n\overline{q} \to \tilde{\lambda}$ and $p_0 = 1$, then the limit distribution for $1+S_{n-1}$ is the convolution $G_1 \exp\{\tilde{\lambda}(G_1 - I)\}$, where G_1 is a geometric distribution with parameter $\tilde{p}$, that is, $\widehat{G}_1(t) = (1-\tilde{p})\mathrm{e}^{\mathrm{i}t}/(1-\tilde{p}\mathrm{e}^{\mathrm{i}t})$. This gives us an idea that approximation of S_n for arbitrary p_0 also should be based on the expansions in powers of $G - I$.

Let F be concentrated on $\mathbb{Z}$ and have all moments finite. We can write formally (that is, without investigating conditions needed for the convergence of series):

$$\widehat{F}(t) = 1 + \sum_{j=1}^{\infty} \frac{\nu_j}{j!}(\mathrm{e}^{\mathrm{i}t} - 1)^j = 1 + \sum_{m=1}^{\infty} \frac{\tilde{\nu}_m}{m!}(\widehat{G}(t) - 1)^m.$$

Here ν_j $(j = 1, 2, \dots)$ are factorial moments of F, and $\tilde{\nu}_m$ can be called the geometric factorial moments. Since,

$$\mathrm{e}^{\mathrm{i}t} - 1 = \frac{q(\widehat{G}(t) - 1)}{1 + p(\widehat{G}(t) - 1)},$$

it is not difficult to establish relation between ν_j and $\tilde{\nu}_m$:

$$\frac{\tilde{\nu}_m}{m!} = (-p)^m \sum_{j=1}^{m} \frac{\nu_j}{j!}\Big(-\frac{q}{p}\Big)^j \binom{m-1}{m-j}, \quad m = 1, 2, \dots . \tag{6}$$

Similar relation holds for factorial cumulants and geometric factorial cumulants. For the MB distribution, we have

$$\tilde{\nu}_1 = q\nu_1 = q\mathrm{E}S_n = n\gamma_1 + \varkappa_1 + \varkappa_2 - (\varkappa_1 + \varkappa_2)(p - \overline{q})^n. \tag{7}$$

(For the formula of the mean, see [7]). Since we will assume the smallness of p and $\overline{q}$, the last summand in (7) will be neglected. As it turns out, F_n is close to some convolution $W_1\Lambda_1^n$, see (32) below. We use (6) for choosing approximating measure for W_1. Cumulant analogue of (6) is used for Λ_1^n.

3.2 Compound Poisson approximation

In this paper, we usually assume that

$$p \leqslant \frac{1}{2}, \quad \overline{q} \leqslant \frac{1}{30}. \tag{8}$$

The smallness of the absolute constants is determined by the method of proof. We expand F_n as series of convolutions of measures. The remainder term is usually estimated by series containing powers of $\overline{q}+p$. If the sum $\overline{q}+p$ is sufficiently small, series converges. Thus, though we have some freedom in the choice of magnitude of p and $\overline{q}$, the sum $p+\overline{q}$ must be small. The choice of condition (8) is determined by the fact that CP limit occurs, when $n\overline{q} \to \tilde{\lambda}$, see, for example, [12, Table 1]. Therefore, if we expect CP approximation to be accurate, then $\overline{q}$ should be small. On the other hand, we have included the case $\overline{q} = Const$, that is, the case which is usually associated with the normal approximation. We choose the assumption $p \leqslant 1/2$ instead of (4) just to make our proofs more clear.

Theorem 3.1 *Let $p \leqslant 1/2$. Then*

$$\|F_n - HD_1^{\lambda}\| \leqslant C\overline{q}(p+\overline{q})\min\Big(1, \frac{1}{\sqrt{n\overline{q}}}\Big) + C\min(\overline{q}, n\overline{q}^2) + C(p+\overline{q})\mathrm{e}^{-C_1 n}. \tag{9}$$

If, in addition, $\overline{q} \leqslant 1/30$, then

$$\begin{aligned} \|F_n - HD_1^{\lambda}\|_\infty &\leqslant C\overline{q}(p+\overline{q})\min\Big(1, \frac{1}{n\overline{q}}\Big) + C\min\Big(\sqrt{\frac{\overline{q}}{n}}, n\overline{q}^2\Big) \\ &\quad + C(p+\overline{q})\mathrm{e}^{-C_1 n}, \qquad (10) \\ \|F_n - HD_1^{\lambda}\|_{\mathrm{W}} &\leqslant C\overline{q}(p+\overline{q}) + C\min(\overline{q}\sqrt{n\overline{q}}, n\overline{q}^2) + C(p+\overline{q})\mathrm{e}^{-C_1 n}. \qquad (11) \end{aligned}$$

Corollary 3.1 *If (8) is satisfied and $n\overline{q} \geqslant 1$, then*

$$\|F_n - HD_1^{n}\| \leqslant C\overline{q}, \quad \|F_n - HD_1^{n}\|_\infty \leqslant C\sqrt{\frac{\overline{q}}{n}}, \quad \|F_n - HD_1^{n}\|_{\mathrm{W}} \leqslant C\overline{q}\sqrt{n\overline{q}}.$$

Corollary 3.2 *If (8) is satisfied, then*

$$\|F_n - HH_1^{\lambda}\| \leqslant C(\overline{q} + p\mathrm{e}^{-C_1 n}).$$

Remarks 3.1.

(i) H is compound Poisson distribution, see Lemma 5.3 below. If $\overline{q} \leqslant p$, then H_1^{λ} is also a CP distribution. Thus, we see that there exist quite different forms of CP approximations with similar order of accuracy.

(ii) The estimate (9) is slightly better than (3) for $p, \overline{q} \leqslant \exp\{-C_1 n\}$, and more accurate than (2) for $\overline{q} \leqslant 1/\sqrt{n\overline{q}}$, and $p_0 \geqslant C$.

(iii) For the closeness of F_n and $H_1 D_1^{\lambda}$, it suffices to assume $\overline{q} \to 0$, much in contrast to $n\overline{q} \to \tilde{\lambda}$ which is needed for the convergence to the limit CP law.

(iv) We can write $\min(\overline{q}, n\overline{q}^2) = n\overline{q}^2\min(1, (n\overline{q})^{-1})$. The last factor, in terms of [2, p. 5], can be called the 'magic' factor.

The accuracy of approximation can be improved by the second-order expansion.

Theorem 3.2 *If $p \leqslant 1/2$, then*

$$\|F_n - HD_1^{\lambda}(I + n\gamma_2(G-I)^2)\| \leqslant C\Big\{\overline{q}^2 + p\overline{q}\min\Big(1, \frac{1}{\sqrt{n\overline{q}}}\Big) + (p+\overline{q})\mathrm{e}^{-C_1 n}\Big\}.$$

If, in addition, $\overline{q} \leqslant 1/30$, then

$$\begin{aligned}
&\|F_n - HD_1^{\lambda}(I + n\gamma_2(G-I)^2)\|_\infty \\
&\qquad \leqslant C\Big\{\overline{q}^2\min\Big(1, \frac{1}{\sqrt{n\overline{q}}}\Big) + p\overline{q}\min\Big(1, \frac{1}{n\overline{q}}\Big) + (p+\overline{q})\mathrm{e}^{-C_1 n}\Big\}, \\
&\|F_n - HD_1^{\lambda}(I + n\gamma_2(G-I)^2)\|_{\mathrm{W}} \\
&\qquad \leqslant C\{\overline{q}^2\max(1, \sqrt{n\overline{q}}) + p\overline{q} + (p+\overline{q})\mathrm{e}^{-C_1 n}\}.
\end{aligned}$$

Note that the last estimate contains $\max(1, \sqrt{n\overline{q}})$, reflecting the fact that the estimates for the Wasserstein distance are less accurate than the ones for the total variation norm. It is even more evident when $n\overline{q} \geqslant 1$.

Corollary 3.3 *If (8) is satisfied and $n\overline{q} \geqslant 1$, then the estimates in Theorem 3.2 are*

$$C\overline{q}\Big(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\Big), \quad C\sqrt{\frac{\overline{q}}{n}}\Big(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\Big), \quad C\overline{q}\sqrt{n\overline{q}}\Big(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\Big),$$

respectively.

We see that, in general, even the second order estimates in total variation are meaningful for $\overline{q} = o(1)$ only.

3.3 Signed Compound Poisson approximations

The choice of the signed CP approximations, in general, means that the first member of asymptotic expansion, unlike Theorem 3.2, is omitted in the exponent.

Theorem 3.3 *If condition (8) is satisfied, then*

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\| \leqslant C(p+\overline{q})\Big\{\min\Big(\overline{q}, \sqrt{\frac{\overline{q}}{n}}\Big) + \mathrm{e}^{-C_1 n}\Big\}, \tag{12}$$

$$\begin{aligned}
\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_\infty &\leqslant C(p+\overline{q})\Big\{\min\Big(\overline{q}, \frac{1}{n}\Big) + \mathrm{e}^{-C_1 n}\Big\}, \\
\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_{\mathrm{W}} &\leqslant C(p+\overline{q})\{\overline{q} + \mathrm{e}^{-C_1 n}\}.
\end{aligned}$$

Note that, for $n\overline{q} \leqslant 1$ and $p_0 = Const$, (12) is more accurate that (5). More importantly, when $p = Const$ and $\overline{q} = Const$, the estimate (12) is of order $O(n^{-1/2})$. In this sense, the signed CP approximation is comparable to the normal one and moreover, it holds in the total variation metric. Meanwhile, for discrete distributions, the normal approximation holds in the uniform metric only. Just like in the CP case, the second order expansions can be used.

Theorem 3.4 *If (8) holds, then*

$$\begin{aligned}
&\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n(I+n\gamma_3(G-I)^3)\| \\
&\qquad\qquad\leqslant\ C(p+\overline{q})\Big\{\min\Big(\overline{q},\frac{1}{n}\Big)+\mathrm{e}^{-C_1 n}\Big\}, \qquad (13)\\
&\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n(I+n\gamma_3(G-I)^3)\|_\infty \\
&\qquad\qquad\leqslant\ C(p+\overline{q})\Big\{\min\Big(\overline{q},\frac{1}{n\sqrt{n\overline{q}}}\Big)+\mathrm{e}^{-C_1 n}\Big\}, \qquad (14)\\
&\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n(I+n\gamma_3(G-I)^3)\|_{\mathrm{W}} \\
&\qquad\qquad\leqslant\ C(p+\overline{q})\Big\{\min\Big(\overline{q},\sqrt{\frac{\overline{q}}{n}}\Big)+\mathrm{e}^{-C_1 n}\Big\}. \qquad (15)
\end{aligned}$$

Corollary 3.4 *Let $n\overline{q}\geqslant 1$. Then the estimates (13)-(15) are at least of order*

$$\frac{C(p+\overline{q})}{n},\quad \frac{C(p+\overline{q})}{n\sqrt{n\overline{q}}},\quad \frac{C(p+\overline{q})\sqrt{\overline{q}}}{\sqrt{n}},$$

respectively.

In Theorem 3.4, only a part of asymptotic expansion is in the exponent. Therefore, the following natural question arises. Is it possible to find a signed CP measure which up to a constant provides the same accuracy as in Theorem 3.4? As it follows from the following result, such a measure indeed exists.

Theorem 3.5 *If (8) holds, then*

$$\begin{aligned}
\|F_n - H\exp\{\varkappa_1(G-I)\}D_3^n\| &\leqslant C(p+\overline{q})\Big\{\min\Big(\overline{q},\frac{1}{n}\Big)+\mathrm{e}^{-C_1 n}\Big\},\\
\|F_n - H\exp\{\varkappa_1(G-I)\}D_3^n\|_\infty &\leqslant C(p+\overline{q})\Big\{\min\Big(\overline{q},\frac{1}{n\sqrt{n\overline{q}}}\Big)+\mathrm{e}^{-C_1 n}\Big\},\\
\|F_n - H\exp\{\varkappa_1(G-I)\}D_3^n\|_{\mathrm{W}} &\leqslant C(p+\overline{q})\Big\{\min\Big(\overline{q},\sqrt{\frac{\overline{q}}{n}}\Big)+\mathrm{e}^{-C_1 n}\Big\}.
\end{aligned}$$

3.4 Lower bound estimates

In this section we show that, in some cases, the estimates in Theorems 3.1 and 3.3 are of the right order. We concentrate our attention on the case $n\overline{q}\geqslant 1$.

Theorem 3.6 *Let condition (8) be satisfied and let $n\overline{q}\geqslant 1$. Then, for some absolute positive constants C_2 and C_3:*

$$\|F_n - HD_1^\lambda\| \geqslant C_2\overline{q}\Big(1-C_3\Big(\overline{q}+\frac{p}{\sqrt{n\overline{q}}}\Big)\Big), \qquad (16)$$

$$\|F_n - HD_1^\lambda\|_\infty \geqslant C_2\sqrt{\frac{\overline{q}}{n}}\Big(1-C_3\Big(\overline{q}+\frac{p}{\sqrt{n\overline{q}}}\Big)\Big), \qquad (17)$$

$$\|F_n - HD_1^\lambda\|_{\mathrm{W}} \geqslant C_2\overline{q}\sqrt{n\overline{q}}\Big(1-C_3\Big(\overline{q}+\frac{p}{\sqrt{n\overline{q}}}\Big)\Big). \qquad (18)$$

It is obvious, that estimates (16)–(18) are non-trivial only when expression in the brackets is positive. Let $p \leqslant 1/2$, $n\overline{q} \to \infty$ and $\overline{q} \to 0$. Combining Theorems 3.1 and 3.6, for sufficiently large n, we obtain

$$\begin{aligned} C_4\overline{q} \leqslant \quad & \|F_n - HD_1^{\lambda}\| & \leqslant C_5\overline{q}, \\ C_4\sqrt{\frac{\overline{q}}{n}} \leqslant \quad & \|F_n - HD_1^{\lambda}\|_\infty & \leqslant C_5\sqrt{\frac{\overline{q}}{n}}, \\ C_4\overline{q}\sqrt{n\overline{q}} \leqslant \quad & \|F_n - HD_1^{\lambda}\|_{\mathrm{W}} & \leqslant C_5\overline{q}\sqrt{n\overline{q}}. \end{aligned}$$

Of course, the last estimate, as well as the one in (18), is of interest if $\overline{q}\sqrt{n\overline{q}} \to 0$, only. Similar results can be obtained for the signed CP approximations.

Theorem 3.7 *Let condition (8) be satisfied and let $n\overline{q} \geqslant 1$. Then, for some absolute positive constants C_6 and C_7:*

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\| \geqslant C_6\sqrt{\frac{\overline{q}}{n}}\Big(|\tilde{\gamma}_3| - C_8\frac{p+\overline{q}}{\sqrt{n\overline{q}}}\Big), \tag{19}$$

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_\infty \geqslant \frac{C_6}{n}\Big(|\tilde{\gamma}_3| - C_8\frac{p+\overline{q}}{\sqrt{n\overline{q}}}\Big), \tag{20}$$

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_{\mathrm{W}} \geqslant C_6\overline{q}\Big(|\tilde{\gamma}_3| - C_8\frac{p+\overline{q}}{\sqrt{n\overline{q}}}\Big). \tag{21}$$

Let $n\overline{q} \to \infty$, as $\overline{q} \to 0$ and $p \to \tilde{p}$. Also, assume n is sufficiently large so that the right-hand estimates of (19)-(21) are positive. Then we have,

$$\begin{aligned} C_8\sqrt{\frac{\overline{q}}{n}} \leqslant \quad & \|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\| & \leqslant C_9\sqrt{\frac{\overline{q}}{n}}, \\ \frac{C_8}{n} \leqslant \quad & \|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_\infty & \leqslant \frac{C_9}{n}, \\ C_8\overline{q} \leqslant \quad & \|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_{\mathrm{W}} & \leqslant C_9\overline{q}. \end{aligned}$$

3.5 Asymptotically sharp constants

In the previous section, we proved that upper and lower bound estimates are of the same order, provided $n\overline{q}$ is large and $\overline{q}$ is small. As it turns out, if in addition p is small, then it is possible to obtain asymptotically sharp constants.

Theorem 3.8 *Let $p \leqslant 1/4$, $\overline{q} \leqslant 1/30$, $n\overline{q} \geqslant 1$. Then*

$$\Big|\|F_n - HD_1^{\lambda}\| - A_{11}\Big| \leqslant C\overline{q}\Big(p + \overline{q} + \frac{1}{\sqrt{n\overline{q}}}\Big), \tag{22}$$

$$\Big|\|F_n - HD_1^{\lambda}\|_\infty - A_{12}\Big| \leqslant C\sqrt{\frac{\overline{q}}{n}}\Big(p + \overline{q} + \frac{1}{\sqrt{n\overline{q}}}\Big), \tag{23}$$

$$\Big|\|F_n - HD_1^{\lambda}\|_{\mathrm{W}} - A_{13}\Big| \leqslant C\overline{q}\sqrt{n\overline{q}}\Big(p + \overline{q} + \frac{1}{\sqrt{n\overline{q}}}\Big), \tag{24}$$

where

$$A_{11} = \frac{4|\gamma_2|}{\gamma_1 q\sqrt{2\pi\mathrm{e}}}, \quad A_{12} = \frac{|\gamma_2|}{\gamma_1\sqrt{\gamma_1}\sqrt{2\pi nq}}, \quad A_{13} = \frac{|\gamma_2|\sqrt{2n}}{q\sqrt{\gamma_1\pi q}}.$$

As a consequence of (22), we note that if $p \to 0$, $\overline{q} \to 0$, and $n\overline{q} \to \infty$, then

$$\|F_n - HD_1^{\lambda}\| \sim \frac{6\overline{q}}{\sqrt{2\pi \mathrm{e}}}.$$

Similar relations can be obtained for the local and Wasserstein norms as well.

4 Applications of Markov Binomial Models

In this section, we discuss some areas where the results of our paper can be applied.

(i) Aggregate Claim Distribution in the Individual Model. Consider a portfolio of n risks. Each risk produces a positive claim amount during a certain reference period. Then the aggregate claim of the portfolio is

$$S^{ind} = X_1 + X_2 + \cdots + X_n.$$

It is usually assumed that all X_j are independent. However, the independence of claims does not always reflect reality. For example, an accident to a tourist group, life insurance of husband and wife, pensions for workers of the same company etc. are likely to produce dependent risks. For discussion on the dependence of risks and further examples, see [16], [9], [26].

Compound Poisson and signed compound Poisson approximations in the independent case of individual model are quite thoroughly investigated, see, for example, [15], [17]. On the other hand, there are only a few results for the total variation metric for dependent risks. Dhaene and Goovaerts [9] investigated similar model (though not explicitly Markovian) under assumption, which in our notation is equivalent to $\overline{q}_m = 0$. However, under such assumption one can not expect the limiting law to be compound Poisson. Therefore, we excluded this bordering case from our paper, assuming $\overline{q}$ to be small, but not identically zero. In [16], Poisson approximation in the general setting of dependent risks was discussed. However, in our case, their result is not aplicable, since for small $\overline{q}$ the distribution of approximated sum is not close to the Poisson distribution but rather to compound Poisson law.

Let us assume that aggregated claim amount S^{ind} of the portfolio consists of N independent groups of risks. We assume homogeneous model for each group of risks with the Markovian dependence. Let each risk to have a two-point distribution. More precisely, let

$$S^{ind} = \sum_{m=1}^{N} \sum_{j=1}^{n_m} X_j^m.$$

Here X_j^m and X_k^l are independent if $m \neq l$. We assume that each risk of the mth group can produce a claim of the size a_m. Moreover, the dependence of risks of the same group is

Markovian: $\mathrm{P}(X_1^m = a_m) = \overline{q}_m$, $\mathrm{P}(X_1^m = 0) = \overline{p}_m$ and

$$\mathrm{P}(X_j^m = a_m \,|\, X_{j-1}^m = 0) = \overline{q}_m, \qquad \mathrm{P}(X_j^m = 0 \,|\, X_{j-1}^m = 0) = \overline{p}_m,$$
$$\mathrm{P}(X_j^m = a_m \,|\, X_{j-1}^m = a_m) = p_m < 1/2, \qquad \mathrm{P}(X_j^m = 0 \,|\, X_{j-1}^m = a_m) = q_m,$$
$$p_m + q_m = \overline{q}_m + \overline{p}_m = 1, \qquad p_m, \overline{q}_m \in (0,1), \quad m = 1,2,\dots,N,\ j = 2,\dots,n_m.$$

The results of previous sections can now easily be applied. We illustrate this by just one example. Let us define a compound Poisson variable in the following way:

$$S^{cp} = \sum_{m=1}^{N} a_m \sum_{j=0}^{N_m} Y_{jm}.$$

Here, Y_{jm} are i.i.d. geometric random variables, $\mathrm{P}(Y_{jm} = k) = q_m p_m^{k-1}$, $k = 1,2,\dots$ and N_m is a Poisson random variables with parameter $n_m q_m \overline{q}_m/(q_m + \overline{q}_m)$. The random variables N_m, $m = 1,2,\dots,N$ are independent and also do not depend on Y_{jm}. Denote the distributions of S^{ind} and S^{cP} by F^{ind} and F^{cp}, respectively. Then the characteristic function of F^{cp} is given by

$$\widehat{F}^{cp}(t) = \exp\Big\{\sum_{m=1}^{N} \frac{n_m q_m \overline{q}_m(\mathrm{e}^{\mathrm{i}ta_m} - 1)}{(q_m + \overline{q}_m)(1 - p_m \mathrm{e}^{\mathrm{i}ta_m})}\Big\}.$$

Also, we have the following estimate of approximation:

$$\begin{aligned} \|F^{ind} - F^{cp}\| \;&\leqslant\; C\sum_{m=1}^{N} \big[\overline{q}_m(p_m + \overline{q}_m)\min(1, (n_m\overline{q}_m)^{-1/2}) \\ &\qquad + \min(\overline{q}_m, n_m\overline{q}_m^2) + (p_m + \overline{q}_m)\mathrm{e}^{-C_1 n_m}\big]. \end{aligned} \tag{25}$$

Note that the approximation is closer, if all $\overline{q}_m$ are small.

For the proof of (25), one should use the triangle inequality, thus reducing problem to N estimates of Markov binomial distributions concentrated on $0, a_m, 2a_m, \dots$. The total variation metric is invariant with respect to norming. Therefore, without loss of generality, one can switch to integer numbers and N times apply (9) with $p_0 = 0$.

It is obvious that the second order estimates and estimates in Wasserstein metric can be obtained in a similar way.

(ii) System Failure Models. Markov binomial distribution naturally arises in weather and stock market trends. It is also a natural model for system failure situations. As an example, we present one model from Sahinoglu [29], who considered electric power supply system with operating and non-operating states throughout a yearly long period of operation discretized in hours. Let M_i be the margin values at hourly steps, that is

$$M_i = TPG - X - L_i,$$

where TPG denotes total power generation, L_i denotes power demand (hourly peak load forecast) and X denotes unplanned forced outages. Let Y_i be an indicator of $\{M_i < 0\}$. Then $S = Y_1 + Y_2 + \cdots + Y_n$ is cumulated hours of negative-margin hours, *i.e.,* the unavailability of power at n-th hour. It is natural to assume that S has a Markov-binomial distribution. Notably, the Markov chain $Y_1, Y_2, \dots$ is non-stationary. Therefore, many known results about the compound Poisson approximation can not be applied directly. Meanwhile the results of our paper relax assumptions on transition probabilities of Sahinoglu model and give estimates of the accuracy of approximations. Further, as shown in Sahinoglu [29, p.49], the probabilities of the compounding geometric law under certain assumptions can be viewed as probabilities for number of trials required to repair system.

(iii) Industrial Applications: Sampling Plans. A basic assumption in standard acceptance plans for attributes is that the characteristics of items in the lots are iid Bernoulli variables. However, recently the focus is on monitoring the ongoing production process, by inspecting the items sequentially. In such cases, the quality of successive items are statistically dependent and it has been found in practice that Markov-dependent model is a very useful one, see [24]. Indeed, Bhat et. al. [3](1990) modified the standard acceptance sampling plans and proposed sequential single sampling plans for monitoring Markov-dependent production process. Vellaisamy and Sankar [35] proposed optimal systematic sampling plans for Markov-dependent process. Our paper outlines some possible new research directions in this field.

5 Auxiliary results

We introduce further the following notations:

$$a_1 = \gamma_1, \quad a_2 = \gamma_2 + \frac{a_1^2}{2}, \quad a_3 = \gamma_3 + a_1 a_2 - \frac{a_1^3}{3}, \tag{26}$$

$$Y = G - I, \quad B = \sum_{j=0}^{\infty} (pI_1 - \overline{q}I)^j, \quad K = \sum_{j=0}^{\infty} (pI_1 - \overline{q}I - 2\gamma_1 Y)^j, \tag{27}$$

$$L = \frac{4\overline{q}^2}{(q+\overline{q})^2} Y^2 \Big[q^2 I + p(q+\overline{q})(I - pI_1)\Big] K^2. \tag{28}$$

In the following two lemmas, $C(k)$ denotes an absolute positive constant depending on k. Throughout this paper, we set $0^0 = 1$.

Lemma 5.1 *Let $t > 0$, $k \in \{0, 1, \cdots\}$ and $0 < p < 1$. Also, let M be a finite (signed) measure concentrated at $\mathbb{Z}$. Then, for Y defined in (27),*

$$\|Y^2 \mathrm{e}^{tY}\| \leqslant \frac{3}{t\mathrm{e}}, \qquad \|Y^k \mathrm{e}^{tY}\| \leqslant \Big(\frac{2k}{t\mathrm{e}}\Big)^{k/2}. \tag{29}$$

If $p \leqslant 1/2$, then

$$\|Y^k \mathrm{e}^{tY}\|_\infty \leqslant \frac{C(k)}{t^{(k+1)/2}}, \quad \|YM\|_{\mathrm{W}} \geqslant \frac{2}{3}\|M\|, \quad \|YM\| \geqslant \frac{2}{3}\|(I_1 - I)M\|. \tag{30}$$

Proof. Estimates in (29) follow from the properties of the total variation norm and results in [27] and [11]. The first estimate in (30) is a consequence of the formula of inversion and inequalities:

$$Re\widehat{Y}(t) \leqslant -\frac{2}{1+p}\sin^2\frac{t}{2}, \quad |\widehat{Y}(t)| \leqslant \frac{2}{q}\Big|\sin\frac{t}{2}\Big|.$$

Here $Re\{\cdot\}$ means the real part of the complex number. In view of the relation between total variation and Wasserstein norms (see Introduction), we get

$$\begin{aligned}
&\|MY\|_{\mathrm{W}} = \|(I_1 - I)M\sum_{j=0}^{\infty} p^j I_j\|_{\mathrm{W}} = \|M\sum_{j=0}^{\infty} p^j I_j\|,\\
&\|M(I_1 - I)\| = \|MY(I - pI_1)\| \leqslant \|MY\|(1+p),\\
&\|M\| = \|M\sum_{j=0}^{\infty} p^j I_j(I - pI_1)\| \leqslant \|M\sum_{j=0}^{\infty} p^j I_j\|(1+p).
\end{aligned}$$

The results in (30) now follow easily.

For our asymptotically sharp results, we need the following lemma. Set

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\,\mathrm{e}^{-x^2/2}, \quad \|\varphi_k\|_1 = \int_{\mathbb{R}} |\varphi_k(x)|\,\mathrm{d}x, \quad \|\varphi_k\|_\infty = \sup_{x\in\mathbb{R}} |\varphi_k(x)| \quad (k = 0, 1, \dots).$$

Lemma 5.2 *Let $t > 0$ and $k = 0, 1, 2 \dots$. Then we have*

$$\begin{aligned}
\Big|\|(I_1 - I)^k \mathrm{e}^{t(I_1-I)}\| - \frac{\|\varphi_k\|_1}{t^{k/2}}\Big| &\leqslant \frac{C(k)}{t^{(k+1)/2}},\\
\Big|\|(I_1 - I)^k \mathrm{e}^{t(I_1-I)}\|_\infty - \frac{\|\varphi_k\|_\infty}{t^{(k+1)/2}}\Big| &\leqslant \frac{C(k)}{t^{k/2+1}},\\
\Big|\|(I_1 - I)^k \mathrm{e}^{t(I_1-I)}\|_{\mathrm{W}} - \frac{\|\varphi_{k-1}\|_1}{t^{(k-1)/2}}\Big| &\leqslant \frac{C(k)}{t^{k/2}} \qquad (k \neq 0).
\end{aligned}$$

The proof follows from a more general Proposition 4 in [28].

Lemma 5.3 *Let $N > 0$, $0 < \alpha \leqslant p < 1$. Then $(I + \alpha Y)^N$ is a CP distribution.*

Proof. Note that

$$(I + \alpha Y)^N = \exp\Big\{-N\ln(1-\alpha)(F - I)\Big\}.$$

Here F is a distribution concentrated on $\{1, 2, 3, \dots\}$ with

$$F\{j\} = -\frac{1}{\ln(1-\alpha)}\frac{1}{j}\bigg(p^j - \Big(\frac{p-\alpha}{1-\alpha}\Big)^j\bigg).$$

The last relation obviously completes the proof.

Before we proceed to our main lemma, we need some additional facts about F_n. Similarly to [5] (see also [7]), it is possible to check that under assumption (8)

$$\widehat{F}_n(t) = \widehat{\Lambda}_1^n(t)\,\widehat{W}_1(t) + \widehat{\Lambda}_2^n(t)\,\widehat{W}_2(t), \tag{31}$$

where

$$\begin{aligned}
\widehat{\Lambda}_{1,2}(t) &= \frac{p\mathrm{e}^{\mathrm{i}t}+\overline{p}\pm\widehat{D}^{1/2}(t)}{2},\\
\widehat{W}_{1,2}(t) &= \frac{p_0}{2}\Big(1\pm\frac{q+\overline{q}+p(\mathrm{e}^{\mathrm{i}t}-1)}{\widehat{D}^{1/2}(t)}\Big)+\frac{1-p_0}{2}\Big(1\pm\frac{q+\overline{q}+(2\overline{q}-p)(\mathrm{e}^{\mathrm{i}t}-1)}{\widehat{D}^{1/2}(t)}\Big),\\
\widehat{D}(t) &= (p\mathrm{e}^{\mathrm{i}t}+\overline{p})^2+4\mathrm{e}^{\mathrm{i}t}(\overline{q}-p).
\end{aligned}$$

This allows us to write F_n as

$$F_n=\Lambda_1^nW_1+\Lambda_2^nW_2 \tag{32}$$

and to express $\Lambda_{1,2}$ and $W_{1,2}$ as the following series:

$$\Lambda_1 = I+a_1Y+\frac{1}{2}\Big\{(1+\overline{q})I-pI_1+2a_1Y\Big\}\sum_{j=1}^{\infty}\binom{1/2}{j}(-1)^jL^j, \tag{33}$$

$$\Lambda_2 = pI_1-\overline{q}I+(I-\Lambda_1), \tag{34}$$

$$\begin{aligned}
W_{1,2} &= \frac{1}{2}\Big\{I\pm[(q+\overline{q})I+p(I_1-I)]K\sum_{j=0}^{\infty}\binom{-1/2}{j}(-1)^jL^j\Big\}\\
&\quad\pm(1-p_0)(\overline{q}-p)(I_1-I)K\sum_{j=0}^{\infty}\binom{-1/2}{j}(-1)^jL^j.
\end{aligned} \tag{35}$$

The following lemma is used as the main tool in the proofs.

Lemma 5.4 *Let condition* (8) *be satisfied. Then*

$$\Lambda_1 = I+\sum_{j=1}^{3}a_jY^j+C\overline{q}^3(p+\overline{q})Y^4\Theta, \tag{36}$$

$$\ln\Lambda_1 = \sum_{j=1}^{3}\gamma_jY^j+C\overline{q}^3(p+\overline{q})Y^4\Theta, \tag{37}$$

$$\ln\Lambda_1 = \gamma_1Y+\frac{19}{60}\gamma_1Y^2\Theta, \tag{38}$$

$$\|\Lambda_2\| \leqslant \frac{19}{30},\quad \|\Lambda_1-I\|\leqslant 0.1, \tag{39}$$

$$W_1 = I+(\varkappa_1+\varkappa_2)Y+C\overline{q}(p+\overline{q})Y^2\Theta, \tag{40}$$

$$W_2 = C(p+\overline{q})(I_1-I)\Theta,\qquad \|W_2\|\leqslant 7. \tag{41}$$

For any finite signed measure M on $\mathbb{Z}$ and any $t>0$, we have

$$\|M\exp\{t\ln\Lambda_1\}\| \leqslant C\|M\exp\{(t\gamma_1/30)Y\}\|, \tag{42}$$

$$\|MD_j^t\| \leqslant C\|M\exp\{(t\gamma_1/30)Y\}\|,\quad j=1,2,3. \tag{43}$$

Estimates (42)-(43) hold for the local norm also.

Proof. We have

$$a_1=\gamma_1\leqslant\frac{1}{30},\quad \frac{1}{q+\overline{q}}\leqslant\frac{1}{1-p}\leqslant 2,\quad \|Y\|\leqslant\|G\|+1=2,$$

$$\|K\| \leqslant \sum_{j=0}^{\infty}(p+\bar{q}+4a_1)^j \leqslant 3, \tag{44}$$

$$\|L\| \leqslant 9\cdot 4\cdot \bar{q}^2\cdot 4\biggl(1+\frac{p}{q+\bar{q}}(1+p)\biggr) \leqslant 0.4. \tag{45}$$

Note that

$$\left|\binom{1/2}{2}\right| = \frac{1}{8}, \quad \left|\binom{1/2}{3}\right| = \frac{1}{16}, \quad \left|\binom{1/2}{j}\right| \leqslant \frac{5}{128}, \quad j \geqslant 4.$$

We have

$$\sum_{j=1}^{\infty}\left|\binom{1/2}{j}\right|\|L\|^{j-1} \leqslant \frac{1}{2}+\frac{0.4}{8}+\frac{(0.4)^2}{16}+\frac{5}{128}\frac{(0.4)^3}{0.6} \leqslant 0.5642$$

and

$$\begin{aligned}
[I(1+\bar{q})-pI_1+2a_1Y]L &= 4\frac{\bar{q}^2Y^2}{(q+\bar{q})^2}[q^2I+p(q+\bar{q})(I-pI_1)]K \\
&= \gamma_1Y^2\frac{12\bar{q}}{q(q+\bar{q})}(q^2+p(q+\bar{q})(1+p))\Theta = \gamma_1Y^2\Theta.
\end{aligned}$$

Consequently,

$$\Lambda_1 = I+\gamma_1Y+\frac{1}{2}0.5642\gamma_1Y^2\Theta = I+1.2821\gamma_1Y\Theta = I+0.1\Theta,$$

and

$$\begin{aligned}
\ln\Lambda_1 &= \Lambda_1 - I + \sum_{j=2}^{\infty}\frac{(-1)^{j+1}}{j}(\Lambda_1-I)^j \\
&= \gamma_1Y+0.2821\gamma_1Y^2\Theta+\frac{1}{2}1.2821\gamma_1^2Y^2\sum_{j=2}^{\infty}(0.1)^{j-2}\Theta \ = \gamma_1Y+\frac{19}{60}\gamma_1Y^2\Theta.
\end{aligned}$$

Moreover, $\|\Lambda_2\| \leqslant p+\bar{q}+\|\Lambda_1-I\| \leqslant 19/30$. Thus, we have proved proved (38). We use this estimate for obtaining (42). By the properties of the total variation norm,

$$\|Me^{t\ln\Lambda_1}\| \leqslant \Big\|M\exp\Big\{\frac{t\gamma_1}{30}Y\Big\}\Big\|\Big\|\exp\Big\{\frac{29t\gamma_1}{30}Y+\frac{19t\gamma_1}{60}Y^2\Theta\Big\}\Big\|.$$

Applying lemma 5.1, we prove that the second norm is majorized by

$$1+\sum_{r=1}^{\infty}\frac{1}{r!}\Big\|\frac{19}{60}t\gamma_1Y^2\exp\Big\{\frac{29}{30r}t\gamma_1Y\Big\}\Big\|^r \leqslant 1+\sum_{r=1}^{\infty}\frac{\mathrm{e}^r}{r^r\sqrt{2\pi r}}\Big(\frac{57r}{58\mathrm{e}}\Big)^r \leqslant C.$$

The last two estimates obviously lead to (42). The estimate (43) is proved similarly. For the proof of (36) note that

$$\begin{aligned}
\Lambda_1 &= I+\gamma_1Y-\frac{\bar{q}^2}{(q+\bar{q})^2}Y^2[q^2I+p(q+\bar{q})(I-pI_1)]K+C\bar{q}^4Y^4\Theta, && (46)\\
I_1-I &= Y(I-pI_1), \quad (q+\bar{q})B = I+p(I_1-I)B, && (47)\\
I_1-I &= 2Y\Theta, \quad (q+\bar{q})(I-pI_1)B = qI-p\bar{q}(I-pI_1)BY, && (48)\\
B &= \frac{1}{q+\bar{q}}I+\frac{pq}{(q+\bar{q})^2}Y-\frac{p^2\bar{q}}{(q+\bar{q})^2}Y(I_1-I)B, && (49)\\
K &= B+2\gamma_1YKB = B-2\gamma_1B^2Y+4\gamma_1^2B^2Y^2B^2K. && (50)
\end{aligned}$$

Substituting (48)-(50) into (46) we obtain (36). Taking into account (39) we obtain

$$\ln \Lambda_1 = \sum_{j=1}^{3} \frac{(-1)^{j+1}}{j}(\Lambda_1 - I)^j + C(\Lambda_1 - I)^4\Theta.$$

Now for the proof of (37), it suffices to use (36). From (50) and the first relation in (48) we get $p(I_1 - I)K = p(I_1 - I)B + Cp\overline{q}Y^2\Theta$. Moreover,

$$p(I_1 - I)B = \frac{pq}{q+\overline{q}}Y - \frac{p^2\overline{q}(I_1 - I)BY}{q+\overline{q}} = \frac{pqY}{q+\overline{q}} + Cp\overline{q}Y^2\Theta.$$

The last two equations and (35) allow us to prove (40). Since $W_1 + W_2 = I$, we easily obtain the first relation in (41). Now,

$$\|W\|_2 \leqslant \frac{1}{2}\Big(1 + (q+\overline{q})3\sum_{j=0}^{\infty}\Big|\binom{-1/2}{j}\Big|0.4^j\Big) + |\overline{q} - p|\cdot 2\cdot 3\sum_{j=0}^{\infty}\Big|\binom{-1/2}{j}\Big|0.4^j < 7.$$

Thus, Lemma 5.4 is proved. For the lower bound estimates, we need the following result.

Lemma 5.5 *Let M be concentrated on $\mathbb{Z}$, $\alpha \in \mathbb{R}$ and $b > 1$. Then,*

$$\|M\| \geqslant C\Big|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\widehat{M}\Big(\frac{t}{b}\Big)\mathrm{e}^{-\mathrm{i}t\alpha}\mathrm{d}t\Big|, \tag{51}$$

$$\|M\|_\infty \geqslant \frac{C}{b}\Big|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\widehat{M}\Big(\frac{t}{b}\Big)\mathrm{e}^{-\mathrm{i}t\alpha}\mathrm{d}t\Big|. \tag{52}$$

The estimates (51) and (52) remain valid if $\mathrm{e}^{-t^2/2}$ *is replaced by* $t\mathrm{e}^{-t^2/2}$.

Lemma 5.5 with $\|M\|$ replaced by the uniform norm of M was proved in [32, p. 400]. Since the uniform norm is majorized by the total variation one, (51) also holds.

Lemma 5.6 *Let (8) be satisfied. Then, for all $|t| \leqslant \pi$,*

$$|\exp\{n\varkappa_1(\widehat{Y}(t) - it/q)\} - 1| \leqslant Cn\overline{q}t^2, \tag{53}$$

$$|\widehat{D}_2^n(t)| \leqslant 1, \quad |\widehat{D}_2^n \exp\{-itn\gamma_1/q\} - 1| \leqslant Cn\overline{q}t^2. \tag{54}$$

Proof of Lemma 5.6 is straightforward and therefore omitted.

Finally, let us introduce an inverse compound measure to H. Let

$$H^{-1} = \exp\Big\{-\sum_{j=1}^{\infty}\frac{p^j}{j}\Big(1 - \Big(\frac{1 - p_0q/(q+\overline{q})}{1-\varkappa_2}\Big)^j\Big)(I_j - I)\Big\}.$$

Lemma 5.7 *Let (8) be satisfied. Then*

$$\|H^{-1}\| \leqslant \mathrm{e}^2, \tag{55}$$

and, for any (signed) finite measure M concentrated at $\mathbb{Z}$,

$$\|MH\| \geqslant \mathrm{e}^{-2}\|M\|, \quad \|M\exp\{-p_0\gamma_1 Y\}\| \geqslant \|M\|. \tag{56}$$

The estimates in (56) remain valid if the total variation norm is replaced by the local one.

Proof. Estimate (55) easily follows from the property $\|\mathrm{e}\|^M \leqslant \mathrm{e}^{\|M\|}$, see Introduction. Now $\|M\| = \|MHH^{-1}\| \leqslant \|MH\|\,\|H^{-1}\| \leqslant \mathrm{e}^2\|MH\|$. Since $\exp\{p_0\gamma_1 Y\}$ is a distribution, its total variation is 1. Therefore, $\|M\| = \|M\exp\{-p_0\gamma_1 Y\}\exp\{p_0\gamma_1 Y\}\| \leqslant \|M\exp\{-p_0\gamma_1 Y\}\|$. Estimates for the local norm are proved similarly.

6 Proofs

For upper bound estimates, we use an adaptation of Le Cam's [23] approach which deals with convolutions of measures.

Proof of Theorem 3.1. Without loss of generality, we can assume that (8) holds. We have

$$\|F_n - HD_1^\lambda\| \leqslant \|\Lambda_1^n - D_1^n\|\|W_1\| + \|D_1^n(W_1 - H\exp\{-p_0\gamma_1 Y\})\| + \|\Lambda_2\|^n\|W_2\|.$$

Further, in view of Lemma 5.4,

$$\begin{aligned}\|\Lambda_1^n - D_1^n\| &\leqslant \left\|D_1^n\int_0^1(\exp\{\tau[n\ln\Lambda_1 - n\gamma_1 Y\})'_\tau\mathrm{d}\tau\right\| \\ &\leqslant n\int_0^1\|[\ln\Lambda_1 - \gamma_1 Y]\exp\{\tau n\ln\Lambda_1 + (1-\tau)n\gamma_1 Y\}\|\mathrm{d}\tau \\ &\leqslant Cn\|[\ln\Lambda_1 - \gamma_1 Y]\exp\{(n\gamma_1/30)Y\}\| \leqslant Cn\overline{q}^2\|Y^2\exp\{(n\gamma_1/30)Y\}\|.\end{aligned}$$

By Lemma 5.4,

$$\begin{aligned}W_1 - H\exp\{-p_0\gamma_1 Y\} &= [W_1 - I - (\varkappa_1+\varkappa_2)Y] + [I + (\varkappa_1+\varkappa_2)Y - (I - p_0\gamma_1 Y)H] \\ &\quad + [H(I - p_0\gamma_1 Y - \exp\{-p_0\gamma_1 Y\})] = C\overline{q}(p+\overline{q})Y\Theta.\end{aligned}$$

Taking into account the last two estimates, applying Lemma 5.1 and estimating $\|W_2\|$ and $\|\Lambda_2\|$ by (41) and (39), we complete the proof of (9). The estimates in (10) and (11) are proved similarly.

Proof of Corollary 3.2. Following the proof of (42), one can prove the same property for H_1. Also,

$$\|HD_1^\lambda - HH_1^\lambda\| \leqslant C\lambda\|(D_1 - H_1)\exp\{(n\gamma_1/30)Y\}\| \leqslant Cn\overline{q}^2\|Y^2\exp\{(n\gamma_1/30)Y\}\|.$$

The rest of the proof is obvious.

Proof of Theorem 3.2. We have

$$\begin{aligned}\|F_n - HD_1^\lambda(I + n\gamma_2 Y^2)\| &\leqslant \|\Lambda_2\|^n\|W_2\| + \|W_1\|\,\|\Lambda_1^n - D_2^n\| \\ &\quad + \|W_1\|\,\|D_1^n(\mathrm{e}^{n\gamma_2 Y^2} - I - n\gamma_2 Y^2)\| + \|D_1^n(I + n\gamma_2 Y^2)(W_1 - H\mathrm{e}^{-p_0\gamma_1 Y})\|.\end{aligned}$$

Similar to the proof of Theorem 3.1 and using (43), we obtain

$$\begin{aligned}\|\Lambda_1^n - D_2^n\| &\leqslant Cn\Big\|[\ln\Lambda_1 - \gamma_1 Y - \gamma_2 Y^2]\int_0^1 \exp\{\tau n\ln\Lambda_1 + (1-\tau)[n\gamma_1 Y + n\gamma_2 Y^2]\}\mathrm{d}\tau\Big\| \\ &\leqslant Cn\|[\ln\Lambda_1 - \gamma_1 Y - \gamma_2 Y^2]\exp\{(n\gamma_1/30)Y\}\| \\ &\leqslant Cn\overline{q}^2(\overline{q}+p)\|Y^3\exp\{(n\gamma_1/30)Y\}\|\end{aligned}$$

and

$$\begin{aligned}&\Big\|D_1^n(\mathrm{e}^{n\gamma_2 Y^2} - I - n\gamma_2 Y^2)\Big\| \\ &\qquad \leqslant \Big\|(n\gamma_2 Y^2)^2\int_0^1 D_1^n \mathrm{e}^{\tau n\gamma_2 Y^2}(1-\tau)\mathrm{d}\tau\Big\| \leqslant C(n\gamma_2)^2\|Y^4\exp\{(n\gamma_1/30)Y\}\|.\end{aligned}$$

Note that for any signed finite measure M,

$$\|D_1^n(I + n\gamma_2 Y^2)M\| \leqslant \|D_1^{n/2}\|M(1 + n|\gamma_2|\|Y^2 D_1^{n/2}\|) \leqslant C\|D_1^{n/2}M\|.$$

The rest of the proof is very similar to the proof of Theorem 3.1 and hence omitted.

Proof of Theorems 3.3, 3.4 and 3.5. The proofs are very similar to those of Theroem 3.1 and 3.2. From Lemma 5.4 and the definition of the exponent measure, it is not difficult to obtain

$$\begin{aligned}W_1 - \mathrm{e}^{\varkappa_1 Y}H &= [W_1 - I - (\varkappa_1 + \varkappa_2)Y] + [I + (\varkappa_1+\varkappa_2)Y - (I + \varkappa_1 Y)H] \\ &\quad + H(I + \varkappa_1 Y - \mathrm{e}^{\varkappa_1 Y}) = C\overline{q}(p+\overline{q})Y^2\Theta, \\ \|\Lambda_1^n W_1 - D_2^n H\mathrm{e}^{\varkappa_1 Y}\| &\leqslant \|\Lambda_1^n - D_2^n\|\,\|W_1\| + \|D_2^n(W_1 - H\mathrm{e}^{\varkappa_1 Y})\|.\end{aligned}$$

Now it is not difficult to prove Theorem 3.3. Theorem 3.5 is proved similarly. For the proof of Theorem 3.4 one should use Theorem 3.5, the triangle inequality and

$$\begin{aligned}\|D_2^n(I + n\gamma_3 Y^3) - D_3^n\| &= \Big\|D_2^n\int_0^1(1-\tau)\mathrm{e}^{\tau n\gamma_3 Y^3}(n\gamma_3 Y^3)^2\mathrm{d}\tau\Big\| \\ &\leqslant C(n\gamma_3)^2\|Y^6\exp\{n\gamma_1 Y^2/30\}\|.\end{aligned}$$

For the last estimate, we have used the same argument as in the proof of (43).

Proof of Theorem 3.6. Taking into account Theorem 3.2, (30) and (56), we get

$$\begin{aligned}\|F_n - HD_1^\lambda\| &\geqslant n|\gamma_2|\|HD_1^\lambda Y^2\| - C\overline{q}\Big(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\Big) \\ &\geqslant C_{10}n|\gamma_2|\|D_1^n(I_1 - I)^2\| - C_{11}\overline{q}\Big(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\Big). \qquad (57)\end{aligned}$$

Let $z = t/(h\sqrt{n\overline{q}})$ and $\mu = n\gamma_1/q$. The constant $h > 1$ will be chosen later. Then applying Lemma 5.6 we obtain

$$J = \Big|\int_{\mathbb{R}} \mathrm{e}^{-t^2/2}\widehat{D}_1^n(z)\mathrm{e}^{-\mathrm{i}z\mu}(\mathrm{e}^{\mathrm{i}z} - 1)^2\mathrm{d}t\Big| \geqslant \Big|\int_{\mathbb{R}}\mathrm{e}^{-t^2/2}z^2\mathrm{d}.t\Big| - J_1 - J_2. \qquad (58)$$

Here

$$J_1=\int_{\mathbb{R}}\mathrm{e}^{-t^2/2}z^2|\widehat{D}_1^n(z)\mathrm{e}^{-\mathrm{i}z\mu}-1|\mathrm{d}t\leqslant Cn\overline{q}\int_{\mathbb{R}}z^4\mathrm{e}^{-t^2/2}\mathrm{d}t=\frac{C}{h^4n\overline{q}},$$

$$J_2=\int_{\mathbb{R}}\mathrm{e}^{-t^2/2}|\widehat{D}_1^n(z)\mathrm{e}^{-\mathrm{i}z\mu}||(\mathrm{e}^{\mathrm{i}z}-1)^2-(\mathrm{i}z)^2|\mathrm{d}t\leqslant\frac{C}{h^3n\overline{q}\sqrt{n\overline{q}}}.$$

Combining the last two estimates with (58) and choosing h to be sufficiently large absolute constant, we obtain

$$J\geqslant\frac{C_{12}}{h^2n\overline{q}}\left(1-\frac{C_{13}}{h^2}-\frac{C_{14}}{h\sqrt{n\overline{q}}}\right)\geqslant\frac{C_{15}}{n\overline{q}}.$$

Applying Lemma 5.5 and substituting the result into (57), we get (16). Estimates (17) and (18) are proved similarly.

For the proof of Theorem 3.7, one should use Theorem 3.4 and take $t\exp\{-t^2/2\}$ instead of $\exp\{-t^2/2\}$. Now the proof is almost identical to that of Theorem 3.6 and is omitted.

Proof of Theorem 3.8. We have

$$\begin{aligned}\Big|\|F_n-HD_1^{\lambda}\|-A_{11}\Big|&\leqslant\|F_n-HD_1^{\lambda}(I+n\gamma_2Y^2)\|\\&+\Big\|(H\mathrm{e}^{-p_0\gamma_1Y}-I)D_1^n n\gamma_2Y^2\Big\|+n|\gamma_2|\Big\|\Big(Y^2-\frac{1}{q^2}(I_1-I)^2\Big)D_1^n\Big\|\\&+\frac{n|\gamma_2|}{q^2}\Big\|(I_1-I)^2\Big(D_1^n-\exp\Big\{\frac{n\gamma_1}{q}(I_1-I)\Big\}\Big)\Big\|\\&+\Big|\frac{n|\gamma_2|}{q^2}\Big\|(I_1-I)^2\exp\Big\{\frac{n\gamma_1}{q}(I_1-I)\Big\}\Big\|-A_{11}\Big|.\end{aligned}$$

Now one should apply Theorem 3.2, (48), Lemmas 5.2 and 5.1 and the easily verifiable relations:

$$Y=\frac{(I_1-I)}{q}\sum_{j=0}^{\infty}\Big(\frac{p}{q}\Big)^j(I_1-I)^j=\frac{(I_1-I)}{q}+\frac{3p}{q^2}(I_1-I)^2\Theta$$

and

$$\begin{aligned}&D_1-\exp\Big\{\frac{\gamma_1}{q}(I_1-I)\Big\}\\&=\exp\Big\{\frac{\gamma_1}{q}(I_1-I)\Big\}\Big(\exp\Big\{\frac{3p\gamma_1}{q^2}(I_1-I)^2\Theta\Big\}-I\Big)=Cp\overline{q}(I_1-I)^2\Theta.\end{aligned}$$

Note that

$$\begin{aligned}&\Big\|(I_1-I)^2\Big(D_1^n-\exp\Big\{\frac{n\gamma_1}{q}(I_1-I)\Big\}\Big)\Big\|\\&=\Big\|(I_1-I)^2\Big(D_1-\exp\Big\{\frac{\gamma_1}{q}(I_1-I)\Big\}\Big)\sum_{j=1}^{n}D_1^{n-j}\exp\Big\{(j-1)\frac{\gamma_1}{q}(I_1-I)\Big\}\Big\|\\&\leqslant Cnp\overline{q}\bigg(\Big\|(I_1-I)^4D_1^{n/3}\Big\|+\Big\|(I_1-I)^4\exp\Big\{\frac{n\gamma_1}{3q}(I_1-I)\Big\}\Big\|\bigg).\end{aligned}$$

All other estimates are obtained similarly.

Acknowledgment

The main part of the work was accomplished during the first author's stay at the Department of Mathematics, IIT Bombay, during November, 2006. The first author would like to thank the members of Department for their hospitality. Also, we are grateful to the referees for some useful remarks.

References

[1] Barbour, A.D. and Čekanavičius, V. (2002). Total variation asymptotics for sums of independent integer random variables. *Ann. Probab.* **30,** 509–545.

[2] Barbour, A.D., Holst, L. and Janson, S. (1992). *Poisson Approximation.* Clarendon Press, Oxford.

[3] Bhat, U. N., Lal, R. and Karunaratne, M. (1990). A sequential inspection plan for Markov-dependent production process. *IIE Trans.* **22,** 56–64.

[4] Campbell, S.A., Godbole, A.P. and Schaller, S. (1994). Discriminating between sequences of Bernoulli and Markov-Bernoulli trials. *Comm. Statist. Theory Methods.* **23,** 2787–2814.

[5] Čekanavičius, V. and Mikalauskas, M. (1999). Signed Poisson approximations for Markov chains. *Stochastic. Process. Appl.* **82,** 205–227.

[6] Čekanavičius, V. and Mikalauskas, M. (2001). Local theorems for the Markov binomial distribution. *Lithuanian Math. J.* **41,** 219–231.

[7] Čekanavičius, V. and Roos, B. (2007). Binomial approximation to the Markov binomial distribution. *Acta Appl. Math.* **96,** 137–146.

[8] Chryssaphinou, O. and Vaggelatou, E. (2002). Compound Poisson approximation for multiple runs in a Markov chain. *Ann. Inst. Statist. Math.* **54,** 411–424.

[9] Dhaene, J. Goovaerts, M.J. (1997). On the dependency of risks in the individual life model. *Insurance Math. Econ.* **19**, 243–253.

[10] Dhaene, J. and De Pril, N. (1994). On a class of approximative computation methods in individual risk model. *Insurance Math. Econom.* **14,** 181–196.

[11] Deheuvels, P. and Pfeifer, D. (1988). On a relationship between Uspensky's theorem and Poisson approximations. *Ann. Inst. Statist. Math.* **40,** 671–681.

[12] Dobrushin, R.L. (1953). Limit theorems for a Markov chain of two states. *Izv. Akad. Nauk. USSR Ser. mat.* **17,** 291–330. (Russian). English translation in *Select. Transl. Math. Stat. and Probab.* **1,** Inst. Math. Stat. and Amer. Math. Soc. 97–134 (1961).

[13] Erhardsson, T. (1999). Compound Poisson approximation for Markov chains using Stein's method. *Ann. Probab.* **27,** 565–596.

[14] Gani, J. (1982). On the probability generating function of the sum of Markov-Bernoulli random variables. *J. Appl. Probab.* (Special vol.) **19A,** 321–326.

[15] Gerber, H.U. (1984). Error bounds for the compound Poisson approximation. *Insurance Math. Econ.* **3,** 191–194.

[16] Goovaerts, M.J. and Dhaene, J. (1996). The compound Poisson approximation for a portfolio of dependent risks. *Insurance Math. Econ.* **18,** 81–85.

[17] Hipp, C. (1986). Improved approximations for the aggregate claims distribution in the individual model. *Astin Bull.* **16,** 89–100.

[18] Hirano, K. and Aki, S. (1993). On number of success runs of specified length in a two-state Markov chain. *Statist. Sinica.* **3,** 313–320.

[19] Hsiau, S.R. (1997). Compound Poisson limit theorems for Markov chains. *J. Appl. Probab.* **34,** 24–34.

[20] Koopman, B.O. (1950). A generalization of Poisson's distribution for Markoff chains. *Proc. Nat. Acad. Sci. USA* **36,** 202–207.

[21] Kornya, P. (1983). Distribution of aggregate claims in the individual risk theory model. *Trans. Soc. Actuaries.* **35,** 823–858.

[22] Kruopis, J. (1986). Precision of approximation of the generalized binomial distribution by convolutions of Poisson measures. *Lithuanian Math. J.* **26,** 37–49.

[23] Le Cam, L. (1960). An approximation theorem for the Poisson binomial distribution. *Pacific J. Math.* **10,** 1181–1197.

[24] Nelson, B.L. (1993). Estimating acceptance sampling plans for dependent production processes. *IIE Trans.* **25,** 11–18.

[25] Presman, É.L. (1983). Approximation of binomial distributions by infinitely divisible ones. *Theory Probab. Appl.* **28,** 393–403.

[26] Ribas, C., Marin-Solano, J. and Alegre, A. (2003). On the computation of the aggregate claims distribution in the individual life model with bivariate dependencies. *Insurance Math. Econ.* **32,** 201– 215.

[27] Roos, B. (2001). Sharp constants in the Poisson approximation, *Statist. Probab. Lett.* **52,** 155–168.

[28] Roos, B. (2003). Poisson approximation via the convolution with Kornya-Presman signed measures. *Theory Probab. Appl.* **48,** 555–560.

[29] Sahinoglu, M. (1990). The limit of sum of Markov Bernoulli variables in system reliability evaluation. *IEE Trans. Reliability* **39,** 46–50.

[30] Serfling, R.J. (1975). A general Poisson approximation theorem. *Ann. Probab.* **3,** 726–731.

[31] Serfozo, R.F. (1986). Compound Poisson approximations for sums of random variables. *Ann. Probab.* **14,** 1391–1398. Correction: *Ann. Probab.* **16,** 429–430, (1988).

[32] Šiaulys, J. and Čekanavičius, V. (1988). Approximation of distributions of integer-valued additive functions by discrete charges I. *Lithuanian Math. J.* **28,** 392–401.

[33] Vellaisamy, P. (2004). Poisson approximation for (k_1, k_2)-events via the Stein-Chen method. *J. Appl. Probab.* **41,** 1081–1092.

[34] Vellaisamy, P. and Chaudhuri, B. (1999). On compound Poisson approximation for sums of random variables. *Statist. Probab. Lett.* **41,** 179–189 .

[35] Vellaisamy, P. and Sankar, S. (2001). Sequetntial and systematic sampling plans for the Markov-dependent production process. *Naval Res. Logist.* **48,** 451–467.

[36] Wang, Y.H. (1981). On the limit of the Markov binomial distribution. *J. Appl. Probab.* **18,** 937–942.

[37] Wang, Y.H. (1992). Approximating kth-order two-state Markov chains. *J. Appl. Probab.* **29,** 861–868.

[38] Wang, Y.H., Chang, H.F, and Chen, S.Y. (2003). Convergence theorems for the lengths of consecutive successes of Markov Bernoulli sequences. *J. Appl. Probab.* **40,** 741–749.