

## ADJUSTED EMPIRICAL LIKELIHOOD WITH HIGH-ORDER PRECISION

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Empirical likelihood is a popular nonparametric or semi-parametric statistical method with many nice statistical properties. Yet when the sample size is small, or the dimension of the accompanying estimating function is high, the application of the empirical likelihood method can be hindered by low precision of the chisquare approximation and by non-existence of solutions to the estimating equations. In this paper, we show that the adjusted empirical likelihood is effective at addressing both problems. With a specific level of adjustment, the adjusted empirical likelihood achieves the high-order precision of the Bartlett correction, in addition to the advantage of a guaranteed solution to the estimating equations. Simulation results indicate that the confidence regions constructed by the adjusted empirical likelihood have coverage probabilities comparable to or substantially more accurate than the original empirical likelihood enhanced by the Bartlett correction.

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**1. Introduction.** In applications such as econometrics, statistical finance, and biostatistics, general estimating equations (GEE) in the form  $E\{g(X; \theta)\} = 0$ , where  $g(x; \theta)$  is a vector-valued function of the observation vector  $x$  and the parameter vector  $\theta$ , are often used to define the parameters of interest (Hansen, 1982; Liang and Zeger, 1986; Kitamura and Stutzer, 1997; Imbens et al., 1998). With a semi-parametric setup, scientists run a low risk of mis-specifying a probability model for the population under investigation. Particularly when the parameter is over-identified, i.e., the dimension of  $g$  is larger than the dimension of  $\theta$ , the generalized moment method (GMM), the empirical likelihood (EL) method or its variations can be used for statistical inference (Hansen, 1982; Owen, 1988; Newey and McFadden, 1994; Qin and Lawless, 1994; Imbens, 1997; Smith, 1997; Newey and Smith, 2004). Many researchers however find that the finite sample properties of the statistics based on GMM or EL are often very different from the asymptotic properties at sample sizes common in applications (Hall and La Scala, 1990; DiCiccio et al., 1991; Corcoran et al., 1995; Burnside and Eichenbaum, 1996; Corcoran, 1998; Tsao, 2004). High-order approximations to the finite sample distribution based on Bartlett correction or bootstrapping can be helpful (DiCiccio et al., 1991; Hall and Horowitz 1996; Brown and Newey, 2002; Newey and Smith, 2004; Chen and Cui, 2007). Yet they do not always live up to their promise, particularly for high-dimensional data (Corcoran et al, 1995; Tsao, 2004).

We propose a novel approach via adjusted empirical likelihood (AEL) (Chen et al, 2008) to achieve the high-order precision promised by the Bartlett correction. The AEL is obtained by adding a pseudo-observation into the data set. Its principal utility is to overcome the difficulty arising when the estimating equations have no solution; a solution is required in

the EL approach. By using a conventional level of adjustment, Chen et al. (2008) found the AEL improves the approximation precision of the chisquare limiting distribution. However, the optimal level of adjustment remains unknown. In this paper, we derive a high-order expansion of the adjusted empirical likelihood ratio statistic; specify an optimal level of adjustment that enables the high-order approximation; prove that the resulting AEL shares the same high-order precision as the Bartlett corrected EL (BEL); and construct a less biased estimator of the Bartlett correction factor that effectively improves the approximation precision.

Although the AEL and the BEL have the same high-order precision, their finite sample performances differ. Simulation studies show that the AEL has better precision than the BEL in general, and especially under linear and asset-pricing models. The AEL with conventional level of adjustment,  $AEL_0$ , is found to have comparable precisions to the AEL under many models considered, but it lacks some generality. In particular, the AEL improves over the  $AEL_0$  under linear and asset-pricing models.

## 2. The EL and the Bartlett Correction.

2.1. *The empirical likelihood.* To convey the idea, suppose we have  $x_1, x_2, \dots, x_n$  as a random sample from a nonparametric population  $F(x)$  such that  $x \in \mathbb{R}^m$  with dimension  $m$ . Assume that the GEE model is defined by

$$Eg(X; \theta) = 0$$

for a  $q$ -dimensional estimating function  $g$  and a  $p$ -dimensional parameter  $\theta$ .

The profile empirical likelihood function of  $\theta$  is defined as

$$(1) \quad L_n(\theta) = \sup \left\{ \prod_{i=1}^n p_i : p_i \geq 0, \sum_{i=1}^n p_i = 1, \sum_{i=1}^n p_i g(x_i; \theta) = 0 \right\}.$$

The empirical log-likelihood ratio function is defined by  $R_n(\theta) = -2 \log(n^n L_n(\theta))$ ; see Owen (2001) and Qin and Lawless (1994). One celebrated property of the empirical likelihood is that under some general conditions,

$$\text{PR}\{R_n(\theta_0) \leq x\} = \text{PR}\{\chi_q^2 \leq x\} + O(n^{-1})$$

as  $n \rightarrow \infty$ , where  $\theta_0$  is the true parameter value. This property is most convenient for the construction of confidence regions of  $\theta$ :

$$(2) \quad \{\theta : R_n(\theta) \leq c(1 - \alpha; q)\}$$

with  $c(1 - \alpha; q)$  being the  $(1 - \alpha)$ th quantile of the chisquare distribution with  $q$  degrees of freedom, and  $1 - \alpha$  the pre-selected confidence level. Such confidence regions are renowned for their data-driven shape and there is no need to estimate any scalar parameters. For other results, such as when  $\theta_0$  is replaced by its nonparametric maximum EL estimate  $\hat{\theta}$ , we refer to Qin and Lawless (1994).

*2.2. The Bartlett correction of the EL.* The precision of the confidence region constructed by (2) can be poor particularly when the sample size is small. To improve the precision of the coverage probability, we may calibrate the distribution of  $R_n(\theta_0)$  by bootstrapping or by high-order approximations. We now review high-order approximation via the Bartlett correction.

The Bartlett correction for a smooth function of means was first established by DiCiccio et al. (1991), while for estimating questions by Chen and Cui (2006, 2007). For ease of illustration, we consider the situation where  $p = q = 1$  and  $g(x; \theta) = x - \theta$ . Under this model, the parameter  $\theta$  is the population mean. The chisquare approximation has precision  $O(n^{-1})$  and the confidence interval of  $\theta$  based on the chisquare approximation may not

have accurate coverage probabilities. The Bartlett correction can improve the approximation precision to  $O(n^{-2})$ .

By the Lagrange method, when the solution to  $\sum_{i=1}^n p_i g(x_i; \theta) = 0$  exists, we have

$$R_n(\theta) = \sum_{i=1}^n \log\{1 + \lambda g(x_i; \theta)\}$$

for a Lagrange multiplier  $\lambda$  that is the solution to

$$(3) \quad \sum_{i=1}^n \frac{g(x_i; \theta)}{1 + \lambda g(x_i; \theta)} = 0.$$

Let  $\alpha_r = E\{g(X; \theta)\}^r$  and  $A_r = n^{-1} \sum_{i=1}^n \{g(x_i; \theta)\}^r - \alpha_r$ . Without loss of generality, we assume that either  $\alpha_2 = 1$  or we can replace  $g(x; \theta)$  with  $\alpha_2^{-1/2} g(x; \theta)$ . Assuming that  $\theta$  is the true parameter value, we can write

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 + O_p(n^{-2})$$

with

$$\lambda_1 = A_1, \quad \lambda_2 = \alpha_3 A_1^2 - A_1 A_2,$$

$$\lambda_3 = A_1 A_2^2 + A_1^2 A_3 + 2\alpha_3^2 A_1^3 - 3\alpha_3 A_1^2 A_2 - \alpha_4 A_1^3.$$

Under some moment conditions,  $\lambda_r = O_p(n^{-r/2})$  for  $r = 1, 2, 3$ . Substituting these expansions into the expression for  $R_n(\theta)$ , we get

$$(4) \quad R_n(\theta) = n\{R_1 + R_2 + R_3\}^2 + O_p(n^{-3/2})$$

with

$$R_1 = A_1,$$

$$R_2 = \frac{1}{3}\alpha_3 A_1^2 - \frac{1}{2}A_1 A_2,$$

$$R_3 = \frac{3}{8}A_1 A_2^2 + \frac{4}{9}\alpha_3^2 A_1^3 - \frac{5}{6}\alpha_3 A_1^2 A_2 + \frac{1}{3}A_1^2 A_3 - \frac{1}{4}\alpha_4 A_1^3.$$

DiCiccio et al. (1991) find that the cumulants of

$$n(1 - b/n)(R_1 + R_2 + R_3)^2$$

match those of the  $\chi_1^2$  distribution to the order of  $n^{-3/2}$  when

$$(5) \quad b = \frac{1}{2}\alpha_4 - \frac{1}{3}\alpha_3^2.$$

Furthermore, since  $R_1 + R_2 + R_3$  are smooth functions of general sample means, the result of Bhattacharya and Ghosh (1978) implies that

$$\text{PR}\{n(1 - b/n)(R_1 + R_2 + R_3)^2 \leq x\} = \text{PR}\{\chi_1^2 \leq x\} + O(n^{-2}).$$

More details are in the Appendix.

In applications, the value  $b$  must be replaced by some root- $n$  consistent estimator and the replacement does not affect the high-order asymptotic conclusion in theory. Naturally,  $b$  is often replaced by a moment estimate.

Another way to improve the finite sample performance is to use bootstrap calibration, i.e., to estimate the sample distribution of the  $R_n(\theta)$  via a bootstrap resampling scheme. See, for example, Hall and Horowitz (1996). There are situations where the solution  $p_i$ 's to the constraints in (1) at  $\theta = \theta_0$  do not exist with non-negligible probability. A convention adopted in this situation is to define  $R_n(\theta) = \infty$ . However, if  $\text{PR}\{R_n(\theta_0) = \infty\} > \alpha$ , then  $\text{PR}\{R_n(\theta_0) < c\} < 1 - \alpha$  for any finite  $c$ . Consequently, a bootstrap scheme can at most boost the coverage probability to  $1 - \text{PR}\{R_n(\theta_0) = \infty\}$  which is still below the nominal level  $1 - \alpha$ . This problem is clearly also shared by the Bartlett correction. See also Tsao (2004).

### 3. The AEL and the High-Order Approximation.

3.1. *The adjusted empirical likelihood.* For each given  $\theta$ , the likelihood ratio function  $R_n(\theta)$  is well defined only if the convex hull of

$$(6) \quad \{g(x_i; \theta) : i = 1, 2, \dots, n\}$$

contains the  $q$  dimensional vector  $\mathbf{0}$ . When  $n$  is not large, or when a good candidate (vector) value of  $\theta$  is not available, this convex hull often fails to contain  $\mathbf{0}$ . See Chen et al. (2008) for examples. Blindly setting  $L_n(\theta) = 0$  as suggested in the literature fails to provide information on whether  $\theta$  is grossly unfit to the data or is in fact only slightly off an appropriate value. Let  $g_i = g(x_i; \theta)$ ,  $i = 1, \dots, n$  and

$$g_{n+1} = -a_n \bar{g}_n = -a_n n^{-1} \sum_{i=1}^n g_i$$

for some  $a_n > 0$ . The adjusted (profile) empirical likelihood is defined as

$$(7) \quad L_n(\theta; a_n) = \sup \left\{ \prod_{i=1}^{n+1} p_i : p_i \geq 0, \sum_{i=1}^{n+1} p_i = 1, \sum_{i=1}^{n+1} p_i g_i = 0 \right\},$$

and the adjusted empirical likelihood ratio function as

$$R_n(\theta; a_n) = -2 \log \{ (n+1)^{n+1} L_n(\theta; a_n) \}.$$

Because  $\bar{g}_n$  and  $g_{n+1}$  are on opposite sides of  $\mathbf{0}$ , the AEL is always well defined. Namely, its value is always non-zero. When  $a_n = o_p(n^{2/3})$ , Chen et al. (2008) showed that the first-order asymptotic properties of the EL are retained by the AEL, and a conventional  $a_n = \max\{1, \log n/2\}$  was found useful in a number of examples. However, an optimal choice of  $a_n$  remains unsolved. We next recommend a specific  $a_n$  and show that the resulting AEL achieves the goal attained by the Bartlett correction.

3.2. *AEL with high-order precision.* The level of adjustment at which the AEL has high-order precision is  $a_n = b/2$ , where  $b$  is the Bartlett correction

factor for the usual EL. This surprising relationship reveals an intrinsic relationship between the AEL and the Bartlett correction. Indeed, the proof of the following result is built on the Bartlett correction.

**THEOREM 1.** *Suppose that  $x_1, x_2, \dots, x_n$  is a random sample from an  $m$ -variate nonparametric population  $F(x)$ . Assume that the GEE model is defined by*

$$Eg(X; \theta) = 0$$

where  $\theta$  is a  $p$ -dimensional parameter,  $g(X; \theta)$  is a  $q$ -dimensional estimating function, and its characteristic function satisfies Cramér's condition:

$$\limsup_{\|t\| \rightarrow \infty} |E \exp\{it^T g(X; \theta)\}| < 1.$$

Assume also that  $E \|g(X; \theta)\|^{18} < \infty$  and  $\text{var}(g(X; \theta))$  is positive definite.

Let  $\theta_0$  be the true parameter value and  $a_n = a + O_p(n^{-1/2})$ . Then

$$R_n(\theta_0; a_n) = n\{R_1 + R_2 + R_{3a}\}^T \{R_1 + R_2 + R_{3a}\} + O_p(n^{-3/2}),$$

where  $R_1$ ,  $R_2$ , and  $R_{3a}$  will be given in (14) and (16). When  $a = b/2$  where  $b$  is the Bartlett correction factor for the usual empirical likelihood,

$$\text{PR}\{n\{R_1 + R_2 + R_{3a}\}^T \{R_1 + R_2 + R_{3a}\} \leq x\} = \text{PR}(\chi_q^2 \leq x) + O(n^{-2}).$$

Adding a pseudo-observation  $g_{n+1}$  results in a slightly different  $R_{3a}$  as compared to  $R_3$  in Section 2.2. This explains the choice of the notation.

When  $q = 1$ , the Bartlett correction factor  $b = \alpha_4/2 - \alpha_3^2/3 > 0$  unless  $g(X; \theta)$  degenerates. Hence, the pseudo-observation obtained by setting  $a_n = b/2$  or its suitable estimator satisfies the condition  $a_n > 0$  required by the AEL. When  $q > 1$ , it is uncertain whether  $b > 0$  or not. While Theorem 1 remains valid, there is a small probability that the AEL is not defined when

$b < 0$ . We can easily avoid this problem by adding two pseudo-observations.

Let

$$(8) \quad L_n(\theta; a_{1n}, a_{2n}) = \sup \left\{ \prod_{i=1}^{n+2} p_i : p_i \geq 0, \sum_{i=1}^{n+2} p_i = 1, \sum_{i=1}^{n+2} p_i g_i = 0 \right\},$$

and the adjusted empirical likelihood ratio function be

$$R_n(\theta; a_{1n}, a_{2n}) = -2 \log \{ (n+2)^{n+2} L_n(\theta; a_{1n}, a_{2n}) \}$$

with  $g_{n+1} = -a_{1n}\bar{g}$  and  $g_{n+2} = a_{2n}\bar{g}$ . When  $a_{2n} - a_{1n} = b$ , the result of Theorem 1 remains.

In general, the Bartlett correction factor  $b$  can be written as the difference of two positive values. This decomposition gives us natural choices of  $a_{1n}$  and  $a_{2n}$  for multidimensional estimating functions. In simulations, we added a single pseudo-observation when  $q = 1$  and two pseudo-observations when  $q \geq 2$ . We also recommend this practice in applications. More detailed discussions about the Bartlett correction factor  $b$  are given in the next subsection.

When  $q > p$  where the parameter is over-identified, it is more efficient to construct confidence regions with

$$\Delta_n(\theta; a_n) = R_n(\theta; a_n) - \inf_{\theta} R_n(\theta; a_n).$$

When  $a_n = 0$ , Chen and Cui (2007) show that  $\Delta_n(\theta_0; 0)$  is also Bartlett correctable. The result of Theorem 1 remains valid as follows.

**THEOREM 2.** *Assume the same conditions as in Theorem 1, and that there exists a neighborhood of  $\theta_0$ ,  $N(\theta_0)$ , and an integrable function,  $h(x)$ , such that*

$$\sup_{\theta \in N(\theta_0)} \|\partial^3 g(x; \theta) / \partial \theta^3\|^3 \leq h(x).$$

Then at the level of adjustment  $a_n = a + O_p(n^{-1/2})$ ,

$$\Delta_n(\theta_0; a_n) = n\{R_1 + R_2 + R_{3a}\}^T\{R_1 + R_2 + R_{3a}\} + O_p(n^{-3/2}),$$

for some  $R_1$ ,  $R_2$ , and  $R_{3a}$ , and there exists a Bartlett correction factor  $b$  such that when  $a = b/2$

$$\text{PR}\{n\{R_1 + R_2 + R_{3a}\}^T\{R_1 + R_2 + R_{3a}\} \leq x\} = \text{PR}(\chi_p^2 \leq x) + O(n^{-2}).$$

The expressions of the Bartlett correction factor  $b$  and  $R_j$ ,  $j = 1, 2$  in Theorem 2 are the same as in Chen and Cui (2007). When  $a_n = 0$ ,  $R_{3a}$  also becomes their  $R_3$ . More details and a brief proof are given in the Appendix.

*3.3. Estimation of the Bartlett correction factor  $b$ .* We first consider the estimation of  $b$  in the case of Theorem 1. Even for the simplistic one-sample problem, Bartlett-corrected ordinary EL confidence intervals for the population mean often have lower than nominal coverage probabilities when the Bartlett correction factor  $b$  is replaced by its moment estimator. The Bartlett-corrected EL intervals with theoretical  $b$  are often much more satisfactory. Our investigation reveals that the moment estimator of  $b$  usually grossly under estimates particularly when  $n$  is small, say  $n = 20, 30$ . See the simulation results presented in the next section.

Let us first examine the case of  $q = p = 1$ , where the Bartlett correction factor is given by

$$b = \frac{\alpha_4}{2\alpha_2^2} - \frac{\alpha_3^2}{3\alpha_2^3}.$$

Note that we no longer assume  $\alpha_2 = 1$ . The moment estimators of  $\alpha_r$  are given by  $\hat{\alpha}_r = n^{-1} \sum_{i=1}^n (g_i - \bar{g})^r$ . Since  $E\hat{\alpha}_2 = (n-1)\alpha_2/n$ , we estimate  $\alpha_2$  by  $\tilde{\alpha}_2 = n\hat{\alpha}_2/(n-1)$  to reduce bias. In summary, we use the estimators given in the following table to construct a less-biased estimator of  $b$ ,

Parameter	Estimator	Expression
$\alpha_2$	$\tilde{\alpha}_2$	$n\hat{\alpha}_2/(n-1)$
$\alpha_4$	$\tilde{\alpha}_4$	$(n\hat{\alpha}_4 - 6\tilde{\alpha}_2^2)/(n-4)$
$\alpha_2^2$	$\tilde{\alpha}_{22}$	$\tilde{\alpha}_2^2 - \tilde{\alpha}_4/n$
$\alpha_3$	$\tilde{\alpha}_3$	$n\hat{\alpha}_3/(n-3)$
$\alpha_3^2$	$\tilde{\alpha}_{33}$	$\tilde{\alpha}_3^2 - (\hat{\alpha}_6 - \tilde{\alpha}_3^2)/n$
$\alpha_2^3$	$\tilde{\alpha}_{222}$	$\tilde{\alpha}_2^3$

The above choices are motivated as follows. Since

$$\begin{aligned} E\hat{\alpha}_4 &= \alpha_4 - \frac{4\alpha_4}{n} + \frac{6\alpha_2^2}{n} + O(n^{-2}), \\ E\tilde{\alpha}_2^2 &= \alpha_2^2 + \frac{\alpha_4}{n} + O(n^{-2}), \\ E\hat{\alpha}_3 &= \alpha_3 - \frac{3\alpha_3}{n} + O(n^{-2}), \end{aligned}$$

we estimate  $\alpha_4$ ,  $\alpha_2^2$  and  $\alpha_3$  by  $\tilde{\alpha}_4 = (n\hat{\alpha}_4 - 6\tilde{\alpha}_2^2)/(n-4)$ ,  $\tilde{\alpha}_{22} = \tilde{\alpha}_2^2 - \tilde{\alpha}_4/n$  and  $\tilde{\alpha}_3 = n\hat{\alpha}_3/(n-3)$ , respectively. The biases of  $\tilde{\alpha}_4$ ,  $\tilde{\alpha}_{22}$  and  $\tilde{\alpha}_3$  are of order  $O(n^{-2})$  compared to the  $O(n^{-1})$  biases of the corresponding moment estimators. Precise form of the  $O(n^{-1})$  bias of  $\hat{\alpha}_3^2$  is complex. Hence, we aim to reduce rather than completely eliminate the  $O(n^{-1})$  bias. Since  $\tilde{\alpha}_3 \approx \frac{1}{n} \sum_{i=1}^n g_i^3$ , we have approximately  $E\tilde{\alpha}_3 = \alpha_3$  and  $E\tilde{\alpha}_3^2 = \alpha_3^2 + \text{var}(\tilde{\alpha}_3)$ , and approximately  $\text{var}(\tilde{\alpha}_3) = (\alpha_6 - \alpha_3^2)/n$ .

When  $q = p > 1$ , the expression for  $b$  is more complex. Let  $V(\theta) = \text{var}\{g(X; \theta)\}$  be the covariance matrix. By eigenvalue decomposition, we may write

$$V(\theta_0) = P \text{diag}\{\xi_1, \dots, \xi_q\} P^T$$

such that  $PP^T = I$  and  $\xi_1, \dots, \xi_q$  are eigenvalues of  $V(\theta_0)$ . Furthermore, let  $Y = P^T g(X; \theta_0)$ , and for any positive integers  $(r, s, \dots, t)$ , define

$$(9) \quad \alpha^{rs\dots t} = E\{Y^r Y^s \dots Y^t\},$$

where  $Y^t$  is the  $t$ th component of vector  $Y$ .

It can be seen that after  $g$  is transformed by multiplying  $P$ ,  $\alpha^{rr} = \xi_r$  and  $\alpha^{rs} = 0$  for  $r \neq s$ . The Bartlett correction factor can then be written as

$$\begin{aligned}
b &= \frac{1}{q} \left\{ \frac{1}{2} \sum_{r,s} \frac{\alpha^{rrss}}{\alpha^{rr}\alpha^{ss}} - \frac{1}{3} \sum_{r,s,t} \frac{\alpha^{rst}\alpha^{rst}}{\alpha^{rr}\alpha^{ss}\alpha^{tt}} \right\} \\
&= \frac{1}{q} \left\{ \sum_r \frac{\alpha^{rrrr}}{2(\alpha^{rr})^2} + \sum_{r \neq s} \frac{\alpha^{rrss}}{2\alpha^{rr}\alpha^{ss}} \right\} \\
&\quad - \frac{1}{q} \left\{ \sum_r \frac{(\alpha^{rrr})^2}{3(\alpha^{rr})^3} + \sum_{r \neq s} \frac{(\alpha^{rss})^2}{\alpha^{rr}(\alpha^{ss})^2} + 2 \sum_{r < s < t} \frac{(\alpha^{rst})^2}{\alpha^{rr}\alpha^{ss}\alpha^{tt}} \right\} \\
&= \frac{1}{q} \sum_r \left\{ \frac{\alpha^{rrrr}}{2(\alpha^{rr})^2} - \frac{(\alpha^{rrr})^2}{3(\alpha^{rr})^3} \right\} + \frac{1}{2q} \sum_{r \neq s} \left\{ \frac{\alpha^{rrss}}{\alpha^{rr}\alpha^{ss}} - \frac{(\alpha^{rss})^2}{\alpha^{rr}(\alpha^{ss})^2} \right\} \\
&\quad - \frac{1}{q} \left\{ \frac{1}{2} \sum_{r \neq s} \frac{(\alpha^{rss})^2}{\alpha^{rr}(\alpha^{ss})^2} + 2 \sum_{r < s < t} \frac{(\alpha^{rst})^2}{\alpha^{rr}\alpha^{ss}\alpha^{tt}} \right\}.
\end{aligned}$$

Let

$$\begin{aligned}
b_1 &= \frac{1}{q} \sum_r \left\{ \frac{\alpha^{rrrr}}{2(\alpha^{rr})^2} - \frac{(\alpha^{rrr})^2}{3(\alpha^{rr})^3} \right\} + \frac{1}{q} \sum_{r < s} \left\{ \frac{\alpha^{rrss}}{\alpha^{rr}\alpha^{ss}} - \frac{(\alpha^{rss})^2}{\alpha^{rr}(\alpha^{ss})^2} \right\}, \\
b_2 &= \frac{1}{q} \sum_{r < s} \frac{(\alpha^{rss})^2}{\alpha^{rr}(\alpha^{ss})^2} + \frac{2}{q} \sum_{r < s < t} \frac{(\alpha^{rst})^2}{\alpha^{rr}\alpha^{ss}\alpha^{tt}}.
\end{aligned}$$

Clearly, both  $b_1$  and  $b_2$  are positive and  $b = b_1 - b_2$ . There can be other ways to decompose  $b$ . We have chosen the above decomposition so that both  $b_1$  and  $b_2$  are of moderate size.

Note that the Bartlett correction factor(s) depends on the unknown  $\theta_0$ . In applications, we first compute a maximum adjusted empirical likelihood estimate  $\hat{\theta}$  at  $a_n = \log n/2$ , and use it as a tentative replacement of  $\theta_0$  for estimating  $b$  or  $b_1$  and  $b_2$ . We decompose the sample variance of  $g(x; \theta)$  at  $\theta = \hat{\theta}$  to obtain the orthogonal matrix  $P$ . We then obtain  $Y_i = P^T g(X_i; \hat{\theta})$  and define the moment estimators as

$$(10) \quad \hat{\alpha}^{rs \cdots t} = n^{-1} \sum_{i=1}^n Y_i^r Y_i^s \cdots Y_i^t.$$

To reduce the bias in the estimation of  $b_1$  and  $b_2$ , we use the estimators given in the following table:

Parameter	Estimator	Expression
$\alpha^{rr}$	$\tilde{\alpha}^{rr}$	$n\hat{\alpha}^{rr}/(n-1)$
$\alpha^{rrss}$	$\tilde{\alpha}^{rrss}$	$\{n\hat{\alpha}^{rrss} - 2\tilde{\alpha}^{rr}\tilde{\alpha}^{ss} - 4I(r=s)\tilde{\alpha}^{rr}\tilde{\alpha}^{rr}\}/(n-4)$
$\alpha^{rst}$	$\tilde{\alpha}^{rst}$	$n\hat{\alpha}^{rst}/(n-3)$
$\alpha^{rst}\alpha^{rst}$	$\tilde{\alpha}^{rst,rst}$	$\tilde{\alpha}^{rst}\tilde{\alpha}^{rst} - (\hat{\alpha}^{rrsstt} - \tilde{\alpha}^{rst}\tilde{\alpha}^{rst})/n$
$\alpha^{rr}\alpha^{ss}$	$\tilde{\alpha}^{rr,ss}$	$\tilde{\alpha}^{rr}\tilde{\alpha}^{ss} - \tilde{\alpha}^{rrss}/n$
$\alpha^{rr}\alpha^{ss}\alpha^{tt}$	$\tilde{\alpha}^{rr,ss,tt}$	$\tilde{\alpha}^{rr}\tilde{\alpha}^{ss}\tilde{\alpha}^{tt}$

for all  $1 \leq r, s, t \leq q$  and  $I(r=s)$  is the indicator function. We denote the resulting estimates as  $\tilde{b}_1$  and  $\tilde{b}_2$ . For  $q > 1$ , we add two pseudo-observations with  $a_{1n} = \tilde{b}_1/2$  and  $a_{2n} = \tilde{b}_2/2$  in the simulations.

To examine the bias properties of the new estimator, we generated 10,000 sets of random samples from a number of selected univariate, bivariate and trivariate distributions. The population distributions are not important at this stage, and they will be specified in the simulation section. We computed the Bartlett correction factors and their average estimates for constructing confidence regions of the population mean. The outcomes are given in Tables 1 and 2. The moment estimators are denoted as  $b_n$  and the new estimators as  $\tilde{b}_n$ . Clearly, the new estimators are much less biased under the normal, exponential and chisquare distributions. Under mixture distributions,  $\tilde{b}_n$  overestimates  $b$ , but the resulting AEL confidence intervals still have good coverage properties. We also examined the bias properties under a number of linear models. The results are given in Table 3. Again,  $\tilde{b}_n$  is much less biased. The model specifications are relegated to the simulation section.

Tables 1, 2, and 3 about here

When  $q > p$ , we prefer  $\Delta_n(a)$  for constructing confidence intervals as in Theorem 2. However, as indicated in Chen and Cui (2007), it is impractical to estimate  $b$  by the method of moments as it involves many terms and high order moments. In simulations, we used a robustified bootstrap estimate of  $b$  suggested by Chen and Cui (2007).

#### 4. Applications.

4.1. *Confidence regions for population mean.* A classical problem is the construction of confidence regions or testing a hypothesis about a specific value of the population mean based on a set of  $n$  independent and identically distributed observations. Particularly for scalar observations, the standard approach is to use the studentised sample mean

$$T_n(\theta) = \frac{\sqrt{n}(\bar{x}_n - \theta)}{s_n}$$

for both purposes where  $\bar{x}_n$  is the sample mean, and  $s_n^2$  is the sample variance. When the population distribution is normal,  $T_n(\theta)$  has a t-distribution with  $n - 1$  degrees of freedom. The confidence interval or hypothesis test calibrated by the t-distribution is found to be accurate even for non-normal population distributions and for moderate sample size  $n$ . For multivariate observations, the t-statistic is replaced by Hotelling's  $T^2$  defined as

$$T_n^2(\theta) = n(\bar{X}_n - \theta)^T S_n^{-1}(\bar{X}_n - \theta)$$

with  $\bar{X}_n$  the vector sample mean and  $S_n$  the sample covariance matrix. When the observations have a  $p$ -dimensional multivariate normal distribution,  $(n-p)T_n^2(\theta)/\{p(n-1)\}$  has an  $F$ -distribution with  $p$  and  $n-p$  degrees of freedom. The  $F$ -distribution often serves as a reference distribution for

both hypothesis tests and constructing confidence regions, whether or not the normality assumption holds. Surprisingly, the normal-theory-based confidence regions have reasonably accurate coverage probabilities even when the sample sizes are small and the population distributions deviate from the normal. Thus, they serve as a good barometer to gauge the performance of a new method.

The EL and AEL counterparts are obtained by letting  $g(x; \theta) = x - \theta$ . For the sake of comparison, we use the same simulation set-ups as in DiCiccio et al. (1991). We investigate the coverage probabilities of 90%, 95%, and 99% confidence intervals based on the following methods:

- (1) Hotelling's  $T^2$  (including the univariate case),  $T^2$ ;
- (2) The usual empirical likelihood, EL;
- (3) Bartlett-corrected empirical likelihood with moment estimate  $b_n$ , BEL;
- (4) Adjusted empirical likelihood with moment estimate  $b_n$ , AEL;
- (5) Bartlett-corrected empirical likelihood with  $\tilde{b}_n$ , BEL\*;
- (6) Adjusted empirical likelihood with  $\tilde{b}_n$ , AEL\*;
- (7) Bartlett corrected empirical likelihood with known  $b$  value,  $BEL_t$ ;
- (8) Adjusted empirical likelihood with known  $b$  value,  $AEL_t$ ;
- (9) Adjusted empirical likelihood with level of adjustment  $a_n = \frac{1}{2} \log n$ ,  $AEL_0$ .

We generated 10,000 samples from four distributions: (a) the standard normal; (b) an exponential distribution with mean 1; (c) a normal mixture  $0.2N(5, 1) + 0.8N(-1.25, 1)$ ; and (d) the  $\chi_1^2$  distribution. The results are presented in Table 4 where  $0.2N_1 + 0.8N_2$  denotes the normal mixture distribution.

Table 4 about here

Under the normal model,  $T^2$  is optimal. Yet we find that the AEL\* is as good within simulation error. The accuracy of the AEL\* is consistently better than that of the BEL and BEL\*. This is particularly true when the population distribution is exponential or chisquare. Under the mixture model, the AEL\* has a slightly higher than nominal coverage probability. Finally, we remark that under the chisquare distribution, all the methods still have room for improvement when  $n = 20$ . Our simulation results on EL and BEL are comparable to those reported in the literature.

In the multivariate case, we conducted simulation experiments for  $p = 2$  and  $p = 3$ . We used the following strategy to generate correlated trivariate observations. We first generated a random observation  $D$  from the uniform distribution on the interval  $[1, 2]$ . Given  $D$ , we generated  $X_1$ ,  $X_2$  and  $X_3$  from the distributions specified as follows:

- (a)  $X_1 \sim N(0, D^2)$ ,  $X_2 \sim \text{Gamma}(D^{-1}, 1)$ ,  $X_3 \sim \chi_D^2$ ;
- (b)  $X_1 \sim \text{Gamma}(D, 1)$ ,  $X_2 \sim \text{Gamma}(D^{-1}, 1)$ ,  $X_3 \sim \text{Gamma}(4 - D, 1)$ ;
- (c)  $X_1 \sim 0.2N(5, D^2) + 0.8N(-1.25, D^{-2})$ ,  
 $X_2 \sim 0.2N(5, D^{-2}) + 0.8N(-1.25, D^2)$ ,  $X_3 \sim N(0, D^2)$ ;
- (d)  $X_1 \sim \text{Poisson}(D)$ ,  $X_2 \sim \text{Poisson}(D^{-1})$ ,  $X_3 \sim \text{Poisson}(4 - D)$ .

When  $p = 2$ , we used  $X_1$  and  $X_2$  in our simulation and generated 10,000 data sets with sample sizes  $n = 20$  and 30. When  $p = 3$ , we also generated 10,000 data sets but increased sample sizes to  $n = 30$  and 50 to accommodate the higher dimension. Table 5 presents the simulation results.

Table 5 about here

We observe that the AEL\* outperforms all other methods, often substantially. Under the bivariate mixture model (c) at nominal level 95% and sample size  $n = 20$ , the AEL\* has 93.7% coverage probability compared to

91.1% for the BEL and 92.3% for the BEL\*. This is significant because the AEL\*, the BEL and the BEL\* are known to be precise up to the same order  $n^{-2}$ . The difference in performances presumably comes from higher orders.

We remark here that the above discussion has not taken  $\text{AEL}_t$  and  $\text{AEL}_0$  into account. The  $\text{AEL}_t$  is only for theoretical interest and its performance indicates how far AEL can be improved by choosing a better estimator of  $b$ . The  $\text{AEL}_0$  is the AEL with a conventional level of adjustment suggested in Chen et al. (2008). It has comparable performance to AEL\*. Due to a lack of theoretical justification, the observed good performance is hard to generalize. We will continue to keep an eye on its performance.

4.2. *Linear regression.* The empirical likelihood method can also be used to construct confidence regions for the regression coefficient  $\beta$  in the following linear regression model:

$$(11) \quad y = \mathbf{x}^T \beta + \epsilon,$$

where  $\beta$  is a  $p$ -dimensional parameter,  $\mathbf{x}$  a  $p$ -dimensional fixed design point, and  $y$  the scalar response. Chen (1993) showed that the empirical likelihood confidence regions for  $\beta$  are also Bartlett correctable. In comparison, by letting  $g(y, \mathbf{x}; \beta) = \mathbf{x}(y - \mathbf{x}^T \beta)$  the proposed AEL method (AEL\*) directly applies.

Table 6 about here

In this simulation study, we examined the performance of the AEL\* method based on model (11) with  $p = 2$ , the true parameter value  $\beta_0 = (1, 1)^T$ , and the errors  $\epsilon_i$  were generated from either a normal distribution or from a centralized exponential distribution as specified in Table 6. The design matrix of  $\mathbf{x}$  of size  $n \times 2$  was taken from the first  $n$  rows in Table 1 of Chen (1993). The simulation results also are given in Table 6. The improve-

ment of the AEL\* over the EL, BEL or BEL\* is universal and substantial, particularly under the non-normal models when the sample sizes are small.

4.3. *An example where  $q > p$ .* In this subsection, we examine the AEL through an asset-pricing model investigated by Hall and Horowitz (1996) and also by Imbens et al. (1998), expanded with  $q$  ( $q \geq 2$ ) moment restrictions by Schennach (2007). The parameter of interest is defined through the following estimating equations:

$$(12) \quad Eg(X; \theta) \equiv E \begin{pmatrix} r(X, \theta) \\ X_2 r(X, \theta) \\ (X_3 - 1)r(X, \theta) \\ \vdots \\ (X_q - 1)r(X, \theta) \end{pmatrix} = 0$$

where  $r(X, \theta) = \exp\{-0.72 - \theta(X_1 + X_2) + 3X_2\} - 1$ ,  $X = (X_1, X_2, \dots, X_q)$  and  $\theta$  is a scalar parameter. Components of  $X$  are mutually independent and  $X_1, X_2 \stackrel{\text{i.i.d.}}{\sim} N(0, 0.16)$ ,  $X_3, \dots, X_q \stackrel{\text{i.i.d.}}{\sim} \chi_1^2$ . We generated data from the models with  $\theta_0 = 3$ ,  $q = 2$  and  $q = 3$  respectively.

Although Theorem 2 is applicable, precisely estimating  $b$  is not easy due to its complex expression. Instead, Chen and Cui (2007) proposed a bootstrap estimate. We adopted their strategy with a robust modification. Let  $\Delta_m$  be the sample median of  $\Delta_n^*(\hat{\theta}; 0)$  based on  $B = 300$  bootstrap samples. We estimate of  $b$  by

$$\hat{b} = n(\Delta_m/0.4549 - 1)$$

where 0.4549 is the median of the  $\chi_1^2$  distribution. We generated samples of sizes  $n = 100$  and 200. The average bootstrap estimates of  $\hat{b}$  are 31 and 58 for  $q = 2$  and  $q = 3$  over 1000 repetitions. We call them off-line estimates

of  $b$  and carried out the corresponding simulations side-by-side with the bootstrap estimator  $\hat{b}$  for each sample generated.

In Table 7, we report the coverage probabilities of the nominal 90%, 95% and 99% confidence intervals of the empirical likelihood (EL), the Bartlett corrected empirical likelihood (BEL), the adjusted empirical likelihood (AEL(5)), and the adjusted empirical likelihood with conventional  $a_n = \log(n)/2$  (AEL<sub>0</sub>). Due to the exponential nature of  $g$  in  $\theta$  in this example, the sample mean  $\bar{g}$  is unstable. For robustness, we computed  $g_{n+1}$  with the trimmed mean by removing five largest  $\|g_i\|$  values.

In terms of the precision of the coverage probabilities, the AEL is better than the BEL which is better than the EL and the AEL<sub>0</sub>, and the latter two have similar performances. Even after the robustification, the bootstrap estimation of  $b$  ranges from  $-27$  to  $376$  when  $n = 100$ . This observation indicates that neither the BEL nor the AEL is ready to be applied to models similar to the one in this example. The simulation results have instead shown the potential of the AEL approach. We hope to further investigate this problem in the future.

Tables 7 about here

**Appendix. Proof of Theorem 1.** We now present the proof for the general case where  $g(x; \theta)$  is vector valued.

In addition to the notation introduced earlier, we further define

$$A^{rs\dots t} = \frac{1}{n} \sum_i^n Y^r Y^s \dots Y^t - \alpha^{rs\dots t},$$

where  $\alpha^{rs\dots t}$  is defined in (10). Without loss of generality, we assume that  $\alpha^{rs} = I(r = s)$  at  $\theta = \theta_0$ . By DiCiccio et al. (1991), the solution to (3), before any adjustment, can be expanded as

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 + O_p(n^{-2})$$

with

$$\lambda_1^r = A^r, \quad \lambda_2^r = -A^{rs}A^s + \alpha^{rst}A^sA^t,$$

and

$$\begin{aligned} \lambda_3^r = & A^{rs}A^{tu}A^u + A^{rst}A^sA^t + 2\alpha^{rst}\alpha^{tuv}A^sA^uA^v \\ & - 3\alpha^{rst}A^{tu}A^sA^u - \alpha^{rstu}A^sA^tA^u. \end{aligned}$$

Here we have used the summation convention according to which, if an index occurs more than once in an expression, summation over the index is understood. Substituting these expansions into the expression for  $R_n(\theta_0)$ , we get

$$(13) \quad R_n(\theta_0) = n\{R_1 + R_2 + R_3\}^T\{R_1 + R_2 + R_3\} + O_p(n^{-3/2})$$

with

$$(14) \quad \begin{aligned} R_1^r &= A^r, \quad R_2^r = \frac{1}{3}\alpha^{rst}A^sA^t - \frac{1}{2}A^{rs}A^s, \\ R_3^r &= \frac{3}{8}A^{rs}A^{st}A^t - \frac{5}{12}\alpha^{rst}A^{tu}A^sA^u - \frac{5}{12}\alpha^{stu}A^{rs}A^tA^u \\ &+ \frac{4}{9}\alpha^{rst}\alpha^{tuv}A^sA^uA^v + \frac{1}{3}A^{rst}A^sA^t - \frac{1}{4}\alpha^{rstu}A^sA^tA^u. \end{aligned}$$

Recall the the usual Lagrange multiplier  $\lambda$  solves  $f(\lambda) = 0$  where

$$f(\zeta) = n^{-1} \sum_{i=1}^n \frac{g(x_i; \theta)}{1 + \zeta^T g(x_i; \theta)}.$$

Now we work on the Lagrange multiplier after an adjustment at level  $a_n = a + O_p(n^{-1/2})$ . Since  $\lambda_a = O_p(n^{-1/2})$ , it must solves

$$f(\lambda_a) - \frac{a}{n}\bar{g} = O_p(n^{-2}).$$

Taylor expansion of  $f(\lambda_a)$  gives

$$f(\lambda_a) = f(\lambda) + \frac{\partial f(\lambda)}{\partial \lambda}(\lambda_a - \lambda) + O((\lambda_a - \lambda)^2).$$

Since  $f(\lambda) = 0$ , it simplifies to

$$\lambda_a - \lambda = \frac{a}{n} \left( \frac{\partial f(\lambda)}{\partial \lambda} \right)^{-1} \bar{g} + O_p(n^{-2}).$$

Note that

$$\frac{\partial f(\lambda)}{\partial \lambda} = -E\{g(X; \theta_0)g^T(X; \theta_0)\} + O_p(n^{-1/2})$$

and by assumption  $E\{g(X; \theta_0)g^T(X; \theta_0)\} = I$ , we arrive at

$$\lambda_a = \lambda - n^{-1}a\bar{g} + O_p(n^{-2}) = (1 - n^{-1}a)\lambda + O_p(n^{-2}).$$

That is, the two Lagrange multipliers are nearly equal.

Next, we quantify the effect of slightly different Lagrange multipliers on the expansion of  $R_n(\theta_0; a_n)$ . We have

$$\begin{aligned} R_n(\theta_0; a_n) &= 2 \sum_{i=1}^n \log\{1 + (1 - n^{-1}a)\lambda^T g_i\} \\ &\quad + 2 \log\{1 - (1 - n^{-1}a)a\lambda^T \bar{g}\} + O(n^{-3/2}). \end{aligned}$$

Note that

$$\log\{1 - (1 - n^{-1}a)a\lambda^T \bar{g}\} = -a\lambda^T \bar{g} + O_p(n^{-2})$$

and surprisingly

$$2 \sum_{i=1}^n \log\{1 + (1 - n^{-1}a)\lambda^T g_i\} = R_n(\theta_0) + O_p(n^{-3/2}).$$

Therefore, we must have

$$(15) \quad R(\theta_0; a_n) = R_n(\theta_0) - 2aR_1^T R_1 + O_p(n^{-3/2}),$$

where  $R_1$  is defined in (14), and consequently

$$R_n(\theta_0; a_n) = n\{R_1 + R_2 + R_{3a}\}^T \{R_1 + R_2 + R_{3a}\} + O_p(n^{-3/2})$$

with

$$(16) \quad R_{3a} = R_3 - n^{-1}aR_1.$$

Denote

$$\begin{aligned} Q_n &= \sqrt{n}(R_1 + R_2 + R_{3a}), \\ U_n &= (A^1, \dots, A^q, A^{11}, A^{12}, \dots, A^{qq}, A^{111}, A^{112}, \dots, A^{qqq})^T, \end{aligned}$$

such that the super-indices in  $A^{rst}$  satisfy  $1 \leq r \leq s \leq t \leq q$ . Hence,  $U_n$  has  $q(q+1)(q+2)/6$  components, and each component is a centralized sample mean. Furthermore,  $Q_n$  is a smooth vector-valued function of  $U_n$ . According to Bhattacharya and Ghosh (1978), the Edgeworth expansion of a smooth function of the sample mean (vector valued) is given by its formal Edgeworth expansion based on its cumulants. Depending on the required order of the expansion, the appropriate lower-order cumulants must exist.

In this theorem, we look for an expansion of the density function of  $Q_n$  up to order  $o(n^{-2})$ . This expansion is determined by the first six cumulants of  $U_n$  and the derivative of  $Q_n$  with respect to  $U_n$ . Note that we assumed that the 18th moment of  $g(x; \theta)$  exists and the highest order in  $U_n$  is three, hence all cumulants of  $U_n$  up to order 6 exist. The cumulants of  $Q_n$  can then be obtained through those of  $U_n$ .

Let  $\kappa_{r,s,\dots,t}(Q_n)$  denote the joint cumulant of the  $r$ th,  $s$ th,  $\dots$ ,  $t$ th components of  $Q_n$ . After some lengthy but routine algebraic work, we get

$$\begin{aligned} \kappa_r(Q_n) &= -n^{-1/2}\mu^r + n^{-3/2}c_1^r + o(n^{-2}), \\ \kappa_{r,s}(Q_n) &= I(r=s) + n^{-1}\gamma^{rs} + n^{-2}c_2^{rs} + o(n^{-2}), \\ \kappa_{r,s,t}(Q_n) &= n^{-3/2}c_3^{rst} + o(n^{-2}), \\ \kappa_{r,s,t,u}(Q_n) &= n^{-2}c_4^{rstu} + o(n^{-2}), \end{aligned}$$

where

$$\begin{aligned} \mu^r &= \frac{1}{6}\alpha^{rss}, \\ \gamma^{rs} &= \frac{1}{2}\alpha^{rstt} - \frac{1}{3}\alpha^{rtu}\alpha^{stu} - \frac{1}{36}\alpha^{rst}\alpha^{tuu} - 2aI(r=s), \end{aligned}$$

and  $c_1^r, c_2^{rs}, c_3^{rst}, c_4^{rstu}$  are some non-random constants. Cumulants of orders five and six are  $o(n^{-2})$ .

Let  $f_{Q_n}(\mathbf{z})$  and  $\phi(\mathbf{z})$  be the density functions of  $Q_n$  and the  $q$ -variate standard normal distribution. The key consequence of the above computation is the resultant formal Edgeworth expansion

$$f_{Q_n}(\mathbf{z}) = \left\{ 1 + \sum_{i=1}^4 n^{-i/2} \pi_i(\mathbf{z}) + o(n^{-2}) \right\} \phi(\mathbf{z})$$

with

$$\begin{aligned} \pi_1(\mathbf{z}) &= \mu^r \mathbf{z}^T, \\ \pi_2(\mathbf{z}) &= \frac{1}{2} (\gamma^{rs} + \mu^r \mu^s) \{ \mathbf{z}^T \mathbf{z}^s - I(r=s) \} \end{aligned}$$

and for some polynomials  $\pi_3(\mathbf{z})$  and  $\pi_4(\mathbf{z})$  which are of order no more than four, the former is odd and the latter is even. Their specific forms are not needed further and so are omitted.

The above expansion implies that

$$\text{PR}\{Q_n^T Q_n \leq x\} = \int_{\mathbf{z}^T \mathbf{z} < x} \left\{ 1 + \sum_{i=1}^4 n^{-i/2} \pi_i(\mathbf{z}) \right\} \phi(\mathbf{z}) d\mathbf{z} + o(n^{-2}).$$

Because  $\pi_1(\mathbf{z})$  and  $\pi_3(\mathbf{z})$  are odd functions, their integrations over the symmetric region are zero. For the same reason, the integrations of the  $\mathbf{z}^r \mathbf{z}^s$  terms in  $\pi_2(\mathbf{z})$  when  $r \neq s$  over a symmetric region are also zero. We further note that the expression of  $\gamma^{rs}$  involves  $a$ , and it is simple to get

$$\int_{\mathbf{z}^T \mathbf{z} < x} \pi_2(\mathbf{z}) \phi(\mathbf{z}) d\mathbf{z} = \frac{1}{2} (b - 2a) \int_{\mathbf{z}^T \mathbf{z} < x} (\mathbf{z}^T \mathbf{z} - q) \phi(\mathbf{z}) d\mathbf{z}$$

where

$$b = \frac{1}{q} \left( \frac{1}{2} \alpha^{rrss} - \frac{1}{3} \alpha^{rst} \alpha^{rst} \right).$$

This  $b$  is the Bartlett correction factor given in DiCiccio et al. (1991). Its expression is simpler than the earlier one because we assumed  $\alpha^{rs} = I(r = s)$ . Hence, when  $a = b/2$ , we have

$$\text{PR}\{Q_n^T Q_n \leq x\} = \int_{\mathbf{z}^T \mathbf{z} < x} \phi(\mathbf{z}) d\mathbf{z} + O(n^{-2}) = \text{PR}\{\chi_q^2 \leq x\} + O(n^{-2}).$$

This completes the proof.

The conclusion for  $R_n(\theta_0; a_{1n}, a_{2n})$  is obtained similarly.

**Proof of Theorem 2.** Expanding  $\Delta_n(\theta_0; a_n)$  and then computing its cumulants are by far the most demanding parts of the proof of Theorem 2. The tasks are formidable. Fortunately, we find a short-cut by relating  $\Delta_n(\theta_0; a_n)$  to  $\Delta_n(\theta_0; 0)$ . By Chen and Cui (2007),

$$\begin{aligned} \Delta_n(\theta_0; 0) &= R_n(\theta_0; 0) - \inf_{\theta} R_n(\theta; 0) \\ &= n\{R_1 + R_2 + R_3\}^T \{R_1 + R_2 + R_3\} + O_p(n^{-3/2}) \end{aligned}$$

for some  $R_1$ ,  $R_2$  and  $R_3$ , some of which are different from those in DiCiccio et al. (1991). They have the same fundamental properties that enable the Bartlett correction. In addition,  $R_1$  equals the first  $p$  components of  $n^{-1} \sum_{i=1}^n g(X_i; \theta_0)$  after  $g$  is standardized in some way.

With some relatively routine algebra, we find

$$R_n(\theta_0; a_n) = R_n(\theta_0; 0) - 2a \sum_{r=1}^q \{n^{-1} \sum_{i=1}^n g^r(X_i; \theta_0)\}^2 + O_p(n^{-3/2})$$

and

$$\inf_{\theta} R_n(\theta; a_n) = \inf_{\theta} R_n(\theta; 0) - 2a \sum_{r=p+1}^q \{n^{-1} \sum_{i=1}^n g^r(X_i; \theta_0)\}^2 + O_p(n^{-3/2}).$$

Hence,

$$\begin{aligned} \Delta_n(\theta_0; a_n) &= \Delta_n(0) - 2a \sum_{r=1}^p \{n^{-1} \sum_{i=1}^n g^r(X_i; \theta_0)\}^2 \\ &= n\{R_1 + R_2 + R_{3a}\}^T \{R_1 + R_2 + R_{3a}\} + O_p(n^{-3/2}) \end{aligned}$$

where

$$R_{3a} = R_3 - \frac{a}{n}R_1.$$

This proves the first part of Theorem 2.

Again, according to Chen and Cui (2007),  $R_1 + R_2 + R_3$  have cumulants such that  $(1 - b/n)\Delta_n(\theta_0; 0)$  is approximated by  $\chi_p^2$  to  $n^{-2}$  precision. Taking advantage of their proof and using a similar derivation to the proof of Theorem 1, we find  $\Delta_n(\theta_0; a_n)$  with  $a_n = b/2 + O_p(n^{-1/2})$  is approximated by  $\chi_p^2$  to  $n^{-2}$  precision. This completes the proof.

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TABLE 1

*Bartlett correction factors and their average estimates for univariate population mean.*

$n$		$N(0, 1)$	$\text{Exp}(1)$	$0.2N_1 + 0.8N_2$	$\chi_1^2$
20	$b$	1.50	3.17	1.11	4.83
	$b_n$	1.16	1.40	1.14	1.59
	$\tilde{b}_n$	1.57	3.19	2.08	5.56
30	$b_n$	1.26	1.66	1.15	1.96
	$\tilde{b}_n$	1.56	3.17	1.63	5.12

TABLE 2

*Bartlett correction factors and their average estimates for multivariate ( $q = 2, 3$ ) population mean.*

$n$		(a)	(b)	(c)	(d)
$q = 2$	$b$	3.21	3.71	1.68	2.21
20	$b_n$	1.63	1.67	1.48	1.46
	$\tilde{b}_n$	2.93	3.34	2.55	2.14
30	$b_n$	1.90	1.98	1.56	1.64
	$\tilde{b}_n$	3.06	3.47	2.18	2.20
$q = 3$	$b$	4.07	3.84	2.36	2.67
30	$b_n$	2.27	2.24	1.98	2.00
	$\tilde{b}_n$	3.72	3.47	2.62	2.62
50	$b_n$	2.67	2.61	2.13	2.22
	$\tilde{b}_n$	3.89	3.64	2.52	2.67

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TABLE 3

*Bartlett correction factors and their average estimates under linear regression models.*

$n$	$N(0, 1)$			$\text{Exp}(1)$		
	$b$	$b_n$	$\tilde{b}_n$	$b$	$b_n$	$\tilde{b}_n$
30	3.55	2.39	3.56	7.98	2.61	5.39
50	3.53	2.74	3.61	7.92	3.35	6.16
100	3.90	3.28	3.86	9.00	4.58	7.07

TABLE 4  
*Coverage probabilities for one-sample population mean*

	$n$	level	$T^2$	EL	BEL	AEL	BEL*	<b>AEL*</b>	BEL <sub>t</sub>	AEL <sub>t</sub>	AEL <sub>0</sub>
$N(0, 1)$	20	90	90.1	88.2	89.0	89.1	89.3	<b>89.5</b>	89.3	89.4	91.0
		95	95.1	93.2	94.0	94.0	94.2	<b>94.4</b>	94.2	94.3	95.4
		99	98.9	97.9	98.2	98.3	98.3	<b>98.4</b>	98.3	98.4	98.9
	30	90	90.2	89.0	89.7	89.8	90.0	<b>90.0</b>	89.9	89.9	91.1
		95	95.5	94.3	94.9	94.9	95.0	<b>95.0</b>	95.0	95.0	95.8
		99	99.1	98.7	98.8	98.8	98.8	<b>98.8</b>	98.9	98.9	99.1
Exp(1)	20	90	87.5	85.6	86.8	87.0	87.6	<b>88.2</b>	88.2	88.9	88.7
		95	92.0	91.2	91.8	91.9	92.3	<b>92.8</b>	92.8	93.5	93.4
		99	96.6	96.7	97.0	97.1	97.2	<b>97.4</b>	97.5	98.0	97.9
	30	90	87.6	86.7	87.7	87.8	88.2	<b>88.5</b>	88.6	88.9	89.0
		95	92.8	92.3	92.9	93.0	93.3	<b>93.6</b>	93.7	93.9	94.0
		99	97.1	97.6	97.9	97.9	98.0	<b>98.0</b>	98.2	98.3	98.4
$0.2N_1 + 0.8N_2$	20	90	88.4	88.4	89.5	89.5	91.0	<b>91.8</b>	89.2	89.2	90.9
		95	92.8	93.3	94.3	94.3	95.0	<b>95.5</b>	94.1	94.1	95.2
		99	97.0	97.8	98.0	98.0	98.1	<b>98.2</b>	98.0	98.0	98.4
	30	90	88.7	89.1	89.9	89.9	90.3	<b>90.4</b>	89.7	89.8	91.2
		95	93.7	94.4	94.9	94.9	95.3	<b>95.5</b>	94.7	94.7	95.6
		99	97.8	98.8	99.1	99.1	99.2	<b>99.3</b>	99.0	99.0	99.3
$\chi_1^2$	20	90	84.8	83.7	85.0	85.2	86.4	<b>87.3</b>	87.2	89.2	86.7
		95	89.2	89.3	90.4	90.5	91.3	<b>92.0</b>	92.2	93.8	91.7
		99	94.4	95.4	96.0	96.0	96.4	<b>96.8</b>	96.9	98.5	96.9
	30	90	85.9	85.4	86.5	86.7	87.7	<b>88.2</b>	88.2	88.9	87.8
		95	90.2	91.1	91.9	91.9	92.4	<b>92.7</b>	93.0	93.6	92.8
		99	95.2	96.5	96.8	96.8	97.0	<b>97.2</b>	97.3	97.7	97.3

TABLE 5  
*Coverage probabilities for one-sample multivariate ( $q = 2, 3$ ) population mean*

	$n$	level	$T^2$	EL	BEL	AEL	BEL*	<b>AEL*</b>	BEL <sub>t</sub>	AEL <sub>t</sub>	AEL <sub>0</sub>	
$q = 2$	(a)	20	90	86.0	81.7	83.8	84.3	84.8	<b>86.2</b>	85.4	87.0	86.6
			95	91.3	87.7	89.3	89.8	90.1	<b>91.6</b>	90.6	92.2	91.8
			99	96.5	94.5	95.3	95.9	95.9	<b>96.7</b>	96.1	97.9	97.4
		30	90	87.2	85.1	86.5	86.7	87.0	<b>87.8</b>	87.4	88.0	88.2
			95	92.2	90.8	91.7	92.0	92.2	<b>92.8</b>	92.4	93.0	93.2
			99	97.0	96.5	97.0	97.2	97.2	<b>97.5</b>	97.4	97.6	97.8
	(b)	20	90	84.5	80.8	82.7	83.4	84.2	<b>86.2</b>	84.9	87.2	85.6
			95	89.5	86.8	88.4	89.1	89.6	<b>91.1</b>	90.0	92.5	91.1
			99	95.4	93.6	94.5	95.0	94.9	<b>96.2</b>	95.3	98.5	96.7
		30	90	85.9	84.5	86.0	86.3	86.8	<b>87.6</b>	87.1	88.0	87.6
			95	90.7	90.4	91.6	91.8	92.2	<b>92.7</b>	92.5	93.1	92.9
			99	96.1	96.3	96.8	97.0	97.1	<b>97.4</b>	97.3	97.8	97.6
(c)	20	90	85.7	84.6	86.2	86.4	87.7	<b>89.4</b>	86.2	86.5	88.8	
		95	90.6	89.9	91.1	91.5	92.3	<b>93.7</b>	91.2	91.4	93.2	
		99	95.8	95.2	95.7	96.0	96.0	<b>97.2</b>	95.7	96.0	97.1	
	30	90	87.9	87.4	88.9	89.0	89.6	<b>90.0</b>	88.9	89.0	90.7	
		95	92.9	93.2	94.2	94.3	94.7	<b>95.1</b>	94.1	94.2	95.4	
		99	97.2	98.0	98.3	98.4	98.5	<b>98.8</b>	98.3	98.4	98.7	
(d)	20	90	88.5	84.2	85.9	86.2	86.6	<b>87.4</b>	86.8	87.6	89.0	
		95	93.3	90.2	91.3	91.6	91.8	<b>92.5</b>	91.8	92.4	93.7	
		99	97.6	95.8	96.2	96.5	96.4	<b>97.0</b>	96.5	97.0	98.1	
	30	90	88.4	86.4	87.6	87.7	87.9	<b>88.2</b>	88.0	88.3	89.8	
		95	93.6	92.3	93.0	93.1	93.3	<b>93.5</b>	93.3	93.5	94.2	
		99	98.0	97.2	97.6	97.7	97.8	<b>97.9</b>	97.8	97.9	98.4	
$q = 3$	(a)	30	90	85.2	81.5	83.8	84.6	84.9	<b>86.5</b>	85.3	87.2	86.0
			95	90.6	88.1	89.8	90.7	90.6	<b>91.9</b>	91.0	92.5	91.6
			99	96.2	94.8	95.6	96.4	96.1	<b>97.1</b>	96.2	97.9	97.0
		50	90	85.8	84.4	85.8	86.2	86.5	<b>86.9</b>	86.5	87.0	86.9
			95	91.2	90.7	91.9	92.2	92.2	<b>92.7</b>	92.4	92.8	92.7
			99	96.6	96.6	97.2	97.5	97.5	<b>97.7</b>	97.5	97.8	97.7
	(b)	30	90	85.3	81.4	83.6	84.4	84.8	<b>86.1</b>	85.2	86.7	86.0
			95	90.8	87.8	89.7	90.3	90.4	<b>91.7</b>	90.8	92.3	91.6
			99	96.4	95.1	95.9	96.5	96.3	<b>97.1</b>	96.4	97.6	97.2
		50	90	86.7	85.7	87.1	87.4	87.6	<b>88.0</b>	87.7	88.1	88.1
			95	92.0	91.1	92.2	92.5	92.6	<b>92.8</b>	92.8	93.1	93.1
			99	97.1	97.5	97.8	97.9	97.9	<b>98.0</b>	98.0	98.1	98.2
	(c)	30	90	88.0	84.7	86.7	87.0	87.2	<b>88.0</b>	86.9	87.4	88.8
			95	93.0	90.5	91.9	92.3	92.4	<b>93.1</b>	92.1	92.5	93.7
			99	97.6	96.5	97.0	97.3	97.2	<b>98.0</b>	97.1	97.3	98.1
		50	90	88.7	87.4	88.7	88.8	89.0	<b>89.1</b>	88.8	88.9	90.0
			95	93.5	93.2	94.1	94.2	94.3	<b>94.4</b>	94.2	94.2	94.9
			99	98.2	98.3	98.6	98.6	98.6	<b>98.7</b>	98.6	98.6	98.9
(d)	30	90	88.4	84.2	86.1	86.6	86.7	<b>87.3</b>	86.7	87.3	88.4	
		95	93.7	90.5	91.9	92.3	92.3	<b>93.0</b>	92.4	93.0	93.7	
		99	98.1	96.4	97.2	97.4	97.3	<b>97.7</b>	97.3	97.7	98.3	
	50	90	88.7	86.8	88.2	88.3	88.4	<b>88.5</b>	88.4	88.5	89.4	
		95	94.0	92.9	93.7	93.8	93.8	<b>93.9</b>	93.8	93.9	94.4	
		99	98.4	97.9	98.2	98.3	98.3	<b>98.3</b>	98.3	98.3	98.6	

TABLE 6  
Coverage probabilities for the regression coefficient  $\beta$ .

	$n$	level	$F$ -test	EL	BEL	AEL	BEL*	<b>AEL*</b>	BEL <sub><math>t</math></sub>	AEL <sub><math>t</math></sub>	AEL <sub>0</sub>
$N(0, 1)$	30	90	90.0	84.0	85.7	86.1	86.6	<b>87.7</b>	86.6	87.5	87.4
		95	94.9	90.1	91.5	92.0	92.2	<b>93.3</b>	92.3	93.2	93.0
		99	99.3	96.6	97.3	97.5	97.4	<b>98.2</b>	97.5	98.2	98.1
	50	90	89.7	86.9	88.4	88.5	88.7	<b>88.9</b>	88.7	89.0	89.2
		95	95.0	92.7	93.6	93.7	93.8	<b>94.0</b>	93.8	94.0	94.2
		99	99.0	97.7	98.1	98.2	98.2	<b>98.2</b>	98.2	98.3	98.4
	100	90	89.6	88.3	89.1	89.1	89.2	<b>89.2</b>	89.2	89.3	89.4
		95	94.8	93.8	94.3	94.4	94.4	<b>94.5</b>	94.4	94.5	94.5
		99	99.0	98.5	98.6	98.7	98.7	<b>98.7</b>	98.7	98.7	98.8
$\text{Exp}(1)$	30	90	87.9	79.6	81.9	82.4	83.6	<b>86.1</b>	85.7	92.6	83.5
		95	92.8	86.4	88.2	88.8	89.4	<b>91.6</b>	91.0	98.5	89.6
		99	97.7	93.7	94.7	95.2	95.3	<b>97.0</b>	96.3	100.0	95.9
	50	90	88.7	83.7	85.4	85.6	86.4	<b>87.5</b>	87.4	89.0	86.0
		95	93.8	90.0	91.3	91.5	92.1	<b>92.8</b>	92.9	94.2	91.8
		99	98.3	96.3	96.9	97.1	97.3	<b>97.8</b>	97.7	98.9	97.2
	100	90	88.9	86.2	87.3	87.3	87.8	<b>88.1</b>	88.4	88.8	87.4
		95	94.2	92.2	93.0	93.0	93.3	<b>93.6</b>	93.8	94.2	93.1
		99	98.5	97.8	98.1	98.1	98.2	<b>98.3</b>	98.3	98.5	98.1

TABLE 7  
Simulation results under the expanded asset-pricing model.

$q = 2$	level	EL	BEL	AEL(5) bootstrapped $b$	BEL	AEL(5) off-line $b = 31$	AEL <sub>0</sub>
$n = 100$	90	82.6	86.5	85.3	87.4	89.8	82.7
	95	88.4	91.2	92.6	92.8	95.4	88.8
	99	95.8	96.7	97.3	97.3	99.5	95.9
$n = 200$	90	83.9	86.6	85.1	87.8	87.2	84.3
	95	91.2	92.6	91.9	93.1	93.3	91.4
	99	96.9	97.4	97.6	97.8	98.2	96.9
$q = 3$	level	EL	BEL	AEL(5) bootstrapped $b$	BEL	AEL(5) off-line $b = 58$	AEL <sub>0</sub>
$n = 100$	90	78.4	84.9	84.1	87.4	90.5	79.8
	95	85.7	90.8	90.4	93.1	96.7	86.1
	99	94.0	96.1	97.9	97.7	99.8	94.0
$n = 200$	90	82.5	86.9	86.5	87.4	89.8	82.5
	95	89.7	92.7	92.9	93.3	95.3	89.8
	99	96.1	97.2	98.5	97.6	99.2	96.1