

BATCH MEANS AND SPECTRAL VARIANCE ESTIMATORS IN MARKOV CHAIN MONTE CARLO

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Calculating a Monte Carlo standard error (MCSE) is an important step in the statistical analysis of the simulation output obtained from a Markov chain Monte Carlo experiment. For example, it can be used to provide a rigorous method for terminating the simulation. An MCSE is usually based on an estimate of the variance of the asymptotic normal distribution. We consider spectral and batch means methods for estimating this variance. In particular, we establish conditions which guarantee that these estimators are strongly consistent as the simulation effort increases. In addition, for the batch means and overlapping batch means methods we establish conditions ensuring consistency in the mean-square sense which in turn allows us to calculate the optimal batch size up to a constant of proportionality. Finally, we examine the empirical finite-sample properties of spectral variance and batch means estimators and provide recommendations for practitioners.

1. Introduction. Suppose π is a probability distribution with support X and the goal is to calculate $E_\pi g := \int_X g(x) \pi(dx)$ where g is a real-valued, π -integrable function. In many situations, π is sufficiently complex so that we may have to rely on Markov chain Monte Carlo (MCMC) methods to estimate $E_\pi g$. As is now widely recognized (Liu [30], Robert and Casella [41]), we can often simulate a Harris ergodic (i.e. aperiodic, π -irreducible, positive Harris recurrent) Markov chain on X having invariant distribution π and easily estimate $E_\pi g$. Specifically, letting $X = \{X_1, X_2, X_3, \dots\}$ denote such a Markov chain, with probability one,

$$(1) \quad \bar{g}_n := \frac{1}{n} \sum_{i=1}^n g(X_i) \rightarrow E_\pi g \quad \text{as } n \rightarrow \infty.$$

The approximate sampling distribution of the Monte Carlo error, $\bar{g}_n - E_\pi g$, is available via a Markov chain central limit theorem (CLT) if there exists

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a constant $\sigma_g^2 \in (0, \infty)$ such that

$$(2) \quad \sqrt{n}(\bar{g}_n - E_\pi g) \xrightarrow{d} N(0, \sigma_g^2) \quad \text{as } n \rightarrow \infty .$$

The conditions we will require in our theoretical work below are sufficient to guarantee (1) and (2) for any initial distribution. In fact, our conditions will imply the stronger Markov chain functional central limit theorem. See Jones [23] and Roberts and Rosenthal [44] for broader discussion of the conditions for a Markov chain CLT.

Obtaining a good estimate, say $\hat{\sigma}_n^2$, of σ_g^2 is an important step in the statistical analysis of the observed sample path for at least two reasons: (1) it can be used to construct asymptotically valid confidence intervals for $E_\pi g$ and (2) it is a key step in deciding when to terminate the simulation. We describe these approaches more fully below but the interested reader is directed to Flegal et al. [12] and Jones et al. [24] for more detail and comparisons with other methods. In particular, these papers demonstrate that terminating the simulation based on $\hat{\sigma}_n^2$ is superior to the common practice of terminating based on so-called convergence diagnostics.

The simplest approach to stopping an MCMC experiment is a *fixed-time* rule. Specifically, the Markov chain simulation is run for a predetermined number of iterations, using \bar{g}_n to estimate $E_\pi g$. If $\hat{\sigma}_n^2$ is a consistent estimator of σ_g^2 , a valid Monte Carlo standard error (MCSE) of \bar{g}_n is given by $\hat{\sigma}_n/\sqrt{n}$ and an asymptotically valid interval estimator of $E_\pi g$ is given in the usual way

$$\bar{g}_n \pm t_* \frac{\hat{\sigma}_n}{\sqrt{n}}$$

where t_* is an appropriate Student's t quantile. Reporting an interval estimate of $E_\pi g$, or at least the MCSE, will allow independent evaluation of the quality of the reported point estimate. Of course, it is possible that the interval estimate is undesirably wide and hence the simulation should continue. This naturally leads to a sequential approach to terminating the simulation the first time the interval is sufficiently narrow, i.e. a *fixed-width* rule, where the total simulation effort is random. Formally, the user specifies a desired half-width ϵ and the simulation terminates the first time the following inequality is satisfied

$$(3) \quad t_* \frac{\hat{\sigma}_n}{\sqrt{n}} + p(n) < \epsilon$$

where t_* is an appropriate Student's t quantile and $p(n)$ is a positive function such that as $n \rightarrow \infty$, $p(n) = o(n^{-1/2})$. Glynn and Whitt [17] established that

if a functional central limit theorem holds and $\hat{\sigma}_n^2 \rightarrow \sigma_g^2$ with probability 1 as $n \rightarrow \infty$, then the interval at (3) is asymptotically valid in the sense that the desired coverage probability is obtained as $\epsilon \rightarrow 0$. Moreover, the role of $p(n)$ is to ensure that for small values of ϵ the simulation effort n is large. Letting n^* be the desired minimum simulation effort an often useful choice is $p(n) = \epsilon I(n \leq n^*) + n^{-1}$ where $I(\cdot)$ is the usual indicator function.

Generally, $\sigma_g^2 \neq \text{Var}_\pi g$ and, due to the inherent serial correlation in the Markov chain, estimating it requires specialized techniques such as non-overlapping batch means (BM), overlapping batch means (OBM), spectral variance (SV) methods, and regenerative simulation (RS). We study asymptotic, specifically strong consistency and mean-square consistency, and finite-sample properties of the BM, OBM and SV procedures. Strong consistency of the BM and RS estimators was addressed by Jones et al. [24] and Hobert et al. [20], respectively. In the current work we develop conditions for the strong consistency of OBM and SV procedures. More specifically, we require that the Markov chain be geometrically ergodic while existing results on the consistency of OBM and SV methods require the chain to be uniformly ergodic; for a definition of geometric and uniform ergodicity see Tierney [50]. Consistency of RS and BM also require geometric ergodicity, however, SV methods require a slightly stronger moment condition on g compared to RS, BM and OBM. Overall, as we discuss in Sections 2 and 3, our results significantly weaken the existing regularity conditions guaranteeing strong consistency for SV and OBM. It is also worth emphasizing that the results on strong consistency of BM, OBM, RS and SV do not require a stationary Markov chain and hence from a theoretical perspective burn-in is not required. Of course, the initial distribution should be carefully chosen since it will impact the finite sample properties of the chain and the resulting output analysis.

Establishing that a given Markov chain is geometrically or uniformly ergodic can be challenging. On the other hand, there has been a substantial amount of effort directed towards doing just this in the context of MCMC. For example, we know that Metropolis-Hastings samplers with state-independent proposals can be uniformly ergodic (Tierney [50]) but such situations are uncommon in realistic settings. Standard random walk Metropolis-Hastings chains on \mathbb{R}^d , $d \geq 1$ cannot be uniformly ergodic but may still be geometrically ergodic; see Mengersen and Tweedie [35]. An incomplete list of other research on establishing convergence rates of Markov chains used in MCMC is given by Geyer [13], Jarner and Hansen [21], Meyn and Tweedie [37] and Neath and Jones [39] who considered Metropolis-Hastings algorithms and Hobert and Geyer [19], Hobert et al. [20], Johnson

and Jones [22], Jones and Hobert [26], Marchev and Hobert [33], Roberts and Polson [42], Roberts and Rosenthal [43], Rosenthal [45, 46], Roy and Hobert [47], Tan and Hobert [49] and Tierney [50] who examined Gibbs samplers.

Optimal batch size selection for BM and OBM is a long-standing open problem. Song and Schmeiser [48] propose an approach that minimizes the asymptotic mean-squared error. Thus we also prove the BM and OBM estimators are *mean-square consistent*, or

$$(4) \quad \text{MSE}(\hat{\sigma}_n^2) := E_{\pi}(\hat{\sigma}_n^2 - \sigma_g^2)^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Our work on establishing (4) allows us to argue that the asymptotically optimal batch size in terms of MSE is proportional to $n^{1/3}$. This is similar to the conclusions of others (Chien et al. [6], Song and Schmeiser [48]), however, our mixing and moment conditions are much weaker. Specifically, the previous work requires a uniformly ergodic Markov chain and an absolute twelfth moment where we require only geometric ergodicity and a bit more than a fourth moment.

The BM, OBM, RS and SV procedures are all easy to implement. RS is sometimes viewed as the standard by which others should be judged (see for example Bratley et al. [4], Jones and Hobert [25]) and matches the weakest regularity conditions of the other methods for asymptotic validity. However, RS may require additional theoretical work which could be an obstacle to some, and moreover, has been found to be problematic in high-dimensional settings (Gilks et al. [14]) and in variable-at-a-time Metropolis-Hastings settings (Neath and Jones [39]). Of the alternatives to RS, OBM and SV procedures have the reputation of being more efficient than BM. For example, results in Section 3.3 show the ratio of the variance of the OBM estimator to the variance of the BM estimator converges to 2/3 as the simulation effort increases. However, asymptotics alone do not appear to give a clear picture of which method should be preferred. Hence we investigate the empirical finite-sample properties of these methods in the context of two examples. Discussion of our findings is given in Section 4.3 but, on balance, some of the SV methods appear to be superior.

The rest of this paper is organized as follows. In Section 2 we establish strong consistency of SV procedures. Next, Section 3 contains a discussion of the asymptotic properties of BM and OBM procedures. The finite sample properties of the various methods are studied in two examples in Section 4 where we also give some general recommendations. Finally, most of the technical details and proofs are presented in appendices.

2. Spectral Estimation. We define a class of estimators of σ_g^2 and establish conditions which guarantee their strong consistency. Define the lag s autocovariance $\gamma(s) = \gamma(-s) := E_\pi [Y_t Y_{t+s}]$ where $Y_i := g(X_i) - E_\pi g$ for $i = 1, 2, 3, \dots$. Consider estimating $\gamma(s)$ with

$$\gamma_n(s) = \gamma_n(-s) := n^{-1} \sum_{t=1}^{n-s} (Y_t - \bar{Y}_n)(Y_{t+s} - \bar{Y}_n)$$

where $\bar{Y}_n := n^{-1} \sum_{i=1}^n Y_i$. If $E_\pi g^2 < \infty$, then for fixed s , $\gamma_n(s) \rightarrow \gamma(s)$ w.p. 1 as $n \rightarrow \infty$.

One could use the sum of the $\gamma_n(s)$ to estimate σ_g^2 , though this turns out to be a poor estimator; see Anderson [1] and Bratley et al. [4]. Instead we will investigate a truncated and weighted estimator version called the *spectral variance estimator*

$$\hat{\sigma}_S^2 := \sum_{s=-(b_n-1)}^{b_n-1} w_n(s) \gamma_n(s),$$

where $w_n(\cdot)$ is the *lag window* and b_n is the *truncation point*.

Our theoretical results require the following assumptions on the lag window, truncation point, and one-step Markov transition kernel associated with X , denoted $P(x, A)$.

ASSUMPTION 1. The lag window $w_n(\cdot)$ is an even function defined on the integers such that

$$\begin{aligned} |w_n(s)| &\leq 1 \quad \text{for all } n \text{ and } s, \\ w_n(0) &= 1 \quad \text{for all } n, \\ w_n(s) &= 0 \quad \text{for all } |s| \geq b_n. \end{aligned}$$

ASSUMPTION 2. Let b_n be an integer sequence such that $b_n \rightarrow \infty$ and $n/b_n \rightarrow \infty$ as $n \rightarrow \infty$ where b_n and n/b_n are monotonically nondecreasing.

ASSUMPTION 3. There exists a function $s : \mathsf{X} \rightarrow [0, 1]$ and a probability measure Q such that $P(x, \cdot) \geq s(x)Q(\cdot)$ for all $x \in \mathsf{X}$.

The main result of this section follows.

THEOREM 1. *Let X be a geometrically ergodic Markov chain with invariant distribution π and $g : \mathsf{X} \rightarrow \mathbb{R}$ be a Borel function with $E_\pi |g|^{4+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$. Suppose Assumptions 1, 2, and 3 hold and define*

$\Delta_1 w_n(k) = w_n(k-1) - w_n(k)$ and $\Delta_2 w_n(k) = w_n(k-1) - 2w_n(k) + w_n(k+1)$. Further suppose (a) $b_n n^{-1} \sum_{k=1}^{b_n} k |\Delta_1 w_n(k)| \rightarrow 0$ as $n \rightarrow \infty$; (b) there exists a constant $c \geq 1$ such that $\sum_n (b_n/n)^c < \infty$; (c) $b_n n^{-1} \log n \rightarrow 0$ as $n \rightarrow \infty$; (d)

$$b_n n^{2\alpha} (\log n)^3 \left(\sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \right)^2 \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ and}$$

$$n^{2\alpha} (\log n)^2 \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0 \text{ as } n \rightarrow \infty$$

where $\alpha = 1/(4 + \delta)$; and (e) $b_n^{-1} n^{2\alpha} \log n \rightarrow 0$ as $n \rightarrow \infty$. Then for any initial distribution, with probability 1, $\hat{\sigma}_g^2 \rightarrow \sigma_g^2$ as $n \rightarrow \infty$.

PROOF. See Appendix B.1. □

REMARK 1. It is convenient in applications to take $b_n = \lfloor n^\nu \rfloor$ for some $0 < \nu < 1$ in which case conditions (b) and (c) of Theorem 1 are automatically satisfied.

REMARK 2. Under stationarity, σ_g^2 is a special case of the Fourier transform of the covariance sequence, or spectral density function, given by

$$f(\phi) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \gamma(s) \cos(\phi s)$$

where the parameter $\phi \in [0, \pi/2]$ is the *frequency*.

REMARK 3. Assumption 3 is not critical. Firstly, we do not require the actual value of s or Q at any point in this paper. Thus, unlike RS which is entirely based on Assumption 3, there is no practical point in searching for a good minorization. Secondly, recall that fundamental Markov chain theory (see Chapter 5 of Meyn and Tweedie [36]) ensures the existence of an integer n_0 for which a minorization condition holds for the n_0 -step kernel, i.e. P^{n_0} . If we cannot establish the one-step minorization in Assumption 3, but we can establish an n_0 -step minorization, then we would just use the chain with kernel P^{n_0} . This is reasonable since the P^{n_0} inherits the stability properties of P .

REMARK 4. Anderson [1] gives an extensive collection of lag windows satisfying Assumption 1. It is useful to consider the applicability of the conditions (a) and (d) of Theorem 1 for some of these windows.

Simple Truncation: Set $w_n(k) = I(|k| < b_n)$, then it is easy to see that condition (d) requires $4b_n n^{2\alpha} (\log n)^3 \rightarrow 0$ which obviously cannot hold.

Blackman-Tukey: Let $w_n(k) = [1 - 2a + 2a \cos(\pi|k|/b_n)]I(|k| < b_n)$ where $a > 0$. When $a = 1/4$ this is the *Tukey-Hanning* window. That condition (a) holds if $b_n^2 n^{-1} \rightarrow 0$ as $n \rightarrow \infty$ while condition (d) is satisfied if $b_n^{-1} n^{2\alpha} (\log n)^3 \rightarrow 0$ as $n \rightarrow \infty$ follows easily from Lemma 7 in Appendix B.2.

Parzen: Let $w_n(k) = [1 - |k|^q/b_n^q]I(|k| < b_n)$ for $q \in \mathbb{Z}_+ = \{1, 2, 3, \dots\}$. When $q = 1$ this window is the *modified Bartlett* window and deserves to be singled out because of its connection to the method of overlapping batch means which we will consider later. In this case $\Delta_1 w_n(k) = \Delta_2 w_n(b_n) = b_n^{-1}$ for $k = 1, \dots, b_n$ so that condition (a) requires $b_n^2 n^{-1} \rightarrow 0$ as $n \rightarrow \infty$. Next, note that $\Delta_2 w_n(k) = 0$ for $k = 1, 2, \dots, b_n - 1$. Thus condition (d) is satisfied if $b_n^{-1} n^{2\alpha} (\log n)^3 \rightarrow 0$ as $n \rightarrow \infty$.

When $q \geq 2$ it is easy to show that the conditions of Lemma 7 in Appendix B.2 hold and hence that conditions (a) and (d) will be satisfied under exactly the same conditions as required for the modified Bartlett lag window. That is, if $b_n^2 n^{-1} \rightarrow 0$ and $b_n^{-1} n^{2\alpha} (\log n)^3 \rightarrow 0$ as $n \rightarrow \infty$.

Scale-Parameter Modified Bartlett: Let $w_n(k) = [1 - \lambda|k|/b_n]I(|k| < b_n)$ where $\lambda \neq 1$ is a positive constant. Then $\Delta_1 w_n(k) = \lambda b_n^{-1}$ for $k = 1, \dots, b_n - 1$ and $\Delta_1 w_n(b_n) = 1 - \lambda + \lambda b_n^{-1}$ so that condition (a) becomes $(1 - \lambda)b_n n^{-1} + \lambda b_n(b_n + 1)(2n)^{-1} \rightarrow 0$, that is, $b_n^2 n^{-1} \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, there is trouble with condition (d). Note that $\Delta_2 w_n(b_n) = 1 - \lambda + \lambda b_n^{-1}$ and $\Delta_2 w_n(b_n - 1) = -1 + \lambda$ but $\Delta_2 w_n(k) = 0$ for $k = 1, 2, \dots, b_n - 2$. Hence, as $n \rightarrow \infty$, $\sum_{k=1}^{b_n} |\Delta_2 w_n(k)|$ does not converge to 0. Thus condition (d) cannot hold.

REMARK 5. Damerdji [9, 10] has previously addressed strong consistency of σ_S^2 . However, our result substantially weakens the regularity conditions for Harris ergodic Markov chains. In particular, Damerdji's approach requires a uniformly ergodic chain whereas Theorem 1 requires only geometric ergodicity. Also, instead of condition (d), Damerdji's result requires

$$(5) \quad b_n n^{1-2\alpha'} (\log n) \left(\sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \right)^2 \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ and}$$

$$n^{1-2\alpha'} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0 \text{ as } n \rightarrow \infty$$

where $0 < \alpha' \leq (\delta - 2 + \epsilon)/(24 + 12(\delta - 2 + \epsilon))$. This requirement is not particularly useful when $b_n = \lfloor n^\nu \rfloor$. For example, consider using the modified Bartlett lag window. Then just as in Theorem 1, Damerdji requires $n^{2\nu-1} \rightarrow$

0 as $n \rightarrow \infty$ but (5) requires $n^{1-2\alpha'-\nu}(\log n) \rightarrow 0$ as $n \rightarrow \infty$ and there is no ν value that satisfies both of these requirements.

Finally, Damerdji required what we view to be an unnatural regularity condition. Specifically, for large enough n , $b_n^{-1} \sum_{i=n-b_n+1}^n Y_i^2$ is almost surely bounded above. An inspection of our proof will show that we can weaken the moment condition to $E_\pi |g|^{2+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$ by making the same assumption as Damerdji.

3. Batch Means. In non-overlapping batch means the output is broken into blocks of equal size. Suppose the algorithm is run for a total of $n = a_n b_n$ iterations and for $k = 0, \dots, a_n - 1$ define $\bar{Y}_k := b_n^{-1} \sum_{i=1}^{b_n} Y_{kb_n+i}$. The BM estimate of σ_g^2 is

$$(6) \quad \hat{\sigma}_{BM}^2 = \frac{b_n}{a_n - 1} \sum_{k=0}^{a_n-1} (\bar{Y}_k - \bar{Y}_n)^2.$$

It is well known that generally (6) is not a consistent estimator of σ_g^2 (Glynn and Iglehart [15], Glynn and Whitt [16]). On the other hand, Jones et al. [24] show that if the batch size and number of batches are allowed to increase as the overall length of the simulation does (e.g., by setting $a_n = b_n = \lfloor n^{1/2} \rfloor$) then $\hat{\sigma}_{BM}^2 \rightarrow \sigma_g^2$ with probability one as $n \rightarrow \infty$. However, Jones et al. [24] found that the finite sample properties can be less desirable than expected, thus we consider OBM.

OBM is a generalization of BM but it is also well known that the OBM estimator is equal, except for some end-effect terms, to the SV estimator arising from the modified Bartlett lag window—a relationship we exploit later. Note that there are $n - b_n + 1$ batches of length b_n indexed by j running from 0 to $n - b_n$ defined as $\bar{Y}_j(b_n) = b_n^{-1} \sum_{i=1}^{b_n} Y_{j+i}$. The OBM estimator of σ_g^2 results from averaging across all batches and is defined as

$$(7) \quad \hat{\sigma}_{OBM}^2 = \frac{nb_n}{(n - b_n)(n - b_n + 1)} \sum_{j=0}^{n-b_n} (\bar{Y}_j(b_n) - \bar{Y}_n)^2.$$

The next result establishes strong consistency of the OBM estimator.

THEOREM 2. *Let X be a geometrically ergodic Markov chain with invariant distribution π but any initial distribution and $g : \mathbf{X} \rightarrow \mathbb{R}$ be a Borel function with $E_\pi |g|^{2+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$. Suppose Assumptions 2 and 3 hold. Further suppose (a) there exists a constant $c \geq 1$ such that $\sum_n (b_n/n)^c < \infty$; (b) $b_n n^{-1} \log n \rightarrow 0$ as $n \rightarrow \infty$; (c) $n^{2\alpha} (\log n)^3 / b_n \rightarrow 0$ as $n \rightarrow \infty$ where $\alpha = 1/(2+\delta)$; (d) there exists an integer n_0 and a constant c_1*

such that for all $n \geq n_0$ we have $\log n/b_n \leq c_1$; and (e) $b_n^2 n^{-2} \log \log n \rightarrow 0$ and $b_n^4 n^{-3} \log \log n \rightarrow 0$ as $n \rightarrow \infty$, then as $n \rightarrow \infty$, $\hat{\sigma}_{OBM}^2 \rightarrow \sigma_g^2$ w.p. 1.

PROOF. See Appendix B.3. □

REMARK 6. It is possible to obtain strong consistency of $\hat{\sigma}_{OBM}^2$ directly from Theorem 1 using the modified Bartlett lag window. However, Theorem 1 requires a stronger moment condition on g . In addition, to meet condition (a) of Theorem 1 would require that $b_n^2 n^{-1} \rightarrow 0$ as $n \rightarrow \infty$, hence the conditions of Theorem 2 are weaker.

REMARK 7. Chan and Geyer [5] established a CLT as at (2) under the same moment and mixing conditions stated in our Theorem 2. Moreover, this moment condition cannot be weakened to just a second moment for geometrically ergodic chains; see Häggström [18].

COROLLARY 1. *Let X be a geometrically ergodic Markov chain with invariant distribution π and $g : \mathcal{X} \rightarrow \mathbb{R}$ be a Borel function with $E_\pi |g|^{2+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$. Suppose Assumption 3 holds and $b_n = \lfloor n^\nu \rfloor$ where $3/4 > \nu > (1 + \delta/2)^{-1}$, then $\hat{\sigma}_{OBM}^2 \rightarrow \sigma_g^2$ w.p. 1.*

PROOF. This follows easily by verifying the conditions of Theorem 2. □

REMARK 8. Damerджи [10] and Jones et al. [24] show the nonoverlapping batch means estimator $\hat{\sigma}_{BM}^2$ is strongly consistent for uniformly and geometrically ergodic chains, respectively, under similar conditions as those required for Theorem 2. For example, Jones et al. [24] show that if $b_n = \lfloor n^\nu \rfloor$, then strong consistency requires $1 > \nu > (1 + \delta/2)^{-1} > 0$ for $\delta > 0$ resulting in weaker conditions on ν than those required in Corollary 1.

3.1. *Mean-Square Consistency.* We now turn our attention to showing that $\hat{\sigma}_n^2$ is consistent in the mean-square sense, i.e. (4). Recall that strong consistency and mean-square consistency do not generally imply each other, hence we cannot directly appeal to the results in the previous section, although they will be useful in our proofs.

There has been some work on this problem in the operations research literature. For example, consider the following results from Chien et al. [6]. If X is a stationary uniformly ergodic Markov chain, Assumption 2 holds, and $E_\pi g^{12} < \infty$, then as $b_n \rightarrow \infty$ (whether $a_n \rightarrow \infty$ or not)

$$(8) \quad b_n \text{Bias} \left[\hat{\sigma}_{BM}^2 \right] = \Gamma + o(1) ,$$

where $\Gamma := -2 \sum_{s=1}^{\infty} s\gamma(s)$ which is well defined and finite. Also, if $a_n \rightarrow \infty$ and $b_n \rightarrow \infty$

$$(9) \quad \frac{n}{b_n} \text{Var}(\hat{\sigma}_{BM}^2) = 2\sigma_g^4 + o(1) .$$

Combining (8) and (9) imply (4) for $\hat{\sigma}_{BM}^2$ as $a_n \rightarrow \infty$ and $b_n \rightarrow \infty$. The next result establishes (4) where $\hat{\sigma}_n^2$ is $\hat{\sigma}_{BM}^2$ or $\hat{\sigma}_{OBM}^2$ under weaker mixing and moment conditions.

THEOREM 3. *Let $\hat{\sigma}_n^2$ be either the BM or OBM estimator of σ_g^2 and let X be a stationary geometrically ergodic Markov chain with invariant distribution π and $g : X \rightarrow \mathbb{R}$ be a Borel function with $E_{\pi}|g|^{4+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$. Suppose Assumptions 2 and 3 hold and $E_{\pi}C^4 < \infty$ where C is defined at (14). If $b_n^{-1}n^{2\alpha}(\log n)^3 \rightarrow 0$ as $n \rightarrow \infty$ where $\alpha = 1/(4 + \delta)$, then $\text{MSE}(\hat{\sigma}_n^2) \rightarrow 0$ as $n \rightarrow \infty$.*

REMARK 9. The proof of Theorem 3 with results in Damerdji [11] show that the conclusions also hold for uniformly ergodic chains if we replace the above condition on b_n with $b_n^{-1}n^{1-2\alpha'}(\log n) \rightarrow 0$ as $n \rightarrow \infty$ where $\alpha' \leq (\delta + 2 + \epsilon)/(24 + 12(\delta + 2 + \epsilon))$.

REMARK 10. Results in Damerdji [11] and Philipp and Stout [40] show that when the Markov chain is uniformly ergodic

$$C(\omega) = 2 + \sum_{\tau_1(\omega)}^{\tau_2(\omega)-1} |g(X_j(\omega))|$$

where τ_i , $i = 1, 2$ are the first two times at which regenerations occur in the split-chain; see Hobert et al. [20], Jones and Hobert [25] and Mykland et al. [38] for an introduction to the split chain. Lemma 2 in Bednorz and Latuszyński [2] shows that, in this case, $E_{\pi}C^4 < \infty$. When the chain is only geometrically ergodic the representation of C is not as clear but we suspect that the results of Bednorz and Latuszyński [2] can again be used to establish the moment condition on C since it is still defined in terms of the split chain; see Csáki and Csörgő [7].

3.2. Optimal Batch Sizes in Terms of MSE. In this section, we will use the previous results to calculate optimal batch sizes. Chien et al. [6] and Song and Schmeiser [48] study the case of BM. Combining (8) and (9) yields

$$\text{MSE}(\hat{\sigma}_{BM}^2) = \frac{\Gamma^2}{b_n^2} + \frac{2b_n\sigma_g^4}{n} + o\left(\frac{1}{b_n^2}\right) + o\left(\frac{b_n}{n}\right) .$$

It is easy to use the above expression to see that $\text{MSE}(\hat{\sigma}_{BM}^2)$ will be minimized asymptotically by selecting the optimal batch size of

$$\hat{b}^* := \left(\frac{\Gamma^2 n}{\sigma_g^4} \right)^{1/3} .$$

Notice that this optimal batch size is dependent on Γ^2/σ_g^4 which is typically an unknown parameter relating to the process. However, this result implies that the optimal batch size should increase proportional to $n^{1/3}$.

The main result of this section follows.

THEOREM 4. *Let X be a stationary geometrically ergodic Markov chain with invariant distribution π and $g : \mathsf{X} \rightarrow \mathbb{R}$ be a Borel function with $E_\pi |g|^{4+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$. Suppose Assumptions 2 and 3 hold and $E_\pi C^4 < \infty$ where C is defined at (14). If $b_n^{-1} n^{1/2+\alpha} (\log n)^{3/2} \rightarrow 0$ as $n \rightarrow \infty$ where $\alpha = 1/(4 + \delta)$, then*

$$(10) \quad \frac{n}{b_n} \text{Var}(\hat{\sigma}_n^2) = c\sigma_g^4 + o(1) .$$

where $c = 2$ for BM and $c = 4/3$ for OBM.

REMARK 11. The proof of Theorem 4 coupled with results from Damerджи [11] can be applied to uniformly ergodic Markov chains. Specifically, the condition on b_n would be changed to $b_n^{-1} n^{1-\alpha'} (\log n)^{1/2} \rightarrow 0$ as $n \rightarrow \infty$ where $\alpha' \leq (\delta+2+\epsilon)/(24+12(\delta+2+\epsilon))$. However, if $b_n = \lfloor n^\nu \rfloor$, then $\nu \in (11/12, 1)$ whereas Theorem 4 requires $\nu \in (1/2 + \alpha, 1)$.

Combining (8) and (10) yields

$$\text{MSE}(\hat{\sigma}_{OBM}^2) = \frac{\Gamma^2}{b_n^2} + \frac{4b_n\sigma_g^4}{3n} + o\left(\frac{1}{b_n^2}\right) + o\left(\frac{b_n}{n}\right) ,$$

which will be minimized asymptotically by selecting a batch size of

$$\hat{b}^* := \left(\frac{8\Gamma^2 n}{3\sigma_g^4} \right)^{1/3} .$$

3.3. OBM versus BM. Comparing the conditions of Theorem 2 with the conditions of Proposition 3 in Jones et al. [24] which addresses the strong consistency of the BM estimator we see that conditions (d) and (e) of Theorem 2 are not required for BM. For mean-square consistency with

OBM, Theorems 3 and 4 require a moment condition on C which is not necessary in Chien et al. [6] and Song and Schmeiser [48], however, the moment conditions on g and mixing conditions on the Markov chain much weaker in our results. Moreover, from an implementation point of view it is clear that OBM will require more computational resources than BM.

Why might we use OBM? Often $\hat{\sigma}_{OBM}^2$ has a lower asymptotic variance compared to $\hat{\sigma}_{BM}^2$. Specifically, note that (10) yields

$$\frac{\text{Var}(\hat{\sigma}_{OBM}^2)}{\text{Var}(\hat{\sigma}_{BM}^2)} \rightarrow \frac{2}{3}$$

as $n \rightarrow \infty$; see Meketon and Schmeiser [34] for the same result under different assumptions. Also, in an effort to reduce the computational demands, Welch [52] argues that most of this benefit can be achieved by a modest amount of overlapping. For example, using a batch of size 64 and splitting the batch into 4 sub-batches, then we only need the overlapping batches (of length 64) starting at $X_1, X_{17}, X_{33}, X_{49}, X_{65}, \dots$

4. Examples. In this section, we compare BM, OBM, RS, and SV via their finite sample properties in two examples one of which is a simple AR(1) model while the other is a more realistic Bayesian probit regression model.

4.1. *AR(1) Model.* Consider the following AR(1) model

$$X_i = \rho X_{i-1} + \epsilon_i \quad \text{for } i = 1, 2, \dots,$$

where the ϵ_i are i.i.d. $N(0,1)$. As long as $|\rho| < 1$ this Markov chain is geometrically ergodic with invariant distribution $N(0, 1/(1 - \rho^2))$. Also, $\text{cov}_\pi(X_1, X_i) = \rho^{i-1}/(1 - \rho^2)$.

Consider estimating $E_\pi X$ with \bar{x}_n . Clearly, a CLT holds and, indeed, $\sigma_x^2 = 1/(1 - \rho^2)$. Thus, if $\hat{\sigma}_n^2$ is a strongly consistent estimate of σ_x^2 , an asymptotically valid confidence interval is given by

$$(11) \quad \bar{x}_n \pm t_* \frac{\hat{\sigma}_n}{\sqrt{n}},$$

where t_* is an appropriate Student's t quantile. Our goal is to evaluate the finite sample properties of this interval when $\hat{\sigma}_n$ is produced by BM, OBM and SV methods with $b_n = \lfloor n^\nu \rfloor$ for some ν and $\rho \in \{0.5, 0.95\}$. For SV estimators, we will consider the Tukey-Hanning window (TH) and the modified Bartlett window (Brt). For BM the degrees of freedom for t_* are $a_n - 1$ and for OBM, TH, and Brt the degrees of freedom for t_* are $n - b_n$.

Next, we compare the effects of using different batch sizes and variance estimation techniques on the coverage probabilities of a nominal 95% interval as at (11) for the two settings for ρ . This comparison is based upon the results of 2000 independent replications of the following procedure. In each replication we simulated the AR(1) chain for $1e5$ iterations. We then constructed the interval estimate using BM, OBM, Brt, and TH with three different sampling plans, $b_n = \lfloor n^\nu \rfloor$ where $\nu \in \{1/3, 1/2, 2/3\}$ at five different points ($1e3$, $5e3$, $1e4$, $5e4$, and $1e5$ iterations).

The results with $\rho = 0.5$ are summarized in Table 1. In the calculations with $\nu = 1/3$ and $\nu = 1/2$, all of the calculated coverage probabilities are within 2 standard errors of the nominal .95 level when at least $5e3$ iterations are used. We can also see that for all the settings, the coverage probabilities improve as the number of iterations increase. The choice of $\nu = 2/3$ seems to slightly underestimate the coverage probabilities for small numbers of iterations. Basically, when $\rho = 0.5$ the estimation problem is relatively easy, and all the methods and settings seem to perform well.

Consider the case with $\rho = 0.95$. Table 2 shows the observed coverage probabilities. We can see the coverage probabilities get closer to the nominal .95 level as the number of iterations increases. Also, as ν increases the confidence intervals become more accurate because the strong correlation in the observations is better captured with the larger batch size or truncation point. In this case, the choice of $\nu = 1/3$ performs much worse than the other options analyzed.

Comparing the results in Tables 1 and 2 we see that for highly correlated chains, a larger simulation effort and b_n are required to achieve good coverage. However, with a lower correlation, larger values for b_n can result in worse coverage probabilities, especially for smaller simulation efforts.

4.2. Bayesian Probit Regression. Consider the Lupus Data from van Dyk and Meng [51]. This example is concerned with the occurrence of latent membranous lupus nephritis using y_i , an indicator of the disease (1 for present), x_{i1} , the difference between IgG3 and IgG4 (immunoglobulin G), and x_{i2} , IgA (immunoglobulin A) where $i = 1, \dots, 55$. Suppose

$$\Pr(Y_i = 1) = \Phi(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2})$$

and assign a flat prior (3-dimensional Lebesgue measure) on $\beta := (\beta_0, \beta_1, \beta_2)$. Roy and Hobert [47] verify that the resulting posterior distribution is proper. Our goal is estimating the posterior expectation of β , $E_\pi \beta$.

We will sample from $\pi(\beta|y)$ using the PX-DA algorithm of Liu and Wu [31]. Let X be the 55×3 design matrix whose i th row is $x_i^T = (1, x_{i1}, x_{i2})$

and let $\text{TN}(\mu, \sigma^2, w)$ denote a normal distribution with mean μ and variance σ^2 that is truncated to be positive if $w = 1$ and negative if $w = 0$. Then one iteration $\beta \rightarrow \beta'$ requires:

1. Draw z_1, \dots, z_{55} independently with $z_i \sim \text{TN}(x_i^T \beta, 1, y_i)$.
2. Draw $g^2 \sim \text{Gamma}\left(\frac{55}{2}, \frac{1}{2} \sum_{i=1}^{55} [z_i - x_i^T (X^T X)^{-1} X^T z]^2\right)$ and set $z' = (gz_1, \dots, gz_{55})^T$.
3. Draw $\beta' \sim \text{N}\left((X^T X)^{-1} X^T z', (X^T X)^{-1}\right)$.

Roy and Hobert [47] prove that this sampler is geometrically ergodic.

Just as with the AR(1) example, we want to compare observed coverage probabilities based on different methods of estimating the variance of the asymptotic normal distribution. Of course, we need the true value of the unknown posterior expectation $E_\pi \beta$ to do this. To solve this problem, we calculated an estimate of $E_\pi \beta$ from a long simulation (1e8 iterations) of the PX-DA chain and declared the observed sample averages to be the truth. Table 3 shows the observed sample averages along with MCSEs which were calculated using BM with a batch size of $b_n = n^{1/2}$.

We now turn our attention to comparing coverage probabilities in the context of fixed-width methodology, described in Section 1, for each combination of BM, OBM, Brt, and TH with sampling plans, $b_n = \lfloor n^\nu \rfloor$ where $\nu \in \{1/3, 1/2\}$. Suppose we want to estimate each component of $E_\pi \beta$ to within $\epsilon \in \{.1, .2, .3\}$ while requiring a minimum simulation effort of $n^* \in \{5e4, 1e4, 5e3\}$ respectively. (Recall that the asymptotics require the desired half-width $\epsilon \rightarrow 0$ to achieve an asymptotically valid interval.) That is, using the PX-DA algorithm started from the maximum likelihood estimate $\hat{\beta} = (-1.778, 4.374, 2.482)$, we terminate the simulation the first time the following inequality holds

$$(12) \quad \max \left\{ t_* \frac{\hat{\sigma}_{\beta_j}}{\sqrt{n}}, \text{ for } j = 0, 1, 2 \right\} + \epsilon I(n \leq n^*) + n^{-1} \leq \epsilon$$

where t_* is the appropriate critical value for a nominal 95% interval and $\hat{\sigma}_{\beta_j}$ is an estimate for σ_{β_j} obtained under the settings described above. If (12) was not satisfied, then an additional 10% of the current number of iterations were simulated before checking the criterion again. To estimate the coverage probabilities we obtained 1000 independent replications of this procedure. Table 4 shows the estimated coverage probabilities and the mean number of iterations at termination.

When $\nu = 1/3$, the results are terrible with any method for estimating σ_β^2 or value of ϵ . We can see that even when the simulations are run for

more than $1e5$ iterations (when $\epsilon = .1$), the resulting coverage probabilities are poor. When $\nu = 1/2$, all of the methods result in coverage probabilities slightly lower than the nominal 0.95 level. It also appears that using TH results in slightly better coverage probabilities while requiring slightly more simulation effort. Somewhat surprisingly, the observed coverage probabilities did not uniformly improve as ϵ decreased.

To this point we have examined the performance of multiple confidence intervals individually but there is an obvious inherent multiplicity issue. Thus, we use a Bonferroni correction to calculate simultaneous confidence intervals. We maintain all settings described above with $\epsilon = .2$, except that instead of using nominal 95% we use nominal 98 1/3% confidence intervals. Table 5 shows the estimated coverage probabilities for $E_\pi\beta$ and mean number of iterations at termination based on 1000 independent replications. With $\nu = 1/3$, the results have improved but are still quite poor. However, in the $\nu = 1/2$ case, all individual confidence intervals perform well with observed coverage probabilities close to the nominal 0.9833 level. In addition, the simultaneous intervals have observed coverage probabilities greater than the 0.95 nominal level.

4.2.1. *Comparison to Regeneration.* Finally, we compare BM, Brt, OBM, and TH with regenerative simulation (RS) in terms of coverage probabilities. RS is often viewed as the gold standard; see the references in Section 1. We are still using the model given above and sampling via the PX-DA algorithm, but we are not concerned with fixed-width methodology in this section. Roy and Hobert [47] implement RS for this example and we use their settings exactly except that we use only 50 regenerations. We obtained 1000 independent replications resulting in a mean simulation effort of $7.12e5$ ($3.2e3$). For a fair comparison in terms of simulation effort, confidence intervals for $E_\pi\beta$ were calculated using BM, OBM, and SV using $b_n = \lfloor n^{1/2} \rfloor$ from 1000 independent replications with $7e5$ iterations each. Table 6 shows the resulting coverage probabilities for $E_\pi\beta$. The coverage probabilities from BM, Brt, OBM and TH are all slightly larger than those of RS, however, the results from all the methods are within two standard errors of the nominal 0.95 level.

4.3. *Summary.* In our examples, we consider different estimators of σ_g^2 . Roughly, all of the methods considered resulted in similar performance in terms of estimated coverage probabilities. Recall, OBM and Brt are asymptotically equivalent and the simulation results show there is little difference between the two in finite samples. However, the TH estimator tends to perform slightly better than OBM and Brt. In our experience, Brt (and SV

methods in general) tended to be slightly faster than OBM from a computational perspective. As suggested by the theoretical results in Section 3.3, the estimator from BM was observed to be more variable than OBM. This was consistent in both examples in multiple realizations of the simulation.

Using the Bayesian probit regression model we compared our methods to RS. The resulting simulation showed all of the methods performed very well. The advantage of RS is that in the fixed-width setting the actual chain does not need to be stored as the simulation progresses. However, RS requires a theoretical cost that, while not overly burdensome in our view, may dissuade the typical practitioner. The resulting simulation is also dependent on the length of the regeneration tours which can be extremely long in even moderately large finite state spaces or as the dimension of the Markov chain increases (Gilks et al. [14]) or in variable-at-a-time Metropolis-Hastings implementations (Neath and Jones [39]). In contrast, BM, OBM, and SV are relatively simple to implement though they can require saving the entire chain if fixed-width methods are employed. Given the current price of computer memory, this is clearly not the obstacle it was in the past.

Another simulation goal was to investigate the finite sample behavior of different batch size selection. Theoretically, we showed that the batch size should increase at a rate proportional to $n^{1/3}$ but the proportionality constant is unknown. In our examples, using $b_n = \lfloor n^{1/3} \rfloor$ seemed to give very poor results because the batch size or truncation point was too small. In realistic examples with higher correlations, the larger batch size $b_n = \lfloor n^{1/2} \rfloor$ worked well agreeing with the previous work of Jones et al. [24]. Our investigation of $b_n = \lfloor n^{2/3} \rfloor$ worked well in high correlation settings, though for long chains more computational effort was required.

On balance, we would recommend using the SV method with the Tukey-Hanning window with $b_n = \lfloor n^{1/2} \rfloor$ as a default method. If the required moment conditions for the SV methods were too much, then we would employ OBM.

APPENDIX A: BROWNIAN MOTION

Let $B = \{B(t), t \geq 0\}$ denote a standard Brownian motion. Define $\bar{B}_j(k) := k^{-1}(B(j+k) - B(j))$ and $\bar{B}_n := n^{-1}B(n)$. We will require the following two results from Csörgő and Révész [8] on the increments of Brownian motion.

LEMMA 1. *For all $\epsilon > 0$ and for almost all sample paths there exists $n_0(\epsilon)$ such that for all $n \geq n_0$*

$$|B(n)| < (1 + \epsilon) [2n \log \log n]^{1/2} .$$

LEMMA 2. *Suppose Assumption 2 holds, then for all $\epsilon > 0$ and for almost all sample paths, there exists $n_0(\epsilon)$ such that for all $n \geq n_0$*

$$\sup_{0 \leq t \leq n-b_n} \sup_{0 \leq s \leq b_n} |B(t+s) - B(t)| < (1 + \epsilon) \left[2b_n \left(\log \frac{n}{b_n} + \log \log n \right) \right]^{1/2}.$$

A *strong invariance principle* holds if there exists a nonnegative increasing function $\psi(n)$ on the positive integers, a constant $0 < \sigma_g < \infty$, and a sufficiently rich probability space Ω such that

$$(13) \quad \left| \sum_{i=1}^n g(X_i) - nE_\pi g - \sigma_g B(n) \right| = O(\psi(n)) \quad \text{w.p. 1 as } n \rightarrow \infty$$

where the w.p. 1 in (13) means for almost all sample paths. Alternatively, (13) can be expressed as there exists n_0 and a finite random variable C such that for almost all $\omega \in \Omega$,

$$(14) \quad \left| \sum_{i=1}^n Y_i - nE_\pi g - \sigma_g B(n) \right| < C(\omega)\psi(n)$$

for all $n > n_0$. A strong invariance principle is enough to guarantee both a strong law (1), a central limit theorem (2) and a functional central limit theorem among other properties; see Philipp and Stout [40] and Damerdjji [9]. We will rely on the following result to connect the strong invariance principle to the convergence rate of a Harris ergodic Markov chain; see Bednorz and Latuszyński [2] and Jones et al. [24] for a proof.

LEMMA 3. *Let $g : X \mapsto \mathbb{R}$ be a Borel function and let X be a geometrically ergodic Markov chain with invariant distribution π . Suppose Assumption 3 holds and $E_\pi |g|^{2+\delta+\epsilon} < \infty$ for some $\delta > 0$ and some $\epsilon > 0$, then a strong invariance principle holds with $\psi(n) = n^\alpha \log n$ where $\alpha = 1/(2 + \delta)$.*

APPENDIX B: STRONG CONSISTENCY PROOFS

B.1. Proof of Theorem 1. The proof will be constructed in 3 stages given in lemmas below but first we define some notation. Recall that $X = \{X_1, X_2, \dots\}$ is a Harris ergodic Markov chain with invariant distribution π and $Y_i = g(X_i) - E_\pi g$ for $i = 1, 2, 3, \dots$. Further define $\bar{Y}_j(k) = k^{-1} \sum_{i=1}^k Y_{j+i}$ for $j = 0, \dots, n - b_n$ and $k = 1, \dots, b_n$ and $\bar{Y}_n = n^{-1} \sum_{i=1}^n Y_i$. Next let

$$\begin{aligned} \hat{\sigma}_{w,n}^2 &:= \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} k^2 \Delta_2 w_n(k) [\bar{Y}_j(k) - \bar{Y}_n]^2, \quad \text{and} \\ \tilde{\sigma}_*^2 &:= \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} k^2 \Delta_2 w_n(k) [\bar{B}_j(k) - \bar{B}_n]^2. \end{aligned}$$

LEMMA 4. *Suppose (13) holds with $\psi(n) = n^\alpha \log n$ where $\alpha = 1/(2 + \delta)$ and Assumptions 1 and 2 hold. If, as $n \rightarrow \infty$,*

$$(15) \quad b_n^{1/2} n^\alpha (\log n)^{3/2} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0 \quad \text{and}$$

$$(16) \quad n^{2\alpha} (\log n)^2 \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0 ,$$

then $\hat{\sigma}_{w,n}^2 - \sigma_g^2 \tilde{\sigma}_*^2 \rightarrow 0$ w.p. 1.

PROOF. Notice

$$\hat{\sigma}_{w,n}^2 - \sigma_g^2 \tilde{\sigma}_*^2 = \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} k^2 \Delta_2 w_n(k) \left([\bar{Y}_j(k) - \bar{Y}_n]^2 - \sigma_g^2 [\bar{B}_j(k) - \bar{B}_n]^2 \right) .$$

Let $A_k = k [\bar{Y}_j(k) - \sigma_g \bar{B}_j(k)]$, $D_k = B(j+k) - B(j)$, $E_{n,k} = k \bar{B}_n$ and $F_{n,k} = k [\bar{Y}_n - \sigma_g \bar{B}_n]$. Then

$$\begin{aligned} k[\bar{Y}_j(k) - \bar{Y}_n] &= k [\bar{Y}_j(k) - \bar{Y}_n \pm \sigma_g \bar{B}_j(k) \pm \sigma_g \bar{B}_n] \\ &= k [\bar{Y}_j(k) - \sigma_g \bar{B}_j(k)] + \sigma_g k \bar{B}_j(k) \\ &\quad - \sigma_g k \bar{B}_n - k [\bar{Y}_n - \sigma_g \bar{B}_n] \\ &= k [\bar{Y}_j(k) - \sigma_g \bar{B}_j(k)] + \sigma_g [B(j+k) - B(j)] \\ &\quad - \sigma_g k \bar{B}_n - k [\bar{Y}_n - \sigma_g \bar{B}_n] \\ &= A_k + \sigma_g (D_k - E_{n,k}) - F_{n,k} . \end{aligned}$$

Hence

$$\begin{aligned} \left| \hat{\sigma}_{w,n}^2 - \sigma_g^2 \tilde{\sigma}_*^2 \right| &\leq \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} \left| \Delta_2 w_n(k) \left[(A_k + \sigma_g (D_k - E_{n,k}) - F_{n,k})^2 - \sigma_g^2 (D_k - E_{n,k})^2 \right] \right| \\ &\leq \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \left(A_k^2 + F_{n,k}^2 + 2\sigma_g |A_k D_k| + 2\sigma_g |A_k E_{n,k}| \right. \\ (17) \quad &\quad \left. + 2|A_k F_{n,k}| + 2\sigma_g |D_k F_{n,k}| + 2\sigma_g |E_{n,k} F_{n,k}| \right) . \end{aligned}$$

It suffices to show that each of the 7 sums in (17) tend to 0 as $n \rightarrow \infty$. By our assumption of (13) we have that for sufficiently large n

$$(18) \quad \left| \sum_{i=1}^n Y_i - \sigma_g B(n) \right| \leq C n^\alpha \log n .$$

1. From (18), we obtain

$$\begin{aligned} |A_k| &= \left| \left(\sum_{i=1}^{j+k} Y_i - \sigma_g B(j+k) \right) - \left(\sum_{i=1}^j Y_i - \sigma_g B(j) \right) \right| \\ &\leq C(j+k)^\alpha \log(j+k) + C(j)^\alpha \log(j) \end{aligned}$$

and since $j+k \leq n$,

$$(19) \quad |A_k| \leq 2Cn^\alpha \log n .$$

Hence

$$\frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| A_k^2 \leq 4C^2 n^{2\alpha} (\log n)^2 \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0$$

as $n \rightarrow \infty$ by (16).

2. From (18) and the fact that $k \leq b_n \leq n$

$$(20) \quad |F_{n,k}| \leq Ckn^{\alpha-1} \log n ,$$

resulting in

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| F_{n,k}^2 &\leq C^2 n^{2\alpha-2} (\log n)^2 \sum_{k=1}^{b_n} k^2 |\Delta_2 w_n(k)| \\ &\leq C^2 b_n^2 n^{2\alpha-2} (\log n)^2 \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ by (16).

3. From Lemma 2

$$\begin{aligned} |D_k| &= |B(j+k) - B(j)| \\ &\leq \sup_{0 \leq t \leq n-b_n} \sup_{0 \leq s \leq b_n} |B(t+s) - B(t)| \\ &\leq (1+\epsilon) \left(2b_n \left(\log \frac{n}{b_n} + \log \log n \right) \right)^{1/2} \\ (21) \quad &\leq 2(1+\epsilon) b_n^{1/2} (\log n)^{1/2} . \end{aligned}$$

Combining (19) with (21), we obtain

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| 2\sigma_g |A_k D_k| \\ \leq 8C\sigma_g (1+\epsilon) b_n^{1/2} n^\alpha (\log n)^{3/2} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ by (15).

4. From Lemma 1

$$(22) \quad |E_{n,k}| \leq \sqrt{2}(1 + \epsilon)kn^{-1/2}(\log \log n)^{1/2}.$$

Combining (19) with (22), get

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| 2\sigma_g |A_k E_{n,k}| \\ \leq 2^{5/2} C\sigma_g (1 + \epsilon) n^{\alpha-1/2} \log n (\log \log n)^{1/2} \sum_{k=1}^{b_n} k |\Delta_2 w_n(k)| \\ \leq 2^{5/2} C\sigma_g (1 + \epsilon) (b_n/n)^{1/2} b_n^{1/2} n^\alpha (\log n)^{3/2} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ by (15) and Assumption 2.

5. From (19) and (20) we have

$$\frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| 2 |A_k F_{n,k}| \leq 4C^2 n^{2\alpha-1} (\log n)^2 \sum_{k=1}^{b_n} k |\Delta_2 w_n(k)| \rightarrow 0$$

as $n \rightarrow \infty$ by (16) and Assumption 2.

6. From (20) and (21)

$$\frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| 2\sigma_g |D_k F_{n,k}| \leq 4\sigma_g (1 + \epsilon) b_n^{1/2} n^{\alpha-1} (\log n)^{3/2} \sum_{k=1}^{b_n} k |\Delta_2 w_n(k)| \rightarrow 0$$

as $n \rightarrow \infty$ by (15) and Assumption 2.

7. From (20) and (22)

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} |\Delta_2 w_n(k)| 2\sigma_g |E_{n,k} F_{n,k}| \\ \leq 2^{3/2} C\sigma_g (1 + \epsilon) n^{\alpha-3/2} \log n (\log \log n)^{1/2} \sum_{k=1}^{b_n} k^2 |\Delta_2 w_n(k)| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ by (15) and Assumption 2.

□

LEMMA 5. *Let X be a geometrically ergodic Markov chain with invariant distribution π and $g : X \rightarrow \mathbb{R}$ be a Borel function with $E_\pi |g|^{4+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$. Set $h(X_i) = [g(X_i) - E_\pi g]^2$ for $i \geq 1$. If Assumptions 2 and 3 hold and $b_n^{-1} n^{2\alpha} \log n \rightarrow 0$ as $n \rightarrow \infty$ where $\alpha = 1/(4 + \delta)$, then $b_n^{-1} \sum_{i=1}^{b_n} h(X_i)$ and $b_n^{-1} \sum_{i=n-b_n+1}^n h(X_i)$ stay bounded as $n \rightarrow \infty$ w.p. 1.*

PROOF. The Ergodic Theorem implies that, w.p. 1, $b_n^{-1} \sum_{i=1}^{b_n} h(X_i)$ converges to a finite limit and hence stays bounded as $n \rightarrow \infty$ w.p. 1.

Note that if $E_\pi |g|^{4+\delta+\epsilon} < \infty$, then $E_\pi |h|^{2+\delta/2+\epsilon/2} < \infty$ and hence our assumptions with Lemma 3 yield the existence of n_0 such that if $n > n_0$,

$$(23) \quad \left| \sum_{i=1}^n h(X_i) - nE_\pi h(X_1) - \sigma_h B(n) \right| < Cn^{2\alpha} \log n$$

where $\alpha = 1/(4 + \delta)$.

Next, for all $\varepsilon > 0$ and sufficiently large $n(\varepsilon)$

$$\begin{aligned} & \frac{1}{b_n} \left| \sum_{i=n-b_n+1}^n h(X_i) \right| = \frac{1}{b_n} \left| \sum_{i=1}^n h(X_i) - \sum_{i=1}^{n-b_n} h(X_i) \right| \\ &= \frac{1}{b_n} \left| \left(\sum_{i=1}^n h(X_i) - nE_\pi h(X_1) - \sigma_h B(n) \right) \right. \\ & \quad \left. - \left(\sum_{i=1}^{n-b_n} h(X_i) - (n-b_n)E_\pi h(X_1) - \sigma_h B(n-b_n) \right) \right. \\ & \quad \left. + \sigma_h (B(n) - B(n-b_n)) + b_n E_\pi h(X_1) \right| \\ &\leq \frac{1}{b_n} \left[\left| \sum_{i=1}^n h(X_i) - nE_\pi h(X_1) - \sigma_h B(n) \right| \right. \\ & \quad \left. + \left| \sum_{i=1}^{n-b_n} h(X_i) - (n-b_n)E_\pi h(X_1) - \sigma_h B(n-b_n) \right| \right. \\ & \quad \left. + \sigma_h |B(n) - B(n-b_n)| + b_n E_\pi h(X_1) \right] \\ &\leq \frac{1}{b_n} \left[2Cn^{2\alpha} \log n + (1 + \varepsilon) \left(2b_n \left(\log \frac{n}{b_n} + \log \log n \right) \right)^{1/2} \right] + E_\pi h(X_1) \\ &= E_\pi h(X_1) + 2Cb_n^{-1} n^{2\alpha} \log n + O\left((b_n^{-1} \log n)^{1/2} \right) \end{aligned}$$

where the second inequality follows from (23) and Lemma 2. Hence, $b_n^{-1} \left| \sum_{i=n-b_n+1}^n h(X_i) \right|$ stays bounded w.p. 1 since $b_n^{-1} n^{2\alpha} \log n \rightarrow 0$ as $n \rightarrow \infty$. \square

LEMMA 6. *Suppose Assumptions 1 and 2 hold. Further assume the conditions of Lemma 5 and that*

1. *there exists a constant $c \geq 1$ such that $\sum_n (b_n/n)^c < \infty$,*

2.

$$b_n n^{-1} \sum_{k=1}^{b_n} k |\Delta_1 w_n(k)| \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ and}$$

3. $b_n n^{-1} \log n \rightarrow 0$ as $n \rightarrow \infty$.

Then there exists a sequence of random variables d_n such that

$$\hat{\sigma}_{w,n}^2 + d_n = \hat{\sigma}_S^2$$

and $d_n \rightarrow 0$ w.p. 1 as $n \rightarrow \infty$. Also, $\tilde{\sigma}_*^2 \rightarrow 1$ as $n \rightarrow \infty$.

PROOF. This follows immediately from the conclusion of Lemma 5 and results in Damerджи [9], p. 1430. \square

PROOF OF THEOREM 1. The result follows by combining Lemmas 3, 4, and 6. Note that since X is geometrically ergodic and $E_\pi |g|^{4+\delta+\epsilon} < \infty$ for some $\delta > 0$ and $\epsilon > 0$, the conclusion of Lemma 4 holds with $\alpha = 1/(4 + \delta)$. \square

B.2. A Condition for Lag Windows.

LEMMA 7. Suppose w is defined on $[0, 1]$ such that $w(0) = 1$ and $w(1) = 0$. Further assume that w is twice continuously differentiable. Also, assume that D_1 and D_2 are finite constants such that $|w'(x)| \leq D_1$ and $|w''(x)| \leq D_2$. Then, as $n \rightarrow \infty$, condition (a) of Theorem 1 holds if $b_n^2 n^{-1} \rightarrow 0$ while condition (d) of Theorem 1 is satisfied if $b_n^{-1} n^{2\alpha} (\log n)^3 \rightarrow 0$.

PROOF. Suppose $1 \leq k \leq b_n - 1$ and let

$$\begin{aligned} \Delta_2 w \left(\frac{k}{b_n} \right) &= w \left(\frac{k-1}{b_n} \right) - 2w \left(\frac{k}{b_n} \right) + w \left(\frac{k+1}{b_n} \right) \\ &= \left[w \left(\frac{k-1}{b_n} \right) - w \left(\frac{k}{b_n} \right) \right] - \left[w \left(\frac{k}{b_n} \right) - w \left(\frac{k+1}{b_n} \right) \right]. \end{aligned}$$

The mean value theorem guarantees the existence of $c_1 \in \left(\frac{k-1}{b_n}, \frac{k}{b_n} \right)$ and $c_2 \in \left(\frac{k}{b_n}, \frac{k+1}{b_n} \right)$ so that

$$\Delta_2 w \left(\frac{k}{b_n} \right) = -\frac{w'(c_1)}{b_n} + \frac{w'(c_2)}{b_n}.$$

A second application of the mean value theorem yields a constant $c \in (c_1, c_2)$ such that

$$\Delta_2 w \left(\frac{k}{b_n} \right) = \frac{w''(c)}{b_n} (c_2 - c_1).$$

Since $c_2 - c_1 \leq 2/b_n$ and $|w''(x)| \leq D_2$ we have

$$\left| \Delta_2 w \left(\frac{k}{b_n} \right) \right| \leq \frac{2D_2}{b_n^2}.$$

Then

$$\begin{aligned} \sum_{k=1}^{b_n} \left| \Delta_2 w \left(\frac{k}{b_n} \right) \right| &= \sum_{k=1}^{b_n-1} \left| \Delta_2 w \left(\frac{k}{b_n} \right) \right| + \left| w \left(\frac{b_n-1}{b_n} \right) - w(1) \right| \\ &\leq \sum_{k=1}^{b_n-1} \frac{2D_2}{b_n^2} + \frac{D_1}{b_n} \\ &= \frac{2(b_n-1)D_2}{b_n^2} + \frac{D_1}{b_n}. \end{aligned}$$

Next observe that

$$\begin{aligned} b_n n^{2\alpha} (\log n)^3 \left(\sum_{k=1}^{b_n} \left| \Delta_2 w \left(\frac{k}{b_n} \right) \right| \right)^2 &\leq b_n n^{2\alpha} (\log n)^3 \left[\frac{2(b_n-1)D_2}{b_n^2} + \frac{D_1}{b_n} \right]^2 \\ &= b_n^{-1} n^{2\alpha} (\log n)^3 \left[\frac{2(b_n-1)D_2}{b_n} + D_1 \right]^2 \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

since $b_n^{-1} n^{2\alpha} (\log n)^3$ as $n \rightarrow \infty$ by assumption. The proofs that our conditions also imply the remaining portion of condition (d) and condition (a) are similar and hence are omitted. \square

B.3. Proof of Theorem 2. Consider the modified Bartlett lag window defined in Remark 4. Then $\Delta_2 w_n(b_n) = b_n^{-1}$ and $\Delta_2 w_n(k) = 0$ for $k = 1, 2, \dots, b_n - 1$ so that

$$(24) \quad \hat{\sigma}_{w,n}^2 = n^{-1} \sum_{j=0}^{n-b_n} \sum_{k=1}^{b_n} k^2 \Delta_2 w_n(k) [\bar{Y}_j(k) - \bar{Y}_n]^2 = \frac{b_n}{n} \sum_{j=0}^{n-b_n} [\bar{Y}_j(b_n) - \bar{Y}_n]^2$$

and

$$(25) \quad \tilde{\sigma}_*^2 = \frac{b_n}{n} \sum_{j=0}^{n-b_n} [\bar{B}_j(b_n) - \bar{B}_n]^2.$$

Further, (15) and (16) are satisfied if $b_n^{-1} n^{2\alpha} (\log n)^3 \rightarrow 0$ as $n \rightarrow \infty$ where $\alpha = 1/(2 + \delta)$, and hence our assumptions imply the conclusion of Lemma 4.

Since

$$\hat{\sigma}_{OBM}^2 = \frac{nb_n}{(n-b_n)(n-b_n+1)} \sum_{j=0}^{n-b_n} [\bar{Y}_j(b_n) - \bar{Y}_n]^2,$$

the conclusion would follow from Lemmas 3 and 4 if $\tilde{\sigma}_*^2 \rightarrow 1$ w.p. 1 as $n \rightarrow \infty$ because $\hat{\sigma}_{w,n}^2$ is asymptotically equivalent to $\hat{\sigma}_{OBM}^2$: as $n \rightarrow \infty$

$$\frac{\hat{\sigma}_{w,n}^2}{\hat{\sigma}_{OBM}^2} = \left(1 - \frac{b_n}{n}\right) \left(1 - \frac{b_n}{n} + \frac{1}{b_n}\right) \rightarrow 1.$$

Define $U_i := B(i) - B(i-1)$, the increments of Brownian motion, so that U_1, \dots, U_n are i.i.d. $N(0, 1)$. Further define $T_i = U_i - \bar{B}_n$ for $i = 1, \dots, n$ and set

$$\begin{aligned} \tilde{d}_n &= \frac{1}{n} \left[\sum_{l=1}^{b_n} \frac{1}{b_n} \left(\sum_{i=1}^{l-1} T_i^2 + \sum_{i=n-b_n+l+1}^n T_i^2 \right) \right] \\ &+ 2 \sum_{s=1}^{b_n-1} \left[\sum_{l=1}^{b_n-s} \frac{1}{b_n} \left(\sum_{i=1}^{l-1} T_i T_{s+i} + \sum_{i=n-b_n+l+1}^{n-s} T_i T_{s+i} \right) \right] \end{aligned}$$

where any empty sums are defined to be zero. Letting $\tilde{\gamma}_n(i) = \tilde{\gamma}_n(-i) := n^{-1} \sum_{t=1}^{n-i} (U_t - \bar{B}_n)(U_{t+i} - \bar{B}_n)$ Damerdji [9] shows that under Assumption 2

$$\tilde{\sigma}_*^2 + \tilde{d}_n = \sum_{s=-(b_n-1)}^{b_n-1} \left(1 - \frac{|k|}{b_n}\right) \tilde{\gamma}_n(s).$$

Moreover, under conditions (a) and (b) of Theorem 2, $\tilde{\sigma}_*^2 + \tilde{d}_n \rightarrow 1$ as $n \rightarrow \infty$ w.p. 1. Thus the proof will be complete if we can show that $\tilde{d}_n \rightarrow 0$ as $n \rightarrow \infty$ w.p. 1.

LEMMA 8. *Suppose Assumption 2 holds. If*

1. *there exists a constant $c \geq 1$ such that $\sum_n (b_n/n)^{2c} < \infty$;*
2. *there exists an integer n_0 and a constant c_1 such that for all $n \geq n_0$ we have $\log n/b_n \leq c_1$; and*
3. *as $n \rightarrow \infty$ we have $b_n^4 n^{-3} \log \log n \rightarrow 0$ and $b_n^2 n^{-2} \log \log n \rightarrow 0$,*

then $\tilde{d}_n \rightarrow 0$ w.p. 1 as $n \rightarrow \infty$.

Before we present the proof we recall two results which will be used below.

LEMMA 9. *(Kendall and Stuart [27]) If $Z \sim \chi_v^2$, then for all positive integers r there exists a constant $K := K(r)$ such that $E[(Z-v)^{2r}] \leq K v^r$.*

LEMMA 10. (Whittle [53]) Let U_1, \dots, U_n be i.i.d. standard normal variables and $A = \sum_j \sum_k a_{jk} U_j U_k$ where a_{jk} are real coefficients, then for $c \geq 1$ we have

$$E \left[|A - EA|^{2c} \right] \leq K(c) \left(\sum_j \sum_k a_{jk}^2 \right)^c,$$

for some constant $K_c < \infty$ depending only on c .

PROOF OF LEMMA 8. Note

$$(26) \quad |\tilde{d}_n| \leq \frac{1}{n} \left| \sum_{l=1}^{b_n} b_n^{-1} \left(\sum_{i=1}^{l-1} T_i^2 + \sum_{i=n-b_n+l+1}^n T_i^2 \right) \right|$$

$$(27) \quad + \frac{1}{n} \left| 2 \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} b_n^{-1} \left(\sum_{i=1}^{l-1} T_i T_{s+i} + \sum_{i=n-b_n+l+1}^{n-s} T_i T_{s+i} \right) \right|.$$

We will show that (26) and (27) each converge almost surely to zero implying the desired result.

First consider (26),

$$\begin{aligned} \frac{1}{n} \left| \sum_{l=1}^{b_n} \frac{1}{b_n} \left(\sum_{i=1}^{l-1} T_i^2 + \sum_{i=n-b_n+l+1}^n T_i^2 \right) \right| &\leq \frac{1}{n} \left| \sum_{l=1}^{b_n} \frac{1}{b_n} \left(\sum_{i=1}^{b_n-1} T_i^2 + \sum_{i=n-b_n+2}^n T_i^2 \right) \right| \\ &= \frac{1}{n} \left| \sum_{i=1}^{b_n-1} T_i^2 + \sum_{i=n-b_n+2}^n T_i^2 \right| \\ &= \frac{1}{n} \left| \sum_{i=1}^{b_n-1} (U_i - \bar{B}_n)^2 + \sum_{i=n-b_n+2}^n (U_i - \bar{B}_n)^2 \right| \\ &\leq \frac{1}{n} \left| \sum_{i=1}^{b_n-1} U_i^2 + \sum_{i=n-b_n+2}^n U_i^2 \right| \\ &\quad + \frac{1}{n} \left| 2\bar{B}_n \left(\sum_{i=1}^{b_n-1} U_i + \sum_{i=n-b_n+2}^n U_i \right) \right| \\ &\quad + \frac{1}{n} \left| 2(b_n - 1)\bar{B}_n^2 \right|. \end{aligned}$$

Now we show that each of the three terms above tend to zero.

1. Since U_1^2, \dots, U_n^2 are i.i.d. χ_1^2 , $\sum_{i=1}^{b_n-1} U_i^2 + \sum_{i=n-b_n+2}^n U_i^2 \sim \chi_{2(b_n-1)}^2$. By

Lemma 9 we have for every positive integer r that

$$E \left\{ \left[\frac{1}{n} \left(\sum_{i=1}^{b_n-1} U_i^2 + \sum_{i=n-b_n+2}^n U_i^2 \right) - \frac{2(b_n-1)}{n} \right]^{2r} \right\} \leq K \left(\frac{2(b_n-1)}{n^2} \right)^r .$$

Now choose $r > 1$ so that

$$\sum_n K \left(\frac{2(b_n-1)}{n^2} \right)^r \leq K \sum_n \left(\frac{2}{n} \right)^r < \infty .$$

Let $\epsilon > 0$ be arbitrary. Then by Markov's inequality we have

$$\epsilon \sum_n \Pr \left\{ \left[\frac{1}{n} \left(\sum_{i=1}^{b_n-1} U_i^2 + \sum_{i=n-b_n+2}^n U_i^2 \right) - \frac{2(b_n-1)}{n} \right]^{2r} \geq \epsilon \right\} < \infty .$$

Now a standard Borel-Cantelli argument (pp 59-60 Billingsley [3]) yields

$$\left[\frac{1}{n} \left(\sum_{i=1}^{b_n-1} U_i^2 + \sum_{i=n-b_n+2}^n U_i^2 \right) - \frac{2(b_n-1)}{n} \right]^{2r} \rightarrow 0 \quad \text{w.p. 1 as } n \rightarrow \infty .$$

Hence

$$\frac{1}{n} \left| \sum_{i=1}^{b_n-1} U_i^2 + \sum_{i=n-b_n+2}^n U_i^2 \right| \rightarrow 0 \quad \text{w.p. 1 as } n \rightarrow \infty$$

since $b_n/n \rightarrow 0$ as $n \rightarrow \infty$.

2. Notice

$$\frac{1}{n} \left| 2\bar{B}_n \left(\sum_{i=1}^{b_n-1} U_i + \sum_{i=n-b_n+2}^n U_i \right) \right| \leq \frac{2|\bar{B}_n|}{n} \sum_{i=1}^n |U_i| .$$

Since the $|U_i|$ are i.i.d. following a half-normal distribution the classical strong law implies $n^{-1} \sum_{i=1}^n |U_i| \rightarrow \sqrt{2/\pi}$ w.p. 1. (Recall that $E|U_i| = \sqrt{2/\pi}$ and $\text{Var}|U_i| = 1 - 2/\pi$.) Combining this with Lemma 1 yields for every $\epsilon > 0$ and sufficiently large n

$$\frac{2|\bar{B}_n|}{n} \sum_{i=1}^n |U_i| \leq 2^{3/2}(1+\epsilon) \left[\frac{\log \log n}{n} \right]^{1/2} \frac{1}{n} \sum_{i=1}^n |U_i| \rightarrow 0$$

with probability 1 as $n \rightarrow \infty$.

3. Using Lemma 1 shows that for every $\epsilon > 0$ and large enough n

$$\frac{1}{n} \left| 2(b_n - 1) \bar{B}_n^2 \right| \leq 4(1 + \epsilon)^2 \frac{(b_n - 1) \log \log n}{n^2} \rightarrow 0$$

since $b_n/n \rightarrow 0$ as $n \rightarrow \infty$.

This establishes that the term in (26) converges almost surely to 0 as $n \rightarrow \infty$.

Next, consider (27)

$$\begin{aligned} & \frac{1}{n} \left| 2 \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \frac{1}{b_n} \left(\sum_{i=1}^{l-1} T_i T_{s+i} + \sum_{i=n-b_n+l+1}^{n-s} T_i T_{s+i} \right) \right| \\ &= \frac{1}{n} \left| 2 \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \frac{1}{b_n} \left(\sum_{i=1}^{l-1} (U_i - \bar{B}_n) (U_{s+i} - \bar{B}_n) + \sum_{i=n-b_n+l+1}^{n-s} (U_i - \bar{B}_n) (U_{s+i} - \bar{B}_n) \right) \right| \\ &\leq \frac{1}{n} \left| 2 \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \frac{1}{b_n} \left(\sum_{i=1}^{l-1} U_i U_{s+i} + \sum_{i=n-b_n+l+1}^{n-s} U_i U_{s+i} \right) \right| \\ &\quad + \frac{1}{n} \left| 2 \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \frac{\bar{B}_n}{b_n} \left(\sum_{i=1}^{l-1} (-U_i - U_{s+i}) + \sum_{i=n-b_n+l+1}^{n-s} (-U_i - U_{s+i}) \right) \right| \\ &\quad + \frac{1}{n} \left| 4 \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \frac{\bar{B}_n^2}{b_n} (b_n - 1) \right|. \end{aligned}$$

We show that each of the three terms above tend to zero.

1. Letting

$$A(b_n) = \frac{2}{nb_n} \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \left(\sum_{i=1}^{l-1} U_i U_{s+i} + \sum_{i=n-b_n+l+1}^{n-s} U_i U_{s+i} \right)$$

it is straightforward to see that $A(b_n)$ can be written in the form $\sum_j \sum_k a_{jk} U_j U_k$ where the coefficients satisfy

$$0 \leq a_{jk} \leq \frac{2(b_n - 1)}{nb_n}.$$

Since $U_1, \dots, U_{b_n-1}, U_{n-b_n+2}, \dots, U_n$ are i.i.d. $N(0, 1)$ we have $EU_i U_{s+i} = 0$ and hence $EA(b_n) = 0$. Thus, we can apply Lemma 10 to obtain for $c \geq 1$ and a constant $K_c < \infty$

$$EA(b_n)^{2c} \leq K_c \left(\sum_j \sum_k a_{jk}^2 \right)^c \leq K_c \left(\frac{4(b_n - 1)}{n} \right)^{2c}.$$

Then, similar to the argument given above, Markov's inequality and Borel-Cantelli yields $|A(b_n)| \rightarrow 0$ as $n \rightarrow \infty$ since by assumption there exists a constant $c \geq 1$ such that $\sum_n (b_n/n)^{2c} < \infty$.

2. Notice

$$\begin{aligned} & \frac{2}{nb_n} \left| \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \bar{B}_n \left(\sum_{i=1}^{l-1} (-U_i - U_{s+i}) + \sum_{i=n-b_n+l+1}^{n-s} (-U_i - U_{s+i}) \right) \right| \\ & \leq \frac{2}{nb_n} |\bar{B}(n)| \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \left(\sum_{i=1}^{l-1} (|U_i| + |U_{s+i}|) + \sum_{i=n-b_n+l+1}^{n-s} (|U_i| + |U_{s+i}|) \right) \\ & \leq \frac{2}{nb_n} |\bar{B}(n)| \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \left(2 \sum_{i=1}^{b_n} |U_i| + 2 \sum_{i=n-b_n+1}^n |U_i| \right) \\ & = \frac{2(b_n-1)}{n^2} |B(n)| \left(\sum_{i=1}^{b_n} |U_i| + \sum_{i=n-b_n+1}^n |U_i| \right). \end{aligned}$$

Then for n large enough and $\epsilon > 0$ Lemma 1 implies the right-hand side is bounded above by

$$2^{3/2}(1+\epsilon) \left[\frac{(b_n-1)^2 \log \log n}{n^3} \right]^{1/2} \left(\sum_{i=1}^{b_n} |U_i| + \sum_{i=n-b_n+1}^n |U_i| \right).$$

Since $b_n^4 n^{-3} \log \log n \rightarrow 0$ as $n \rightarrow \infty$ by assumption it suffices to show that

$$\frac{1}{b_n} \left(\sum_{i=1}^{b_n} |U_i| + \sum_{i=n-b_n+1}^n |U_i| \right)$$

stays bounded with probability 1 as $n \rightarrow \infty$.

The classical SLLN implies $b_n^{-1} \sum_{i=1}^{b_n} |U_i|$ stays bounded w.p. 1. All that remains is to show that $b_n^{-1} \sum_{i=n-b_n+1}^n |U_i|$ stays bounded almost surely as $n \rightarrow \infty$. Since the half-normal distribution enjoys a moment generating function we can appeal to the classical strong invariance principle (Komlós et al. [28, 29], Major [32]) to obtain that for sufficiently large n with probability 1 there is a constant C' such that

$$(28) \quad \left| \sum_{i=1}^n |U_i| - n\sqrt{2/\pi} - (1 - 2/\pi)B(n) \right| \leq C' \log n.$$

Now observe that

$$\begin{aligned}
 \frac{1}{b_n} \sum_{i=n-b_n+1}^n |U_i| &= \frac{1}{b_n} \left| \sum_{i=1}^n |U_i| - \sum_{i=1}^{n-b_n} |U_i| \right| \\
 &= \frac{1}{b_n} \left| \left(\sum_{i=1}^n |U_i| - n\sqrt{2/\pi} - (1-2/\pi)B(n) \right) \right. \\
 &\quad \left. - \left(\sum_{i=1}^{n-b_n} |U_i| - (n-b_n)\sqrt{2/\pi} - (1-2/\pi)B(n-b_n) \right) \right. \\
 &\quad \left. + (1-2/\pi)(B(n) - B(n-b_n)) + b_n\sqrt{2/\pi} \right|.
 \end{aligned}$$

Hence, by (28) and Lemma 2, for sufficiently large n

$$\begin{aligned}
 \frac{1}{b_n} \sum_{i=n-b_n+1}^n |U_i| &\leq \frac{1}{b_n} \left(2C' \log n + (1+\epsilon) \left[2b_n \left(\log \frac{n}{b_n} + \log \log n \right) \right]^{1/2} + b_n\sqrt{2/\pi} \right) \\
 &= \sqrt{2/\pi} + 2C'b_n^{-1} \log n + O\left((b_n^{-1} \log n)^{1/2}\right).
 \end{aligned}$$

Hence, $b_n^{-1} \sum_{i=n-b_n+1}^n |U_i|$ stays bounded w.p. 1 since, by assumption, $b_n^{-1} \log n$ is bounded for all sufficiently large n .

3. Using Lemma 1 we have for every $\epsilon > 0$ and sufficiently large n

$$\begin{aligned}
 \frac{4}{n} \left| \sum_{s=1}^{b_n-1} \sum_{l=1}^{b_n-s} \frac{\bar{B}_n^2}{b_n} (b_n-1) \right| &= \frac{2(b_n-1)^2 \bar{B}_n^2}{n} \\
 &\leq 4(1+\epsilon)^2 \frac{(b_n-1)^2 [\log \log n]}{n^2} \\
 &\rightarrow 0
 \end{aligned}$$

as $n \rightarrow \infty$ since we assumed $b_n^2 n^{-2} \log \log n \rightarrow 0$ as $n \rightarrow \infty$.

□

APPENDIX C: MEAN-SQUARE CONSISTENCY PROOFS

C.1. Preliminaries. Recall that $\tilde{\sigma}_{BM}^2$ is defined at (6) while $\hat{\sigma}_{OBM}^2$ is defined at (7). Next define the Brownian motion version of BM by

$$\tilde{\sigma}_{BM}^2 = \frac{b_n}{a_n - 1} \sum_{k=0}^{a_n-1} (\bar{B}_k - \bar{B}_n)^2,$$

where $\bar{B}_k := b_n^{-1} (B((k+1)b_n) - B(kb_n + 1))$ for $k = 0, \dots, a_n - 1$. Further define the Brownian motion version of OBM by

$$\tilde{\sigma}_{OBM}^2 = \frac{nb_n}{(n-b_n)(n-b_n+1)} \sum_{j=0}^{n-b_n} (\bar{B}_j(b_n) - \bar{B}_n)^2.$$

LEMMA 11. (*Damerdji [11]*) *Suppose Assumption 2 holds, then*

$$\begin{aligned} E \left[\tilde{\sigma}_{OBM}^2 \right] &= E \left[\tilde{\sigma}_{BM}^2 \right] = 1, \\ \frac{n}{b_n} \text{Var} \left[\tilde{\sigma}_{OBM}^2 \right] &= \frac{4}{3} + o(1), \text{ and} \\ \frac{n}{b_n} \text{Var} \left[\tilde{\sigma}_{BM}^2 \right] &= 2 + o(1) \end{aligned}$$

where the limits are taken as $n \rightarrow \infty$.

The next claim follows from a careful examination of the proof of Lemma B.4 in Jones et al. [24].

LEMMA 12. *Suppose (13) holds with $\gamma(n) = n^\alpha \log n$ where $\alpha = 1/(2+\delta)$ and Assumption 2 holds. Then for sufficiently large n , there exist functions $g_1 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ and $g_2 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ such that*

$$\left| \hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right| \leq C^2 g_1(n) + C g_2(n)$$

where the random variable C is defined at (14). Moreover, if, as $n \rightarrow \infty$, $b_n^{-1} n^{2\alpha} [\log n]^3 \rightarrow 0$, then $g_1(n) \rightarrow 0$ and $g_2(n) \rightarrow 0$.

We also require an analogous result for OBM.

LEMMA 13. *Suppose (13) holds with $\gamma(n) = n^\alpha \log n$ where $\alpha = 1/(2+\delta)$ and Assumption 2 holds. Then for sufficiently large n , there exist functions $g_1 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$, $g_2 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$, and $g_3 : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ such that*

$$\left| \hat{\sigma}_{OBM}^2 - \sigma_g^2 \tilde{\sigma}_{OBM}^2 \right| \leq C^2 g_1(n) + C g_2(n) + \left(\hat{\sigma}_{OBM}^2 + \sigma_g^2 \tilde{\sigma}_{OBM}^2 \right) g_3(n)$$

where the random variable C is defined at (14). Moreover, if, as $n \rightarrow \infty$, $b_n^{-1} n^{2\alpha} [\log n]^3 \rightarrow 0$, then $g_1(n) \rightarrow 0$, $g_2(n) \rightarrow 0$, and $g_3(n) \rightarrow 0$.

PROOF. Let w be the modified Bartlett lag window. Notice

$$(29) \quad \left| \hat{\sigma}_{OBM}^2 - \sigma_g^2 \tilde{\sigma}_{OBM}^2 \right| \leq \left| \hat{\sigma}_{w,n}^2 - \sigma_g^2 \tilde{\sigma}_*^2 \right| + \left| \hat{\sigma}_{OBM}^2 - \hat{\sigma}_{w,n}^2 \right| + \left| \sigma_g^2 \tilde{\sigma}_*^2 - \sigma_g^2 \tilde{\sigma}_{OBM}^2 \right|$$

where $\hat{\sigma}_{w,n}^2$ and $\tilde{\sigma}_*^2$ are defined at (24) and (25) respectively. Our assumptions imply (15) and (16). Careful examination of the proof of Lemma 4 shows that for sufficiently large n

$$\left| \hat{\sigma}_{w,n}^2 - \sigma_g^2 \tilde{\sigma}_*^2 \right| \leq C^2 g_1(n) + C g_2(n) .$$

Let $g_3(n) = 1 - (n - b_n)(n - b_n + 1)/n^2$ and observe that

$$\begin{aligned} \left| \hat{\sigma}_{OBM}^2 - \hat{\sigma}_{w,n}^2 \right| &= \left| \left(\frac{nb_n}{(n - b_n)(n - b_n + 1)} - \frac{b_n}{n} \right) \sum_{j=0}^{n-b_n} [\bar{Y}_j(b_n) - \bar{Y}_n]^2 \right| \\ &= \frac{nb_n g_3(n)}{(n - b_n)(n - b_n + 1)} \sum_{j=0}^{n-b_n} [\bar{Y}_j(b_n) - \bar{Y}_n]^2 \\ &= g_3(n) \hat{\sigma}_{OBM}^2 \end{aligned}$$

and, similarly, $\left| \sigma_g^2 \tilde{\sigma}_*^2 - \sigma_g^2 \tilde{\sigma}_{OBM}^2 \right| = \sigma_g^2 \tilde{\sigma}_{OBM}^2 g_3(n)$. Combining these results with (29) yields the claim. \square

LEMMA 14. *Let $\hat{\sigma}_n^2$ be either the BM or OBM estimator of σ_g^2 and suppose (13) holds with $\gamma(n) = n^\alpha \log n$ where $\alpha = 1/(2 + \delta)$ and Assumption 2 holds. Also, assume that $EC^4 < \infty$ where C is defined at (14) and $E_\pi g^4 < \infty$. If $b_n^{-1} n^{2\alpha} [\log n]^3 \rightarrow 0$ as $n \rightarrow \infty$, then $E \left[\left| \hat{\sigma}_n^2 - \sigma_g^2 \tilde{\sigma}^2 \right| \right] \rightarrow 0$ and $E \left[\left(\hat{\sigma}_n^2 - \sigma_g^2 \tilde{\sigma}^2 \right)^2 \right] \rightarrow 0$ while if as $n \rightarrow \infty$, $b_n^{-1} n^{1/2+\alpha} [\log n]^{3/2} \rightarrow 0$, then $E \left[\frac{n}{b_n} \left(\hat{\sigma}_n^2 - \sigma_g^2 \tilde{\sigma}^2 \right)^2 \right] \rightarrow 0$.*

PROOF. We will prove only the first claim for BM as the other proofs are quite similar. The omitted proofs for OBM require the use of Lemma 13 in place of Lemma 12 and Lemma 4 with the Ergodic Theorem in place of Lemma B.4 from Jones et al. [24].

From Lemma 12 there exists an integer N_0 and functions g_1 and g_2 such that

$$\begin{aligned} \left| \hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right| &= \left| \hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right| I(0 \leq n \leq N_0) + \left| \hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right| I(N_0 < n) \\ &\leq \left| \hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right| I(0 \leq n \leq N_0) + [C^2 g_1(n) + C g_2(n)] I(N_0 < n) \\ &:= g_n(X_0, \dots, X_n, B(0), \dots, B(n)) . \end{aligned}$$

Now

$$E |g_n(X_0, \dots, X_n, B(0), \dots, B(n))| \leq E \left| \hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right| + [g_1(n) E(C^2) + g_2(n) E(C)]$$

and since Lemma 11 and our assumptions on the moments of g imply

$$E|\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2| \leq E\hat{\sigma}_{BM}^2 + \sigma_g^2 E\tilde{\sigma}_{BM}^2 = E\hat{\sigma}_{BM}^2 + \sigma_g^2 < \infty$$

it follows from our assumptions on the moments of C that $E|g_n| < \infty$. Next observe that as $n \rightarrow \infty$, we have $g_n \rightarrow 0$ w.p. 1 and $Eg_n \rightarrow 0$ by Lemma 12. From Lemma B.4 in Jones et al. [24] we have that $|\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2| \rightarrow 0$ w.p. 1 as $n \rightarrow \infty$. An application of the generalized majorized convergence theorem (pp 1015 Zeidler [54]) implies that, as $n \rightarrow \infty$, $E\left[|\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2|\right] \rightarrow 0$. \square

C.2. Proof of Theorem 3. We will prove only the claim for BM as the proof for OBM is nearly identical.

Recall, the mean-square error (MSE) of the estimator $\hat{\sigma}_{BM}^2$ of σ_g^2 is given by

$$\text{MSE}(\hat{\sigma}_{BM}^2) = \text{Var}(\hat{\sigma}_{BM}^2) + \text{Bias}^2(\hat{\sigma}_{BM}^2).$$

First, consider the bias term. Recall from Lemma 11 that $E\tilde{\sigma}_{BM}^2 = 1$ so that

$$\text{Bias}(\hat{\sigma}_{BM}^2) = E(\hat{\sigma}_{BM}^2) - \sigma_g^2 \leq E\left[|\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2|\right] \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

by Lemma 14. Hence $\text{Bias}^2(\hat{\sigma}_{BM}^2) = o(1)$ as $n \rightarrow \infty$.

Note that the claim will follow if we can show that $\text{Var}(\hat{\sigma}_{BM}^2) = o(1)$ as $n \rightarrow \infty$. Thus, we now consider the variance term but we begin with a preliminary result. Recall from Lemma 11 that

$$\frac{n}{b_n} \text{Var}\left[\tilde{\sigma}_{BM}^2\right] = 2 + o(1).$$

Define

$$(30) \quad \eta(\hat{\sigma}_{BM}^2, \tilde{\sigma}_{BM}^2) = \text{Var}\left(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2\right) + 2\sigma_g^2 E\left[(\tilde{\sigma}_{BM}^2 - E\tilde{\sigma}_{BM}^2)(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2)\right].$$

Using the Cauchy-Schwarz inequality and the the fact that $\text{Var}(X) \leq EX^2$ obtain

$$\begin{aligned} |\eta| &= \left| \text{Var}\left(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2\right) + 2\sigma_g^2 E\left[(\tilde{\sigma}_{BM}^2 - E\tilde{\sigma}_{BM}^2)(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2)\right] \right| \\ &\leq E\left[\left(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2\right)^2\right] + 2\sigma_g^2 \left(E\left[\left(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2\right)^2\right] \text{Var}\left(\tilde{\sigma}_{BM}^2\right)\right)^{1/2} \\ &= o(1) + 2\sigma_g^2 \left(\frac{b_n}{n}\right)^{1/2} [o(1)(2 + o(1))]^{1/2} \quad \text{as } n \rightarrow \infty \end{aligned}$$

and hence $\eta(\hat{\sigma}_{BM}^2, \tilde{\sigma}_{BM}^2) = o(1)$ since $b_n/n \rightarrow 0$ as $n \rightarrow \infty$. Now

$$\begin{aligned}
 \text{Var}(\hat{\sigma}_{BM}^2) &= E \left[\left((\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2) + \sigma_g^2 (\tilde{\sigma}_{BM}^2 - E\tilde{\sigma}_{BM}^2) - (E\hat{\sigma}_{BM}^2 - \sigma_g^2 E\tilde{\sigma}_{BM}^2) \right)^2 \right] \\
 &= \sigma_g^4 E \left[(\tilde{\sigma}_{BM}^2 - E\tilde{\sigma}_{BM}^2)^2 \right] + E \left[\left((\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2) - E(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2) \right)^2 \right] \\
 &\quad + 2\sigma_g^2 E \left[(\tilde{\sigma}_{BM}^2 - E\tilde{\sigma}_{BM}^2) \left((\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2) - E(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2) \right) \right] \\
 (31) \quad &= \sigma_g^4 \text{Var}(\tilde{\sigma}_{BM}^2) + \eta(\hat{\sigma}_{BM}^2, \tilde{\sigma}_{BM}^2) \\
 &= 2\sigma_g^4 \frac{b_n}{n} + o\left(\frac{b_n}{n}\right) + o(1) \quad \text{as } n \rightarrow \infty.
 \end{aligned}$$

Therefore, we conclude that $\text{Var}(\hat{\sigma}_{BM}^2) = o(1)$ as $n \rightarrow \infty$.

C.3. Proof of Theorem 4. We only prove the claim for BM as the proof for OBM is nearly identical.

From (31) and Lemma 11 we obtain

$$\frac{n}{b_n} \text{Var}(\hat{\sigma}_{BM}^2) = 2\sigma_g^4 + o(1) + \frac{n}{b_n} \eta$$

where η is defined at (30). Note that the claim will follow if we show that $\frac{n}{b_n} \eta = o(1)$ as $n \rightarrow \infty$. As in the proof of Theorem 3 we have

$$\begin{aligned}
 \frac{n}{b_n} |\eta| &= \frac{n}{b_n} \left| \text{Var} \left(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right) + 2\sigma_g^2 E \left[(\tilde{\sigma}_{BM}^2 - E\tilde{\sigma}_{BM}^2) (\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2) \right] \right| \\
 &\leq E \left[\frac{n}{b_n} \left(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right)^2 \right] + 2\sigma_g^2 \left(E \left[\frac{n}{b_n} \left(\hat{\sigma}_{BM}^2 - \sigma_g^2 \tilde{\sigma}_{BM}^2 \right)^2 \right] \frac{n}{b_n} \text{Var} \left(\tilde{\sigma}_{BM}^2 \right) \right)^{1/2} \\
 &= o(1) + 2\sigma_g^2 (o(1)(2 + o(1)))^{1/2} \quad \text{as } n \rightarrow \infty
 \end{aligned}$$

using the results of Lemmas 11 and 14. Hence $\frac{n}{b_n} \eta = o(1)$ as $n \rightarrow \infty$.

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References.

- [1] Anderson, T. W. (1994). *The Statistical Analysis of Time Series*. Wiley-Interscience.
- [2] Bednorz, W. and Latuszyński, K. (2007). A few remarks on ‘Fixed-width output analysis for Markov chain Monte Carlo’ by Jones et al. *Journal of the American Statistical Association*, 102:1485–1486.
- [3] Billingsley, P. (1995). *Probability and Measure*. Wiley, New York, third edition.

- [4] Bratley, P., Fox, B. L., and Schrage, L. E. (1987). *A Guide to Simulation*. Springer-Verlag, New York.
- [5] Chan, K. S. and Geyer, C. J. (1994). Comment on “Markov chains for exploring posterior distributions”. *The Annals of Statistics*, 22:1747–1758.
- [6] Chien, C.-H., Goldsman, D., and Melamed, B. (1997). Large-sample results for batch means. *Management Science*, 43:1288–1295.
- [7] Csáki, E. and Csörgő, M. (1995). On additive functionals of Markov chains. *Journal of Theoretical Probability*, 8:905–919.
- [8] Csörgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press.
- [9] Damerджи, H. (1991). Strong consistency and other properties of the spectral variance estimator. *Management Science*, 37:1424–1440.
- [10] Damerджи, H. (1994). Strong consistency of the variance estimator in steady-state simulation output analysis. *Mathematics of Operations Research*, 19:494–512.
- [11] Damerджи, H. (1995). Mean-square consistency of the variance estimator in steady-state simulation output analysis. *Operations Research*, 43:282–291.
- [12] Flegal, J. M., Haran, M., and Jones, G. L. (2008). Markov chain Monte Carlo: Can we trust the third significant figure? *Statistical Science*, 23:250–260.
- [13] Geyer, C. J. (1999). Likelihood inference for spatial point processes. In Barndorff-Nielsen, O. E., Kendall, W. S., and van Lieshout, M. N. M., editors, *Stochastic Geometry: Likelihood and Computation*, pages 79–140. Chapman & Hall/CRC, Boca Raton.
- [14] Gilks, W. R., Roberts, G. O., and Sahu, S. K. (1998). Adaptive Markov chain Monte Carlo through regeneration. *Journal of the American Statistical Association*, 93:1045–1054.
- [15] Glynn, P. W. and Iglehart, D. L. (1990). Simulation output analysis using standardized time series. *Mathematics of Operations Research*, 15:1–16.
- [16] Glynn, P. W. and Whitt, W. (1991). Estimating the asymptotic variance with batch means. *Operations Research Letters*, 10:431–435.
- [17] Glynn, P. W. and Whitt, W. (1992). The asymptotic validity of sequential stopping rules for stochastic simulations. *The Annals of Applied Probability*, 2:180–198.
- [18] Häggström, O. (2004). On the central limit theorem for geometrically ergodic Markov chains. *Probability Theory and Related Fields*, 132:74–82.
- [19] Hobert, J. P. and Geyer, C. J. (1998). Geometric ergodicity of Gibbs and block Gibbs samplers for a hierarchical random effects model. *Journal of Multivariate Analysis*, 67:414–430.
- [20] Hobert, J. P., Jones, G. L., Presnell, B., and Rosenthal, J. S. (2002). On the applicability of regenerative simulation in Markov chain Monte Carlo. *Biometrika*, 89:731–743.
- [21] Jarner, S. F. and Hansen, E. (2000). Geometric ergodicity of Metropolis algorithms. *Stochastic Processes and Their Applications*, 85:341–361.
- [22] Johnson, A. A. and Jones, G. L. (2008). Gibbs sampling for a Bayesian hierarchical version of the general linear mixed model. Technical report, University of Minnesota, School of Statistics.
- [23] Jones, G. L. (2004). On the Markov chain central limit theorem. *Probability Surveys*, 1:299–320.
- [24] Jones, G. L., Haran, M., Caffo, B. S., and Neath, R. (2006). Fixed-width output analysis for Markov chain Monte Carlo. *Journal of the American Statistical Association*, 101:1537–1547.
- [25] Jones, G. L. and Hobert, J. P. (2001). Honest exploration of intractable probability distributions via Markov chain Monte Carlo. *Statistical Science*, 16:312–334.
- [26] Jones, G. L. and Hobert, J. P. (2004). Sufficient burn-in for Gibbs samplers for a

- hierarchical random effects model. *The Annals of Statistics*, 32:784–817.
- [27] Kendall, M. G. and Stuart, A. (1977). *The Advanced Theory of Statistics. Vol. I: Distribution Theory (4th Ed); Vol. 2: Inference and Relationship (3rd Ed)*. Charles Griffin & Co.
- [28] Komlós, J., Major, P., and Tusnády, G. (1975). An approximation of partial sums of independent RVs and the sample DF. I (Ref: 76V34 p33-58). *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 32:111–131.
- [29] Komlós, J., Major, P., and Tusnády, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II (Ref: 75V32 p111-131). *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 34:33–58.
- [30] Liu, J. S. (2001). *Monte Carlo Strategies in Scientific Computing*. Springer, New York.
- [31] Liu, J. S. and Wu, Y. N. (1999). Parameter expansion for data augmentation. *Journal of the American Statistical Association*, 94:1264–1274.
- [32] Major, P. (1976). The approximation of partial sums of independent RV's. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 35:213–220.
- [33] Marchev, D. and Hobert, J. P. (2004). Geometric ergodicity of van Dyk and Meng's algorithm for the multivariate Student's t model. *Journal of the American Statistical Association*, 99:228–238.
- [34] Meketon, M. S. and Schmeiser, B. (1984). Overlapping batch means: something for nothing? In *WSC '84: Proceedings of the 16th conference on Winter simulation*, pages 226–230, Piscataway, NJ, USA. IEEE Press.
- [35] Mengersen, K. and Tweedie, R. L. (1996). Rates of convergence of the Hastings and Metropolis algorithms. *The Annals of Statistics*, 24:101–121.
- [36] Meyn, S. P. and Tweedie, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer-Verlag, London.
- [37] Meyn, S. P. and Tweedie, R. L. (1994). Computable bounds for geometric convergence rates of Markov chains. *The Annals of Applied Probability*, 4:981–1011.
- [38] Mykland, P., Tierney, L., and Yu, B. (1995). Regeneration in Markov chain samplers. *Journal of the American Statistical Association*, 90:233–241.
- [39] Neath, R. and Jones, G. L. (2009). Variable-at-a-time implementations of Metropolis-Hastings. Technical report, University of Minnesota, School of Statistics.
- [40] Philipp, W. and Stout, W. (1975). Almost sure invariance principles for partial sums of weakly dependent random variables. *Memoirs of the American Mathematical Society*, 2:1–140.
- [41] Robert, C. P. and Casella, G. (1999). *Monte Carlo Statistical Methods*. Springer, New York.
- [42] Roberts, G. O. and Polson, N. G. (1994). On the geometric convergence of the Gibbs sampler. *Journal of the Royal Statistical Society, Series B*, 56:377–384.
- [43] Roberts, G. O. and Rosenthal, J. S. (1999). Convergence of slice sampler Markov chains. *Journal of the Royal Statistical Society, Series B*, 61:643–660.
- [44] Roberts, G. O. and Rosenthal, J. S. (2004). General state space Markov chains and MCMC algorithms. *Probability Surveys*, 1:20–71.
- [45] Rosenthal, J. S. (1995). Minorization conditions and convergence rates for Markov chain Monte Carlo. *Journal of the American Statistical Association*, 90:558–566.
- [46] Rosenthal, J. S. (1996). Analysis of the Gibbs sampler for a model related to James-Stein estimators. *Statistics and Computing*, 6:269–275.
- [47] Roy, V. and Hobert, J. P. (2007). Convergence rates and asymptotic standard errors for Markov chain Monte Carlo algorithms for Bayesian probit regression. *Journal of the Royal Statistical Society, Series B: Statistical Methodology*, 69(4):607–623.

- [48] Song, W. T. and Schmeiser, B. W. (1995). Optimal mean-squared-error batch sizes. *Management Science*, 41:110–123.
- [49] Tan, A. and Hobert, J. P. (2008). Block Gibbs sampling for Bayesian random effects models with improper priors: Convergence and regeneration. Technical report, University of Florida, Department of Statistics.
- [50] Tierney, L. (1994). Markov chains for exploring posterior distributions (with discussion). *The Annals of Statistics*, 22:1701–1762.
- [51] van Dyk, D. A. and Meng, X.-L. (2001). The art of data augmentation (with discussion). *Journal of Computational and Graphical Statistics*, 10:1–50.
- [52] Welch, P. D. (1987). On the relationship between batch means, overlapping means and spectral estimation. In *WSC '87: Proceedings of the 19th conference on Winter simulation*, pages 320–323, New York, NY, USA. ACM.
- [53] Whittle, P. (1960). Bounds for the moments of linear and quadratic forms in independent variables. *Theory of Probability and its Applications*, 5(3):302–305.
- [54] Zeidler, E. (1990). *Nonlinear functional analysis and its applications. II/B: Nonlinear monotone operators*. Springer-Verlag, New York, second edition.

Method	$b_n =$	Number of Iterations				
		1e3	5e3	1e4	5e4	1e5
BM	$\lfloor n^{1/3} \rfloor$	0.9315	0.939	0.937	0.942	0.943
Brt		0.93	0.9395	0.936	0.9415	0.944
OBM		0.9305	0.9395	0.936	0.942	0.944
TH		0.936	0.9465	0.9395	0.9465	0.947
BM	$\lfloor n^{1/2} \rfloor$	0.9415	0.948	0.939	0.947	0.949
Brt		0.933	0.946	0.935	0.947	0.9475
OBM		0.9385	0.947	0.9355	0.9475	0.9475
TH		0.9365	0.9465	0.9365	0.948	0.948
BM	$\lfloor n^{2/3} \rfloor$	0.9475	0.9445	0.9385	0.95	0.9465
Brt		0.9105	0.9265	0.9275	0.9445	0.9425
OBM		0.9245	0.935	0.932	0.947	0.944
TH		0.9115	0.927	0.927	0.9435	0.9425

TABLE 1

Table of coverage probabilities for 2000 replications using the AR(1) example with $\rho = 0.5$. All calculations were based on the nominal level of 0.95. The standard errors for these numbers are easily calculated as $\sqrt{\hat{p}(1-\hat{p})/2000}$ which results in a largest standard error of $6.4e-3$.

Method	$b_n =$	Number of Iterations				
		1e3	5e3	1e4	5e4	1e5
BM	$[n^{1/3}]$	0.614	0.738	0.766	0.842	0.872
Brt		0.606	0.736	0.764	0.841	0.871
OBM		0.61	0.736	0.764	0.842	0.872
TH		0.61	0.74	0.77	0.854	0.886
BM	$[n^{1/2}]$	0.838	0.903	0.9155	0.94	0.9425
Brt		0.807	0.893	0.911	0.9365	0.9385
OBM		0.821	0.895	0.913	0.937	0.9395
TH		0.822	0.9055	0.9235	0.943	0.945
BM	$[n^{2/3}]$	0.927	0.9385	0.933	0.948	0.9465
Brt		0.872	0.916	0.9185	0.944	0.942
OBM		0.89	0.925	0.924	0.9455	0.943
TH		0.885	0.92	0.924	0.9435	0.9425

TABLE 2

Table of coverage probabilities for 2000 replications using the AR(1) example with $\rho = 0.95$. All calculations were based on the nominal level of 0.95. The standard errors for these numbers are easily calculated as $\sqrt{\hat{p}(1-\hat{p})}/2000$ which results in a largest standard error of 0.011.

	β_0	β_1	β_2
$\hat{\beta}_i$	-3.0166	6.9107	3.9792
$\hat{\sigma}_{\beta_i}$	11.85	22.60	14.74
MCSE	1.18e-3	2.26e-3	1.47e-3

TABLE 3

Values treated as the “truth” for estimating confidence interval coverage probabilities based on 1e8 iterations.

Method	ϵ	$b_n = \lfloor n^{1/3} \rfloor$				$b_n = \lfloor n^{1/2} \rfloor$			
		β_0	β_1	β_2	n	β_0	β_1	β_2	n
BM	.3	0.699	0.709	0.704	6.43e3 (33)	0.921	0.925	0.927	1.85e4 (89)
Brt		0.699	0.708	0.705	6.34e3 (32)	0.926	0.923	0.930	1.81e4 (81)
OBM		0.700	0.710	0.706	6.37e3 (32)	0.926	0.922	0.932	1.83e4 (81)
TH		0.699	0.710	0.703	6.53e3 (34)	0.936	0.938	0.941	1.95e4 (85)
BM	.2	0.794	0.781	0.782	1.91e4 (60)	0.922	0.929	0.930	4.54e4 (154)
Brt		0.797	0.775	0.781	1.89e4 (60)	0.927	0.928	0.934	4.48e4 (141)
OBM		0.796	0.777	0.781	1.90e4 (60)	0.928	0.928	0.936	4.50e4 (140)
TH		0.804	0.790	0.794	1.97e4 (61)	0.940	0.944	0.944	4.79e4 (144)
BM	.1	0.854	0.849	0.853	1.11e5 (187)	0.923	0.920	0.917	1.94e5 (454)
Brt		0.851	0.850	0.850	1.11e5 (185)	0.925	0.922	0.917	1.93e5 (410)
OBM		0.853	0.850	0.850	1.11e5 (185)	0.925	0.922	0.917	1.93e5 (412)
TH		0.860	0.858	0.850	1.18e5 (195)	0.932	0.929	0.932	2.02e5 (419)

TABLE 4

Summary of results for using fixed-width methods for the Lupus data Bayesian probit regression. Coverage probabilities using calculated half-width have MCSEs between $1.1e-2$ and $1.5e-2$ when $b_n = \lfloor n^{1/3} \rfloor$ and between $7.3e-3$ and $8.7e-3$ when $b_n = \lfloor n^{1/2} \rfloor$. The table also shows the mean (s.e.) simulation effort at termination in terms of number of iterations.

Method	$b_n =$	β_0	β_1	β_2	Simultaneous	n
BM	$\lfloor n^{1/3} \rfloor$	0.911	0.904	0.906	0.878	3.21e4 (85)
Brt		0.909	0.904	0.909	0.879	3.19e4 (85)
OBM		0.910	0.905	0.909	0.879	3.20e4 (85)
TH		0.917	0.909	0.911	0.884	3.34e4 (87)
BM	$\lfloor n^{1/2} \rfloor$	0.973	0.975	0.972	0.965	6.97e4 (210)
Brt		0.973	0.971	0.974	0.965	6.86e4 (189)
OBM		0.973	0.970	0.972	0.963	6.90e4 (191)
TH		0.973	0.976	0.972	0.969	7.30e4 (195)

TABLE 5

Summary of results for using fixed-width methods with a Bonferroni correction for the Lupus data Bayesian probit regression. Coverage probabilities using calculated half-width have MCSEs of between $9e-3$ and $1e-2$ when $b_n = \lfloor n^{1/3} \rfloor$ and between $4.9e-3$ and $6e-3$ when $b_n = \lfloor n^{1/2} \rfloor$. The table also shows the mean (s.e.) simulation effort at termination in terms of number of iterations.

Method	β_0	β_1	β_2
BM	0.945	0.945	0.950
Brt	0.941	0.942	0.948
OBM	0.942	0.942	0.948
TH	0.946	0.945	0.950
RS	0.938	0.938	0.938

TABLE 6

Coverage probabilities comparing BM, OBM, and SV using $7e5$ iterations and $b_n = \lfloor n^{1/2} \rfloor$ to RS. MCSEs vary between $6.9e-3$ and $7.6e-3$.

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