

FIDUCIAL THEORY AND OPTIMAL INFERENCE

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It is shown that the fiducial distribution in a group model, or more generally a quasigroup model, determines the optimal equivariant frequentist inference procedures. The proof does not rely on existence of invariant measures, and generalizes results corresponding to the choice of the right Haar measure as a Bayesian prior. Classical and more recent examples show that fiducial arguments can be used to give good candidates for exact or approximate confidence distributions. It is here suggested that the fiducial algorithm can be considered as an alternative to the Bayesian algorithm for the construction of good frequentist inference procedures more generally.

1. Introduction. Fiducial theory was introduced by Fisher (1930) to avoid the problems related to the choice of a prior distribution. Fiducial inference has not gained much popularity as such, but the related theory has been historically influential (Efron, 1998), and is still important in the current flow of statistical developments (Efron, 2006; Lidong et al., 2008; Ghosh et al., 2010; Fraser et al., 2010; Wang et al., 2012). Fisher's own view on fiducial inference evolved over the years as can be inferred from a reading of his initial (Fisher, 1930, 1935) and more final formulation of the theory (Fisher, 1973). He was in particular very positive to the developments by Fraser (1961a,b, 1962, 1964), and we most certainly share this point of view. Fraser (1968, 1979) develops the theory and uses the label *structural inference* for this. A strongly related theory was presented under the label of *functional models* by Bunke (1975) and Dawid and Stone (1982). The term *fiducial* will here be used more generally so that it includes structural, functional, and the original fiducial arguments given by Fisher.

The original idea of Fisher was to obtain the fiducial distribution directly from the cumulative distribution, but this line of argument runs into problems when similar arguments are tried in the multivariate case. The definition used here is based on the solution of a fiducial equation, and is in this sense similar to the approaches of Fraser (1968), Dawid and Stone (1982), and Hannig (2009, 2012). A more precise definition of the term *fiducial model* as used here is given in Section 2 in Definition 1. A brief review of alternative, but strongly related definitions found in the literature is given in the final Section 4.

Let $l = \gamma(\theta, a)$ denote the realized loss for an action $a \in \Omega_A$ given the model parameter $\theta \in \Omega_\Theta$. Let Ω_X be the sample space equipped with the σ -field \mathcal{E}_X of events. The risk ρ of a decision rule $\delta : \Omega_X \rightarrow \Omega_A$ is by definition equal to the expected value $\rho = E^\theta \gamma(\theta, \delta(X))$.

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This is determined by the statistical model given by the family $\{(\Omega_X, \mathcal{E}_X, P_X^\theta) \mid \theta \in \Omega_\Theta\}$ of probability spaces.

Consider now the more special case where $\Omega_X = \Omega_\Theta = \Omega_A = G$, possibly after a suitable change of variables. Assume that G is a measurable quasigroup with a unit e , and product $(g_1, g_2) \mapsto g_1 g_2$ written like ordinary multiplication (Smith, 2006). This includes the more common case of a group, but it is more general since the associative law is not assumed to hold. Assume furthermore that $X \sim \theta U$ conditionally on $\Theta = \theta$ and that the law of $(U \mid \Theta = \theta)$ does not depend on θ .

This gives an example of a fiducial model for the statistical model as defined more generally on page 4. The fiducial distribution is obtained by solving the fiducial equation $x = \theta u$ for θ when u is sampled from P_U^θ . Existence and uniqueness is ensured since G is a quasi-group. A variable Θ^x is uniquely determined by $x = \Theta^x U$. The fiducial distribution is then the conditional law of Θ^x given $\Theta = \theta$.

Assume that the loss is invariant in the sense that $\gamma(\theta, a) = \gamma(g\theta, ga)$, and that the decision rule is equivariant in the sense that $\delta(gx) = g\delta(x)$. The assumptions ensure the validity of the following calculation

(1a)	$\rho = E^\theta \gamma(\theta, \delta(X))$	Risk
(1b)	$= E^\theta \gamma(\theta, \delta(\theta U))$	Fiducial model for P_X^θ
(1c)	$= E^\theta \gamma(\theta, \theta \delta(U))$	Equivariance of δ
(1d)	$= E^\theta \gamma(e, \delta(U))$	Invariance of γ
(1e)	$= E^\theta \gamma(\Theta^x, \Theta^x \delta(U))$	Invariance of γ
(1f)	$= E^\theta \gamma(\Theta^x, \delta(\Theta^x U))$	Equivariance of δ
(1g)	$= E^\theta \gamma(\Theta^x, \delta(x))$	Fiducial equation

A variation of the above argument gives that Θ^x can be replaced by xU_r^{-1} in the conclusion. In the group case the law of Θ^x will coincide with the law of xU_r^{-1} , but in general not since the defining equation $e = UU_r^{-1}$ of the right inverse U_r^{-1} does not provide the solution of the fiducial equation. It follows from this that an optimal equivariant rule, if it exists, is determined by the fiducial distribution of Θ^x or by the distribution of xU_r^{-1} from the right inverse. The first part of the claims in the abstract has hence been established.

It is hoped that the reader can appreciate the simplicity and consequence of the calculation given in equation (1), but it could also be considered to be essentially Greek. The required theory of decisions and fiducial theory will be explained in some more detail in Section 2, and examples are presented in Section 3. The presentation is essentially as given in standard textbooks (Lehmann and Casella, 1998; Lehmann and Romano, 2005; Schervish, 1995; Berger, 1985; Stuart et al., 1999), but with the simplifications given by a fiducial model. The monographs by Eaton (1989) and Wijsman (1990) are recommended as excellent sources for theory and examples beyond the standard textbooks.

The presentation in the following will be mostly restricted to the group case, but it will be more general than the previous in the sense that the assumption of equality of

the involved spaces will be abandoned. It will be more general than standard theory since, as above, the arguments will not depend on existence of invariant measures.

2. Optimal inference. Consider the case where the loss of an action $a \in \Omega_A$ is of the form $l = \gamma(\theta, a)$ corresponding to a statistical model $\{P_X^\theta \mid \theta \in \Omega_\Theta\}$. It is here assumed that the model parameter Θ is a σ -finite random quantity and this and all other random quantities are defined based on the underlying conditional probability space (Ω, \mathcal{E}, P) as explained by Taraldsen and Lindqvist (2010). This means in particular that $P_X^\theta(B) = P(X \in B \mid \Theta = \theta)$, and $X : \Omega \rightarrow \Omega_X$, $\Theta : \Omega \rightarrow \Omega_\Theta$ are measurable functions. It means also that all expectations that occur are defined by integration over Ω . As an example $E(\phi(Z) \mid T = t) = \int \phi(Z(\omega)) P^t(d\omega)$ by definition. It is here assumed that $\phi : \Omega_Z \rightarrow \mathbb{R}$, $Z : \Omega \rightarrow \Omega_Z$, and $T : \Omega \rightarrow \Omega_T$ are measurable. The conditional law P^t is well defined if P_T is σ -finite. The consequence $E(\phi(Z) \mid T = t) = \int \phi(z) P_Z^t(dz)$ is a theorem.

The law P_Θ of Θ is not assumed known and is not needed in the arguments which follow. The reason for the assumption of existence of Θ , X , and indeed any random quantity involved in the arguments, as functions defined on the conditional probability space (Ω, P, \mathcal{E}) is as in the formulation of probability theory given by Kolmogorov (1933): Any collection of random quantities gives a new random quantity with a well defined law, and measurable functions of random quantities give new random quantities. The theory is completely based on the underlying abstract space Ω .

A *group invariant problem* is given by a group G that has a transformation group action on the sample space Ω_X , the model parameter space Ω_Θ , and the action space Ω_A . The problem is group invariant if $P_{gX}^\theta = P_X^{g\theta}$ and $\gamma(g\theta, ga) = \gamma(\theta, a)$. An inference rule δ with a corresponding action $A = \delta(X)$ is *equivariant* if $\delta(gx) = g\delta(x)$. The restriction to the class of equivariant actions can be interpreted as a consistency requirement: An observation x from P_X^θ corresponds to an observation gx from $P_X^{g\theta}$. The two corresponding problems are formally identical and the use of an equivariant action ensures consistency.

The problem considered here is to determine an equivariant δ such that the risk

$$(2) \quad \rho = E^\theta \gamma(\Theta, \delta(X)) = E(\gamma(\Theta, \delta(X)) \mid \Theta = \theta)$$

is minimized. It will be assumed that $G = \Omega_\Theta$ with the action given by the group multiplication $g\theta$ directly. The *orbit* of x in Ω_X is defined by $Gx = \{gx \mid g \in G\}$, and likewise for orbits in Ω_Θ and Ω_A . The action of G is *free* on Ω_X if the mapping $g \mapsto gx$ is injective for all x . The group action is *transitive* on Ω_X if $Gx = \Omega_X$. If the group action is both transitive and free, then it is said to be *regular* and the corresponding space is then a principal homogeneous space for G . It follows in particular that the model parameter space Ω_Θ is a principal homogeneous space for G , but there has also been an identification of the identity element e in Ω_Θ .

Let U be a random quantity such that $P(U \in A \mid \Theta = \theta) = P(X \in A \mid \Theta = e)$ holds identically for all A and θ . It follows then that

$$(3) \quad (X \mid \Theta = \theta) \sim (\theta U \mid \Theta = \theta)$$

since the group invariance of the statistical model justifies $P_X^\theta = P_{\theta X}^e = P_{\theta U}^\theta$. This construction proves that (U, χ) with

$$(4) \quad \chi(u, \theta) = \theta u$$

is a fiducial model for P_X^θ . The concept of a fiducial model is defined as follows.

DEFINITION 1 (Fiducial model). *Let Θ be a σ -finite random quantity. A fiducial model (U, ζ) is given by a random quantity U and a measurable function $\zeta : \Omega_U \times \Omega_\Theta \rightarrow \Omega_Z$. This is a fiducial model for the statistical model $\{P_Z^\theta | \theta \in \Omega_\Theta\}$ if*

$$(5) \quad (\zeta(U, \Theta) | \Theta = \theta) \sim (Z | \Theta = \theta)$$

The notation $(W_1 | \Theta = \theta) \sim (W_2 | \Theta = \theta)$ means that $P_{W_1}^\theta = P_{W_2}^\theta$ so Definition 1 can be compared with equation (3). It is allowed in the above that P_U^θ does depend on θ . Interesting examples where this occurs are discussed by Fraser (1979) in the form of dependence on shape parameters in addition to pure group parameters. In the following it will, however, be assumed throughout that the fiducial model is *conventional* in the sense that P_U^θ does not depend on θ .

It is important to notice that many different fiducial models are possible for a given statistical model. A fiducial model provides a different basis for statistical inference than a statistical model. The choice of a particular fiducial model can be compared with the choice of a Bayesian prior together with a statistical model. Fiducial inference is then initially different from frequentist and Bayesian inference since the inferential basis is given by a fiducial model which is assumed known. Fiducial inference as such will not be considered here, but the corresponding fiducial algorithms will be used as vehicles for the construction of frequentist procedures.

A fiducial model (U, ζ) is *simple* if the fiducial equation $\zeta(u, \theta) = z$ has a unique solution $\theta^z(u)$ when solved for θ for all u, z . In the simple and conventional case the fiducial distribution is defined as the distribution of $\Theta^z = \theta^z(U)$ conditional on $\Theta = \theta$.

DEFINITION 2 (Fiducial distribution in the simple and conventional case). *Let (U, ζ) be a conventional simple fiducial model. Define the random quantity Θ^z by $z = \zeta(U, \Theta^z)$. The fiducial distribution is the conditional law of Θ^z given $\Theta = \theta$.*

The fiducial model (U, χ) given by equation (4) is simple if and only if Ω_X is a principal homogeneous space for G . In this case, by the choice of a unit element in Ω_X , the identification $G = \Omega_\Theta = \Omega_X$ can be done. It follows then that $\theta^x(u) = xu^{-1}$ is the unique solution of $x = \theta u$, and the fiducial distribution is the conditional distribution of $\Theta^x = xU^{-1}$ as it appears in equation (1g).

The remainder of this section will be on the analysis of the group model by means of the constructed fiducial model given by equation (4) in the case where Ω_X is not assumed to be a principal homogeneous space for G . The aim is to determine an equivariant inference rule δ so that the risk given by equation (2) is minimized. A definition of a fiducial distribution will also be presented for this group case. The resulting distribution

coincides with the distribution described with many more examples, explicit calculation of densities, and illustrative figures by Fraser (1968, 1979).

A first observation is that the calculations given by equations (1a)-(1d) are valid and the risk is given by $\rho = E(\gamma(e, \delta(U)) | \Theta = \theta)$. The construction of the fiducial model has hence given a simple proof that gives that the risk does not depend on the model parameter since P_U^θ does not depend on θ .

Let $Y = \phi(X)$ be an invariant statistic in the sense that $\phi(\theta x) = \phi(x)$ for all θ, x . This is equivalent with the requirement that ϕ is constant on all orbits in the sample space Ω_X . The fiducial model in equation (4) gives that $Y = \phi(X) \sim \phi(\Theta U) = \phi(U)$ conditionally on $\Theta = \theta$. The conclusion is that P_Y^θ does not depend on θ and has a known distribution. This proves that an invariant statistic Y is ancillary.

Assume furthermore that $Y = \phi(X)$ is a *maximal invariant statistic*. This means that the family of level sets of ϕ coincides with the family of orbits in Ω_X . Let x be given and assume that $y = \phi(x) = \phi(u)$. The maximality ensures that $x \in Gu = \Omega_\Theta u$, so $x = \theta^x u$ for some θ^x . This θ^x will be unique if G acts freely on Ω_X , but here it will more generally be assumed that θ^x is determined by the choice of a measurable selection. The measurable selection theorem (Castaing and Valadier, 1977) ensures existence under mild conditions. The fiducial distribution of the corresponding variable Θ^x can be described as follows.

DEFINITION 3 (Fiducial distribution in the group case). *Let u be a sample from the distribution of $(U | \Theta = \theta, \phi(U) = \phi(x))$ where ϕ is a maximal invariant. Let θ^x be a measurable selection solution of $x = \theta^x u$. This θ^x is a sample from a fiducial distribution.*

The solution θ^x exists since $y = \phi(x) = \phi(u)$ ensures that x and u are on the same orbit. Definition 3 is a special case of Definition 2 if Ω_X is a principal homogeneous space for G , and the definitions are hence consistent. It is possible to define a fiducial distribution for more general cases. One version is as presented by Hannig (2009), but there are also other possibilities available. This will not be discussed further here since the given definitions of the fiducial distribution are sufficient for the purposes in this paper.

Let $Y = \phi(X)$ be a maximal invariant statistic. The calculation that gave equation (1d) can now be continued to give

$$(6) \quad \rho = \int \left[E^{\theta, y} \gamma(e, \delta(U)) \right] P_Y^\theta(dy)$$

The expression $[\cdot]$ does only depend on y . The optimal rule δ , if it exists, is found by minimization for each given y . Assume that x is such that $y = \phi(x)$. It follows then that

$$(7) \quad E^{\theta, y} \gamma(e, \delta(U)) = E^{\theta, y} \gamma(\Theta^x, \Theta^x \delta(U)) = E^{\theta, y} \gamma(\Theta^x, \delta(x))$$

and the optimal rule δ is determined by the fiducial distribution of Θ^x . The variable Θ^x is defined as a measurable selection solution of $x = \Theta^x U$. This result can be summarized as the main technical result in this paper.

THEOREM 1. *The risk of an equivariant rule in a group invariant problem is determined by a fiducial distribution if the model parameter space is a principal homogeneous space for the group.*

It should be noted that the statement assumes existence of a fiducial distribution as described above, but uniqueness of a fiducial distribution is not assumed. Optimal inference procedures are determined by the fiducial distribution regardless of the choice of a measurable selection for the determination of a fiducial distribution. The optimal δ is found, if it exists, as the minimizer $\delta(x)$ of the expression

$$(8) \quad E^{\theta,y} \gamma(\Theta^x, \delta(x))$$

where the conditional distribution of Θ^x is a fiducial distribution.

Theorem 1 generalizes directly to the larger class of randomized equivariant actions. This is obtained by a replacement of the equivariant action $\delta(X)$ by the randomized equivariant action $\delta(X, V) = \delta(\theta U, V)$ in the calculations. It is here assumed that U and V are conditionally independent in the sense that $P_{U,V}^{\theta}(du, dv) = P_U^{\theta}(du) P_V^{\theta}(dv)$, and both conditional distributions do not depend on θ . The equivariance is defined by the identity $\delta(gX, V) = g\delta(X, V)$.

A randomized action corresponds to the assignment of a probability measure on the action space Ω_A . The set of randomized actions is hence always a convex set, and this gives theoretical advantages to the problem of minimization of the risk. If, however, the loss function $l(\theta, a)$ is convex on Ω_A for each θ , then the Jensen inequality gives that it is sufficient to consider non-random actions (Lehmann and Casella, 1998, p.48).

Theorem 1 generalizes also directly to the case where G is only assumed to act transitively on Ω_{Θ} . The construction is as above, and starts with fixing a θ_0 and the construction of a random variable U such that $P_U^{\theta} = P_X^{\theta_0}$. All the arguments given above can then be repeated with G playing the role of a new and possibly larger parameter space. The result is then first a fiducial distribution on G , but this is pushed forward to a fiducial distribution on Ω_{Θ} by the mapping $g \mapsto g\theta_0$.

It is known that the fiducial coincides with the posterior from a right Haar prior, and for these cases Theorem 1 is a known result with the posterior used in the formulation instead. There are, however, groups where no Haar prior exists, and in this case Theorem 1 and its extensions given by the above comments is a novel result. The derivation given in the introduction also gives a similar result in the more general case of a quasi-group, and the existence of invariant measures is then also not automatic.

3. Examples. The examples presented next are chosen to illustrate the concepts. Many more examples and thorough discussions are found in the previously quoted textbooks and monographs. A complete treatment of the given examples - including in particular simulation studies of the resulting procedures - will not be pursued since this would tend to take attention away from the main issue. The purpose is simply to indicate the usefulness of fiducial theory.

3.1. *The Bernoulli distribution.* A random sample of size n from the Bernoulli distribution provides an example where the results related to Theorem 1 can not be applied directly. Fiducial theory can, nonetheless, be used to obtain optimal inference.

The largest possible group G equals $\{e, g_1\}$ corresponding to the group of permutations of the set $\{0, 1\}$. The action on $\Omega_\Theta = (0, 1)$ is determined by $g_1 p = 1 - p$, and the set of orbits in the parameter space is uncountable. The conclusion is that conditioning on the maximal invariant as in the arguments leading to Theorem 1 does not provide any essential simplification of the problem.

This example is, however, very important from the point of view that fiducial distributions can still provide optimal procedures. Blank (1956) has constructed a randomized most powerful unbiased confidence interval, and this is related to a fiducial distribution (Anscombe, 1948; Stevens, 1950, 1957).

The empirical mean is the unbiased estimator of p with minimum variance. It can, however, be argued that neither unbiasedness nor minimum variance are natural concepts in this particular case. The parameter space Ω_Θ can alternatively be identified with the circular arc $\{(\sqrt{p}, \sqrt{q}) \mid p, q > 0, p + q = 1\}$ in the (\sqrt{p}, \sqrt{q}) -plane. This has the advantage that the Fisher information metric distance between two distributions in this parametric family equals the distance along the arc (Rao, 1945; Atkinson and Mitchell, 1981; Amari, 1990). The distance squared provides a loss that is invariant with respect to G . A natural task is to investigate on existence of an optimal equivariant estimator of p with respect to the distance squared on the arc. A reasonable candidate arises from the previously referenced fiducial distribution, but this will not be discussed further here.

3.2. *The Octonions.* The purpose here is to give an example which does not involve a group and where the argument given in equation (1) provides a fiducial distribution that can be used for the determination of the possibility of an optimal decision rule. The octonions is here used as an example since it is one of the more interesting examples of a group-like structure where the associative law fails. It has also a natural invariant loss that can be used in the arguments that follow. A more familiar example without associativity can be constructed for the original model of Fisher for the correlation coefficient, but we have not been able to identify a natural invariant loss in that case.

The Cayley-Dickson construction defines a multiplication $(a, b)(c, d) = (ac - d^*b, da + bc^*)$ and an involution $(a, b)^* = (a^*, -b)$ on $\mathcal{A} \times \mathcal{A}$ where \mathcal{A} is an algebra with an involution. Starting with the real numbers \mathbb{R} this gives the complex numbers \mathbb{C} . Repeated application of the construction gives then next the quaternions \mathbb{H} and then next the octonions \mathbb{O} . The octonions is hence equal to the 8-dimensional vector space \mathbb{R}^8 equipped with a particular multiplication operation so that \mathbb{O} is an algebra (Baez, 2002).

The number 1 is the unit for multiplication, and every nonzero element x has a multiplicative inverse x^{-1} with $1 = xx^{-1} = x^{-1}x$. The usual norm on \mathbb{R}^8 is also given by the product and involution as $\|x\|^2 = x^*x = xx^*$, and the identity $\|xy\| = \|x\| \|y\|$ holds. It follows in particular that $x^{-1} = x^*/\|x\|^2$. The multiplication is not associative, but the algebra \mathbb{O} is alternative: The subalgebra generated by any two elements is associative.

Consider next a fiducial model (U, χ) where $x = \chi(u, \theta) = \theta u$ is given by the product in \mathbb{O} , and where the conditional law P_U^θ is specified and does not depend on θ . Assume

that $\Omega_X = \Omega_U = \Omega_\Theta = \Omega_A = G$ where G is a subset of \mathbb{O} that contains 1, the product of any two elements, and the inverse of any element. The particular examples where G is the nonzero octonions or where G is the octonions with unit norm provide examples where G is not a group since the associative law fails. This is then a fiducial model for a statistical model $\{P_X^\theta | \theta \in \Omega_\Theta\}$ where $(X | \Theta = \theta) \sim (\theta U | \Theta = \theta)$. The corresponding fiducial distribution is the conditional distribution of $\Theta^x = xU^{-1}$ given $\Theta = \theta$. Consider the case where the loss of an action a is given by $\gamma(\theta, a) = \|\theta - a\|^2 / \|\theta\|^2$. This loss is invariant, so the calculation in equation (1) gives that the risk of an equivariant decision rule is given by $E^\theta \gamma(\Theta^x, \delta(x))$.

Existence of an optimal estimator depends on P_U^θ or equivalently on Θ^x , and this will not be discussed further here. It can, however, be noted that any optimal equivariant decision rule is determined by $\delta(x) = x\delta(1)$, and $\delta(1)$ is the minimizer of $E^\theta \gamma(\Theta^1, \delta(1))$. A rule on this form will be equivariant if $\delta(1)$ belongs to the set $\{a \in G | (g_1 g_2)a = g_1(g_2 a) \ \forall g_1, g_2 \in G\}$.

There are many other examples of binary operations that are not associative. A generic family of examples are produced by a relationship $x = \chi(u, \theta)$ that has the property (*): It gives a one-one correspondence between the domains of any two of the variables when the value of the third is fixed. Corresponding fiducial models based on χ defines the class of *simple pivotal models* in accordance with the terminology of Dawid and Stone (1982, p.1057). Concrete elementary examples are provided by $x = u - \theta$, $x = u\theta^{-1}$, and $x = u^\theta$ on suitable domains.

The property (*) is conserved by a change of variables by one-one transformations resulting in $\phi_x(\tilde{x}) = \chi(\phi_u(\tilde{u}), \phi_\theta(\tilde{\theta}))$. For the given elementary examples there exists a change of variables so that the result is a relation $\tilde{x} = \tilde{u}\tilde{\theta}$ given by a group multiplication. This is not possible in general. Simple counter examples arises for the Fisherian simple pivotal models determined by the relation $u = F(x | \theta)$ where F is a suitable cumulative distribution function. The prototypical example used by Fisher (1930, p.534) when he introduced the fiducial distribution is given by the sample correlation coefficient from a bivariate normal distribution. In this case a reduction to a group model as for the given elementary examples is not possible.

In the general case starting from the property (*) there exists, however, a change of variables that results in a relation given by a quasi-group with a unit: A loop. The important conclusion of this short discussion is that the theory of simple pivotal models is linked naturally to the theory of loops. The nonzero octonions provides an example of a loop which is not reducible to a group by a change of variables.

3.3. Hilbert space. One purpose of this example is to demonstrate existence of a case where Theorem 1 can be used, but where an invariant measure does not exist.

Let $\Omega_\Theta = \Omega_A = G$ and $\Omega_X = \Omega_U = G^n$ where G is a complex or real Hilbert space. The Hilbert space G is a group where the addition of vectors is the group operation, and an invariant loss is given by the squared distance between vectors as $\gamma(\theta, a) = \|\theta - a\|^2$. A conventional fiducial model (χ, U) is given by $x_i = \chi_i(u, \theta) = \theta + u_i$ for $i = 1, \dots, n$ and a specification of a distribution P_U^θ that does not depend on θ . A maximal invariant is given by $y = (x_2 - x_1, \dots, x_n - x_1)$. The fiducial distribution is given as the distribution

of $\Theta^x = x_1 - U_1$ from the conditional law $(U | \Theta = \theta, (U_2 - U_1, \dots, U_n - U_1) = y)$. The optimal estimator of θ is given as $\delta(x) = x_1 - E(U_1 | \Theta = \theta, (U_2 - U_1, \dots, U_n - U_1) = y)$.

It will be demonstrated in the next subsection that it is not necessary to assume independence of $\{U_i\}$ in the previous argument, and this assumption has indeed not been mentioned above. More important is the fact that a right Haar prior does not exist in the case where G is an infinite dimensional Hilbert space. An explicit example is given by $G = l^2(\mathbb{N}) = \{(a_i) | \|a\|^2 = \sum_{i=1}^{\infty} |a_i|^2 < \infty\}$.

The previous example has an infinite dimensional parameter space, and this feature is quite common in applications as exemplified by non-parametric statistics. The example does also include data that are infinite dimensional, and this can be considered to be unrealistic in applications. There are, however, applications where it is nonetheless convenient to assume that the observations are also infinite dimensional. An important source of examples is given by the statistical signal processing literature (Van Trees, 2003). Explicitly, it can be convenient to assume that a signal is observed not only at discretely sampled points, but for all points. Similarly, it can be convenient to assume that a complete infinite sequence of sampled points is observed, even though only a finite number of samples are actually observed. In both cases this can lead to a sample space that is not finite dimensional. A related and very common convenience is to assume that observations are given by real numbers, even though the majority of concrete examples actually only involves a finite set of observable values due to limited instrument resolution (Taraldsen, 2006). Explicit consideration of the limit from discretized data to continuous data gives, incidentally, a most promising route for the definition of fiducial distributions more generally than considered in Section 2 as demonstrated recently by Hannig (2012).

If one observes the real random variables X_1, \dots, X_n independently normally distributed with unknown mean $\theta = (\mu_1, \dots, \mu_n)$ and variance 1, it is customary to estimate μ_i by X_i . If the loss is the sum of squares of the errors, this estimator is admissible for $n \leq 2$, but inadmissible for $n > 3$ (Stein, 1956). The optimal estimator derived above coincides with the customary estimator. This exemplifies that the optimal estimator can be inadmissible. The optimality is only ensured within the class of equivariant estimators. Equivariance can be a most natural demand, but this depends on the particular concrete modeling case at hand. In certain situations (Efron and Morris, 1977) it can be natural to give away the equivariance demand in order to obtain more precise estimates. In other cases, especially in the context of physics, the equivariance demand can be closer to the foundation of the subject matter and will be an absolute demand.

3.4. Uniform distribution. A particular case of the previous Hilbert space example is given by assuming $G = \mathbb{R}$ and where P_U^θ corresponds to a random sample of size n from the uniform distribution on $(0, 1)$. This gives then a fiducial model for a random sample from the uniform distribution on $(\theta, \theta + 1)$. A fiducial distribution and a corresponding optimal estimator of θ follows from the Hilbert space argument. An alternative and more geometrically tractable argument follows as explained next from the use of the sufficient statistic given by the maximum and minimum observation.

Let $x_i = \theta + u_i$ where the joint distribution of (u_1, u_2) conditional on $\Theta = \theta$ is given

by the density $f(u|\theta) = n(n-1)(u_2 - u_1)^{n-2}$ on $\{(u_1, u_2) | 0 < u_1 < u_2 < 1\}$. This is then a fiducial group model for the sufficient statistic given by the smallest and largest observation from a random sample from the uniform distribution on $(\theta, \theta + 1)$. The model is also a special case of the Hilbert space example with $n = 2$ and where $\{U_i\}$ are conditionally dependent given $\Theta = \theta$. Reduction by sufficiency has here simplified the problem, but the fiducial equation is still overdetermined so a further reduction by the maximal invariant $y = x_2 - x_1$ is necessary. The resulting conditional distribution $(U_1 | \Theta = \theta, U_2 - U_1 = y)$ becomes the uniform distribution on $(0, 1 - y)$, and the fiducial distribution of Θ^x becomes the uniform distribution on $(x_2 - 1, x_1)$. This is also a confidence distribution for θ . The optimal estimator for θ given the invariant loss $|\theta - a|^2$ is $\delta(x) = (x_1 + x_2)/2 - 1/2$.

We choose to add a few comments on this model and estimator since it has some unusual features. A first observation is that the Fisher information metric fails to exist due to nonexistence of the required derivative. The corresponding distance between two distributions can, however, still be defined through the length of the parametric curve $\theta \mapsto \sqrt{f(\cdot|\theta)}$ in the Hilbert space of square integrable functions. This curve is continuous, but the length from θ_1 to θ_2 is infinite: It is larger than $2\sqrt{n}\sqrt{|\theta_2 - \theta_1|}$ for any integer n .

The squared distance $|\theta_1 - \theta_2|^2$ is the squared distance from the Fisher information metric for any location family where the density is smooth. Based on this we consider the invariant loss $|\theta - a|^2$ to be a natural choice also in the non-smooth example considered here.

The optimal estimator δ found above is unbiased and has hence minimum variance in the class of unbiased and equivariant estimators. Nonetheless, according to Lehmann and Casella (1998, p.87), there exists no uniformly minimum variance unbiased estimator of θ . The statistic (X_1, X_2) is a minimal sufficient statistic, but it is not complete. The estimator δ is, however, the uniformly minimum variance unbiased estimator in the larger parametric family which also includes a scale parameter (Johnson et al., 1994, vol.2, p.292). This later reference is also a very good source for further references and peculiarities regarding the uniform law.

3.5. Exponential. The following example is a scale example, and can be reduced to be a special case of the Hilbert space location problem by the logarithmic transformation. A direct solution is equally elementary and is presented to illustrate the derivation of an optimal estimator. The explicit formula for the estimator is possibly a novelty.

Assume that Y_1, \dots, Y_n is a random sample of size n from the exponential distribution with scale parameter β . A fiducial model is given by $Y_i = \beta V_i$ where the law P_V^β is as for a random sample from the standard exponential distribution. The arithmetic mean $X = \bar{Y}$ is a minimal sufficient statistic. A corresponding fiducial model is given by $X = \beta \bar{V} = \beta U$ where P_U^β is the law of a gamma variable with scale equal to $1/n$ and shape equal to n . This follows from well known properties of the gamma distribution. The model is both simple and conventional, and the fiducial distribution for an observation $x = \bar{y}$ is hence the conditional distribution of $\Theta^x = x/U$. The conclusion is that the fiducial distribution is the inverse-gamma with scale xn and shape n .

A direct - but more lengthy - calculation of the Bayesian posterior corresponding to the right Haar prior $d\beta/\beta$ gives a posterior that coincides with the fiducial distribution found here. It is well known more generally that the Bayesian posterior from a right Haar prior in a group model coincides with the fiducial. The calculation demonstrates then that a fiducial model and the solution of the fiducial equation gives an alternative and in many cases simpler route for the calculation of the Bayesian posterior. The multivariate normal gives another example where the fiducial calculation is done in a few lines, but the corresponding Bayesian calculation is much more cumbersome.

An added advantage of the fiducial calculation is that it shows directly that the corresponding fiducial distribution is a confidence distribution. This is not easily obtained from the Bayesian calculation. The confidence distribution can alternatively be found by the likelihood ratio test, and this has the advantage of giving proof of optimality and corresponding optimal choices of confidence interval endpoints. An alternative approach is to also derive optimal intervals based on Theorem 1 as exemplified by Berger (1985).

An alternative fiducial calculation can be done without the reduction to the complete sufficient statistic. A maximal invariant ϕ is given by $\phi(y) = y/\|y\|$. The conditional law $(V | \Theta = \theta, \phi(V) = \phi(y))$ will be concentrated on the ray $Gy = \{\alpha\phi(y) | \alpha > 0\}$ with a distribution from a density for α proportional to $f_V(\alpha\phi(y))\alpha^{n-1}$. The assumption of a random sample from the standard exponential gives a particularly simple f_V , and the fiducial is found explicitly as before. The alternative calculation has the advantage that it can be used in the more general case where reduction by sufficiency is not available.

Consider now the problem of estimation of $\theta = \beta$ with a loss given by $\gamma(\theta, a) = |\ln \theta - \ln a|^2$. This loss is a natural generalization of the squared error loss, but with the ordinary distance replaced by the distance $|\ln \theta - \ln a|$ which is the distance given by the Fisher information metric in the case of the given scale model. In this case $G = \Omega_X = \Omega_\Theta = \Omega_A = \mathbb{R}^+$ with multiplication as the group operation. The loss is equivariant, and it follows that the optimal rule δ based on the sufficient statistic X is given as the minimizer of $\rho = E^\theta |\ln \Theta^x - \ln \delta(x)|^2$. This gives that the optimal rule is determined from $\ln \delta(x) = E^\theta \ln \Theta^x$. Evaluation of the corresponding integral gives an explicit formula for the optimal rule. It is

$$(9) \quad \delta(x) = x \exp(\ln n - \psi(n))$$

where ψ is the digamma function. The estimator given by equation (9) is possibly known in some contexts, but we have not found this explicit expression in any of the textbooks in the list of references or elsewhere.

3.6. *Gamma distribution.* Assume that Y_1, \dots, Y_n is a random sample of size n from the gamma distribution with scale parameter β and shape parameter α . The model parameter is $\theta = (\alpha, \beta)$. This gives an example as in subsection 3.1 where the results related to Theorem 1 can not be applied directly. Fiducial theory can be used to obtain candidates for good frequentist inference procedures as indicated next. Particular results include an exact joint confidence distribution for (α, β) , an exact confidence distribution for α , and a recipe which produces estimators for functions of (α, β) that depends on the choice of a loss.

A fiducial model is given by $Y_i = \beta F^{-1}(U_i; \alpha)$, where the law P_U^θ corresponds to a random sample of size n from the uniform distribution on the unit interval $(0, 1)$, and $F^{-1}(u, \alpha)$ is the inverse cumulative distribution function of a gamma variable with shape α and scale 1.

Let $X = (\bar{Y}, \tilde{Y}/\bar{Y})$ where \bar{Y} and \tilde{Y} are the arithmetic and geometric means. The Bartlett statistic $W = \tilde{Y}/\bar{Y}$ depends only on α , and is independent of \bar{Y} as a consequence of the Basu theorem. A corresponding fiducial model (χ, U) for P_X^θ is given by $\chi_1(u, \theta) = \beta \overline{F^{-1}(u; \alpha)}$ and $\chi_2(u, \theta) = \overline{F^{-1}(u; \alpha)}/\overline{F^{-1}(u; \alpha)}$. It can be noted that (χ_2, U) gives separately a fiducial model for P_W^α . The corresponding fiducial distribution for α is hence a confidence distribution.

An alternative fiducial model (η_2, V_2) for P_W^α is given by inversion of the cumulative distribution function for W . An alternative to $\overline{F^{-1}(u; \alpha)}$ is given by inversion for a gamma variable with shape $n\alpha$ and scale $1/n$. The combination gives an alternative fiducial model (η, V) for P_X^θ with the property that $x = \eta(v, \theta)$ defines a one-one correspondence between any two variables when the third is kept fixed. The law P_V^θ is the uniform law on the unit square $(0, 1)^2$. Coordinate transformations can be used to identify $G = \Omega_\Theta = \Omega_V = \Omega_X$ as sets with a quasigroup structure with a unit.

Both fiducial models (χ, U) and (η, V) are simple and conventional and determine a fiducial distribution. For concreteness let Θ^x be the fiducial corresponding to (η, V) . The quasigroup structure ensures that Θ^x gives a joint exact confidence distribution for (α, β) .

Consider the problem of estimation of a function $\tau = h(\alpha, \beta)$ of the model parameter $\theta = (\alpha, \beta)$. It can be allowed that h is vector valued, but assume that each component is positive. Three examples that are included are then given by $\tau = \alpha$, $\tau = \beta$, and $\tau = (\alpha, \beta)$. A possible loss in these three cases is given by squared error loss on a logarithmic scale. A candidate estimator δ is then given naturally by

$$(10) \quad \delta(x) = \exp \left(E^\theta \ln h(\Theta^x) \right)$$

This can be evaluated by Monte-Carlo simulation from P_V^θ which is the uniform distribution on the unit square $[0, 1]^2$. Another possibility is given by squared distance loss defined by the Fisher information metric on Ω_Θ in the case $h(\theta) = (\alpha, \beta)$.

4. Discussion. The foundations of Bayesian and frequentist modeling and inference are well established both in terms of mathematical theory and interpretation. We do not think that the same can be said about fiducial theory, but some readers may object to this. A brief discussion of alternative formulations and naming conventions found in the literature seems hence to be in place.

Definition 1 identifies a fiducial model with a pair (U, ζ) . The fiducial model is by definition *conventional* if P_U^θ does not depend on θ . In this case we suggest to denote U as the *Monte Carlo variable* and the measurable function ζ as the *fiducial relation*. The corresponding equation $z = \zeta(u, \theta)$ is the fiducial equation, but it may also equivalently be denoted as the fiducial relation. A fiducial model (U, ζ) is hence defined by a Monte Carlo variable U and a fiducial relation ζ .

If u is a sample from the Monte Carlo distribution P_U^θ , then $z = \zeta(u, \theta)$ is a sample from the statistical model as in Definition 1. The inversion method gives the prototypical example with $\zeta(u, \theta) = F_Z^{-1}(u | \theta)$ and P_U^θ equal to the uniform law on the interval $[0, 1]$. This gives the link to the original definition of Fisher, and also a justification of the choice of the term *Monte Carlo variable* since this represents a standard method for simulation from a statistical model on a computer.

The ingredients above given by the pair (U, ζ) are also the starting point for Dempster-Shafer theory (Dempster, 1968; Shafer, 1982). Martin et al. (2010) refer to U as the *auxiliary variable* and the probability measure μ as the *pivotal measure*, where $U \sim \mu$. The equation $Z = \zeta(U, \Theta)$ is denoted the *a-equation*. The whole set-up is referred to as an *inferential model*, and this is identified as something which is not equivalent to a statistical model. Except for differences in naming conventions it can be concluded that an *inferential model* is essentially the same as a *conventional fiducial model* as summarized in the previous paragraphs. The Dempster-Shafer calculus gives an alternative route for inference based on a fiducial model, but this is not discussed further here.

The discussion of fiducial theory we have presented is close to the presentation given by Dawid and Stone (1982). Dawid and Stone (1982, p.1055) use the term *fiducial model* for the combination of $\Theta^z = \theta^z(U)$ and $U \sim P_U^\theta$, and use the term *functional model* to describe the more general relation $Z = \zeta(U, \Theta)$. We chose to avoid the term *functional model* since the term *functional data analysis* is now the name of a branch of statistics. Dawid and Stone (1982) denote the variable U as the *error variable*, and uses the symbol E instead. This corresponds to the naming convention used by Fraser (1968). Fraser (1968, p.50) uses the terms *structural model* and *structural equation* in the case where group theory is an essential ingredient. Hannig (2009) uses the term *structural equation* in stead of the term *fiducial equation* as used by us. We have avoided the term *structural* here since there is an active and well developed different theory which goes under the label of *structural equations modeling*. Our preference for the term *fiducial* as used here is mainly based on economy of language, and since this gives the direct link to the original papers of Fisher.

The mathematically inclined reader may claim that Definition 1 is not precise. This, and the fact that this definition is a novelty compared with previous writers, motivate us to state in more detail the assumptions that are taken as implicitly given from the context in the statement of Definition 1. The main difference is that every concept is based on an underlying abstract conditional probability space (Ω, \mathcal{E}, P) as stated initially in Section 2. The fiducial relation ζ is a measurable function $\zeta : \Omega_U \times \Omega_\Theta \rightarrow \Omega_Z$. This means, as usual, that $(\zeta \in A) = \{(u, \theta) | \zeta(u, \theta) \in A\}$ is a measurable set in the product σ -algebra of $\Omega_U \times \Omega_\Theta$ whenever A is a measurable set in Ω_Z . A consequence is that $\zeta(U, \Theta)$ is a random element in Ω_Z defined by the mapping $\omega \mapsto \zeta(U(\omega), \Theta(\omega))$. This is measurable since it is assumed that $\Theta : \Omega \rightarrow \Omega_\Theta$ and $U : \Omega \rightarrow \Omega_U$ are measurable. The conditional law P_U^θ of the Monte Carlo variable U is known and does not depend on θ in the case of a conventional fiducial model. If the considerations were limited to the case where (Ω, \mathcal{E}, P) is a probability space, then this would imply $P_U^\theta = P_U$. This fails generally as explained in more detail by Taraldsen and Lindqvist (2010) since P_U^θ is a

probability measure, but P_U is unbounded if P is unbounded. The reason for allowing unbounded measures is the need to include improper priors P_Θ . This has proved fruitful in related ongoing research by the authors. It gives in particular natural conditions that imply equality of Bayesian posteriors and fiducial distributions. In specific modeling cases the spaces Ω_U , Ω_Z , Ω_Θ , the fiducial relation ζ , and the conditional law P_U^θ are all explicitly given. This is as demonstrated by the examples in Section 3. The other ingredients mentioned above are not given explicitly since they rely on the underlying unspecified space Ω . This is as in the ordinary formulation of probability theory by Kolmogorov (1933) where the whole theory is built upon the underlying abstract space Ω . Existence must be proved in each specific modeling case, but follows trivially in many cases from the construction of a suitable product space.

Optimal inference for the scale, the location, and the location-scale problems were investigated using fiducial theory by Pitman (1939). His presentation is most readable and is a good alternative to the presentations found in standard textbooks. It can, however, be noted that he concludes that the confidence and the fiducial theories are essentially the same. This is in contrast to the views of Neyman and Fisher. They seemed to agree that in principle the fiducial distribution as described by Fisher is not connected to the concept of confidence intervals as described by Neyman and co-workers. The content and aims of these two theories are different. It seems clear that Fisher never intended to get confidence intervals as the result of his fiducial arguments.

It is true that the fiducial distributions found in the location-scale problems, and more general group problem as in Theorem 1, are confidence distributions, but we do consider the concepts to be essentially different in general. The interpretation of the fiducial distribution, according to Fisher (1973, p.54, p.59) is identical with the interpretation of the Bayesian posterior: It represents the state of knowledge regarding the model parameter as a result of the model assumptions and the observation in the experiment. It follows then in particular that the fiducial distribution of a function $\phi(\theta)$ of the model parameter θ equals the distribution of $\phi(\Theta^x)$ where Θ^x has the fiducial distribution. This property does not hold for confidence distributions in general. In addition, the fiducial distribution for a simple fiducial model as in Definition 2 is not a confidence distribution in general (Dawid and Stone, 1982).

The possibly most famous fiducial distribution is the fiducial distribution of the difference of means $\mu_1 - \mu_2$ corresponding to two independent samples from two different normal distributions. This fiducial distribution gives Fisher's solution to the Behrens-Fisher problem, but it can be shown by simulation that it is not a confidence distribution in the sense of having exact coverage probabilities. A more general class of confidence distributions is defined by requiring not exact but conservative coverage probabilities. This is in conformity with the definition of confidence sets in general. Exactness is often misguidedly taken as a measure of goodness, but it is not. Power of the associated tests gives one natural measure of goodness. Examples demonstrate that this more general concept of a confidence distribution does not coincide with fiducial distributions in general, but it seems to be an open question whether the Behrens-Fisher fiducial distribution is a confidence distribution in this more extended sense. Numerical simulations indicate

that the claim holds (Robinson, 1976; Barnard, 1984, p.269).

The more general problem of obtaining a confidence interval for the linear combination of several means from different normal distributions is of considerable practical importance (ISO/IEC, 2008). The ISO recommended solution is in terms of a Welch-Satterthwaite solution, but a continuation of the arguments given by Barnard (1984) leads to the conclusion that the fiducial solution is a most competitive alternative solution.

The main virtue of the location-scale models in the context here is that they illustrate very well the reduction given by a maximal invariant in cases where a reduction by sufficiency is not possible. This is also true for the multivariate models treated by Fraser (1979). In this case the multivariate normal can be reduced by sufficiency, but more general models can again be treated by a reduction through maximal invariants. It seems that optimal, or good, inference procedures in these multivariate cases deserves further study guided by fiducial theory. A recent example of this is given by Lidong et al. (2008), but there are a multitude of different possible examples as indicated by Fraser (1979). The suggestion given by Theorem 1 is that not only confidence intervals, but also other kinds of inference such as estimation should be considered.

Eaton (1989, p.89-91) considers the estimation of the covariance matrix from a multivariate normal sample. He gives two possible candidates to use as a loss $\gamma(\theta, a)$. This exemplifies that in the multivariate cases, and in more complicated group cases, it can be difficult to decide upon which equivariant loss to use. It can even be difficult to come up with a good candidate. In our examples it has been indicated that the squared distance from the Fisher information metric is a natural choice. This will be invariant under mild conditions. For a statistical model $f(x|\theta)\mu(dx)$ the distance is defined via the length of paths $t \mapsto x(t) = \sqrt{f(\cdot|\theta(t))}$ in the Hilbert space $L_2(\mu)$. The nonparametric case given by a parameter space equal to all densities with respect to μ gives the distance $d(f, g) = \cos^{-1}(\int \sqrt{fg} d\mu)$. The other end of the scale is given by a smooth finite dimensional parametric model. In this case the previous leads to the Fisher information metric: $ds^2 = (1/4)g_{ij}d\theta^i d\theta^j$ where $g_{ij} = E_X^\theta(\partial_i \ln f(X|\theta))(\partial_j \ln f(X|\theta))$. In either case it gives the model parameter space as a manifold equipped with a distance derived intrinsically from the statistical model.

The focus of fiducial theory has initially and currently most often been on the fiducial distribution by itself and the related possibility of construction of approximate or exact confidence intervals. The relation to other kinds of optimal inference such as estimation or prediction was considered by Hora and Buehler (1966, 1967). The proofs they presented rely on the existence of an invariant measure, and it was clear that the fiducial in the case they considered corresponded to a Bayesian posterior from the right Haar prior. Since then it has been established in a variety of problems that the Bayesian algorithm can be used quite generally to obtain good or optimal frequentist procedures. The calculation given in equation (1) can be taken as a strong indication that the fiducial algorithm can be used similarly to not only obtain confidence intervals, but also possibly good or optimal frequentist procedures more generally. This statement is too general to be provable, but we consider nonetheless this to be the main content and message in this

paper. The point of view in this paper does not rely on any particular interpretation of the fiducial. It is here simply viewed as a very convenient vehicle for the derivation of good, and sometimes optimal as in Theorem 1, frequentist inference procedures.

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