On the Uniform Asymptotic Validity of Subsampling and the Bootstrap

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Abstract

This paper provides conditions under which subsampling and the bootstrap can be used to construct estimators of the quantiles of the distribution of a root that behave well uniformly over a large class of distributions $\mathbf{P}$. These results are then applied (i) to construct confidence regions that behave well uniformly over $\mathbf{P}$ in the sense that the coverage probability tends to at least the nominal level uniformly over $\mathbf{P}$ and (ii) to construct tests that behave well uniformly over $\mathbf{P}$ in the sense that the size tends to no greater than the nominal level uniformly over $\mathbf{P}$. Without these stronger notions of convergence, the asymptotic approximations to the coverage probability or size may be poor even in very large samples. Specific applications include the multivariate mean, testing moment inequalities, multiple testing, the empirical process, and U-statistics.

KEYWORDS: Bootstrap, Empirical Process, Moment Inequalities, Multiple Testing, Subsampling, Uniformity, U-Statistic

1 Introduction

Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ and denote by $J_n(x, P)$ the distribution of a real-valued root $R_n = R_n(X^{(n)}, P)$ under P. In statistics and econometrics, it is often of interest to estimate certain quantiles of $J_n(x, P)$. Two commonly used methods for this purpose are subsampling and the bootstrap. This paper provides conditions under which these estimators behave well uniformly over $\mathbf{P}$. More precisely, we provide conditions under which subsampling and the bootstrap may be used to construct estimators $\hat{c}_n(\alpha_1)$ of the α_1 quantiles of $J_n(x, P)$ and $\hat{c}_n(1-\alpha_2)$ of the $1-\alpha_2$ quantiles of $J_n(x, P)$ satisfying

$$\liminf_{n\to\infty} \inf_{P\in\mathbf{P}} P\{\hat{c}_n(\alpha_1) \leq R_n \leq \hat{c}_n(1-\alpha_2)\} \geq 1 - \alpha_1 - \alpha_2 \tag{1}$$

Here, $\hat{c}_n(0)$ is understood to be $-\infty$ and $\hat{c}_n(1)$ is understood to be $+\infty$. For the construction of two-sided confidence intervals of nominal level $1-2\alpha$ for a real-valued parameter, we typically would consider $\alpha_1 = \alpha_2 = \alpha$, while for a one-sided confidence interval of nominal level $1-\alpha$ we would consider either $\alpha_1 = 0$ and $\alpha_2 = \alpha$, or $\alpha_1 = \alpha$ and $\alpha_2 = 0$. In many cases, it is possible to replace the $\liminf_{n\to\infty}$ and $\geq$ in (1) with $\lim_{n\to\infty}$ and $=$, respectively. These results differ from those usually stated in the literature in that they require the convergence to hold uniformly over $\mathbf{P}$ instead of just pointwise over $\mathbf{P}$. The importance of this stronger notion of convergence when applying these results is discussed further below.

As we will see, the result (1) may hold with $\alpha_1 = 0$ and $\alpha_2 = \alpha \in (0,1)$, but it may fail if $\alpha_2 = 0$ and $\alpha_1 = \alpha \in (0,1)$, or the other way round. This phenomenon arises when it is not possible to estimate $J_n(x, P)$ uniformly well with respect to a suitable metric, but, in a sense to be made precise by our results, it is possible to estimate it sufficiently well to ensure that (1) still holds for certain choices of α_1 and α_2. Note that metrics compatible with the weak topology are not sufficient for our purposes. In particular, closeness of distributions with respect to such a metric does not ensure closeness of quantiles. See Remark 2.7 for further discussion of this point. In fact, closeness of distributions with respect to even stronger metrics, such as the Kolmogorov metric, does not ensure closeness of quantiles either. For this reason, our results rely heavily on Lemma 4.1 that relates closeness of distributions with respect to a suitable metric and coverage statements.

In contrast, the usual arguments for the pointwise asymptotic validity of subsampling and the bootstrap rely on showing for each $P \in \mathbf{P}$ that $\hat{c}_n(1-\alpha)$ tends in probability under P to the $1-\alpha$ quantile of the limiting distribution of R_n under P. Because our results are uniform in $P \in \mathbf{P}$, we must consider the behavior of R_n and $\hat{c}_n(1-\alpha)$ under arbitrary sequences $\{P_n \in \mathbf{P} : n \geq 1\}$,

under which the quantile estimators need not even settle down. Thus, the results are not trivial extensions of the usual pointwise asymptotic arguments.

The construction of $\hat{c}_n(\alpha)$ satisfying (1) is useful for constructing confidence regions that behave well uniformly over $\mathbf{P}$. More precisely, our results provide conditions under which subsampling and the bootstrap can be used to construct confidence regions $C_n = C_n(X^{(n)})$ of level α for a parameter $\theta(P)$ that are uniformly consistent in level in the sense that

$$\liminf_{n\to\infty} \inf_{P\in\mathbf{P}} P\{\theta(P) \in C_n\} \geq 1-\alpha \ . \tag{2}$$

Our results are also useful for constructing tests $\phi_n = \phi_n(X^{(n)})$ of level α for a null hypothesis $P \in \mathbf{P}_0 \subseteq \mathbf{P}$ against the alternative $P \in \mathbf{P}_1 = \mathbf{P} \setminus \mathbf{P}_0$ that are uniformly consistent in level in the sense that

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}_0} E_P[\phi_n] \leq \alpha \ . \tag{3}$$

In some cases, it is possible to replace the $\liminf_{n\to\infty}$ and $\geq$ in (2) or the $\limsup_{n\to\infty}$ and $\leq$ in (3) with $\lim_{n\to\infty}$ and $=$, respectively.

Confidence regions satisfying (2) are desirable because they ensure that for every $\epsilon > 0$ there is an N such that for $n > N$ we have that $P\{\theta(P) \in C_n\}$ is no less than $1-\alpha-\epsilon$ for all $P \in \mathbf{P}$. In contrast, confidence regions that are only pointwise consistent in level in the sense that

$$\liminf_{n\to\infty} P\{\theta(P) \in C_n\} \geq 1-\alpha$$

for each fixed $P \in \mathbf{P}$ have the feature that there exists some $\epsilon > 0$ and $\{P_n \in \mathbf{P} : n \geq 1\}$ such that $P_n\{\theta(P_n) \in C_n\}$ is less than $1-\alpha-\epsilon$ infinitely often. Likewise, tests satisfying (3) are desirable for analagous reasons. For this reason, inferences based on confidence regions or tests that fail to satisfy (2) or (3) may be very misleading in finite samples. Of course, as pointed out by Bahadur and Savage (1956), there may be no nontrivial confidence region or test satisfying (2) or (3) when $\mathbf{P}$ is sufficiently rich. For this reason, we will have to restrict $\mathbf{P}$ appropriately in our examples. In the case of confidence regions for or tests about the mean, for instance, we will have to impose a very weak uniform integrability condition. See also Kabaila (1995), Pötscher (2002), Leeb and Pötscher (2006a), Leeb and Pötscher (2006b), Pötscher (2009) for related results in more complicated settings, including post-model selection, shrinkage-estimators, and ill-posed problems.

Some of our results on subsampling are closely related to results in Andrews and Guggenberger (2010), which were developed independently and at about the same time as our results. See the discussion on p. 431 of Andrews and Guggenberger (2010). Our results show that the question of whether subsampling can be used to construct estimators $\hat{c}_n(\alpha)$ satisfying (1) reduces to a

single, succinct requirement on the asymptotic relationship between the distribution of $J_n(x, P)$ and $J_b(x, P)$, where b is the subsample size, whereas the results of Andrews and Guggenberger (2010) require the verification of a larger number of conditions. Moreover, we also provide a converse, showing this requirement on the asymptotic relationship between the distribution of $J_n(x, P)$ and $J_b(x, P)$ is also necessary in the sense that, if the requirement fails, then for some nominal coverage level the uniform coverage statements fail. Thus, our results are stated under essentially the weakest possible conditions, yet are verifiable in a large class of examples. On the other hand, the results of Andrews and Guggenberger (2010) further provide a means of calculating the limiting value of $\inf_{P \in \mathbf{P}} P\{\hat{c}_n(\alpha_1) \leq R_n \leq \hat{c}_n(1 - \alpha_2)\}$ in the case where it may not satisfy (1). To the best of our knowledge, our results on the bootstrap are the first to be stated at this level of generality. An important antecedent is Romano (1989), who studies the uniform asymptotic behavior of confidence regions for a univariate cumulative distribution function. See also Mikusheva (2007), who analyzes the uniform asymptotic behavior of some tests that arise in the context of an autoregressive model.

The remainder of the paper is organized as follows. In Section 2, we present the conditions under which $\hat{c}_n(\alpha)$ satisfying (1) may be constructed using subsampling or the bootstrap. We then provide in Section 3 several applications of our general results. These applications include the multivariate mean, testing moment inequalities, multiple testing, the empirical process and U-statistics. The discussion of U-statistics is especially noteworthy because it highlights the fact that the assumptions required for the uniform asymptotic validity of subsampling and the bootstrap may differ. In particular, subsampling may be uniformly asymptotically valid under conditions where, as noted by Bickel and Freedman (1981), the bootstrap fails even to be pointwise asymptotically valid. The application to multiple testing is also noteworthy because, despite the enormous recent literature in this area, our results appear to be the first that provide uniformly asymptotically valid inference. Proofs of the main results (Theorems 2.1 and 2.4) can be found in the Appendix; proofs of all other results can be found in Romano and Shaikh (2012), which contains supplementary material. Many of the intermediate results may be of independent interest, including uniform weak laws of large numbers for U-statistics and V-statistics (Lemmas S.17.3 and S.17.4 in Romano and Shaikh (2012), respectively) as well as the aforementioned Lemma 4.1.

2 General Results

2.1 Subsampling

Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$. Denote by $J_n(x, P)$ the distribution of a real-valued root $R_n = R_n(X^{(n)}, P)$ under P. The goal is to construct procedures which are valid uniformly in P. In order to describe the subsampling approach to approximate $J_n(x, P)$, let $b = b_n < n$ be a sequence of positive integers tending to infinity, but satisfying $b/n \to 0$ and define $N_n = \binom{n}{b}$. For $i = 1, \ldots, N_n$, denote by $X^{n,(b),i}$ the ith subset of data of size b. Below, we present results for two subsampling-based estimators of $J_n(x, P)$. We first consider the estimator given by

$$L_n(x, P) = \frac{1}{N_n} \sum_{1 \le i \le N_n} I\{R_b(X^{n,(b),i}, P) \le x\} \ . \tag{4}$$

More generally, we will also consider feasible estimators $\hat{L}_n(x)$ in which R_b is replaced by some estimator $\hat{R}_b$, i.e.,

$$\hat{L}_n(x) = \frac{1}{N_n} \sum_{1 \le i \le N_n} I\{\hat{R}_b(X^{n,(b),i}) \le x\} \ . \tag{5}$$

Typically, $\hat{R}_b(\cdot) = R_b(\cdot, \hat{P}_n)$, where $\hat{P}_n$ is the empirical distribution, but this is not assumed below. Even though the estimator of $J_n(x, P)$ defined in (4) is infeasible because of its dependence on P, which is unknown, it is useful both as an intermediate step towards establishing some results for the feasible estimator of $J_n(x, P)$ and, as explained in Remarks 2.2 and 2.3, on its own in the construction of some feasible tests and confidence regions.

Theorem 2.1. *Let $b = b_n < n$ be a sequence of positive integers tending to infinity, but satisfying $b/n \to 0$ and define $L_n(x, P)$ as in (4). Then, the following statements are true:*

(i) If $\limsup_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}} \{J_b(x, P) - J_n(x, P)\} \le 0$, then

$$\liminf_{n\to\infty} \inf_{P\in\mathbf{P}} P\{L_n^{-1}(\alpha_1, P) \le R_n \le L_n^{-1}(1 - \alpha_2, P)\} \ge 1 - \alpha_1 - \alpha_2 \tag{6}$$

holds for $\alpha_1 = 0$ and any $0 \le \alpha_2 < 1$.

(ii) If $\limsup_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}} \{J_n(x, P) - J_b(x, P)\} \le 0$, then (6) holds for $\alpha_2 = 0$ and any $0 \le \alpha_1 < 1$.

(iii) If $\lim_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}} |J_b(x, P) - J_n(x, P)| = 0$, then (6) holds for any $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$ satisfying $0 \le \alpha_1 + \alpha_2 < 1$.

Remark 2.1. It is typically easy to deduce from the conclusions of Theorem 2.1 stronger results in which the $\liminf_{n\to\infty}$ and $\geq$ in (6) are replaced by $\lim_{n\to\infty}$ and $=$, respectively. For example, in order to assert that (6) holds with $\liminf_{n\to\infty}$ and $\geq$ replaced by $\lim_{n\to\infty}$ and $=$, respectively, all that is required is that

$$\lim_{n\to\infty} P\{L_n^{-1}(\alpha_1, P) \leq R_n \leq L_n^{-1}(1-\alpha_2, P)\} = 1 - \alpha_1 - \alpha_2$$

for some $P \in \mathbf{P}$. This can be verified using the usual arguments for the pointwise asymptotic validity of subsampling. Indeed, it suffices to show for some $P \in \mathbf{P}$ that $J_n(x, P)$ tends in distribution to a limiting distribution $J(x, P)$ that is continuous at the appropriate quantiles. See Politis et al. (1999) for details. ■

Remark 2.2. As mentioned earlier, $L_n(x, P)$ defined in (4) is infeasible because it still depends on P, which is unknown, through $R_b(X^{n,(b),i}, P)$. Even so, Theorem 2.1 may be used without modification to construct feasible confidence regions for a parameter of interest $\theta(P)$ provided that $R_n(X^{(n)}, P)$, and therefore $L_n(x, P)$, depends on P only through $\theta(P)$. If this is the case, then one may simply invert tests of the null hypotheses $\theta(P) = \theta$ for all $\theta \in \Theta$ to construct a confidence region for $\theta(P)$. More concretely, suppose $R_n(X^{(n)}, P) = R_n(X^{(n)}, \theta(P))$ and $L_n(x, P) = L_n(x, \theta(P))$. Whenever we may apply part (i) of Theorem 2.1, we have that

$$C_n = \{\theta \in \Theta : R_n(X^{(n)}, \theta) \leq L_n^{-1}(1-\alpha, \theta)\}$$

satisfies (2). Similar conclusions follow from parts (ii) and (iii) of Theorem 2.1. ■

Remark 2.3. It is worth emphasizing that even though Theorem 2.1 is stated for roots, it is, of course, applicable in the special case where $R_n(X^{(n)}, P) = T_n(X^{(n)})$. This is especially useful in the context of hypothesis testing. See Example 3.3 for one such instance. ■

Next, we provide some results for feasible estimators of $J_n(x, P)$. The first result, Corollary 2.1, handles the case of the most basic root, while Theorem 2.2 applies to more general roots needed for many of our applications.

Corollary 2.1. *Suppose $R_n = R_n(X^{(n)}, P) = \tau_n(\hat{\theta}_n - \theta(P))$, where $\{\tau_n \in \mathbf{R} : n \geq 1\}$ is a sequence of normalizing contants, $\theta(P)$ is a real-valued parameter of interest, and $\hat{\theta}_n = \hat{\theta}_n(X^{(n)})$ is an estimator of $\theta(P)$. Let $b = b_n < n$ be a sequence of positive integers tending to infinity, but satisfying $b/n \to 0$ and define*

$$\hat{L}_n(x) = \frac{1}{N_n} \sum_{1\leq i\leq N_n} I\{\tau_b(\hat{\theta}_b(X^{n,(b),i}) - \hat{\theta}_n) \leq x\} \ .$$

Then, statements (i) - (iii) of Theorem 2.1 hold when $L_n^{-1}(\cdot, P)$ is replaced by $\frac{\tau_n}{\tau_n+\tau_b}\hat{L}_n^{-1}(\cdot)$.

Theorem 2.2. *Let $b = b_n < n$ be a sequence of positive integers tending to infinity, but satisfying $b/n \to 0$. Define $L_n(x, P)$ as in (4) and $\hat{L}_n(x)$ as in (5). Suppose for all $\epsilon > 0$ that*

$$\sup_{P \in \mathbf{P}} P\left\{\sup_{x \in \mathbf{R}} |\hat{L}_n(x) - L_n(x, P)| > \epsilon\right\} \to 0 \ . \tag{7}$$

Then, statements (i) - (iii) of Theorem 2.1 hold when $L_n^{-1}(\cdot, P)$ is replaced by $\hat{L}_n^{-1}(\cdot)$.

As a special case, Theorem 2.2 can be applied to studentized roots.

Corollary 2.2. *Suppose*

$$R_n = R_n(X^{(n)}, P) = \frac{\tau_n(\hat{\theta}_n - \theta(P))}{\hat{\sigma}_n} \ ,$$

where $\{\tau_n \in \mathbf{R} : n \geq 1\}$ is a sequence of normalizing constants, $\theta(P)$ is a real-valued parameter of interest, and $\hat{\theta}_n = \hat{\theta}_n(X^{(n)})$ is an estimator of $\theta(P)$, and $\hat{\sigma}_n = \hat{\sigma}_n(X^{(n)}) \geq 0$ is an estimator of some parameter $\sigma(P) \geq 0$. Suppose further that

(i) The family of distributions $\{J_n(x, P) : n \geq 1, P \in \mathbf{P}\}$ is tight, and any subsequential limiting distribution is continuous.

(ii) For any $\epsilon > 0$,

$$\sup_{P \in \mathbf{P}} P\left\{\left|\frac{\hat{\sigma}_n}{\sigma(P)} - 1\right| > \epsilon\right\} \to 0 \ .$$

Let $b = b_n < n$ be a sequence of positive integers tending to infinity, but satisfying $b/n \to 0$ and $\tau_b/\tau_n \to 0$. Define

$$\hat{L}_n(x) = \frac{1}{N_n} \sum_{1 \leq i \leq N_n} I\left\{\frac{\tau_b(\hat{\theta}_b(X^{n,(b),i}) - \hat{\theta}_n)}{\hat{\sigma}_b(X^{n,(b),i})} \leq x\right\} \ .$$

Then, statements (i) - (iii) of Theorem 2.1 hold when $L_n^{-1}(\cdot, P)$ is replaced by $\hat{L}_n^{-1}(\cdot)$.

Remark 2.4. One can take $\hat{\sigma}_n = \sigma(P)$ in Corollary 2.2. Since $\sigma(P)$ effectively cancels out from both sides of the inequality in the event $\{R_n \leq \hat{L}_n^{-1}(1 - \alpha)\}$, such a root actually leads to a computationally feasible construction. However, Corollary 2.2 still applies and shows that we can obtain a positive result without the correction factor $\tau_n/(\tau_n + \tau_b)$ present in Corollary 2.1, provided the conditions of Corollary 2.2 hold. For example, if for some $\sigma(P)$, we have that $\tau_n(\hat{\theta}_n - \theta(P_n))/\sigma(P_n)$ is asymptotically standard normal under any sequence $\{P_n \in \mathbf{P} : n \geq 1\}$, then the conditions hold. ■

Remark 2.5. In Corollaries 2.1 and 2.2, it is assumed that the rate of convergence τ_n is known. This assumption may be relaxed using techniques described in Politis et al. (1999). ■

We conclude this section with a result that establishes a converse for Theorems 2.1 and 2.2.

Theorem 2.3. *Let $b = b_n < n$ be a sequence of positive integers tending to infinity, but satisfying $b/n \to 0$ and define $L_n(x, P)$ as in (4) and $\hat{L}_n(x)$ as in (5). Then, the following statements are true:*

(i) If $\limsup_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}}\{J_b(x, P) - J_n(x, P)\} > 0$, then (6) fails for $\alpha_1 = 0$ and some $0 \le \alpha_2 < 1$.

(ii) If $\limsup_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}}\{J_n(x, P) - J_b(x, P)\} > 0$, then (6) fails for $\alpha_2 = 0$ and some $0 \le \alpha_1 < 1$.

(iii) If $\liminf_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}} |J_b(x, P) - J_n(x, P)| > 0$, then (6) fails for some $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$ satisfying $0 \le \alpha_1 + \alpha_2 < 1$.

If, in addition, (7) holds for any $\epsilon > 0$, then statements (i) - (iii) above hold when $L_n^{-1}(\cdot, P)$ is replaced by $\hat{L}_n^{-1}(\cdot)$.

2.2 Bootstrap

As before, let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$. Denote by $J_n(x, P)$ the distribution of a real-valued root $R_n = R_n(X^{(n)}, P)$ under P. The goal remains to construct procedures which are valid uniformly in P. The bootstrap approach is to approximate $J_n(\cdot, P)$ by $J_n(\cdot, \hat{P}_n)$ for some estimator $\hat{P}_n$ of P. Typically, $\hat{P}_n$ is the empirical distribution, but this is not assumed in Theorem 2.4 below. Because $\hat{P}_n$ need not *a priori* even lie in $\mathbf{P}$, it is necessary to introduce a family $\mathbf{P}'$ in which $\hat{P}_n$ lies (at least with high probability). In order for the bootstrap to succeed, we will require that $\rho(\hat{P}_n, P)$ be small for some measure (perhaps a metric) $\rho(\cdot, \cdot)$ defined on $\mathbf{P}' \times \mathbf{P}$. For any given problem in which the theorem is applied, $\mathbf{P}$, $\mathbf{P}'$ and ρ must be specified.

Theorem 2.4. *Let $\rho(\cdot, \cdot)$ be a function on $\mathbf{P}' \times \mathbf{P}$ and let $\hat{P}_n$ be a (random) sequence of distributions. Then, the following are true:*

(i) Suppose $\limsup_{n\to\infty} \sup_{x\in\mathbf{R}}\{J_n(x, Q_n) - J_n(x, P_n)\} \le 0$ for any sequences $\{Q_n \in \mathbf{P}' : n \ge 1\}$ and $\{P_n \in \mathbf{P} : n \ge 1\}$ satisfying $\rho(Q_n, P_n) \to 0$. If

$$\rho(\hat{P}_n, P_n) \overset{P_n}{\to} 0 \text{ and } P_n\{\hat{P}_n \in \mathbf{P}'\} \to 1 \tag{8}$$

for any sequence $\{P_n \in \mathbf{P} : n \geq 1\}$*, then*

$$\liminf_{n\to\infty} \inf_{P\in\mathbf{P}} P\{J_n^{-1}(\alpha_1, \hat{P}_n) \leq R_n \leq J_n^{-1}(1-\alpha_2, \hat{P}_n)\} \geq 1 - \alpha_1 - \alpha_2 \tag{9}$$

holds for $\alpha_1 = 0$ *and any* $0 \leq \alpha_2 < 1$*.*

(ii) Suppose $\limsup_{n\to\infty} \sup_{x\in\mathbf{R}}\{J_n(x, P_n) - J_n(x, Q_n)\} \leq 0$ *for any sequences* $\{Q_n \in \mathbf{P}' : n \geq 1\}$ *and* $\{P_n \in \mathbf{P} : n \geq 1\}$ *satisfying* $\rho(Q_n, P_n) \to 0$*. If* (8) *holds for any sequence* $\{P_n \in \mathbf{P} : n \geq 1\}$*, then* (9) *holds for* $\alpha_2 = 0$ *and any* $0 \leq \alpha_1 < 1$*.*

(iii) Suppose $\lim_{n\to\infty} \sup_{x\in\mathbf{R}} |J_n(x, Q_n) - J_n(x, P_n)| = 0$ *for any sequences* $\{Q_n \in \mathbf{P}' : n \geq 1\}$ *and* $\{P_n \in \mathbf{P} : n \geq 1\}$ *satisfying* $\rho(Q_n, P_n) \to 0$*. If* (8) *holds for any sequence* $\{P_n \in \mathbf{P} : n \geq 1\}$*, then* (9) *holds for any* $\alpha_1 \geq 0$ *and* $\alpha_2 \geq 0$ *satisfying* $0 \leq \alpha_1 + \alpha_2 < 1$*.*

Remark 2.6. It is typically easy to deduce from the conclusions of Theorem 2.4 stronger results in which the $\liminf_{n\to\infty}$ and $\geq$ in (9) are replaced by $\lim_{n\to\infty}$ and $=$, respectively. For example, in order to assert that (9) holds with $\liminf_{n\to\infty}$ and $\geq$ replaced by $\lim_{n\to\infty}$ and $=$, respectively, all that is required is that

$$\lim_{n\to\infty} P\{J_n^{-1}(\alpha_1, \hat{P}_n) \leq R_n \leq J_n^{-1}(1-\alpha_2, \hat{P}_n)\} = 1 - \alpha_1 - \alpha_2$$

for some $P \in \mathbf{P}$. This can be verified using the usual arguments for the pointwise asymptotic validity of the bootstrap. See Politis et al. (1999) for details. ■

Remark 2.7. In some cases, it is possible to construct estimators $\hat{J}_n(x)$ of $J_n(x, P)$ that are uniformly consistent over a large class of distributions $\mathbf{P}$ in the sense that for any $\epsilon > 0$

$$\sup_{P\in\mathbf{P}} P\{\rho(\hat{J}_n(\cdot), J_n(\cdot, P)) > \epsilon\} \to 0 \ , \tag{10}$$

where ρ is the Levy metric or some other metric compatible with the weak topology. Yet a result such as (10) is not strong enough to yield uniform coverage statements such as those in Theorem 2.1 and Theorem 2.4. In other words, such conclusions do not follow from uniform approximations of the distribution of interest if the quality of the approximation is measured in terms of metrics metrizing weak convergence. To see this, consider the following simple example.

Example 2.1. Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P_\theta = \text{Bernoulli}(\theta)$. Denote by $J_n(x, P_\theta)$ the distribution of root $R_n = \sqrt{n}(\hat{\theta}_n - \theta)$ under P_θ, where $\hat{\theta}_n = \bar{X}_n$. Let $\hat{P}_n$ be the empirical distribution of $X^{(n)}$ or, equivalently, $P_{\hat{\theta}_n}$. Lemma S.1.1 in Romano and Shaikh (2012) implies for any $\epsilon > 0$ that

$$\sup_{0\leq\theta\leq 1} P_\theta\{\rho(J_n(\cdot, \hat{P}_n), J_n(\cdot, P_\theta)) > \epsilon\} \to 0 \tag{11}$$

whenever ρ is a metric compatible with the weak topology. Nevertheless, it follows from the argument on p. 78 of Romano (1989) that the coverage statements in Theorem 2.4 fail to hold provided that both α_1 and α_2 do not equal zero. Indeed, consider part (i) of Theorem 2.4. Suppose $\alpha_1 = 0$ and $0 < \alpha_2 < 1$. For a given n and $\delta > 0$, let $\theta_n = (1-\delta)^{\frac{1}{n}}$. Under P_{θ_n}, the event $X_1 = \cdots = X_n = 1$ has probability $1-\delta$. Moreover, whenever such an event occurs, $R_n > J_n^{-1}(1-\alpha_2, \hat{P}_n) = 0$. Therefore, $P_{\theta_n}\{J_n^{-1}(\alpha_1, \hat{P}_n) \leq R_n \leq J_n^{-1}(1-\alpha_2, \hat{P}_n)\} \leq \delta$. Since the choice of δ was arbitrary, it follows that

$$\liminf_{n\to\infty} \inf_{0\leq\theta\leq 1} P_\theta\{J_n^{-1}(\alpha_1, \hat{P}_n) \leq R_n \leq J_n^{-1}(1-\alpha_2, \hat{P}_n)\} = 0 \ .$$

A similar argument establishes the result for parts (ii) and (iii) of Theorem 2.4.

On the other hand, when ρ is the Kolmogorov metric, (11) holds when the supremum over $0 \leq \theta \leq 1$ is replaced with a supremum over $\delta < \theta < 1-\delta$ for some $\delta > 0$. Moreover, when θ is restricted to such an interval, the coverage statements in Theorem 2.4 hold as well. ■

3 Applications

Before proceeding, it is useful to introduce some notation that will be used frequently throughout many of the examples below. For a distribution P on $\mathbf{R}^k$, denote by $\mu(P)$ the mean of P, by $\Sigma(P)$ the covariance matrix of P, and by $\Omega(P)$ the correlation matrix of P. For $1 \leq j \leq k$, denote by $\mu_j(P)$ the jth component of $\mu(P)$ and by $\sigma_j^2(P)$ the jth diagonal element of $\Sigma(P)$. In all of our examples, $X^{(n)} = (X_1, \ldots, X_n)$ will be an i.i.d. sequence of random variables with distribution P and $\hat{P}_n$ will denote the empirical distribution of $X^{(n)}$. As usual, we will denote by $\bar{X}_n = \mu(\hat{P}_n)$ the usual sample mean, by $\hat{\Sigma}_n = \Sigma(\hat{P}_n)$ the usual sample covariance matrix, and by $\hat{\Omega}_n = \Omega(\hat{P}_n)$ the usual sample correlation matrix. For $1 \leq j \leq k$, denote by $\bar{X}_{j,n}$ the jth component of $\bar{X}_n$ and by $S_{j,n}^2$ the jth diagonal element of $\hat{\Sigma}_n$. Finally, we say that a family of distributions $\mathbf{Q}$ on the real line satisfies the standardized uniform integrability condition if

$$\lim_{\lambda\to\infty} \sup_{Q\in\mathbf{Q}} E_Q\left[\left(\frac{Y-\mu(Q)}{\sigma(Q)}\right)^2 I\left\{\left|\frac{Y-\mu(Q)}{\sigma(Q)}\right| > \lambda\right\}\right] = 0 \ . \tag{12}$$

In the preceding expression, Y denotes a random variable with distribution Q. The use of the term standardized to describe (12) reflects that fact that the variable Y is centered around its mean and normalized by its standard deviation.

3.1 Subsampling

Example 3.1. *(Multivariate Nonparametric Mean)* Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}^k$. Suppose one wishes to construct a rectangular confidence region for $\mu(P)$. For this purpose, a natural choice of root is

$$R_n(X^{(n)}, P) = \max_{1 \le j \le k} \frac{\sqrt{n}(\bar{X}_{j,n} - \mu_j(P))}{S_{j,n}} \tag{13}$$

In this setup, we have the following theorem:

Theorem 3.1. *Denote by $\mathbf{P}_j$ the set of distributions formed from the jth marginal distributions of the distributions in $\mathbf{P}$. Suppose $\mathbf{P}$ is such that (12) is satisfied with $\mathbf{Q} = \mathbf{P}_j$ for all $1 \le j \le k$. Let $J_n(x, P)$ be the distribution of the root (13). Let $b = b_n < n$ be a sequence of positive integers tending to infinity, but satisfying $b/n \to 0$ and define $L_n(x, P)$ by (4). Then,*

$$\lim_{n\to\infty} \inf_{P \in \mathbf{P}} P\left\{ L_n^{-1}(\alpha_1, P) \le \max_{1 \le j \le k} \frac{\sqrt{n}(\bar{X}_{j,n} - \mu_j(P))}{S_{j,n}} \le L_n^{-1}(1 - \alpha_2, P) \right\} = 1 - \alpha_1 - \alpha_2 \tag{14}$$

for any $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$ such that $0 \le \alpha_1 + \alpha_2 < 1$. Furthermore, (14) remains true if $L_n^{-1}(\cdot, P)$ is replaced by $\hat{L}_n^{-1}(\cdot)$, where $\hat{L}_n(x)$ is defined by (5) with $\hat{R}_b(X^{n,(b),i}) = R_b(X^{n,(b),i}, \hat{P}_n)$.

Under suitable restrictions, Theorem 3.1 generalizes to the case where the root is given by

$$R_n(X^{(n)}, P) = f(Z_n(P), \hat{\Omega}_n) \ , \tag{15}$$

where f is a continuous, real-valued function and

$$Z_n(P) = \left(\frac{\sqrt{n}(\bar{X}_{1,n} - \mu_1(P))}{S_{1,n}}, \ldots, \frac{\sqrt{n}(\bar{X}_{k,n} - \mu_k(P))}{S_{k,n}} \right)' \ . \tag{16}$$

In particular, we have the following theorem:

Theorem 3.2. *Let $\mathbf{P}$ be defined as in Theorem 3.1. Let $J_n(x, P)$ be the distribution of the root (15), where f is continuous.*

(i) Suppose further that for all $x \in \mathbf{R}$ that

$$P_n\{f(Z_n(P_n), \Omega(\hat{P}_n)) \le x\} \ \to \ P\{f(Z, \Omega) \le x\} \tag{17}$$

$$P_n\{f(Z_n(P_n), \Omega(\hat{P}_n)) < x\} \ \to \ P\{f(Z, \Omega) < x\} \tag{18}$$

for any sequence $\{P_n \in \mathbf{P} : n \ge 1\}$ such that $Z_n(P_n) \xrightarrow{d} Z$ under P_n and $\Omega(\hat{P}_n) \xrightarrow{P_n} \Omega$, where $Z \sim N(0, \Omega)$. Then,

$$\liminf_{n\to\infty} \inf_{P \in \mathbf{P}} P\{L_n^{-1}(\alpha_1, P) \le f(Z_n(P), \hat{\Omega}_n) \le L_n^{-1}(1 - \alpha_2, P)\} \ge 1 - \alpha_1 - \alpha_2 \tag{19}$$

for any $\alpha_1 \ge 0$ and $\alpha_2 \ge 0$ such that $0 \le \alpha_1 + \alpha_2 < 1$.

(ii) Suppose further that if $Z \sim N(0,\Omega)$ for some Ω satisfying $\Omega_{j,j} = 1$ for all $1 \leq j \leq k$, then $f(Z,\Omega)$ is continuously distributed. Then, (19) *remains true if $L_n^{-1}(\cdot, P)$ is replaced by $\hat{L}_n^{-1}(\cdot)$, where $\hat{L}_n(x)$ is defined by* (5) *with $\hat{R}_b(X^{n,(b),i}) = R_b(X^{n,(b),i}, \hat{P}_n)$. Moreover, the $\liminf_{n\to\infty}$ and $\geq$ may be replaced by $\lim_{n\to\infty}$ and $=$, respectively.*

In order to verify (17) and (18) in Theorem 3.2, it suffices to assume that $f(Z,\Omega)$ is continuously distributed. Under the assumptions of the theorem, however, $f(Z,\Omega)$ need not be continuously distributed. In this case, (17) and (18) hold immediately for any x at which $P\{(Z,\Omega) \leq x\}$ is continuous, but require a further argument for x at which $P\{(Z,\Omega) \leq x\}$ is discontinuous. See, for example, the proof of Theorem 3.9, which relies on Theorem 3.8, where the same requirement appears. ■

Example 3.2. *(Constrained Univariate Nonparametric Mean)* Andrews (2000) considers the following example. Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}$. Suppose it is known that $\mu(P) \geq 0$ for all $P \in \mathbf{P}$ and one wishes to construct a confidence interval for $\mu(P)$. A natural choice of root in this case is

$$R_n = R_n(X^{(n)}, P) = \sqrt{n}(\max\{\bar{X}_n, 0\} - \mu(P)) \ .$$

This root differs from the one considered in Theorem 3.1 and the ones discussed in Theorem 3.2 in the sense that under weak assumptions on $\mathbf{P}$

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}} \{J_b(x,P) - J_n(x,P)\} \leq 0 \tag{20}$$

holds, but

$$\limsup_{n\to\infty} \sup_{P\in\mathbf{P}} \sup_{x\in\mathbf{R}} \{J_n(x,P) - J_b(x,P)\} \leq 0 \tag{21}$$

fails to hold. To see this, suppose (12) holds with $\mathbf{Q} = \mathbf{P}$. Note that

$$\begin{aligned} J_b(x,P) &= P\{\max\{Z_b(P), -\sqrt{b}\mu(P)\} \leq x\} \\ J_n(x,P) &= P\{\max\{Z_n(P), -\sqrt{n}\mu(P)\} \leq x\} \ , \end{aligned}$$

where $Z_b(P) = \sqrt{b}(\bar{X}_b - \mu(P))$ and $Z_n(P) = \sqrt{n}(\bar{X}_n - \mu(P))$. Since $\sqrt{b}\mu(P) \leq \sqrt{n}\mu(P)$ for any $P \in \mathbf{P}$, $J_b(x,P) - J_n(x,P)$ is bounded from above by

$$P\{\max\{Z_b(P), -\sqrt{n}\mu(P)\} \leq x\} - J_n(x,P) \ .$$

It now follows from the uniform central limit theorem established by Lemma 3.3.1 of Romano and Shaikh (2008) and Theorem 2.11 of Bhattacharya and Rao (1976) that (20) holds. It therefore follows from

Theorem 2.1 that (6) holds with $\alpha_1 = 0$ and any $0 \leq \alpha_2 < 1$. To see that (21) fails, suppose further that $\{Q_n : n \geq 1\} \subseteq \mathbf{P}$, where $Q_n = N(h/\sqrt{n}, 1)$ for some $h > 0$. For $Z \sim N(0,1)$,

$$\begin{aligned} J_n(x, Q_n) &= P\{\max(Z, -h) \leq x\} \\ J_b(x, Q_n) &= P\{\max(Z, -h\sqrt{b}/\sqrt{n}) \leq x\} \ . \end{aligned}$$

The lefthand-side of (21) is therefore greater than or equal to

$$\limsup_{n\to\infty} \Big(P\{\max(Z, -h) \leq x\} - P\{\max(Z, -h\sqrt{b}/\sqrt{n}) \leq x\} \Big) \ ,$$

for any x. In particular, if $-h < x < 0$, then the second term is zero for large enough n, and so the limiting value is $P\{Z \leq x\} = \Phi(x) > 0$. It therefore follows from Theorem 2.3 that (6) fails for $\alpha_2 = 0$ and some $0 \leq \alpha_1 < 1$. On the other hand, (6) holds with $\alpha_2 = 0$ and any $.5 < \alpha_1 < 1$. To see this, consider any sequence $\{P_n \in \mathbf{P} : n \geq 1\}$ and the event $\{L_n^{-1}(\alpha_1, P_n) \leq R_n\}$. For the root in this example, this event is scale invariant. So, in calculating the probability of this event, we may without loss of generality assume $\sigma^2(P_n) = 1$. Since $\mu(P_n) \geq 0$, we have for any $x \geq 0$ that

$$J_n(x, P_n) = P\{\max\{Z_n(P_n), -\sqrt{n}\mu(P_n)\} \leq x\} = P\{Z_n(P_n) \leq x\} \to \Phi(x) \ ,$$

and similarly for $J_b(x, P_n)$. Using the usual subsampling arguments, it is thus possible to show for $.5 < \alpha_1 < 1$ that

$$L_n^{-1}(\alpha_1, P_n) \stackrel{P_n}{\to} \Phi^{-1}(\alpha_1) \ .$$

The desired conclusion therefore follows from Slutsky's Theorem. Arguing as the the proof of Corollary 2.2 and Remark 2.4, it can be shown that the same results hold when $L_n^{-1}(\cdot, P)$ is replaced by $\hat{L}_n^{-1}(\cdot)$, where $\hat{L}_n(x)$ is defined as $L_n(x, P)$ is defined but with $\mu(P)$ replaced by $\bar{X}_n$. ■

Example 3.3. *(Moment Inequalities)* The generality of Theorem 2.1 illustrated in Example 3.2 is also useful when testing multisided hypotheses about the mean. To see this, let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}^k$. Define $\mathbf{P}_0 = \{P \in \mathbf{P} : \mu(P) \leq 0\}$ and $\mathbf{P}_1 = \mathbf{P} \backslash \mathbf{P}_0$. Consider testing the null hypothesis that $P \in \mathbf{P}_0$ versus the alternative hypothesis that $P \in \mathbf{P}_1$ at level $\alpha \in (0,1)$. Such hypothesis testing problems have recently received considerable attention in the "moment inequality" literature in econometrics. See, for example, Andrews and Soares (2010), Andrews and Guggenberger (2010), Andrews and Barwick (2012), Bugni (2010), Canay (2010), Romano and Shaikh (2008), and Romano and Shaikh (2010). Theorem 2.1 may be used to construct tests that are uniformly consistent in level in the sense that (3) holds under weak assumptions on $\mathbf{P}$. Formally, we have the following theorem:

Theorem 3.3. *Let* $\mathbf{P}$ *be defined as in Theorem 3.1. Let* $J_n(x, P)$ *be the distribution of*

$$T_n(X^{(n)}) = \max_{1 \le j \le k} \frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}} \ .$$

Let $b = b_n < n$ *be a sequence of positive integers tending to infinity, but satisfying* $b/n \to 0$ *and define* $L_n(x)$ *by the righthand-side of* (4) *with* $R_n(X^{(n)}, P) = T_n(X^{(n)})$. *Then, the test defined by*

$$\phi_n(X^{(n)}) = I\{T_n(X^{(n)}) > L_n^{-1}(1-\alpha)\}$$

satisfies (3) *for any* $0 < \alpha < 1$.

The argument used to establish Theorem 3.3 is essentially the same as the one presented in Romano and Shaikh (2008) for

$$T_n(X^{(n)}) = \sum_{1 \le j \le k} \max\{\sqrt{n}\bar{X}_{j,n}, 0\}^2 \ ,$$

though Lemma S.6.1 in Romano and Shaikh (2012) is needed for establishing (20) here because of Studentization. Related results are obtained by Andrews and Guggenberger (2009). ■

Example 3.4. *(Multiple Testing)* We now illustrate the use of Theorem 2.1 to construct tests of multiple hypotheses that behave well uniformly over a large class of distributions. Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}^k$ and consider testing the family of null hypotheses

$$H_j : \mu_j(P) \le 0 \text{ for } 1 \le j \le k \tag{22}$$

versus the alternative hypotheses

$$H_j' : \mu_j(P) > 0 \text{ for } 1 \le j \le k \ . \tag{23}$$

in a way that controls the familywise error rate at level $0 < \alpha < 1$ in the sense that

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}} FWER_P \le \alpha \ , \tag{24}$$

where

$$FWER_P = P\{\text{reject some } H_j \text{ with } \mu_j(P) \le 0\} \ .$$

For $K \subseteq \{1, \ldots, k\}$, define $L_n(x, K)$ according to the righthand-side of (4) with

$$R_n(X^{(n)}, P) = \max_{j \in K} \frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}}$$

and consider the following stepwise multiple testing procedure:

Algorithm 3.1.

Step 1: Set $K_1 = \{1, \ldots, k\}$. If

$$\max_{j \in K_1} \frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}} \leq L_n^{-1}(1-\alpha, K_1) \ ,$$

then stop. Otherwise, reject any H_j with

$$\frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}} > L_n^{-1}(1-\alpha, K_1)$$

and continue to Step 2 with

$$K_2 = \left\{ j \in K_1 : \frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}} \leq L_n^{-1}(1-\alpha, K_1) \right\} \ .$$

$\vdots$

Step s: If

$$\max_{j \in K_s} \frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}} \leq L_n^{-1}(1-\alpha, K_s) \ ,$$

then stop. Otherwise, reject any H_j with

$$\frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}} > L_n^{-1}(1-\alpha, K_s)$$

and continue to Step $s+1$ with

$$K_{s+1} = \left\{ j \in K_s : \frac{\sqrt{n}\bar{X}_{j,n}}{S_{j,n}} \leq L_n^{-1}(1-\alpha, K_s) \right\} \ .$$

$\vdots$

We have the following theorem:

Theorem 3.4. *Let* $\mathbf{P}$ *be defined as in Theorem 3.1. Let* $b = b_n < n$ *be a sequence of positive integers tending to infinity, but satisfying* $b/n \to 0$*. Then, Algorithm 3.1 satisfies*

$$\limsup_{n \to \infty} \sup_{P \in \mathbf{P}} FWER_P \leq \alpha \tag{25}$$

for any $0 < \alpha < 1$*.*

It is, of course, possible to extend the analysis in a straightforward way to two-sided testing. See also Romano and Shaikh (2010) for related results about a multiple testing problem involving an infinite number of null hypotheses. ■

Example 3.5. *(Empirical Process on* $\mathbf{R}$*)* Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}$. Suppose one wishes to construct a confidence region for the cumulative distribution function associated with P, i.e., $P\{(-\infty, t]\}$. For this purpose a natural choice of root is

$$\sup_{t \in \mathbf{R}} \sqrt{n} |\hat{P}_n\{(-\infty, t]\} - P\{(-\infty, t]\}| \ . \tag{26}$$

In this setting, we have the following theorem:

Theorem 3.5. *Fix any* $\epsilon \in (0, 1)$ *and let*

$$\mathbf{P} = \{P \ on \ \mathbf{R} : \epsilon < P\{(-\infty, t]\} < 1 - \epsilon \ for \ some \ t \in \mathbf{R}\} \ . \tag{27}$$

Let $J_n(x, P)$ *be the distribution of the root* (26)*. Then*

$$\lim_{n \to \infty} \inf_{P \in \mathbf{P}} P \left\{ L_n^{-1}(\alpha_1, P) \leq \sup_{t \in \mathbf{R}} \sqrt{n} |\hat{P}_n\{(-\infty, t]\} - P\{(-\infty, t]\}| \leq L_n^{-1}(1 - \alpha_2, P) \right\} = 1 - \alpha_1 - \alpha_2 \tag{28}$$

for any $\alpha_1 \geq 0$ *and* $\alpha_2 \geq 0$ *such that* $0 \leq \alpha_1 + \alpha_2 < 1$*. Furthermore,* (28) *remains true if* $L_n^{-1}(\cdot, P)$ *is replaced by* $\hat{L}_n^{-1}(\cdot)$*, where* $\hat{L}_n(x)$ *is defined by* (5) *with* $\hat{R}_b(X^{n,(b),i}) = R_b(X^{n,(b),i}, \hat{P}_n)$*.* ■

Example 3.6. *(One Sample* U*-Statistics)* Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}$. Suppose one wishes to construct a confidence region for

$$\theta(P) = \theta_h(P) = E_P[h(X_1, \ldots, X_m)] \ , \tag{29}$$

where h is a symmetric kernel of degree m. The usual estimator of $\theta(P)$ in this case is given by the U-statistic

$$\hat{\theta}_n = \hat{\theta}_n(X^{(n)}) = \frac{1}{\binom{n}{m}} \sum_c h(X_{i_1}, \ldots, X_{i_m}) \ .$$

Here, $\sum_c$ denotes summation over all $\binom{n}{m}$ subsets $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$. A natural choice of root is therefore given by

$$R_n(X^{(n)}, P) = \sqrt{n}(\hat{\theta}_n - \theta(P)) \ . \tag{30}$$

In this setting, we have the following theorem:

Theorem 3.6. *Let*

$$g(x, P) = g_h(x, P) = E_P[h(x, X_2, \ldots, X_m)] - \theta(P) \ , \tag{31}$$

and

$$\sigma_h^2(P) = m^2 \, Var_P[g(X_i, P)] \ . \tag{32}$$

Suppose $\mathbf{P}$ *satisfies the uniform integrability condition*

$$\lim_{\lambda\to\infty}\sup_{P\in\mathbf{P}} E_P\left[\frac{g^2(X_i,P)}{\sigma_h^2(P)} I\left\{\left|\frac{g(X_i,P)}{\sigma_h(P)}\right| > \lambda\right\}\right] = 0 \tag{33}$$

and

$$\sup_{P\in\mathbf{P}} \frac{Var_P[h(X_1,\ldots,X_m)]}{\sigma^2(P)} < \infty\ . \tag{34}$$

Let $J_n(x,P)$ *be the distribution of the root* (30)*. Let* $b = b_n < n$ *be a sequence of positive integers tending to infinity, but satisfying* $b/n \to 0$ *and define* $L_n(x,P)$ *by* (4)*. Then,*

$$\lim_{n\to\infty}\inf_{P\in\mathbf{P}} P\left\{L_n^{-1}(\alpha_1,P) \le \sqrt{n}(\hat{\theta}_n - \theta(P)) \le L_n^{-1}(1-\alpha_2,P)\right\} = 1-\alpha_1-\alpha_2 \tag{35}$$

for any $\alpha_1 \ge 0$ *and* $\alpha_2 \ge 0$ *such that* $0 \le \alpha_1+\alpha_2 < 1$*. Furthermore,* (35) *remains true if* $L_n^{-1}(\cdot,P)$ *is replaced by* $\hat{L}_n^{-1}(\cdot)$*, where* $\hat{L}_n(x)$ *is defined by* (5) *with* $\hat{R}_b(X^{n,(b),i}) = R_b(X^{n,(b),i},\hat{P}_n)$*.* ■

3.2 Bootstrap

Example 3.7. *(Multivariate Nonparametric Mean)* Let $X^{(n)} = (X_1,\ldots,X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}^k$. Suppose one wishes to construct a rectangular confidence region for $\mu(P)$. As described in Example 3.1, a natural choice of root in this case is given by (13). In this setting, we have the following theorem, which is a bootstrap counterpart to Theorem 3.1:

Theorem 3.7. *Let* $\mathbf{P}$ *be defined as in Theorem 3.1. Let* $J_n(x,P)$ *be the distribution of the root* (13)*. Then,*

$$\lim_{n\to\infty}\inf_{P\in\mathbf{P}} P\left\{J_n^{-1}(\alpha_1,\hat{P}_n) \le \max_{1\le j\le k}\frac{\sqrt{n}(\bar{X}_{j,n}-\mu_j(P))}{S_{j,n}} \le J_n^{-1}(1-\alpha_2,\hat{P}_n)\right\} = 1-\alpha_1-\alpha_2 \tag{36}$$

for any $\alpha_1 \ge 0$ *and* $\alpha_2 \ge 0$ *such that* $0 \le \alpha_1+\alpha_2 < 1$*.*

Theorem 3.7 generalizes in the same way that Theorem 3.1 generalizes. In particular, we have the following result:

Theorem 3.8. *Let* $\mathbf{P}$ *be defined as in Theorem 3.1. Let* $J_n(x,P)$ *be the distribution of the root* (15)*. Suppose* f *is continuous. Suppose further that for all* $x \in \mathbf{R}$

$$P_n\{f(Z_n(P_n),\Omega(\hat{P}_n)) \le x\} \quad\to\quad P\{f(Z,\Omega) \le x\} \tag{37}$$

$$P_n\{f(Z_n(P_n),\Omega(\hat{P}_n)) < x\} \quad\to\quad P\{f(Z,\Omega) < x\} \tag{38}$$

for any sequence $\{P_n \in \mathbf{P} : n \geq 1\}$ *such that* $Z_n(P_n) \xrightarrow{d} Z$ *under* P_n *and* $\Omega(\hat{P}_n) \xrightarrow{P_n} \Omega$*, where* $Z \sim N(0, \Omega)$*. Then,*

$$\liminf_{n\to\infty} \inf_{P\in\mathbf{P}} P\{J_n^{-1}(\alpha_1, \hat{P}_n) \leq f(Z_n(P), \hat{\Omega}_n) \leq J_n^{-1}(1-\alpha_2, \hat{P}_n)\} \geq 1 - \alpha_1 - \alpha_2 \tag{39}$$

for any $\alpha_1 \geq 0$ *and* $\alpha_2 \geq 0$ *such that* $0 \leq \alpha_1 + \alpha_2 < 1$*.* ■

Example 3.8. *(Moment Inequalities)* Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}^k$ and define $\mathbf{P}_0$ and $\mathbf{P}_1$ as in Example 3.3. Andrews and Barwick (2012) propose testing the null hypothesis that $P \in \mathbf{P}_0$ versus the alternative hypothesis that $P \in \mathbf{P}_1$ at level $\alpha \in (0, 1)$ using an "adjusted quasi-likelihood ratio" statistic $T_n(X^{(n)})$ defined as follows:

$$T_n(X^{(n)}) = \inf_{t\in\mathbf{R}^k : t\leq 0} W_n(t)'\tilde{\Omega}_n^{-1} W_n(t) \ .$$

Here, $t \leq 0$ is understood to mean that the inequality holds component-wise,

$$W_n(t) = \left(\frac{\sqrt{n}(\bar{X}_{1,n} - t_1)}{S_{1,n}}, \ldots, \frac{\sqrt{n}(\bar{X}_{k,n} - t_k)}{S_{k,n}} \right)' \ ,$$

and

$$\tilde{\Omega}_n = \max\{\epsilon - \det(\hat{\Omega}_n), 0\} I_k + \hat{\Omega}_n \ , \tag{40}$$

where $\epsilon > 0$ and I_k is the k-dimensional identity matrix. Andrews and Barwick (2012) propose a procedure for constructing critical values for $T_n(X^{(n)})$ that they term "refined moment selection." For illustrative purposes, we instead consider in the following theorem a simpler construction.

Theorem 3.9. *Let* $\mathbf{P}$ *be defined as in Theorem 3.1. Let* $J_n(x, P)$ *be the distribution of the root*

$$R_n(X^{(n)}, P) = \inf_{t\in\mathbf{R}^k : t\leq 0} (Z_n(P) - t)'\tilde{\Omega}_n^{-1}(Z_n(P) - t) \ , \tag{41}$$

where $Z_n(P)$ *is defined as in* (16)*. Then, the test defined by*

$$\phi_n(X^{(n)}) = I\{T_n(X^{(n)}) > J_n^{-1}(1-\alpha, \hat{P}_n)\}$$

satisfies (3) *for any* $0 < \alpha < 1$*.*

Theorem 3.9 generalizes in a straightforward fashion to other choices of test statistics, including the one used in Theorem 3.3. On the other hand, even when the underlying choice of test statistic is the same, the first-order asymptotic properties of the tests in Theorem 3.9 and Theorem 3.3 will differ. For other ways of constructing critical values that are more similar to the construction given in Andrews and Barwick (2012), see Romano et al. (2012). ■

Example 3.9. *(Multiple Testing)* Theorem 2.4 may be used in the same way that Theorem 2.1 was used in Example 3.4 to construct tests of multiple hypotheses that behave well uniformly over a large class of distributions. To see this, let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}^k$ and again consider testing the family of null hypotheses (22) versus the alternative hypotheses (23) in a way that satisfies (24) for $\alpha \in (0,1)$. For $K \subseteq \{1, \ldots, k\}$, let $J_n(x, K, P)$ be the distribution of the root

$$R_n(X^{(n)}, P) = \max_{j \in K} \frac{\sqrt{n}(\bar{X}_{j,n} - \mu_j(P))}{S_{j,n}}$$

under P and consider the stepwise multiple testing procedure given by Algorithm 3.1 with $L_n^{-1}(1-\alpha, K_j)$ replaced by $J_n^{-1}(1-\alpha, K_j, \hat{P}_n)$. We have the following theorem, which is a bootstrap counterpart to Theorem 3.4:

Theorem 3.10. *Let $\mathbf{P}$ be defined as in Theorem 3.1. Then, Algorithm 3.1 with $L_n^{-1}(1-\alpha, K_j)$ replaced by $J_n^{-1}(1-\alpha, K_j, \hat{P}_n)$ satisfies* (25) *for any $0 < \alpha < 1$.*

It is, of course, possible to extend the analysis in a straightforward way to two-sided testing. ■

Example 3.10. *(Empirical Process on $\mathbf{R}$)* Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}$. Suppose one wishes to construct a confidence region for the cumulative distribution function associated with P, i.e., $P\{(-\infty, t]\}$. As described in Example 3.5, a natural choice of root in this case is given by (26). In this setting, we have the following theorem, which is a bootstrap counterpart to Theorem 3.5:

Theorem 3.11. *Fix any $\epsilon \in (0,1)$ and let $\mathbf{P}$ be defined as in Theorem 3.5. Let $J_n(x, P)$ be the distribution of the root* (26)*. Denote by $\hat{P}_n$ the empirical distribution of $X^{(n)}$. Then,*

$$\lim_{n \to \infty} \inf_{P \in \mathbf{P}} P\left\{ J_n^{-1}(\alpha_1, \hat{P}_n) \leq \sup_{t \in \mathbf{R}} \sqrt{n} |\hat{P}_n\{(-\infty, t]\} - P\{(-\infty, t]\}| \leq J_n^{-1}(1-\alpha_2, \hat{P}_n) \right\} = 1 - \alpha_1 - \alpha_2$$

for any $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ such that $0 \leq \alpha_1 + \alpha_2 < 1$.

Some of the conclusions of Theorem 3.11 can be found in Romano (1989), though the method of proof given in Romano and Shaikh (2012) is quite different. ■

Example 3.11. *(One Sample U-Statistics)* Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$ on $\mathbf{R}$ and let h be a symmetric kernel of degree m. Suppose one wishes to construct a confidence region for $\theta(P) = \theta_h(P)$ given by (29). As described in Example 3.6, a natural choice of root in this case is given by (30). Before proceeding, it is useful

to introduce the following notation. For an arbitrary kernel $\tilde{h}$, $\epsilon > 0$ and $B > 0$, denote by $\mathbf{P}_{\tilde{h},\epsilon,B}$ the set of all distributions P on $\mathbf{R}$ such that

$$E_P[|\tilde{h}(X_1, \ldots, X_m) - \theta_{\tilde{h}}(P)|^\epsilon] \leq B \ . \tag{42}$$

Similarly, for an arbitrary kernel $\tilde{h}$ and $\delta > 0$, denote by $\mathbf{S}_{\tilde{h},\delta}$ the set of all distributions P on $\mathbf{R}$ such that

$$\sigma^2_{\tilde{h}}(P) \geq \delta \ , \tag{43}$$

where $\sigma^2_{\tilde{h}}(P)$ is defined as in (32). Finally, for an arbitrary kernel $\tilde{h}$, $\epsilon > 0$ and $B > 0$, let $\bar{\mathbf{P}}_{\tilde{h},\epsilon,B}$ be the set of distributions P on $\mathbf{R}$ such that

$$E_P[|\tilde{h}(X_{i_1}, \ldots, X_{i_m})] - \theta_{\tilde{h}}(P)|^\epsilon] \leq B$$

whenever $1 \leq i_j \leq n$ for all $1 \leq j \leq m$. Using this notation, we have the following theorem:

Theorem 3.12. *Define the kernel h' of degree $2m$ according to the rule*

$$h'(x_1, \ldots, x_{2m}) = h(x_1, \ldots, x_m)h(x_1, x_{m+2}, \ldots, x_{2m}) - h(x_1, \ldots, x_m)h(x_{m+1}, \ldots, x_{2m}) \ . \tag{44}$$

Suppose

$$\mathbf{P} \subseteq \mathbf{P}_{h,2+\delta,B} \cap \mathbf{S}_{h,\delta} \cap \bar{\mathbf{P}}_{h',1+\delta,B} \cap \bar{\mathbf{P}}_{h,2+\delta,B}$$

for some $\delta > 0$ and $B > 0$. Let $J_n(x, P)$ be the distribution of the root R_n defined by (30)*. Then*

$$\lim_{n\to\infty} \inf_{P\in\mathbf{P}} P\left\{J_n^{-1}(\alpha_1, \hat{P}_n) \leq \sqrt{n}(\hat{\theta}_n - \theta(P)) \leq J_n^{-1}(1 - \alpha_2, \hat{P}_n)\right\} = 1 - \alpha_1 - \alpha_2$$

for any α_1 and α_2 such that $0 \leq \alpha_1 + \alpha_2 < 1$.

Note that the kernel h' defined in (44) arises in the analysis of the estimated variance of the U-statistic. Note further that the conditions on $\mathbf{P}$ in Theorem 3.12 are stronger than the conditions on $\mathbf{P}$ in Theorem 3.6. While it may be possible to weaken the restrictions on $\mathbf{P}$ in Theorem 3.12 some, it is not possible to establish the conclusions of Theorem 3.12 under the conditions on $\mathbf{P}$ in Theorem 3.6. Indeed, as shown by Bickel and Freedman (1981), the bootstrap based on the root R_n defined by (30) need not be even pointwise asymptotically valid under the conditions on $\mathbf{P}$ in Theorem 3.6. ■

4 Appendix

4.1 Proof of Theorem 2.1

Lemma 4.1. *If F and G are (nonrandom) distribution functions on $\mathbf{R}$, then we have that:*

(i) If $\sup_{x\in\mathbf{R}}\{G(x)-F(x)\}\le\epsilon$*, then* $G^{-1}(1-\alpha_2)\ge F^{-1}(1-(\alpha_2+\epsilon))$.

(ii) If $\sup_{x\in\mathbf{R}}\{F(x)-G(x)\}\le\epsilon$*, then* $G^{-1}(\alpha_1)\le F^{-1}(\alpha_1+\epsilon)$.

Furthermore, if $X\sim F$*, it follows that:*

(iii) If $\sup_{x\in\mathbf{R}}\{G(x)-F(x)\}\le\epsilon$*, then* $P\{X\le G^{-1}(1-\alpha_2)\}\ge 1-(\alpha_2+\epsilon)$.

(iv) If $\sup_{x\in\mathbf{R}}\{F(x)-G(x)\}\le\epsilon$*, then* $P\{X\ge G^{-1}(\alpha_1)\}\ge 1-(\alpha_1+\epsilon)$.

(v) If $\sup_{x\in\mathbf{R}}|G(x)-F(x)|\le\frac{\epsilon}{2}$*, then* $P\{G^{-1}(\alpha_1)\le X\le G^{-1}(1-\alpha_2)\}\ge 1-(\alpha_1+\alpha_2+\epsilon)$.

If $\hat{G}$ *is a random distribution function on* $\mathbf{R}$*, then we have further that:*

(vi) If $P\{\sup_{x\in\mathbf{R}}\{\hat{G}(x)-F(x)\}\le\epsilon\}\ge 1-\delta$*, then* $P\{X\le\hat{G}^{-1}(1-\alpha_2)\}\ge 1-(\alpha_2+\epsilon+\delta)$.

(vii) If $P\{\sup_{x\in\mathbf{R}}\{F(x)-\hat{G}(x)\}\le\epsilon\}\ge 1-\delta$*, then* $P\{X\ge\hat{G}^{-1}(\alpha_1)\}\ge 1-(\alpha_1+\epsilon+\delta)$.

(viii) If $P\{\sup_{x\in\mathbf{R}}|\hat{G}(x)-F(x)|\le\frac{\epsilon}{2}\}\ge 1-\delta$*, then* $P\{\hat{G}^{-1}(\alpha_1)\le X\le\hat{G}^{-1}(1-\alpha_2)\}\ge 1-(\alpha_1+\alpha_2+\epsilon+\delta)$.

Proof: To see (i), first note that $\sup_{x\in\mathbf{R}}\{G(x)-F(x)\}\le\epsilon$ implies that $G(x)-\epsilon\le F(x)$ for all $x\in\mathbf{R}$. Thus, $\{x\in\mathbf{R}:G(x)\ge 1-\alpha_2\}=\{x\in\mathbf{R}:G(x)-\epsilon\ge 1-\alpha_2-\epsilon\}\subseteq\{x\in\mathbf{R}:F(x)\ge 1-\alpha_2-\epsilon\}$, from which it follows that $F^{-1}(1-(\alpha_2+\epsilon))=\inf\{x\in\mathbf{R}:F(x)\ge 1-\alpha_2-\epsilon\}\le\inf\{x\in\mathbf{R}:G(x)\ge 1-\alpha_2\}=G^{-1}(1-\alpha_2)$. Similarly, to prove (ii), first note that $\sup_{x\in\mathbf{R}}\{F(x)-G(x)\}\le\epsilon$ implies that $F(x)-\epsilon\le G(x)$ for all $x\in\mathbf{R}$, so $\{x\in\mathbf{R}:F(x)\ge\alpha_1+\epsilon\}=\{x\in\mathbf{R}:F(x)-\epsilon\ge\alpha_1\}\subseteq\{x\in\mathbf{R}:G(x)\ge\alpha_1\}$. Therefore, $G^{-1}(\alpha_1)=\inf\{x\in\mathbf{R}:G(x)\ge\alpha_1\}\le\inf\{x\in\mathbf{R}:F(x)\ge\alpha_1+\epsilon\}=F^{-1}(\alpha_1+\epsilon)$. To prove (iii), note that because $\sup_{x\in\mathbf{R}}\{G(x)-F(x)\}\le\epsilon$, it follows from (i) that $\{X\le G^{-1}(1-\alpha_2)\}\supseteq\{X\le F^{-1}(1-(\alpha_2+\epsilon))\}$. Hence, $P\{X\le G^{-1}(1-\alpha_2)\}\ge P\{X\le F^{-1}(1-(\alpha_2+\epsilon))\}\ge 1-(\alpha_2+\epsilon)$. Using the same reasoning, (iv) follows from (ii) and the assumption that $\sup_{x\in\mathbf{R}}\{F(x)-G(x)\}\le\epsilon$. To see (v), note that

$$\begin{aligned}P\{G^{-1}(\alpha_1)\le X\le G^{-1}(1-\alpha_2)\} &\ge 1-P\{X<G^{-1}(\alpha_1)\}-P\{X>G^{-1}(1-\alpha_2)\}\\ &\ge 1-(\alpha_1+\alpha_2+\epsilon)\ ,\end{aligned}$$

where the first inequality follows from the Bonferroni inequality and the second inequality follows

from (iii) and (iv). To prove (vi), note that

$$\begin{aligned} P\{X \le \hat{G}^{-1}(1-\alpha_2)\} &\ge P\left\{X \le \hat{G}^{-1}(1-\alpha_2) \cap \sup_{x\in\mathbf{R}}\{\hat{G}(x) - F(x)\} \le \epsilon\right\} \\ &\ge P\left\{X \le F^{-1}(1-(\alpha_2+\epsilon)) \cap \sup_{x\in\mathbf{R}}\{\hat{G}(x) - F(x)\} \le \epsilon\right\} \\ &\ge P\left\{X \le F^{-1}(1-(\alpha_2+\epsilon))\} - P\{\sup_{x\in\mathbf{R}}\{\hat{G}(x) - F(x)\} > \epsilon\right\} \\ &= 1-\alpha_2-\epsilon-\delta \ , \end{aligned}$$

where the second inequality follows from (i). A similar argument using (ii) establishes (vii). Finally, (viii) follows from (vi) and (vii) by an argument analogous to the one used to establish (v). ■

Lemma 4.2. *Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution P. Denote by $J_n(x, P)$ the distribution of a real-valued root $R_n = R_n(X^{(n)}, P)$ under P. Let $N_n = \binom{n}{b}$, $k_n = \lfloor \frac{n}{b} \rfloor$ and define $L_n(x, P)$ according to (4). Then, for any $\epsilon > 0$, we have that*

$$P\left\{\sup_{x\in\mathbf{R}} |L_n(x,P) - J_b(x,P)| > \epsilon\right\} \le \frac{1}{\epsilon}\sqrt{\frac{2\pi}{k_n}} \ . \tag{45}$$

Proof: Let $\epsilon > 0$ be given and define $S_n(x, P; X_1, \ldots, X_n)$ by

$$\frac{1}{k_n} \sum_{1\le i\le k_n} I\{R_b((X_{b(i-1)+1}, \ldots, X_{bi}), P) \le x\} - J_b(x, P) \ .$$

Denote by $\mathcal{S}_n$ the symmetric group with n elements. Note that using this notation, we may rewrite $L_n(x, P) - J_b(x, P)$ as

$$Z_n(x, P; X_1, \ldots, X_n) = \frac{1}{n!} \sum_{\pi\in\mathcal{S}_n} S_n(x, P; X_{\pi(1)}, \ldots, X_{\pi(n)}) \ .$$

Note further that

$$\sup_{x\in\mathbf{R}} |Z_n(x, P; X_1, \ldots, X_n)| \le \frac{1}{n!} \sum_{\pi\in\mathcal{S}_n} \sup_{x\in\mathbf{R}} |S_n(x, P; X_{\pi(1)}, \ldots, X_{\pi(n)})| \ ,$$

which is a sum of $n!$ identically distributed random variables. Let $\epsilon > 0$ be given. It follows that $P\{\sup_{x\in\mathbf{R}} |Z_n(x, P; X_1, \ldots, X_n)| > \epsilon\}$ is bounded above by

$$P\left\{\frac{1}{n!} \sum_{\pi\in\mathcal{S}_n} \sup_{x\in\mathbf{R}} |S_n(x, P; X_{\pi(1)}, \ldots, X_{\pi(n)})| > \epsilon\right\} \ . \tag{46}$$

Using Markov's inequality, (46) can be bounded by

$$\frac{1}{\epsilon} E_P\left[\sup_{x\in\mathbf{R}} |S_n(x, P; X_1, \ldots, X_n)|\right] = \frac{1}{\epsilon}\int_0^1 P\left\{\sup_{x\in\mathbf{R}} |S_n(x, P; X_1, \ldots, X_n)| > u\right\} du \ . \tag{47}$$

We may use the Dvoretsky-Kiefer-Wolfowitz inequality to bound the righthand-side of (47) by

$$\frac{1}{\epsilon}\int_0^1 2\exp\{-2k_n u^2\}du = \frac{2}{\epsilon}\sqrt{\frac{2\pi}{k_n}}\left[\Phi(2\sqrt{k_n}) - \frac{1}{2}\right] < \frac{1}{\epsilon}\sqrt{\frac{2\pi}{k_n}} \ ,$$

which establishes (45). ■

Lemma 4.3. *Let $X^{(n)} = (X_1, \ldots, X_n)$ be an i.i.d. sequence of random variables with distribution $P \in \mathbf{P}$. Denote by $J_n(x, P)$ the distribution of a real-valued root $R_n = R_n(X^{(n)}, P)$ under P. Let $k_n = \lfloor \frac{n}{b} \rfloor$ and define $L_n(x, P)$ according to (4). Let*

$$\begin{aligned}
\delta_{1,n}(\epsilon,\gamma,P) &= \frac{1}{\gamma\epsilon}\sqrt{\frac{2\pi}{k_n}} + I\left\{\sup_{x\in\mathbf{R}}\{J_b(x,P) - J_n(x,P)\} > (1-\gamma)\epsilon\right\} \\
\delta_{2,n}(\epsilon,\gamma,P) &= \frac{1}{\gamma\epsilon}\sqrt{\frac{2\pi}{k_n}} + I\left\{\sup_{x\in\mathbf{R}}\{J_n(x,P) - J_b(x,P)\} > (1-\gamma)\epsilon\right\} \\
\delta_{3,n}(\epsilon,\gamma,P) &= \frac{1}{\gamma\epsilon}\sqrt{\frac{2\pi}{k_n}} + I\left\{\sup_{x\in\mathbf{R}}|J_b(x,P) - J_n(x,P)| > (1-\gamma)\epsilon\right\} \ .
\end{aligned}$$

Then, for any $\epsilon > 0$ and $\gamma \in (0,1)$, we have that

(i) $P\{R_n \le L_n^{-1}(1-\alpha_2, P)\} \ge 1 - (\alpha_2 + \epsilon + \delta_{1,n}(\epsilon,\gamma,P))$.

(ii) $P\{R_n \ge L_n^{-1}(\alpha, P)\} \ge 1 - (\alpha_1 + \epsilon + \delta_{2,n}(\epsilon,\gamma,P))$.

(iii) $P\{L_n^{-1}(\alpha_1, P) \le R_n \le L_n^{-1}(1-\alpha_2, P)\} \ge 1 - (\alpha_1 + \alpha_2 + \epsilon + \delta_{3,n}(\epsilon,\gamma,P))$.

Proof: Let $\epsilon > 0$ and $\gamma \in (0,1)$ be given. Note that

$$\begin{aligned}
& P\left\{\sup_{x\in\mathbf{R}}\{L_n(x,P) - J_n(x,P)\} > \epsilon\right\} \\
\le\ & P\left\{\sup_{x\in\mathbf{R}}\{L_n(x,P) - J_b(x,P)\} + \sup_{x\in\mathbf{R}}\{J_b(x,P) - J_n(x,P)\} > \epsilon\right\} \\
\le\ & P\left\{\sup_{x\in\mathbf{R}}\{L_n(x,P) - J_b(x,P)\} > \gamma\epsilon\right\} + I\left\{\sup_{x\in\mathbf{R}}\{J_b(x,P) - J_n(x,P)\} > (1-\gamma)\epsilon\right\} \\
\le\ & \frac{1}{\gamma\epsilon}\sqrt{\frac{2\pi}{k_n}} + I\left\{\sup_{x\in\mathbf{R}}\{J_b(x,P) - J_n(x,P)\} > (1-\gamma)\epsilon\right\} \ ,
\end{aligned}$$

where the final inequality follows from Lemma 4.2. Assertion (i) thus follows from the definition of $\delta_{1,n}(\epsilon,\gamma,P)$ and part (vi) of Lemma 4.1. Assertions (ii) and (iii) are established similarly. ■

Proof of Theorem 2.1: To prove (i), note that by part (i) of Lemma 4.3, we have for any $\epsilon > 0$ and $\gamma \in (0,1)$ that

$$\sup_{P\in\mathbf{P}} P\{R_n \le L_n^{-1}(1-\alpha_2, P)\} \ge 1 - \left(\alpha_2 + \epsilon + \inf_{P\in\mathbf{P}} \delta_{1,n}(\epsilon,\gamma,P)\right) \ ,$$

where

$$\delta_{1,n}(\epsilon,\gamma,P) = \frac{1}{\gamma\epsilon}\sqrt{\frac{2\pi}{k_n}} + I\left\{\sup_{x\in\mathbf{R}}\{J_b(x,P) - J_n(x,P)\} > (1-\gamma)\epsilon\right\} .$$

By the assumption on $\sup_{P\in\mathbf{P}}\sup_{x\in\mathbf{R}}\{J_b(x,P) - J_n(x,P)\}$, we have that $\inf_{P\in\mathbf{P}}\delta_{1,n}(\epsilon,\gamma,P) \to 0$ for every $\epsilon > 0$. Thus, there exists a sequence $\epsilon_n > 0$ tending to 0 so that $\inf_{P\in\mathbf{P}}\delta_{1,n}(\epsilon_n,\gamma,P) \to 0$. The desired claim now follows from applying part (i) of Lemma 4.3 to this sequence. Assertions (ii) and (iii) follow in exactly the same way. ■

4.2 Proof of Theorem 2.4

We prove only (i). Similar arguments can be used to establish (ii) and (iii). Let $\alpha_1 = 0$, $0 \le \alpha_2 < 1$ and $\eta > 0$ be given. Choose $\delta > 0$ so that

$$\sup_{x\in\mathbf{R}}\{J_n(x,P') - J_n(x,P)\} < \frac{\eta}{2}$$

whenever $\rho(P',P) < \delta$ for $P' \in \mathbf{P}'$ and $P \in \mathbf{P}$. For n sufficiently large, we have that

$$\sup_{P\in\mathbf{P}} P\{\rho(\hat{P}_n,P) > \delta\} < \frac{\eta}{4} \text{ and } \sup_{P\in\mathbf{P}} P\{\hat{P}_n \notin \mathbf{P}'\} < \frac{\eta}{4} .$$

For such n, we therefore have that

$$1 - \frac{\eta}{2} \le \inf_{P\in\mathbf{P}} P\{\rho(\hat{P}_n,P) \le \delta \cap \hat{P}_n \in \mathbf{P}'\} \le \inf_{P\in\mathbf{P}} P\left\{\sup_{x\in\mathbf{R}}\{J_n(x,\hat{P}_n) - J_n(x,P)\} \le \frac{\eta}{2}\right\} .$$

It follows from part (vi) of Lemma 4.1 that for such n

$$\inf_{P\in\mathbf{P}} P\{R_n \le J_n^{-1}(1-\alpha_2,\hat{P}_n)\} \ge 1 - (\alpha_2 + \eta) .$$

Since the choice of η was arbitrary, the desired result follows. ■

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