BUSEMANN FUNCTIONS AND THE SPEED OF A SECOND CLASS PARTICLE IN THE RAREFACTION FAN

ERIC CATOR AND LEANDRO P. R. PIMENTEL

Abstract. In this paper we will show how the results found in [7], about the Busemann functions in last-passage percolation, can be used to calculate the asymptotic distribution of the speed of a single second class particle starting from an arbitrary deterministic configuration which has a rarefaction fan, in either the totally asymetric exclusion process, or the Hammersley interacting particle process. The method will be to use the well known last-passage percolation description of the exclusion process and of the Hammersley process, and then the well known connection between second class particles and competition interfaces.

1. Introduction

The macroscopic behavior of the density profile of the totally asymmetric nearest neighbor exclusion process (TASEP) is governed by the Burgers equation, which corresponds to the "shape theorem" in last-passage percolation [18]. The second class particles, that follow roughly the behavior of a perturbation of the system, are asymptotically governed by the characteristics of the Burgers equation. When there is only one characteristic, the second class particle follows it; when there are infinitely many (rarefaction fan), the particle chooses one of them at random to follow. Until recently, there was only one method, introduced by Kipnis and Ferrari [11], to compute the distribution of the asymptotic speed when the initial distribution is a product measure with densities $p \in [0, 1)$ and $p' \in (p, 1]$, to the right and left of the origin, respectively. They proved that the rescaled position of the second class particle converges in distribution, as time goes to infinity, to a random variable uniformly distributed in the interval $[1 - 2p', 1 - 2p]$. Their method is based on the hydrodynamic description of the macroscopic behavior and it strongly relies on the product structure of the initial profile.

Later, following initial results by Guiol and Mountford [14] and Ferrari and Pimentel [13], Ferrari, Martin and Pimentel in [12] proved that for initial configurations exhibiting a rarefaction fan, the second class particles almost surely have an asymptotic speed, but no new results were found about the distribution of this speed.

Recently, in a paper by Amir, Angel and Valkó [2], a new method has been introduced to study the behavior of a multiclass TASEP, which allows the calculation of joint distributions of multiclass particles under a specific initial configuration. However, as far as we can see, this does not shed new light on how to calculate the asymptotic distribution of a single second class particle in an arbitrary initial configuration exhibiting a rarefaction fan.

We will also consider the Hammersley interacting particle process [1], where the situation concerning the second class particle in a rarefaction fan is very similar to TASEP: almost sure existence of the asymptotic speed is proved by Coletti and Pimentel in [9] (they even prove this for more general objects than second class particles), but the distribution of the second class particle in a

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rarefaction fan was only determined for a very specific family of initial conditions, involving Poisson processes ([4, 9]).

In this paper we will show how the results found in [7] about the Busemann functions in last-passage percolation (LPP) can be used to calculate the asymptotic law of the speed of a single second class particle starting from an arbitrary deterministic configuration which has a rarefaction fan, in either the totally asymetric exclusion process (TASEP), or the Hammersley process. The method will be to use the well known LPP description of the TASEP and of the Hammersley process, and then the well known connection between second class particles and competition interfaces (see [13] for the TASEP case, and [7] for the Hammersley case) to show that the limit law can be expressed in terms of the supremum of two independent random walks in an environment that depends upon the initial configuration. This method allows us to recover the classical results, as well as to get explicit formulas for the law of the asymptotic speed starting from periodic profiles. Busemann functions in first passage percolation were first introduced by Newman [15], and these were later extended to LPP by Wüthrich in [20], by Ferrari and Pimentel in [13] and by the authors in [7].

To give some intuition for the main idea in the paper, we present a heuristic outline of it in the lattice LPP competition model. Suppose that to each $\mathbf{x} \in \mathbb{Z}^2$ we attach an exponential random variable $X_{\mathbf{x}}$ of intensity one. For $\mathbf{x} \leq \mathbf{y}$ (coordinate-wise) let $L^\ell(\mathbf{x}, \mathbf{y})$ denote the maximum, over all up-right nearest-neighbor paths ϖ connecting $\mathbf{x}$ to $\mathbf{y}$, of the sum of $X_{\mathbf{z}}$ along ϖ, leaving out $\mathbf{x}$ (see (2.6)). We adopt the notation in [7] and, to indicate that it refers to the lattice model, we use the superscript ℓ. Now choose two non-ordered points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ with $y_1 < x_1$ and $y_2 > x_2$. We will assign each point $\mathbf{z} \in \mathbb{R}^2$ with $\mathbf{x}, \mathbf{y} \leq \mathbf{z}$ to either $\mathbf{x}$ or $\mathbf{y}$ in the following way: $\mathbf{x}$ "conquers" $\mathbf{z}$ if $L^\ell(\mathbf{x}, \mathbf{z}) > L^\ell(\mathbf{y}, \mathbf{z})$, and otherwise $\mathbf{y}$ conquers $\mathbf{z}$. The point $\mathbf{z}$ gets the color blue if it is conquered by $\mathbf{x}$ and otherwise it gets the color red. This example corresponds to the two-corner profile discussed in Section 4.1. The *competition interface* is the up right path that separates the blue and the red regions. It has been shown in [13] that a competition interface has an (a.s.) asymptotic angle Θ, which will be random in this case (rarefaction regime). To check whether $\Theta \leq \alpha$, for some fixed $\alpha \in (0, \pi/2)$, one can consider $\mathbf{z}_\alpha$ far away on the line with angle α, and compare the length $L^\ell(\mathbf{x}, \mathbf{z}_\alpha)$ with $L^\ell(\mathbf{y}, \mathbf{z}_\alpha)$. It is at this point that the Busemann function comes into play: for $\alpha \in (0, \pi/2)$,

$$\text{a.s.} \quad \exists\, B^\ell_\alpha(\mathbf{x}, \mathbf{y}) := \lim_{|\mathbf{z}_\alpha| \to +\infty} L^\ell(\mathbf{y}, \mathbf{z}_\alpha) - L^\ell(\mathbf{x}, \mathbf{z}_\alpha)\,.$$

This means that checking whether $L^\ell(\mathbf{x}, \mathbf{z}_\alpha) \leq L^\ell(\mathbf{y}, \mathbf{z}_\alpha)$ (this corresponds to $\Theta \leq \alpha$) eventually, corresponds to checking whether $B^\ell_\alpha(\mathbf{x}, \mathbf{y}) \geq 0$ [1]. In [7] we derived the distribution of B^ℓ_α, which is why we are in fact able to calculate the asymptotic distribution of Θ. This argument can be extended for any deterministic initial profile in the rarefaction regime.

Overview. In Section 2 we state our three main results, the first describing the distribution of the speed of the second class particle in a rarefaction fan for TASEP, the second describing the distribution of the angle of a competition interface in a rarefaction fan for the lattice last-passage percolation model (strongly related to the first result), and the third Theorem describing the distribution of the speed of the second class particle in a rarefaction fan for the Hammersley process. In Section 3 we recall the equilibrium description of the Busemann functions from [7] and establish its connection with second class particles and competition interfaces to prove all the theorems. Section 4 and Section 5 are devoted to compute the law of the asymptotic speed, for

[1] This idea was also used to study the almost sure behavior of competition interfaces [13, 16]

some initial configurations in the rarefaction regime, in the TASEP and in the Hammersley models, respectively.

2. Main Results

In this section we state the general results about the asymptotic distribution of the second class particle in a rarefaction fan in the TASEP case and the Hammersley case. The TASEP result is formulated in two ways: the first using the particle description, the second using the last passage description. In Section 3 we will point out that these two descriptions are strongly related (which is already well known). Applications of these general results to particular initial conditions can be found in Section 4 and Section 5.

2.1. **Second-class particles in the exclusion process.** The one-dimensional nearest neighbor totally asymmetric simple exclusion process is a Markov process $(\eta_t\,,\, t \geq 0)$ in the state space $\{0,1\}^{\mathbb{Z}}$. In our notation, $\eta_t(x) = 1$ indicates that there is a particle at site x at time t; only one particle is allowed at each site. At rate one, if there is a particle at site $x \in \mathbb{Z}$, it attempts to jump to $x+1$; if there is no particle in $x+1$ the jump occurs, otherwise nothing happens. To construct a realization of this process, one considers independent one dimensional Poisson processes $\mathcal{N} = (N_x(\cdot), x \in \mathbb{Z})$ of intensity 1. The process $(\eta_t\,,\, t \geq 0)$ can be constructed as a deterministic function of the initial configuration η and the Poisson processes $\mathcal{N}$ as follows: if s is a Poisson epoch of N_x and there is a particle at x and no particle at $x+1$ in the configuration η_{s-}, then at time s the new configuration is obtained by making the particle jump from x to $x+1$. Let Φ be the function that takes η and $\mathcal{N}$ to $(\eta_t\,,\, t \geq 0)$. Let η and η' be two arbitrary configurations. The *basic* coupling between two exclusion processes with initial configurations η and η' respectively is the joint realization $(\Phi(\eta,\mathcal{N}), \Phi(\eta',\mathcal{N})) = ((\eta_t, \eta'_t),\, t \geq 0)$ obtained by using the same Poisson epochs for the two different initial conditions. Liggett (1985, 1999) are the default references for the exclusion process.

Let η be an initial configuration and let η' be the configuration that differs from η only at site 0. With the basic coupling, the configurations at time $t > 0$ differ only at one site $X^\eta(t)$ defined by

$$X^\eta(t) := \sum_{x\in\mathbb{Z}} x\mathbf{1}\{\eta_t(x) \neq \eta'_t(x)\}$$

($\mathbf{1}$ denotes the indicator function). The process $(X^\eta(t),\, t \geq 0)$ is the trajectory of a "second class particle". This particle jumps one unit to the right at rate one if there is no η particle in its right nearest neighbor and it jumps one unit to the left at rate one if there is an η particle in its left nearest neighbor site, interchanging positions with it.

To state our results, we need some more notation. For each η we associate a new configuration $\bar{\eta}$ as follows:

$$\bar{\eta}(k) := \begin{cases} \eta(k) & \text{for } k < 0 \\ 0 & \text{for } k = 0 \\ 1 & \text{for } k = 1 \\ \eta(k-1) & \text{for } k > 1. \end{cases} \tag{2.1}$$

The hole-particle pair at the origin will be important to formulate the LPP competition model and to establish the connection between second class particles and competition interfaces [13]. Note that $\eta(0)$ is not relevant for $\bar{\eta}$; this corresponds to the idea of having a second class particle in 0. For $\rho \in (0,1)$ let $\{\mathrm{Exp}_k(\rho) \,:\, k \in \mathbb{Z}\}$ and $\{\mathrm{Exp}'_k(1-\rho) \,:\, k \in \mathbb{Z}\}$ be independent collections of

i.i.d. exponential random variables of intensity ρ and $1-\rho$, respectively. These collections are also assumed to be independent of η, whenever η is random. For $k>0$ let

$$Z_k^{\rho,\eta} := \begin{cases} -\mathrm{Exp}_k(\rho) & \text{if } \bar\eta(k)=0 \\ \mathrm{Exp}'_k(1-\rho) & \text{if } \bar\eta(k)=1\,, \end{cases} \tag{2.2}$$

and for $k\le 0$ let

$$Z_k^{\rho,\eta} := \begin{cases} \mathrm{Exp}_k(\rho) & \text{if } \bar\eta(k)=0 \\ -\mathrm{Exp}'_k(1-\rho) & \text{if } \bar\eta(k)=1\,. \end{cases} \tag{2.3}$$

Define the random walks

$$S_{n,+}^{\rho,\eta} := \sum_{k=1}^{n} Z_k^{\rho,\eta} \text{ and } S_{n,-}^{\rho,\eta} := \sum_{k=0}^{-n} Z_k^{\rho,\eta}\,.$$

We assume that the particle system is in the rarefaction regime, i.e.:

$$p_\eta := \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\eta(k) < \liminf_{n\to\infty}\frac{1}{n}\sum_{k=-n}^{-1}\eta(k) =: p'_\eta\,. \tag{2.4}$$

Theorem 1. *For each $u\in[1-2p'_\eta, 1-2p_\eta]$ let*

$$\rho_u = \frac{1+u}{2}\,.$$

If (2.4) *holds then*

$$\frac{X^\eta(t)}{t} \xrightarrow{\mathcal{D}} U^\eta \text{ as } t\to\infty\,,$$

where U^η is a random variable with support $[1-2p'_\eta, 1-2p_\eta]$ and for $u\in(1-2p'_\eta, 1-2p_\eta)$

$$\mathbb{P}\left(U^\eta \ge u\right) = \mathbb{P}\left(\sup_{n\ge 0} S_{n,-}^{\rho_u,\eta} \ge \sup_{n\ge 1} S_{n,+}^{\rho_u,\eta}\right)\,. \tag{2.5}$$

To give some intuition for this result, note that the asymptotic drift of the random walk $S_{n,-}^{\rho_u,\eta}$ is lesser than or equal to $-p'_\eta/(1-\rho_u)+(1-p'_\eta)/\rho_u$, which is strictly negative if $u>1-2p'_\eta$ and it is a decreasing function of u. Also, the asymptotic drift of the random walk $S_{n,+}^{\rho_u,\eta}$ is lesser than or equal to $-(1-p_\eta)/\rho_u + p_\eta/(1-\rho_u)$, which is strictly negative if $u<1-2p_\eta$ and it is an increasing function of u. Note that when $u=1-2p'_\eta$ or $u=1-2p_\eta$, (2.5) may not hold in general! An example of this can be found in Section 3 (see Remark 3.1). The left-hand side of (2.5) is still determined at the boundary by left-continuity.

2.2. **Competition interfaces in the lattice last-passage percolation model.** Let $X := \{X_{\mathbf{z}}\}_{\mathbf{x}\in\mathbb{Z}^2}$ be a collection of i.i.d. random variables with an exponential distribution of parameter one. For $\mathbf{x},\mathbf{y}\in\mathbb{Z}^2$, with $\mathbf{x}\le\mathbf{y}$ (coordinate-wise), let

$$L^\ell(\mathbf{x},\mathbf{y}) := \max_{\varpi:\mathbf{x}\nearrow\mathbf{y}} \sum_{\mathbf{z}\in\varpi} X_{\mathbf{z}}\,, \tag{2.6}$$

where $\varpi:\mathbf{x}\nearrow\mathbf{y}$ denotes an increasing (or up-right) path connecting $\mathbf{x}$ to $\mathbf{y}$, not including $\mathbf{x}$. The random variable L^ℓ is called the last-passage (percolation) time from $\mathbf{x}$ to $\mathbf{y}$.

Using the well known connection between the totally asymmetric exclusion process (TASEP) and the last-passage percolation model, introduced by Rost [18], we identify the initial configuration $\bar\eta$ of particles and holes, that includes a hole at 0 and a particle at 1, with an initial (growth) *profile* σ

in $\mathbb{Z}^2$, in the following way: we define the profile $\sigma = \sigma_\eta$ (as a function of η), which is a down-right bi-infinite sequence of points in $\mathbb{Z}^2$ containing the origin, such that

$$\sigma(0) = (0,0) \ \text{ and } \ \sigma(k) - \sigma(k-1) = \begin{cases} (1,0) & \text{if } \bar{\eta}(k) = 0, \\ (0,-1) & \text{if } \bar{\eta}(k) = 1. \end{cases}$$

Given such an initial profile σ, we can define a competition interface, following [13] and [12]. We introduce the notation

$$\sigma^- := \{\sigma(k) \ : \ k < 0\} \ \text{ and } \ \sigma^+ = \{\sigma(k) \ : \ k > 0\}.$$

The competition for territory goes as follows: choose $\mathbf{t} \in \mathbb{R}^2$ such that $\{\mathbf{x} \ : \ \mathbf{x} \leq \mathbf{t}\} \cap \sigma \neq \emptyset$. Define

$$L^\ell(\sigma^-, \mathbf{t}) := \sup\{L^\ell(\sigma(k), \mathbf{t}) \ : \ k < 0 \text{ and } \sigma(k) \leq \mathbf{t}\}$$

and

$$L^\ell(\sigma^+, \mathbf{t}) := \sup\{L^\ell(\sigma(k), \mathbf{t}) \ : \ k > 0 \text{ and } \sigma(k) \leq \mathbf{t}\}.$$

If $\{\mathbf{x} \ : \ \mathbf{x} \leq \mathbf{t}\} \cap \sigma^\pm = \emptyset$, we define $L^\ell(\sigma^\pm, \mathbf{t}) = 0$. The point $\mathbf{t}$ gets the color blue if it is further (the longest path is longer) from the negative part of σ than from the positive part, so if

$$L^\ell(\sigma^-, \mathbf{t}) > L^\ell(\sigma^+, \mathbf{t}),$$

and otherwise it gets the color red. Note that because of the hole-particle pair at the origin, $\sigma(0)$ can never be the most "distant" point to $\mathbf{t}$. Also, the coloring is well defined for every $\mathbf{t}$ above σ if the initial condition η has infinitely many particles to the left of the origin, and infinitely many holes to the right; we will always assume this. The competition interface $\Phi^\eta := (\phi_n^\eta)_{n \geq 0}$ is the up right path through the lattice points starting at $\mathbf{0}$ that separates the blue and the red region (see Section 2 of [12] for an extensive description).

Under (2.4), the rarefaction condition, let

$$\alpha_\eta := \arctan\left(\left(\frac{p_\eta}{1-p_\eta}\right)^2\right) \ < \ \arctan\left(\left(\frac{p'_\eta}{1-p'_\eta}\right)^2\right) =: \alpha'_\eta \,.$$

Note that $p_\eta/(1-p_\eta)$ is the (negative) asymptotic slope of σ at $+\infty$, whereas $p'_\eta/(1-p'_\eta)$ is the slope of σ at $-\infty$.

Theorem 2. *For each $\alpha \in [\alpha_\eta, \alpha'_\eta]$ let*

$$\rho_\alpha := \frac{1}{1+\sqrt{\tan\alpha}} \,.$$

If (2.4) *holds then the competition interface $(\phi_n^\eta)_{n \geq 0}$ satisfies*

$$\frac{\phi_n^\eta}{|\phi_n^\eta|} \xrightarrow{\mathcal{D}} (\cos\Theta^\eta, \sin\Theta^\eta) \ \text{ as } n \to \infty \,, \tag{2.7}$$

where Θ^η is a random variable with support $[\alpha_\eta, \alpha'_\eta]$ and for $\alpha \in (\alpha_\eta, \alpha'_\eta)$

$$\mathbb{P}\left(\Theta^\eta \leq \alpha\right) = \mathbb{P}\left(\sup_{n \geq 0} S_{n,-}^{\rho_\alpha,\eta} \ \geq \ \sup_{n \geq 1} S_{n,+}^{\rho_\alpha,\eta}\right) \,. \tag{2.8}$$

Theorem 2 and Theorem 1, although each of separate interest, are of course intimately linked through the connection between TASEP and the lattice LPP, which enables us to directly translate the competition interface in the LPP into the position of the second class particle in TASEP; see Lemma 3.5 in Section 3 for the details. Our proof will use the LPP description, as explained in the Introduction. Again we have that at the boundary, (2.8) may fail.

Remark 2.1. Ferrari, Martin and Pimentel in [12] show that the competition interface in the rarefaction fan has in fact almost surely an asymptotic direction (which implies that the second class particle in TASEP has an asymptotic speed). They assume an asymptotic inclination of the initial profile σ in the positive and in the negative direction (which corresponds to an asymptotic particle density to the left and to the right of zero in TASEP). However, it is not hard to adapt their proof only slightly to get the same results under our rarefaction fan condition (2.4).

2.3. **Second class particles in the Hammersley particle system.** Let $\mathbf{P} \subseteq \mathbb{R}^2$ be a two-dimensional Poisson random set of intensity one. The last-passage time (or longest increasing path) between $\mathbf{z}, \mathbf{v} \in \mathbb{R}^2$, with $\mathbf{z} \leq \mathbf{v}$, is defined by

$$L(\mathbf{z}, \mathbf{v}) := \max_{\varpi : \mathbf{z} \nearrow \mathbf{v}} |\varpi| \,, \tag{2.9}$$

where $\varpi : \mathbf{z} \nearrow \mathbf{v}$ denotes an increasing path from $\mathbf{z}$ to $\mathbf{v}$, consisting of points in $\mathbf{P}$ that are not on the same horizontal nor vertical level as $\mathbf{z}$. Furthermore, $|\varpi|$ denotes the size (the number of Poisson points) of such a path. The Hammersley model can be seen as a Markov process on the space $\mathcal{N}$ consisting of the set of all counting processes ν on $\mathbb{R}$ such that

$$\liminf_{y \to -\infty} \frac{\nu(y)}{y} > 0 \,, \tag{2.10}$$

where

$$\nu(y) := \begin{cases} \nu((0, y]) & \text{for } y \geq 0 \\ -\nu((y, 0]) & \text{for } y < 0. \end{cases} \tag{2.11}$$

(We allow $\nu(y) = -\infty$ for all $y < 0$.) Note that $\nu(0) = 0$. For each $\nu \in \mathcal{N}$, the process

$$L_\nu(x, t) := \sup_{z \leq x} \{\nu(z) + L((z, 0), (x, t))\} \qquad \text{for} \quad x \in \mathbb{R} \,, \text{ and } \quad t \geq 0 \tag{2.12}$$

is well defined and the time evolution defined by

$$M_t^\nu((x, y]) := L_\nu(y, t) - L_\nu(x, t) \,, \tag{2.13}$$

is a Markov process in $\mathcal{N}$, called the Hammersley interacting particle system [1]. With a slight abuse of notation, we also refer to L_ν as the Hammersley process. One can think of M_t as a particle system (or process in $\mathcal{N}$) that starts with the particles (or atoms) of ν, and that moves according to the (informal) rule: for $(x, t) \in \mathbf{P}$, at time $t \geq 0$ the nearest particle to the right of x is moved to x, with a new particle created at x if no such particle exists.

To introduce a second class particle in the system, we put an atom at 0 in the process ν by defining

$$\bar{\nu}([0, x]) = \nu([0, x]) + 1 \quad \text{for} \quad x \geq 0 \,.$$

With this new process, and using the same $\mathbf{P}$, we construct $M_t^{\bar{\nu}}$, and then define the location of the second class particle $Y^\nu(t)$ as

$$Y^\nu(t) = \inf\{x \geq 0 : M_t^{\bar{\nu}}(x) = M_t^\nu(x) + 1\} \,.$$

We note that $Y^\nu(t)$ is a non-decreasing function of t, meaning that the second class particle moves to the right. We use the notation of [7], where relevant details can also be found, including the following important connection between the longest path description and the second class particle. For $\nu \in \mathcal{N}$, let ν^+, ν^- be the processes defined by

$$\nu^+(x) = \begin{cases} \nu(x) & \text{for } x \geq 0 \\ -\infty & \text{for } x < 0. \end{cases} \quad \text{and } \nu^-(x) = \begin{cases} 0 & \text{for } x \geq 0 \\ \nu(x) & \text{for } x < 0. \end{cases} \tag{2.14}$$

Then

$$\{Y^\nu(t) \le x\} = \{L_{\nu^+}(x,t) \ge L_{\nu^-}(x,t)\}\,. \tag{2.15}$$

The above equation resembles the situation where one should check if the competition interface is to the left or to the right of some angle α. This indicates that the speed of a second class particle can be related to the respective Busemann functions in a similar way. We require the following rarefaction assumption (recall (2.11) and (2.14))

$$a_\nu := \limsup_{x\to\infty} \frac{\nu^+(x)}{x} < \liminf_{y\to-\infty} \frac{\nu^-(y)}{y} =: b_\nu\,. \tag{2.16}$$

For $\rho > 0$ let $\bar\nu_\rho$ be a one-dimensional Poisson counting process of intensity $\rho > 0$, that is independent of ν, whenever ν is random.

Theorem 3. *For each $v \in [b_\nu^{-2}, a_\nu^{-2}]$ let*

$$\rho_v := \frac{1}{\sqrt{v}}\,.$$

If (2.16) *holds then*

$$\frac{Y^\nu(t)}{t} \xrightarrow{\mathcal{D}} V^\nu \ \text{ as } t\to\infty\,,$$

where V^ν is a random variable with support $[b_\nu^{-2}, a_\nu^{-2}]$ and for $v \in (b_\nu^{-2}, a_\nu^{-2})$,

$$\mathbb{P}\left(V^\nu \le v\right) = \mathbb{P}\left(\sup_{z\ge 0}\{\nu(z) - \bar\nu_{\rho_v}(z)\} \ge \sup_{z<0}\{\nu(z) - \bar\nu_{\rho_v}(z)\}\right)\,. \tag{2.17}$$

Condition (2.16) ensures that with probability 1, at least one of the two suprema is finite, and hence the right hand side probability is well defined for every $v \in (b_\nu^{-2}, a_\nu^{-2})$. Again we have that at the boundary, (2.17) may fail.

3. Proof of the Theorems

We will start with the proof for the Hammersley process, since the ideas are most clearly represented in that case. In the TASEP case, we will use very similar methods.

3.1. **Busemann functions and equilibrium measures in the Hammersley model.** In the Hammersley last-passage percolation model, Wüthrich [20] proved that, for $\alpha \in (0, \pi/2)$,

$$\text{a.s.}\ \ \exists\, B_\alpha(\mathbf{x},\mathbf{y}) := \lim_{|\mathbf{z}_\alpha|\to+\infty} L(\mathbf{y},\mathbf{z}_\alpha) - L(\mathbf{x},\mathbf{z}_\alpha)\,.$$

Here, $\mathbf{z}_\alpha$ is any deterministic path with $|\mathbf{z}_\alpha| \to \infty$ that follows the ray through $(\cos\alpha, \sin\alpha)$:

$$\frac{\mathbf{z}_\alpha}{|\mathbf{z}_\alpha|} \to (\cos\alpha, \sin\alpha).$$

By the invariance under the map $\alpha \to \alpha + \pi$, it is straightforward to see that the analogous result holds for $\alpha \in (\pi, 3\pi/2)$. In fact, Wüthrich shows that with probability 1, for every $\mathbf{x} \in \mathbb{R}^2$ there exists a unique *α-ray* $\varpi_\alpha(\mathbf{x})$, which is a semi-infinite path starting at $\mathbf{x}$ and following the ray through $(\cos\alpha, \sin\alpha)$, such that this path is a geodesic: for each two points $\mathbf{y}$ and $\mathbf{z}$ on $\varpi_\alpha(\mathbf{x})$, the lowest longest path between $\mathbf{y}$ and $\mathbf{z}$ coincides with $\varpi_\alpha(\mathbf{x})$. This implies that $L(\cdot,\cdot)$ is additive (rather than super-additive) when restricted to $\varpi_\alpha(\mathbf{x})$. Wüthrich also showed that for any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, the two α-rays $\varpi_\alpha(\mathbf{x})$ and $\varpi_\alpha(\mathbf{y})$ coalesce: there exists a point $\mathbf{c} \in \varpi_\alpha(\mathbf{x}) \cap \varpi_\alpha(\mathbf{y})$ such that beyond this point $\mathbf{c}$, the two α-rays coincide. This also implies that

$$B_\alpha(\mathbf{x},\mathbf{y}) = L(\mathbf{c},\mathbf{y}) - L(\mathbf{c},\mathbf{x}).$$

In fact, the above equality holds for any $\mathbf{c} \in \varpi\alpha(\mathbf{x}) \cap \varpi_\alpha(\mathbf{y})$ (this follows from the additivity of $L(\cdot,\cdot)$ when restricted to $\varpi_\alpha(\mathbf{x})$ and the fact that when two α-rays touch, they will necessarily coalesce from that point on).

The most important property of the Busemann functions for us is the following connection with the associated interacting particle process, established by the authors in Corollary 4.4, Theorem 5.3 and Corollary 5.5 of [7]): Let $\alpha \in (\pi, 3\pi/2)$ and define

$$\nu_\alpha((x,y]) := B_\alpha(\mathbf{0},(y,0)) - B_\alpha(\mathbf{0},(x,0)) = B_\alpha((x,0),(y,0)) \quad \text{for} \quad x < y\,. \tag{3.1}$$

For $\alpha \in (0, \pi/2)$, we define

$$\nu_\alpha((x,y]) := B_\alpha(\mathbf{0},(x,0)) - B_\alpha(\mathbf{0},(y,0)) = B_\alpha((y,0),(x,0)) \quad \text{for} \quad x < y\,. \tag{3.2}$$

The reason for these two slightly different definitions is that in this way, $\nu_\alpha((x,y])$ is positive for $x < y$. We have that ν_α and $\nu_{\alpha+\pi}$ are the unique spatially ergodic counting processes with intensity

$$\mathbb{E}\nu_\alpha(1) = \sqrt{\tan\alpha}\,,$$

that satisfy (time invariance)

$$M_t^{\nu_\alpha} \stackrel{\mathcal{D}}{=} \nu_\alpha\,, \text{ for all } t \geq 0\,. \tag{3.3}$$

A consequence of this result is that we can compute the law of the Busemann function along the horizontal axis. For $\rho > 0$ let $\bar{\nu}_\rho$ be an one-dimensional Poisson counting process of intensity $\rho > 0$. Then

$$\nu_\alpha \stackrel{\mathcal{D}}{=} \bar{\nu}_{\sqrt{\tan\alpha}}\,. \tag{3.4}$$

We also need the following lemma for the Hammersley process L_ν (see (2.12)). This result, in a completely different formulation, would also follow from Lemma 2.2 in [5].

Lemma 3.1. *Consider the Hammersley process L_ν, and suppose that for $t \geq 0$, $0 \leq x \leq y < Y^\nu(t)$. Then*

$$L_\nu(y,t) - L_\nu(x,t) \leq L(\mathbf{0},(y,t)) - L(\mathbf{0},(x,t)).$$

Proof: The fact that $x \leq y < Y^\nu(t)$, implies that (see (2.15)) there exists $t_x \leq t_y < 0$ such that

$$L_\nu(x,t) = L((t_x,0),(x,t)) + \nu(t_x) \quad \text{and} \quad L_\nu(y,t) = L((t_y,0),(y,t)) + \nu(t_y).$$

Define $\mathbf{z}$ as the (an) intersection point of the longest upright path from $(t_y,0)$ to (y,t) and the longest upright path from $\mathbf{0}$ to (x,t). Then we can see that

$$\begin{aligned} L_\nu(x,t) &\geq L((t_y,0),(x,t)) + \nu(t_y) \\ &\geq L((t_y,0),\mathbf{z}) + L(\mathbf{z},(x,t)) + \nu(t_y). \end{aligned}$$

This implies

$$\begin{aligned} L_\nu(y,t) - L_\nu(x,t) &\leq L((t_y,0),(y,t)) - L((t_y,0),\mathbf{z}) - L(\mathbf{z},(x,t)) \\ &= L(\mathbf{z},(y,t)) - L(\mathbf{z},(x,t)) \\ &= L(\mathbf{z},(y,t)) - L(\mathbf{0},(x,t)) + L(\mathbf{0},\mathbf{z}) \\ &\leq L(\mathbf{0},(y,t)) - L(\mathbf{0},(x,t)). \end{aligned}$$

□

3.2. **Proof of Theorem 3.** For each $v, t > 0$ let $\alpha = \alpha_v = \arctan(1/v)$, $x = x_t := t/\tan\alpha$, $\mathbf{x}_\alpha := (x, x\tan\alpha)$. By (2.15),

$$\begin{aligned}\{Y^\nu(t) \le x\} &= \{L_{\nu^+}(\mathbf{x}_\alpha) \ge L_{\nu^-}(\mathbf{x}_\alpha)\} \\ &= \Big\{ \sup_{0\le z\le x} \big[\nu^+(z) + L((z,0),\mathbf{x}_\alpha)\big] \ge \sup_{z<0} \big[\nu^-(z) + L((z,0),\mathbf{x}_\alpha)\big] \Big\} \\ &= \Big\{ \sup_{0\le z\le x} \big[\nu^+(z) + L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\big] \\ &\qquad \ge \sup_{z<0} \big[\nu^-(z) + L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\big] \Big\}. \end{aligned} \tag{3.5}$$

The last equality simply follows from subtracting $L(\mathbf{0},\mathbf{x}_\alpha)$ left and right. The reason for doing this is that for any compact set $K \subset \mathbb{R}^2$

$$(L(\mathbf{z},\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\,;\, \mathbf{z}\in K) \xrightarrow{\mathcal{D}} (B_\alpha(\mathbf{0},\mathbf{z})\,;\, \mathbf{z}\in K) \quad \text{as} \quad x\to\infty\,. \tag{3.6}$$

This remark is the core of the proof. It follows from Theorem 2.2 in [7]: we can take $n \ge 1$ big enough, so that $K \subset [-n,n]\times[-n,n]$. The α-rays starting at $(-n,n)$ and $(n,-n)$ will coalesce at some point $\mathbf{c}\in\mathbb{R}^2$. Furthermore, the longest paths to $\mathbf{x}_\alpha$, starting at $(-n,n)$ and $(n,-n)$, will converge to the respective α-rays in a bigger bounded square, containing $\mathbf{c}$. From that time on, for all $\mathbf{z}\in K$, we will have

$$L(\mathbf{z},\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha) = B_\alpha(\mathbf{0},\mathbf{z}).$$

What follows are some technicalities to show that the location of the maximum behaves well.

Lemma 3.2. *If $\sqrt{\tan\alpha}\in(a_\nu,b_\nu)$ (recall (2.16)) then, as $x\to\infty$,*

$$\sup_{0\le z\le x} \big[\nu^+(z) + L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\big] \xrightarrow{\mathcal{D}} \sup_{z\ge 0} \big[\nu^+(z) - \nu_\alpha^+(z)\big]\,,$$

and

$$\sup_{z<0} \big[\nu^-(z) + L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\big] \xrightarrow{\mathcal{D}} \sup_{z<0} \big[\nu^-(z) - \nu_\alpha^-(z)\big]\,.$$

Proof : We will prove that $a_\nu < \sqrt{\tan\alpha}$ implies the first statement. The proof of the second statement follows exactly the same reasoning.

Pick $a < \sqrt{\tan(\alpha)}$ and $C > 0$ such that $\nu^+(x) \le ax + C$ for all $x \ge 0$. For fixed x we know that

$$(L(\mathbf{0},\mathbf{x}_\alpha) - L((z,0),\mathbf{x}_\alpha)\,;\, 0\le z\le x) \overset{\mathcal{D}}{=} (L(\mathbf{0},\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha - (z,0))\,;\, 0\le z\le x)\,. \tag{3.7}$$

Choose α' and a' such that $a < a' = \sqrt{\tan\alpha'} < \sqrt{\tan\alpha}$. Start a stationary Hammersley system with a Poisson process of intensity a', $\bar\nu_{a'}$, using the same Poisson process in the upper half-plane as L. Define as before

$$L_{\bar\nu_{a'}}(x,t) = \sup_{z\le x} \{L((z,0),(x,t)) + \bar\nu_{a'}(z)\}\,.$$

We know that the characteristic direction of this process is α', and from [6] we conclude that the second class particle of this process will be to the right of $\mathbf{x}_\alpha$ with probability greater than $1 - O(x^{-3})$; call this event E_x. Fix $0\le z\le x$. Because of Lemma 3.1, under the event E_x, we have

$$L(\mathbf{0},\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha - (z,0)) \ge L_{\bar\nu_{a'}}(\mathbf{x}_\alpha) - L_{\bar\nu_{a'}}(\mathbf{x}_\alpha - (z,0)).$$

As a process in z, the right hand side is just a Poisson process with intensity a'. This process will be above the line $az + C$ eventually, and therefore be larger than $\nu^+(z)$. Using (3.7), this means that for each $\varepsilon > 0$, we can find $R > 0$ such that for all x large enough,

$$\mathbb{P}\left(\sup_{0\le z\le x}\left[\nu^+(z) + L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\right] > \sup_{0\le z\le R}\left[\nu^+(z) + L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\right]\right) < \varepsilon.$$

Now we can use (3.6) to finish the proof of the first statement of Lemma 3.2. Note that

$$L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha) \to B_\alpha(\mathbf{0},(z,0)) = -\nu_\alpha(z).$$

□

Notice that, since ν_α is a function of the underlying two dimensional Poisson random set $\mathbf{P}$, it is independent of ν. Together with (3.4), (3.5) and Lemma 3.2, this proves the theorem for $\tan\alpha \in (a_\nu, b_\nu)$. Now, if $b_\nu < \sqrt{\tan\alpha}$ then a.s. $\sup_{y<0}\nu^-(y) - \nu_\alpha^-(y) = \infty$. By the same reasoning as in the proof of the last lemma,

$$\sup_{z\le 0}\left[\nu^-(z) + L((z,0),\mathbf{x}_\alpha) - L(\mathbf{0},\mathbf{x}_\alpha)\right] \xrightarrow{\mathcal{D}} \infty\ ,\ \text{as}\ \ x\to\infty\,.$$

Hence, both functions in (2.17) are zero. The case where $\sqrt{\tan\alpha} < a_\nu$ follows from a similar argument.

Remark 3.1. Note that the limiting distribution of $t^{-1}Y^\nu(t)$, for a deterministic ν, may only have an atom at α such that $\sqrt{\tan\alpha} = a_\nu$ or $\sqrt{\tan\alpha} = b_\nu$. If this is the case, then the function (2.17) is not continuous at these choices of α, but it is continuous everywhere else. An example would be if $\nu^+(y) = \lfloor y\rfloor - \lfloor \lfloor y\rfloor^{2/3}\rfloor$. In this case, when we take $\alpha = \pi/4$, which corresponds to $a_\nu = \sqrt{\tan\alpha}$, and we make sure that $a_\nu < b_\nu$, we see that

$$\mathbb{P}\left(\sup_{y\ge 0}\left[\nu^+(y) - \nu_\alpha(y)\right] \ge \sup_{y<0}\left[\nu^-(y) - \nu_\alpha(y)\right]\right) < 1,$$

whereas $\mathbb{P}(\lim_{t\to\infty} Y^\nu(t)/t \le 1/\tan\alpha) = 1$. This means that $\mathbb{P}(\lim_{t\to\infty} Y^\nu(t)/t = 1) > 0$.

3.3. **Busemann functions and equilibrium measures in the TASEP-LPP context.** For the lattice last-passage percolation model with exponential weights, the existence of the Busemann functions was proved in [10, 13]: for $\alpha \in (0,\pi/2)$,

$$\text{a.s.}\ \ \exists\ B_\alpha^\ell(\mathbf{x},\mathbf{y}) := \lim_{|\mathbf{z}_\alpha|\to+\infty} L^\ell(\mathbf{y},\mathbf{z}_\alpha) - L^\ell(\mathbf{x},\mathbf{z}_\alpha)\,.$$

By the invariance under the map $\alpha \to \alpha + \pi$, it is straightforward to see that the analogous result holds for $\alpha \in (\pi, 3\pi/2)$. In this set up one can also define last-passage times L_{ν^ℓ} similar to (2.12), with respect to an initial configuration ν^ℓ defined on $\mathbb{Z}$, and an evolution M similar to (2.13). This evolution will be a Markov process and

$$\nu_\alpha^\ell((x,y]) := B_\alpha^\ell(\mathbf{0},(y,0)) - B_\alpha^\ell(\mathbf{0},(x,0)) \quad \text{for} \quad x<y\,,$$

will be the unique spatially ergodic process with intensity

$$\mathbb{E}\nu_\alpha^\ell(1) = \frac{1}{1+\sqrt{\tan\alpha}}\,(=\rho_\alpha)\,,$$

that is time invariant (Theorem 8.1 of [7]). This allow us to compute the distribution of B^ℓ as follows:

Lemma 3.3. *The Busemann function is additive,*

$$\mathbf{B}^\ell_\alpha(\mathbf{x},\mathbf{y}) = \mathbf{B}^\ell_\alpha(\mathbf{x},\mathbf{z}) + \mathbf{B}^\ell_\alpha(\mathbf{z},\mathbf{y})\,,$$

and anti-symmetric,

$$\mathbf{B}^\ell_\alpha(\mathbf{x},\mathbf{y}) = -\mathbf{B}^\ell_\alpha(\mathbf{y},\mathbf{x})\,.$$

Suppose $\alpha \in (0,\pi/2)$. Define $Z_k = B^\ell_\alpha(\sigma(k-1),\sigma(k))$. Then all Z_k's are independent and

$$Z_k \overset{\mathcal{D}}{=} \begin{cases} -\mathrm{Exp}\,\{\rho_\alpha\} & \text{if } \bar\eta(k)=0 \\ \mathrm{Exp}\,\{1-\rho_\alpha\} & \text{if } \bar\eta(k)=1\,, \end{cases} \tag{3.8}$$

where $\mathrm{Exp}\,\{\rho\}$ *denotes a exponential random variable of intensity ρ.*

Proof : The first two statements are general properties of the Busemann function and follow from Proposition 3.1 of [7]. Property (3.8) follows from Theorem 8.1 of [7] (see also [3]) together with anti-symmetry. □

3.4. **Proof of Theorem 2.** For each $t>0$ let $\varphi^\eta(t)$ be the intersection of the competition interface $\Phi^\eta = (\phi_n)_{n\ge1}$ with the line $\{(x,t) : x \ge 0\}$. For each $\alpha \in (0,\pi/2)$ let $x = x_t := t/\tan\alpha$, $\mathbf{x}_\alpha := (x, x\tan\alpha)$. Then

$$\begin{aligned}
\{\varphi^\eta(t) \le x\} &= \{L^\ell(\sigma^+,\mathbf{x}_\alpha) \ge L^\ell(\sigma^-,\mathbf{x}_\alpha)\} \\
&= \Big\{ \sup_{k>0,\sigma(k)\le\mathbf{x}_\alpha} L^\ell(\sigma(k),\mathbf{x}_\alpha) \ge \sup_{k<0,\sigma(k)\le\mathbf{x}_\alpha} L^\ell(\sigma(k),\mathbf{x}_\alpha)\Big\} \\
&= \Big\{ \sup_{k>0,\sigma(k)\le\mathbf{x}_\alpha} \Big[L^\ell(\sigma(k),\mathbf{x}_\alpha) - L^\ell(\mathbf{0},\mathbf{x}_\alpha)\Big] \\
&\qquad\qquad \ge \sup_{k<0,\sigma(k)\le\mathbf{x}_\alpha} \Big[L^\ell(\sigma(k),\mathbf{x}_\alpha) - L^\ell(\mathbf{0},\mathbf{x}_\alpha)\Big] \Big\}\,.
\end{aligned}$$

We now need to control the location of the maximum on the profile. For this we use the following lemma:

Lemma 3.4. *Suppose $\alpha < \alpha'_\eta$. Almost surely, there exists $M>0$ such that for all $x>0$,*

$$\sup_{k<0} L^\ell(\sigma(k),\mathbf{x}_\alpha) = \sup_{-M\le k<0} L^\ell(\sigma(k),\mathbf{x}_\alpha).$$

Similarly, suppose $\alpha > \alpha_\eta$. Almost surely, there exists $M>0$ such that for all $x>0$,

$$\sup_{k>0} L^\ell(\sigma(k),\mathbf{x}_\alpha) = \sup_{0<k\le M} L^\ell(\sigma(k),\mathbf{x}_\alpha).$$

Proof : This result follows more or less directly from Proposition 3.1 of [12]. That result is stated for profiles that have an asymptotic direction, whereas we only have (2.4). However, the proof in the more general case is exactly the same. The proposition states (translated to our notation), that if α and α' are chosen such that $\tan\alpha > \tan\alpha' > \tan\alpha_\eta$, and $\{\mathbf{z}_i\}$ is a sequence of points going to infinity with asymptotic direction α', then there are only finitely many $\mathbf{z}_i$'s that belong to a longest path from σ_+ to any of the $\mathbf{x}_\alpha$'s. If we take $M_+>0$ such that the x-coordinates of those finitely many $\mathbf{z}_i$'s are smaller than the x-coordinate of $\sigma(M_+)$, then it is clear that all the longest paths from σ_+ to any of the $\mathbf{x}_\alpha$'s must start before $\sigma(M_+)$. We apply Proposition 3.1 of [12] again to the negative part of σ, obtaining $M_->0$. Now put $M = \max\{M_-, M_+\}$. □

The proof now follows as in the Hammersley case, using the convergence to the Busemann function on a compact set. Therefore,

$$\mathbb{P}(\Theta^\eta \leq \alpha) = \mathbb{P}\left(\sup_{k<0} B^\ell_\alpha(\mathbf{0}, \sigma(k)) \geq \sup_{k>0} B^\ell_\alpha(\mathbf{0}, \sigma(k))\right), \tag{3.9}$$

and Theorem 2 is a consequence of Lemma 3.3 together with (3.9), for $\alpha \in (\alpha_\eta, \alpha'_\eta)$.

We need an argument for $\alpha < \alpha_\eta$ (the argument for $\alpha > \alpha'_\eta$ will be analogous). When considering $L^l(\sigma_-, \mathbf{x}_\alpha)$, Lemma 3.4 tells us that almost surely, there exists $M > 0$ such that

$$L^l(\sigma_-, \mathbf{x}_\alpha) \leq L^l((-M, 0), \mathbf{x}_\alpha).$$

Choose $0 < p < p_\eta$ such that

$$\rho_\alpha := \frac{1}{1 + \sqrt{\tan \alpha}} > 1 - p.$$

This is possible, since $\alpha < \alpha_\eta$ implies that $\rho_\alpha > 1 - p_\eta$. Define a sequence $\tilde{\sigma}_+(k) = (k, \lfloor -\lambda_p k \rfloor)$, with $\lambda_p = p/(1-p)$. Because of the rarefaction assumption (2.4), we know that there are infinitely many points on σ_+ below $\tilde{\sigma}_+$. We first show that with probability 1, there exists $k_0 \geq 1$ such that for $k \geq k_0$,

$$B_\alpha((-M, 0), \tilde{\sigma}_+(k)) > 0. \tag{3.10}$$

This follows from the fact that

$$B_\alpha((-M,0),(k,\lfloor -\lambda_p k\rfloor)) \quad = \quad B_\alpha((-M,0),(k,0)) + B_\alpha((k,0),(k,\lfloor -\lambda_p k\rfloor)).$$

From Lemma 3.3, we know that $B_\alpha((-M,0),(k,0))$ is distributed as the sum of $k+M$ independent exponentials with expectation $-1/\rho_\alpha$, whereas $B_\alpha((k,0),(k,\lfloor -\lambda_p k\rfloor)$ is distributed as the sum of $\lceil \lambda_p k \rceil$ independent exponentials with expectation $1/(1-\rho_\alpha)$. Now (3.10) follows for k big enough from the observation that

$$-\frac{1}{\rho_\alpha} + \frac{\lambda_p}{1-\rho_\alpha} > -\frac{1}{1-p} + \frac{\lambda_p}{p} = 0.$$

Now choose $\mathbf{z} \in \sigma_+$ such that $\mathbf{z} \leq \tilde{\sigma}_+(k)$, for some $k \geq k_0$. Then $B_\alpha((-M,0),\mathbf{z}) > 0$. Therefore, for x large enough, we will have that $L^l(\mathbf{z}, \mathbf{x}_\alpha) > L^l((-M,0),\mathbf{x}_\alpha) \geq L^l(\sigma_-, \mathbf{x}_\alpha)$. Since $\mathbf{z} \in \sigma_+$, this means that with probability 1, the angle of the competition interface will eventually be greater than α.

3.5. **Proof of Theorem 1.** Together with Lemma 3.5 below, Theorem 2 implies Theorem 1. This lemma introduces the hole-particle representation of the second class particle to make the link with the competition interface.

Lemma 3.5. *For $u \in (-1, 1)$ let*

$$\alpha_u := \arctan\left(\frac{1-u}{1+u}\right)^2.$$

Then

$$\mathbb{P}\left(U^\eta \geq u\right) = \mathbb{P}\left(\Theta^\eta \leq \alpha_u\right).$$

Proof : We label the particles sequentially from right to left and the holes from left to right, with the convention that the particle at site one and the hole at site zero are both labeled 0. Let $P_j(0)$ and $H_j(0)$, $j \in \mathbb{Z}$ be the positions of the particles and holes respectively at time 0 of the initial configuration $\bar{\eta}$. The position at time t of the jth particle $P_j(t)$ and the ith hole $H_i(t)$ are functions of the variables $L(\sigma, \mathbf{z})$: at time $L(\sigma, (i,j))$, the jth particle and the ith hole interchange positions.

Disregarding labels and defining $\bar{\eta}_t(P_j(t)) = 1$, $\bar{\eta}_t(H_j(t)) = 0$, $j \in \mathbb{Z}$, the process $\bar{\eta}_t$ indeed realizes the exclusion dynamics (see [18]). Now, let $\tau_0 = 0$ and

$$\tau_n := \max\{L^l(\sigma^-, \phi_n), L^l(\sigma^+, \phi_n)\}\,.$$

Define the process $((I^\sigma(t), J^\sigma(t))\,,\, t \geq 0)$ by

$$(I^\sigma(t), J^\sigma(t)) = \phi_n \ \text{ for } t \in [\tau_n, \tau_{n+1})\,.$$

This process is the hole-particle representation of the second class particle. By Proposition 3 of [13],

$$(X^\eta(t)\,,\, t \geq 0) \stackrel{\mathcal{D}}{=} (I^\sigma(t) - J^\sigma(t)\,,\, t \geq 0)\,.$$

By using a similar argument as in the proof of Proposition 5 of [13], one has that

$$\frac{I^\sigma(t) - J^\sigma(t)}{t} \stackrel{\mathcal{D}}{\to} f(\Theta) \ \text{ as } t \to \infty$$

where

$$f(\alpha) := \frac{1 - \sqrt{\tan\alpha}}{1 + \sqrt{\tan\alpha}}\,,$$

which implies Lemma 3.5. □

4. Computing the law of the asymptotic speed in the TASEP

4.1. **Two-corner initial profile.** Choose integers $x, y \geq 1$ and let $\eta = \eta_{x,y}$ be defined as follows:

$$\eta_{x,y}(k) := \begin{cases} 0 & \text{if } -(x-1) \leq k < 0 \ \text{ or } k \geq y \\ 1 & \text{if } 0 < k \leq y-1 \ \text{ or } k \leq -x. \end{cases}$$

For this initial configuration the supremum is attained at deterministic points and it is easy to see that

$$\sup_{n\geq 0} S^{\rho,\eta}_{n,-} \stackrel{\mathcal{D}}{=} \Gamma_{x,\rho} \ \text{ and } \ \sup_{n\geq 0} S^{\rho,\eta}_{n,+} \stackrel{\mathcal{D}}{=} \Gamma_{y,1-\rho}\,,$$

where $\Gamma_{x,\rho}$ and $\Gamma_{y,1-\rho}$ are two independent gamma random variables of parameters (x, ρ) and $(y, 1-\rho)$, respectively. This is because the corresponding profile σ has y steps down starting at $(0, 0)$ (1 from the second class particle in 0 and $y-1$ from η), and it has x steps to the left (1 from the second class particle in 0 and $x-1$ from η). On the other hand,

$$\mathbb{P}\left(\Gamma_{x,\rho} \geq \Gamma_{y,1-\rho}\right) = \frac{1}{\Gamma(x)\Gamma(y)} \int_0^\infty \Gamma\left(x, \frac{\rho}{1-\rho} z\right) z^{y-1} e^{-z} dz\,,$$

where, for $x, z \in [0, \infty)$,

$$\Gamma(x, z) := \int_z^\infty u^{x-1} e^{-u} du = \frac{\Gamma(x)}{e^z} \sum_{j=0}^{x-1} \frac{z^j}{j!} \ \text{ and } \ \Gamma(x) := \Gamma(x, 0) = (x-1)!\,.$$

By calculating the integral, one finds that

$$\mathbb{P}\left(\Gamma_{x,\rho} \geq \Gamma_{y,1-\rho}\right) = \frac{1}{(y-1)!}(1-\rho)^y \sum_{j=0}^{x-1} \frac{(j+y-1)!}{j!} \rho^j\,,$$

which implies that

$$\mathbb{P}\left(U \geq u\right) = \frac{1}{(y-1)!}\left(\frac{1-u}{2}\right)^y \sum_{j=0}^{x-1} \frac{(j+y-1)!}{j!}\left(\frac{1+u}{2}\right)^j ,$$

and that

$$\mathbb{P}\left(\Theta \leq \alpha\right) = \frac{1}{(y-1)!}\left(\frac{\sqrt{\tan\alpha}}{1+\sqrt{\tan\alpha}}\right)^y \sum_{j=0}^{x-1} \frac{(j+y-1)!}{j!}\left(\frac{1}{1+\sqrt{\tan\alpha}}\right)^j .$$

Compare with the results found in [11, 13] for $x = y = 1$. We also remark that, for $x = L$ and $y = 1$, $U \xrightarrow{\mathcal{D}} 1$ $(L \to \infty)$, while for $x = 1$ and $y = L$, $U \xrightarrow{\mathcal{D}} -1$ $(L \to \infty)$. Furthermore, if $x = y = L$, then $U \xrightarrow{\mathcal{D}} 0$ $(L \to \infty)$; this follows immediately from the fact that $\mathbb{E}(\Gamma(L,\rho) - \Gamma(L, 1-\rho)) = O(L)$ if $\rho \neq 1/2$, whereas the variance is also of order L. This answers a question posed in [12] (see the end of Section 2.2 there).

4.2. **Periodic initial configuration.** We consider the following initial configuration η for TASEP: choose $k_+, k_- \geq 1$, and suppose to the right of zero we start with k_+ holes, then one particle, then k_+ holes, etc. So for the configuration $\bar{\eta}$ with a hole-particle pair at zero we have, for $i \geq 1$,

$$\bar{\eta}(i) = \begin{cases} 1 & \text{if } i \bmod (k_+ + 1) = 1, \\ 0 & \text{if } i \bmod (k_+ + 1) \neq 1. \end{cases}$$

To the left of zero, we have a similar, dual pattern: we start with k_- particles, then one hole, then k_- particles, etc. So for $i \leq 0$,

$$\bar{\eta}(i) = \begin{cases} 0 & \text{if } i \bmod (k_- + 1) = 0, \\ 1 & \text{if } i \bmod (k_- + 1) \neq 0. \end{cases}$$

If $\max\{k_+, k_-\} \geq 2$, then we are in the rarefaction regime. The profile σ corresponding to η starts at $(0,0)$ with a jump down, then k_+ jumps to the right, then a jump down, etc. Going to the left, it starts with a jump to the left, then k_- jumps up, then a jump to the left, etc.

The G/M/1 queue. The key to solving the distribution of

$$S_+ := \sup_{n \geq 0} S^{\rho,\eta}_{n,+} \quad \text{and} \quad S_- := \sup_{n \leq -1} S^{\rho,\eta}_{n,-}$$

is the following connection to a G/M/1 queue: suppose the arrival times A_i of the queue have a distribution

$$A_i \sim \text{Exp}\{\rho\} + \ldots + \text{Exp}\{\rho\} \qquad (\text{i.i.d. } k_+ \text{ times}) .$$

The server time B_i is exponential, $B_i \sim \text{Exp}\{1-\rho\}$. If we define $X_i = B_i - A_i$, and $Z_+ \sim \text{Exp}\{1-\rho\}$ independent of the X_i's, then

$$S_+ \sim Z_+ + \max\left\{0\,,\, \sup_{n \geq 2} \sum_{i=1}^{n-1} X_i\right\} .$$

It is a well known fact (see for example Part II of [8]) that the second term of the right-hand side has the same distribution as the waiting time W_∞ of the stationary G/M/1 queue (in our case we are considering a stationary $E^{k^+}/M/1$ queue), which exists if $\mathbb{E}A_i > \mathbb{E}B_i$, or

$$\frac{1-\rho}{\rho} > \frac{1}{k_+} \Leftrightarrow 1 - \rho > \frac{1}{1+k_+} .$$

Note that $1/(1+k_+)$ equals the fraction of particles to the right of zero. The distribution of W_∞ is explicitly known in terms of the Laplace transform of A_i: define

$$\phi(s) = \mathbb{E}\left(e^{-sA_1}\right) = \left(\frac{\rho}{\rho+s}\right)^{k_+} .$$

Now define $\lambda_+ = \lambda_+(\rho)$ as the smallest positive zero of

$$z \mapsto z - \phi((1-\rho)(1-z)) .$$

Then λ_+ is the smallest positive solution of

$$\lambda_+(1-(1-\rho)\lambda_+)^{k_+} = \rho^{k_+} . \tag{4.1}$$

Note that $\lambda_+ \uparrow 1$ as $1-\rho \downarrow 1/(1+k_+)$. Given this constant, we know that for all $s \geq 0$,

$$\mathbb{P}(W_\infty > s) = \lambda_+ e^{-(1-\rho)(1-\lambda_+)s} .$$

It follows by a direct calculation of the Laplace transform that, surprisingly,

$$S_+ \sim \mathrm{Exp}\{(1-\rho)(1-\lambda_+)\} .$$

In the case of S_-, we would have the arrival time

$$A_i \sim \mathrm{Exp}'\{1-\rho\} + \ldots + \mathrm{Exp}'\{1-\rho\} \qquad (\text{i.i.d.}\, k_- \text{ times}) ,$$

and the server time $B_i \sim \mathrm{Exp}\{\rho\}$. This means that for

$$1-\rho < \frac{k_-}{1+k_-}$$

(note that $k_-/(1+k_-)$ is the fraction of particles to the left of zero), we define $\lambda_- = \lambda_-(\rho)$ as the smallest positive solution of

$$\lambda_-(1-\rho\lambda_-)^{k_-} = (1-\rho)^{k_-} , \tag{4.2}$$

and we get that

$$S_- \sim \mathrm{Exp}\{\rho(1-\lambda_-)\} .$$

This easily implies that

$$\mathbb{P}(S_- \geq S_+) = \frac{(1-\rho)(1-\lambda_+)}{\rho(1-\lambda_-)+(1-\rho)(1-\lambda_+)} . \tag{4.3}$$

Therefore

$$\mathbb{P}(U \geq u) = \frac{(1-\rho_u)(1-\lambda_{+,u})}{\rho_u(1-\lambda_{-,u})+(1-\rho_u)(1-\lambda_{+,u})} ,$$

and

$$\mathbb{P}(\Theta \leq \alpha) = \frac{(1-\rho_\alpha)(1-\lambda_{+,\alpha})}{\rho_\alpha(1-\lambda_{-,\alpha})+(1-\rho_\alpha)(1-\lambda_{+,\alpha})} ,$$

(recall (4.1) and (4.2)). We remark that $k_+ = k_- = \infty$ corresponds to $x = y = 1$ in the previous example. In this case, $\lambda_+ = \lambda_- = 0$ and we get $(1-\rho)$ in the right-hand side of (4.3), which agrees with the previous calculation.

4.3. **Bernoulli initial configuration.** Choose $p_1 \in [0,1)$ $p_2 \in (p_1, 1]$, and assume that $\{\eta(k) : k > 0\}$ is a collection of i.i.d. Bernouilli random variables of parameter p_1, while $\{\eta(k) : k < 0\}$, $\eta(k)$ is a collection of i.i.d. Bernouilli random variables of parameter p_2. In this case, we see that the profile σ performs a two-sided random walk in $\mathbb{Z}^2$: when going down-right, σ starts by moving down, and then it has probability p_1 to move down, and probability $1-p_1$ to move to the right. When going left-up, σ starts by moving left, and then it has probability p_2 to move up, and probability $1-p_2$ to move to the left. Note that this is a random initial configuration, independent of evolution of the process. Since with probability 1 our initial configuration will satisfy rarefaction condition (2.4), we can still use our results. We again make the connection to a queue. Using the notation of Section 4.2, when considering S^+, we would get that Z^+ and B_i (the first step and the server time) are distributed as a geometric sum of i.i.d. exponentials (the number of steps between two right-steps, so the expected number of down-steps is $1/(1-p_1)$), which is again exponential:

$$Z_+, B_i \sim \mathrm{Exp}\{(1-p_1)(1-\rho)\}.$$

Furthermore, the arrival time A_i is also a geometric sum of i.i.d. exponentials, so

$$A_i \sim \mathrm{Exp}\{p_1\rho\}.$$

Note that in fact, in this case we have an M/M/1 queue. We need that $1-\rho > p_1$ for S^+ to be finite. It is not hard to repeat the argument of Section 4.2 (other arguments for this classical problem are also possible): we find

$$\lambda_+ = \frac{p_1}{1-p_1}\frac{\rho}{1-\rho} < 1,$$

and from this

$$S^+ \sim \mathrm{Exp}\{1-\rho-p_1\}.$$

For S_- we get the analogous result for $1-\rho < p_2$ by replacing p_1 by $1-p_2$ and ρ by $1-\rho$:

$$S^- \sim \mathrm{Exp}\{p_2-(1-\rho)\}.$$

This means that

$$\mathbb{P}(S^- \geq S^+) = \frac{1-\rho-p_1}{p_2-p_1}.$$

Using that $\rho_u = (1+u)/2$, we find

$$\mathbb{P}\left(U \geq u\right) = \frac{(1-2p_1)-u}{2(p_2-p_1)},$$

which is the uniform distribution on $(1-2p_2, 1-2p_1)$, exactly as was found in [11]. The reader can also check that the above computation leads to the same limit distribution for the angle of the competition interface found in [12].

5. Computing the law of the asymptotic speed in the Hammersley process

5.1. **Periodic initial configuration.** Choose $\lambda > 0$ and $\mu > \lambda$, and assume that $(\nu^+(y)\,;\, y \geq 0)$ and $(\nu^-(y)\,;\, y < 0)$ are deterministic periodic configurations of intensity λ and μ, i.e., ν^+ is concentrated on $\{k/\lambda : k \geq 1\}$ while ν^- is concentrated on $\{k/\mu : k \leq -1\}$. Suppose $\lambda < \rho < \mu$. Theorem 3 then tells us that we should consider

$$\mathbb{P}\left(\sup_{z\geq 0}\{\nu(z)-\bar{\nu}_\rho(z)\} \geq \sup_{z<0}\{\nu(z)-\bar{\nu}_\rho(z)\}\right),$$

where $\bar{\nu}_\rho$ is a Poisson process on $\mathbb{R}$. Define

$$S^+ = \sup_{z\geq 0}\{\nu(z) - \bar{\nu}_\rho(z)\} \quad \text{and} \quad S^- = \sup_{z<0}\{\nu(z) - \bar{\nu}_\rho(z)\}.$$

Clearly, for $z \geq 0$, we have $\nu(z) = \lfloor \lambda z \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. Therefore,

$$\begin{aligned} S^+ &= \sup_{z\geq 0}\{-\bar{\nu}_\rho(z) + \lfloor \lambda z \rfloor\} \\ &= \lfloor \sup_{z\geq 0}\{-\bar{\nu}_\rho(z) + \lambda z\} \rfloor \\ &= -\lceil \inf_{z\geq 0}\{\bar{\nu}_\rho(z) - \lambda z\} \rceil. \end{aligned}$$

This infimum of a Poisson process minus a linear function is studied in [17]. Theorem 3 in [17] entails that for $k \geq 0$

$$\begin{aligned} \mathbb{P}(S^+ \geq k) &= \mathbb{P}(\lceil \inf_{z\geq 0}\{\bar{\nu}_\rho(z) - \lambda z\} \rceil \leq -k) \\ &= \mathbb{P}(\inf_{z\geq 0}\{\bar{\nu}_\rho(z) - \lambda z\} \leq -k) \\ &= (1-p_+)^k, \end{aligned}$$

where $p_+ = p_+(\lambda, \rho) \in (0,1)$ is the positive solution of

$$p_+ = 1 - e^{-p_+\rho/\lambda}.$$

For $z < 0$ we have $\nu(z) = -\lfloor \mu |z| \rfloor$. So when we switch to positive z, we get

$$\begin{aligned} S^- &\stackrel{\mathcal{D}}{=} \sup_{z\geq 0}\{\bar{\nu}_\rho(z) - \lfloor \mu z \rfloor\} \\ &= \lceil \sup_{z\geq 0}\{\bar{\nu}_\rho(z) - \mu z\} \rceil. \end{aligned}$$

Now we can use Equation (7) of [17]: for $k \geq 0$

$$\begin{aligned} \mathbb{P}(S^- \leq k) &= \mathbb{P}(\sup_{z\geq 0}\{\bar{\nu}_\rho(z) - \mu z\} \leq k) \\ &= (1-\rho/\mu)\sum_{i=0}^{k}(-\rho/\mu)^i \frac{(k-i)^i}{i!} e^{\rho(k-i)/\mu}. \end{aligned}$$

In particular, $\mathbb{P}(S^- = 0) = 1 - \rho/\mu$. Therefore,

$$\begin{aligned} \mathbb{P}(S^+ \geq S^-) &= \sum_{k=0}^{\infty} \mathbb{P}(S^- \leq k) p_+ (1-p_+)^k \\ &= p_+(1-\rho/\mu)\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-\rho/\mu)^i \frac{(k-i)^i}{i!} e^{\rho(k-i)/\mu}(1-p_+)^k. \end{aligned}$$

We were not able to simplify this formula significantly. For $\lambda = 1$ and $\mu = 2$, Figure 1 gives the graph of $P(S^+ \geq S^-)$ as a function of $\rho \in (1,2)$. We compare it with the probability $(\mu-\rho)/(\mu-\lambda)$, which we would get if we would take a Poisson process of intensity λ for $x \geq 0$, and of intensity μ for $x \leq 0$ (see Section 5.2).

An interesting limit is $\lambda \to \infty$. This means that $S^+ = 0$, and we get

$$\mathbb{P}(S^+ \geq S_-) = 1 - \rho/\mu.$$

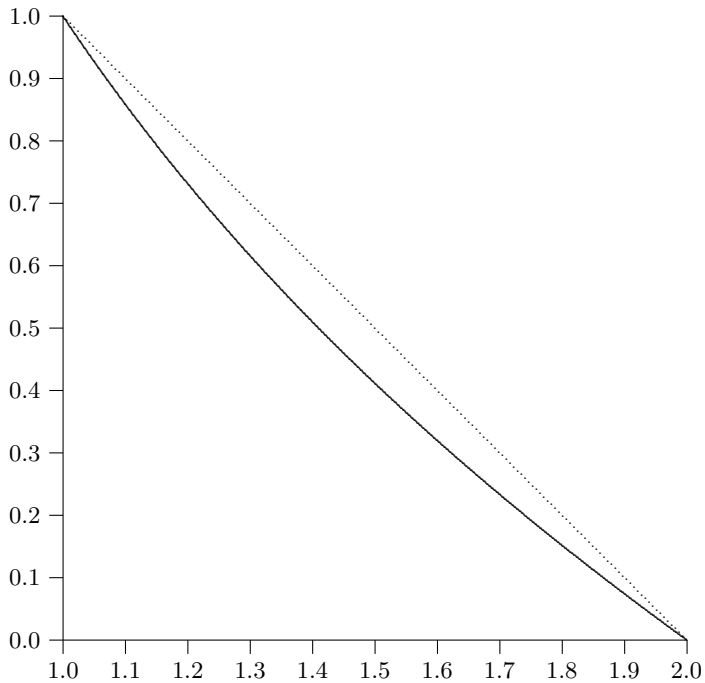

FIGURE 1. Picture of $P(S^+ \geq S^-)$, together with uniform distribution.

This is exactly the same probability as when we would have a Poisson process of intensity μ left of 0, and no particles to the right of 0 (see Section 5.2)!

5.2. **Poisson initial configuration.** Choose $\lambda > 0$ and $\mu > \lambda$, and assume that $(\nu^+(y)\,;\, y \geq 0)$ and $(\nu^-(y)\,;\, y < 0)$ are independent Poisson counting processes of intensity λ and μ, respectively. We will only consider the case $\rho \in (\lambda, \mu)$, the other cases are trivial. Define two asymmetric simple random walks W^+ and W^-, with

$$\mathbb{P}(W^+(n+1) - W^+(n) = +1) = p^+ := \frac{\lambda}{\lambda + \rho}$$

and

$$\mathbb{P}(W^-(n+1) - W^-(n) = +1) = p^- := \frac{\rho}{\mu + \rho}.$$

Since ν^+, ν^+_ρ, ν^- and ν^-_ρ are independent Poisson counting process, it is not hard to see that

$$\sup_{z \geq 0}\{\nu^+(z) - \bar{\nu}^+_\rho(z)\} \stackrel{\mathcal{D}}{=} \sup_{n \geq 0} W^+(n) \quad \text{and that} \quad \sup_{z<0}\{\nu^-(z) - \bar{\nu}^-_\rho(z)\} \stackrel{\mathcal{D}}{=} \sup_{n \geq 0} W^-(n).$$

Furthermore, it is well known (and easy to see) that $S^i := \sup_{n \geq 0} W^i(n) \sim \mathrm{Geo}(r^i)$, where $r^i = p^i/(1-p^i)$, since $p^i < 0.5$ (for $i = +, -$). We find

$$\begin{aligned} \mathbb{P}(S^+ \geq S^-) &= \sum_{k=0}^{\infty} \mathbb{P}(S^+ \geq k)\mathbb{P}(S^- = k) \\ &= \sum_{k=0}^{\infty} (r^+)^k (r^-)^k (1 - r_-) = \frac{1 - r^-}{1 - r^+ r^-} = \frac{\mu - \rho}{\mu - \lambda}\,. \end{aligned}$$

Therefore, by Theorem 3 (and taking $\rho_v = 1\sqrt{v}$), this proves that

$$\mathbb{P}(V^\nu \geq v) = \begin{cases} 0 & \text{if } \ v \geq 1/\lambda^2 \\ \frac{\frac{1}{\sqrt{v}}-\lambda}{\mu-\lambda} & \text{if } \ 1/\mu^2 < v < 1/\lambda^2 \\ 1 & \text{if } \ v \leq 1/\mu^2. \end{cases}$$

This agrees with the results found in [4, 9].

6. Final comments

One interesting aspect of our method is that (2.8) also holds in the lattice last-passage model with general i.i.d. weights, as soon as we can construct Busemann functions. We define the shape function in the lattice last-passage model as

$$f(\mathbf{x}) := \lim_{n\to\infty} \frac{L^l(\mathbf{0}, n\mathbf{x})}{n}.$$

In the exponential case, the limit shape is given by $f(x_1, x_2) = (\sqrt{x_1} + \sqrt{x_2})^2$ [18]. Also in the case of geometric weights, so

$$\mathbb{P}(X_{\mathbf{z}} = k) = (1-p)p^{k-1} \quad (k \geq 1),$$

the shape function is explicitly known, and given by (see Theorem 3A of [19])

$$f(\mathbf{x}) = p^{-1}\left\{x_1 + x_2 + 2[(1-p)x_1x_2]^{1/2}\right\}.$$

With our methods we could determine the distribution of the asymptotic speed of a second class particle. Busemann functions can be constructed under a curvature hypotheses for the respective limit shape [15]. However, to prove that this curvature hypotheses holds in a great generality is by far one the most challenging problem in lattice last- (or first-) passage percolation models.

In the Hammersley last-passage model with random weights [7] (each Poissonian point $\mathbf{x}$ has a weight $X_{\mathbf{x}}$), the limit shape (due to the invariance of the model under volume preserving maps) $f(x,t) = \gamma\sqrt{xt}$ allows us to construct Busemann functions, and then prove (2.17) for the respective second class particle. However, we do not know the equilibrium distribution and so we can not go much further than that.

Part of the results concerning the distributional behavior of second class particles and competition interfaces in the rarefaction regime were already known [4, 9, 11, 12, 13, 14]. The genuine contributions are (2.5), (2.8) and (2.17), and how it can be used to compute the distribution of the asymptotic speed, as soon as we have a good candidate for the equilibrium measure.

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References

[1] Aldous, D. and Diaconis, P. (1995). Hammersley's interacting particle system and longest increasing subsequences. *Probab. Theory Related Fields* Vol. 103, 199–213.

[2] Amir, G., Angel, O. and Valkó, B. (2008). The TASEP speed process. Available from `http://arxiv.org/abs/0811.3706`.

[3] Balazs, M., Cator, E. and Seppalainen, T. (2006). Cube root fluctuations for the corner growth model associated to the exclusion process. *Elect. J. Probabi.* Vol. 11, pp. 1094-1132

[4] Cator, E.A. and Dobrynin, S. (2006). Behavior of a second class particle in Hammersley's process. *Elect. J. Probab.* Vol. 11, pp. 670-685.

[5] Cator, E. A., Groeneboom, P. (2005). Hammersley's process with sources and sinks. *Ann. Probab.* Vol. 33, No. 3, 879-903

[6] Cator, E. A., Groeneboom, P. (2006). Second class particles and cube root asymptotics for Hammersley's process. *Ann. Probab.* Vol. 34, pp. 1273-1295.

[7] Cator, E. A. and Pimentel, L.P.R. (2009). Busemann functions and equilibrium measures in last-passage percolation. To appear in *Probab. Theory Related Fields.* Available from Probability Theory and Related Fields Online First (Open Acces): `http://www.springerlink.com/content/t276w266k8805303/`.

[8] Cohen, J.W. (1969). The Single Server Queue. North-Holland Publishing Company, Amsterdam.

[9] Coletti, C.F. and Pimentel, L.P.R. (2007). On the collision between two PNG droplets. *J. Stat. Phisycs* Vol 126, pp. 1145-1164.

[10] Coupier, D. (2011). Multiple geodesics with the same direction. Available from `http://arxiv.org/abs/1104.1321`.

[11] Ferrari, P.A. and Kipnis, C. (1995). Second class particles in the rarefaction fan. *Ann. Inst. H. Poincaré Probab. Statist.* Vol. 31, pp. 143-154.

[12] Ferrari, P.A., Martin, J.B. and Pimentel, L.P.R. (2009). A phase transition for competition interfaces. *Ann. Appl. Probab.* Vol. 19, pp. 281-317.

[13] Ferrari, P.A. and Pimentel, L.P.R. (2005). Competition interfaces and second class particles. *Ann. Probab.* Vol. 33, pp. 1235-1254.

[14] Guiol, H. and Mountford (2005). The motion of a second class particle for the TASEP starting from a decreasing shock profile. *Ann. Appl. Probab.* Vol. 15, pp. 1227-1259.

[15] Newman, C. M., (1995) A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, 1017–1023. Birkhäuser, Basel.

[16] Pimentel, L.P.R. (2007). A multi-type shape theorem for FPP models. *Adv. Appl. Probab.* Vol. 39, pp. 53-76.

[17] Pyke, R. (1959). The Supremum and Infimum of the Poisson Process. *Ann. Math. Statist.* Vol. 30, pp. 568-576.

[18] Rost, H. (1981). Non equilibrium behavior of a many particle process: Density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete* Vol. 58, pp. 41-53.

[19] Seppäläinen, T. (1998). Hydrodynamic scaling, convex duality and asymptotic shapes of growth models. *Markov Process. Related Fields* Vol. 4, no. 1, pp. 1-26.

[20] Wüthrich, M., (2002). Asymptotic behavior of semi-infinite geodesics for maximal increasing subsequences in the plane. In V. Sidoravicius, ed., *In and Out of Equilibrium*, 205–226. Birkhäuser, Basel.

Delft Institute of Applied Mathematics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

E-mail address: `E.A.Cator@tudelft.nl`

Institute of Mathematics, Federal University of Rio de Janeiro, Caixa Postal 68530, Rio de Janeiro, RJ, 21941-909, Brazil

E-mail address: `leandro@im.ufrj.br`