On the chaotic character of the stochastic heat equation, before the onset of intermitttency*

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Abstract

We consider a nonlinear stochastic heat equation $\partial_t u = \frac{1}{2}\partial_{xx} u + \sigma(u)\partial_{xt} W$, where $\partial_{xt} W$ denotes space-time white noise and $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous. We establish that, at every fixed time $t > 0$, the global behavior of the solution depends in a critical manner on the structure of the initial function u_0: Under suitable technical conditions on u_0 and σ, $\sup_{x\in\mathbf{R}} u_t(x)$ is a.s. finite when u_0 has compact support, whereas with probability one, $\limsup_{|x|\to\infty} u_t(x)/(\log|x|)^{1/6} > 0$ when u_0 is bounded uniformly away from zero. The mentioned sensitivity to the initial data of the stochastic heat equation is a way to state that the solution to the stochastic heat equation is *chaotic* at fixed times, well before the onset of intermittency.

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1 Introduction and main results

Let $\boldsymbol{W} := \{W(t\,,x)\}_{t\geqslant 0, x\in\mathbf{R}}$ denote a real-valued Brownian sheet indexed by two parameters $(t\,,x) \in \mathbf{R}_+ \times \mathbf{R}$. That is, $\boldsymbol{W}$ is a centered Gaussian process with covariance

$$\operatorname{Cov}\left(W(t\,,x)\ ,\ W(s\,,y)\right) = \min(t\,,s) \times \min(|x|\,,|y|) \times \mathbf{1}_{(0,\infty)}(xy). \tag{1.1}$$

And let us consider the nonlinear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} u_t(x) + \sigma(u_t(x))\frac{\partial^2}{\partial t\,\partial x} W(t\,,x), \tag{1.2}$$

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where $x \in \mathbf{R}$, $t > 0$, $\sigma : \mathbf{R} \to \mathbf{R}$ is a nonrandom and Lipschitz continuous function, $\kappa > 0$ is a fixed viscosity parameter, and the initial function $u_0 : \mathbf{R} \to \mathbf{R}$ is bounded, nonrandom, and measurable. The mixed partial derivative $\partial^2 W(t\,,x)/(\partial t\,\partial x)$ is the so-called "space-time white noise," and is defined as a generalized Gaussian random field; see Chapter 2 of Gelfand and Vilenkin [17, §2.7], for example.

It is well known that the stochastic heat equation (1.2) has a [weak] solution $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ that is jointly continuous; it is also unique up to evanescence; see, for example, Chapter 3 of Walsh [24, (3.5), p 312]. And the solution can be written in mild form as the [a.s.] solution to the following stochastic integral equation:

$$u_t(x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\,W(\mathrm{d}s\,\mathrm{d}y), \tag{1.3}$$

where

$$p_t(z) := \frac{1}{(2\pi\kappa t)^{1/2}} \exp\left(-\frac{z^2}{2\kappa t}\right) \qquad (t > 0,\ z \in \mathbf{R}) \tag{1.4}$$

denotes the free-space heat kernel, and the final integral in (1.3) is a stochastic integral in the sense of Walsh [24, Chapter 2]. Chapter 1 of the minicourse by Dalang, Khoshnevisan, Mueller, Nualart, and Xiao [11] contains a quick introduction to the topic of stochastic PDEs of the type considered here.

We are interested solely in the physically interesting case that $u_0(x) \geqslant 0$ for all $x \in \mathbf{R}$. In that case, a minor variation of Mueller's comparison principle [21] implies that if in addition $\sigma(0) = 0$, then with probability one $u_t(x) \geqslant 0$ for all $t > 0$ and $x \in \mathbf{R}$; see also Theorem 5.1 of Dalang et al [11, p. 130], as well as Theorem 2.1 below.

We follow Foondun and Khoshnevisan [16], and say that the solution $u := \{u_t(x)\}_{t>0,x\in\mathbf{R}}$ to (1.2) is [weakly] *intermittent* if

$$0 < \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}\left(|u_t(x)|^\nu\right) < \infty \qquad \text{for all } \nu \geqslant 2, \tag{1.5}$$

where "log" denotes the natural logarithm, to be concrete. Here, we refer to the property (1.5), if and when it holds, as *mathematical intermittency* [to be distinguished from *physical intermittency*, which is a phenomenological property of an object that (1.2) is modeling].

If $\sigma(u) = \text{const} \cdot u$ and u_0 is bounded from above and below uniformly, then the work of Bertini and Cancrini [2] and Mueller's comparison principle [21] together imply (1.5). In the fully nonlinear case, Foondun and Khoshnevisan [16] discuss a connection to nonlinear renewal theory, and use that connection to establish (1.5) under various conditions; for instance, they have shown that (1.5) holds provided that $\liminf_{|x|\to\infty} |\sigma(x)/x| > 0$ and $\inf_{x\in\mathbf{R}} u_0(x)$ is sufficiently large.

If the lim sup in (1.5) is a bona fide limit, then we arrive at the usual description of intermittency in the literature of mathematics and theoretical physics;

see for instance Molchanov [20] and Zeldovich, Molchanov, Ruzmaikin, and Sokolov [25, 26].

Mathematical intermittency is motivated strongly by a vast physics literature on [physical] intermittency and *localization*, and many of the references can be found in the combined bibliographies of [16, 20, 25, 26]. Let us say a few more words about "localization" in the present context.

It is generally accepted that if (1.5) holds, then $u := \{u_t(x)\}_{t>0,x\in\mathbf{R}}$ ought to undergo a separation of scales [or "pattern/period breaking"]. In fact, one can argue that the property (1.5) implies that, as $t \to \infty$, the random function $x \mapsto u_t(x)$ starts to develop very tall peaks, distributed over small x-intervals [see §2.4 of Bertini and Cancrini [2] and the Introduction of the monograph by Carmona and Molchanov [6].] This "peaking property" is called *localization*, and is experienced with very high probability, provided that: (i) the intermittency property (1.5) holds; and (ii) $t \gg 1$.

Physical intermittency is expected to hold both in space and time, and not only when $t \gg 1$. And it is also expected that there are physically-intermittent processes, not unlike those studied in the present paper, which however do not satisfy the [mathematical] intermittency condition (1.5) on Liapounov exponents; see, for example, the paper by Cherktov, Falkovich, Kolokolov, and Lebedev [8].

Our wish is to better understand "physical intermittency" in the setting of the stochastic heat equation (1.2). We are motivated strongly by the literature on smooth finite-dimensional dynamical systems [23, §1.3] which ascribes intermittency in part to "chaos," or slightly more precisely, sensitive dependence on the initial state of the system.

In order to describe the contributions of this paper, we first recall a consequence of a more general theorem of Foondun and Khoshnevisan [15]: If $\sigma(0) = 0$, and if u_0 is Hölder continuous of index $> \frac{1}{2}$ and has compact support, then for every $t > 0$ fixed,

$$\limsup_{z\to\infty} \frac{1}{\log z} \log \mathrm{P}\left\{\sup_{x\in\mathbf{R}} u_t(x) > z\right\} = -\infty. \tag{1.6}$$

It follows in particular that the global maximum of the solution [at a fixed time] is a finite [nonnegative] random variable.

By contrast, one expects that if

$$\inf_{x\in\mathbf{R}} u_0(x) > 0, \tag{1.7}$$

then the solution u_t is unbounded for all $t > 0$. Here we prove that fact and a good deal more; namely, we demonstrate here that there in fact exists a minimum rate of "blowup" that applies regardless of the parameters of the problem.

A careful statement requires a technical condition that turns out to be necessary as well as sufficient. In order to discover that condition, let us consider the case that $u_0(x) \equiv \rho > 0$ is a constant for all $x \in \mathbf{R}$. Then, (1.7) clearly

holds; but there can be no blowup if $\sigma(\rho) = 0$. Indeed, in that case the unique solution to the stochastic heat equation is $u_t(x) \equiv \rho$, which is bounded. Thus, in order to have an unbounded solution, we need at the very least to consider the case that $\sigma(x) \neq 0$ for all $x > 0$. [Note that $\sigma(0) = 0$ is permitted.] Instead, we will assume the following seemingly stronger, but in fact more or less equivalent, condition from now on:

$$\sigma(x) > 0 \quad \text{for all } x \in \mathbf{R} \setminus \{0\}. \tag{1.8}$$

We are ready to present the first theorem of this paper. Here and throughout we write "$f(R) \gtrsim g(R)$ as $R \to \infty$" in place of the more cumbersome "there exists a nonrandom $C > 0$ such that $\liminf_{R\to\infty} f(R)/g(R) \geqslant C$." The largest such C is called "the constant in $\lesssim$." We might sometimes also write "$g(R) \lesssim f(R)$" in place of "$f(R) \gtrsim g(R)$." And there is a corresponding "constant in $\gtrsim$."

Theorem 1.1. *The following hold:*

1. *If $\inf_{x\in\mathbf{R}} \sigma(x) \geqslant \epsilon_0 > 0$ and $t > 0$, then a.s.:*

$$\sup_{x\in[-R,R]} u_t(x) \gtrsim \frac{(\log R)^{1/6}}{\kappa^{1/12}} \qquad \text{as } R \to \infty; \tag{1.9}$$

   and the constant in $\gtrsim$ does not depend on κ.

2. *If $\sigma(x) > 0$ for all $x \in \mathbf{R}$ and there exists $\gamma \in (0\,,1/6)$ such that*

$$\lim_{|x|\to\infty} \sigma(x) \log(|x|)^{(1/6)-\gamma} = \infty, \tag{1.10}$$

   then for all $t > 0$ the following holds almost surely:

$$\sup_{x\in[-R,R]} |u_t(x)| \gtrsim \frac{(\log R)^{\gamma}}{\kappa^{1/12}} \qquad \text{as } R \to \infty; \tag{1.11}$$

   and the constant in $\gtrsim$ does not depend on κ.

Note in particular that if σ is uniformly bounded below then a.s.,

$$\limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/6}} \geqslant \frac{\text{const}}{\kappa^{1/12}}. \tag{1.12}$$

We believe that it is a somewhat significant fact that a rate $(\log R)^{1/6}$ of blowup exists that is valid for all u_0 and σ in the first part of Theorem 1.1. However, the actual numerical estimate—i.e., the (1/6)th power of the logarithm—appears to be less significant, as the behavior in κ might suggest (see Remark 1.5). In fact, we believe that the actual blowup rate might depend critically on the fine properties of the function σ. Next, we highlight this assertion in one particularly interesting case. Here and throughout, we write "$f(R) \asymp g(R)$ as $R \to \infty$" as shorthand for "$f(R) \gtrsim g(R)$ and $g(R) \gtrsim f(R)$ as $R \to \infty$." The two constants in the preceding two $\gtrsim$'s are called the "constants in $\asymp$."

Theorem 1.2. *If σ is uniformly bounded away from 0 and ∞ and $t > 0$, then*

$$\sup_{x\in[-R,R]} u_t(x) \asymp \frac{(\log R)^{1/2}}{\kappa^{1/4}} \qquad \text{a.s. as } R \to \infty, \tag{1.13}$$

and the constants in $\asymp$ do not depend on κ.

In particular, we find that if σ is bounded uniformly away from 0 and ∞ then there exist constants $c_*, c^* \in \infty(0\,,\infty)$ such that

$$\frac{c_*}{\kappa^{1/4}} \leqslant \limsup_{|x|\to\infty} \frac{u_t(x)}{(\log|x|)^{1/2}} \leqslant \frac{c^*}{\kappa^{1/4}} \qquad \text{a.s.} \tag{1.14}$$

The preceding discusses the behavior in case σ is bounded uniformly away from 0; that is, a uniformly-noisy stochastic heat equation (1.2). In general, we can say little about the remaining case that $\sigma(0) = 0$. Nevertheless in the well-known parabolic Anderson model, namely (1.2) with $\sigma(x) = cx$ for some constant $c > 0$, we are able to obtain some results (Theorem 1.3) that parallel Theorems 1.1 and 1.2.

Theorem 1.3. *If u solves (1.2) with $\sigma(x) = cx$ for some $c > 0$, then a.s.,*

$$\log \sup_{x\in[-R,R]} u_t(x) \asymp \frac{(\log R)^{2/3}}{\kappa^{1/3}} \quad \text{as } R \to \infty, \tag{1.15}$$

and the constants in $\asymp$ do not depend on κ.

Hence, when $\sigma(x) = cx$ we can find constants $C_*, C^* \in (0\,,\infty)$ such that

$$0 < \limsup_{|x|\to\infty} \frac{u_t(x)}{\exp\left\{C_*(\log|x|)^{2/3}/\kappa^{1/3}\right\}} \leqslant \limsup_{|x|\to\infty} \frac{u_t(x)}{\exp\left\{C^*(\log|x|)^{2/3}/\kappa^{1/3}\right\}} < \infty,$$

almost surely.

Remark 1.4. Thanks to (1.3), and since Walsh stochastic integrals have zero mean, it follows that $\mathrm{E}u_t(x) = (p_t * u_0)(x)$. In particular, $\mathrm{E}u_t(x) \leqslant \sup_{x\in\mathbf{R}} u_0(x)$ is uniformly bounded. Since $u_t(x)$ is nonnegative, it follows from Fatou's lemma that $\liminf_{|x|\to\infty} u_t(x) < \infty$ a.s. Thus, the behavior described by Theorem 1.1 is one about the highly-oscillatory nature of $x \mapsto u_t(x)$, valid for every fixed time $t > 0$. We will say a little more about this topic in Appendix B below. □

Remark 1.5. We pay some attention to the powers of the viscosity parameter κ in Theorems 1.2 and 1.3. Those powers suggest that at least two distinct universality classes can be associated to (1.2): (i) When σ is bounded uniformly away from zero and infinity, the solution behaves as random walk in weakly-interacting random environment; and (ii) When $\sigma(x) = cx$ for some $c > 0$, then the solution behaves as objects that arise in some random matrix models.

Remark 1.6. In [18, (2)], M. Kardar, G. Parisi, and Y.-C. Zhang consider the solution u to (1.2) and apply formally the Hopf–Cole transformation $u_t(x) := \exp(\lambda h_t(x))$ to deduce that $\boldsymbol{h} := \{h_t(x)\}_{t\geqslant 0, x\in\mathbf{R}}$ satisfies the following "SPDE": For $t > 0$ and $x \in \mathbf{R}$,

$$\frac{\partial}{\partial t} h_t(x) = \frac{\kappa}{2}\frac{\partial^2}{\partial x^2} h_t(x) + \frac{\kappa\lambda}{2}\left(\frac{\partial}{\partial x} h_t(x)\right)^2 + \frac{\partial^2}{\partial t\,\partial x} W(t\,,x). \tag{1.16}$$

This is the celebrated "KPZ equation," named after the authors of [18], and the random field $\boldsymbol{h}$ is believed to be a universal object [e.g., it is expected to arise as a continuum limit of a large number of interacting particle systems]. Theorem 1.3 implies that

$$0 < \limsup_{|x|\to\infty} \frac{h_t(x)}{(\log|x|)^{2/3}} < \infty \qquad \text{a.s. for all } t > 0. \tag{1.17}$$

This is purely formal, but only because the construction of $\boldsymbol{h}$ via u is not rigorous. More significantly, our proofs suggest strongly a kind of asymptotic space-time scaling "$|\log x| \approx t^{\pm 1/2}$." If so, then the preceding verifies that fluctuation exponent $1/z$ of $\boldsymbol{h}$ is $2/3$ under quite general conditions on the h_0. The latter has been predicted by Kardar et al [18, p. 890] and proved by M. Balazs, J. Quastel, and T. Seppäläinen [1] for a special choice of u_0 [hence h_0]. □

The proofs of our three theorems involve a fairly long series of technical computations. Therefore, we conclude the Introduction with a few remarks on the methods of proofs for the preceding three theorems in order to highlight the "pictures behind the proofs."

Theorem 1.1 relies on two well-established techniques from interacting particle systems [14, 22]: Namely, comparison and coupling. Comparison reduces our problem to the case that u_0 is a constant; at a technical level this uses Mueller's comparison principle [21]. And we use coupling on a few occasions: First, we describe a two-step coupling of $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ to the solution $\{v_t(x)\}_{t>0,x\in\mathbf{R}}$ of (1.2)—using the same space-time white noise $\partial^2 W/(\partial t\,\partial x)$—in the case that σ is bounded below uniformly on $\mathbf{R}$. The latter quantity [i.e., $\{v_t(x)\}_{t>0,x\in\mathbf{R}}$] turns out to be more amenable to moment analysis than $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$, and in this way we obtain the following *a priori* estimate, valid for every $t > 0$ fixed:

$$\log \inf_{x\in\mathbf{R}} \mathrm{P}\left\{u_t(x) \geqslant \lambda\right\} \gtrsim -\sqrt{\kappa}\,\lambda^6 \qquad \text{as } \lambda \to \infty. \tag{1.18}$$

Theorem 1.1 follows immediately from this and the Borel–Cantelli lemma, provided that we prove that if x and x' are "$O(1)$ distance apart," then $u_t(x)$ and $u_t(x')$ are "approximately independent." A quantitative version of this statement follows from coupling $\{u_t(x)\}_{t>0,x\in\mathbf{R}}$ to the solution $\{w_t(x)\}_{t>0,x\in\mathbf{R}}$ of a random evolution equation that can be thought of as the "localization" of the original stochastic heat equation (1.2). The localized approximation $\{w_t(x)\}_{t>0,x\in\mathbf{R}}$ has the property that $w_t(x)$ and $w_t(x')$ are [exactly] independent for "most" values of x and x' that are $O(1)$ distance apart. And this turns out to be adequate for our needs.

Theorem 1.2 requires establishing separately a lower and an upper bound on $\sup_{x\in[-R,R]} u_t(x)$. Both bounds rely heavily on the following quantitative improvement of (1.18): If σ is bounded, then

$$\log \inf_{x\in\mathbf{R}} \mathrm{P}\left\{u_t(x) \geqslant \lambda\right\} \gtrsim -\sqrt{\kappa}\,\lambda^2 \qquad \text{as } \lambda \to \infty. \tag{1.19}$$

And, as it turns out, the preceding lower bound will per force imply a corresponding upper estimate,

$$\log \inf_{x\in\mathbf{R}} \mathrm{P}\left\{u_t(x) \geqslant \lambda\right\} \lesssim -\sqrt{\kappa}\,\lambda^2 \qquad \text{as } \lambda \to \infty. \tag{1.20}$$

The derivation of the lower bound on $\sup_{x\in[-R,R]} u_t(x)$ follows closely the proof of Theorem 1.1, after (1.19) and (1.20) are established. Therefore, the remaining details will be omitted.

The upper bound on $\sup_{x\in[-R,R]} u_t(x)$ requires only (1.20) and a well-known quantitative version of the Kolmogorov continuity theorem.

Our proof of Theorem 1.3 has a similar flavor to that of Theorem 1.1, for the lower bound, and Theorem 1.2, for the upper bound. We make strong use of the moments formulas of Bertini and Cancrini [2, Theorem 2.6]. [This is why we are only able to study the linear equation in the case that $\sigma(0) = 0$.]

Throughout this paper, we use the following abbreviation:

$$u_t^*(R) := \sup_{x\in[-R,R]} u_t(x) \qquad (R > 0). \tag{1.21}$$

We will also need the following elementary facts about the heat kernel:

$$\|p_s\|^2_{L^2(\mathbf{R})} = (4\pi\kappa s)^{-1/2} \qquad \text{for every } s > 0; \tag{1.22}$$

and therefore,

$$\int_0^t \|p_s\|^2_{L^2(\mathbf{R})}\,\mathrm{d}s = \sqrt{t/(\pi\kappa)} \qquad \text{for all } t \geqslant 0. \tag{1.23}$$

We will tacitly write Lip_σ for the optimal Lipschitz constant of σ; that is,

$$\mathrm{Lip}_\sigma := \sup_{-\infty<x\neq x'<\infty} \left|\frac{\sigma(x)-\sigma(x')}{x-x'}\right|. \tag{1.24}$$

Of course, Lip_σ is finite because σ is Lipschitz continuous. Finally, we use the following notation for the $L^\nu(\mathrm{P})$ norm of a random variable $Z \in L^\nu(\mathrm{P})$:

$$\|Z\|_\nu := \left\{\mathrm{E}\left(|Z|^\nu\right)\right\}^{1/\nu}. \tag{1.25}$$

2 Mueller's comparison principle, and a reduction

Mueller's comparison principle [21] is one of the cornerstones of the theory of stochastic PDEs. In its original form, Mueller's comparison principle is stated

for an equation that is similar to (1.2), but for two differences: (i) $\sigma(z) := \kappa z$ for some $\kappa > 0$, and (ii) The variable x takes values in a compact interval such as $[0\,,1]$. In his Utah Minicourse [11, Theorem 5.1, p. 130], C. Mueller outlines how one can include also the more general functions σ of the type studied here. And in both cases, the proofs assume that the initial function u_0 has compact support. Below we state and prove a small variation of the preceding comparison principles that shows that Mueller's theory continues to work when: (i) The variable x takes values in $\mathbf{R}$; and (ii) The initial function u_0 is not necessarily compactly supported.

Theorem 2.1 (Mueller's comparison principle)**.** *Let $u_0^{(1)}$ and $u_0^{(2)}$ denote two nonnegative bounded continuous functions on $\mathbf{R}$ such that $u_0^{(1)}(x) \geqslant u_0^{(2)}(x)$ for all $x \in \mathbf{R}$. Let $u_t^{(1)}(x)$, $u_t^{(2)}(x)$ be solutions to* (1.2) *with respective initial functions $u_0^{(1)}$ and $u_0^{(2)}$. Then,*

$$\mathrm{P}\left\{u_t^{(1)}(x) \geqslant u_t^{(2)}(x) \quad \textit{for all } t > 0 \textit{ and } x \in \mathbf{R}\right\} = 1. \tag{2.1}$$

Proof. Because the solution to (1.2) is continuous in $(t\,,x)$, it suffices to prove that

$$\mathrm{P}\left\{u_t^{(1)}(x) \geqslant u_t^{(2)}(x)\right\} = 1 \quad \text{for all } t > 0 \text{ and } x \in \mathbf{R}. \tag{2.2}$$

In the case that $u_0^{(1)}$ and $u_0^{(2)}$ both have bounded support, the preceding is proved almost exactly as in Theorem 3.1 of Mueller [21]. For general $u_0^{(1)}$ and $u_0^{(2)}$, we proceed as follows.

Let $v_0 : \mathbf{R} \to \mathbf{R}_+$ be a bounded and measurable initial function, and define a new initial function $v_0^{[N]} : \mathbf{R} \to \mathbf{R}_+$ as

$$v_0^{[N]}(x) := \begin{cases} v_0(x) & \text{if } |x| \leqslant N, \\ v_0(N)(-x+N+1) & \text{if } N < x < N+1, \\ v_0(-N)(x+N+1) & \text{if } -(N+1) < x < -N, \\ 0 & \text{if } |x| \geqslant N+1. \end{cases} \tag{2.3}$$

Then, let $v_t^{[N]}(x)$ be the solution to (1.2) with initial condition $v_0^{[N]}$. We claim that

$$\delta_t^{[N]}(x) := v_t(x) - v_t^{[N]}(x) \to 0 \qquad \text{in probability as } N \to \infty. \tag{2.4}$$

Let $u_t^{(1),[N]}$ and $u_t^{(2),[N]}$ denote the solutions to (1.2) with initial conditions $u_0^{(1),[N]}$ and $u_0^{(2),[N]}$ respectively, where the latter are defined similarly as $v_0^{[N]}$ above. Now, (2.4) has the desired result because it shows that $u_t^{(1),[N]}(x) \to u_t^{(1)}(x)$ and $u_t^{(2),[N]}(x) \to u_t^{(2)}(x)$ in probability as $N \to \infty$. Since $u_t^{(1),[N]}(x) \geqslant u_t^{(2),[N]}(x)$ a.s. for all $t > 0$ and $x \in \mathbf{R}$, (2.2) follows from taking limits.

In order to conclude we establish (2.4); in fact, we will prove that

$$\sup_{x\in\mathbf{R}} \sup_{t\in(0,T)} \mathrm{E}\left(|\delta_t^{[N]}(x)|^2\right) = O(1/N) \qquad \text{as } N \to \infty, \tag{2.5}$$

for all $T > 0$ fixed. Recall that

$$\delta_t^{[N]}(x) = \left(p_t * \delta_0^{[N]}\right)(x) + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\left\{\sigma\left(v_s(y)\right) - \sigma\left(v_s^{[N]}(y)\right)\right\} W(\mathrm{d}s\,\mathrm{d}y). \tag{2.6}$$

Because $(p_t * \delta_0^{[N]})(x) \leqslant 2\left(\sup_{z\in\mathbf{R}} v_0(z)\right)\int_{|y|>N} p_t(y)\,\mathrm{d}y$, a direct estimate of the latter stochastic integral yields

$$\begin{aligned}
\mathrm{E}\left(\left|\delta_t^{[N]}(x)\right|^2\right) &\leqslant \mathrm{const}\cdot t^{-1/2}\mathrm{e}^{-N^2/(2t)} \\
&\quad + \mathrm{const}\cdot \mathrm{Lip}_\sigma^2 \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; p_{t-s}^2(y-x)\mathrm{E}\left(\left|\delta_s^{[N]}(y)\right|^2\right) \\
&\leqslant \mathrm{const}\cdot t^{-1/2}\mathrm{e}^{-N^2/(2t)} \\
&\quad + \mathrm{const}\cdot \mathrm{Lip}_\sigma^2 \mathrm{e}^{\beta t} M_t^{[N]}(\beta)\int_0^\infty \mathrm{e}^{-\beta r}\|p_r\|_{L^2(\mathbf{R})}^2\,\mathrm{d}r,
\end{aligned} \tag{2.7}$$

where $\beta > 0$ is, for the moment, arbitrary and

$$M_t^{[N]}(\beta) := \sup_{s\in(0,t),y\in\mathbf{R}}\left[\mathrm{e}^{-\beta s}\mathrm{E}\left(\left|v_s(y) - v_s^{[N]}(y)\right|^2\right)\right]. \tag{2.8}$$

We multiply both sides of (2.7) by $\exp(-\beta t)$ and take the supremum over all $t \in (0\,,T)$ where $T > 0$ is fixed. An application of (1.22) yields

$$M_T^{[N]}(\beta) \leqslant \mathrm{const}\cdot\left[\sup_{t\in(0,T)}\left\{t^{-1/2}\mathrm{e}^{-N^2/(2t)}\right\} + \beta^{-1/2}M_T^{[N]}(\beta)\right]. \tag{2.9}$$

The quantity in $\sup_{t\in(0,T)}\{\cdots\}$ is proportional to $1/N$ [with the constant of proportionality depending on T], and the implied constant does not depend on β. Therefore, it follows that if β were selected sufficiently large, then $M_T^{[N]}(\beta) = O(1/N)$ as $N \to \infty$ for that choice of β. This implies (2.4). □

Next we apply Mueller's comparison principle to make a helpful simplification to our problem.

Because $\underline{B} := \inf_{x\in\mathbf{R}} u_0(x) > 0$ and $\overline{B} := \sup_{x\in\mathbf{R}} u_0(x) < \infty$, it follows from Theorem 2.1 that almost surely,

$$\underline{u}_t(x) \leqslant u_t(x) \leqslant \overline{u}_t(x) \qquad \text{for all } t > 0 \text{ and } x \in \mathbf{R}, \tag{2.10}$$

where $\overline{u}$ solves the stochastic heat equation (1.2) starting from initial function $\overline{u}_0(x) := \overline{B}$, and $\underline{u}$ solves (1.2) starting from $\underline{u}_0(x) := \underline{B}$. This shows that it suffices to prove Theorems 1.1, 1.2, and 1.3 with $u_t(x)$ everywhere replaced by $\underline{u}_t(x)$ and $\overline{u}_t(x)$. In other words, we can assume without loss of generality that u_0 is identically a constant. In order to simplify the notation, we will assume

from now on that the mentioned constant is one. A quick inspection of the ensuing proofs reveals that this assumption is harmless. Thus, from now on, we consider in place of (1.2), the following parabolic stochastic PDE:

$$\left|\begin{array}{l} \dfrac{\partial}{\partial t}u_t(x) = \dfrac{\kappa}{2}\dfrac{\partial^2}{\partial x^2}u_t(x) + \sigma(u_t(x))\,\dfrac{\partial^2}{\partial t\partial x}W(t\,,x) \qquad (t>0, x\in\mathbf{R}), \\ u_0(x) = 1. \end{array}\right. \tag{2.11}$$

We can write its solution in mild form as follows:

$$u_t(x) = 1 + \int_{[0,t]\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\,W(\mathrm{d}s\,\mathrm{d}y). \tag{2.12}$$

3 Tail probability estimates

In this section we derive the following corollary which estimates the tails of the distribution of $u_t(x)$, where $u_t(x)$ solves (2.11) and (2.12). In fact, Corollary 3.5, Proposition 3.7 and Proposition 3.8 below imply the following readily:

Corollary 3.1. *If* $\inf_{z\in\mathbf{R}}[\sigma(z)] > 0$*, then for all* $t>0$*,*

$$-\sqrt{\kappa}\,\lambda^6 \precsim \log \mathrm{P}\left\{|u_t(x)| \geqslant \lambda\right\} \precsim -\sqrt{\kappa}\,(\log\lambda)^{3/2}, \tag{3.1}$$

uniformly for $x\in\mathbf{R}$ *and* $\lambda > \mathrm{e}$*. And the constants in* $\precsim$ *do not depend on* κ*.*

If (1.10) *holds for some* $\gamma\in(0\,,1/6)$*, then for all* $t>0$*,*

$$-\kappa^{1/(12\gamma)}\,\lambda^{1/\gamma} \precsim \log \mathrm{P}\left\{|u_t(x)| \geqslant \lambda\right\} \precsim -\sqrt{\kappa}\,(\log\lambda)^{3/2}, \tag{3.2}$$

uniformly for all $x\in\mathbf{R}$ *and* $\lambda > \mathrm{e}$*. And the constants in* $\precsim$ *do not depend on* κ*.*

3.1 An upper-tail estimate

We begin by working toward the upper bound in Corollary 3.1.

Lemma 3.2. *Choose and fix* $T>0$*, and define* $a := T(\mathrm{Lip}_\sigma \vee 1)^4/(2\kappa)$*. Then, for all real numbers* $k\geqslant 1$*,*

$$\sup_{x\in\mathbf{R}}\sup_{t\in[0,T]} \mathrm{E}\left(|u_t(x)|^k\right) \leqslant C^k \mathrm{e}^{ak^3}, \textit{ where } C := 8\left(1 + \frac{|\sigma(0)|}{2^{1/4}(\mathrm{Lip}_\sigma\vee 1)}\right).$$

Proof. We follow closely the proof of Theorem 2.1 of [16], but matters simplify considerably in the present, more specialized, setting.

First of all, we note that because $u_0 \equiv 1$ is spatially homogeneous, the distribution of $u_t(x)$ does not depend on x; this property was observed earlier by Dalang [10] for example.

Therefore, an application of Burkholder's inequality, using the Carlen–Kree bound [7] on Davis's optimal constant [12] in the Burkholder–Davis–Gundy

inequality [3–5] and Minkowski's inequality imply the following: For all $t \geqslant 0$, $\beta > 0$, and $x \in \mathbf{R}$,

$$\begin{aligned}
&\|u_t(x)\|_k \\
&\leqslant 1 + \left\| \int_{[0,t]\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y) \right\|_k \\
&\leqslant 1 + \mathrm{e}^{\beta t} 2\sqrt{k} \left(|\sigma(0)| + \mathrm{Lip}_\sigma \sup_{r\geqslant 0} \left[\mathrm{e}^{-2\beta r} \|u_r(x)\|_k \right] \right) \left(\int_0^\infty \mathrm{e}^{-2\beta s} \|p_s\|_2^2 \,\mathrm{d}s \right)^{1/2} \\
&= 1 + \frac{\sqrt{k}\mathrm{e}^{\beta t}}{(8\kappa\beta)^{1/4}} \left(|\sigma(0)| + \mathrm{Lip}_\sigma \sup_{r\geqslant 0} \left[\mathrm{e}^{-2\beta r} \|u_r(x)\|_k \right] \right), \qquad (3.3)
\end{aligned}$$

See Foondun and Khoshnevisan [16, Lemma 3.3] for the details of the derivation of such an estimate. [Although Lemma 3.3 of [16] is stated for even integers $k \geqslant 2$, a simple variation on the proof of that lemma implies the result for general $k \geqslant 1$; see Conus and Khoshnevisan [9].] It follows that

$$\psi(\beta\,, k) := \sup_{t\geqslant 0} \left[\mathrm{e}^{-\beta t} \|u_t(x)\|_k \right] \qquad (3.4)$$

satisfies

$$\psi(\beta\,, k) \leqslant 1 + \frac{\sqrt{k}}{(4\kappa\beta)^{1/4}} \left(|\sigma(0)| + \mathrm{Lip}_\sigma \psi(\beta\,, k) \right). \qquad (3.5)$$

If $\mathrm{Lip}_\sigma = 0$, then clearly $\psi(\beta\,, k) < \infty$. If $\mathrm{Lip}_\sigma > 0$, then $\psi(\beta\,, k) < \infty$ for all $\beta > k^2 \mathrm{Lip}_\sigma^4/(4\kappa)$; therefore, the preceding proves that if $\beta > k^2 \mathrm{Lip}_\sigma^4/(4\kappa)$ then

$$\psi(\beta\,, k) \leqslant \frac{1}{1 - (\sqrt{k}\mathrm{Lip}_\sigma/(4\kappa\beta)^{1/4})} \cdot \left(1 + \frac{\sqrt{k}|\sigma(0)|}{(4\kappa\beta)^{1/4}} \right). \qquad (3.6)$$

We apply this with $\beta := k^2(\mathrm{Lip}_\sigma \vee 1)^4/(2\kappa)$ to obtain the lemma. □

Remark 3.3. In the preceding results, the term $\mathrm{Lip}_\sigma \vee 1$ appears in place of the more natural quantity Lip_σ only because it can happen that $\mathrm{Lip}_\sigma = 0$. In the latter case, σ is a constant function, and the machinery of Lemma 3.2 is not needed since $u_t(x)$ is a centered Gaussian process with a variance that can be estimated readily. [We remind the reader that the case where σ is a constant is covered by Theorem 1.2; see Section 6.] □

Next we describe a real-variable lemma that shows how to transform the moment estimate of Lemma 3.2 into subexponential moment estimates.

Lemma 3.4. *Suppose X is a nonnegative random variable that satisfies the following: There exist finite numbers $a, C > 0$ and $b > 1$ such that*

$$\mathrm{E}(X^k) \leqslant C^k \mathrm{e}^{ak^b} \qquad \textit{for all real numbers } k \geqslant 1. \qquad (3.7)$$

Then, $\mathrm{E}\exp\{\alpha(\log_+ X)^{b/(b-1)}\} < \infty$ *—for* $\log_+ u := \log(u \vee \mathrm{e})$ *—provided that*

$$0 < \alpha < \frac{1 - b^{-1}}{(ab)^{1/(b-1)}}. \qquad (3.8)$$

Lemma 3.2, Lemma 3.4 and Chebyshev's inequality together imply the following result.

Corollary 3.5. *Choose and fix $T > 0$, and define $c_0 := \sqrt{6}/3 \approx 0.81650$. Then for all $\alpha < c_0\sqrt{\kappa}/(\sqrt{T}\big(\mathrm{Lip}_\sigma \vee 1\big))$,*

$$\sup_{x\in\mathbf{R}} \sup_{t\in[0,T]} \mathrm{E}\left(\mathrm{e}^{\alpha(\log_+ u_t(x))^{3/2}}\right) < \infty. \tag{3.9}$$

Consequently,

$$\limsup_{\lambda\uparrow\infty} \frac{1}{(\log\lambda)^{3/2}} \sup_{x\in\mathbf{R}} \sup_{t\in[0,T]} \log \mathrm{P}\left\{u_t(x) > \lambda\right\} \leqslant \frac{-c_0\sqrt{\kappa}}{\sqrt{T}(\mathrm{Lip}_\sigma \vee 1)}. \tag{3.10}$$

We skip the derivation of Corollary 3.5 from Lemma 3.4, as it is immediate. The result holds uniformly in $t \in [0,T]$ and $x \in \mathbf{R}$ as the constants a and C in Lemma 3.2 are independent of t and x. Instead we verify Lemma 3.4.

Proof of Lemma 3.4. Because

$$\left[\log_+\left(\frac{X}{C}\right)\right]^{b/(b-1)} \leqslant 2^{b/(b-1)} \cdot \left\{(\log_+ X)^{b/(b-1)} + (\log_+ C)^{b/(b-1)}\right\}, \tag{3.11}$$

we can assume without loss of generality that $C = 1$; for otherwise we may consider X/C in place of X from here on.

For all $z > \mathrm{e}$, Chebyshev's inequality implies that

$$\mathrm{P}\left\{\mathrm{e}^{\alpha(\log_+ X)^{b/(b-1)}} > z\right\} \leqslant \mathrm{e}^{-\max_k g(k)}, \tag{3.12}$$

where

$$g(k) := k\left(\frac{\log z}{\alpha}\right)^{(b-1)/b} - ak^b. \tag{3.13}$$

One can check directly that $\max_k g(k) = c\log z$, where

$$c := \frac{1 - b^{-1}}{\alpha\cdot(ab)^{1/(b-1)}}. \tag{3.14}$$

Thus, it follows that $\mathrm{P}\{\exp[\alpha(\log_+ X)^{b/(b-1)}] > z\} = O(z^{-c})$ for $z \to \infty$. Consequently, $\mathrm{E}\exp\{\alpha(\log_+ X)^{b/(b-1)}\} < \infty$ as long as $c > 1$; this is equivalent to the statement of the lemma. □

3.2 Lower-tail estimates

In this section we proceed to estimate the tail of the distribution of $u_t(x)$ from below. We first consider the simplest case in which σ is bounded uniformly from below, away from zero.

Proposition 3.6. *If $\epsilon_0 := \inf_{z\in\mathbf{R}} \sigma(z) > 0$, then for all $t > 0$,*

$$\inf_{x\in\mathbf{R}} \mathrm{E}\left(|u_t(x)|^{2k}\right) \geqslant \left(\sqrt{2} + o(1)\right)(\mu_t k)^k \qquad (\text{as } k \to \infty), \tag{3.15}$$

where the "$o(1)$" term depends only on k, and

$$\mu_t := \frac{2}{\mathrm{e}} \cdot \epsilon_0^2 \sqrt{\frac{t}{\pi\kappa}}. \tag{3.16}$$

Proof. Because the initial function in (2.11) is $u_0(x) \equiv 1$, it follows that the distribution of $u_t(x)$ does not depend on x (see Dalang [10]). Therefore, the "inf" in the statement of the Proposition is superfluous.

Throughout, let us fix $x \in \mathbf{R}$ and $t > 0$. Now we may consider a mean-one martingale $\{M_\tau\}_{0\leqslant\tau\leqslant t}$ defined as follows:

$$M_\tau := 1 + \int_{(0,\tau)\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y) \qquad (0 \leqslant \tau \leqslant t). \tag{3.17}$$

The quadratic variation of this martingale is

$$\langle M\rangle_\tau = \int_0^\tau \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; p_{t-s}^2(y-x)\sigma^2(u_s(y)) \qquad (0 \leqslant \tau \leqslant t). \tag{3.18}$$

Therefore, by Itô's formula, for all positive integers k, and for every $\tau \in [0\,,t]$,

$$\begin{aligned} M_\tau^{2k} &= 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2}\int_0^\tau M_s^{2(k-1)}\,\mathrm{d}\langle M\rangle_s \\ &= 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s \\ &\qquad + \binom{2k}{2}\int_0^\tau M_s^{2(k-1)}\mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; p_{t-s}^2(y-x)\sigma^2(u_s(y)). \end{aligned} \tag{3.19}$$

By the assumption of the lemma, $\sigma(u_s(y)) \geqslant \epsilon_0$ a.s. Therefore,

$$\begin{aligned} M_\tau^{2k} &\geqslant 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2}\epsilon_0^2\int_0^\tau M_s^{2(k-1)} \cdot \|p_{t-s}\|^2_{L^2(\mathbf{R})}\,\mathrm{d}s \\ &= 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2}\epsilon_0^2\int_0^\tau \frac{M_s^{2(k-1)}}{(4\pi\kappa(t-s))^{1/2}}\,\mathrm{d}s. \end{aligned} \tag{3.20}$$

We set $\tau := t$ and then take expectations to find that

$$\begin{aligned} \mathrm{E}\left(M_t^{2k}\right) &\geqslant 1 + \binom{2k}{2}\epsilon_0^2\int_0^t \mathrm{E}\left(M_s^{2(k-1)}\right)\frac{\mathrm{d}s}{(4\pi\kappa(t-s))^{1/2}} \\ &= 1 + \binom{2k}{2}\epsilon_0^2\int_0^t \mathrm{E}\left(M_s^{2(k-1)}\right)\nu(t\,,\mathrm{d}s), \end{aligned} \tag{3.21}$$

where the measures $\{\nu(t\,,\cdot)\}_{t>0}$ are defined as

$$\nu(t\,,\mathrm{d}s) := \frac{\mathbf{1}_{(0,t)}(s)}{(4\pi\kappa(t-s))^{1/2}}\,\mathrm{d}s. \tag{3.22}$$

We may iterate the preceding in order to obtain

$$\begin{aligned} &\mathrm{E}\left(M_t^{2k}\right) \\ &\geqslant 1 + \sum_{l=0}^{k-1} a_{l,k}\epsilon_0^{2(l+1)} \cdot \int_0^t \nu(t\,,\mathrm{d}s_1)\int_0^{s_1}\nu(s_1\,,\mathrm{d}s_2)\cdots\int_0^{s_l}\nu(s_l\,,\mathrm{d}s_{l+1}), \end{aligned} \tag{3.23}$$

where

$$a_{l,k} := \prod_{j=0}^{l}\binom{2k-2j}{2} \qquad \text{for } 0 \leqslant l < k \tag{3.24}$$

and $s_0 := t$. The right-hand side of (3.23) is exactly equal to $\mathrm{E}(M_t^{2k})$ in the case where $\sigma(z) \equiv \epsilon_0$ for all $z \in \mathbf{R}$. Indeed, the same computation as above works with identities all the way through. In other words,

$$\mathrm{E}\left(|u_t(x)|^{2k}\right) = \mathrm{E}\left(M_t^{2k}\right) \geqslant \mathrm{E}\left(\eta_t(x)^{2k}\right), \tag{3.25}$$

where

$$\eta_t(x) := 1 + \epsilon_0 \cdot \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,W(\mathrm{d}s\,\mathrm{d}y). \tag{3.26}$$

We define

$$\zeta_t(x) := \epsilon_0 \cdot \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\,W(\mathrm{d}s\,\mathrm{d}y), \tag{3.27}$$

so that $\eta_t(x) = 1 + \zeta_t(x)$. Clearly,

$$\mathrm{E}\left(\eta_t(x)^{2k}\right) \geqslant \mathrm{E}\left(\zeta_t(x)^{2k}\right). \tag{3.28}$$

Since ζ is a centered Gaussian process,

$$\mathrm{E}\left(\zeta_t(x)^{2k}\right) = \left[\mathrm{E}\left(\zeta_t(x)^2\right)\right]^k \cdot \frac{(2k)!}{k!\cdot 2^k} \tag{3.29}$$

and

$$\mathrm{E}\left(\zeta_t(x)^2\right) = \epsilon_0^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; p_{t-s}^2(y-x) = \epsilon_0^2 \cdot \sqrt{\frac{t}{\pi\kappa}}; \tag{3.30}$$

see (1.23). The proposition follows from these observations and Stirling's formula. □

We can now use Proposition 3.6 to obtain a lower estimate on the tail of the distribution of $u_t(x)$.

Proposition 3.7. *If there exists $\epsilon_0 > 0$ such that $\sigma(x) \geqslant \epsilon_0$ for all $x \in \mathbf{R}$, then there exists a universal constant $C \in (0\,,\infty)$ such that for all $t > 0$,*

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda^6} \inf_{x\in\mathbf{R}} \log \mathrm{P}\left\{|u_t(x)| \geqslant \lambda\right\} \geqslant -C\frac{(\mathrm{Lip}_\sigma \vee 1)^4\sqrt{\kappa}}{\epsilon_0^6\sqrt{t}}. \tag{3.31}$$

Proof. Choose and fix $t > 0$ and $x \in \mathbf{R}$. We apply the celebrated Paley–Zygmund inequality in the following form: For every integer $k \geqslant 1$,

$$\begin{aligned}
&\mathrm{E}\left(|u_t(x)|^{2k}\right)\\
&\quad\leqslant \mathrm{E}\left(|u_t(x)|^{2k}\,;\,|u_t(x)| \geqslant \frac{1}{2}\|u_t(x)\|_{2k}\right) + \frac{1}{2}\mathrm{E}\left(|u_t(x)|^{2k}\right)\\
&\quad\leqslant \sqrt{\mathrm{E}\left(|u_t(x)|^{4k}\right)\mathrm{P}\left\{|u_t(x)| \geqslant \frac{1}{2}\|u_t(x)\|_{2k}\right\}} + \frac{1}{2}\mathrm{E}\left(|u_t(x)|^{2k}\right).
\end{aligned} \tag{3.32}$$

This yields the following bound:

$$\begin{aligned}
\mathrm{P}\left\{|u_t(x)| \geqslant \frac{1}{2}\|u_t(x)\|_{2k}\right\} &\geqslant \frac{\left[\mathrm{E}\left(|u_t(x)|^{2k}\right)\right]^2}{4\mathrm{E}\left(|u_t(x)|^{4k}\right)}\\
&\geqslant \exp\left(-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\kappa}\,k^3(1+o(1))\right),
\end{aligned} \tag{3.33}$$

as $k \to \infty$; see Lemma 3.2 and Proposition 3.6. Another application of Proposition 3.6 shows that $\|u_t(x)\|_{2k} \geqslant (1+o(1))\,(\mu_t k)^{1/2}$ as $k \to \infty$ where μ_t is defined in (3.16). This implies as $k \to \infty$,

$$\mathrm{P}\left\{|u_t(x)| \geqslant \frac{1}{2}(\mu_t k)^{1/2}\right\} \geqslant \exp\left[-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\kappa}\,k^3(1+o(1))\right]. \tag{3.34}$$

The proposition follows from this by setting k to be the smallest possible integer that satisfies $(\mu_t k)^{1/2} \geqslant \lambda$. □

Now, we study the tails of the distribution of $u_t(x)$ under the conditions of part 2 of Theorem 1.1.

Proposition 3.8. *Suppose $\sigma(x) > 0$ for all $x \in \mathbf{R}$ and (1.10) holds for some $\gamma \in (0\,,1/6)$. Then*

$$\liminf_{\lambda\to\infty} \inf_{x\in\mathbf{R}} \frac{\log \mathrm{P}\left\{|u_t(x)| > \lambda\right\}}{\lambda^{1/\gamma}} \geqslant -C\left(\frac{(\mathrm{Lip}_\sigma \vee 1)^{2/3}\kappa^{1/12}}{t^{1/12}}\right)^{1/\gamma}, \tag{3.35}$$

where $C \in (0\,,\infty)$ is a constant that depends only on γ.

Proof. For every integer $N \geqslant 1$ define

$$\sigma^{(N)}(x) := \begin{cases} \sigma(x) & \text{if } |x| \leqslant N,\\ \sigma(-N) & \text{if } x < -N,\\ \sigma(N) & \text{if } x > N. \end{cases} \tag{3.36}$$

It can be checked directly that $\sigma^{(N)}$ is a Lipschitz function, and that in fact: (i) $\mathrm{Lip}_{\sigma^{(N)}} \leqslant \mathrm{Lip}_\sigma$; and (ii) and $\inf_{z\in\mathbf{R}} \sigma^{(N)}(z) > 0$.

Let $u_t^{(N)}(x)$ denote the solution to (2.11), when σ is replaced by $\sigma^{(N)}$. We first establish the bound

$$\mathrm{E}\left(\left|u_t^{(N)}(x) - u_t(x)\right|^2\right) = O\left(N^{-2}\right) \qquad \text{as } N \to \infty. \tag{3.37}$$

Let us observe, using the mild representation of the solution to (2.11), that

$$\mathrm{E}\left(\left|u_t^{(N)}(x) - u_t(x)\right|^2\right) \leqslant 2(\mathcal{T}_1 + \mathcal{T}_2), \tag{3.38}$$

where

$$\begin{aligned}\mathcal{T}_1 :=& \mathrm{E}\left(\left|\int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\left[\sigma^{(N)}\left(u_s^{(N)}(y)\right) - \sigma\left(u_s^{(N)}(y)\right)\right] W(\mathrm{d}s\,\mathrm{d}y)\right|^2\right)\\ =& \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; p_{t-s}^2(y-x)\mathrm{E}\left(\left|\sigma^{(N)}\left(u_s^{(N)}(y)\right) - \sigma\left(u_s^{(N)}(y)\right)\right|^2\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_2 :=& \mathrm{E}\left(\left|\int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\left[\sigma\left(u_s^{(N)}(y)\right) - \sigma\left(u_s(y)\right)\right] W(\mathrm{d}s\,\mathrm{d}y)\right|^2\right)\\ =& \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; p_{t-s}^2(y-x)\mathrm{E}\left(\left|\sigma\left(u_s^{(N)}(y)\right) - \sigma\left(u_s(y)\right)\right|^2\right).\end{aligned} \tag{3.39}$$

We can estimate the integrand of $\mathcal{T}_1$ by the following:

$$\begin{aligned}&\mathrm{E}\left(\left|\sigma^{(N)}\left(u_s^{(N)}(y)\right) - \sigma\left(u_s^{(N)}(y)\right)\right|^2\right)\\ &\qquad\leqslant \mathrm{Lip}_\sigma^2 \cdot \mathrm{E}\left(\left|N - u_s^{(N)}(y)\right|^2;\; u_s^{(N)}(y) > N\right)\\ &\qquad\qquad + \mathrm{Lip}_\sigma^2 \cdot \mathrm{E}\left(\left|-N - u_s^{(N)}(y)\right|^2;\; u_s^{(N)}(y) < -N\right)\\ &\qquad\leqslant 4\mathrm{Lip}_\sigma^2 \cdot \mathrm{E}\left(\left|u_s^{(N)}(y)\right|^2;\; \left|u_s^{(N)}(y)\right| > N\right).\end{aligned} \tag{3.40}$$

We first apply the Cauchy–Schwarz inequality and then Chebyshev's inequality [in this order] to conclude that

$$\begin{aligned}\mathrm{E}\left(\left|\sigma^{(N)}\left(u_s^{(N)}(y)\right) - \sigma\left(u_s^{(N)}(y)\right)\right|^2\right) &\leqslant \frac{4\mathrm{Lip}_\sigma^2}{N^2} \cdot \mathrm{E}\left(\left|u_s^{(N)}(y)\right|^4\right)\\ &= O(N^{-2}) \qquad \text{as } N \to \infty,\end{aligned} \tag{3.41}$$

uniformly for all $y \in \mathbf{R}$ and $s \in (0\,,t)$. Indeed, Lemma 3.2 ensures that $\mathrm{E}[|u_s^{(N)}(y)|^4]$ is bounded in N, because $\lim_{N\to\infty} \mathrm{Lip}_{\sigma^{(N)}} = \mathrm{Lip}_\sigma$. This implies readily that $\mathcal{T}_1 = O(N^{-2})$ as $N \to \infty$.

Next we turn to $\mathcal{T}_2$; thanks to (1.23), the quantity $\mathcal{T}_2$ can be estimated as follows:

$$\begin{aligned}\mathcal{T}_2 &\leqslant \mathrm{Lip}_\sigma^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\; p_{t-s}^2(y-x)\mathrm{E}\left(\left|u_s^{(N)}(y) - u_s(y)\right|^2\right) \\ &\leqslant \mathrm{const} \cdot \int_0^t \frac{\mathcal{M}(s)}{\sqrt{t-s}},\end{aligned} \tag{3.42}$$

where

$$\mathcal{M}(s) := \sup_{y\in\mathbf{R}} \mathrm{E}\left(\left|u_s^{(N)}(y) - u_s(y)\right|^2\right) \qquad (0 \leqslant s \leqslant t). \tag{3.43}$$

Notice that the implied constant in (3.42) does not depend on t.

We now combine our estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ to conclude that

$$\begin{aligned}\mathcal{M}(s) &\leqslant \frac{\mathrm{const}}{N^2} + \mathrm{const} \cdot \int_0^s \frac{\mathcal{M}(r)}{\sqrt{s-r}}\,\mathrm{d}r \qquad (0 \leqslant s \leqslant t) \\ &\leqslant \mathrm{const} \cdot \left\{\left(\int_0^s [\mathcal{M}(r)]^{3/2}\,\mathrm{d}r\right)^{2/3} + \frac{1}{N^2}\right\},\end{aligned} \tag{3.44}$$

thanks to Hölder's inequality. We emphasize that the implied constant depends only on the Lipschitz constant of σ, the variable t and the diffusion constant κ. Therefore,

$$[\mathcal{M}(s)]^{3/2} \leqslant \mathrm{const} \cdot \left\{\int_0^s [\mathcal{M}(r)]^{3/2}\,\mathrm{d}r + \frac{1}{N^3}\right\}, \tag{3.45}$$

uniformly for $s \in (0\,,t)$. Gronwall's inequality then implies the bound $\mathcal{M}(t) = O(N^{-2})$, valid as $N \to \infty$.

Now we proceed with the proof of Proposition 3.8. For all $N \geqslant 1$, the function $\sigma^{(N)}$ is bounded below. Let $\epsilon(N)$ be such that $\sigma^{(N)}(x) \geqslant \epsilon(N)$ for all $x \in \mathbf{R}$. Let $D := D_t := (4t/(\mathrm{e}^2\pi\kappa))^{1/4}$. According to the proof of Proposition 3.7, specifically (3.34) applied to $u^{(N)}$, we have

$$\begin{aligned}\mathrm{P}\left\{|u_t(x)| \geqslant \frac{D}{4}\epsilon(N)k^{1/2}\right\} &\geqslant \exp\left[-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\kappa}\,k^3(1+o(1))\right] \\ &\quad - \mathrm{P}\left\{\left|u_t(x) - u_t^{(N)}(x)\right| \geqslant \frac{D}{4}\epsilon(N)k^{1/2}\right\}.\end{aligned} \tag{3.46}$$

Thanks to (1.10), we can write

$$\epsilon(N) \gg (\log N)^{-(1/6-\gamma)} \qquad \text{as } N \to \infty, \tag{3.47}$$

using standard notation. Therefore, if we choose

$$N := \left\lfloor \exp\left\{\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4 k^3}{\kappa}\right\}\right\rfloor, \tag{3.48}$$

then we are led to the bound

$$\epsilon(N) \gg \left(\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\kappa}\right)^{-(1/6-\gamma)} k^{3\gamma-(1/2)}. \tag{3.49}$$

We can use Chebyshev's inequality in order to estimate the second term on the right-hand side of (3.46). In this way we obtain the following:

$$\begin{aligned} \mathrm{P}\left\{|u_t(x)| \geqslant \frac{\widetilde{D}}{4}k^{3\gamma}\right\} \geqslant \exp\left\{-\frac{64t(\mathrm{Lip}_\sigma \vee 1)^4}{\kappa}\, k^3(1+o(1))\right\} \\ - \frac{1}{C_1 N^2 k^{6\gamma}}, \end{aligned} \tag{3.50}$$

where

$$\widetilde{D} := D\left\{\frac{\kappa}{64t(\mathrm{Lip}_\sigma \vee 1)^4}\right\}^{(1/6)-\gamma}, \tag{3.51}$$

and C_1 is a constant that depends only on t, Lip_σ, and κ. For all sufficiently large integers N,

$$\frac{1}{C_1 N^2 k^{6\gamma}} \leqslant \exp\left[-\frac{128t(\mathrm{Lip}_\sigma \vee 1)^4}{\kappa} k^3(1+o(1))\right], \tag{3.52}$$

and the proposition follows upon setting $\lambda := \widetilde{D}k^{3\gamma}/4$. □

4 Localization

The next step in the proof of Theorem 1.1 requires us to show that if x and x' are $O(1)$ apart, then $u_t(x)$ and $u_t(x')$ are approximately independent. We show this by coupling $u_t(x)$ to the solution of a localized version—see (4.2) below—of the stochastic heat equation (2.11).

For every $\beta, t > 0$ we define $\mathcal{D}_t^{(\beta)}$ to be the collection of all half-open, half-closed intervals of the form

$$\Big[(j-1)\beta\sqrt{t}\ ,\ j\beta\sqrt{t}\Big) \qquad (j \in \mathbf{Z}). \tag{4.1}$$

In this way we find that for every $x \in \mathbf{R}$ there exists a unique interval $\mathcal{I}_t^{(\beta)}(x) \in \mathcal{D}_t^{(\beta)}$ which contains x. Now consider the following parametric family of random evolution equations [indexed by the parameter $\beta > 0$]:

$$U_t^{(\beta)}(x) = 1 + \int_{(0,t)\times\mathcal{I}_t^{(\beta)}(x)} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right) W(\mathrm{d}s\,\mathrm{d}y), \tag{4.2}$$

subject to $U_0^{(\beta)}(x) = 1$ for all $x \in \mathbf{R}$ and $t > 0$.

Lemma 4.1. *Choose and fix $\beta > 0$. Then, (4.2) has an almost surely unique solution $U^{(\beta)}$ such that for all $T > 0$ and $k \geqslant 1$,*

$$\sup_{\beta>0} \sup_{t\in[0,T]} \sup_{x\in\mathbf{R}} \mathrm{E}\left(|U_t^{(\beta)}(x)|^k\right) \leqslant C^k \mathrm{e}^{ak^3}, \tag{4.3}$$

where a and C are defined in Lemma 3.2.

Proof. A fixed-point argument shows that there exists a unique, up to modification, solution to (4.2) subject to the condition that for all $T > 0$,

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{R}} \mathrm{E}\left(|U_t^{(\beta)}(x)|^k\right) < \infty \qquad \text{for all } k \geqslant 1. \tag{4.4}$$

See Foondun and Khoshnevisan [16] for more details on the ideas of the proof; and the moment estimate follows as in the proof of Lemma 3.2. We omit the numerous remaining details. □

Lemma 4.2. *For every $T > 0$ there exists a finite and positive constant C such that for sufficiently large $\beta > 0$ and $k \geqslant 1$,*

$$\sup_{t\in[0,T]} \sup_{j\in\mathbf{Z}} \mathrm{E}\left(\left|u_t(x_j) - U_t^{(\beta)}(x_j)\right|^k\right) \leqslant C^k k^{k/2} \mathrm{e}^{Fk^3} \beta^{-k/2} \tag{4.5}$$

where $F \in (0\,,\infty)$ depends on T but not on $(k\,,\beta)$, and

$$x_j := x_j(\beta\,,t) := \left(j - \frac{1}{2}\right)\beta\sqrt{t} \qquad (j \in \mathbf{Z}). \tag{4.6}$$

is the middle point of the interval $[(j-1)\beta\sqrt{t}\,, j\beta\sqrt{t}) \in \mathcal{D}_t^{(\beta)}$.

Proof. For all $x \in \mathbf{R}$ and $t > 0$ define

$$V_t(x) := 1 + \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right) W(\mathrm{d}s\,\mathrm{d}y). \tag{4.7}$$

Then,

$$\begin{aligned} &\left\|V_t(x) - U_t^{(\beta)}(x)\right\|_k \\ &\qquad = \left\|\int_{(0,t)\times\left[\mathcal{I}_t^{(\beta)}(x)\right]^c} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right) W(\mathrm{d}s\,\mathrm{d}y)\right\|_k. \end{aligned} \tag{4.8}$$

For each $\beta > 4$ we define the set

$$\mathbf{A}_\beta := \bigcup_{j\in\mathbf{Z}} \left[(j-1)\beta\sqrt{t} + \beta^{1/2}\sqrt{t}\,, j\beta\sqrt{t} - \beta^{1/2}\sqrt{t}\right]. \tag{4.9}$$

Every $\mathbf{A}_\beta$ includes the points $(x_j)_{j\in\mathbf{N}}$, and $\|y-x\| \geqslant \beta^{1/2}\sqrt{t}$ for all $x \in \mathbf{A}_\beta$ and $y \notin \mathcal{I}_t^{(\beta)}(x)$. Consequently, for every $x \in \mathbf{A}_\beta$,

$$\begin{aligned}&\left\|V_t(x) - U_t^{(\beta)}(x)\right\|_k \\ &\qquad \leqslant 2\sqrt{k}\left\|\int_0^t \mathrm{d}s \int_{|y-x|\geqslant\beta^{1/2}\sqrt{t}} \mathrm{d}y\; p_{t-s}^2(y-x)\sigma^2\left(U_s^{(\beta)}(y)\right)\right\|_{k/2}^{1/2}.\end{aligned} \tag{4.10}$$

The preceding hinges on an application of Burkholder's inequality, using the Carlen–Kree bound [7] on Davis's optimal constant [12] in the Burkholder–Davis–Gundy inequality [3–5]; see Foondun and Khoshnevisan [16] for the details of the derivation of such an estimate. Minkowski's inequality tells us then that the preceding quantity is at most

$$\begin{aligned}&2\sqrt{k\int_0^t \mathrm{d}s \int_{|y-x|\geqslant\beta^{1/2}\sqrt{t}} \mathrm{d}y\; p_{t-s}^2(y-x)\left\|\sigma^2\left(U_s^{(\beta)}(y)\right)\right\|_{k/2}} \\ &\qquad \leqslant \text{const}\cdot\sqrt{k\int_0^t \mathrm{d}s \int_{|y-x|\geqslant\beta^{1/2}\sqrt{t}} \mathrm{d}y\; p_{t-s}^2(y-x)\left(1+\left\|U_s^{(\beta)}(y)\right\|_k^2\right)},\end{aligned} \tag{4.11}$$

Eq. (4.11) holds because the Lipschitz continuity of the function σ ensures that it has at-most-linear growth: $|\sigma(x)| \leqslant \text{const}\cdot(1+|x|)$ for all $x \in \mathbf{R}$. The inequality in Lemma 4.1 implies that, uniformly over all $t \in [0\,,T]$ and $x \in \mathbf{R}$,

$$\begin{aligned}\left\|V_t(x) - U_t^{(\beta)}(x)\right\|_k &\leqslant \text{const}\cdot\sqrt{kC^2\mathrm{e}^{2ak^2}\int_0^t \mathrm{d}r \int_{|z|\geqslant\beta^{1/2}\sqrt{t}} \mathrm{d}z\; p_r^2(z)} \\ &\leqslant \text{const}\cdot\frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\kappa}}\sqrt{\int_0^1 \frac{\mathrm{d}s}{\sqrt{s}}\int_{|w|\geqslant\sqrt{2\beta}} \mathrm{d}w\; p_s(w)},\end{aligned} \tag{4.12}$$

where we have used (1.4). Now a standard Gaussian tail estimate yields

$$\int_{|w|\geqslant\sqrt{2\beta}} p_s(w)\,\mathrm{d}w \leqslant 2\mathrm{e}^{-\beta/s\kappa}, \tag{4.13}$$

and the latter quantity is at most $2\exp(-\beta/\kappa)$ whenever $s \in (0\,,1]$. Therefore, on one hand:

$$\left\|V_t(x) - U_t^{(\beta)}(x)\right\|_k \leqslant \text{const}\cdot\frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\kappa}}\,\mathrm{e}^{-\beta/2\kappa}. \tag{4.14}$$

On the other hand,

$$u_t(x) - V_t(x) = \int_{(0,t)\times\mathbf{R}} p_{t-s}(y-x)\left[\sigma\left(u_s(y)\right) - \sigma\left(U_s^{(\beta)}(y)\right)\right] W(\mathrm{d}s\,\mathrm{d}y),$$

whence

$$
\begin{aligned}
&\|u_t(x) - V_t(x)\|_k \\
&\leqslant 2\sqrt{k} \left\| \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \; p_{t-s}^2(y-x) \left[\sigma\left(u_s(y)\right) - \sigma\left(U_s^{(\beta)}(y)\right) \right]^2 \right\|_{k/2}^{1/2} \\
&\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma \left\| \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \; p_{t-s}^2(y-x) \left[u_s(y) - U_s^{(\beta)}(y) \right]^2 \right\|_{k/2}^{1/2} \\
&\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma \cdot \sqrt{\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \; p_{t-s}^2(y-x) \left\| u_s(y) - U_s^{(\beta)}(y) \right\|_k^2}.
\end{aligned}
\tag{4.15}
$$

Consequently, (4.14) implies that for all $x \in \mathbf{A}_\beta$,

$$
\begin{aligned}
&\left\| u_t(x) - U_t^{(\beta)}(x) \right\|_k \\
&\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma \cdot \sqrt{\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \; p_{t-s}^2(y-x) \left\| u_s(y) - U_s^{(\beta)}(y) \right\|_k^2} \\
&\qquad\qquad + \mathrm{const} \cdot \frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\kappa}} \, \mathrm{e}^{-\beta/2\kappa}.
\end{aligned}
\tag{4.16}
$$

We now split the integral $\int_{-\infty}^{\infty} \mathrm{d}y$ as $\int_{\mathbf{A}_\beta} \mathrm{d}y + \int_{\mathbf{R}\setminus\mathbf{A}_\beta} \mathrm{d}y$, and then use the elementary bound $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$—valid for all $a, b > 0$—in order to obtain

$$
\begin{aligned}
&\left\| u_t(x) - U_t^{(\beta)}(x) \right\|_k \\
&\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma \cdot \sqrt{\int_0^t \mathrm{d}s \int_{\mathbf{A}_\beta} \mathrm{d}y \; p_{t-s}^2(y-x) \left\| u_s(y) - U_s^{(\beta)}(y) \right\|_k^2} \\
&\qquad + 2\sqrt{k}\mathrm{Lip}_\sigma \cdot \sqrt{\int_0^t \mathrm{d}s \int_{\mathbf{R}\setminus\mathbf{A}_\beta} \mathrm{d}y \; p_{t-s}^2(y-x) \left\| u_s(y) - U_s^{(\beta)}(y) \right\|_k^2} \\
&\qquad\qquad + \mathrm{const} \cdot \frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\kappa}} \, \mathrm{e}^{-\beta/2\kappa}.
\end{aligned}
\tag{4.17}
$$

Owing to Lemmas 3.2 and 4.1, it follows that whenever $x \in \mathbf{A}_\beta$,

$$
\begin{aligned}
&\left\| u_t(x) - U_t^{(\beta)}(x) \right\|_k \\
&\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma \cdot \sqrt{\int_0^t \mathrm{d}s \int_{\mathbf{A}_\beta} \mathrm{d}y \; p_{t-s}^2(y-x) \left\| u_s(y) - U_s^{(\beta)}(y) \right\|_k^2} \\
&\qquad + 4C\mathrm{Lip}_\sigma \sqrt{k} \, \mathrm{e}^{ak^2} \sqrt{\int_0^t \mathrm{d}s \int_{\mathbf{R}\setminus\mathbf{A}_\beta} \mathrm{d}y \; p_s^2(y-x)} \\
&\qquad\qquad + \mathrm{const} \cdot \frac{k^{1/2}\mathrm{e}^{ak^2}}{\sqrt{\kappa}} \, \mathrm{e}^{-\beta/2\kappa}.
\end{aligned}
\tag{4.18}
$$

Next we estimate $\int_{\mathbf{R}\setminus\mathbf{A}_\beta} p_s^2(y-x)\,\mathrm{d}y$ for all $x \in \mathbf{A}_\beta$.

Without loss of too much generality we can assume that x is in the interval corresponding to $j=1$ in the definition of $\mathbf{A}_\beta$. From this, a direct computation shows that

$$\begin{aligned}\int_{\mathbf{R}\setminus\mathbf{A}_\beta} p_s^2(y-x)\,\mathrm{d}y &\leqslant 2\sum_{i=0}^{\infty}\int_0^{2\sqrt{\beta t}} \frac{1}{2\pi s\kappa}\exp\left(\frac{-\left(i\beta\sqrt{t}+y\right)^2}{s\kappa}\right)\mathrm{d}y \\ &\leqslant 2\sum_{i=0}^{\infty}\int_0^{2\sqrt{\beta t}} \frac{1}{2\pi s\kappa}\exp\left(\frac{-i^2\beta^2 t-y^2}{s\kappa}\right)\mathrm{d}y.\end{aligned} \tag{4.19}$$

We bound the preceding sum over i by a corresponding integral and obtain the following inequality:

$$\sup_{s\in[0,t]}\int_{\mathbf{R}\setminus\mathbf{A}_\beta} p_s^2(y-x)\,\mathrm{d}y \leqslant \frac{\text{const}}{\beta\sqrt{t}} \qquad \text{for all } t>0. \tag{4.20}$$

The same inequality holds [for a generic constant] uniformly for all $x \in \mathbf{A}_\beta$, and for the very same reasons. Therefore, this and (4.18) together imply that, uniformly for $t \in [0\,,T]$,

$$\begin{aligned}&\sup_{x\in\mathbf{A}_\beta}\left\|u_t(x)-U_t^{(\beta)}(x)\right\|_k \\ &\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma\cdot\sqrt{\int_0^t \mathrm{d}s\int_{\mathbf{A}_\beta}\mathrm{d}y\; p_{t-s}^2(y-x)\left\|u_s(y)-U_s^{(\beta)}(y)\right\|_k^2} \\ &\qquad\qquad + \text{const}\cdot\frac{k^{1/2}\mathrm{e}^{ak^2}}{\beta^{1/2}}.\end{aligned} \tag{4.21}$$

Let us introduce a parameter $\delta>0$ and define the seminorms

$$\mathcal{N}_{k,\beta,\delta}(Z) := \sup_{s\geqslant 0}\sup_{y\in\mathbf{A}_\beta}\left[\mathrm{e}^{-\delta s}\|Z_s(y)\|_k\right], \tag{4.22}$$

for every space-time random field $Z := \{Z_s(y)\}_{s>0,y\in\mathbf{R}}$. Then, we have

$$\begin{aligned}&\mathcal{N}_{k,\beta,\delta}\left(u-U^{(\beta)}\right) \\ &\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma\,\mathcal{N}_{k,\beta,\delta}\left(u-U^{(\beta)}\right)\cdot\sqrt{\int_0^\infty \mathrm{e}^{-2\delta r}\|p_r\|_{L^2(\mathbf{R})}^2\,\mathrm{d}r} \\ &\qquad\qquad + \text{const}\cdot\frac{k^{1/2}\mathrm{e}^{ak^2}}{\beta^{1/2}}.\end{aligned} \tag{4.23}$$

Thanks to (1.22), if $\delta := Dk^2$ for some sufficiently large constant D, then the square root is at most $[4\sqrt{k}(\mathrm{Lip}_\sigma\vee 1)]^{-1}$, whence it follows that [for that fixed choice of δ],

$$\mathcal{N}_{k,\beta,\delta}\left(u-U^{(\beta)}\right) \leqslant \text{const}\cdot\frac{k^{1/2}\mathrm{e}^{ak^2}}{\beta^{1/2}}. \tag{4.24}$$

The lemma follows from this. □

Lemma 4.3. *Choose and fix $\beta > 4, t > 0$. Then $\{U_t^{(\beta)}(x_j)\}_{j\in\mathbf{Z}}$ is a collection of i.i.d. random variables, where $x_j = x_j(\beta, t)$ is defined in Lemma 4.2.*

Proof. The fact that the sequence is identically distributed follows from a Picard-iteration argument such as the one that appears in Dalang [10]. Here are a few more details: Let $U_t^{(\beta,0)}(x) := 1$ for all $t > 0$ and $x \in \mathbf{R}$; and define iteratively

$$U_t^{(\beta,n)}(x) := 1 + \int_{(0,t)\times\mathcal{I}_t^{(\beta)}(x)} p_{t-s}(y-x)\sigma\left(U_s^{(\beta,n-1)}(y)\right) W(\mathrm{d}s\,\mathrm{d}y). \quad (4.25)$$

The proof of Lemma 4.1 relies tacitly on the fact that, among other things,

$$\lim_{n\to\infty}\sup_{x\in\mathbf{R}} \mathrm{E}\left(\left|U_t^{(\beta,n)}(x) - U_t^{(\beta)}(x)\right|^2\right) = 0 \qquad \text{for all } t > 0. \quad (4.26)$$

The independence of the sequence $\{U_t^{(\beta,n)}(x_j)\}_{j\in\mathbf{Z}}$ is a ready consequence of Corollary A.2 of the Appendix and implies independence of the $U_t^{(\beta)}(x_j)$'s. The assertion about their identical distribution follows now because induction reveals that $\{U_t^{(\beta,n)}(x_j)\}_{j\in\mathbf{Z}}$ are identically distributed random variables. □

5 Proof of Theorem 1.1

We are ready to combine our efforts thus far in order to verify Theorem 1.1.

Proof of Theorem 1.1. Parts 1 and 2 of Theorem 1.1 are proved similarly. Therefore, we present the details of the second part. For the proof of the first part, we can take $\gamma := 1/6$ in the following argument.

Let $x_1, x_2, \cdots, x_N$ be chosen as in Lemma 4.2. Then,

$$\begin{aligned} &\mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u_t(x_j)| < \lambda\right\} \quad (5.1)\\ &\quad\leqslant \mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|U_t^{(\beta)}(x_j)| < 2\lambda\right\} + \mathrm{P}\left\{\max_{1\leqslant j\leqslant N}\left|U_t^{(\beta)}(x_j) - u_t(x_j)\right| > \lambda\right\}. \end{aligned}$$

The argument that led to Proposition 3.8 and the first inequality in (3.2) can be modified to prove the existence of constants $c_1, c_2 > 0$—independent of β for all sufficiently large values of β—so that

$$\mathrm{P}\left\{|U_t^{(\beta)}(x)| \geqslant \lambda\right\} \geqslant c_1 \mathrm{e}^{-c_2\lambda^{1/\gamma}}. \quad (5.2)$$

(This requires only a slight modification of Proposition 3.6.)

Next, we apply the independence of $\{U_t^{(\beta)}(x_j)\}_{j=1}^N$, and Lemma 4.2, in order to derive the following:

$$\begin{aligned} &\mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u_t(x_j)| < \lambda\right\} \\ &\qquad\leqslant \left(1 - c_1\mathrm{e}^{-c_2\lambda^{1/\gamma}}\right)^N + NC^k k^{k/2}\mathrm{e}^{Fk^3}\beta^{-k/2}\lambda^{-k}. \end{aligned} \quad (5.3)$$

We now select the various parameters judiciously: Choose $\lambda := k$, $N := \lceil k \exp(c_2 k^{1/\gamma}) \rceil$ and $\beta := \exp(\rho k^{-(\gamma-1)/\gamma})$ for a large-enough positive constant $\rho > 2c_2$. In this way, (5.3) simplifies: For all sufficiently large integers k,

$$\begin{aligned} &\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_t(x_j)| < k\right\} \\ &\leqslant \mathrm{e}^{-c_1 k} + \exp\left[c_2 k^{1/\gamma} + \log k + k\log C - \frac{k\log k}{2} + Fk^3 - \frac{\rho k^{1/\gamma}}{2}\right] \\ &\leqslant 2\mathrm{e}^{-c_1 k}. \end{aligned} \tag{5.4}$$

Since x_j denotes the center point of the half-open, half-closed interval $[(j-1)\beta\sqrt{t}\,, j\beta\sqrt{t})$, the preceding implies that for all sufficiently large k,

$$\mathrm{P}\left\{\sup_{x\in[0,N\beta\sqrt{t}]} |u_t(x)| < k\right\} \leqslant 2\mathrm{e}^{-c_1 k}. \tag{5.5}$$

Clearly, as $k \to \infty$,

$$N\beta\sqrt{t} = \left\lceil k\exp\left(c_2 k^{1/\gamma}\right)\right\rceil \exp\left(\rho k^{-1+(1/\gamma)}\right)\sqrt{t} = O\left(\mathrm{e}^{2c_2 k^{1/\gamma}}\right). \tag{5.6}$$

Consequently, the Borel–Cantelli lemma, used in conjunction with a standard monotonicity argument, implies that with probability one $u_t^*(R) \geqslant \text{const}\cdot(\log R)^\gamma$ for all sufficiently-large values of R, where $u_t^*(R)$ is defined in (1.21). By Proposition 3.8, $c_2 = \text{const}\cdot\kappa^{1/12\gamma}$. Therefore, the theorem follows. □

6 Proof of Theorem 1.2

Next we prove Theorem 1.2. In order to obtain the lower bound, the proof requires an estimate of spatial continuity of $x \mapsto u_t(x)$. However, matters are somewhat complicated by the fact that we need a modulus of continuity estimate that holds simultaneously for every $x \in [-R\,, R]$, uniformly for all large values of R. This will be overcome in a few steps. The first is a standard moment bound for the increments of the solution; however we need to pay close attention to the constants in the estimate.

Lemma 6.1. *Choose and fix some $t > 0$, and suppose σ is uniformly bounded. Then there exists a finite and positive constant A such that for all real numbers $k \geqslant 2$,*

$$\sup_{-\infty<x\neq x'<\infty} \frac{\mathrm{E}\left(|u_t(x)-u_t(x')|^{2k}\right)}{|x-x'|^k} \leqslant \left(\frac{Ak}{\kappa}\right)^k. \tag{6.1}$$

Proof. Throughout, let $S_0 := \sup_{x\in\mathbf{R}} |\sigma(x)|$.

If $x, x' \in [-R\,, R]$ and $t > 0$ are held fixed, then we can write $u_t(x) - u_t(x') = N_t$, where $\{N_\tau\}_{\tau\in(0,t)}$ is the continuous mean-one martingale described by

$$N_\tau := \int_{(0,\tau)\times\mathbf{R}} [p_{t-s}(y-x) - p_{t-s}(y-x')]\,\sigma(u_s(y))\,W(\mathrm{d}s\,\mathrm{d}y), \tag{6.2}$$

for $\tau \in (0\,,t)$. The quadratic variation of $\{N_\tau\}_{\tau\in(0,t)}$ is estimated as follows:

$$\begin{aligned}\langle N\rangle_\tau &\leqslant S_0^2\cdot\int_0^\tau \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,[p_s(y-x)-p_s(y-x')]^2\\ &\leqslant \mathrm{e}^\tau S_0^2\cdot\int_0^\infty \mathrm{e}^{-s}\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\;\;[p_s(y-x)-p_s(y-x')]^2\,.\end{aligned}\tag{6.3}$$

For every $s>0$ fixed, we can compute the dy-integral using Plancherel's theorem, and obtain $\pi^{-1}\int_{-\infty}^{\infty}(1-\cos(\xi|x-x'|))\exp(-\kappa s\xi^2)\,\mathrm{d}\xi$. Therefore, there exists a finite and positive constant a such that

$$\langle N\rangle_\tau \leqslant \frac{\mathrm{e}^\tau S_0^2}{\pi}\cdot\int_{-\infty}^{\infty}\frac{1-\cos(\xi|x-x'|)}{1+\kappa\xi^2}\,\mathrm{d}\xi \leqslant \frac{a}{\kappa}\,|x-x'|,\tag{6.4}$$

uniformly for all $\tau\in(0\,,t)$; we emphasize that a depends only on S_0 and t. The Carlen–Kree estimate [7] for the Davis [12] optimal constant in the Burkholder–Davis–Gundy inequality [3–5] implies the lemma. □

The second estimate turns the preceding moment bounds into an maximal exponential estimate. We use a standard chaining argument to do this. However, once again we have to pay close attention to the parameter dependencies in the implied constants.

Lemma 6.2. *Choose and fix $t>0$, and suppose σ is uniformly bounded. Then there exist constants $\zeta\in(0\,,\kappa)$ and $C\in(0\,,\infty)$ such that*

$$\mathrm{E}\left[\sup_{\substack{x,x'\in I:\\|x-x'|\leqslant\delta}}\exp\left(\frac{\zeta|u_t(x)-u_t(x')|^2}{\delta}\right)\right]\leqslant\frac{C}{\delta},\tag{6.5}$$

uniformly for every $\delta\in(0\,,1]$ and every interval $I\subset[0\,,\infty)$ of length at most one.

Proof. Recall [11, (39), p. 11] the Kolmogorov continuity in the following quantitative form: Suppose there exist $\nu>\gamma>1$ for which a stochastic process $\{\xi(x)\}_{x\in\mathbf{R}}$ satisfies the following:

$$\mathrm{E}\left(|\xi(x)-\xi(x')|^\nu\right)\leqslant C|x-x'|^\gamma;\tag{6.6}$$

we assume that the preceding holds for all $x,x'\in\mathbf{R}$, and $C\in(0\,,\infty)$ is independent of x and x'. Then, for every integer $m\geqslant 0$,

$$\mathrm{E}\left(\sup_{\substack{x,x'\in I:\\|x-x'|\leqslant 2^{-m}}}|\xi(x)-\xi(x')|^\nu\right)\leqslant\left(\frac{2^{(2-\gamma+\nu)/\nu}C^{1/\nu}}{1-2^{-(\gamma-1)/\nu}}\right)^\nu\cdot 2^{-m(\gamma-1)}.\tag{6.7}$$

[Ref. [11, (39), p. 11] claims this with $2^{-m(\gamma-1)}$ replaced with $2^{-m\gamma}$ on the right-hand side. But this is a typographical error; compare with [11, (38), p. 11].]

If $\delta \in (0\,,1]$, then we can find an integer $m \geqslant 0$ such that $2^{-m-1} \leqslant \delta \leqslant 2^{-m}$, whence it follows that

$$\begin{aligned} \mathrm{E}\left(\sup_{\substack{x,x'\in I:\\ |x-x'|\leqslant\delta}} |\xi(x)-\xi(x')|^{\nu}\right) &\leqslant \left(\frac{2^{(2-\gamma+\nu)/\nu}C^{1/\nu}}{1-2^{-(\gamma-1)/\nu}}\right)^{\nu}\cdot 2^{-m(\gamma-1)} \\ &\leqslant \left(\frac{2^{(2-\gamma+\nu)/\nu}C^{1/\nu}}{1-2^{-(\gamma-1)/\nu}}\right)^{\nu}\cdot (2\delta)^{\gamma-1}. \end{aligned} \tag{6.8}$$

We apply the preceding with $\xi(x) := u_t(x)$, $\gamma := \nu/2 := k$, and $C := (Ak/\kappa)^k$, where A is the constant of Lemma 6.1. It follows that there exists a positive and finite constant A_* such that for all intervals I of length at most one, all integers $k \geqslant 2$, and every $\delta \in (0\,,1]$,

$$\mathrm{E}\left(\sup_{\substack{x,x'\in I:\\ |x-x'|\leqslant\delta}} |u_t(x)-u_t(x')|^{2k}\right) \leqslant \left(\frac{A_*k}{\kappa}\right)^{k}\delta^{k-1}. \tag{6.9}$$

Stirling's formula tells us that there exists a finite constant $B_* > 1$ such that $(A_*k)^k \leqslant B_*^k k!$ for all integers $k \geqslant 1$. Therefore, for all $\zeta, \delta > 0$,

$$\mathrm{E}\left[\sup_{\substack{x,x'\in I:\\ |x-x'|\leqslant\delta}} \exp\left(\frac{\zeta|u_t(x)-u_t(x')|^2}{\delta}\right)\right] \leqslant \frac{1}{\delta}\sum_{k=0}^{\infty}\left(\frac{\zeta B_*}{\kappa}\right)^{k}. \tag{6.10}$$

And this is finite if $\zeta \in (0\,,\kappa/B_*)$. The result follows. □

Next we obtain another moments bound, this time for the solution rather than its increments.

Lemma 6.3. *Choose and fix $t > 0$, and suppose σ is uniformly bounded. Then for all integers $k \geqslant 1$,*

$$\sup_{x\in\mathbf{R}} \mathrm{E}\left(|u_t(x)|^{2k}\right) \leqslant \left(2\sqrt{2}+o(1)\right)(\tilde{\mu}_t k)^k \qquad (\textit{as } k\to\infty), \tag{6.11}$$

where the "$o(1)$" term depends only on k, and

$$\tilde{\mu}_t := \frac{8}{\mathrm{e}}\cdot S_0^2\sqrt{\frac{t}{\pi\kappa}}. \tag{6.12}$$

Proof. Let us choose and fix a $t > 0$. Define $S_0 := \sup_{x\in\mathbf{R}}|\sigma(x)|$, and recall the martingale $\{M_\tau\}_{\tau\in(0,t)}$ from (3.17). Itô's formula (3.19) tells us that a.s., for all $\tau \in (0\,,t)$,

$$M_\tau^{2k} \leqslant 1 + 2k\int_0^\tau M_s^{2k-1}\,\mathrm{d}M_s + \binom{2k}{2}S_0^2\int_0^\tau \frac{M_s^{2(k-1)}}{(4\pi\kappa(t-s))^{1/2}}\,\mathrm{d}s. \tag{6.13}$$

[Compare with (3.20).] We can take expectations, iterate the preceding, and argue as we did in the proof of Proposition 3.6. To summarize the end result, let us define

$$\eta_t(x) := 1 + S_0 \cdot \int_{(0,\tau)\times\mathbf{R}} p_{t-s}(y-x)\, W(\mathrm{d}s\,\mathrm{d}y) \qquad (0 \leqslant \tau \leqslant t), \tag{6.14}$$

and

$$\zeta_t(x) := S_0 \cdot \int_{(0,\tau)\times\mathbf{R}} p_{t-s}(y-x)\, W(\mathrm{d}s\,\mathrm{d}y) \qquad (0 \leqslant \tau \leqslant t). \tag{6.15}$$

Then we have $\mathrm{E}[M_t^{2k}] \leqslant \mathrm{E}[\eta_t(x)^{2k}] \leqslant 2^{2k}(1+\mathrm{E}[\zeta_t(x)^{2k}])$, and similar computations as those in the proof of Proposition 3.6 yield the lemma. □

Next we turn the preceding moment bound into a sharp Gaussian tail-probability estimate.

Lemma 6.4. *Choose and fix a $t > 0$, and suppose that σ is uniformly bounded. Then there exist finite constants $C > c > 0$ such that simultaneously for all $\lambda > 1$ and $x \in \mathbf{R}$,*

$$c \exp\left(-C\sqrt{\kappa}\,\lambda^2\right) \leqslant \mathrm{P}\left\{|u_t(x)| > \lambda\right\} \leqslant C \exp\left(-c\sqrt{\kappa}\,\lambda^2\right). \tag{6.16}$$

Proof. The lower bound is proved by an appeal to the Paley–Zygmund inequality, in the very same manner that Proposition 3.7 was established. However, we apply the improved inequality in Lemma 6.3 [in place of the result of Lemma 3.2]. As regards the upper bound, note that Lemma 6.3 implies that there exists a positive and finite constant $\tilde{A}$ such that for all integers $m \geqslant 0$, $\sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^{2m}) \leqslant (\tilde{A}/\sqrt{\kappa})^m m!$, thanks to the Stirling formula. Thus,

$$\sup_{x\in\mathbf{R}} \mathrm{E}\exp\left(\zeta|u_t(x)|^2\right) \leqslant \sum_{m=0}^{\infty} \left(\frac{\zeta\tilde{A}}{\sqrt{\kappa}}\right)^m = \frac{1}{1-\zeta\tilde{A}\kappa^{-1/2}} < \infty, \tag{6.17}$$

provided that $\zeta \in (0\,,\sqrt{\kappa}/\tilde{A})$. If we fix such a ζ, then we obtain from Chebyshev's inequality the bound $\mathrm{P}\{u_t(x) > \lambda\} \leqslant (1-\zeta\tilde{A}\kappa^{-1/2})^{-1} \cdot \exp(-\zeta\lambda^2)$, valid simultaneously for all $x \in \mathbf{R}$ and $\lambda > 0$. We write $\zeta := c\sqrt{\kappa}$ to finish. □

We are finally ready to assemble the preceding estimates in order to establish Theorem 1.2.

Proof of Theorem 1.2. Consider the proof of Theorem 1.1: If replace the role of (3.2) by the bounds in Lemma 6.4, modify Lemmas 4.1 and 4.2 and choose $\lambda := k$, $N := \lceil k\exp(c\sqrt{\kappa}\,k^2)\rceil$ and $\beta := \exp((\mathrm{Lip}_\sigma \vee 1)^4 k^2/\kappa)$ in the equivalent of (5.3) with the appropriate estimates, then we obtain the almost-sure bound $\liminf_{R\to\infty} u_t^*(R)/(\log R)^{1/2} > 0$. It remains to derive a corresponding upper bound for the lim sup.

Suppose $R \geqslant 1$ is an integer. We partition the interval $[-R\,,R]$ using a length-1 mesh with endpoints $\{x_j\}_{j=0}^{2R}$ via

$$x_j := -R + j \qquad \text{for } 0 \leqslant j \leqslant 2R. \tag{6.18}$$

Then we write

$$\mathrm{P}\left\{u_t^*(R) > 2\alpha(\log R)^{1/2}\right\} \leqslant \mathcal{T}_1 + \mathcal{T}_2, \tag{6.19}$$

where

$$\begin{aligned} \mathcal{T}_1 &:= \mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R} u_t(x_j) > \alpha(\log R)^{1/2}\right\}, \\ \mathcal{T}_2 &:= \mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R} \sup_{x\in(x_j,x_{j+1})} |u_t(x) - u_t(x_j)| > \alpha(\log R)^{1/2}\right\}. \end{aligned} \tag{6.20}$$

By Lemma 6.4,

$$\mathcal{T}_1 \leqslant 2R \sup_{x\in\mathbf{R}} \mathrm{P}\left\{u_t(x) > \alpha(\log R)^{1/2}\right\} \leqslant \frac{\text{const}}{R^{-1+c\sqrt{\kappa}\,\alpha^2}}. \tag{6.21}$$

Similarly,

$$\mathcal{T}_2 \leqslant 2R \sup_I \mathrm{P}\left\{\sup_{x,x'\in I} |u_t(x) - u_t(x')| > \alpha(\log R)^{1/2}\right\}, \tag{6.22}$$

where "$\sup_I$" designates a supremum over all intervals I of length one. Chebyshev's inequality and Lemma 6.2 together imply that

$$\begin{aligned} \mathcal{T}_2 &\leqslant 2R^{-\zeta\alpha^2+1} \sup_I \mathrm{E}\left[\sup_{x,x'\in I} \exp\left(\zeta|u_t(x) - u_t(x')|^2\right)\right] \\ &\leqslant \frac{\text{const}}{R^{-1+\zeta\alpha^2}}. \end{aligned} \tag{6.23}$$

Let $\beta := \min(\zeta\,, c\sqrt{\kappa})$ to find that

$$\sum_{R=1}^{\infty} \mathrm{P}\left\{u_t^*(R) > 2\alpha(\log R)^{1/2}\right\} \leqslant \text{const} \cdot \sum_{R=1}^{\infty} R^{-\beta\alpha^2+1}, \tag{6.24}$$

and this is finite provided that $\alpha > (2/\beta)^{1/2}$. By the Borel–Cantelli lemma,

$$\limsup_{\substack{R\to\infty:\\ R\in\mathbf{Z}}} \frac{u_t^*(R)}{(\log R)^{1/2}} \leqslant \left(\frac{8}{\beta}\right)^{1/2} < \infty \qquad \text{a.s.} \tag{6.25}$$

By Lemma 6.2, $\zeta < \kappa$, hence $(8/\beta)^{1/2} \leqslant \text{const} \cdot \kappa^{-1/4}$. And we can remove the restriction "$R \in \mathbf{Z}$" in the lim sup by a standard monotonicity argument; namely, we find—by considering in the following $R - 1 \leqslant X \leqslant R$—that

$$\limsup_{X\to\infty} \frac{u_t^*(X)}{(\log X)^{1/2}} \leqslant \limsup_{\substack{R\to\infty:\\ R\in\mathbf{Z}}} \frac{u_t^*(R)}{(\log(R-1))^{1/2}} \leqslant \left(\frac{8}{\beta}\right)^{1/2} \qquad \text{a.s.} \tag{6.26}$$

This proves the theorem. □

7 Proof of Theorem 1.3

This section is mainly concerned with the proof of Theorem 1.3. For that purpose, we start with tail-estimates.

Lemma 7.1. *Consider* (2.11) *with* $\sigma(x) := cx$, *where* $c > 0$ *is fixed. Then,*

$$\log \mathrm{P}\left\{|u_t(x)| \geqslant \lambda\right\} \asymp -\sqrt{\kappa}\,(\log \lambda)^{3/2} \qquad \text{as } \lambda \to \infty. \tag{7.1}$$

Proof. Corollary 3.5 implies the upper bound [the boundedness of σ is not required in the results of §3.1].

As for the lower bound, we know from [2, Theorem 2.6] that there exist nontrivial constants C and D such that

$$\mathrm{E}\left(|u_t(x)|^k\right) = C\mathrm{e}^{Dk(k^2-1)/\kappa}, \tag{7.2}$$

uniformly for all integers $k \geqslant 2$, $t \in [0\,,T]$, and $x \in \mathbf{R}$. Now we follow the same method as in the proof of Proposition 3.7, and use the Paley–Zygmund inequality to obtain

$$\begin{aligned} \mathrm{P}\left\{|u_t(x)| \geqslant \frac{1}{2}\|u_t(x)\|_{2k}\right\} &\geqslant \frac{\left[\mathrm{E}\left(|u_t(x)|^{2k}\right)\right]^2}{4E\left(|u_t(x)|^{4k}\right)} \\ &\geqslant C_1\mathrm{e}^{-D_1k^3/\kappa}, \end{aligned} \tag{7.3}$$

for some nontrivial constants C_1 and D_1 that do not depend on x or k. We then obtain the following: Uniformly for all $x \in \mathbf{R}$ and sufficiently-large integers k,

$$\mathrm{P}\left\{|u_t(x)| \geqslant \frac{C}{2}\mathrm{e}^{4Dk^2/\kappa}\right\} \geqslant C_1\mathrm{e}^{-D_1k^3/\kappa} \tag{7.4}$$

Let $\lambda := (C/2)\exp\{4Dk^2/\kappa\}$ and apply a direct computation to deduce the lower bound. □

We are now ready to prove Theorem 1.3. Our proof is based on roughly-similar ideas to those used in the course of the proof of Theorem 1.1. However, at a technical level, they are slightly different. Let us point out some of the essential differences: Unlike what we did in the proof of Theorem 1.1, we now do not choose the values of N, β and λ as functions of k, but rather as functions of R; the order of the moments k will be fixed; and we will not sum on k, but rather sum on a discrete sequence of values of the parameter R. The details follow.

Proof of Theorem 1.3. First we derive the lower bound by following the same method that was used in the proof of Theorem 1.1; see Section 5. But we now use Lemma 7.1 rather than Corollary 3.1.

The results of §4 can be modified to apply to the parabolic Anderson model, provided that we again apply Lemma 7.1 in place of Corollary 3.1. In this way

we obtain the following:

$$
\begin{aligned}
&\mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u_t(x_j)|<\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\\
&\leqslant \mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|U_t^{(\beta)}(x_j)|<2\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\\
&\qquad+\mathrm{P}\left\{|u_t(x_j)-U_t^{(\beta)}|>\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\text{ for some }1\leqslant j\leqslant N\right\}\\
&\leqslant\left(1-\mathrm{P}\left\{|U_t^{(\beta)}(x_j)|\geqslant 2\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\right)^N\\
&\qquad+N\beta^{-k/2}C^k k^{k/2}\mathrm{e}^{Fk^3}\mathrm{e}^{-kC_1(\log R)^{2/3}/\sqrt{\kappa}}
\end{aligned}
\tag{7.5}
$$

Note that we do not yet have a lower bound on $\mathrm{P}\{|U_t^{(\beta)}(x)|\geqslant\lambda\}$. However, we have

$$
\begin{aligned}
&\mathrm{P}\left\{|U_t^{(\beta)}(x_j)|\geqslant 2\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\\
&\geqslant \mathrm{P}\left\{|u_t(x_j)|\geqslant 3\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\\
&\qquad\qquad-\mathrm{P}\left\{|u_t(x_j)-U_t^{(\beta)}(x_j)|\geqslant \mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\\
&\geqslant \alpha_1 R^{-\alpha_2 C_1^{3/2}}-N\beta^{-k/2}C^k k^{k/2}\mathrm{e}^{Fk^3}\mathrm{e}^{-kC_1(\log R)^{2/3}/\sqrt{\kappa}},
\end{aligned}
\tag{7.6}
$$

valid for some positive constants α_1 and α_2. Now let us choose $N:=\lceil R^a\rceil$ and $\beta:=R^{1-a}$ for a fixed $a\in(0\,,1)$. With these values of N and β and the lower bound in (7.6), the upper bound in (7.5) becomes

$$
\begin{aligned}
&\mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u_t(x_j)|<\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\\
&\leqslant\left(1-\alpha_1 R^{-\alpha_2 C_1^{3/2}}+\frac{C^k k^{k/2}\mathrm{e}^{Fk^3}}{R^{k(1-a)/(2-a)}\,\mathrm{e}^{kC_1(\log R)^{2/3}/\sqrt{\kappa}}}\right)^N\\
&\qquad\qquad+\frac{C^k k^{k/2}\mathrm{e}^{Fk^3}}{R^{k(1-a)/(2-a)}\mathrm{e}^{kC_1(\log R)^{2/3}/\sqrt{\kappa}}}.
\end{aligned}
\tag{7.7}
$$

Let us consider k large enough so that $k(1-a)/(2-a)>2$. Notice that k will not depend on R; this is in contrast with what happened in the proof of Theorem 1.1.

We can choose the constant C_1 to be small enough to satisfy $\alpha_2 C_1^{3/2}<a/2$. Using these, we obtain

$$
\mathrm{P}\left\{\sup_{x\in[0,R]}|u_t(x)|<\mathrm{e}^{C_1(\log R)^{2/3}/\kappa^{1/3}}\right\}\leqslant\exp\left(-\alpha_1 R^{a/2}\right)+\frac{\text{const}}{R^2}.
\tag{7.8}
$$

The Borel–Cantelli Lemma yields the lower bound of the theorem.

We can now prove the upper bound. Our derviation is modeled after the proof of Theorem 1.2.

First, we need a continuity estimate for the solution of (2.11) in the case that $\sigma(x) := cx$. In accord with (7.2),

$$\begin{aligned}
&\mathrm{E}\left(|u_t(x) - u_t(y)|^{2k}\right) \\
&\leqslant \left(2\sqrt{2k}\right)^{2k} \left[\int_0^t \mathrm{d}r\, \|u(r\,,0)\|_{2k}^2 \int_{\mathbf{R}} \mathrm{d}z\ \ |p_{t-r}(x-z) - p_{t-r}(y-z)|^2\right]^k \\
&\leqslant \left(2\sqrt{2k}\right)^{2k} \left[\int_0^t \mathrm{d}r\ C^{1/k}\mathrm{e}^{8Dk^2/\kappa} \int_{\mathbf{R}} \mathrm{d}z\ |p_{t-r}(x-z) - p_{t-r}(y-z)|^2\right]^k .
\end{aligned} \tag{7.9}$$

Consequently (see the derivation of (6.4)),

$$\mathrm{E}\left(|u_t(x) - u_t(y)|^{2k}\right) \leqslant \frac{|y-x|^k}{\kappa}\ \exp\left(\frac{Bk^3}{\kappa}\right), \tag{7.10}$$

for some constant $B \in (0\,,\infty)$ that does not depend on k. We apply an argument, similar to one we used in the proof of Lemma 6.2, in order to deduce that for simultaneously all intervals I of length 1,

$$\mathrm{E}\left(\sup_{x,x'\in I:|x-x'|\leqslant 1} |u_t(x) - u_t(x')|^{2k}\right) \leqslant \mathrm{e}^{Ck^3/\kappa}, \tag{7.11}$$

for a constant $C \in (0\,,\infty)$ that does not depend on k. Now, we follow the proof of Theorem 1.2 and partition $[-R\,,R]$ into intervals of length 1. Let $b > 0$ to deduce the following:

$$\mathrm{P}\left\{u_t^*(R) > 2\mathrm{e}^{b(\log R)^{2/3}/\kappa^{1/3}}\right\} \leqslant \mathcal{T}_1 + \mathcal{T}_2, \tag{7.12}$$

where

$$\mathcal{T}_1 := \mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R} u_t(x_j) > \mathrm{e}^{b(\log R)^{2/3}/\kappa^{1/3}}\right\}, \tag{7.13}$$

and

$$\mathcal{T}_2 := \mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R}\ \sup_{x\in(x_j,x_{j+1})} |u_t(x) - u_t(x_j)| > \mathrm{e}^{b(\log R)^{2/3}/\kappa^{1/3}}\right\}. \tag{7.14}$$

[Compare with (6.19)].

On one hand, Lemma 7.1 implies that

$$\begin{aligned}
\mathcal{T}_1 &\leqslant 2R\cdot \mathrm{P}\left\{u_t(x_j) > \mathrm{e}^{b(\log R)^{2/3}/\kappa^{1/3}}\right\} \\
&\leqslant \frac{2c_3R}{R^{c_4b^{3/2}/\sqrt{\kappa}}}.
\end{aligned} \tag{7.15}$$

for some constants $c_3, c_4 > 0$. On the other hand (7.11) and Chebyshev's inequality imply that

$$\begin{aligned}
\mathcal{T}_2 &\leqslant 2R\,\mathrm{P}\left\{\sup_{x,x'\in I:|x-x'|\leqslant 1} |u_t(x) - u_t(x')| \geqslant \mathrm{e}^{b(\log R)^{2/3}/\kappa^{1/3}}\right\} \\
&\leqslant \frac{2R\mathrm{e}^{Ck^3/\kappa}}{\mathrm{e}^{2kb(\log R)^{2/3}/\kappa^{1/3}}}.
\end{aligned} \tag{7.16}$$

Now we choose $k := \lceil \kappa (\log R)^{1/3} \rceil$ in order to obtain $\mathcal{T}_2 \leqslant 2R^{1+B-2b}$. With these choices of parameters we deduce from (7.15) and (7.16) that if b were sufficiently large, then

$$\sum_{R=1}^{\infty} \mathrm{P}\left\{ u_t^*(R) > 2\mathrm{e}^{b(\log R)^{2/3}/\kappa^{1/3}} \right\} < \infty. \tag{7.17}$$

The Borel-Cantelli lemma and a monotonicity argument together complete the proof. □

A Appendix on Walsh stochastic integrals

Throughout this appendix, $(\Omega\,, \mathcal{F}, \mathrm{P})$ denotes [as is usual] the underlying probability space. We state and prove some elementary properties of Walsh stochastic integrals [24].

Let $\mathcal{L}^d$ denote the collection of all Borel-measurable sets in $\mathbf{R}^d$ that have finite d-dimensional Lebesgue measure. [We could work with Lebesgue-measurable sets, also.]

Let us follow Walsh [24] and define for every $t > 0$ and $A \in \mathcal{L}^d$ the random field

$$W_t(A) := \int_{[0,t]\times A} W(\mathrm{d}s\,\mathrm{d}y). \tag{A.1}$$

The preceding stochastic integral is defined in the same sense as N. Wiener.

Let $\mathcal{F}_t(A)$ denote the sigma-algebra generated by all random variables of the form

$$\left\{ W_s(B) :\ s \in (0\,, t],\ B \in \mathcal{L}^d,\ B \subseteq A \right\}. \tag{A.2}$$

We may assume without loss of generality that, for all $A \in \mathcal{L}^d$, $\{\mathcal{F}_t(A)\}_{t>0}$ is a right-continuous P-complete filtration [i.e., satisfies the "usual hypotheses" of Dellacherie and Meyer [13]]. Otherwise, we augment $\{\mathcal{F}_t(A)\}_{t>0}$ in the usual way. Let

$$\mathcal{F}_t := \bigvee_{A\in\mathcal{L}^d} \mathcal{F}_t(A) \qquad (t > 0). \tag{A.3}$$

Let $\mathcal{P}$ denote the collection of all processes that are predictable with respect to $\{\mathcal{F}_t\}_{t>0}$. The elements of $\mathcal{P}$ are precisely the "predictable random fields" of Walsh [24].

For us, the elements of $\mathcal{P}$ are of interest because if $Z \in \mathcal{P}$ and

$$\|Z\|^2_{L^2(\mathbf{R}_+\times\mathbf{R}^d\times\Omega)} := \mathrm{E}\int_0^\infty \mathrm{d}t \int_{\mathbf{R}^d} \mathrm{d}x\ \left[Z_t(x)\right]^2 < \infty, \tag{A.4}$$

then the Walsh stochastic integral $I_t := \int_{[0,t]\times\mathbf{R}} Z_s(y)\, W(\mathrm{d}s\,\mathrm{d}y)$ is defined properly, and has good mathematical properties. Chief among those good properties are the following: $\{I_t\}_{t>0}$ is a continuous mean-zero L^2-martingale with quadratic variation $\langle I\rangle_t := \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [Z_s(y)]^2$.

Let us define $\mathcal{P}(A)$ to be the collection of all processes that are predictable with respect to $\{\mathcal{F}_t(A)\}_{t>0}$. Clearly, $\mathcal{P}(A) \subseteq \mathcal{P}$ for all $A \in \mathcal{L}^d$.

Proposition A.1. *If $Z \in \mathcal{P}(A)$ for some $A \in \mathcal{L}^d$, and $\|Z\|_{L^2(\mathbf{R}_+ \times \mathbf{R}^d \times \Omega)} < \infty$, then the martingale defined by $J_t := \int_{[0,t]\times A} Z_s(y)\, W(\mathrm{d}s\,\mathrm{d}y)$ is in $\mathcal{P}(A)$.*

Proof. It suffices to prove this for a random field Z that has the form

$$Z_s(y)(\omega) = \mathbf{1}_{[a,b]}(s) X(\omega) \mathbf{1}_A(y) \qquad (s > 0,\, y \in \mathbf{R},\, \omega \in \Omega), \tag{A.5}$$

where $0 \leqslant a < b$, and X is a bounded $\mathcal{F}_a(A)$-measurable random variable. But in that case, $J_t(\omega) = X(\omega) \cdot \int_{[0,t]\cap[a,b]\times A} W(\mathrm{d}s\,\mathrm{d}y)$, whence the result follows easily from the easy-to-check fact that the stochastic process defined by $I_t := \int_{[0,t]\cap[a,b]\times A} W(\mathrm{d}s\,\mathrm{d}y)$ is continuous [up to a modification]. The latter assertion follows from the Kolmogorov continuity theorem; namely, we check first that $\mathrm{E}(|I_t - I_r|^2) = |A| \cdot |t - r|$, where $|A|$ denotes the Lebesgue measure of A. Then use the fact, valid for all Gaussian random variables including $I_t - I_r$, that $\mathrm{E}(|I_t - I_r|^k) = \mathrm{const} \cdot \{\mathrm{E}(|I_t - I_r|^2)\}^{k/2}$ for all $k \geqslant 2$. □

Proposition A.1 is a small variation on Walsh's original construction of his stochastic integrals. We need this minor variation for the following reason:

Corollary A.2. *Let $A^{(1)}, \ldots, A^{(N)}$ be fixed and nonrandom disjoint elements of $\mathcal{L}^d$. If $Z^{(1)}, \ldots, Z^{(N)}$ are respectively in $\mathcal{P}(A^{(1)}), \ldots, \mathcal{P}(A^{(N)})$ and $\|Z^{(j)}\|_{L^2(\mathbf{R}_+ \times \mathbf{R}^d \times \Omega)} < \infty$ for all $j = 1, \ldots, N$, then $J^{(1)}, \ldots, J^{(N)}$ are independent processes, where*

$$J_t^{(j)} := \int_{[0,t]\times A_j} Z_s(y)\, W(\mathrm{d}s\,\mathrm{d}y) \qquad (j = 1, \ldots, N,\, t > 0). \tag{A.6}$$

Proof. Owing to Proposition A.1, it suffices to prove that if some sequence of random fields $X^{(1)}, \ldots, X^{(N)}$ satisfies $X^{(j)} \in \mathcal{P}(A^{(j)})$ $(j = 1, \ldots, N)$, then $X^{(1)}, \ldots, X^{(N)}$ are independent. It suffices to prove this in the case that the $X^{(j)}$'s are simple predictable processes; i.e., in the case that

$$X_t^{(j)}(\omega) = \mathbf{1}_{[a_j,b_j]}(s) Y_j(\omega) \mathbf{1}_{A^{(j)}}(y), \tag{A.7}$$

where $0 < a_j < b_j$ and Y_j is a bounded $\mathcal{F}_{a_j}(A^{(j)})$-measurable random variable. In turn, we may restrict attention to Y_j's that have the form

$$Y_j(\omega) := \varphi_j\left(\int_{[\alpha_j,\beta_j]\times A^{(j)}} W(\mathrm{d}s\,\mathrm{d}y)\right), \tag{A.8}$$

where $0 < \alpha_j < \beta_j \leqslant a_j$ and $\varphi_j : \mathbf{R} \to \mathbf{R}$ is bounded and Borel measurable. But the assertion is now clear, since $Y_1, \ldots, Y_N$ are manifestly independent. In order to see this we need only verify that the covariance between $\int_{[\alpha_j,\beta_j]\times A^{(j)}} W(\mathrm{d}s\,\mathrm{d}y)$ and $\int_{[\alpha_k,\beta_k]\times A^{(k)}} W(\mathrm{d}s\,\mathrm{d}y)$ is zero when $j \neq k$; and this is a ready consequence of the fact that $A^{(j)} \cap A^{(k)} = \varnothing$ when $j \neq k$. □

B Some final remarks

Recall that $\|U\|_k$ denotes the usual $L^k(\mathrm{P})$-norm of a random variable U for all $k \in (0\,,\infty)$. According to Lemma 3.2,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbf{R}} \mathrm{E}\left(|u_t(x)|^k\right) \leqslant aCk^3 \quad \text{if } k \geqslant 2. \tag{B.1}$$

This and Jensen's inequality together imply that

$$\overline{\gamma}(\nu) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbf{R}} \mathrm{E}\left(|u_t(x)|^\nu\right) < \infty \quad \text{for all } \nu > 0. \tag{B.2}$$

And Chebyshev's inequality implies that for all $\nu > 0$,

$$\begin{aligned} \sup_{x\in\mathbf{R}} \mathrm{P}\left\{u_t(x) \geqslant \mathrm{e}^{-qt}\right\} &\leqslant \exp\left(\nu t\left[q + \frac{1}{\nu t}\log\sup_{x\in\mathbf{R}} \mathrm{E}\left(|u_t(x)|^\nu\right)\right]\right) \\ &= \exp\left(\nu t\left[q + \frac{\overline{\gamma}(\nu)}{\nu} + o(1)\right]\right) \qquad (t\to\infty). \end{aligned} \tag{B.3}$$

Because $u_t(x) \geqslant 0$, it follows that $\nu \mapsto \overline{\gamma}(\nu)/\nu$ is nondecreasing on $(0\,,\infty)$, whence

$$\ell := \lim_{\nu\downarrow 0} \frac{\overline{\gamma}(\nu)}{\nu} = \inf_{\nu>0} \frac{\overline{\gamma}(\nu)}{\nu} \text{ exists and is finite.} \tag{B.4}$$

Therefore, in particular,

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbf{R}} \mathrm{P}\left\{u_t(x) \geqslant \mathrm{e}^{-qt}\right\} < 0 \text{ for all } q \in (-\infty\,,-\ell). \tag{B.5}$$

Now consider the case that $\sigma(0) = 0$, and recall that in that case $u_0(x) \geqslant 0$ for all $x \in \mathbf{R}$. Mueller's comparison principle tells us that $u_t(x) \geqslant 0$ a.s. for all $t \geqslant 0$ and $x \in \mathbf{R}$, whence it follows that $\|u_t(x)\|_1 = \mathrm{E}[u_t(x)] = (p_t * u_0)(x)$ is bounded in t. This shows that $\overline{\gamma}(1) = 0$, and hence $\ell \leqslant 0$. We have proved the following:

Proposition B.1. *If $\sigma(0) = 0$, then there exists $q \geqslant 0$ such that*

$$\frac{1}{t}\log u_t(x) \leqslant -q + o_{\mathrm{P}}(1) \qquad \text{as } t\to\infty, \text{ for every } x \in \mathbf{R}, \tag{B.6}$$

where $o_{\mathrm{P}}(1)$ is a term that converges to zero in probability as $t \to \infty$.

Bertini and Giacomin [5] have studied the case that $\sigma(x) = cx$, and have shown that in that case, there exists a special choice of u_0 such that for all compactly supported probability densities $\psi \in C^\infty(\mathbf{R})$,

$$\frac{1}{t}\log\int_{-\infty}^{\infty} u_t(x)\psi(x)\,\mathrm{d}x = -\frac{c^2}{24\kappa} + o_{\mathrm{P}}(1) \quad \text{as } t\to\infty. \tag{B.7}$$

Eq. (B.7) and more generally Proposition B.1 show that the typical behavior of the sample-function of the solution to (1.2) is subexponential in time, as one might expect from the unforced linear heat equation. And yet, it frequently is the case that $u_t(x)$ grows in time exponentially rapidly in $L^k(\mathrm{P})$ for $k \geqslant 2$ [2, 6, 16]. This phenomenon is further evidence of physical intermittency in the sort of systems that are modeled by (1.2).

Standard predictions suggest that the typical behavior of $u_t(x)$ [in this and related models] is that it decays exponentially rapidly with time. [(B.7) is proof of this fact in one special case.] In other words, one might expect that typically $q > 0$. We are not able to resolve this matter here, and therefore ask the following questions:

Open problem 1. Is $q > 0$ in Proposition B.1? Equivalently, is $\ell < 0$ in (B.4)?

Open problem 2. Can the $o_{\mathrm{P}}(1)$ in (B.6) be replaced by a term that converges almost surely to zero as $t \to \infty$?

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