SUPREMA OF LÉVY PROCESSES

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Abstract. In this paper we study the supremum functional $M_t = \sup_{0\le s\le t} X_s$, where $X_t, t \ge 0$, is a one-dimensional Lévy process. Under very mild assumptions we provide a simple uniform estimate of the cumulative distribution function of M_t. In the symmetric case we find an integral representation of the Laplace transform of the distribution of M_t if the Lévy-Khintchin exponent of the process increases on $(0,\infty)$.

1. Introduction

By a classical reflection argument, the supremum functional $M_t = \sup_{0\le s\le t} X_s$ of the Brownian motion X_t has truncated normal distribution, $\mathbf{P}(M_t \ge x) = 2\mathbf{P}(X_t \ge x)$ $(x \ge 0)$. A similar question for symmetric α-stable processes was first studied by Darling [11], and the case of general Lévy processes X_t was addressed by Baxter and Donsker [3]. Theorem 1 therein gives a formula for the double Laplace transform of the distribution of M_t, which for a *symmetric* Lévy process X_t with Lévy-Khintchin exponent $\Psi(\xi)$ reads

$$\int_0^\infty \int_0^\infty e^{-\xi x - zt}\mathbf{P}(M_t \in dx)dt = \frac{1}{\sqrt{z}} \exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi \log(z+\Psi(\zeta))}{\xi^2+\zeta^2}\, d\zeta\right). \tag{1.1}$$

Inversion of the double Laplace transform is typically a very difficult task. Apart from the Brownian motion case, an explicit formula for the distribution of M_t was found for the Cauchy process (the symmetric 1-stable process) by Darling [11], for a compound Poisson process with $\Psi(\xi) = 1 - \cos\xi$ by Baxter and Donsker [3] and for the Poisson process with drift by Pyke [32].

The development of the fluctuation theory for Lévy processes resulted in many new identities involving the supremum functional M_t, see, for example, [5, 13, 31, 33]. There are numerous other representations for the distribution of M_t, at least in the stable case, see [4, 7, 11, 12, 15, 16, 19, 20, 27, 28, 30, 36]. The main goal of this article is to give a more explicit formula for $\mathbf{P}(M_t < x)$ and simple sharp bounds for $\mathbf{P}(M_t < x)$ in terms of the Lévy-Khintchin exponent $\Psi(\xi)$ for a class of Lévy processes. Most estimates of the cumulative distribution function of M_t are proved for very general Lévy processes, without symmetry assumptions.

Let τ_x denote the first passage time through a barrier at the level x for the process X_t,

$$\tau_x = \inf\{t \ge 0 : X_t \ge x\}, \qquad x \ge 0,$$

with the infimum understood to be infinity when the set is empty. We always assume that $X_0 = 0$. Since $\mathbf{P}(M_t < x) = \mathbf{P}(\tau_x > t)$, the problems of finding the cumulative distribution functions of M_t and τ_x are the same. The supremum functional and first passage time statistics are important in various areas of applied probability ([1, 2]), as well as in mathematical physics ([21, 26]). The recent progress in the potential theory of Lévy processes is, in part, due to the application of fluctuation theory, see [9, 10, 18, 22, 23, 24, 25].

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The paper is organized as follows. Section 2 contains some preliminary material related to Bernstein functions, Stieltjes functions and estimates for the Laplace transform. In Section 3 (Theorem 3.1 and Corollary 3.2) we prove, under mild assumptions, the estimate

$$\mathbf{P}(M_t < x) \approx \min\left(1, \kappa(1/t, 0)V(x)\right), \qquad t, x > 0,$$

where $V(x)$ and $\kappa(z, 0)$ are the renewal function for the ascending ladder-height process, and the Laplace exponent of the the ascending ladder-time process corresponding to X_t, respectively. Here $f(x) \approx g(x)$ means that there are constants $c_1, c_2 > 0$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$. In Section 4 we show that in the symmetric case, given some regularity of $\Psi(\xi)$, we have

$$V(x) \approx \frac{1}{\sqrt{\Psi(1/x)}}, \qquad x > 0,$$

see Theorem 4.4. Therefore the estimate of the above cumulative distribution function of M_t takes a very explicit form

$$\mathbf{P}(M_t < x) \approx \min\left(1, \frac{1}{\sqrt{t\Psi(1/x)}}\right), \qquad t, x > 0.$$

The other main result of Section 4 is an explicit formula for the (single, in the space variable) Laplace transform of the distribution of M_t (Theorem 4.1), under the assumption that X_t is symmetric and $\Psi(\xi)$ is increasing on $[0, \infty)$.

When $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function $\psi(\xi)$, the above resuls can be significantly improved. Following the approach of [30], a (rather complicated) explicit formula for $\mathbf{P}(M_t < x)$ can be given, and estimates and asymptotic formulae for $\mathbf{P}(M_t < x)$ extend to $(d/dt)^n \mathbf{P}(M_t < x)$ when x is small or t is large. These results will be covered in a forthcoming paper.

Notation. We denote by C, C_1, C_2 etc. constants in theorems, and by c, c_1, c_2 etc. temporary constants in proofs. Any dependence of a constant on some parameters is always indicated by writing, for example, $c(n, \varepsilon)$. We write $f(x) \sim g(x)$ when $f(x)/g(x) \to 1$. We use the terms *increasing*, *decreasing*, *concave*, *convex function* etc. in the weak sense.

2. Preliminaries

2.1. **Complete Bernstein and Stieltjes functions.** A function $\psi(\xi)$ is said to be a *complete Bernstein function* (CBF) if

$$\psi(\xi) = c_1 + c_2\xi + \frac{1}{\pi}\int_{0+}^{\infty} \frac{\xi}{\xi + \zeta}\,\frac{\mu(d\zeta)}{\zeta}, \qquad \xi \in \mathbf{C} \setminus (-\infty, 0), \tag{2.1}$$

where $c_1, c_2 \ge 0$, and μ is a measure on $(0, \infty)$ such that $\int \min(\zeta^{-1}, \zeta^{-2})\mu(d\zeta) < \infty$. A function $\tilde{\psi}(\xi)$ is said to be a *Stieltjes functions* if

$$\tilde{\psi}(\xi) = \frac{\tilde{c}_1}{\xi} + \tilde{c}_2 + \frac{1}{\pi}\int_{0+}^{\infty} \frac{1}{\xi + \zeta}\,\tilde{\mu}(d\zeta), \qquad \xi \in \mathbf{C} \setminus (-\infty, 0], \tag{2.2}$$

for some $\tilde{c}_1, \tilde{c}_2 \ge 0$ and some measure $\tilde{\mu}$ on $(0, \infty)$ such that $\int \min(1, \zeta^{-1})\tilde{\mu}(d\zeta) < \infty$. See [34] for a general account on complete Bernstein functions, Stieltjes functions and related notions.

It is known that $\psi(\xi)$ is a CBF if and only if $\psi(\xi)$ is nonnegative and increasing on $(0, \infty)$, holomorphic in $\mathbf{C} \setminus (-\infty, 0]$, and $\operatorname{Im}\psi(\xi) > 0$ when $\operatorname{Im}\xi > 0$. Furthermore, if $\psi(\xi)$ is a CBF, then $\xi/\psi(\xi)$ is a CBF, and $1/\psi(\xi)$ and $\psi(\xi)/\xi$ are Stieltjes functions

The function $\tilde{\psi}(\xi)$ given by (2.2) is the Laplace transform of $\tilde{c}_2\delta_0(dx) + (\tilde{c}_1 + \mathcal{L}\tilde{\mu}(x))dx$ ([34], Theorem 2.2). Furthermore, $\pi\tilde{c}_1\delta_0(d\zeta) + \tilde{\mu}(d\zeta)$ is the limit of measures $-\operatorname{Im}(\tilde{\psi}(-\zeta +$

$i\varepsilon))d\zeta$ as $\varepsilon \to 0^+$ ([34], Corollary 6.3 and Comments 6.12), so, in a sense, it is the boundary value of $\tilde{\psi}$. Therefore, we use a shorthand notation $-\operatorname{Im}(\tilde{\psi}^+(-\zeta))d\zeta$ for $\tilde{\mu}(d\zeta)$. Furthermore, we have $\tilde{c}_1 = \lim_{\xi\to 0}(\xi\tilde{\psi}(\xi))$ and $\tilde{c}_2 = \lim_{\xi\to\infty}\tilde{\psi}(\xi)$.

Following [30], we define

$$\psi^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi\log\psi(\zeta^2)}{\xi^2+\zeta^2}\,d\zeta\right), \qquad \operatorname{Re}\xi > 0, \tag{2.3}$$

for any function $\psi(\xi)$ such that $\min(1,\zeta^{-2})\log\psi(\zeta^2)$ is integrable in $\zeta > 0$. By a simple substitution,

$$\psi^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log\psi(\xi^2\zeta^2)}{1+\zeta^2}\,d\zeta\right), \qquad \xi > 0. \tag{2.4}$$

By [30], Lemma 4, if $\psi(\xi)$ is a CBF, then also $\psi^\dagger(\xi)$ is a CBF (this was independently proved in [24], Proposition 2.4), and

$$\psi^\dagger(\xi)\psi^\dagger(-\xi) = \psi(-\xi^2), \qquad \xi \in \mathbf{C}\setminus\mathbf{R}. \tag{2.5}$$

Proposition 2.1. *If $\psi(\xi)$ is nonnegative on $(0,\infty)$ and both $\psi(\xi)$ and $\xi/\psi(\xi)$ are increasing on $(0,\infty)$, then*

$$e^{-2\mathcal{C}/\pi}\sqrt{\psi(\xi^2)} \le \psi^\dagger(\xi) \le e^{2\mathcal{C}/\pi}\sqrt{\psi(\xi^2)}, \tag{2.6}$$

where $\mathcal{C} \approx 0.916$ is the Catalan constant. Note that $e^{2\mathcal{C}/\pi} \le 2$.

If, in addition, $\psi(\xi)$ is regularly varying at ∞, then

$$\psi^\dagger(\xi) \sim \sqrt{\psi(\xi^2)}, \qquad \xi \to \infty. \tag{2.7}$$

An analogous statement for $\xi \to 0$ holds for $\psi(\xi)$ regularly varying at 0.

In particular, (2.6) holds for any CBF. Likewise, (2.7) holds for any regularly varying CBF.

A result similar to (2.6) was obtained independently in [25], Proposition 3.7, while (2.7) for CBFs was derived in [22], Proposition 2.2.

Proof. By the assumptions, we have

$$\psi(\xi^2)\min(1,\zeta^2) \le \psi(\xi^2\zeta^2) \le \psi(\xi^2)\max(1,\zeta^2), \qquad \xi,\zeta > 0. \tag{2.8}$$

It follows that

$$\psi^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log\psi(\xi^2\zeta^2)}{1+\zeta^2}\,d\zeta\right) \le \sqrt{\psi(\xi^2)}\,\exp\left(\frac{1}{\pi}\int_1^\infty \frac{\log\zeta^2}{1+\zeta^2}\,d\zeta\right) = e^{2\mathcal{C}/\pi}\sqrt{\psi(\xi^2)}.$$

The lower bound is obtained in a similar manner.

The second statement of the proposition is proved in a very similar manner to Lemma 15 in [30]. Define an auxiliary function $h(\xi,\zeta) = \psi(\xi^2\zeta^2)/\psi(\xi^2)$. By (2.8) we have $|\log h(\xi,\zeta)| \le 2|\log\zeta|$, $\xi,\zeta > 0$. Since ψ is regularly varying at infinity, for some α, $\lim_{\xi\to\infty} h(\xi,\zeta) = \zeta^{2\alpha}$ for each $\zeta > 0$. Hence, by dominated convergence,

$$\lim_{\xi\to\infty}\int_0^\infty \frac{\log h(\xi,\zeta)}{1+\zeta^2}\,d\zeta = \int_0^\infty \frac{\log\zeta^{2\alpha}}{1+\zeta^2}\,d\zeta = 0.$$

It follows that

$$\lim_{\xi\to\infty}\left(\int_0^\infty \frac{\log\psi(\xi^2\zeta^2)}{1+\zeta^2}\,d\zeta - \frac{\pi}{2}\log\psi(\xi^2)\right) = 0,$$

and so finally $\lim_{\xi\to\infty}\psi^\dagger(\xi)/\sqrt{\psi(\xi^2)} = 1$, as desired. Regular variation at 0 is proved in a similar way. □

As in [30], for differentiable functions $\psi(\xi)$ with positive derivative we define

$$\psi_\lambda(\xi) = \frac{1-\xi/\lambda^2}{1-\psi(\xi)/\psi(\lambda^2)}, \qquad \lambda > 0,\ \xi \in \mathbf{C} \setminus (-\infty, 0). \tag{2.9}$$

This definition is extended continuously by $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2\psi'(\lambda^2))$. Note that if $\psi(0) = 0$, then $\psi_\lambda(0) = 1$. For simplicity, we denote $\psi_\lambda^\dagger(\xi) = (\psi_\lambda)^\dagger(\xi)$. By [30], Lemma 2, if $\psi(\xi)$ is a CBF, then $\psi_\lambda(\xi)$ is a CBF for any $\lambda > 0$.

2.2. **Estimates for the Laplace transform.** This short section contains some rather standard estimates for the inverse Laplace transform.

Proposition 2.2. *Let $a > 0$, $c \geq 1$. If f is nonnegative and $f(x) \leq cf(a)\max(1, x/a)$ $(x > 0)$, then for any $\xi > 0$,*

$$f(a) \geq \frac{\xi \mathcal{L}f(\xi)}{c(1+(a\xi)^{-1}e^{-a\xi})}.$$

Proof. We have

$$\begin{aligned}
\xi\mathcal{L}f(\xi) &= \int_0^a \xi e^{-\xi x} f(x)dx + \int_a^\infty \xi e^{-\xi x} f(x)dx \\
&\leq cf(a)\int_0^a \xi e^{-\xi x}dx + \frac{cf(a)}{a}\int_a^\infty \xi x e^{-\xi x}dx \\
&= cf(a)(1-e^{-a\xi}) + \frac{cf(a)}{a\xi}(1+a\xi)e^{-a\xi} = cf(a)(1+(a\xi)^{-1}e^{-a\xi}),
\end{aligned}$$

as desired. □

Proposition 2.3. *If f is nonnegative and increasing, then for $a, \xi > 0$,*

$$f(a) \leq e^{a\xi}\xi\mathcal{L}f(\xi).$$

Proof. As before,

$$\begin{aligned}
\xi\mathcal{L}f(\xi) &= \int_0^a \xi e^{-\xi x} f(x)dx + \int_a^\infty \xi e^{-\xi x} f(x)dx \\
&\geq f(a)\int_a^\infty \xi e^{-\xi x}dx = f(a)e^{-a\xi},
\end{aligned}$$

as claimed. □

Proposition 2.4. *If f is nonnegative and decreasing, then for $a, \xi > 0$,*

$$f(a) \leq \frac{\xi\mathcal{L}f(\xi)}{1-e^{-a\xi}}.$$

Proof. Again,

$$\begin{aligned}
\xi\mathcal{L}f(\xi) &= \int_0^a \xi e^{-\xi x} f(x)dx + \int_a^\infty \xi e^{-\xi x} f(x)dx \\
&\geq f(a)\int_0^a \xi e^{-\xi x}dx = f(a)(1-e^{-a\xi}),
\end{aligned}$$

as claimed. □

3. Suprema of general Lévy processes

We briefly recall the basic notions of the fluctuation theory for Lévy processes. Let L_t be the local time of the process X_t reflected at its supremum M_t, and denote by L_s^{-1} the right-continuous inverse of L_t, the ascending ladder-time process for X_t. This is a (possibly killed) subordinator, and $H_s = X(L_s^{-1}) = M(L_s^{-1})$ is another (possibly killed) subordinator, called the ascending ladder-height process. The Laplace exponent of the ascending ladder process, that is, the (possibly killed) bivariate subordinator (L_s^{-1}, H_s) $(s < L(\infty))$, is denoted by $\kappa(z,\xi)$. By, e.g., [5], Corollary VI.10,

$$\kappa(z,\xi) = c\exp\left(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-zt-\xi x})t^{-1}\mathbf{P}(X_t \in dx)dt\right), \tag{3.1}$$

where c is a normalization constant of the local time. Since our results are not affected by the choice of c, we assume that $c = 1$. We note that $\kappa(z,0)$ is a Bernstein function of z, and also $z/\kappa(z,0)$ is a Bernstein function (this follows from (3.1) by Frullani's integral; see [5], formula (VI.3) for the case when X_t is not a compound Poisson process). For more account on the fluctuation theory we refer the reader to [5, 13, 31]. In general, there is no closed-form formula for $\kappa(z,\xi)$. For a list of special cases, see [29] and the references therein. For a symmetric process which is not a compound Poisson process, we have $\kappa(z,0) = \sqrt{z}$.

As usual, τ_x denotes the first passage time through a barrier at $x \geq 0$ for X_t (or for M_t). Following [5], we define

$$V^z(x) = \mathbf{E}\left(\int_0^\infty \exp(-zL_s^{-1})\mathbf{1}_{[0,x)}(H_s)ds\right) = \mathbf{E}\left(\int_0^\infty e^{-zt}\mathbf{1}_{[0,x)}(M_t)dL_t\right), \quad x,z \geq 0.$$

For $z = 0$, we simply have $V^0(x) = \int_0^\infty \mathbf{P}(H_s < x)ds$, so that $V^0(x) = V(x)$ is the renewal function of the process H_s, studied in more detail for symmetric Lévy processes in Section 4. By [5], formula (VI.8),

$$\int_0^\infty e^{-zt}\mathbf{P}(M_t < x)dt = \frac{\kappa(z,0)V^z(x)}{z}, \qquad x,z \geq 0. \tag{3.2}$$

(Note that in [5] a weak inequality $M_t \leq x$ is used in the definition of $V^z(x)$.) Hence, for a symmetric process X_t which is not a compound Poisson process, we have

$$\int_0^\infty e^{-zt}\mathbf{P}(M_t < x)dt = \frac{V^z(x)}{\sqrt{z}}, \qquad x,z \geq 0. \tag{3.3}$$

This is a partial inverse of the double Laplace transform in (1.1); however, there is no known explicit formula for $V^z(x)$. For a different and, in a sense, more explicit partial inverse, see (4.2) below.

By [5], Section VI.4, the Laplace transform of $V^z(x)$ is $1/(\xi\kappa(z,\xi))$. Hence, when X_t is symmetric and it is not a compound Poisson process, the right hand side of the Baxter-Donsker formula (1.1) can be written as $\sqrt{z}/(z\kappa(z,\xi))$ (see [14], Corollary 9.7).

Theorem 3.1. *Let X_t be a Lévy process, $M_t = \sup_{0\leq s\leq t} X_s$, and let $\kappa(z,\xi)$ be the bivariate Laplace exponent of its ascending ladder process. Suppose that*

$$K(s) = \int_s^\infty \frac{\kappa(z,0)}{z^2}\,dz < \infty, \qquad s > 0, \tag{3.4}$$

and that $\kappa(z,0)/z$ is unbounded (near 0). For $t,x > 0$, we have

$$\min\left(C_1, C_2(\kappa,t)\kappa(1/t,0)V(x)\right) \leq \mathbf{P}(M_t < x) \leq \min\left(1, \frac{e}{e-1}\,\kappa(1/t,0)V(x)\right). \tag{3.5}$$

Here

$$C_1 = \frac{e-1}{8e^2} \quad \textit{and} \quad C_2(\kappa, t) = \frac{zt}{2e}, \quad \textit{where } z \in (0, 1/t) \textit{ solves } \quad \frac{\kappa(z,0)}{z} = \frac{4e^2}{e-1} K(1/t).$$

Proof. The upper bound in (3.5) is a direct consequence of (3.2) and Proposition 2.4 with $\xi = 1/t$.

Following [5], Lemma VI.21, we find a lower bound for $V^z(x)$. We have

$$\begin{aligned} V(x) &= \mathbf{E}\left(\int_0^\infty \mathbf{1}_{[0,x)}(M_t) dL_t\right) \\ &\le e\,\mathbf{E}\left(\int_0^{1/z} e^{-zt}\mathbf{1}_{[0,x)}(M_t) dL_t\right) + \mathbf{E}\left(\int_{1/z}^\infty \mathbf{1}_{[0,x)}(M_t) dL_t\right), \end{aligned}$$

which implies

$$eV^z(x) \ge V(x) - \mathbf{E}\left(\int_{1/z}^\infty \mathbf{1}_{[0,x)}(M_t) dL_t\right). \tag{3.6}$$

Let $\sigma_z = \inf\{t \ge 1/z : X_t = M_t\} = L^{-1}(L_{1/z})$; σ_z is a stopping time. Since the support of the measure dL_t is contained in the set $\{t : X_t = M_t\}$ of zeros of the reflected process, we have

$$\begin{aligned} \mathbf{E}\left(\int_{1/z}^\infty \mathbf{1}_{[0,x)}(M_t) dL_t\right) &= \mathbf{E}\left(\int_{\sigma_z}^\infty \mathbf{1}_{[0,x)}(M_t) dL_t; M_{1/z} < x\right) \\ &\le \mathbf{E}\left(\int_{\sigma_z}^\infty \mathbf{1}_{[0,x)}(M_t - M_{\sigma_z}) dL_t; M_{1/z} < x\right). \end{aligned}$$

Next, observe that $M_{\sigma_z} = X_{\sigma_z}$, so that

$$M_t - M_{\sigma_z} = \sup_{s \le t - \sigma_z} (X_{\sigma_z + s} - X_{\sigma_z}), \qquad t \ge \sigma_z.$$

Hence,

$$\begin{aligned} \mathbf{E}\left(\int_{1/z}^\infty \mathbf{1}_{[0,x)}(M_t) dL_t\right) &\le \mathbf{E}\left(\int_{\sigma_z}^\infty \mathbf{1}_{[0,x)}\left(\sup_{s \le t-\sigma_z} (X_{\sigma_z+s} - X_{\sigma_z})\right) dL_t; M_{1/z} < x\right) \\ &= \mathbf{E}\left(\int_0^\infty \mathbf{1}_{[0,x)}\left(\sup_{s \le u} (X_{\sigma_z+s} - X_{\sigma_z})\right) d(L_{\sigma_z+u} - L_{\sigma_z}); M_{1/z} < x\right). \end{aligned}$$

Since $\sigma_z \ge 1/z$, by the strong Markov property,

$$\mathbf{E}\left(\int_{1/z}^\infty \mathbf{1}_{[0,x)}(M_t) dL_t\right) \le \mathbf{P}(M_{1/z} < x)\mathbf{E}\left(\int_0^\infty \mathbf{1}_{[0,x)}(M_u) dL_u\right) = \mathbf{P}(M_{1/z} < x)V(x),$$

which, by (3.6), yields

$$V^z(x) \ge \frac{(1 - \mathbf{P}(M_{1/z} < x))V(x)}{e} = \frac{\mathbf{P}(M_{1/z} \ge x)V(x)}{e}.$$

Let $k > 0$. By (3.2) and the already proved upper bound of (3.5),

$$\begin{aligned} V^z(x)\kappa(z,0) &= z\int_0^{k/z} e^{-zt}\mathbf{P}(M_t < x) dt + z\int_{k/z}^\infty e^{-zt}\mathbf{P}(M_t < x) dt \\ &\le \frac{e}{e-1} V(x) z \int_0^{k/z} e^{-zt}\kappa(1/t, 0) dt + \mathbf{P}(M_{k/z} < x). \end{aligned}$$

The last two estimates give

$$\begin{aligned}\mathbf{P}(M_{k/z} < x) &\geq \frac{\kappa(z,0)\mathbf{P}(M_{1/z} \geq x)V(x)}{e} - \frac{e}{e-1}V(x)z\int_0^{k/z}\kappa(1/t,0)dt \\ &= \frac{V(x)\kappa(z,0)}{e}\left(\mathbf{P}(M_{1/z} \geq x) - \frac{e^2}{e-1}\,\frac{zK(z/k)}{\kappa(z,0)}\right).\end{aligned} \tag{3.7}$$

Fix $\varepsilon \in (0,1)$ (later we choose $\varepsilon = 1/4$). Note that the function $\kappa(z,0)/z$ is continuous, decreasing and unbounded. Hence, it maps the interval $(0,1/t]$ onto the interval $[t\kappa(1/t,0),\infty)$. Furthermore, $\kappa(z,0)$ is increasing, so that $K(z) \geq \kappa(z,0)/z$. In particular, $\frac{e^2}{\varepsilon(e-1)}K(1/t) > K(1/t) \geq t\kappa(1/t,0)$. It follows that we can choose $z = z(t) < 1/t$ such that

$$\frac{\kappa(z,0)}{z} = \frac{e^2}{\varepsilon(e-1)}K(1/t).$$

Setting $k = zt < 1$, the above equality can be rewritten as

$$\frac{e^2}{e-1}\,\frac{zK(z/k)}{\kappa(z,0)} = \varepsilon. \tag{3.8}$$

Suppose now that $V(x)\kappa(z,0) \leq \varepsilon(e-1)/e$. Then, by the upper bound of (3.5), we have $\mathbf{P}(M_{1/z} \geq x) = 1 - \mathbf{P}(M_{1/z} < x) \geq 1 - \varepsilon$. This, (3.7) and (3.8) give

$$\mathbf{P}(M_t < x) = \mathbf{P}(M_{k/z} < x) \geq \frac{V(x)\kappa(z,0)}{e}(1-2\varepsilon).$$

This estimate holds for $t \geq t_0$, where $V(x)\kappa(z(t_0),0) = \varepsilon(e-1)/e$ (here we use continuity of $\kappa(z(t),0)$ as a function of t). Hence, by monotonicity of $\mathbf{P}(M_t < x)$ in t,

$$\mathbf{P}(M_t < x) \geq \min\left(\frac{\varepsilon(1-2\varepsilon)(e-1)}{e^2}, \frac{(1-2\varepsilon)V(x)\kappa(z,0)}{e}\right).$$

The lower bound in (3.5) follows by taking $\varepsilon = 1/4$ and using the inequality $\kappa(z,0) = \kappa(k/t,0) \geq k\kappa(1/t,0)$. □

To formulate the next result we define *upper scaling conditions*:

$$\text{for some } \varrho \in (0,1) \text{ and } c > 0, \quad \frac{\kappa(z_2,0)}{\kappa(z_1,0)} \leq c\,\frac{z_2^\varrho}{z_1^\varrho} \quad \text{when } 0 < z_1 < z_2 < 1, \tag{3.9}$$

$$\text{for some } \varrho \in (0,1) \text{ and } c > 0, \quad \frac{\kappa(z_2,0)}{\kappa(z_1,0)} \leq c\,\frac{z_2^\varrho}{z_1^\varrho} \quad \text{when } 1 < z_1 < z_2. \tag{3.10}$$

Observe that the condition (3.10) implies that for any $z^* > 0$ there is c^* such that

$$\frac{\kappa(z_2,0)}{\kappa(z_1,0)} \leq c^*\,\frac{z_2^\varrho}{z_1^\varrho} \qquad \text{when } z^* < z_1 < z_2. \tag{3.11}$$

Corollary 3.2. *Let X_t be a Lévy process, $M_t = \sup_{0\leq s\leq t} X_s$, and let $\kappa(z,\xi)$ be the bivariate Laplace exponent of its ascending ladder process. If $\kappa(z,0)$ satisfies condition* (3.9) *with $0 < \varrho < 1$ and the integral $\int_1^\infty \kappa(z,0)z^{-2}dz$ is finite then*

$$C(\kappa)\min\left(1, \kappa(1/t,0)V(x)\right) \leq \mathbf{P}(M_t < x) \leq \min\left(1, 2\kappa(1/t,0)V(x)\right), \tag{3.12}$$

for every $x > 0$ and $t \geq 1$. If $\kappa(z,0)$ satisfies (3.10) *with $0 < \varrho < 1$ and $\lim_{z\to 0} z/\kappa(z,0) = 0$ then* (3.12) *holds for $x > 0$ and $t \leq 1$.*

In particular, if $\kappa(z,0)$ satisfies both (3.9) *and* (3.10)*, that is, there are $c > 0$ and $\varrho \in (0,1)$ such that $\kappa(\lambda z,0) \leq c\lambda^\varrho\kappa(z,0)$ for $\lambda \geq 1$ and $z > 0$, then* (3.12) *is true for every $x > 0$ and $t > 0$.*

Proof. We begin with the first part of the statement. By the condition (3.9),

$$\kappa(z,0) \le c_1(\kappa)\left(\frac{z}{s}\right)^{\varrho}\kappa(s,0), \qquad s \le z \le 1.$$

In particular, $\kappa(s,0)/s$ is unbounded. Furthermore, using also finiteness of the integral $\int_1^\infty \kappa(z,0)z^{-2}dz$, we obtain

$$K(s) \le c_2(\kappa)\frac{\kappa(s,0)}{s}, \qquad s \le 1. \tag{3.13}$$

This implies that the assumptions of Theorem 3.1 are satisfied.

Let $t \ge 1$ and define $z = z(t) \in (0, 1/t)$ as in Theorem 3.1. By the condition (3.9) we have

$$\frac{\kappa(1/t,0)}{\kappa(z,0)} \le \frac{c_3(\kappa)}{(zt)^{\varrho}}.$$

By definition of z and (3.13) (with $s = 1/t$), we have

$$\frac{1}{z} = \frac{4e^2}{e-1}\,\frac{K(1/t)}{\kappa(z,0)} \le \frac{4e^2 c_2(\kappa)c_3(\kappa)}{e-1}\,\frac{t}{(tz)^{\varrho}},$$

which gives $zt \ge c_4(\kappa)$. Hence, the constant C_2 in Theorem 3.1 satisfies $C_2 = zt/(2e) \ge c_4(\kappa)/(2e)$. This ends the proof of the first part.

The second part can be justified in a similar way, since the condition (3.10) implies that

$$K(s) \le c_5(\kappa)\frac{\kappa(s,0)}{s}, \qquad s \ge 1.$$

Moreover, for $t < 1$ and $z = z(t)$ selected according to Theorem 3.1 we have $z(1) \le z(t) < 1/t$. Applying (3.10) (with $z^* = z(1)$) we obtain

$$\frac{\kappa(1/t,0)}{\kappa(z,0)} \le \frac{c_6(\kappa)}{(zt)^{\varrho}}, \qquad z \le \frac{1}{t}.$$

Finally, the last statement is a direct consequence of the previous ones. □

Remark 3.3. Due to Potter's theorem ([8], Theorem 1.5.6) the condition (3.9) is implied by regular variation of $\kappa(z,0)$ at zero with index $0 < \varrho^* < 1$. Likewise, the condition (3.10) is implied by regular variation of $\kappa(z,0)$ at ∞ with index $0 < \varrho^* < 1$.

In the second part of the above corollary the assumption $\lim_{z\to 0} z/\kappa(z,0) = 0$ can be removed at the expence that the lower bound holds for $t \le t_0$, where $t_0 = t_0(\kappa)$ is sufficiently small. This is due to the fact that since $\lim_{t\searrow 0} K(1/t) = 0$, $z = z(t)$ in Theorem 3.1 is well defined for t small enough.

By the results of [5], Theorem VI.14, and [6], the regular variation of order $\varrho \in (0,1)$ of $\kappa(z,0)$ at 0 or at ∞ is equivalent to the existence of the limit of $\mathbf{P}(X_t \ge 0)$ as $t \to \infty$ or $t \to 0^+$, respectively. Hence, Corollary 3.2 implies the following result.

Corollary 3.4. *Let X_t be a Lévy process and $M_t = \sup_{0\le s\le t} X_s$. If*

$$\lim_{t\to\infty}\mathbf{P}(X_t \ge 0) \in (0,1) \qquad \textit{and} \qquad \limsup_{t\to 0^+}\mathbf{P}(X_t \ge 0) < 1,$$

then (3.12) *holds for $x > 0$ and $t \ge 1$. If*

$$\lim_{t\to 0^+}\mathbf{P}(X_t \ge 0) \in (0,1) \qquad \textit{and} \qquad \limsup_{t\to\infty}\mathbf{P}(X_t \ge 0) < 1,$$

then (3.12) *is true for $x > 0$ and $t \le 1$. Finally, if*

$$\lim_{t\to\infty}\mathbf{P}(X_t \ge 0) \in (0,1) \qquad \textit{and} \qquad \lim_{t\to 0^+}\mathbf{P}(X_t \ge 0) \in (0,1),$$

then (3.12) *holds for every* $x > 0$ *and* $t > 0$.

Proof. We only need to verify that $\kappa(z,0)/z^2$ is integrable at infinity, and that $\lim_{z\to 0^+}(z/\kappa(z,0)) = 0$. In each of the cases, there is $\varepsilon > 0$ such that $\mathbf{P}(X_t \geq 0) \leq 1-\varepsilon$ for all $t > 0$. Therefore, by (3.1) and the Frullani integral, $\kappa(z,0) \leq z^{1-\varepsilon}$ for $z \geq 1$, and $\kappa(z,0) \geq z^{1-\varepsilon}$ when $0 < z < 1$. The result follows. □

Remark 3.5. The uniform estimates of Corollary 3.4 complement the existing results from [17] about the asymptotic behavior of $\mathbf{P}(M_t < x)$, where it was shown that

$$\lim_{t\to\infty} \frac{\sqrt{\pi}}{\kappa(1/t,0)} \mathbf{P}(M_t < x) = V(x),$$

under the assumption that $\kappa(z,0)$ is regularly varying at zero with index $\varrho \in (0,1)$.

4. Suprema of symmetric Lévy processes

In this section we assume that X_t is a *symmetric* Lévy process with Lévy-Khintchin exponent $\Psi(\xi)$. In a rather general setting, we can invert the Laplace transform in time variable in (1.1).

Theorem 4.1. *Suppose that* X_t *is a symmetric Lévy process with Lévy-Khintchin exponent* $\Psi(\xi)$. *Suppose that* $\Psi(\xi)$ *is increasing in* $\xi > 0$. *If* $M_t = \sup_{0\leq s\leq t} X_s$, *then*

$$\mathbf{E}e^{-\xi M_t} = \frac{1}{\pi}\int_0^\infty \frac{\xi\Psi'(\lambda)}{(\lambda^2+\xi^2)\sqrt{\Psi(\lambda)}} \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log\frac{\lambda^2-\zeta^2}{\Psi(\lambda)-\Psi(\zeta)}}{\xi^2+\zeta^2}\,d\zeta\right) e^{-t\Psi(\lambda)}d\lambda. \tag{4.1}$$

Since $\mathbf{P}(M_t < x) = \mathbf{P}(\tau_x > t)$, the following integrated form of (4.1) is sometimes more convenient.

Corollary 4.2. *With the notation and assumptions of Theorem 4.1,*

$$\begin{aligned}\int_0^\infty e^{-\xi x}\mathbf{P}(\tau_x > t)dx &= \frac{\mathbf{E}e^{-\xi M_t}}{\xi} \\ &= \frac{1}{\pi}\int_0^\infty \frac{\Psi'(\lambda)}{(\lambda^2+\xi^2)\sqrt{\Psi(\lambda)}} \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log\frac{\lambda^2-\zeta^2}{\Psi(\lambda)-\Psi(\zeta)}}{\xi^2+\zeta^2}\,d\zeta\right) e^{-t\Psi(\lambda)}d\lambda.\end{aligned} \tag{4.2}$$

Proof of Theorem 4.1. Let $\psi(\xi) = \Psi(\sqrt{\xi})$ for $\xi > 0$. For any $z \in \mathbf{C}\setminus(-\infty,0]$ and $\xi > 0$, we define (see (1.1) and (2.3))

$$\varphi(\xi,z) = \sqrt{z}\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi\,\log(z+\Psi(\zeta))}{\xi^2+\zeta^2}\,d\zeta\right) = \exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi\log(1+\frac{\psi(\zeta^2)}{z})}{\xi^2+\zeta^2}\,d\zeta\right).$$

For any $\xi > 0$, the function $\varphi(\xi,z)$ is positive and increasing in $z \in (0,\infty)$. As $z \to 0$ or $z \to \infty$, $\varphi(\xi,z)$ converges to 0 and 1, respectively. Furthermore, if $\operatorname{Im} z > 0$, then $\arg(1+\psi(\zeta^2)/z) \in (-\pi,0)$ for all $\zeta > 0$, and therefore

$$\arg\varphi(\xi,z) = -\frac{1}{\pi}\int_0^\infty \frac{\xi\arg(1+\frac{\psi(\zeta^2)}{z})}{\xi^2+\zeta^2}\,d\zeta \in (0,\pi/2).$$

Hence, for any $\xi > 0$, $\varphi(\xi,z)$ (and even $(\varphi(\xi,z))^2$) is a complete Bernstein function of z. Note that the continuous boundary limit $\varphi^+(\xi,-z)$ exists for $z > 0$: if $z = \psi(\lambda^2)$, or

$\lambda = \sqrt{\psi^{-1}(z)}$, then

$$\begin{aligned}\varphi^+(\xi,-z) &= \exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi \log^-(1-\frac{\psi(\zeta^2)}{\psi(\lambda^2)})}{\xi^2+\zeta^2}\,d\zeta\right)\\ &= \exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi \log|1-\frac{\psi(\zeta^2)}{\psi(\lambda^2)}|}{\xi^2+\zeta^2}\,d\zeta + i\int_\lambda^\infty \frac{\xi}{\xi^2+\zeta^2}\,d\zeta\right)\\ &= \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi(\log\psi_\lambda(\zeta^2)-\log|1-\frac{\zeta^2}{\lambda^2}|)}{\xi^2+\zeta^2}\,d\zeta + i\arctan\frac{\xi}{\lambda}\right);\end{aligned}$$

see (2.9) for the notation. Here $\log^-$ denotes the boundary limit on $(-\infty,0)$ approached from below, $\log^-(-\zeta) = -i\pi/2+\log\zeta$ for $\zeta>0$. The function $\log|1-\zeta^2/\lambda^2|$ is harmonic in the upper half-plane $\operatorname{Im}\zeta>0$, so that

$$\frac{1}{\pi}\int_0^\infty \frac{\xi\log|1-\frac{\zeta^2}{\lambda^2}|}{\xi^2+\zeta^2}\,d\zeta = \frac{1}{2}\log\left(1+\frac{\xi^2}{\lambda^2}\right).$$

Furthermore, $\exp(i\arctan(\xi/\lambda)) = (\lambda+i\xi)/\sqrt{\lambda^2+\xi^2}$. Therefore, with $z=\psi(\lambda^2)$,

$$\varphi^+(\xi,-z) = \frac{\lambda(\lambda+i\xi)}{\lambda^2+\xi^2}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi\log\psi_\lambda(\zeta^2)}{\xi^2+\zeta^2}\,d\zeta\right) = \frac{\lambda(\lambda+i\xi)\psi_\lambda^\dagger(\xi)}{\lambda^2+\xi^2}; \tag{4.3}$$

see (2.3) for the notation. Note that if $\psi(\xi)$ is bounded on $(0,\infty)$ and $z\ge\sup_{\xi>0}\psi(\xi)$, then $\varphi^+(\xi,-z)$ is real.

By (1.1), $\varphi(\xi,z)/z$ is the double Laplace transform of the distribution of M_t. But for all $\xi>0$, $\varphi(\xi,z)/z$ is a Stieltjes function of z. Therefore, by (2.2),

$$\begin{aligned}\frac{\varphi(\xi,z)}{z} &= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\frac{\varphi^+(\xi,-\zeta)}{\zeta}\,\frac{1}{z+\zeta}\,d\zeta\\ &= \frac{1}{\pi}\int_0^\infty 2\lambda\psi'(\lambda^2)\operatorname{Im}\frac{\varphi^+(\xi,-\psi(\lambda^2))}{\psi(\lambda^2)}\,\frac{1}{z+\psi(\lambda^2)}\,d\lambda\\ &= \frac{2}{\pi}\int_0^\infty \frac{\lambda\psi'(\lambda^2)}{\psi(\lambda^2)}\,\frac{\lambda\xi\psi_\lambda^\dagger(\xi)}{\lambda^2+\xi^2}\,\frac{1}{z+\psi(\lambda^2)}\,d\lambda.\end{aligned}$$

Note that the second equality holds true also when $\psi(\xi)$ is bounded. Since $1/(z+\psi(\lambda^2)) = \int_0^\infty e^{-t\psi(\lambda^2)}e^{-zt}dt$, we have

$$\frac{\varphi(\xi,z)}{z} = \int_0^\infty\left(\frac{2}{\pi}\int_0^\infty \frac{\lambda\psi'(\lambda^2)}{\psi(\lambda^2)}\,\frac{\lambda\xi\psi_\lambda^\dagger(\xi)}{\lambda^2+\xi^2}\,e^{-t\psi(\lambda^2)}d\lambda\right)e^{-zt}dt.$$

The theorem follows by the uniqueness of the Laplace transform. □

Let $V(x)=V^0(x)$ be the renewal function for the ascending ladder-height process H_s corresponding to X_t; see Section 3 for the definition. When X_t satisfies the *absolute continuity condition* (for example, if $1/(1+\Psi(\xi))$ is integrable in ξ), then $V(x)$ is the (unique up to a multiplicative constant) increasing harmonic function for X_t on $(0,\infty)$, and $V'(x)$ is the decreasing harmonic function for X_t on $(0,\infty)$, cf. [35]. It is known ([5], formula (VI.6))) that for $\xi>0$,

$$\mathcal{L}V(\xi) = \frac{1}{\xi\kappa(0,\xi)},$$

Moreover, if X_t is not a compound Poisson process, then by [14], Corollary 9.7,

$$\kappa(0,\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi \log \Psi(\zeta)}{\xi^2+\zeta^2}\,d\zeta\right) = \psi^\dagger(\xi),$$

where $\Psi(\xi) = \psi(\xi^2)$, see (2.3) for the notation. Clearly, we have $\mathcal{L}V'(\xi) = \xi\mathcal{L}V(\xi) = 1/\psi^\dagger(\xi)$; here V' is the distributional derivative of V on $[0,\infty)$. We remark that when X_t is a compound Poisson process, then, also by [14], Corollary 9.7,

$$\kappa(0,\xi) = c\psi^\dagger(\xi), \qquad \text{with} \qquad c = \exp\left(-\frac{1}{2}\int_0^\infty \frac{1-e^{-t}}{t}\,\mathbf{P}(X_t=0)dt\right). \tag{4.4}$$

For simplicity, we state the next three results only for the case when X_t is not a compound Poisson process. However, extensions for compound Poisson processes are straightforward due to (4.4).

As an immediate consequence of Proposition 2.1 and Karamata's Tauberian theorem ([8], Theorem 1.7.1), we obtain the following result, which in the case of complete Bernstein functions was derived in Proposition 2.7 of [22].

Proposition 4.3. *Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process X_t, which is not a compound Poisson process, and suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. If $\Psi(\xi)$ is regularly varying at ∞, then V is regularly varying at 0 and $\Gamma(1+\alpha)V(x) \sim 1/\sqrt{\Psi(1/x)}$ as $x \to 0$. Similarly, if $\Psi(\xi)$ is regularly varying at 0, then $\Gamma(1+\alpha)V(x) \sim 1/\sqrt{\Psi(1/x)}$ as $x \to \infty$.*

Another consequence of Proposition 2.1 is a uniform estimate of the renewal function (see also Proposition 3.9 of [25]).

Theorem 4.4. *Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process X_t, which is not a compound Poisson process, and suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$. Then*

$$\frac{1}{5}\frac{1}{\sqrt{\Psi(1/x)}} \le V(x) \le 5\frac{1}{\sqrt{\Psi(1/x)}}, \qquad x > 0.$$

Proof. Let $\psi(\xi) = \Psi(\sqrt{\xi})$ for $\xi > 0$. By Proposition 2.1 we get $e^{-2\mathcal{C}/\pi}/\sqrt{\xi^2\psi(\xi^2)} \le \mathcal{L}V(\xi) \le e^{2\mathcal{C}/\pi}/\sqrt{\xi^2\psi(\xi^2)}$, $\xi > 0$. Since V is increasing, Proposition 2.3 gives

$$V(x) \le \frac{e\mathcal{L}V(1/x)}{x} \le \frac{e^{1+2\mathcal{C}/\pi}}{\sqrt{\psi(1/x^2)}} \le \frac{5}{\sqrt{\psi(1/x^2)}}.$$

Furthermore, using subadditivity and monotonicity of V (see [5], Section III.1), for $x = ka + r$ ($k \ge 0$, $r \in [0,a)$) we obtain $V(x) \le kV(a) + V(r) \le (k+1)V(a)$. It follows that $V(x) \le 2V(a)\max(1, x/a)$ for all $a, x > 0$, and so, by Proposition 2.2,

$$V(x) \ge \frac{\mathcal{L}V(1/x)}{2x(1+e^{-1})} \ge \frac{1}{2(1+e^{-1})e^{2\mathcal{C}/\pi}\sqrt{\psi(1/x^2)}} \ge \frac{1}{5\sqrt{\psi(1/x^2)}},$$

as desired. □

We remark that when V is a concave function on $(0,\infty)$ (for example, when ψ is a complete Bernstein function, see below), then clearly $V(x) \le \max(1, x/a)V(a)$, so that the lower bound in Theorem 4.4 holds with constant $2/5$ instead of $1/5$.

If $\psi(\xi)$ is a complete Bernstein function (CBF, see (2.1)), then $\psi^\dagger(\xi)$ and $\xi/\psi^\dagger(\xi)$ are CBFs, and hence $1/\psi^\dagger(\xi)$ is a Stieltjes function (see (2.2)). Therefore, $V'(x)$ is a completely monotone function on $(0,\infty)$, and $V(x)$ is a Bernstein function (see [34] for the relation between completely monotone, Bernstein, complete Bernstein and Stieltjes functions). More precisely, we have the following result.

Proposition 4.5. *Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process X_t, which is not a compound Poisson process, and suppose that $\Psi(\xi) = \psi(\xi^2)$ for a complete Bernstein function ψ. Then V is a Bernstein function, and*

$$V(x) = bx + \frac{1}{\pi}\int_{0+}^{\infty} \operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right)\frac{\psi^\dagger(\xi)}{\xi}\,(1-e^{-x\xi})d\xi, \qquad x>0, \tag{4.5}$$

$$V'(x) = b + \frac{1}{\pi}\int_{0+}^{\infty} \operatorname{Im}\left(-\frac{1}{\psi^+(-\xi^2)}\right)\psi^\dagger(\xi)e^{-x\xi}d\xi, \qquad x>0, \tag{4.6}$$

where $b = \lim_{\xi\to 0^+}(\xi/\sqrt{\psi(\xi^2)})$.

As explained after formula (2.2), the expression $\operatorname{Im}(-1/\psi^+(-\xi^2))d\xi$ in (4.5) and (4.6) should be understood in the distributional sense, as a weak limit of measures $\operatorname{Im}(-1/\psi(-\xi^2+i\varepsilon))d\xi$ on $(0,\infty)$ as $\varepsilon\to 0^+$. The measure $\operatorname{Im}(-1/\psi^+(-\xi))d\xi$ has an atom of mass πb at 0, and this atom is not included in the integrals from 0^+ to ∞ in (4.5) and (4.6).

Proof. Since $1/\psi^\dagger(\xi)$ is a Stieltjes function, it has the form (2.2),

$$\mathcal{L}V'(\xi) = \frac{1}{\psi^\dagger(\xi)} = a + \frac{b}{\xi} + \frac{1}{\pi}\int_{0+}^{\infty}\frac{1}{\xi+\zeta}\,\tilde{\mu}(d\zeta), \qquad \xi\in\mathbf{C}\setminus(-\infty,0],$$

where, using (2.5),

$$\tilde{\mu}(d\xi) = -\operatorname{Im}\left(\frac{1}{(\psi^\dagger)^+(-\xi)}\right)d\xi = -\operatorname{Im}\left(\frac{\psi^\dagger(\xi)}{\psi^+(-\xi^2)}\right)d\xi,$$

and

$$a = \lim_{\xi\to\infty}\frac{1}{\psi^\dagger(\xi)}, \qquad b = \lim_{\xi\to 0^+}\frac{\xi}{\psi^\dagger(\xi)}.$$

Using Proposition 2.1, we can express a and b in terms of ψ. Since ψ is unbounded, also $\psi^\dagger$ is unbounded (by (2.6)), and so in fact $a=0$. In a similar way, if $\xi/\psi(\xi)$ converges to 0 as $\xi\to 0^+$, then (2.6) gives $\xi/\psi^\dagger(\xi)\to 0$, so that $b=0$. When the limit of $\xi/\psi(\xi)$ is positive (since $\xi/\psi(\xi)$ is a CBF, the limit always exists), then ψ is regularly varying at 0, and so $b=\lim_{\xi\to 0^+}(\xi/\sqrt{\psi(\xi^2)})$, as desired. By the uniqueness of the Laplace transform,

$$V'(x) = b + \frac{1}{\pi}\int_{0+}^{\infty} e^{-x\xi}\tilde{\mu}(d\xi), \qquad x>0.$$

The result follows by integration in x. □

Note that for a compound Poisson process, we have $a>0$, so there is an extra positive constant in (4.5).

As a combination of Theorem 3.1 and Theorem 4.4, we obtain the following result.

Theorem 4.6. *Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process X_t. Suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi>0$. If $M_t = \sup_{0\le s\le t} X_s$, then*

$$\frac{1}{100}\min\left(1, \frac{1}{200\sqrt{t\Psi(1/x)}}\right) \le \mathbf{P}(M_t<x) \le \min\left(1, \frac{10}{\sqrt{t\Psi(1/x)}}\right), \qquad t,x>0.$$

Proof. When X_t is not a compound Poisson process, then the result follows from Theorems 3.1 and 4.4, and from $\kappa(z,0)=\sqrt{z}$. Suppose that X_t is a compound Poisson process. For $\varepsilon>0$ consider $X_t^\varepsilon = \varepsilon B_t + X_t$, where the Brownian motion B_t is independent of X_t. Then the Lévy-Khintchin exponent of X_t^ε equals to $\Psi_\varepsilon(\xi) = (\varepsilon\xi)^2 + \Psi(\xi)$. It

is easy to check that $\xi^2/\Psi_\varepsilon(\xi)$ is increasing. Moreover, M_t^ε converges in distribution to M_t as $\varepsilon \to 0$. The result follows by the continuity of $\Psi(\xi)$. □

Remark 4.7. Clearly, the condition '$\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$' in Theorem 4.4, Proposition 4.3 and Theorem 4.6 can be replaced with

$$0 < \Psi'(\xi) < \frac{2\Psi(\xi)}{\xi}, \qquad \xi > 0. \tag{4.7}$$

If $\Psi(\xi) = \psi(\xi^2)$, then (4.7) reads

$$0 < \psi'(\xi) < \frac{\psi(\xi)}{\xi}, \qquad \xi > 0. \tag{4.8}$$

Using the standard representation of Bernstein functions, it is easy to check that any Bernstein function $\psi(\xi)$ (not necessarily a complete one) satisfies (4.8). Hence, Theorem 4.6 applies to any *subordinate Brownian motion*: a process $X_t = B_{\eta_t}$, where $B(s)$ is the standard Brownian motion (with $\mathbf{E}(B_s) = 0$ and $\mathrm{Var}(B_s) = 2s$), η_t is a subordinator (with $\mathbf{E}(e^{-\xi\eta_t}) = e^{-t\psi(\xi)}$), and B_s and η_t are independent processes.

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