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REGULARITY OF SOLUTIONS TO QUANTUM MASTER EQUATIONS: A STOCHASTIC APPROACH

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Applying probabilistic techniques we study regularity properties of quantum master equations (QMEs) in the Lindblad form with unbounded coefficients; a density operator is regular if, roughly speaking, it describes a quantum state with finite energy. Using the linear stochastic Schrödinger equation we deduce that solutions of QMEs preserve the regularity of the initial states under a general non-explosion condition. To this end, we develop the probabilistic representation of QMEs and we prove the uniqueness of solutions for adjoint quantum master equations. By means of the non-linear stochastic Schrödinger equation we obtain the existence of regular stationary solutions for QMEs, under a Lyapunov-type condition.

1. Introduction. In order to establish the well-posedness of the mean values of quantum observables represented by unbounded operators, we investigate the regularity of solutions of quantum master equations (with unbounded coefficients) in stationary and transient regimes. For this purpose, we use classical stochastic analysis.

1.1. *Gorini-Kossakowski-Lindblad-Sudarshan equations.* In many open quantum systems, the states of a small quantum system with Hamiltonian $H : \mathfrak{h} \to \mathfrak{h}$ evolve according to the operator equation:

$$\frac{d}{dt}\rho_t(\varrho) = \mathcal{L}_*(\rho_t(\varrho)), \qquad \rho_0(\varrho) = \varrho, \tag{1.1}$$

where $\mathcal{L}_*(\rho) = G\rho + \rho G^* + \sum_{k=1}^{\infty} L_k \rho L_k^*$ (see, e.g., [8, 22, 38]). Here, $(\mathfrak{h}, \langle\cdot,\cdot\rangle)$ is a separable complex Hilbert space, $G, L_1, L_2, \ldots$ are given linear operators in $\mathfrak{h}$ satisfying $G = -iH - \frac{1}{2}\sum_{k=1}^{\infty} L_k^* L_k$ on suitable common domain, and the unknown density operator $\rho_t(\varrho)$ is a non-negative operator in $\mathfrak{h}$ with unit trace. The operators $L_1, L_2, \ldots$ describe the weak interaction between the small quantum system and a heat bath.

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The measurable physical quantities of the small quantum system are represented by self-adjoint operators in $\mathfrak{h}$, which are called observables. Very important observables are unbounded, like position, momentum and kinetic energy operators. In the Schrödinger picture, the mean value of the observable A at time t is given by $tr\left(\rho_t\left(\varrho\right)A\right)$, the trace of $\rho_t\left(\varrho\right)A$.

In the Heisenberg picture, the initial density operator ϱ is fixed; we use, for instance, (1.1) to obtain the equation of motion for the observable A:

$$\frac{d}{dt}\mathcal{T}_t\left(A\right)=\mathcal{L}\left(\mathcal{T}_t\left(A\right)\right),\qquad\qquad \mathcal{T}_0\left(A\right)=A, \tag{1.2}$$

where $\mathcal{L}\left(\mathcal{T}_t\left(A\right)\right)=\mathcal{T}_t\left(A\right)G+G^*\mathcal{T}_t\left(A\right)+\sum_{k=1}^{\infty}L_k^*\mathcal{T}_t\left(A\right)L_k$ (see, e.g., [8, 22]). The expected valued of A at time time t is given by $tr\left(\varrho\mathcal{T}_t\left(A\right)\right)$.

1.2. *Stochastic Schrödinger equations (SSEs).* The evolution of the state of a quantum system conditioned on continuous measurement is governed (see, e.g., [3, 6, 38]) by the stochastic evolution equation on $\mathfrak{h}$:

$$Y_t=Y_0+\int_0^t G\left(Y_s\right)ds+\sum_{k=1}^{\infty}\int_0^t L_k\left(Y_s\right)dB_s^k. \tag{1.3}$$

Here $G\left(y\right)=Gy+\sum_{k=1}^{\infty}\left(\Re\left\langle y,L_ky\right\rangle L_ky-\frac{1}{2}\Re^2\left\langle y,L_k\ y\right\rangle y\right)$, $L_k\left(y\right)=L_ky-\Re\left\langle y,L_ky\right\rangle y$, and $B^1,B^2,\ldots$ are real valued independent Wiener processes.

Example 1. *Set $\mathfrak{h}=L^2\left(\mathbb{R},\mathbb{C}\right)$. Let $Q,P:\mathfrak{h}\rightarrow\mathfrak{h}$ be defined by $Qf\left(x\right)=xf\left(x\right)$ and $Pf\left(x\right)=-if'\left(x\right)$. In (1.3), take $H=\frac{1}{2m}P^2+cQ^2$, $L_1=\alpha Q$ and $L_2=\beta P$, with $m>0$, $\alpha,\beta\geq 0$ and $c\in\mathbb{R}$. For all $k\geq 3$, fix $L_k=0$.*

Example 1 with $\alpha,\beta,c>0$ describes the simultaneous monitoring of position and momentum of a linear harmonic oscillator (see, e.g., [24, 37]). Taking instead $\alpha>0$ and $\beta=c=0$ we get a well-studied model for the continuous measurement of position of a free particle (see, e.g., [16, 24], [5, 27] and references therein).

Our main tool for studying (1.1) and (1.2) is the linear SSE on $\mathfrak{h}$:

$$X_t\left(\xi\right)=\xi+\int_0^t GX_s\left(\xi\right)ds+\sum_{k=1}^{\infty}\int_0^t L_kX_s\left(\xi\right)dW_s^k, \tag{1.4}$$

where $W^1,W^2,\ldots$ are real valued independent Wiener processes on a filtered complete probability space $\left(\Omega,\mathfrak{F},\left(\mathfrak{F}_t\right)_{t\geq 0},\mathbb{P}\right)$. In fact, the basic assumption of this paper is:

(H) There exists a non-negative self-adjoint operator C in $\mathfrak{h}$ such that: (i) G is relatively bounded with respect to C; and (ii) (1.4) has a unique C-solution for any initial condition ξ satisfying $\mathbb{E}\left(\|C\xi\|^2 + \|\xi\|^2\right) < \infty$.

Here, a strong solution $X(\xi)$ of (1.4) is called C-solution if $\mathbb{E}\|X_t(\xi)\|^2 \leq \mathbb{E}\|\xi\|^2$ and the function $t \mapsto \mathbb{E}\|CX_t(\xi)\|^2$ is uniformly bounded on compact time intervals (see Definition 2.1 for details).

The law of $X_t(Y_0) / \|X_t(Y_0)\|$ with respect to $\|X_T(Y_0)\|^2 \cdot \mathbb{P}$ coincides with the law of Y_t for all $t \in [0, T]$ (see [31]). The main technical and conceptual advantage of (1.3) over (1.4) is that the norm of Y_t is equal to 1.

1.3. *Principal objectives.* Our main goal is to make progress in the understanding of the evolution of $tr(\rho_t(\varrho) A)$ when A is unbounded.

Given a self-adjoint non-negative operator C in $\mathfrak{h}$, we denote by $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ the set of all non-negative operators $\varrho : \mathfrak{h} \to \mathfrak{h}$ for which, loosely speaking, $C\varrho$ is a trace-class operator (see Definition 3.1). From Section 3 we have that the expected value of A with respect to $\varrho \in \mathfrak{L}^+_{1,C}(\mathfrak{h})$ is well defined whenever $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, where $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ is the space of all operators relatively bounded with respect to C. Our first objective is:

(O1) To prove that the solution $\rho_t(\varrho)$ of (1.1) belongs to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ (for all $t > 0$) provided that C satisfies Hypothesis H and that $\varrho \in \mathfrak{L}^+_{1,C}(\mathfrak{h})$.

The key condition for guarantee the uniqueness of solution of (1.1) is the existence of a self-adjoint non-negative operator C in $\mathfrak{h}$ such that formally

$$\mathcal{L}\left(C^2\right) \leq K\left(C^2 + I\right), \tag{1.5}$$

where I is the identity operator in $\mathfrak{h}$ and $K \in [0, \infty[$. This condition, introduced by Chebotarev and Fagnola [11] (see also [13, 17, 21]), is a quantum analogue of the Lyapunov condition for non-explosion of classical Markov processes (see [13] for heuristic arguments). Since Hypothesis H holds under a weak version of (1.5) (see [18, 30] and Remark 6.1), the inequality (1.5) is the underlying assumption of Objective O1. In many physical examples, relevant observables belong to $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ for some C satisfying (1.5). In Example 1, for instance, $C = P^2 + Q^2$ satisfies Hypothesis H (see, e.g., [18, 30]) and the position and momentum operators Q and P are $(P^2 + Q^2)$-bounded.

Previously, the regularity of the solutions to (1.1) has been treated in [1, 12, 14] using methods from the operator theory. Exploiting the characteristics of a model describing a variable number of neutrons moving in a translation invariant external reservoir of unstable atoms, Davies [14] established that $\rho_t(\varrho) \in \mathfrak{L}^+_{1,C}(\mathfrak{h})$ whenever C is the particle number operator on

an adequate Fermion Fock space. Arnold and Sparber [1] obtained the same property with C being essentially the energy operator for a linear quantum master equation associated to a diffusion models with Hartree interaction.

The second objective presents the first attempt (to the best of my knowledge) to show the existence of stationary solutions of (1.1) with finite energy.

(O2) To prove the existence of a stationary solution of (1.1) belonging to $\mathfrak{L}_{1,D}^{+}(\mathfrak{h})$ provided essentially that:

  (L) There exist two non-negative self-adjoint operators C and D and a constant $K > 0$ such that $\left\{x \in \mathfrak{h} : \|Dx\|^2 + \|x\|^2 \leq 1\right\}$ is compact in $\mathfrak{h}$, and $\mathcal{L}(C^2) \leq -D^2 + K(1 + I)$.

Hypothesis L is a quantum version of the Lyapunov criterion for the existence of invariant probability measures for stochastic differential equations, which applies to many open quantum systems (see, e.g., [20] and Section 6). Let $\mathfrak{L}(\mathfrak{h})$ be the set of all bounded operators from $\mathfrak{h}$ to $\mathfrak{h}$. In the case where G is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathfrak{h}$ and, loosely speaking, $\mathcal{T}_t(I) = I$ for all $t \geq 0$, Fagnola and Rebolledo [20] proved that under Hypothesis L, there exists at least one density operator ϱ_∞ satisfying $tr(\varrho_\infty \mathcal{T}_t(A)) = tr(\varrho_\infty A)$ for all $t \geq 0$ and $A \in \mathfrak{L}(\mathfrak{h})$. The main point of Objective O2 is that among such stationary states ϱ_∞ we can select a finite-energy density operator belonging to the domain of $\mathcal{L}_*$, under the same Hypothesis L.

The third main objective develops the rigorous probabilistic representation of the solution of (1.1), the key step to achieve Objectives O1 and O2.

(O3) Assume Hypothesis H and let $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$, where ξ is a $\mathfrak{h}$-valued random variable such that $\mathbb{E}\|\xi\|^2 = 1$ and $\mathbb{E}\|C\xi\|^2 < \infty$. We wish to prove that (1.1) has a unique solution, which is

$$\rho_t(\varrho) = \mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)|. \tag{1.6}$$

In Dirac notation, $|x\rangle\langle y| : \mathfrak{h} \to \mathfrak{h}$ is defined by $|x\rangle\langle y|(z) = \langle y, z\rangle x$, with $x, y \in \mathfrak{h}$. Using (1.6) we can assert that

$$\rho_t(\varrho) = \mathbb{E}|Y_t\rangle\langle Y_t|, \tag{1.7}$$

with $Y_0 = \xi$ (see [31]). Objective O3, together with (1.7), shows that physical models based on the stochastic Schrödinger equations are in good agreement with their formulations in terms of quantum master equations.

In the physical literature, the probabilistic representations (1.6) and (1.7) of the density operator at time t have been obtained by means of formal computations (see, e.g., [2, 8, 23]). Barchielli and Holevo [4] established essentially (1.6) and (1.7) in situations where $G, L_1, L_2, \ldots$ are bounded.

1.4. *Approach.* In the perspective of the operator theory, methods based on the Hille-Yosida theorem and perturbations of linear operators [26, 33] present severe limitations for studying linear functionals of the solutions of (1.1) and (1.2). For example, it is very difficult to decompose $\mathcal{L}_*$ into $\mathcal{L}_*^1+\mathcal{L}_*^2$ for a dissipative operator $\mathcal{L}_*^1$ in $\mathfrak{L}_1(\mathfrak{h})$ and an infinitesimal generator $\mathcal{L}_*^2$ of a C_0 semigroup of contractions on $\mathfrak{L}_1(\mathfrak{h})$, which together satisfy: $\left\|\mathcal{L}_*^1(\varrho)\right\|_1 \leq \alpha\left\|\mathcal{L}_*^2(\varrho)\right\|_1 + K\|\varrho\|_1$ whenever $\varrho \in \mathcal{D}\left(\mathcal{L}_*^2\right)$. Here, $0 \leq \alpha < 1$, $K \geq 0$ and $\mathfrak{L}_1(\mathfrak{h})$ is the Banach space of trace-class operators on $\mathfrak{h}$ equipped with the trace norm $\|\cdot\|_1$. Another difficulty is that $\mathcal{L}$ and $\mathcal{L}_*$ are defined formally; indeed $\mathcal{L}$ and $\mathcal{L}_*$ can be interpreted as sesquilinear forms, but without having a priori knowledge about their cores.

When G is the generator of a C_0 semigroup of contractions on $\mathfrak{h}$, E.B. Davis [15] provided solutions of (1.1) by means of semigroups. Modifying Davis' ideas, Chebotarev constructed a quantum dynamical semigroup $\mathcal{T}^{(min)}$ that is weak solution of (1.2) by generalizing the K.L. Chung construction of the minimal solution of Feller-Kolmogorov equations for countable state Markov chains (see Remark 2.5). Under certain conditions involving (1.5) and invariant sets for $\exp(Gt)$, Chebotarev and Fagnola [11] proved the uniqueness of $\mathcal{T}^{(min)}$ (see Remark 2.5). This property implies that $\mathcal{L}_*$ is the infinitesimal generator of the predual semigroup $\rho^{(min)}$ of $\mathcal{T}^{(min)}$ and a core for $\mathcal{L}_*$ is formed by the linear span of all $|x\rangle\langle y|$ with x, y belonging to $\mathcal{D}(G)$, the domain of G (see Remark 4.2). In Remark 4.2, we outline how to obtain $\rho^{(min)}\left(\mathfrak{L}_{1,C}^+(\mathfrak{h})\right) \subset \mathfrak{L}_{1,C}^+(\mathfrak{h})$ under various assumptions including $\exp(Gt)\,\mathcal{D}(C) \subset \mathcal{D}(C)$. It is a hard problem, in general, to find C satisfying (1.5) whose domain $\mathcal{D}(C)$ is invariant under the action of $\exp(Gt)$.

In contrast to closed quantum systems, solutions of (1.1) are not decomposable as dyadic products of solutions of evolution equations in $\mathfrak{h}$. Nevertheless, the solution of (1.1) is unraveled into stochastic quantum trajectories; more precisely, Objective O3 establishes that $\rho_t(\varrho)$ is expressed as the mean value of quadratic functionals of the solutions of SSEs in a general context. This property allows us to achieve Objectives O1 and O2 by using SSEs, without serious difficulties and without assumptions involving invariant sets for $\exp(Gt)$. Applying (1.6) we also deduce that $\rho_t(\varrho)$ satisfies (1.1) in both sense integral and $\mathfrak{L}_1(\mathfrak{h})$-weak. This leads to prove rigorously some dynamical properties of $\rho_t(\varrho)$ given in physics (see, e.g., Theorem 4.6).

We now focus on Objective O1. By Section 3, $\varrho \in \mathfrak{L}_{1,C}^+(\mathfrak{h})$ iff there exists a $\mathfrak{h}$-valued random variable ξ satisfying $\mathbb{E}\left(\|C\xi\|^2 + \|\xi\|^2\right) < \infty$ and $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$. Therefore (1.6) leads directly to Objective O1 since $\mathbb{E}\,\|CX_t(\xi)\|^2 + \mathbb{E}\,\|X_t(\xi)\|^2 < \infty$. The assumption (1.5) is natural in the context of (1.4),

because (1.5) is essentially the dissipative condition for (1.4).

We turn to Objective O2. Here, Hypothesis L is a classical Lyapunov condition for (1.4). Relation (1.6) suggests us that $\int_{\mathfrak{h}} |x\rangle\langle x| \, \mu(dx)$ is a good candidate for being stationary solution for (1.1) when μ is an invariant probability measure for (1.4) such that $\int_{\mathfrak{h}} \|x\|^2 \mu(dx) = 1$. This reduces Objective O2 to prove that there exists an invariant probability measure for (1.4) different from the Dirac measure at 0, which is a difficult problem. We instead use (1.7). Under a weak version of Hypothesis L, there exists an invariant probability measure Γ for (1.3) such that $\int_{\mathfrak{h}} \|x\|^2 \Gamma(dx) = 1$ and $\int_{\mathfrak{h}} \|Dx\|^2 \Gamma(dx) < \infty$ (see [31]). Then, using (1.7) we deduce that $\varrho_\infty = \int_{\mathfrak{h}} |x\rangle\langle x| \, \Gamma(dx)$ is a stationary solution to (1.1) that belongs to $\mathfrak{L}^+_{1,D}(\mathfrak{h})$ (see Section 5). This is a step forward in the study of the long time behavior of unbounded observables.

1.5. *Technical ideas: Unraveling.* Fix $\varrho \in \mathfrak{L}^+_{1,C}(\mathfrak{h})$. Then $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$ for some $\mathfrak{h}$-valued random variable ξ satisfying $\mathbb{E}\left(\|C\xi\|^2 + \|\xi\|^2\right) < \infty$ (see Section 3). We can define

$$\rho_t(\varrho) := \mathbb{E}\,|X_t(\xi)\rangle\langle X_t(\xi)|, \tag{1.8}$$

because $\rho_t(\varrho)$ does not depend on the choice of ξ (see Theorem 4.1). We next outline how to establish that $\rho_t(\varrho)$ is a solution of (1.1).

Applying Itô's formula we obtain

$$\begin{aligned}\langle X_t(\xi), x\rangle X_t(\xi) =& \langle \xi, x\rangle \xi + \int_0^t \left(\langle X_s(\xi), x\rangle G X_s(\xi) + \langle G X_s(\xi), x\rangle X_s(\xi)\right) ds \\ &+ \sum_{k=1}^\infty \int_0^t \langle L_k X_s(\xi), x\rangle L_k X_s(\xi)\, ds + M_t,\end{aligned}$$

with $M_t = \sum_{k=1}^\infty \int_0^t \left(\langle X_s(\xi), x\rangle L_k X_s(\xi) + \langle L_k X_s(\xi), x\rangle X_s(\xi)\right) dW_s^k$. Since M_t is a local martingale, we use stopping times and the dominated convergence theorem to deduce that

$$\begin{aligned}\mathbb{E}\langle X_t(\xi), x\rangle X_t(\xi) &= \mathbb{E}\langle \xi, x\rangle \xi + \int_0^t \mathbb{E}\langle X_s(\xi), x\rangle G X_s(\xi)\, ds \\ &+ \int_0^t \mathbb{E}\langle G X_s(\xi), x\rangle X_s(\xi)\, ds + \sum_{k=1}^\infty \int_0^t \mathbb{E}\langle L_k X_s(\xi), x\rangle L_k X_s(\xi)\, ds.\end{aligned} \tag{1.9}$$

Define the operator $\mathcal{L}_*(\xi, s) : \mathfrak{h} \to \mathfrak{h}$ to be

$$\mathbb{E}\,|G X_s(\xi)\rangle\langle X_s(\xi)| + \mathbb{E}\,|X_s(\xi)\rangle\langle G X_s(\xi)| + \sum_{k=0}^\infty \mathbb{E}\,|L_k X_s(\xi)\rangle\langle L_k X_s(\xi)|.$$

We now face the major technical difficulties; we have to prove that $\mathcal{L}_*(\xi,s)$ is a trace-class operator such that: (i) $\mathcal{L}_*(\xi,t)=\mathcal{L}_*(\rho_s(\varrho))$; (ii) the function $s\mapsto\|\mathcal{L}_*(\xi,s)\|_1$ is locally bounded; and (iii) $s\mapsto\mathcal{L}_*(\xi,s)$ is weakly continuous in $\mathfrak{L}_1(\mathfrak{h})$. Then, applying (1.9) yields

$$\rho_t(\varrho)=\varrho+\int_0^t\mathcal{L}_*(\rho_s(\varrho))\,ds, \tag{1.10}$$

where we understand the integral of (1.10) in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$. Thus, we can deduce that for any $A\in\mathfrak{L}(\mathfrak{h})$,

$$\frac{d}{dt}tr\left(A\rho_t(\varrho)\right)=tr\left(A\mathcal{L}_*(\rho_t(\varrho))\right). \tag{1.11}$$

1.6. *Technical ideas: Uniqueness.* Recall that $\rho_t(\varrho)$ is defined by (1.8) for any $\varrho\in\mathfrak{L}_{1,C}^+(\mathfrak{h})$. In order to establish the uniqueness of the solution of (1.1) under Hypothesis H, Theorem 4.3 extends $\rho_t(\varrho)$ to a strongly continuous semigroup $(\rho_t)_{t\geq0}$ of bounded operators on $\mathfrak{L}_1(\mathfrak{h})$. Thus, $(\rho_t)_{t\geq0}$ belongs to the class $\mathcal{S}$ formed by all the locally bounded semigroups $(\widehat{\rho}_t)_{t\geq0}$ on $\mathfrak{L}_1(\mathfrak{h})$ such that for any $x\in\mathcal{D}(C)$: (i) $t\mapsto\widehat{\rho}_t(|x\rangle\langle x|)$ is weakly continuous in $\mathfrak{L}_1(\mathfrak{h})$); and (ii) $\widehat{\rho}_t(|x\rangle\langle x|)$ satisfies (1.11) in $t=0$ (see Theorem 4.5 and Lemma 7.20 for details). We next outline the proof that $(\rho_t)_{t\geq0}$ is the unique element in $\mathcal{S}$, and so (1.1) has a unique solution (in the semigroup sense).

Let $(\widehat{\rho}_t)_{t\geq0}\in\mathcal{S}$. Taking in mind that $\mathfrak{L}(\mathfrak{h})$ is the dual of $\mathfrak{L}_1(\mathfrak{h})$, we consider the semigroup $(\mathcal{T}_t)_{t\geq0}$ on $\mathfrak{L}(\mathfrak{h})$ which is the adjoint semigroup of $(\widehat{\rho}_t)_{t\geq0}$. Using techniques from operator theory we obtain in Lemma 7.21 that $(\mathcal{T}_t)_{t\geq0}$ is a weak solution of (1.2), namely, for all $t\geq0$, $A\in\mathfrak{L}(\mathfrak{h})$ and $x\in\mathcal{D}(C)$ we have

$$\frac{d}{dt}\langle x,\mathcal{T}_t(A)\,x\rangle=\langle x,\mathcal{L}(\mathcal{T}_t(A))\,x\rangle. \tag{1.12}$$

Now, we wish to prove that $(\mathcal{T}_t)_{t\geq0}$ is the unique weak solution of (1.2), which is an important problem itself (see, e.g., [10, 11, 12, 13, 17, 29]). Suppose for a moment that $\mathfrak{h}$ is finite-dimensional and $L_k\neq0$ for only a finite number of k. Applying the Itô formula to $\langle X_s(x),\mathcal{T}_{t-s}(A)\,X_s(x)\rangle$ we deduce that

$$\begin{aligned}&\langle X_t(x),AX_t(x)\rangle=\langle x,\mathcal{T}_t(A)\,x\rangle+M_t\\&+\int_0^t\left(\langle X_s(x),\mathcal{L}(\mathcal{T}_{t-s}(A))\,X_s(x)\rangle-\left\langle X_s(x),\frac{d\mathcal{T}_r(A)}{dr}\Big|_{r=t-s}X_s(x)\right\rangle\right)ds,\end{aligned}$$

with

$$M_t=\sum_{k=1}^\infty\int_0^t\left(\langle L_kX_s(x),\mathcal{T}_{t-s}(A)\,X_s(x)\rangle+\langle X_s(x),\mathcal{T}_{t-s}(A)\,L_kX_s(x)\rangle\right)dW_s^k.$$

From (1.12) we obtain $\langle X_t(x), AX_t(x)\rangle = \langle x, \mathcal{T}_t(A)x\rangle + M_t$, and so the martingale property of M_t leads to $\mathbb{E}\langle X_t(x), AX_t(x)\rangle = \langle x, \mathcal{T}_t(A)x\rangle$, hence all the elements in $\mathcal{S}$ are the same semigroups, which implies $\widehat{\rho} = \rho$.

In the general case, G and L_k are unbounded operators. Therefore

$$(s,x) \mapsto \frac{d}{ds}\langle x, \mathcal{T}_{t-s}(A)x\rangle \left(= \langle x, \mathcal{L}(\mathcal{T}_{t-s}(A))x\rangle\right)$$

is not continuous on $[0,t]\times\mathfrak{h}$, and consequently we cannot apply directly Itô's formula to $\langle X_s(x), \mathcal{T}_{t-s}(A)X_s(x)\rangle$. We overcome this difficulty in Subsection 7.1 by applying Itô's formula to a regularized version of $\langle x, \mathcal{T}_{t-s}(A)x\rangle$; the resulting stochastic integrals (similar to those in M_t) are only local martingales, and so we have to use stopping times.

1.7. *Outline.* Section 2 addresses the existence and uniqueness of solutions for the adjoint quantum master equation, as well as its probabilistic representation. Section 3 deals with the probabilistic interpretations of regular density operators. In Section 4 we construct Schrödinger evolutions by means of stochastic Schrödinger equations and study the regularity of solutions to (1.1). Section 5 focusses on the existence of regular stationary solutions for (1.1). In Section 6 we apply our results to a quantum oscillator. Section 7 is devoted to proofs.

1.8. *Notation.* Throughout this paper, the scalar product $\langle\cdot,\cdot\rangle$ is linear in the second variable and anti-linear in the first one. We write $\mathfrak{B}(\mathfrak{h})$ for the Borel σ-algebra on $\mathfrak{h}$. Suppose that A is a linear operator in $\mathfrak{h}$. Then A^* denotes the adjoint of A. If A has a unique bounded extension to $\mathfrak{h}$, then we continue to write A for the closure of A.

Let $\mathfrak{X}$, $\mathfrak{Z}$ be normed spaces. We write $\mathfrak{L}(\mathfrak{X},\mathfrak{Z})$ for the set of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Z}$ (together with norm $\|\cdot\|_{\mathfrak{L}(\mathfrak{X},\mathfrak{Z})}$). We abbreviate $\|\cdot\|_{\mathfrak{L}(\mathfrak{X},\mathfrak{Z})}$ to $\|\cdot\|$, if no misunderstanding is possible, and define $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X},\mathfrak{X})$. By $\mathfrak{L}_1^+(\mathfrak{h})$ we mean the subset of all non-negative trace-class operators on $\mathfrak{h}$.

Let C be a self-adjoint positive operator in $\mathfrak{h}$. Then, for any $x, y \in \mathcal{D}(C)$ we set $\langle x,y\rangle_C = \langle x,y\rangle + \langle Cx, Cy\rangle$ and $\|x\|_C = \sqrt{\langle x,x\rangle_C}$. As usual, $L^2(\mathbb{P},\mathfrak{h})$ stands for the set of all square integrable random variables from $(\Omega,\mathfrak{F},\mathbb{P})$ to $(\mathfrak{h},\mathfrak{B}(\mathfrak{h}))$. We write $L^2_C(\mathbb{P},\mathfrak{h})$ for the set of all $\xi \in L^2(\mathbb{P},\mathfrak{h})$ satisfying $\xi \in \mathcal{D}(C)$ a.s. and $\mathbb{E}\|\xi\|_C^2 < \infty$. The function $\pi_C : \mathfrak{h} \to \mathfrak{h}$ is defined by $\pi_C(x) = x$ if $x \in \mathcal{D}(C)$ and $\pi_C(x) = 0$ whenever $x \notin \mathcal{D}(C)$. In the sequel, the letter K denotes generic constants.

2. Adjoint quantum master equation. We begin by presenting in detail the notion of C-solution to (1.4).

HYPOTHESIS 1. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$ such that $\mathcal{D}(C)$ is a subset of the domains of $G, L_1, L_2, \ldots$ and the maps $G \circ \pi_C, L_1 \circ \pi_C, L_2 \circ \pi_C, \ldots$ are measurable.*

DEFINITION 2.1. *Let Hypothesis 1 hold. Assume that $\mathbb{I}$ is either $[0, \infty[$ or $[0, T]$, with $T \in \mathbb{R}_+$. An $\mathfrak{h}$-valued adapted process $(X_t(\xi))_{t \in \mathbb{I}}$ with continuous sample paths is called strong C-solution of (1.4) on $\mathbb{I}$ with initial datum ξ if and only if for all $t \in \mathbb{I}$:*

- $\mathbb{E}\|X_t(\xi)\|^2 \leq \mathbb{E}\|\xi\|^2$, $X_t(\xi) \in \mathcal{D}(C)$ *a.s.*, $\sup_{s \in [0,t]} \mathbb{E}\|CX_s(\xi)\|^2 < \infty$.
- $X_t(\xi) = \xi + \int_0^t G\pi_C(X_s(\xi))\, ds + \sum_{k=1}^{\infty} \int_0^t L_k \pi_C(X_s(\xi))\, dW_s^k \quad \mathbb{P}$-*a.s.*

NOTATION 2.1. *The symbol $X(\xi)$ will be reserved for the strong C-solution of (1.4) with initial datum ξ.*

REMARK 2.1. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$, together with $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. Then $A \circ \pi_C : \mathfrak{h} \to \mathfrak{h}$ is measurable whenever $\mathfrak{h}$ is equipped with its Borel σ-algebra (see, e.g., [18] for details).*

We now make more precise our basic assumptions, that is Hypothesis H.

HYPOTHESIS 2. *Suppose that Hypothesis 1 holds. In addition, assume:*

(H2.1) *The operator G belongs to $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$.*
(H2.2) *For all $x \in \mathcal{D}(C)$, $2\Re\langle x, Gx\rangle + \sum_{k=1}^{\infty} \|L_k x\|^2 = 0$.*
(H2.3) *Let $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$ be $\mathfrak{F}_0$-measurable. Then for all $T > 0$, (1.4) has a unique strong C-solution on $[0, T]$ with initial datum ξ.*

REMARK 2.2. *Let A be a closable operator in $\mathfrak{h}$ whose domain is contained in $\mathcal{D}(C)$, where C is a self-adjoint positive operator in $\mathfrak{h}$. Applying the closed graph theorem we obtain $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, which leads to a sufficient condition for H2.1.*

REMARK 2.3. *Let C be a self-adjoint positive operator in $\mathfrak{h}$ such that $\mathcal{D}(C) \subset \mathcal{D}(G)$. Assume that $2\Re\langle x, Gx\rangle + \sum_{k=1}^{\infty} \|L_k x\|^2 \leq 0$ for all $x \in \mathcal{D}(G)$. Then the numerical range of G is contained in the left half-plane of $\mathbb{C}$, and so G is closable. Therefore $G \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ by Remark 2.2.*

Using arguments given in Subsection 1.6 we prove the following theorem, establishing the uniqueness of the solution of (1.2).

DEFINITION 2.2. *Suppose that $A \in \mathfrak{L}(\mathfrak{h})$ and that C is a self-adjoint positive operator in $\mathfrak{h}$. A family of operators $(\mathcal{A}_t)_{t \geq 0}$ belonging to $\mathfrak{L}(\mathfrak{h})$ is a C-solution of (1.2) with initial datum A iff $\mathcal{A}_0 = A$ and for all $t \geq 0$:*

(a) $\frac{d}{dt}\langle x,\mathcal{A}_t y\rangle=\langle x,\mathcal{A}_t Gy\rangle+\langle Gx,\mathcal{A}_t y\rangle+\sum_{k=1}^{\infty}\langle L_k x,\mathcal{A}_t L_k y\rangle$ *for all* $x,y\in\mathcal{D}(C)$.

(b) $\sup_{s\in[0,t]}\|\mathcal{A}_s\|_{\mathfrak{L}(\mathfrak{h})}<\infty$.

THEOREM 2.1. *Suppose that Hypothesis 2 holds. Let A belong to $\mathfrak{L}(\mathfrak{h})$. Then, for every non-negative real number t there exists a unique $\mathcal{T}_t(A)$ in $\mathfrak{L}(\mathfrak{h})$ such that for all x,y in $\mathcal{D}(C)$,*

$$\langle x,\mathcal{T}_t(A)\,y\rangle=\mathbb{E}\langle X_t(x),AX_t(y)\rangle. \tag{2.1}$$

Moreover, any C-solution of (1.2) with initial datum A coincides with $\mathcal{T}(A)$, and $\|\mathcal{T}_t(A)\|_{\mathfrak{L}(\mathfrak{h})}\leq\|A\|_{\mathfrak{L}(\mathfrak{h})}$ for all $t\geq 0$.

PROOF. The proofs fall naturally into Lemmata 7.1 and 7.2. □

As a by-product of our proof of the existence of solutions to (1.1), we "construct" a solution to (1.2), and so Theorem 2.1 leads to Theorem 2.2.

THEOREM 2.2. *Let Hypothesis 2 holds. Suppose that $A\in\mathfrak{L}(\mathfrak{h})$ and that $\mathcal{T}_t(A)$ is as in Theorem 2.1. Then $(\mathcal{T}_t(A))_{t\geq 0}$ is the unique C-solution of (1.2) with initial datum A.*

PROOF. Lemmata 7.20 and 7.21 shows that $(\mathcal{T}_t(A))_{t\geq 0}$ is a C-solution of (1.2) with initial datum A. Theorem 2.1 now completes the proof. □

REMARK 2.4. *In [29], C.M.M. developed the existence and uniqueness of the solution to (1.2) with A unbounded, as well as its probabilistic representation. Thus taking $A\in\mathfrak{L}(\mathfrak{h})$, Corollary 14 of [29] established the statement of Theorem 2.2 under assumptions including the existence of an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{h}$ that satisfies, for example, $Ge_n,L_ke_n\in\mathcal{D}(C)$ and $\sup_{n\in\mathbb{Z}_+}\|CP_nx\|\leq\|Cx\|$ for all $x\in\mathcal{D}(C)$, where P_n is the orthogonal projection of $\mathfrak{h}$ over the linear manifold spanned by $e_0,\ldots e_n$. In Theorem 2.2 we remove such basis, extending the range of applications.*

REMARK 2.5. *Suppose that $2\Re\langle x,Gx\rangle+\sum_{k=1}^{\infty}\|L_kx\|^2\leq 0$ for all $x\in\mathcal{D}(G)$. Let G be the infinitesimal generator of a C_0-semigroup of contractions. Define the sequence $\left(\mathcal{T}^{(n)}\right)_{n\geq 0}$ of linear contractions on $\mathfrak{L}(\mathfrak{h})$ by*

$$\left\langle u,\mathcal{T}_t^{(n+1)}(A)\,v\right\rangle=\left\langle e^{Gt}u,Ae^{Gt}v\right\rangle+\sum_{k=1}^{\infty}\int_0^t\left\langle L_ke^{G(t-s)}u,\mathcal{T}_s^{(n)}(A)L_ke^{G(t-s)}v\right\rangle ds$$

where $u, v \in \mathcal{D}(G)$, $A \in \mathfrak{L}(\mathfrak{h})$, *and* $\mathcal{T}^{(-1)} = 0$. *A.M. Chebotarev proved that the Picard's successive approximations* $\mathcal{T}^{(n)}$ *converge as* $n \to \infty$ *to a quantum dynamical semigroup* $\mathcal{T}^{(min)}$ *which is a weak solution to (1.2) (see, e.g., [13, 17]). Holevo [25] developed the probabilistic representation of* $\mathcal{T}^{(min)}$ *under restrictions including* G *and* G^* *are the infinitesimal generators of* C_0*-semigroup of contractions. From Chebotarev and Fagnola [11] we have that* $\mathcal{T}_t^{(min)}(I) = I$ *for any* $t \geq 0$ *provided that there exists a self-adjoint positive operator* C *in* $\mathfrak{h}$ *and a linear manifold* $\mathfrak{D} \subset \mathcal{D}(G)$ *which is a core for* C *such that: (i) The semigroup generated by* G *leaves invariant* $\mathfrak{D}$*; and (ii) For some* $\gamma > 0$, $2\Re\langle C^2 x, Gx\rangle + \sum_{k=1}^{\infty} \|CL_k x\|^2 \leq \alpha \|x\|_C^2$ *for all* $x \in (\gamma I - G)^{-1}(\mathfrak{D})$ *(see also [12, 17]). This implies the uniqueness (in the semigroup sense) of the solution to (1.2) with* A *bounded (see, e.g., [17]).*

In addition to its proof, the main novelty of Theorem 2.2 is that we do not assume properties like G *is the infinitesimal generator of a semigroup and Condition (i) which involves the study of invariant sets for* $\exp(Gt)$. *The latter is not an easy problem in general.*

3. Probabilistic representations of regular density operators. The following notion of regular density operator was introduced by Chebotarev, García and Quezada [12] to investigate the identity preserving property of minimal quantum dynamical semigroups.

Definition 3.1. *Let* C *be a self-adjoint positive operator in* $\mathfrak{h}$. *An operator* ϱ *belonging to* $\mathfrak{L}_1^+(\mathfrak{h})$ *is called* C*-regular iff* $\varrho = \sum_{n\in\mathfrak{I}} \lambda_n |u_n\rangle\langle u_n|$ *for some countable set* $\mathfrak{I}$, *summable non-negative real numbers* $(\lambda_n)_{n\in\mathfrak{I}}$ *and family* $(u_n)_{n\in\mathfrak{I}}$ *of elements of* $\mathcal{D}(C)$, *which together satisfy:* $\sum_{n\in\mathfrak{I}} \lambda_n \|Cu_n\|^2 < \infty$. *We write* $\mathfrak{L}_{1,C}^+(\mathfrak{h})$ *for the set of all* C*-regular density operators.*

We next formulate the concept of C-regular operator in terms of random variables. This characterization of $\mathfrak{L}_{1,C}^+(\mathfrak{h})$ complements those given in [12] using operator theory (see also [13]).

Theorem 3.1. *Suppose that* C *is a self-adjoint positive operator in* $\mathfrak{h}$. *Let* ϱ *be a linear operator in* $\mathfrak{h}$. *Then* ϱ *is* C*-regular if and only if* $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$ *for some* $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$. *Moreover,* $\mathbb{E}|\xi\rangle\langle\xi|$ *can be interpreted as a Bochner integral in both* $\mathfrak{L}_1(\mathfrak{h})$ *and* $\mathfrak{L}(\mathfrak{h})$.

Proof. The proof is divided into Lemmata 7.5 and 7.6. □

By the following theorem, the mean values of a large number of unbounded observables are well-posed when the density operators are C-regular. Theorem 3.2 also provides probabilistic interpretations of these expected values.

THEOREM 3.2. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$, and fix $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$ with $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$. Then:*

(a) The range of ϱ is contained in $\mathcal{D}(C)$ and $C\varrho = \mathbb{E}\left|C\xi\right\rangle\left\langle\xi\right|$.
(b) Consider $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, and let B be a densely defined linear operator in $\mathfrak{h}$ such that $\mathcal{D}(C) \subset \mathcal{D}(B^*)$. Then $A\varrho B$ is densely defined and bounded. The unique bounded extension of $A\varrho B$ belongs to $\mathfrak{L}_1(\mathfrak{h})$ and is equal to $\mathbb{E}\left|A\xi\right\rangle\left\langle B^*\xi\right|$, where $\mathbb{E}\left|A\xi\right\rangle\left\langle B^*\xi\right|$ is a well-defined Bochner integral in both $\mathfrak{L}_1(\mathfrak{h})$ and $\mathfrak{L}(\mathfrak{h})$. Moreover,

$$tr\left(A\varrho B\right) = \mathbb{E}\left\langle B^*\xi, A\xi\right\rangle.$$

PROOF. Deferred to Subsection 7.2. □

4. Quantum master equation. We first deduce that (1.8) defines a density operator.

THEOREM 4.1. *Let Hypothesis 2 hold. Then, for every $t \geq 0$ there exists a unique operator $\rho_t \in \mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))$ such that for each C-regular operator ϱ,*

$$\rho_t(\varrho) = \mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|, \tag{4.1}$$

where ξ is an arbitrary random variable in $L^2_C(\mathbb{P}, \mathfrak{h})$ satisfying $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$. Here $X(\xi)$ is the strong C-solution of (1.4) with initial datum ξ, and we can interpret $\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|$ as a Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$ as well as in $\mathfrak{L}(\mathfrak{h})$. Moreover, $\|\rho_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} \leq 1$ for all $t \geq 0$.

PROOF. Deferred to Subsection 7.4. □

NOTATION 4.1. *From now on, ρ_t stands for the operator given by (4.1).*

The next theorem says that the expected value $\mathbb{E}$ commutes with the action of ρ_t on random C-regular pure density operators.

THEOREM 4.2. *Assume that Hypothesis 2 holds. Let $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$, with $\xi \in L^2_C(\mathbb{P}, \mathfrak{h})$. Then $\mathbb{E}\rho_t(\left|\xi\right\rangle\left\langle\xi\right|) = \rho_t(\varrho)$ for all $t \geq 0$.*

PROOF. Deferred to Subsection 7.5. □

We now summarize some relevant properties of the family of linear operators $(\rho_t)_{t\geq 0}$.

THEOREM 4.3. *Adopt Hypothesis 2. Then* $(\rho_t)_{t\geq 0}$ *is a semigroup of contractions such that* $\rho_t\left(\mathfrak{L}_1^+(\mathfrak{h})\right) \subset \mathfrak{L}_1^+(\mathfrak{h})$, $\rho_t\left(\mathfrak{L}_{1,C}^+(\mathfrak{h})\right) \subset \mathfrak{L}_{1,C}^+(\mathfrak{h})$, *and for all* $\varrho \in \mathfrak{L}_{1,C}^+(\mathfrak{h})$*:*

$$\lim_{s\to t} tr\left|\rho_s(\varrho) - \rho_t(\varrho)\right| = 0. \tag{4.2}$$

PROOF. The proof is divided into Lemmata 7.12, 7.13 and 7.14. □

The analysis outlined in Subsection 1.5 leads to our first main theorem, which asserts that $\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|$ satisfies (1.1) in both sense integral and $\mathfrak{L}_1(\mathfrak{h})$-weak whenever $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$ is C-regular.

HYPOTHESIS 3. *The operators* $G, L_1, L_2, \ldots$ *are closable.*

THEOREM 4.4. *Let Hypotheses 2 and 3 hold. Suppose that* ϱ *is* C*-regular. Then for all* $t \geq 0$*:*

$$\rho_t(\varrho) = \varrho + \int_0^t \left(G\rho_s(\varrho) + \rho_s(\varrho)G^* + \sum_{k=1}^{\infty} L_k\rho_s(\varrho)L_k^*\right)ds, \tag{4.3}$$

where we understand the above integral in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$*. Moreover, for any* $A \in \mathfrak{L}(\mathfrak{h})$ *and* $t \geq 0$*:*

$$\frac{d}{dt}tr\left(A\rho_t(\varrho)\right) = tr\left(A\left(G\rho_t(\varrho) + \rho_t(\varrho)G^* + \sum_{k=1}^{\infty} L_k\rho_t(\varrho)L_k^*\right)\right). \tag{4.4}$$

PROOF. Deferred to Subsection 7.7. □

REMARK 4.1. *Let* $G, L_1, L_2, \ldots$ *be densely defined. Then Hypothesis 3 is equivalent to say that* $G^*, L_1^*, L_2^*, \ldots$ *are densely defined.*

The second main theorem of this paper establishes that under Hypothesis 2, $\rho_t(\varrho)$ is the unique solution of (4.4) in the semigroup sense. Its proof is based on arguments given in Subsection 1.6.

DEFINITION 4.1. *A semigroup* $(\widehat{\rho}_t)_{t\geq 0}$ *of bounded operators on* $\mathfrak{L}_1(\mathfrak{h})$ *is called semigroup* C*-solution of (1.1) if and only if:*

(i) For each non-negative real number T, $\sup_{t\in[0,T]}\left\|\widehat{\rho}_t\right\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} < \infty$.
(ii) For any $x \in \mathcal{D}(C)$ *and* $A \in \mathfrak{L}(\mathfrak{h})$*, the function* $t \mapsto tr\left(\widehat{\rho}_t\left(\left|x\right\rangle\left\langle x\right|\right)A\right)$ *is continuous.*

(iii) $\lim_{t\to 0+}\left(tr\left(A\widehat{\rho}_t\left(|x\rangle\langle x|\right)\right)-tr\left(A\,|x\rangle\langle x|\right)\right)/t=\langle x,AGx\rangle+\langle Gx,Ax\rangle+\sum_{k=1}^{\infty}\langle L_kx,AL_kx\rangle$ *whenever* $x\in\mathcal{D}\left(C\right)$ *and* $A\in\mathfrak{L}\left(\mathfrak{h}\right)$.

THEOREM 4.5. *Let Hypothesis 2 hold. Then* $\left(\rho_t\right)_{t\geq 0}$ *is the unique semigroup* C*-solution of (1.1).*

PROOF. Deferred to Subsection 7.8. □

Theorems 4.4 and 4.5, together with Theorem 3.2, show that the mean values of the observables with respect to the solutions of the quantum master equations are well-posed in many physical situations. Moreover, Theorems 3.2, 4.4 and 4.5 allow us to make rigorous some explicit computations concerning the evolution of unbounded observables, like the following Ehrenfest-type theorem.

THEOREM 4.6. *Assume the setting of Example 1. Then* $\left(\rho_t\right)_{t\geq 0}$ *is the unique semigroup* $\left(P^2+Q^2\right)$*-solution of (1.1). If* $\varrho\in\mathfrak{L}^+_{1,P^2+Q^2}\left(L^2\left(\mathbb{R},\mathbb{C}\right)\right)$*, then for all* $t\geq 0$*:*

$$\frac{d}{dt}tr\left(Q\rho_t\left(\varrho\right)\right)=\frac{1}{m}tr\left(P\rho_t\left(\varrho\right)\right),\qquad \frac{d}{dt}tr\left(P\rho_t\left(\varrho\right)\right)=-2c\,tr\left(Q\rho_t\left(\varrho\right)\right). \tag{4.5}$$

PROOF. Deferred to Subsection 7.9. □

REMARK 4.2. *A novelty of this paper lies in the use of probabilistic methods for proving Theorems 4.4 and 4.5. In order to adopt a purely Operator Theory viewpoint, we now return to Remark 2.5. Let* $\left(\mathcal{T}_{*t}\right)_{t\geq 0}$ *be the semigroup on* $\mathfrak{L}_1\left(\mathfrak{h}\right)$ *whose adjoint semigroup is* $\left(\mathcal{T}_t^{(min)}\right)_{t\geq 0}$*, i.e.,* $\mathcal{T}_*$ *is the predual semigroup of* $\mathcal{T}^{(min)}$*. In case* $\mathcal{T}^{(min)}$ *leaves invariant the identity operator, the linear span of* $\{|x\rangle\langle y|:x,y\in\mathcal{D}\left(G\right)\}$ *is a core for the infinitesimal generator of* $\left(\mathcal{T}_{*t}\right)_{t\geq 0}$*, which is denoted by* $\mathcal{L}_*$ *for simplicity of notation (see, e.g., Proposition 3.32 of [17]). Then, under Conditions (i) and (ii) given in Remark 2.5,* $\left(\mathcal{T}_{*t}\right)_{t\geq 0}$ *is the unique strongly continuous semigroup on* $\mathfrak{L}_1\left(\mathfrak{h}\right)$ *satisfying a version of (1.1) for all* $\varrho=|x\rangle\langle y|$ *with* $x,y\in\mathcal{D}\left(G\right)$*. In order to establish* $\mathcal{T}_{*t}\left(\mathfrak{L}^+_{1,C}\left(\mathfrak{h}\right)\right)\subset\mathfrak{L}^+_{1,C}\left(\mathfrak{h}\right)$ *as well as the assertions of Theorem 4.4, we have to prove first that* $\mathfrak{L}^+_{1,C}\left(\mathfrak{h}\right)\subset\mathcal{D}\left(\mathcal{L}_*\right)$*. If we are able to do it, then* $\mathfrak{L}^+_{1,C}\left(\mathfrak{h}\right)$ *is an invariant set for* $\mathcal{T}_{*t}$ *provided that*

$$\sup_{n\in\mathbb{N}}\left|tr\left(\mathcal{T}^{(min)}\left(n\left(n+C^2\right)^{-1}C^2\right)\varrho\right)\right|<\infty \tag{4.6}$$

for any $\varrho \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$. *When* C *is invertible, (4.6) follows from*

$$\left\| C^{-1}\mathcal{T}^{(min)}\left(n\left(n+C^{2}\right)^{-1}C^{2}\right)C^{-1}\right\| \leq K\left\| n\left(n+C^{2}\right)^{-1}\right\|. \tag{4.7}$$

A careful reading of [21] reveals that for any $A \in \mathfrak{L}(\mathfrak{h})$ *we have*

$$\left\| C^{-1}\mathcal{T}^{(min)}(A)\,C^{-1}\right\| \leq K\left\| C^{-1}AC^{-1}\right\| \tag{4.8}$$

under assumptions of type (1.5), together with $\exp(Gt)$ *leaves invariant a core of* C *contained in* $\mathcal{D}(G)$ *(see also [12, 13]). This gives (4.7), and so (4.6) holds. Under the same assumptions, an alternative is to obtain (4.8) by proving* $\left\| C^{-1}\mathcal{T}^{(n)}(A)\,C^{-1}\right\| \leq K\left\| C^{-1}AC^{-1}\right\|$ *directly from the definition of* $\mathcal{T}^{(n)}$, *but with effort. Here* $\mathcal{T}^{(n)}$ *is as in Remark 2.5. Finally, to establish (4.3) and (4.4) we have to get that* $\mathcal{L}_{*}(\varrho) = G\varrho + \varrho G^{*} + \sum_{k=1}^{\infty} L_{k}\varrho L_{k}^{*}$ *for any* $\varrho \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$.

5. Regular stationary solutions of quantum master equations. This section is devoted to Objective O2. In this direction, the next theorem provides the representation of the density operator at time t as the average of all pure states $|Y_t\rangle\langle Y_t|$ associated to the non-linear stochastic Schrödinger equation (1.3). This model has a sound physical basis (see, e.g., [2, 3, 24, 37]).

Definition 5.1. *Let* C *satisfy Hypothesis 1. Suppose that* $\mathbb{I}$ *is either* $[0,+\infty[$ *or* $[0,T]$ *provided* $T \in [0,+\infty[$. *We say that* $\left(\mathbb{Q},(Y_t)_{t\in\mathbb{I}},(B_t)_{t\in\mathbb{I}}\right)$ *is a* C*-solution of (1.3) with initial distribution* θ *on* $\mathbb{I}$ *if and only if:*

- $B = \left(B^{k}\right)_{k\in\mathbb{N}}$ *is a sequence of real valued independent Brownian motions on the filtered complete probability space* $\left(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\in\mathbb{I}},\mathbb{Q}\right)$.
- $(Y_t)_{t\in\mathbb{I}}$ *is an* $\mathfrak{h}$*-valued process with continuous sample paths such that the law of* Y_0 *coincides with* θ *and* $\mathbb{Q}\left(\|Y_t\| = 1 \text{ for all } t\in\mathbb{I}\right) = 1$.
- *For every* $t \in \mathbb{I}$: $Y_t \in \mathcal{D}(C)$ $\mathbb{Q}$*-a.s. and* $\sup_{s\in[0,t]}\mathbb{E}_{\mathbb{Q}}\|CY_s\|^{2} < \infty$.
- $\mathbb{Q}$*-a.s.,* $Y_t = Y_0 + \int_0^t G(\pi_C(Y_s))\,ds + \sum_{k=1}^{\infty}\int_0^t L_k(\pi_C(Y_s))\,dB_s^k$ *for all* $t \in \mathbb{I}$.

Theorem 5.1. *Suppose that Hypothesis 2 holds. Let* $\varrho = \int_{\mathfrak{h}} |y\rangle\langle y|\,\theta(dy)$, *with* θ *probability measure over* $\mathfrak{h}$ *satisfying* $\theta\left(\mathcal{D}(C)\cap\{x\in\mathfrak{h} : \|x\| = 1\}\right) = 1$ *and* $\int_{\mathfrak{h}}\|Cx\|^{2}\,\theta(dx) < \infty$. *Then for all* $t \geq 0$:

$$\rho_t(\varrho) = \mathbb{E}_{\mathbb{Q}}\,|Y_t\rangle\langle Y_t|,$$

where $\rho_t(\varrho)$ *is defined by (4.1) and* $\left(\mathbb{Q},(Y_t)_{t\geq 0},(B_t)_{t\geq 0}\right)$ *is the* C*-solution of (1.3) with initial law* θ.

Proof. Deferred to Subsection 7.10. □

Remark 5.1. *Let θ be as in Theorem 5.1. Suppose that Hypothesis 2 holds. Then, we can use the same arguments as in the proof of Theorem 1 of [31] for establishing that (1.3) has a unique (in the probabilistic sense) C-solution $\left(\mathbb{Q},(Y_t)_{t\geq 0},(B_t)_{t\geq 0}\right)$ with initial law θ.*

Remark 5.2. *From Theorems 3.2 and 5.1 we obtain that the expected value of $A\in\mathfrak{L}\left(\left(\mathcal{D}(C),\|\cdot\|_C\right),\mathfrak{h}\right)$ at time t is equal to $\mathbb{E}\langle Y_t,AY_t\rangle$. This gives theoretical support for the numerical computation of the mean value of an observable A at time t through (1.3), which is the principal method for computing efficiently $tr\left(A\rho_t(\varrho)\right)$ (see, e.g., [8, 28, 34]).*

Our third main theorem deals with the existence of regular stationary states for (1.1). This is a step forward the understanding of the long time behaviour of unbounded observables.

Hypothesis 4. *Let Hypothesis 2 hold. Assume the existence of a probability measure Γ on $\mathfrak{B}(\mathfrak{h})$ such that: $\Gamma\left(Dom(C)\cap\{x\in\mathfrak{h}:\|x\|=1\}\right)=1$, $\int_{\mathfrak{h}}\|Cz\|^2\Gamma(dz)<\infty$, and*

$$\Gamma(A)=\int_{\mathfrak{h}}P_t(x,A)\,\Gamma(dx) \tag{5.1}$$

for any $t\geq 0$ and $A\in\mathfrak{B}(\mathfrak{h})$. Here $P_t(x,A)=\mathbb{Q}_x\left(Y_t^x\in A\right)$ if $x\in Dom(C)$ and $P_t(x,A)=\delta_x(A)$ otherwise; the C-solution of (1.3) with initial data $x\in Dom(C)$ is denoted by $\left(\mathbb{Q}_x,(Y_t^x)_{t\geq 0},(B_t^{\cdot,x})_{t\geq 0}\right)$.

Theorem 5.2. *Under Hypothesis 4, there exists a C-regular operator ϱ_∞ such that $\rho_t(\varrho_\infty)=\varrho_\infty$ for all $t\geq 0$.*

Proof. Deferred to Subsection 7.11. □

Remark 5.3. *Combining the results of Section 4 with Theorem 5.2 yields the existence of a C-regular stationary solution to (1.1).*

6. Quantum oscillator. In this section we illustrate our general results with the following quantum oscillator.

Example 2. *Consider $\mathfrak{h}=l^2(\mathbb{Z}_+)$, together with its canonical orthonormal basis $(e_n)_{n\in\mathbb{Z}_+}$. The closed operators $a^\dagger$, a are given by: for all $n\in\mathbb{Z}_+$ $a^\dagger e_n=\sqrt{n+1}e_{n+1}$, $ae_0=0$, and $ae_n=\sqrt{n}e_{n-1}$ if $n\in\mathbb{N}$. Define $N=a^\dagger a$.*

Choose $H = i\beta_1\left(a^\dagger - a\right) + \beta_2 N + \beta_3\left(a^\dagger\right)^2 a^2$ *with* $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$. *Let* $L_1 = \alpha_1 a$, $L_2 = \alpha_2 a^\dagger$, $L_3 = \alpha_3 N$, $L_4 = \alpha_4 a^2$, $L_5 = \alpha_5\left(a^\dagger\right)^2$ *and* $L_6 = \alpha_6 N^2$, *where* $\alpha_1, \ldots, \alpha_6 \in \mathbb{C}$. *Set* $L_k = 0$ *for any* $k \geq 7$, *and so take* $G = -iH - \sum_{k=1}^{6} L_k^* L_k/2$.

Example 2 describes a laser-driven quantum oscillator in a Kerr medium that interacts with a thermal bath. In addition, Example 2 unifies concrete physical systems such as the following two basic models:

- A mode with natural frequency ω of a electromagnetic field inside of a cavity is described by $\beta_2 = \omega$, $\alpha_1 = \sqrt{A(\nu+1)}$, $\alpha_2 = \sqrt{A\nu}$ and $\beta_1 = \beta_3 = \alpha_k = 0$, with $k = 3, \ldots, 6$. Here, the mode is damped with rate α_1 by a thermal reservoir, and ν is a parametrization of the bath temperature (see, e.g., [8, 22, 38]).
- A simply two-photon absorption and emission process is modeled by $\beta_3 \in \mathbb{R}$, $\alpha_4 > 0$, $\alpha_5 \geq 0$ and $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_6 = 0$ (see, e.g., [9, 19] and references therein).

The next theorem characterizes the well-posedness of the mean values of observables formed by a finite composition of $a^\dagger$ and a in transient and stationary regimes. Important examples of such observables are: $Q = i\left(a^\dagger + a\right)/\sqrt{2}$, $P = i\left(a^\dagger - a\right)/\sqrt{2}$ and N.

THEOREM 6.1. *Assume the setting of Example 2 and let* $\rho_t(\varrho)$ *be as in Theorem 4.1. Suppose that* p *is a natural number greater than or equal to* 4.

(i) Let $|\alpha_4| \geq |\alpha_5|$ *and let* $\varrho \in \mathfrak{L}^+_{1,N^p}\left(l^2\left(\mathbb{Z}_+\right)\right)$. *Then* $\rho_t(\varrho)$ *is a* N^p*-regular operator that satisfies both (4.3) and (4.4). Moreover,* $(\rho_t)_{t\geq 0}$ *is the unique semigroup* N^p*-solution of (1.1).*

(ii) Suppose that either $|\alpha_4| > |\alpha_5|$ *or* $|\alpha_4| = |\alpha_5|$ *with* $|\alpha_2|^2 - |\alpha_1|^2 + 4(2p+1)|\alpha_4|^2 < 0$. *Then, there exists a* N^p*-regular operator* ϱ_∞ *such that* $\rho_t(\varrho_\infty) = \varrho_\infty$ *for any* $t \geq 0$.

PROOF. Deferred to Subsection 7.12. □

REMARK 6.1. *In the proof of Theorem 6.1 we use the following sufficient condition for Condition H2.3, which is developed in [18].*

HYPOTHESIS 5. *Suppose that* C *is a self-adjoint positive operator in* $\mathfrak{h}$ *such that* $G, L_1, L_2, \ldots$ *belong to* $\mathfrak{L}\left(\left(\mathcal{D}(C), \|\cdot\|_C\right), \mathfrak{h}\right)$, *and* $2\Re\langle x, Gx\rangle + \sum_{k=1}^{\infty}\|L_k x\|^2 \leq 0$ *for any* x *in a core of* C. *In addition, assume that for any* x *belonging to a core of* C^2, $2\Re\langle C^2 x, Gx\rangle + \sum_{k=1}^{\infty}\|CL_k x\|^2 \leq K\left(\|x\|_C^2 + 1\right)$.

7. Proofs.

7.1. *Proof of Theorem 2.1.* We first prove that (2.1) define implicitly a bounded operator $\mathcal{T}_t(A)$.

LEMMA 7.1. *Adopt the assumptions of Hypothesis 2 with the exception of Condition H2.2. Consider $A \in \mathfrak{L}(\mathfrak{h})$. Then for every $t \geq 0$ there exists a unique $\mathcal{T}_t(A)$ belonging to $\mathfrak{L}(\mathfrak{h})$ for which (2.1) holds for all x, y in $\mathcal{D}(C)$. Moreover, $\|\mathcal{T}_t(A)\| \leq \|A\|$ for any $t \geq 0$.*

PROOF. By Definition 2.1, $|\mathbb{E}\langle X_t(x), AX_t(y)\rangle| \leq \|A\|\|x\|\|y\|$ for all $x, y \in \mathcal{D}(C)$. Hence the sesquilinear form over $\mathcal{D}(C) \times \mathcal{D}(C)$ given by $(x, y) \mapsto \mathbb{E}\langle X_t(x), AX_t(y)\rangle$ can be extended uniquely to a sesquilinear form $[\cdot,\cdot]$ over $\mathfrak{h} \times \mathfrak{h}$ with the property that $|[x, y]| \leq \|A\|\|x\|\|y\|$ for any $x, y \in \mathfrak{h}$. There exists a unique bounded operator $\mathcal{T}_t(A)$ on $\mathfrak{h}$ such that $|[x, y]| = \langle x, \mathcal{T}_t(A)y\rangle$ for all x, y in $\mathfrak{h}$. Furthermore, $\|\mathcal{T}_t(A)\| \leq \|A\|$. □

Using arguments given in Subsection 1.6 we next establish the uniqueness of solutions for the adjoint quantum master equations.

LEMMA 7.2. *Let Hypothesis 2 hold. Assume that $(\mathcal{A}_t)_{t\geq 0}$ is a C-solution of (1.2) with initial datum $A \in \mathfrak{L}(\mathfrak{h})$. Then $\mathcal{A}_t = \mathcal{T}_t(A)$ for all $t \geq 0$, where $\mathcal{T}_t(A)$ is as in Therorem 2.1.*

PROOF. Using Itô's formula we will prove that for all $x, y \in \mathcal{D}(C)$,

$$\mathbb{E}\langle X_t(x), AX_t(y)\rangle = \langle x, \mathcal{A}_t y\rangle. \tag{7.1}$$

This, together with Lemma 7.1, implies $\mathcal{A}_t = \mathcal{T}_t(A)$.

Motivated by $\mathcal{A}_t$ is only a weak solution, we fix an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{h}$ and consider the function $F_n : [0, t] \times \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ defined by

$$F_n(s, u, v) = \langle R_n\overline{u}, \mathcal{A}_{t-s}R_n v\rangle,$$

where $R_n = n(n + C)^{-1}$ and $\bar{u} = \sum_{n\in\mathbb{N}} \overline{\langle e_n, u\rangle} e_n$. Since the range of R_n is contained in $\mathcal{D}(C)$, Condition (a) of Definition 2.2 yields

$$\frac{d}{ds}F_n(s, u, v) = -g(s, R_n\overline{u}, R_n v), \tag{7.2}$$

with $g(s, x, y) = \langle x, \mathcal{A}_{t-s}Gy\rangle + \langle Gx, \mathcal{A}_{t-s}y\rangle + \sum_{k=1}^{\infty}\langle L_k x, \mathcal{A}_{t-s}L_k y\rangle$. According to Conditions (a), (b) of Definition 2.2, we have that $t \longmapsto \langle u, \mathcal{A}_t v\rangle$ is continuous for all $u, v \in \mathfrak{h}$, and so combining $CR_n \in \mathfrak{L}(\mathfrak{h})$ with Hypothesis 2 we get the uniformly continuity of $(s, u, v) \longmapsto g(s, R_n\overline{u}, R_n v)$ on

bounded subsets of $[0,t]\times\mathfrak{h}\times\mathfrak{h}$. Therefore we can apply Itô's formula to $F_n\left(s\wedge\tau_j,\overline{X_s^{\tau_j}(x)},X_s^{\tau_j}(y)\right)$, with $\tau_j=\inf\{t\geq 0:\|X_t(x)\|+\|X_t(y)\|>j\}$.

Fix $x,y\in\mathcal{D}(C)$. Combining Itô's formula with (7.2) we deduce that

$$F_n\left(t\wedge\tau_j,\overline{X_t^{\tau_j}(x)},X_t^{\tau_j}(y)\right)=F_n\left(0,\overline{X_0(x)},X_0(y)\right)+I^n_{t\wedge\tau_j}+M_t.$$

Here for $s\in[0,t]$: $M_s=\sum_{k=1}^{\infty}\int_0^{s\wedge\tau_j}\left\langle R_nX_r^{\tau_j}(x),\mathcal{A}_{t-r}R_nL_kX_r^{\tau_j}(y)\right\rangle dW_r^k+\sum_{k=1}^{\infty}\int_0^{s\wedge\tau_j}\left\langle R_nL_kX_r^{\tau_j}(x),\mathcal{A}_{t-r}R_nX_r^{\tau_j}(y)\right\rangle dW_r^k$ and

$$I_s^n=\int_0^s\left(-g\left(r,R_nX_r(x),R_nX_r(y)\right)+g_n\left(r,X_r(x),X_r(y)\right)\right)dr,$$

the function $g_n(r,u,v)$ is equal to $\langle R_nu,\mathcal{A}_{t-r}R_nGv\rangle+\langle R_nGu,\mathcal{A}_{t-r}R_nv\rangle+\sum_{k=1}^{\infty}\langle R_nL_ku,\mathcal{A}_{t-r}R_nL_kv\rangle$.

We next establish the martingale property of M_s. For all $r\in[0,t]$ we have

$$\left\|R_nX_r^{\tau_j}(x)\right\|^2\|\mathcal{A}_{t-r}\|^2\left\|R_nL_kX_r^{\tau_j}(y)\right\|^2\leq j^2\sup_{s\in[0,t]}\|\mathcal{A}_s\|^2\left\|L_kX_r^{\tau_j}(y)\right\|^2.$$

By H2.1 and H2.2, $\mathbb{E}\int_0^{t\wedge\tau_j}\sum_{k=1}^{\infty}\left|\left\langle R_nX_r^{\tau_j}(x),\mathcal{A}_{t-r}R_nL_kX_r^{\tau_j}(y)\right\rangle\right|^2ds<\infty$. Thus $\left(\sum_{k=1}^{\infty}\int_0^{s\wedge\tau_j}\left\langle R_nX_r^{\tau_j}(x),\mathcal{A}_{t-r}R_nL_kX_r^{\tau_j}(y)\right\rangle dW_r^k\right)_{s\in[0,t]}$ is a martingale. The same conclusion can be draw for

$$\sum_{k=1}^{\infty}\int_0^{s\wedge\tau_j}\left\langle R_nL_kX_r^{\tau_j}(x),\mathcal{A}_{t-r}R_nX_r^{\tau_j}(y)\right\rangle dW_r^k,$$

and so $(M_s)_{s\in[0,t]}$ is a martingale. Hence

$$\mathbb{E}\left\langle R_nX_t^{\tau_j}(x),\mathcal{A}_{t-t\wedge\tau_j}R_nX_t^{\tau_j}(y)\right\rangle=\langle R_nx,\mathcal{A}_tR_ny\rangle+\mathbb{E}I^n_{t\wedge\tau_j}. \tag{7.3}$$

We will take the limit as $j\to\infty$ in (7.3). Since $\mathbb{E}\left(\sup_{s\in[0,t]}\|X_s(\xi)\|^2\right)<\infty$ for $\xi=x,y$ (see, e.g., Th. 4.2.5 of [36]), using the dominated convergence theorem, together with the continuity of $t\longmapsto\langle u,\mathcal{A}_tv\rangle$, we get

$$\mathbb{E}\left\langle R_nX_t^{\tau_j}(x),\mathcal{A}_{t-t\wedge\tau_j}R_nX_t^{\tau_j}(y)\right\rangle\to_{j\to\infty}\mathbb{E}\left\langle R_nX_t(x),AR_nX_t(y)\right\rangle.$$

Applying again the dominated convergence theorem yields $\mathbb{E}I^n_{t\wedge\tau_j}\longrightarrow_{j\to\infty}\mathbb{E}I^n_t$, and hence letting $j\to\infty$ in (7.3) we deduce that

$$\begin{aligned}&\mathbb{E}\left\langle R_nX_t(x),AR_nX_t(y)\right\rangle-\langle R_nx,\mathcal{A}_tR_ny\rangle\\&=\mathbb{E}\int_0^t\left(-g\left(s,R_nX_s(x),R_nX_s(y)\right)+g_n\left(s,X_s(x),X_s(y)\right)\right)ds.\end{aligned} \tag{7.4}$$

Finally, we take the limit as $n \to \infty$ in (7.4). Since $\|R_n\| \leq 1$ and R_n tends pointwise to I as $n \to \infty$, the dominated convergence theorem yields

$$\lim_{n\to\infty} \mathbb{E} \int_0^t g_n\left(s, X_s\left(x\right), X_s\left(y\right)\right) ds = \mathbb{E} \int_0^t g\left(s, X_s\left(x\right), X_s\left(y\right)\right) ds.$$

For any $x \in \mathcal{D}\left(C\right)$, $\lim_{n\to\infty} CR_n x = Cx$. By $\|CR_n x\| \leq \|Cx\|$, using the dominated convergence theorem gives

$$\lim_{n\to\infty} \mathbb{E} \int_0^t g\left(s, R_n X_s\left(x\right), R_n X_s\left(y\right)\right) ds = \mathbb{E} \int_0^t g\left(s, X_s\left(x\right), X_s\left(y\right)\right) ds.$$

Thus, letting $n \to \infty$ in (7.4) we obtain (7.1). □

7.2. *Proof of Theorem 3.2.* We begin by examining the properties of the Bochner integral $\mathbb{E}\left|\xi\right\rangle\left\langle\chi\right|$ when $\xi, \chi \in L^2\left(\mathbb{P}, \mathfrak{h}\right)$.

Lemma 7.3. *Suppose that ξ and χ belong to $L^2\left(\mathbb{P}, \mathfrak{h}\right)$. Then $\mathbb{E}\left|\xi\right\rangle\left\langle\chi\right|$ defines an element of $\mathfrak{L}_1\left(\mathfrak{h}\right)$, which moreover, is given by*

$$\left\langle x, \mathbb{E}\left|\xi\right\rangle\left\langle\chi\right| y\right\rangle = \mathbb{E}\left\langle x, \xi\right\rangle\left\langle\chi, y\right\rangle \tag{7.5}$$

for all $x, y \in \mathfrak{h}$. Here, $\mathbb{E}\left|\xi\right\rangle\left\langle\chi\right|$ is well-defined as a Bochner integral with values in both $\mathfrak{L}_1\left(\mathfrak{h}\right)$ and $\mathfrak{L}\left(\mathfrak{h}\right)$. In addition, $tr\left(\mathbb{E}\left|\xi\right\rangle\left\langle\chi\right|\right) = \mathbb{E}\left\langle\chi, \xi\right\rangle$.

Proof. We first get $\mathbb{E}\left|\xi\right\rangle\left\langle\chi\right| \in \mathfrak{L}_1\left(\mathfrak{h}\right)$. Since the image of $\left|\xi\right\rangle\left\langle\chi\right|$ lies in the set of all rank-one operators on $\mathfrak{h}$, $\left|\xi\right\rangle\left\langle\chi\right|$ takes values in $\mathfrak{L}_1\left(\mathfrak{h}\right)$. Applying Parseval's equality yields

$$tr\left(A\left|\xi\right\rangle\left\langle\chi\right|\right) = \left\langle\chi, A\xi\right\rangle. \tag{7.6}$$

Hence $\left|\xi\right\rangle\left\langle\chi\right|$ is $\mathfrak{B}\left(\mathfrak{L}_1\left(\mathfrak{h}\right)\right)$-measurable, because the dual of $\mathfrak{L}_1\left(\mathfrak{h}\right)$ is formed by all maps $\varrho \mapsto tr\left(A\varrho\right)$ with $A \in \mathfrak{L}\left(\mathfrak{h}\right)$. Let $x, y \in \mathfrak{h}$. The absolute value of the operator $\left|x\right\rangle\left\langle y\right|$ is equal to the operator $\left|y\right\rangle\left\langle y\right| \|x\| / \|y\|$ in case $y \neq 0$, and coincides with the null operator otherwise. Therefore

$$\left\|\left|x\right\rangle\left\langle y\right|\right\|_1 = \frac{\|x\|}{\|y\|} \|y\|^2 = \|x\| \|y\|. \tag{7.7}$$

Combining $\xi, \chi \in L^2\left(\mathbb{P}, \mathfrak{h}\right)$ with (7.7) gives $\mathbb{E}\left\|\left|\xi\right\rangle\left\langle\chi\right|\right\|_1 < \infty$, and so the Bochner integral $\mathbb{E}\left|\xi\right\rangle\left\langle\chi\right|$ is well-defined in the separable Banach space $\mathfrak{L}_1\left(\mathfrak{h}\right)$.

We now turn to work in $\mathfrak{L}\left(\mathfrak{h}\right)$. The application $\left(x, y\right) \mapsto \left|x\right\rangle\left\langle y\right|$ from $\mathfrak{h} \times \mathfrak{h}$ to $\mathfrak{L}\left(\mathfrak{h}\right)$ is continuous, and in consequence the measurability of ξ and χ implies that $\left|\xi\right\rangle\left\langle\chi\right|$ is $\mathfrak{B}\left(\mathfrak{L}\left(\mathfrak{h}\right)\right)$-measurable. Thus using $\|\cdot\|_{\mathfrak{L}(\mathfrak{h})} \leq \|\cdot\|_1$ we deduce

that $|\xi\rangle\langle\chi|$ is Bochner $\mathbb{P}$-integrable in $\mathfrak{L}(\mathfrak{h})$ (see, e.g., [39] for a treatment of the Bochner integral in Banach spaces which in general are not separable). Since $\mathfrak{L}_1(\mathfrak{h})$ is continuously embedded in $\mathfrak{L}(\mathfrak{h})$, either of the interpretations of $\mathbb{E}|\xi\rangle\langle\chi|$ given above refers to the same operator.

Finally, for any x, y belonging to $\mathfrak{h}$, the linear function $A \mapsto \langle x, Ay\rangle$ is continuous as a map from $\mathfrak{L}(\mathfrak{h})$ to $\mathbb{C}$. This gives (7.5). Similarly, (7.6) yields $tr(\mathbb{E}|\xi\rangle\langle\chi|) = \mathbb{E}tr(|\xi\rangle\langle\chi|) = \mathbb{E}\langle\chi,\xi\rangle$, because $tr(\cdot) \in \mathfrak{L}_1(\mathfrak{h})'$. □

Remark 7.1. *Under the assumptions of Lemma 7.3, $\mathbb{E}|\xi\rangle\langle\chi|$ can also be interpreted as a Bochner integral in the pointwise sense (see, e.g., [35]).*

To prove Theorem 3.2, we need the following lemma.

Lemma 7.4. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$ and $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. Then $A\xi$ belongs to $L^2(\mathbb{P},\mathfrak{h})$.*

Proof. Since $A\xi = A\pi_C(\xi)$ $\mathbb{P}$-*a.s.*, from Remark 2.1 we deduce that $A\xi$ is strongly measurable. Thus $A\xi \in L^2(\mathbb{P},\mathfrak{h})$. □

Proof of Theorem 3.2. We start by proving Statement (a). Let $x \in \mathcal{D}(C)$ and let $y \in \mathfrak{h}$. Using Lemma 7.3 yields

$$\langle Cx, \varrho y\rangle = \mathbb{E}\langle Cx,\xi\rangle\langle\xi,y\rangle = \mathbb{E}\langle x, C\xi\rangle\langle\xi,y\rangle.$$

In Lemma 7.4 we take $A = C$ to obtain $C\xi \in L^2(\mathbb{P},\mathfrak{h})$. Thus, Lemma 7.3 implies $\mathbb{E}\langle x, C\xi\rangle\langle\xi,y\rangle = \langle x, \mathbb{E}|C\xi\rangle\langle\xi|\, y\rangle$, and so $\langle Cx,\varrho y\rangle = \langle x, \mathbb{E}|C\xi\rangle\langle\xi|\, y\rangle$. Then $\varrho y \in \mathcal{D}(C^*) = \mathcal{D}(C)$ and $C\varrho y = \mathbb{E}|C\xi\rangle\langle\xi|\, y$, which is our assertion.

Part (a) yields $\mathcal{D}(B) = \mathcal{D}(A\varrho B)$, and so $A\varrho B$ is densely defined. We next prove that $A\varrho B$ coincides with $\mathbb{E}|A\xi\rangle\langle B^*\xi|$ on $\mathcal{D}(B)$. For this purpose, we approximate A by AR_n, where R_n is the Yosida approximation of $-C$.

Suppose that $x \in \mathfrak{h}$ and $y \in \mathcal{D}(B)$. As in the proof of Lemma 7.2 we consider $R_n = n(n+C)^{-1}$, and so $CR_nz \longrightarrow_{n\to\infty} Cz$ for any $x \in \mathcal{D}(C)$. Therefore $\langle x, AR_n\varrho By\rangle \longrightarrow_{n\to\infty} \langle x, A\varrho By\rangle$, hence Lemma 7.3 gives

(7.8)
$$\langle x, A\varrho By\rangle = \lim_{n\to\infty}\mathbb{E}\langle (AR_n)^* x,\xi\rangle\langle\xi, By\rangle = \lim_{n\to\infty}\mathbb{E}\langle x, AR_n\xi\rangle\langle\xi,By\rangle.$$

Since $\|R_n\| \le 1$ and R_n commutes with C, $\|AR_nz\| \le K\|z\|_C$. Using the dominated convergence theorem we obtain

$$\langle x, A\varrho By\rangle = \lim_{n\to\infty}\mathbb{E}\langle x, AR_n\xi\rangle\langle\xi,By\rangle = \mathbb{E}\langle x, A\xi\rangle\langle B^*\xi, y\rangle. \tag{7.9}$$

By B is densely defined, B^* is closed. Remark 2.2 now shows that $B^* \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, and so applying Lemma 7.4 gives $A\xi, B^*\xi \in L^2(\mathbb{P},\mathfrak{h})$.

Combining (7.9) with Lemma 7.3 we get $\langle x, A\varrho By\rangle = \langle x, \mathbb{E}\left|A\xi\right\rangle\left\langle B^*\xi\right| y\rangle$. Since the closure of $A\varrho B$ is equal to $\mathbb{E}\left|A\xi\right\rangle\left\langle B^*\xi\right|$, we complete the proof of Statement (b) by using Lemma 7.3. □

7.3. *Proof of Theorem 3.1.* First, we easily construct a random variable that represents a given C-regular operator.

Lemma 7.5. *Let $\varrho \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$, with C self-adjoint positive operator in $\mathfrak{h}$. Then there exists $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$ such that $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$ and $\|\xi\|^2 = tr(\varrho)$ a.s.*

Proof. In case $\varrho = 0$, we take $\xi = 0$. Otherwise, consider that ϱ is written as in Definition 3.1. Then, we choose $\Omega = \mathfrak{I}$, and for any $n \in \mathfrak{I}$ we define $\mathbb{P}(\{n\}) = \lambda_n / tr(\varrho)$ and $\xi(n) = \sqrt{tr(\varrho)} u_n$. □

Second, we use part (a) of Theorem 3.2, together with Lemma 7.3, to establish the sufficient condition of Theorem 3.1.

Lemma 7.6. *Let C be a self-adjoint positive operator in $\mathfrak{h}$. Suppose that $\varrho = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$, with $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$. Then ϱ is C-regular.*

Proof. Lemma 7.3 shows that $\varrho \in \mathfrak{L}_1^{+}(\mathfrak{h})$, hence $\varrho = \sum_{n\in\mathfrak{I}} \lambda_n \left|u_n\right\rangle\left\langle u_n\right|$, where $\mathfrak{I}$ is a countable set, $(\lambda_n)_{n\in\mathfrak{I}}$ are summable positive real numbers and $(u_n)_{n\in\mathfrak{I}}$ is a orthonormal family of vectors of $\mathfrak{h}$. Using Statement (a) of Theorem 3.2 yields $u_n \in \mathcal{D}(C)$ for all $n \in \mathfrak{I}$.

We can extend $(u_n)_{n\in\mathfrak{I}}$ to an orthonormal basis $(e_n)_{n\in\mathfrak{I}'}$ of $\mathfrak{h}$ formed by elements of $\mathcal{D}(C)$. From Parseval's equality we obtain

$$\sum_{n\in\mathfrak{I}} \lambda_n \|Cu_n\|^2 = \sum_{n\in\mathfrak{I}} \sum_{k\in\mathfrak{I}'} \lambda_n \left|\langle Cu_n, e_k\rangle\right|^2 = \sum_{k\in\mathfrak{I}'} \sum_{n\in\mathfrak{I}} \lambda_n \left\langle Ce_k, \left|u_n\right\rangle\left\langle u_n\right| Ce_k\right\rangle,$$

and so $\sum_{n\in\mathfrak{I}} \lambda_n \|Cu_n\|^2 = \sum_{k\in\mathfrak{I}'} \langle Ce_k, \varrho Ce_k\rangle$. Combining Lemma 7.3 with Parseval's equality we now get

$$\sum_{n\in\mathfrak{I}} \lambda_n \|Cu_n\|^2 = \sum_{k\in\mathfrak{I}'} \mathbb{E}\left|\langle\xi, Ce_k\rangle\right|^2 = \mathbb{E}\sum_{k\in\mathfrak{I}'} \left|\langle C\xi, e_n\rangle\right|^2 = \mathbb{E}\|C\xi\|^2.$$

This gives $\varrho \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$. □

7.4. *Proof of Theorem 4.1.* We first establish, in our framework, the well know relation between Heisenberg and Schrödinger pictures.

Lemma 7.7. *Suppose that Hypothesis 2 holds, together with $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$. Let $\mathcal{T}_t(A)$ be as in Theorem 2.1. Then for all $A \in \mathfrak{L}(\mathfrak{h})$,*

$$tr\left(A\mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|\right) = tr\left(\mathcal{T}_t(A)\,\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|\right). \tag{7.10}$$

Proof. Fix $A \in \mathfrak{L}(\mathfrak{h})$, and define the function $f_n : \mathfrak{h} \to \mathbb{C}$ by $f_n(x) = \langle x, Ax \rangle$ if $\|x\| \leq n$, and $f_n(x) = 0$ otherwise. Using the Markov property of $X_t(\xi)$, which can be obtained by techniques of well-posed martingale problems, we get

$$\mathbb{E}(f_n(X_t(\xi))) = \mathbb{E}((f_n(X_t(\xi)))/\mathfrak{F}_0) = \mathbb{E}P_t f_n(\xi), \tag{7.11}$$

where $P_t f_n(x) = \mathbb{E}(f_n(X_t(x)))$ for all $x \in \mathcal{D}(C)$.

We will take the limit as $n \to \infty$ in (7.11). The dominated convergence theorem leads to

$$\lim_{n\to\infty} \mathbb{E}(f_n(X_t(\xi))) = \mathbb{E}\langle X_t(\xi), AX_t(\xi)\rangle. \tag{7.12}$$

Combining (7.12) with (2.1) yields $P_t f_n(x) \longrightarrow_{n\to\infty} \langle x, \mathcal{T}_t(A)x\rangle$ whenever $x \in \mathcal{D}(C)$. Since $\|P_t f_n(x)\| \leq \|A\| \|x\|^2$, according to the dominated convergence theorem we have $\mathbb{E}P_t f_n(\xi) \longrightarrow \mathbb{E}\langle \xi, \mathcal{T}_t(A)\xi\rangle$ as $n \to \infty$. Then, letting $n \to \infty$ in (7.11) we get $\mathbb{E}\langle X_t(\xi), AX_t(\xi)\rangle = \mathbb{E}\langle \xi, \mathcal{T}_t(A)\xi\rangle$ by (7.12), and so Theorem 3.2 leads to (7.10). □

We next check that $\rho_t(\varrho)$ is well defined by (4.1).

Lemma 7.8. *Let Hypothesis 2 hold and consider $\xi, \varphi \in L^2_C(\mathbb{P}, \mathfrak{h})$ such that $\mathbb{E}|\xi\rangle\langle\xi| = \mathbb{E}|\varphi\rangle\langle\varphi|$. Then $\mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)| = \mathbb{E}|X_t(\varphi)\rangle\langle X_t(\varphi)|$.*

Proof. Let $A \in \mathfrak{L}(\mathfrak{h})$. Using Lemma 7.7 yields

$$tr(A\mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)|) = tr(\mathcal{T}_t(A)\mathbb{E}|\xi\rangle\langle\xi|) = tr(A\mathbb{E}|X_t(\varphi)\rangle\langle X_t(\varphi)|).$$

Hence $\|\mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)| - \mathbb{E}|X_t(\varphi)\rangle\langle X_t(\varphi)|\|_{\mathfrak{L}_1(\mathfrak{h})} = 0$ (see, e.g., Proposition 9.12 of [32]). □

We now address the contraction property of the restriction of ρ_t to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$.

Lemma 7.9. *Let Hypothesis 2 hold. If $\varrho, \widetilde{\varrho}$ are C-regular, then*

$$tr|\rho_t(\varrho) - \rho_t(\widetilde{\varrho})| \leq tr|\varrho - \widetilde{\varrho}|. \tag{7.13}$$

Proof. Since $tr|\rho_t(\varrho) - \rho_t(\widetilde{\varrho})| = \sup_{\|A\|_{\mathfrak{L}(\mathfrak{h})}=1} |tr(A\rho_t(\varrho)) - tr(A\rho_t(\widetilde{\varrho}))|$, according to Lemma 7.7 we have

$$tr|\rho_t(\varrho) - \rho_t(\widetilde{\varrho})| = \sup_{A\in\mathfrak{L}(\mathfrak{h}), \|A\|=1} |tr(\mathcal{T}_t(A)\varrho) - tr(\mathcal{T}_t(A)\widetilde{\varrho})|.$$

Therefore $tr|\rho_t(\varrho) - \rho_t(\widetilde{\varrho})| \leq tr|\varrho - \widetilde{\varrho}| \sup_{\|A\|_{\mathfrak{L}(\mathfrak{h})}=1} \|\mathcal{T}_t(A)\|$, and so Theorem 2.1 leads to (7.13). □

The following lemma help us to extend ρ_t to all $\mathfrak{L}_1(\mathfrak{h})$.

LEMMA 7.10. *Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$. Then $\mathfrak{L}_{1,C}^{+}(\mathfrak{h})$ is dense in $\mathfrak{L}_1^{+}(\mathfrak{h})$ with respect to the trace norm.*

PROOF. Let $\varrho \in \mathfrak{L}_1^{+}(\mathfrak{h})$. Then there exist a sequence of orthonormal vectors $(u_j)_{j\in\mathbb{N}}$ for which $\varrho = \sum_{j\in\mathbb{N}} \lambda_j |u_j\rangle\langle u_j|$, with $\lambda_j \geq 0$ and $\sum_{j\in\mathbb{N}} \lambda_j < \infty$. For any $x, y \in \mathfrak{h}$ we have

$$tr\,||x\rangle\langle x| - |y\rangle\langle y|\,| = \sup_{\|A\|_{\mathfrak{L}(\mathfrak{h})}=1} |\langle x, Ax\rangle - \langle y, Ay\rangle| \leq \|x-y\|^2 + 2\|y\|\,\|x-y\|,$$

and so $\{|x\rangle\langle x| : x \in \mathcal{D}(C)\}$ is a $\|\cdot\|_{\mathfrak{L}_1(\mathfrak{h})}$-dense subset of $\{|x\rangle\langle x| : x \in \mathfrak{h}\}$ since $\mathcal{D}(C)$ is dense in $\mathfrak{h}$. Now, the lemma follows from $tr\left|\varrho - \sum_{j=1}^{n} \lambda_j |u_j\rangle\langle u_j|\right| = \sum_{j=n+1}^{\infty} \lambda_j \longrightarrow_{n\to\infty} 0$. □

PROOF OF THEOREM 4.1. Combining Theorem 3.1 with Lemma 7.8 we obtain that (4.1) defines unambiguously a linear operator $\rho_t(\varrho)$ for any $\varrho \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$ and $t \geq 0$. Lemma 7.10 guarantees the uniqueness of the operator belonging to $\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))$ for which (4.1) holds. We next extend ρ_t to a bounded linear operator in $\mathfrak{L}_1(\mathfrak{h})$ by means of density arguments.

Suppose that $\varrho \in \mathfrak{L}_1^{+}(\mathfrak{h})$. By Lemma 7.10, there exists a sequence $(\varrho_n)_{n\in\mathbb{N}}$ of C-regular operators for which $\lim_{n\to\infty} \|\varrho - \varrho_n\|_{\mathfrak{L}_1(\mathfrak{h})} \to 0$. We define $\rho_t(\varrho)$ to be the limit in $\mathfrak{L}_1(\mathfrak{h})$ of $\rho_t(\varrho_n)$ as $n \to \infty$; according to Lemma 7.9 this limit exists and does not depend on the choice of $(\varrho_n)_{n\in\mathbb{N}}$. Recall that every $A \in \mathfrak{L}(\mathfrak{h})$ has a unique decomposition of the form $A = \Re(A) + i\,\Im(A)$, with $\Re(A)$ and $\Im(A)$ selfadjoint operators in $\mathfrak{h}$. For each $\varrho \in \mathfrak{L}_1(\mathfrak{h})$ we set

$$\rho_t(\varrho) = \rho_t\left(\Re(\varrho)_+\right) - \rho_t\left(\Re(\varrho)_-\right) + i\left(\rho_t\left(\Im(\varrho)_+\right) - \rho_t\left(\Im(\varrho)_-\right)\right),$$

where A_+, A_- denotes respectively the positive and negative parts of the selfadjoint operator A (see, e.g., [7] for details).

We will verify that $\rho_t \in \mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))$. Let $\varrho = \varrho_1 - \varrho_2 + i(\varrho_3 - \varrho_4)$, with $\varrho_j \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$ for any $j = 1, \ldots, 4$. Since $\|\mathcal{T}_t(A)\| \leq \|A\|$, Lemma 7.7 yields

$$tr\,|\rho_t(\varrho)| = \sup_{\|A\|_{\mathfrak{L}(\mathfrak{h})}=1} |tr\,(A\rho_t(\varrho))| = \sup_{\|A\|_{\mathfrak{L}(\mathfrak{h})}=1} |tr\,(\mathcal{T}_t(A)\,\varrho)| \leq tr\,(|\varrho|).$$

The construction of $\rho_t(\varrho)$ now implies $\|\rho_t(\varrho)\|_{\mathfrak{L}_1(\mathfrak{h})} \leq \|\varrho\|_{\mathfrak{L}_1(\mathfrak{h})}$ for all $\varrho \in \mathfrak{L}_1(\mathfrak{h})$. Consider two C-regular operators $\varrho, \widetilde{\varrho}$ and $\alpha \geq 0$. By Definition 3.1, $\varrho + \alpha\widetilde{\varrho}$ belongs to $\mathfrak{L}_{1,C}^{+}(\mathfrak{h})$. If $A \in \mathfrak{L}(\mathfrak{h})$, then applying Lemma 7.7 we obtain

$$tr\,(\rho_t(\varrho + \alpha\widetilde{\varrho})\,A) = tr\,(\mathcal{T}_t(A)\,\varrho) + \alpha\; tr\,(\mathcal{T}_t(A)\,\widetilde{\varrho}) = tr\,((\rho_t(\varrho) + \alpha\rho_t(\widetilde{\varrho}))\,A).$$

Therefore $\|\rho_t(\varrho+\alpha\widetilde{\varrho})-\rho_t(\varrho)-\alpha\rho_t(\widetilde{\varrho})\|_{\mathfrak{L}_1(\mathfrak{h})}=0$, and so Lemma 7.10 leads to $\rho_t(\varrho+\alpha\widetilde{\varrho})=\rho_t(\varrho)+\alpha\rho_t(\widetilde{\varrho})$ for any $\varrho,\widetilde{\varrho}\in\mathfrak{L}_1^+(\mathfrak{h})$. Careful algebraic manipulations now show the linearity of $\rho_t:\mathfrak{L}_1(\mathfrak{h})\to\mathfrak{L}_1(\mathfrak{h})$. □

7.5. *Proof of Theorem 4.2.* Let us first prove the continuity of the map $\xi\mapsto\rho_t(\mathbb{E}|\xi\rangle\langle\xi|)$.

Lemma 7.11. *Assume that Hypothesis 2 holds. Let ξ and ξ_n, with $n\in\mathbb{N}$, be random variables in $L^2_C(\mathbb{P},\mathfrak{h})$ satisfying $\mathbb{E}\|\xi-\xi_n\|^2\longrightarrow_{n\to\infty}0$. Then $\rho_t(\mathbb{E}|\xi_n\rangle\langle\xi_n|)$ converges in $\mathfrak{L}(\mathfrak{h})$ to $\rho_t(\mathbb{E}|\xi\rangle\langle\xi|)$ as $n\to\infty$.*

Proof. Let $x\in\mathfrak{h}$. Combining (4.1) with the linearity of (1.4) we get

$$\begin{aligned}&\|\rho_t(\mathbb{E}|\xi_n\rangle\langle\xi_n|)x-\rho_t(\mathbb{E}|\xi\rangle\langle\xi|)x\|\\ \le\ &\mathbb{E}|\langle X_t(\xi_n),x\rangle|\,\|X_t(\xi_n-\xi)\|+\mathbb{E}|\langle X_t(\xi-\xi_n),x\rangle|\,\|X_t(\xi)\|\\ \le\ &\|x\|\left(\mathbb{E}\|\xi-\xi_n\|^2+2\sqrt{\mathbb{E}\|\xi-\xi_n\|^2}\sqrt{\mathbb{E}\|\xi\|^2}\right).\end{aligned}$$

In the last inequality we use that $\mathbb{E}\|X_t(\eta)\|^2\le\mathbb{E}\|\eta\|^2$ for $\eta\in L^2_C(\mathbb{P},\mathfrak{h})$. □

Proof of Theorem 4.2. There exits a sequence $(\xi_n)_n$ of $(\mathcal{D}(C),\|\cdot\|)$-valued random variables with finite ranges such that $\|\xi_n-\xi\|$ converges monotonically to 0 (see, e.g., [35]). By Lemma 7.11, $\rho_t(\mathbb{E}|\xi_n\rangle\langle\xi_n|)$ converges to $\rho_t(\mathbb{E}|\xi\rangle\langle\xi|)$ in $\mathfrak{L}(\mathfrak{h})$. Since ρ_t is linear, an easy computation shows that $\mathbb{E}\rho_t(|\xi_n\rangle\langle\xi_n|)=\rho_t(\mathbb{E}|\xi_n\rangle\langle\xi_n|)$, hence

$$\mathbb{E}\rho_t(|\xi_n\rangle\langle\xi_n|)\longrightarrow_{n\to\infty}\rho_t(\mathbb{E}|\xi\rangle\langle\xi|)\qquad\text{in }\mathfrak{L}(\mathfrak{h}).\tag{7.14}$$

We will prove that $\mathbb{E}\rho_t(|\xi_n\rangle\langle\xi_n|)$ converges to $\mathbb{E}\rho_t(|\xi\rangle\langle\xi|)$ in $\mathfrak{L}(\mathfrak{h})$ as $n\to\infty$, which together with (7.14) implies $\rho_t(\mathbb{E}|\xi\rangle\langle\xi|)=\mathbb{E}\rho_t(|\xi\rangle\langle\xi|)$. From Lemma 7.11 we obtain $\|\rho_t(|\xi_n\rangle\langle\xi_n|)-\rho_t(|\xi\rangle\langle\xi|)\|_{\mathfrak{L}(\mathfrak{h})}\longrightarrow_{n\to\infty}0$. For any $x,y\in\mathfrak{h}$ we have $\||x\rangle\langle y|\|_1=\|x\|\,\|y\|$, and so Lemma 7.9 yields

$$\|\rho_t(|\xi_n\rangle\langle\xi_n|)\|\le\|\rho_t(|\xi_n\rangle\langle\xi_n|)\|_1\le\|\xi_n\|^2\le2\left(\|\xi_1-\xi\|^2+\|\xi\|^2\right).$$

Therefore $\mathbb{E}\|\rho_t(|\xi_n\rangle\langle\xi_n|)-\rho_t(|\xi\rangle\langle\xi|)\|_{\mathfrak{L}(\mathfrak{h})}\longrightarrow_{n\to\infty}0$. □

7.6. *Proof of Theorem 4.3.* Our proof is divided into three lemmata. The first two deal with the semigroup property of $(\rho_t)_{t\ge0}$.

Lemma 7.12. *Let Hypothesis 2 hold and let ϱ be C-regular. Then for all $t\ge0$, $\rho_t(\varrho)$ belongs to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$ and $\rho_{t+s}(\varrho)=\rho_t\circ\rho_s(\varrho)$ whenever $s\ge0$.*

Proof. Since $X_t(\xi) \in L^2_C(\mathbb{P},\mathfrak{h})$, combining Theorem 3.1 with (4.1) gives $\rho_t\left(\mathfrak{L}^+_{1,C}(\mathfrak{h})\right) \subset \mathfrak{L}^+_{1,C}(\mathfrak{h})$.

We will establish the semigroup property of the restriction of ρ to $\mathfrak{L}^+_{1,C}(\mathfrak{h})$. Consider $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$ satisfying $\rho = \mathbb{E}\,|\xi\rangle\langle\xi|$, and fix $x, y \in \mathfrak{h}$. For all $z \in \mathfrak{h}$ we define $p_n(z) = \langle z, x\rangle\langle y, z\rangle$ if $|\langle z, x\rangle\langle y, z\rangle| \leq n$, and $p_n(z) = 0$ otherwise. Using the Markov property of $X_t(\xi)$ we deduce that

$$\mathbb{E}\left(p_n\left(X_{t+s}(\xi)\right)\right) = \mathbb{E}\left(\left(p_n\left(X_{t+s}(\xi)\right)\right)\diagup\mathfrak{F}_s\right) = \mathbb{E}P_t(p_n)\left(X_s(\xi)\right), \tag{7.15}$$

where for all $z \in \mathcal{D}(C)$, $P_t(p_n)(z) = \mathbb{E}\left(p_n\left(X_t(z)\right)\right)$.

Let $z \in \mathcal{D}(C)$. Applying the dominated convergence theorem gives

$$\lim_{n\to\infty}\mathbb{E}\left(p_n\left(X_t(z)\right)\right) = \mathbb{E}\langle X_t(z), x\rangle\langle y, X_t(z)\rangle = \langle y, \rho_t(|z\rangle\langle z|)\,x\rangle,$$

hence $\lim_{n\to\infty} P_t(p_n)(z) = \langle y, \rho_t(|z\rangle\langle z|)\,x\rangle$. Then $\mathbb{E}P_t(p_n)(X_s(\xi)) \longrightarrow_{n\to\infty} \mathbb{E}\langle y, \rho_t(|X_s(\xi)\rangle\langle X_s(\xi)|)\,x\rangle$, and so Theorem 4.2 leads to

$$\lim_{n\to\infty}\mathbb{E}P_t(p_n)\left(X_s(\xi)\right) = \langle y, \rho_t(\mathbb{E}\,|X_s(\xi)\rangle\langle X_s(\xi)|)\,x\rangle = \langle y, \rho_t\circ\rho_s(\varrho)\,x\rangle. \tag{7.16}$$

By (7.15), in (7.16) we replace s by 0 and t by $t+s$ to obtain

$$\lim_{n\to\infty}\mathbb{E}\left(p_n\left(X_{t+s}(\xi)\right)\right) = \lim_{n\to\infty}\mathbb{E}P_{t+s}(p_n)\left(X_0(\xi)\right) = \langle y, \rho_{t+s}(\varrho)\,x\rangle.$$

Thus, letting $n \to \infty$ in (7.15) we get $\rho_{t+s}(\varrho) = \rho_t\circ\rho_s(\varrho)$ by (7.16). □

Lemma 7.13. *Under Hypothesis 2, $(\rho_t)_{t\geq 0}$ is a semigroup of contractions which leaves $\mathfrak{L}^+_1(\mathfrak{h})$ invariant.*

Proof. By Theorem 4.1, $\|\rho_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} \leq 1$. Since $\rho_t(\varrho)$ is positive whenever ϱ is C-regular, using Lemma 7.10 yields $\langle x, \rho_t(\varrho)\,x\rangle \geq 0$ for any $\varrho \in \mathfrak{L}^+_1(\mathfrak{h})$ and $x \in \mathfrak{h}$.

Suppose that $\varrho = \varrho_1 - \varrho_2 + i(\varrho_3 - \varrho_4)$, where $\varrho_1, \ldots, \varrho_4$ are C-regular operators. Applying (4.1) gives $\rho_0(\varrho) = \varrho$, and Lemma 7.12 asserts that $\rho_{t+s}(\varrho) = \rho_t\circ\rho_s(\varrho)$ for any $s, t \geq 0$. Then, combining Lemma 7.10 with density arguments we deduce that $(\rho_t)_{t\geq 0}$ is a semigroup. □

We now examine the continuity of the map $t \mapsto \rho_t(\varrho)$ when ϱ is C-regular.

Lemma 7.14. *Adopt Hypothesis 2, together with $\varrho \in \mathfrak{L}^+_{1,C}(\mathfrak{h})$. Then the map $t \mapsto \rho_t(\varrho)$ from $[0,\infty[$ to $\mathfrak{L}_1(\mathfrak{h})$ is continuous.*

Proof. Consider $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$ such that $\varrho = \mathbb{E}\,|\xi\rangle\langle\xi|$. Theorem 3.2 yields $\mathbb{E}\,\|X_t(\xi)\|^2 \le \mathbb{E}\,\|\xi\|^2 = tr(\varrho)$ for all $t \ge 0$, and so combining Theorem 3.2 with the Cauchy-Schwarz inequality yields

$$\begin{aligned} tr\,|\rho_t(\varrho) - \rho_s(\varrho)| &= \sup_{A\in\mathfrak{L}(\mathfrak{h}),\|A\|=1} |\mathbb{E}\,\langle X_t(\xi), AX_t(\xi)\rangle - \langle X_s(\xi), AX_s(\xi)\rangle| \\ &\le 2\,(tr(\varrho))^{1/2}\left(\mathbb{E}\,\|X_t(\xi) - X_s(\xi)\|^2\right)^{1/2}. \end{aligned}$$

Since $\mathbb{E}\left(\sup_{s\in[0,T]}\|X_s(\xi)\|^2\right) < \infty$ for any $T > 0$ (see, e.g., Th. 4.2.5 of [36]), using the dominated convergence theorem we get (4.2). □

7.7. *Proof of Theorem 4.4.* First, we establish the weak continuity of the map $t \mapsto AX_t(\xi)$ when A is relatively bounded by C.

Lemma 7.15. *Assume that Hypothesis 2 holds. If ξ belongs to $L^2_C(\mathbb{P},\mathfrak{h})$ and if $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, then for all $\psi \in L^2(\mathbb{P},\mathfrak{h})$ and $t \ge 0$ we have*

$$\lim_{s\to t} \mathbb{E}\,\langle \psi, AX_s(\xi)\rangle = \mathbb{E}\,\langle \psi, AX_t(\xi)\rangle. \tag{7.17}$$

Proof. Let $(s_n)_n$ be a sequence of non-negative real numbers converging to t. Since $((X_{s_n}(\xi), AX_{s_n}(\xi), CX_{s_n}(\xi)))_n$ is a bounded sequence in $L^2(\mathbb{P},\mathfrak{h}^3)$ with $\mathfrak{h}^3 = \mathfrak{h}\times\mathfrak{h}\times\mathfrak{h}$, there exists a subsequence $\left(s_{n(k)}\right)_k$ for which

$$\left(X_{s_{n(k)}}(\xi), AX_{s_{n(k)}}(\xi), CX_{s_{n(k)}}(\xi)\right) \longrightarrow_{k\to\infty} (Y,U,V) \tag{7.18}$$

weakly in $L^2(\mathbb{P},\mathfrak{h}^3)$.

Set $\mathfrak{M} = \{(\eta, A\eta, C\eta) : \eta \in L^2_C(\mathbb{P},\mathfrak{h})\}$. Then $\mathfrak{M}$ is a linear manifold of $L^2(\mathbb{P},\mathfrak{h}^3)$ closed with respect to the strong topology. In fact, suppose that $((\eta_n, A\eta_n, C\eta_n))_n$ is a sequence of elements of $\mathfrak{M}$ that converges to (η_1,η_2,η_3) in $L^2(\mathbb{P},\mathfrak{h}^3)$. Hence there exists a subsequence $\left(\left(\eta_{n(j)}, A\eta_{n(j)}, C\eta_{n(j)}\right)\right)_j$ converging almost surely to (η_1,η_2,η_3). Therefore $\eta_1 \in \mathcal{D}(C)$ and $\eta_3 = C\eta_1$ by C is closed. Using $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ gives $\eta_2 = A\eta_1$.

For any $k \in \mathbb{N}$, $\left(X_{s_{n(k)}}(\xi), AX_{s_{n(k)}}(\xi), CX_{s_{n(k)}}(\xi)\right)$ belongs to $\mathfrak{M}$. Since $\mathfrak{M}$ is a closed linear manifold of $L^2(\mathbb{P},\mathfrak{h}^3)$, (7.18) implies $(Y,U,V) \in \mathfrak{M}$ (see, e.g., Subsection III.1.6 of [26]). Combining the dominated convergence theorem with $\mathbb{E}\left(\sup_{s\in[0,t+1]}\|X_s(\xi)\|^2\right) < \infty$ we get that $\mathbb{E}\left\|X_{s_{n(k)}}(\xi) - X_t(\xi)\right\|^2$ converges to 0. Thus $Y = X_t(\xi)$, and so $U = AX_t(\xi)$. Hence $AX_{s_{n(k)}}(\xi)$ converges to $AX_t(\xi)$ weakly in $L^2(\mathbb{P},\mathfrak{h})$. □

Second, we show that the probabilistic representation of the right-hand side of (4.4) is continuous as a function from $[0,+\infty[$ to $\mathbb{C}$.

LEMMA 7.16. *Let Hypothesis 2 hold. Fix $\xi \in L^2_C(\mathbb{P},\mathfrak{h})$ and $A \in \mathfrak{L}(\mathfrak{h})$. Then, the function that maps each t in $[0,+\infty[$ to the complex number $\mathbb{E}\langle GX_t(\xi), AX_t(\xi)\rangle + \mathbb{E}\langle X_t(\xi), AGX_t(\xi)\rangle + \sum_{k=1}^{\infty}\mathbb{E}\langle L_kX_t(\xi), AL_kX_t(\xi)\rangle$ is continuous.*

PROOF. Let $(t_n)_n$ be a sequence of non-negative real numbers such that t_n converges to t. Since $\mathbb{E}\left(\sup_{s\in[0,t+1]}\|X_s(\xi)\|^2\right) < \infty$ (see, e.g., Th. 4.2.5 of [36]), $AX_{t_n}(\xi) \longrightarrow_{n\to\infty} AX_t(\xi)$ in $L^2(\mathbb{P},\mathfrak{h})$. Hence Lemma 7.15 yields

$$\lim_{n\to\infty}\mathbb{E}\langle GX_{t_n}(\xi), AX_{t_n}(\xi)\rangle = \mathbb{E}\langle GX_t(\xi), AX_t(\xi)\rangle \tag{7.19}$$

(see, e.g., Subsection III.1.7 of [26]). By (7.19) with A replaced by A^*, $t \mapsto \mathbb{E}\langle A^*X_t(\xi), GX_t(\xi)\rangle$ is continuous, then so is $t \mapsto \mathbb{E}\langle X_t(\xi), AGX_t(\xi)\rangle$.

We now focus on $\sum_{k=1}^{\infty}\mathbb{E}\langle L_kX_t(\xi), AL_kX_t(\xi)\rangle$. Taking $A = I$ in (7.19) we get $\mathbb{E}\Re\langle X_{t_n}(\xi), GX_{t_n}(\xi)\rangle \to_{n\to\infty} \mathbb{E}\Re\langle X_t(\xi), GX_t(\xi)\rangle$. Thus, Condition H2.2 leads to

$$\sum_{k=1}^{\infty}\mathbb{E}\|L_kX_{t_n}(\xi)\|^2 \longrightarrow_{n\to\infty} \sum_{k=1}^{\infty}\mathbb{E}\|L_kX_t(\xi)\|^2. \tag{7.20}$$

Using (7.20) we will deduce that $L_kX_{t_n}(\xi)$ converges strongly in $L^2(\mathbb{P},\mathfrak{h})$ to $L_kX_t(\xi)$ as $n\to\infty$. Conversely, suppose that for a given $j\in\mathbb{N}$,

$$\limsup_{n\to\infty}\mathbb{E}\|L_jX_{t_n}(\xi)\|^2 > \mathbb{E}\|L_jX_t(\xi)\|^2. \tag{7.21}$$

Since $\mathbb{E}\|L_kX_t(\xi)\|^2 \le \liminf_{n\to\infty}\mathbb{E}\|L_kX_{t_n}(\xi)\|^2$, Fatou's lemma shows

$$\sum_{k\neq j}\mathbb{E}\|L_kX_t(\xi)\|^2 \le \liminf_{n\to\infty}\sum_{k\neq j}\mathbb{E}\|L_kX_{t_n}(\xi)\|^2. \tag{7.22}$$

According to (7.20) and (7.21) we have

$$\begin{aligned}\liminf_{n\to\infty}\sum_{k\neq j}\mathbb{E}\|L_kX_{t_n}(\xi)\|^2 &= \sum_{k=1}^{\infty}\mathbb{E}\|L_kX_t(\xi)\|^2 - \limsup_{n\to\infty}\mathbb{E}\|L_jX_{t_n}(\xi)\|^2\\ &< \sum_{k\neq j}\mathbb{E}\|L_kX_t(\xi)\|^2,\end{aligned}$$

contrary to (7.22), and so

$$\limsup_{n\to\infty}\mathbb{E}\|L_jX_{t_n}(\xi)\|^2 \le \mathbb{E}\|L_jX_t(\xi)\|^2. \tag{7.23}$$

Applying Lemma 7.15 we get that $L_j X_{t_n}(\xi)$ converges weakly in $L^2(\mathbb{P},\mathfrak{h})$ to $L_j X_t(\xi)$ as $n\to\infty$, and so (7.23) leads to $L_k X_{t_n}(\xi) \longrightarrow_{n\to\infty} L_k X_t(\xi)$ in $L^2(\mathbb{P},\mathfrak{h})$.

From Condition H2.2 it follows that $\sum_{k=1}^{n}\mathbb{E}\langle L_k X_t(\xi), AL_k X_t(\xi)\rangle$ converges to $\sum_{k=1}^{\infty}\mathbb{E}\langle L_k X_t(\xi), AL_k X_t(\xi)\rangle$ as $n\to\infty$ uniformly on any finite interval. Since $\mathbb{E}\langle L_k X_{t_n}(\xi), AL_k X_{t_n}(\xi)\rangle \longrightarrow_{n\to\infty} \mathbb{E}\langle L_k X_t(\xi), AL_k X_t(\xi)\rangle$, the map $t\mapsto \sum_{k=1}^{\infty}\mathbb{E}\langle L_k X_t(\xi), AL_k X_t(\xi)\rangle$ is continuous. □

Third, we deals with basic properties of the probabilistic representation of the right-hand side of (4.3).

Lemma 7.17. *Let Hypothesis 2 hold. For any $\xi\in L^2_C(\mathbb{P},\mathfrak{h})$, we define*

$$\mathcal{L}_*(\xi,t)=\mathbb{E}|GX_t(\xi)\rangle\langle X_t(\xi)|+\mathbb{E}|X_t(\xi)\rangle\langle GX_t(\xi)|+\sum_{k=1}^{\infty}\mathbb{E}|L_kX_t(\xi)\rangle\langle L_kX_t(\xi)|.$$

Then $\mathcal{L}_*(\xi,t)$ is a trace-class operator on $\mathfrak{h}$ whose trace-norm is uniformly bounded with respect to t on bounded time intervals; the series involved in the definition of $\mathcal{L}_*$ converges in $\mathfrak{L}_1(\mathfrak{h})$.

Proof. By Condition H2.2, using (7.7) and Lemma 7.3 we get

$$\|\mathbb{E}|GX_t(\xi)\rangle\langle X_t(\xi)|\|_1+\|\mathbb{E}|X_t(\xi)\rangle\langle GX_t(\xi)|\|_1+\sum_{k=1}^{\infty}\|\mathbb{E}|L_kX_t(\xi)\rangle\langle L_kX_t(\xi)|\|_1$$
$$\le 4\mathbb{E}\left(\|X_t(\xi)\|\,\|GX_t(\xi)\|\right)\le K\sqrt{\mathbb{E}\|\xi\|^2}\sqrt{\mathbb{E}\|X_t(\xi)\|_C^2},$$

where the last inequality follows from $G\in\mathfrak{L}((\mathcal{D}(C),\|\cdot\|_C),\mathfrak{h})$. □

Applying Lemmata 7.3 and 7.16 we easily obtain Lemma 7.18.

Lemma 7.18. *Adopt the assumptions and notation of Lemma 7.17, together with $A\in\mathfrak{L}(\mathfrak{h})$. Then, the trace of $A\mathcal{L}_*(\xi,t)$ is equal to*

$$\mathbb{E}\langle X_t(\xi), AGX_t(\xi)\rangle+\mathbb{E}\langle GX_t(\xi), AX_t(\xi)\rangle+\sum_{k=1}^{\infty}\mathbb{E}\langle L_kX_t(\xi), AL_kX_t(\xi)\rangle,$$

and $t\mapsto tr(A\mathcal{L}_*(\xi,t))$ is continuous as a function from $[0,\infty[$ to $\mathbb{C}$.

We proceed to prove that $\mathbb{E}|X_t(\xi)\rangle\langle X_t(\xi)|$ satisfies an integral version of (1.1). To this end, we combine the regularity of $X(\xi)$ with Itô's formula.

LEMMA 7.19. *Adopt Hypothesis 2 together with $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$. Then*

$$\rho_t(\mathbb{E}|\xi\rangle\langle\xi|) = \mathbb{E}|\xi\rangle\langle\xi| + \int_0^t \mathcal{L}_*(\xi, s)\, ds, \tag{7.24}$$

where $t \geq 0$ and $\mathcal{L}_*(\xi, s)$ is as in Lemma 7.17; we understand the above integral in the sense of Bochner integral in $\mathfrak{L}_1(\mathfrak{h})$.

PROOF. Our proof is based on arguments given in Subsection 1.5. Fix $x \in \mathfrak{h}$, and choose $\tau_n = \inf\{s \geq 0 : \|X_s(\xi)\| > n\}$, with $n \in \mathbb{N}$. Applying the complex Itô formula we obtain that

$$\langle X_{t\wedge\tau_n}(\xi), x\rangle X_{t\wedge\tau_n}(\xi) = \langle \xi, x\rangle \xi + \mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi))\, ds + M_t, \tag{7.25}$$

where $M_t = \sum_{k=1}^{\infty}\int_0^{t\wedge\tau_n}(\langle X_s(\xi), x\rangle L_k X_s(\xi) + \langle L_k X_s(\xi), x\rangle X_s(\xi))\, dW_s^k$, and $L_x(z) = \langle z, x\rangle Gz + \langle Gz, x\rangle z + \sum_{k=1}^{\infty}\langle L_k z, x\rangle L_k z$ for any $z \in \mathcal{D}(C)$. According to Condition H2.2 we have

$$\begin{aligned}
&\mathbb{E}\sum_{k=1}^{\infty}\int_0^{t\wedge\tau_n}\|\langle X_s(\xi), x\rangle L_k X_s(\xi) + \langle L_k X_s(\xi), x\rangle X_s(\xi)\|^2\, ds \\
&\leq 4n^3\|x\|^2\,\mathbb{E}\int_0^{t\wedge\tau_n}\|GX_s\|\, ds.
\end{aligned}$$

Therefore $\mathbb{E}M_t = 0$ by G belongs to $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, and so (7.25) yields

$$\mathbb{E}\langle X_{t\wedge\tau_n}(\xi), x\rangle X_{t\wedge\tau_n}(\xi) = \mathbb{E}\langle \xi, x\rangle \xi + \mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi))\, ds. \tag{7.26}$$

We will take the limit as $n \to \infty$ in (7.26). Since $X(\xi)$ has continuous sample paths, $\tau_n \nearrow_{n\to\infty} \infty$. By H2.1 and H2.2, applying the dominated convergence yields $\lim_{n\to\infty}\mathbb{E}\int_0^{t\wedge\tau_n} L_x(X_s(\xi))\, ds = \mathbb{E}\int_0^t L_x(X_s(\xi))\, ds$. Combinig $\mathbb{E}\left(\sup_{s\in[0,t+1]}\|X_s(\xi)\|^2\right) < \infty$ with the dominated convergence theorem gives $\lim_{n\to\infty}\mathbb{E}\langle X_{t\wedge\tau_n}(\xi), x\rangle X_{t\wedge\tau_n}(\xi) = \mathbb{E}\langle X_t(\xi), x\rangle X_t(\xi)$. Then, letting first $n \to \infty$ in (7.26) and then using Fubini's theorem we get

$$\mathbb{E}\langle X_t(\xi), x\rangle X_t(\xi) = \mathbb{E}\langle \xi, x\rangle \xi + \int_0^t \mathbb{E}L_x(X_s(\xi)). \tag{7.27}$$

By Condition H2.2, the dominated convergence theorem leads to

$$\mathbb{E}\sum_{k=1}^{\infty}\langle L_k X_s(\xi), x\rangle L_k X_s(\xi) = \sum_{k=1}^{\infty}\mathbb{E}\langle L_k X_s(\xi), x\rangle L_k X_s(\xi),$$

and so Lemmata 7.3 yields $\mathbb{E}L_x\left(X_s\left(\xi\right)\right)=\mathcal{L}_*\left(\xi,s\right)x$, hence

$$\int_0^t \mathbb{E}L_x\left(X_s\left(\xi\right)\right)=\int_0^t \mathcal{L}_*\left(\xi,s\right)xds. \tag{7.28}$$

Since the dual of $\mathfrak{L}_1\left(\mathfrak{h}\right)$ consists in all linear maps $\varrho\mapsto tr\left(A\varrho\right)$ with $A\in\mathfrak{L}\left(\mathfrak{h}\right)$, Lemma 7.18 implies that $t\mapsto\mathcal{L}_*\left(\xi,t\right)$ is measurable as a function from $[0,\infty[$ to $\mathfrak{L}_1\left(\mathfrak{h}\right)$. Furthermore, using Lemma 7.17 we get that $t\mapsto\mathcal{L}_*\left(\xi,t\right)$ is a Bochner integrable $\mathfrak{L}_1\left(\mathfrak{h}\right)$-valued function on bounded intervals. Then (7.27), together with (7.28), gives (7.24). □

We are in position to show (4.3) and (4.4) with the help of Hypothesis 3.

Proof of Theorem 4.4. By Theorem 3.1, $\varrho=\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$ for certain $\xi\in L_C^2\left(\mathbb{P},\mathfrak{h}\right)$. Theorem 3.2 now gives $AG\rho_t\left(\varrho\right)=\mathbb{E}\left|AGX_t\left(\xi\right)\right\rangle\left\langle X_t\left(\xi\right)\right|$. Applying Hypothesis 3 we get that $G^*,L_1^*,L_2^*,\ldots$ are densely defined and G^{**}, $L_1^{**},\ldots$ coincide with the closures of $G,L_1,\ldots$ respectively (see, e.g., Theorem III.5.29 of [26]). Theorem 3.2 yields $A\rho_t\left(\varrho\right)G^*=\mathbb{E}\left|AX_t\left(\xi\right)\right\rangle\left\langle GX_t\left(\xi\right)\right|$ and $AL_k\rho_t\left(\varrho\right)L_k^*=\mathbb{E}\left|AL_kX_t\left(\xi\right)\right\rangle\left\langle L_kX_t\left(\xi\right)\right|$. Therefore

$$\mathcal{L}_*\left(\xi,t\right)=G\rho_t\left(\varrho\right)+\rho_t\left(\varrho\right)G^*+\sum_{k=1}^{\infty}L_k\rho_t\left(\varrho\right)L_k^*, \tag{7.29}$$

where $\mathcal{L}_*\left(\xi,t\right)$ is as in Lemma 7.17. Combining (7.29) with Lemma 7.19 we get (4.3), and so $tr\left(A\rho_t\left(\varrho\right)\right)=tr\left(A\varrho\right)+\int_0^t tr\left(A\mathcal{L}_*\left(\xi,s\right)\right)ds$ for all $t\geq 0$. Using the continuity of $\mathcal{L}_*\left(\xi,\cdot\right)$ we obtain (4.4). □

7.8. *Proof of Theorem 4.5.* We first obtain the existence of a solution of (1.1) in the semigroup sense, without Hypothesis 3.

Lemma 7.20. *Under Hypothesis 2, ρ is a semigroup C-solution of (1.1).*

Proof. By Theorem 4.3, $\left(\rho_t\right)_{t\geq 0}$ is a semigroup of bounded operators on $\mathfrak{L}_1\left(\mathfrak{h}\right)$ that satisfies Property (i) of Definition 4.1. Fix $\varrho=\left|x\right\rangle\left\langle x\right|$, with $x\in\mathcal{D}\left(C\right)$. Thus ϱ is a C-regular operator, and so (4.2) leads to Property (ii). Finally, using Lemmata 7.18 and 7.19 we get Property (iii). □

We next make it legitimate to use in our context the duality relation between quantum master equations and adjoint quantum master equations.

Lemma 7.21. *Let Hypothesis 2 hold. Suppose that $A\in\mathfrak{L}\left(\mathfrak{h}\right)$ and that $\left(\widehat{\rho}_t\right)_{t\geq 0}$ is a semigroup C-solution of (1.1). Then $\left(\widehat{\rho}_t^*\left(A\right)\right)_{t\geq 0}$ is a C-solution*

of (1.2) with initial datum A, where $(\widehat{\rho}_t^*)_{t\geq 0}$ is the adjoint semigroup of $(\widehat{\rho}_t)_{t\geq 0}$ (see, e.g., [33]), that is, $(\widehat{\rho}_t^*)_{t\geq 0}$ is the unique semigroup of bounded operators on $\mathfrak{L}(\mathfrak{h})$ such that for all $B \in \mathfrak{L}(\mathfrak{h})$ and $\varrho \in \mathfrak{L}_1(\mathfrak{h})$,

$$tr\left(\widehat{\rho}_t(\varrho) B\right) = tr\left(\widehat{\rho}_t^*(B)\varrho\right). \tag{7.30}$$

Proof. Using (7.30) we get that for all vectors $x, y \in \mathfrak{h}$ whose norm is 1,

$$\begin{aligned} |\langle y, \widehat{\rho}_t^*(A) x\rangle| &= |tr\left(\widehat{\rho}_t^*(A)|x\rangle\langle y|\right)| \\ &= tr\left(|\widehat{\rho}_t(|x\rangle\langle y|) A|\right) \leq \|A\| \|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} tr\left(|\ |x\rangle\langle y|\ |\right). \end{aligned}$$

We conclude from (7.7) that $tr\left(|\ |x\rangle\langle y|\ |\right) = 1$, hence that $|\langle y, \widehat{\rho}_t^*(A) x\rangle| \leq \|A\| \|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))}$, and finally that

$$\|\widehat{\rho}_t^*(A)\|_{\mathfrak{L}(\mathfrak{h})} \leq \|A\| \|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))}. \tag{7.31}$$

Applying Property (i) of Definition 4.1 gives Property (b) of Definition 2.2.

In order to verify Property (a), we will prove the continuity of $t \mapsto \langle x, \widehat{\rho}_t^*(A) y\rangle$ for any $x, y \in \mathfrak{h}$. As in the proof of Lemma 7.2, we define $R_n = n(n+C)^{-1}$ for $n \in \mathbb{N}$. According to (7.30) we have

$$\langle R_n x, \widehat{\rho}_t^*(A) R_n x\rangle = tr\left(\widehat{\rho}_t^*(A)|R_n x\rangle\langle R_n x|\right) = tr\left(\widehat{\rho}_t(|R_n x\rangle\langle R_n x|) A\right).$$

Since $R_n x \in \mathcal{D}(C)$, Property (ii) of Definition 4.1 implies the continuity of the function $t \mapsto \langle R_n x, \widehat{\rho}_t^*(A) R_n x\rangle$. By (7.31),

$$\begin{aligned} |\langle x, \widehat{\rho}_t^*(A) x\rangle - \langle x, \widehat{\rho}_s^*(A) x\rangle| &\leq |\langle R_n x, \widehat{\rho}_t^*(A) R_n x\rangle - \langle R_n x, \widehat{\rho}_s^*(A) R_n x\rangle| \\ &\quad + 2\|A\| \left(\|\widehat{\rho}_t\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))} + \|\widehat{\rho}_s\|_{\mathfrak{L}(\mathfrak{L}_1(\mathfrak{h}))}\right) \|x\| \|x - R_n x\|. \end{aligned}$$

Using $R_n x \longrightarrow_{n\to\infty} x$ we deduce that the map $t \mapsto \langle x, \widehat{\rho}_t^*(A) x\rangle$ is continuous, so is $t \mapsto \langle x, \widehat{\rho}_t^*(A) y\rangle$ by the polarization identity.

Assume that $x \in \mathcal{D}(C)$. By (7.30), combining $\widehat{\rho}_{t+s}^*(A) = \widehat{\rho}_s^*(\widehat{\rho}_t^*(A))$ with Property (iii) of Definition 4.1 yields

$$\begin{aligned} &\lim_{s\to 0+} \frac{1}{s}\left(\langle x, \widehat{\rho}_{t+s}^*(A) x\rangle - \langle x, \widehat{\rho}_t^*(A) x\rangle\right) \\ &= \lim_{s\to 0+} \frac{1}{s}\left(tr\left(\widehat{\rho}_s(|x\rangle\langle x|)\widehat{\rho}_t^*(A)\right) - tr\left(|x\rangle\langle x|\widehat{\rho}_t^*(A)\right)\right) = \mathcal{L}\left(\widehat{\rho}_t^*(A), x\right), \end{aligned}$$

with $\mathcal{L}\left(\widehat{\rho}_t^*(A), x\right) = \langle x, \widehat{\rho}_t^*(A) G x\rangle + \langle G x, \widehat{\rho}_t^*(A) x\rangle + \sum_{k=1}^{\infty} \langle L_k x, \widehat{\rho}_t^*(A) L_k x\rangle$. Thus

$$\frac{d}{dt}^{+} \langle x, \widehat{\rho}_t^*(A) x\rangle = \mathcal{L}\left(\widehat{\rho}_t^*(A), x\right) \tag{7.32}$$

From (7.31) and Condition H2.2 we get that $\sum_{k=1}^{\infty}\langle L_k x, \widehat{\rho}_t^*(A) L_k x\rangle$ is uniformly convergent on bounded intervals, and so $t \mapsto \sum_{k=1}^{\infty}\langle L_k x, \widehat{\rho}_t^*(A) L_k x\rangle$ is continuous, hence the application $t \mapsto \frac{d}{dt}^{+}\langle x, \widehat{\rho}_t^*(A) x\rangle$ is continuous. Therefore $\langle x, \widehat{\rho}_t^*(A) x\rangle$ is continuously differentiable (see, e.g., Section 2.1 of [33]). Property (a) of Definition 2.2 now follows from (7.32). □

We are in position to show our second main theorem.

Proof of Theorem 4.5. Let $(\widehat{\rho}_t)_{t\geq 0}$ be a semigroup C-solution of (1.1). Consider the adjoint semigroup $(\widehat{\rho}_t^*)_{t\geq 0}$ of $(\widehat{\rho}_t)_{t\geq 0}$, and let $(\mathcal{T}_t(A))_{t\geq 0}$ be given by Theorem 2.1. Combining Lemma 7.21 with Theorem 2.1 we obtain $\widehat{\rho}_t^*(A) = \mathcal{T}_t(A)$ for all $t \geq 0$ and $A \in \mathfrak{L}(\mathfrak{h})$. If $\varrho \in \mathfrak{L}_{1,C}^{+}(\mathfrak{h})$ and $A \in \mathfrak{L}(\mathfrak{h})$, then applying (7.30) and Lemma 7.7 yields

$$tr(\rho_t(\varrho) A) = tr(\mathcal{T}_t(A)\varrho) = tr(\widehat{\rho}_t^*(A)\varrho) = tr(\widehat{\rho}_t(\varrho) A),$$

and so $\rho_t(\varrho) = \widehat{\rho}_t(\varrho)$. Lemma 7.10 now implies that $\rho_t(\varrho) = \widehat{\rho}_t(\varrho)$ for all ϱ belonging to $\mathfrak{L}_1^{+}(\mathfrak{h})$, hence $\rho_t = \widehat{\rho}_t$. Finally, Lemma 7.20 completes the proof. □

7.9. *Proof of Theorem 4.6.*

Proof. From [18] we have that Hypothesis 2 holds with $C = P^2 + Q^2$. Hence Theorem 4.5 yields our first assertion.

Suppose that $A = P$ or $A = Q$. Using, for instance, the spectral theorem we deduce the existence of a sequence A_n of bounded self-adjoint operators in $\mathfrak{h}$ such that for all $f \in \mathcal{D}(A)$ we have $\|A_n f\| \leq \|Af\|$ and $A_n f \longrightarrow_{n\to\infty} Af$. Applying Theorems 3.2 and 4.4 (or better Lemmata 7.18 and 7.19) gives

$$\begin{aligned}(7.33)\ tr(A_n\rho_t(\varrho)) = tr(A_n\varrho) + \int_0^t \Big(&\mathbb{E}\langle A_n X_t(\xi), GX_t(\xi)\rangle \\ &+\mathbb{E}\langle GX_t(\xi), A_n X_t(\xi)\rangle + \sum_{k=1}^{\infty}\mathbb{E}\langle L_k X_t(\xi), A_n L_k X_t(\xi)\rangle\, ds\Big),\end{aligned}$$

where $\varrho = \mathbb{E}|\xi\rangle\langle\xi|$ with $\mathbb{E}\left(\|C\xi\|^2 + \|\xi\|^2\right) < \infty$. By the dominated convergence theorem, letting $n \to \infty$ we obtain

$$\begin{aligned}(7.34)\quad tr(A\rho_t(\varrho)) = tr(A\varrho) + \int_0^t \Big(&\mathbb{E}\langle AX_t(\xi), GX_t(\xi)\rangle \\ &+\mathbb{E}\langle GX_t(\xi), AX_t(\xi)\rangle + \sum_{k=1}^{\infty}\mathbb{E}\langle L_k X_t(\xi), AL_k X_t(\xi)\rangle\, ds\Big).\end{aligned}$$

Let $f \in C_c^\infty(\mathbb{R},\mathbb{C})$. Using $[P,Q]=-iI$ we deduce that $\mathcal{L}(P)f = i[H,P]f$ and $\mathcal{L}(Q)f = i[H,Q]f$. Therefore

$$\langle Af,Gf\rangle+\langle Gf,Af\rangle+\sum_{k=1}^{\infty}\langle L_kf,AL_kf\rangle=\begin{cases}\langle f,Pf\rangle/m, & if\ \text{A=Q}\\ -2c\langle f,Qf\rangle, & if\ \text{A=P}\end{cases}. \tag{7.35}$$

Since $C_c^\infty(\mathbb{R},\mathbb{C})$ is a core for $C=P^2+Q^2$, combining a limit procedure with, for instance, Lemma 12 of [18] we get that (7.35) holds for all $f\in\mathcal{D}(C)$. Then, (7.34) leads to (4.5). □

7.10. *Proof of Theorem 5.1.*

Proof. Let ξ be distributed according to θ. Set $\widetilde{\mathbb{Q}}=\|X_T(\xi)\|^2\cdot\mathbb{P}$, with $T>0$. For any $t\in[0,T]$, we choose $\widetilde{Y}_t = X_t(\xi)/\|X_t(\xi)\|$ if $X_t(\xi)\neq 0$ and $\widetilde{Y}_t=0$ otherwise; let $B_t^k = W_t^k - \int_0^t \frac{1}{\|X_s(\xi)\|^2} d\left[W^k, X(\xi)\right]_s$ for any $k\in\mathbb{N}$. Proceeding along the same lines as in the proof of Proposition 1 of [31] we obtain that $\left(\mathbb{Q},(Y_t)_{t\in[0,T]},\left(B_t^k\right)_{t\in[0,T]}^{k\in\mathbb{N}}\right)$ is a C-solution of (1.3) with initial law θ. By Remark 5.1, (1.3) has a unique C-solution with initial distribution θ. Therefore the distribution of $\widetilde{Y}_t$ with respect to $\widetilde{\mathbb{Q}}$ coincides with the distribution of Y_t under $\mathbb{Q}$. From [18] we have that $\left(\|X_t\|^2\right)_{t\in[0,T]}$ is a martingale, hence for any $x\in\mathfrak{h}$ and $t\in[0,T]$:

$$\mathbb{E}_{\mathbb{Q}}|\langle x,Y_t\rangle|^2=\mathbb{E}_{\widetilde{\mathbb{Q}}}\left|\left\langle x,\widetilde{Y}_t\right\rangle\right|^2=\mathbb{E}_{\mathbb{P}}\left(\left|\left\langle x,\widetilde{Y}_t\right\rangle\right|^2\|X_t(\xi)\|^2\right)=\mathbb{E}_{\mathbb{P}}|\langle x,X_t(\xi)\rangle|^2.$$

Applying (4.1) and the polarization identity gives $\rho_t(\varrho)=\mathbb{E}|Y_t\rangle\langle Y_t|$. □

7.11. *Proof of Theorem 5.2.*

Proof of Theorem 5.2. Let $\left(\mathbb{Q},(Y_t)_{t\geq 0},(B_t)_{t\geq 0}\right)$ be the C-solution of (1.3) with initial distribution Γ (see Remark 5.1). Choose $\varrho_\infty=\mathbb{E}|Y_0\rangle\langle Y_0|$. Then, Theorem 5.1 shows that $\rho_t(\varrho_\infty)=\mathbb{E}|Y_t\rangle\langle Y_t|$ for all $t\geq 0$.

As in the proof of Theorem 3 of [31], applying techniques of well-posed martigale problems we obtain the Markov property of the C-solutions of (1.3) under Hypothesis 2, hence for any $x\in\mathfrak{h}$ and $t\geq 0$:

$$\mathbb{E}\left(\mathbf{1}_{[0,\|x\|^2]}\left(|\langle x,Y_t\rangle|^2\right)\right)=\mathbb{E}\left(\int_{\mathfrak{h}}\mathbf{1}_{[0,\|x\|^2]}\left(|\langle x,y\rangle|^2\right)P_t(Y_0,dy)\right).$$

On the other hand, using (5.1) we deduce that

$$\mathbb{E}\left(\mathbf{1}_{[0,\|x\|^2]}\left(|\langle x,Y_0\rangle|^2\right)\right)=\int_{\mathfrak{h}}\left(\int_{\mathfrak{h}}\mathbf{1}_{[0,\|x\|^2]}\left(|\langle x,y\rangle|^2\right)P_t(z,dy)\right)\Gamma(dz).$$

Now, combining $\|Y_t\| = 1$ with $\int_{\mathfrak{h}} \left(\int_{\mathfrak{h}} \mathbf{1}_{[0,\|x\|^2]} \left(|\langle x, y\rangle|^2 \right) P_t(z, dy) \right) \Gamma(dz) = \mathbb{E} \left(\int_{\mathfrak{h}} \mathbf{1}_{[0,\|x\|^2]} \left(|\langle x, y\rangle|^2 \right) P_t(Y_0, dy) \right)$ we get $\mathbb{E} |\langle x, Y_0\rangle|^2 = \mathbb{E} |\langle x, Y_t\rangle|^2$. This gives $\mathbb{E} |Y_t\rangle\langle Y_t| = \mathbb{E} |Y_0\rangle\langle Y_0|$, and so $\rho_t(\varrho_\infty) = \varrho_\infty$. □

7.12. *Proof of Theorem 6.1.*

Proof of Theorem 6.1. Since $\mathcal{D}(G) = \mathcal{D}(N^4)$, from Remark 2.3 we have that G is a closable operator satisfying $G \in \mathfrak{L}((\mathcal{D}(N^p), \|\cdot\|_{N^p}), \mathfrak{h})$. Fix $x \in \mathfrak{h}$ such that $x_n := \langle e_n, x\rangle$ is equal to 0 for all $n \in \mathbb{Z}_+$ except a finite number. An easy computation shows that $2\Re\langle N^{2p}x, Gx\rangle + \sum_{k=1}^{\infty} \|N^p L_k x\|^2$ is equal to the sum of $4p\left(|\alpha_5|^2 - |\alpha_4|^2\right) \sum_{n=0}^{\infty} n^{2p+1} |x_n|^2$ and

$$2\beta_1 \sum_{n=1}^{\infty} \sqrt{n+1} \left((n+1)^{2p} - n^{2p} \right) \Re\left(x_n \overline{x_{n+1}}\right) + \sum_{n=0}^{\infty} f(n) |x_n|^2,$$

where f is a $2p$-degree polynomial whose coefficients depend of $|\alpha_k|^2$ with $k = 1, 2, 4, 5$. Hence N^p satisfies Hypothesis 5 whenever $|\alpha_4| \geq |\alpha_5|$. From [18] it follows that N^p fulfills Condition H2.3 of Hypothesis 2, and so Theorems 4.4 and 4.5 lead to Statement (i).

From Theorem 8 of [31] we have the existence of an invariant probability measure Γ for (1.3) that satisfies the properties given in Hypothesis 4 with $C = N^p$. Using Theorem 5.2 yields Statement (ii). □

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