

## ON EXPLOSIONS IN HEAVY-TAILED BRANCHING RANDOM WALKS

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Consider a branching random walk on  $\mathbb{R}$ , with offspring distribution  $Z$  and nonnegative displacement distribution  $W$ . We say that *explosion* occurs if an infinite number of particles may be found within a finite distance of the origin. In this paper, we investigate this phenomenon when the offspring distribution  $Z$  is heavy-tailed. Under an appropriate condition, we are able to characterize the pairs  $(Z, W)$  for which explosion occurs, by demonstrating the equivalence of explosion with a seemingly much weaker event: that the sum over generations of the minimum displacement in each generation is finite. Furthermore, we demonstrate that our condition on the tail is best possible for this equivalence to occur.

We also investigate, under additional smoothness assumptions, the behaviour of  $M_n$ , the position of the particle in generation  $n$  closest to the origin, when explosion does not occur (and hence  $\lim_{n \rightarrow \infty} M_n = \infty$ ).

**1. Introduction.** Our aim in this paper is to give a classification of the displacement random variables in heavy-tailed branching random walks in  $\mathbb{R}$  for which explosion—a concept we will define shortly—occurs. Thus consider a branching random walk on  $\mathbb{R}$ . The process begins with a single particle at the origin; this particle moves to another point of  $\mathbb{R}$  according to a displacement distribution  $W$ , where it gives birth to a random number of offspring, according to a distribution  $Z$ . This procedure is then repeated: the particles in a given generation each take a single step according to an independent copy of the same distribution  $W$ , and then give birth to the next generation. We consider the case where  $W$  is nonnegative (in which case the process is also called an *age-dependent* process; the displacement of a particle can also be interpreted as a birthdate). Let  $\Gamma_t$  be the number of particles with displacement at most  $t$ ; then we say that *explosion* occurs if  $\Gamma_t = \infty$  for some finite  $t$ .

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Alternatively, let  $M_n$  be the displacement of the leftmost particle in the  $n$ -th generation. If the process dies out and there are no particles remaining in the  $n$ -th generation, then define  $M_n = \infty$ . Explosion is the event that  $\lim_{n \rightarrow \infty} M_n < \infty$ . Note that, since  $M_n$  is monotone, it has a limit.

Taking a tree view of the above process, denote by  $T_Z$  a random Galton-Watson tree with offspring distribution  $Z$ , and let  $Z_n$  be the number of children at level  $n$ . To avoid the trivial case, we assume throughout that  $\mathbb{P}\{Z = 1\} < 1$ . Each edge of  $T_Z$  is then independently given a weight according to the nonnegative distribution  $W$ . The connection to the above process is that the displacement of a node is simply the sum of the weights on the path from the root to that node. From this perspective, which is the one we will take in this paper, explosion is the event that there exists an infinite path for which the sum of the weights on the path is finite.

In the process of studying the event of explosion, we first consider the case where the offspring distribution has finite mean. The different cases described in the next paragraph show that we can either trivially solve the problem, or reduce to the most interesting case of an infinite mean.

*Reduction to the case of an infinite mean.* Consider a Galton-Watson process with offspring distribution  $Z$  satisfying  $0 < \mathbb{E}\{Z\} < \infty$ . We still assume  $\mathbb{P}\{Z = 1\} < 1$ . Let  $W$  be a weight (or displacement) distribution on the edges of the Galton-Watson tree.

Consider first the case where  $\mathbb{P}\{W = 0\} = 1$ . In this case, explosion is equivalent to the event that the Galton-Watson tree is infinite, i.e., the survival of the Galton-Watson process. In that case, if  $\mathbb{E}\{Z\} \leq 1$ , there is no survival, and if  $\mathbb{E}\{Z\} > 1$ , there is a positive probability of survival [4]. From now on we will assume that  $\mathbb{P}\{W = 0\} < 1$  and assume that the Galton-Watson process is supercritical.

In the case of a supercritical Galton-Watson process, under the assumption  $\mathbb{E}\{Z\} < \infty$ , the results of Hammersley [21], Kingman [26], and Biggins [7] show the existence of a constant  $\gamma$  such that conditional on the non-extinction of the process,  $M_n/n$  tends to  $\gamma$  almost surely. This shows that the random variables  $M_n$ , conditional on survival, behave linearly in  $n$ , i.e.,  $M_n = \gamma n + o(n)$ . One consequence of the Hammersley-Kingman-Biggins theorem is that if  $\gamma > 0$ , then explosion never happens. Now define

$$H := \mathbb{E}\{Z\}\mathbb{P}\{W = 0\}.$$

It can be shown that  $\gamma = 0$  if and only if  $H \geq 1$ . We consider in fact three cases:  $H < 1$ ,  $H > 1$  and  $H = 1$ .

- CASE I:  $H < 1$ .

Here, as stated above, explosion occurs with probability zero. This can be seen more simply as follows: fix an  $\epsilon > 0$  such that  $\mathbb{P}\{W < \epsilon\} < (\mathbb{E}\{Z\})^{-1}$  and mark all edges with weight smaller than  $\epsilon$ . Then each component in the forest of marked edges is a subcritical Galton-Watson tree, and hence has finite size almost surely. Thus any infinite path must contain an infinite number of unmarked edges, and hence cannot be an exploding path.

- CASE II:  $H > 1$ .

In this case, explosion happens with probability one. To see this, take a sub-Galton-Watson tree by keeping only children for which  $W = 0$ . This tree is supercritical and thus survives with some positive probability  $\rho$ . It follows that with positive probability, there is an infinite path of length zero. Since, conditional on survival, explosion is a 0-1 event (for a proof see later in this introduction), we infer that it happens with probability one. A theorem of Dekking and Host [15] ensures the existence of an almost surely finite random variable  $M$  such that  $M_n$  converges a.s. to  $M$ . Under the extra condition  $\mathbb{E} Z^2 < \infty$ , they determine stronger results on the limit distribution  $M$ .

- CASE III:  $H = 1$ .

This threshold case is the most intriguing—it was already considered in an earlier pioneering work of Bramson [10], and the work of Dekking and Host [15]. In this case, the occurrence of explosion is a delicately balanced event that depends upon the behavior of the distribution of  $W$  near the origin and on the distribution of  $Z$ .

Bramson's main theorem is the following result on the behaviour of  $M_n$  under the assumption that there exists a  $\delta > 0$  such that  $\mathbb{E}\{Z^{2+\delta}\} < \infty$ . For any fixed  $\lambda$ , define  $\sigma_{\lambda,n} = p + (1-p)e^{-\lambda^n}$  where  $p = \mathbb{P}\{W = 0\} < 1$ . Then explosion happens if and only if there exists some  $\lambda > 1$  such that  $\sum_{n=1}^{\infty} F_W^{-1}(\sigma_{\lambda,n}) < \infty$ . In the case of no explosion, and conditional on the survival of the branching process, the following convergence result on the asymptotic of  $M_n$  holds. Almost surely, we have

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^{s(n)} F_W^{-1}(\sigma_{2,k})} = 1,$$

where  $s(n) = \lceil \log \log n / \log 2 \rceil$ . We refer to [15] for a generalization of Bramson's theorem to the case of  $\mathbb{E}\{Z^2\} < \infty$ , under some extra mild conditions.

Following Bramson [10], we first transform the tree  $T_Z$  into a new tree  $T'$  as follows. The roots are identical. First consider the sub-Galton-Watson

tree rooted at the root of  $T_Z$  consisting only of children (edges) that have zero weight. This subtree is critical. For any distribution of  $Z$  satisfying the threshold condition, note that the size  $S$  of the sub-Galton-Watson tree is a random variable  $S \geq 1$  with  $\mathbb{E}\{S\} = \infty$ . In some cases, we know more—for example, when  $\text{Var}\{Z\} = \sigma^2 \in (0, \infty)$ , then  $\mathbb{P}\{S \geq k\} \sim \sqrt{2/\pi\sigma^2k}$  as  $k \rightarrow \infty$  (see, e.g., the book of Kolchin [27]). All of the nodes in  $S$  are mapped to the root of the new tree  $T'$ . The children of that root in  $T'$  are all the children of the mapped nodes in  $T_Z$  that did not have  $W = 0$ .

Let  $X_i$  be the number of vertices of degree  $i$  in the sub-Galton-Watson tree. The number of children of the root of  $T_Z$  is distributed as

$$\zeta = \sum_{i=0}^{\infty} \sum_{j=1}^{X_i} \zeta_{i,j},$$

where  $\zeta_{i,1}, \zeta_{i,2}, \dots$  are i.i.d. random variables having distribution of a random variable  $\zeta_i$ . In addition the distribution of  $\zeta_i$  is given by

$$\mathbb{P}\{\zeta_i = k\} = c_i \binom{k+i}{i} (1 - \mathbb{P}\{W = 0\})^k \mathbb{P}\{W = 0\}^i \mathbb{P}\{Z = k+i\},$$

where  $c_i$  is a normalizing constant. Note that  $\sum_{i \geq 0} X_i = S$ .

For each child of the root in  $T'$ , repeat the above collapsing procedure. It is easily seen that  $T'$  itself is a Galton-Watson tree with offspring distribution  $\zeta$ . The moment generating function  $G_\zeta(s)$  of  $\zeta$  is easily seen to satisfy the functional equation

$$(1.2) \quad G_\zeta(s) = G_Z\left((1 - \mathbb{P}\{W = 0\})s + \mathbb{P}\{W = 0\}G_\zeta(s)\right).$$

Furthermore, the displacement distribution is  $W$  conditional on  $W > 0$ . Finally, one can verify that  $\mathbb{E}\{\zeta\} = \infty$ . More importantly, explosion occurs in  $T_Z$  if and only if explosion happens in  $T'$ . We have thus reduced the explosion question to one for a new tree in which the expected number of children is infinite, and in which  $W$  does not have an atom at zero.

Observe that the transformation described in CASE III is valid whenever  $W$  has an atom at the origin. In particular, this construction can also be used to eliminate an atom at the origin when  $\mathbb{P}\{W = 0\} > 0$  and  $\mathbb{E}\{Z\} = \infty$ . In this case, we still have  $\mathbb{E}\{\zeta\} = \infty$ .

It follows from the above discussion that in the study of the the event of explosion, we need to consider only the (most interesting) case where

$$\mathbb{E}\{Z\} = \infty, \quad \mathbb{P}\{W = 0\} = 0.$$

All our results below are concerned only with this case.

*A simple necessary condition for explosion.* There is a rather obvious necessary condition for explosion. Let  $Y_i$  be the minimum weight edge at level  $i$  in the tree. Then the sum of weights along any infinite path is certainly at least  $\sum_{i=1}^{\infty} Y_i$ . We say that a fixed weighted tree is *min-summable* if this sum is bounded; if a tree is not min-summable, it cannot have an exploding path.

For any fixed, infinite, rooted tree  $T$ , and distribution  $W$  on the nonnegative reals, let  $T^W$  denote a random weighted tree obtained by weighting each edge with an independent copy of  $W$ . For a fixed tree  $T$  and weight distribution  $W$ , it follows easily from Kolmogorov's 0-1 law that explosion and min-summability of  $T^W$  are both 0-1 events. Thus, we make the following definitions.

DEFINITION 1.1. For any infinite rooted tree  $T$ ,

- (i) let  $\mathcal{W}_{\text{EX}}(T)$  be the set of weight distributions so that  $T^W$  contains an exploding path almost surely, and
- (ii) let  $\mathcal{W}_{\text{MS}}(T)$  be the set of weight distributions so that  $T^W$  is min-summable almost surely.

In this new notation, the observation above is simply that  $\mathcal{W}_{\text{EX}}(T) \subseteq \mathcal{W}_{\text{MS}}(T)$ , for any tree  $T$ . Unsurprisingly, in general  $\mathcal{W}_{\text{EX}}(T)$  may be strictly contained within  $\mathcal{W}_{\text{MS}}(T)$ . For example, consider an infinite binary tree  $T$  and a uniform weight distribution  $W$  on  $[0, 1]$ . Except with probability at most  $\exp(-2^{i/2})$  the minimum of  $2^i$  copies of  $W$  is at most  $2^{-i/2}$ . Thus, with positive probability  $\sum_{i \geq 1} Y_i \leq \sum_{i \geq 1} 2^{-i/2} < 3$ , and so  $W \in \mathcal{W}_{\text{MS}}(Z)$ . On the other hand, we may easily prove that  $W \notin \mathcal{W}_{\text{EX}}(Z)$ , i.e., that the probability that there exists an exploding path is zero. To see this, consider the event  $A_i$  that there exists a path from the root to level  $i$  of weight less than  $i/128$ . The existence of an exploding path certainly implies that for all sufficiently large  $i$ ,  $A_i$  occurs. We now observe that  $\mathbb{P}\{A_i\} \leq 2^{-i}$ . Indeed, the event  $A_i$  implies that there is a path from the root to level  $i$  at least half of whose edges have weight less than  $\frac{1}{64}$ . Since there are only  $2^i$  paths to level  $i$  and at most  $2^i$  ways to choose a subset of the edges of a fixed path, and since for each path and each fixed subset of at least  $\frac{i}{2}$  edges, the probability that all these edges have weight less than  $\frac{1}{64}$  is at most  $8^{-i}$ , the bound easily follows. The same proof shows that for the exponential distribution  $E$ , no explosion can happen (however,  $E \in \mathcal{W}_{\text{MS}}(T)$ ; this follows from Example (iv) of Section 4).

*Main results.* It may appear that, aside from some trivial cases,  $\mathcal{W}_{\text{MS}}(T)$  should always strictly contain  $\mathcal{W}_{\text{EX}}(T)$ . However, somewhat counter-intuitively, this is not the case; there are examples of trees with generation sizes growing very fast (double exponentially) for which  $\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T)$ . Consider for example the tree  $T$  defined as follows: all nodes of generation  $n$  have  $2^{2^n}$  children. In this case, for a given weight distribution  $W$ , the distribution of the sum of minimum weights of levels is

$$\sum_{n \geq 1} \min_{1 \leq i \leq 2^{(2^n - 1)}} W_n^i,$$

where each  $W_n^i$  is an independent copy of  $W$ . Also the path constructed by the simple greedy algorithm which, starting from root, adds at each step the lowest weight edge from the current node to one of its children, has total weight distributed as

$$\sum_{n \geq 1} \min_{1 \leq i \leq 2^{2^{(n-1)}}} W_n^i.$$

The property of these sums being finite almost surely is clearly equivalent, so that  $\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T)$ . Our main result is that this phenomenon is in fact quite general in trees obtained by a Galton-Watson process with a heavy tailed offspring distribution. We call the distribution  $Z$  *plump* if for some positive constant  $\epsilon$  the inequality

$$(1.3) \quad \mathbb{P}\{Z \geq m^{1+\epsilon}\} \geq \frac{1}{m}$$

holds for all  $m$  sufficiently large. Equivalently,  $Z$  is plump if its distribution function  $F_Z$  satisfies  $F_Z^{-1}(1 - 1/m) \geq m^{1+\epsilon}$  for  $m$  sufficiently large. We remark that  $\mathbb{E} Z = \infty$  for any plump  $Z$ .

**EQUIVALENCE THEOREM.** *Let  $Z$  be a plump distribution. Let  $T$  be a random Galton-Watson tree with offspring distribution  $Z$ , but conditioned on survival. Then*

$$\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T) \quad \text{with probability 1.}$$

We now state a second form of the equivalence theorem. For this, we must extend the definition of  $\mathcal{W}_{\text{EX}}$  and  $\mathcal{W}_{\text{MS}}$  to Galton-Watson offspring distributions. Let  $Z$  be an offspring distribution and  $W$  a weight distribution. We have

CLAIM 1.2. *For a given offspring distribution  $Z$  and weight distribution  $W$ , and conditioning on survival of the Galton-Watson process, explosion and min-summability are 0-1 events.*

PROOF. Let  $(W_i)_{i=1}^\infty$  be a sequence of independent copies of  $W$ , let  $(S_i)_{i=1}^\infty$  be a random walk with jump distribution given by  $Z - 1$ , and let  $(X_i)_{i=1}^\infty$  be the increments. In the usual way, this random walk can be thought of as representing (in breadth-first fashion) a sequence of one or more Galton-Watson trees, with  $X_i + 1$  giving the number of children at step  $i$  and  $W_i$  the weight of the  $i$ -th edge. Since  $\mathbb{E} Z > 1$ , one of these trees  $T'$  will be infinite with probability 1, and this tree is exactly a Galton-Watson tree conditioned on survival. The sequence  $((X_i, W_i))_{i=1}^\infty$  clearly encodes all the information about  $T'$ , and the two events under consideration are tail events with respect to this sequence; thus Kolmogorov's 0-1 law applies. The same argument holds for min-summability.  $\square$

We can thus define  $\mathcal{W}_{\text{EX}}(Z)$  and  $\mathcal{W}_{\text{MS}}(Z)$  for an offspring distribution  $Z$  as follows:

$$\mathcal{W}_{\text{EX}}(Z) := \left\{ W \mid W \in \mathcal{W}_{\text{EX}}(T_Z) \text{ almost surely conditioned on survival} \right\},$$

and

$$\mathcal{W}_{\text{MS}}(Z) := \left\{ W \mid W \in \mathcal{W}_{\text{MS}}(T_Z) \text{ almost surely conditioned on survival} \right\}.$$

The alternative (though slightly weaker) formulation of the Equivalence Theorem can now be stated as follows:

EQUIVALENCE THEOREM—SECOND VERSION. *For a plump distribution  $Z$ ,*

$$\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z).$$

Min-summability is clearly a simpler kind of condition than explosion; in particular, it depends only on the generation sizes  $Z_n$  rather than the full structure of the tree  $T_Z$ . Indeed, the Equivalence Theorem becomes more interesting if one observes that it is possible to derive the following quite explicit necessary and sufficient condition for min-summability.

THEOREM 1.3. *Given a plump offspring distribution  $Z$ , let  $m_0 > 1$  be large enough such that the condition (1.3) holds for all  $m \geq m_0$ . Define the function  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  as follows:*

$$(1.4) \quad h(0) = m_0 \quad \text{and} \quad h(n+1) = F_Z^{-1}(1 - 1/h(n)) \quad \text{for all } n \geq 1.$$

Then for any weight distribution  $W$ ,  $W \in \mathcal{W}_{\text{MS}}(Z)$ , and hence also  $W \in \mathcal{W}_{\text{EX}}(Z)$ , if and only if

$$\sum_n F_W^{-1}(h(n)^{-1}) < \infty.$$

Given the Equivalence Theorem above, one may wonder if there is a way to weaken the condition given in (1.3) such that the theorem still remains valid. We show that this condition is to some extent the best we can ask for. More precisely, we prove

**SHARPNESS OF CONDITION (1.3).** *Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be an increasing function satisfying*

$$g(m) = m^{1+o(1)}.$$

*Then there is an offspring distribution  $Z$  satisfying  $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$  for all  $m \in \mathbb{N}$ , but for which  $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$ .*

So far our results concerned the appearance of the event of explosion; however, it is also natural to ask how fast  $M_n$  tends to infinity in the case there is a.s. no exploding path. Although there is no reason to expect a convergence theorem in the case of no explosion for general plump distributions in the absence of any smoothness condition on the tails of  $Z$ , we show that a stronger plumpness property allows to obtain a precise information on the rate of convergence to infinity of  $M_n$ . To explain this, note that the plumpness assumption on  $Z$  is equivalent to  $1 - F_Z(k) \geq k^{-\eta}$  for  $\eta = \frac{1}{1+\epsilon}$  and for all  $k$  sufficiently large. Consider now the stronger smoothness condition

$$(1.5) \quad 1 - F_Z(k) = k^{-\eta} \ell(k),$$

where  $\ell$  is any continuous and bounded function which is nonzero at infinity.

**LIMIT THEOREM UNDER CONDITION (1.5).** *Let  $Z$  satisfy the smoothness condition, and let  $W$  be any weight distribution with  $W \notin \mathcal{W}_{\text{EX}}(Z)$ . Then a.s. conditional on survival,*

$$\lim_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\left(\exp\left(-(1+\epsilon)^k\right)\right)} = 1$$

for all  $\epsilon > 0$ .



Applying a Tauberian theorem (see Section 6 for more details), we find that Condition (1.5) is equivalent to the condition

$$K_Z(s) := 1 - G_Z(1 - s) \sim a s^\eta \ell\left(\frac{1}{s}\right)$$

near  $s = 0$  for some  $a > 0$ ; recall  $G_Z$  is the moment generating function of  $Z$ . Going back to Case III of the finite mean case, and the transformation described there, we observe that the use of the functional Equation (1.2) allows to translate the smoothness condition above, imposed on the modified offspring distribution  $\zeta$  of infinite mean (obtained after the transformation), to a smoothness condition on  $Z$ , the original distribution of finite mean. In particular,

$$K_\zeta(s) = 1 - G_\zeta(1 - s) \sim a s^{1/(1+\epsilon)} (1 + O(s^\beta)) \quad \text{for } s \text{ near zero,}$$

for some  $a, \epsilon, \beta > 0$  is equivalent to a condition of the form

$$(1.6) \quad K_Z(s) \sim \mathbb{E}\{Z\} s - c s^{1+\epsilon} (1 + O(s^\delta)) \quad \text{for } s \text{ near zero,}$$

for some  $c, \delta > 0$ . We note that Condition (1.6) assumes some regularity on the tails of  $Z$  but the variance could be infinite; thus, the above result can be regarded as a strengthening of Bramson's theorem [10].

*Further related work.* The literature on explosion is partially surveyed by Vatutin and Zubkov [38]. The early work deals with exponentially distributed weights: in this case, there is no explosion almost surely if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n \sum_{r=0}^n \mathbb{P}\{Z > r\}} < \infty$$

(see [22, V. 6], [17, 30]). This condition cannot be simplified; Grey [20] showed that there does not exist any fixed function  $\psi \geq 0$  such that explosion would be equivalent to  $\mathbb{E}\{\psi(Z)\} = \infty$ .

Some general properties of the event of explosion were obtained in [33] by considering the generating functions of the number of particles born before time  $t$ , parametrized by  $t$ , and looking at the nonlinear integral equation satisfied by these generating functions. By using this analytic approach, and under some smoothness conditions on the distribution function  $F_W$  of the displacement  $W$ , Sevast'yanov [33, 34], Gel'fond [19], and Vatutin [35, 36] obtain necessary and sufficient conditions on the event of explosion. The result of Vatutin [36] can be stated as follows. Consider the case  $\mathbb{P}\{W = 0\} = 0$

and suppose that zero is an accumulation point of  $W$ , i.e., the distribution function  $F_W$  of  $W$  satisfies  $F_W(w) > 0$  for all  $w > 0$ . Assume the following regular variation style condition holds: there exists  $\lambda \in (0, 1)$  such that

$$(1.7) \quad 0 < \liminf_{t \downarrow 0} \frac{F_W^{-1}(\lambda t)}{F_W^{-1}(t)} \leq \limsup_{t \downarrow 0} \frac{F_W^{-1}(\lambda t)}{F_W^{-1}(t)} < 1.$$

Then explosion does not occur if and only if for all  $\epsilon > 0$ ,

$$(1.8) \quad \int_0^\epsilon F_W^{-1}\left(\frac{s}{K_Z(s)}\right) \frac{ds}{s} = \infty.$$

Condition (1.7) basically forces  $F_W$  to behave in a polynomial manner near the origin. Indeed, if  $F_W(w) \sim w^\alpha$  for some  $\alpha > 0$  as  $w \downarrow 0$ , then  $F_W^{-1}(t) \sim t^{1/\alpha}$  as  $t \downarrow 0$ , and so (1.7) holds. The exponential law corresponds to  $\alpha = 1$ , for example. The criterion given by (1.8) was earlier proved to be necessary and sufficient for non-explosion by Sevast'yanov [33, 34] and Gel'fond [19] under the slightly more restrictive condition that  $F_W(w)/w^\alpha \in [a, b]$  for all  $w$ , where  $0 < a \leq b < \infty$  and  $\alpha \geq 0$ . As soon as we leave that polynomial oasis, Vatutin's condition is violated. Examples include  $F_W(w) \sim \exp(-1/w^\alpha)$  and  $F_W(w) \sim 1/\log^\alpha(1/w)$  for  $\alpha > 0$ .

A quite general sufficient (but not necessary) condition without any explicit regularity assumption on  $W$  was proved by Vatutin [37] for explosion in non-homogenous branching random walks. In the homogenous case, the result states that if there exists a sequence of nonnegative reals  $(y_n)_{n \in \mathbb{N}}$  such that  $\lim_n y_n = 0$  and

$$\sum_{n=1}^{\infty} F_W^{-1}(y_n/K_{Z_n}(y_n)) < \infty,$$

then explosion occurs. This result is close in spirit to our equivalence theorem, but we stress that the results are distinct—we see no way in which one may be deduced from the other.

More precise information on the behaviour and convergence to infinity of  $M_n$  can be obtained in the finite mean case and under extra conditions. Recall that in the finite mean case,  $M_n = \gamma n + o(n)$  for some  $\gamma \geq 0$ . McDiarmid showed in [28] that  $M_n - \gamma n = O(\log n)$  if  $\mathbb{E}\{Z^2\} < \infty$  and  $W$  has an exponential upper tail. Recently, Hu and Shi [23] proved that if the displacements are bounded and  $\mathbb{E}\{Z^{1+\epsilon}\} < \infty$  for any  $\epsilon > 0$ , then conditional on survival,  $(M_n - \gamma n)/\log n$  converges in probability but, interestingly, not almost surely. (We note in passing that this work and the recent work of Aïdekon-Shi [3] provide Seneta-Heyde norming results [9] in the boundary

case.) Under the extra assumption that  $Z$  is bounded, Addario-Berry and Reed [1] calculate  $\mathbb{E}\{M_n\}$  to within  $O(1)$  and prove exponential tail bounds for  $\mathbb{P}\{|M_n - \mathbb{E}\{M_n\}| > x\}$ . Extending these results, Aïdekon [2] proves the convergence of  $M_n$  centered around its median for a large class of branching random walks. For tightness results in general, under some extra assumptions on the decay of the tail distribution or weight distribution, see Bachmann [5] and Bramson-Zeitouni [11, 12].

*Organization of the paper.* Section 2 will concern some preliminaries, mostly involving what we call the *speed* of an offspring distribution. In Section 3, we prove the Equivalence Theorem. The proof is somewhat algorithmic in nature, and shows that a certain (infinite) algorithm will always find an exploding path under the given conditions. In Section 4, we prove Theorem 1.3, and give some examples calculating the condition for specific cases. In Section 5 we provide a generic counterexample that shows that the equivalence does not hold if we weaken the conditions in any substantial way, proving the sharpness of Condition (1.3). Finally, in Section 6 we prove the limit theorem under Condition (1.5).

**2. Preliminaries.** In this section we present some definitions and results needed for the proof of the Equivalence Theorem. That theorem (in its second form) is concerned with the equivalence of  $\mathcal{W}_{\text{MS}}(Z)$  and  $\mathcal{W}_{\text{EX}}(Z)$  for certain offspring distributions  $Z$ . Thus it will be important to have a good characterization of whether a weight distribution  $W$  belongs to  $\mathcal{W}_{\text{MS}}(Z)$ ; in other words, whether  $\sum_{n \geq 1} \min\{W_n^1, \dots, W_n^{Z_n}\}$  is finite, each  $W_n^i$  being an independent copy of  $W$ . To do this we will introduce two notions. The first is the concept of the *speed* of a branching process, from which we will obtain an understanding of the growth of the generation sizes  $Z_n$ . The second is the concept of *summability with respect to an integer sequence*, which concerns the behaviour of sums of the form  $\sum_{n \geq 1} \min\{W_n^1, \dots, W_n^{\sigma_n}\}$  for a given integer sequence  $(\sigma_n)_{n \in \mathbb{N}}$ .

*Speed of a Galton-Watson branching process.* We introduce the concept immediately and then give a number of examples.

DEFINITION 2.1. An increasing function  $f : \mathbb{N} \rightarrow \mathbb{R}^+$ , taking only strictly positive values, is called a *speed* of a Galton-Watson offspring distribution  $Z$  if there exist positive integers  $a$  and  $b$  such that with positive probability

$$Z_{n/a} \leq f(n) \leq Z_{bn} \quad \text{for all } n \in \mathbb{N}.$$

(Here, we set  $Z_x = Z_{\lfloor x \rfloor}$  for  $x \in \mathbb{R}$ .)

Note that there is a small issue of extinction here, and that is why we insist that  $f$  is strictly positive, otherwise  $f(n) = 0$  would be a speed for any distribution with  $\mathbb{P}\{Z = 0\} > 0$ .

*Examples of speeds.* Here we give examples of speeds for various distributions  $Z$ .

- (i) If  $\mathbb{E}\{Z\} \leq 1$  then almost surely  $Z_n = 0$  for all sufficiently large  $n$ , and so  $Z$  does not have a speed.
- (ii) If  $\mathbb{E}\{Z\} = m \in (1, \infty)$ , then Doob's limit law states that the random variable  $V_n = Z_n/m^n$  form a martingale sequence with  $\mathbb{E} V_n \equiv 1$ , and  $V_n \rightarrow V$  almost surely, where  $V$  is a nonnegative random variable. Furthermore, in the case that  $Z$  is bounded the limit random variable  $V$  has mean 1 (and so in particular  $\mathbb{P}\{V \geq 1\} > 0$ ). From this we may easily verify that  $m^n$  is a speed of  $Z$ . Indeed Doob's limit law implies that the inequality  $Z_n \leq (M+1)m^n$  holds for all  $n$  large enough, with probability at least  $P(V \leq M)$ . Taking  $M$  sufficiently large, this probability may be made arbitrarily close to 1. For the lower bound, one may consider a truncation  $Z'$  of  $Z$  such that  $\mathbb{E}\{Z'\} \geq \sqrt{m}$ . Since  $Z'$  is bounded, we deduce that in the truncated branching process associated with  $Z'$  there is a positive probability that  $Z'_n \geq m^{n/2}/2$  for all sufficiently large  $n$ . Since there is a natural coupling such that  $Z_n \geq Z'_n$  for all  $n$ , this completes our proof that  $m^n$  is a speed of  $Z$ .
- (iii) If  $Z$  is defined by  $\mathbb{P}\{Z \geq m+1\} = m^{-\beta}$  for each  $m \geq 1$ , where  $\beta \in (0, 1)$ , then  $Z$  is plump (one may take  $\epsilon = \beta^{-1} - 1$  in Condition (1.3)) and the double exponential function  $f(n) = 2^{(\beta^{-1})^n}$  is a speed of  $Z$ . Heuristically, this follows from the fact that conditioned on the value of  $Z_n$  one would expect  $Z_{n+1}$  to be of the order  $Z_n^{\beta^{-1}}$ . A formal proof follows from Theorem 2.4 together with the observation that the function  $h$  appearing in that theorem is equivalent to  $f$  as a speed (i.e., there exist  $a', b' \in \mathbb{N}$  such that the inequalities  $f(\lfloor n/a' \rfloor) \leq h(n) \leq f(b'n)$  hold for all  $n$ ). Indeed, as we will explain in Section 6, a much stronger statement holds in this case.
- (iv) If  $Z$  is defined by  $\mathbb{P}\{Z \geq m\} = 1/\log_2 m$  for each  $m \geq 2$ , then  $Z$  is plump. Applying Theorem 2.4 we find that the tower function  $h(n)$  defined by  $h(0) = 2$  and  $h(n+1) = 2^{h(n)}$  for  $n \geq 0$  is a speed of  $Z$ .

*Summable weight distributions with respect to an integer sequence.* Let  $W$  be a random variable with nonnegative values. Let  $\sigma = (\sigma_n)_{n \in \mathbb{N}}$  be a sequence of positive integers and  $W_n^j$  be a family of independent copies of

$W$  for  $n, j \in \mathbb{N}$ . Define the sequence of minima

$$\Lambda_n := \min_{1 \leq j \leq \sigma_n} W_n^j.$$

The random variable  $W$  is called  $\sigma$ -summable if there is a positive probability that  $\sum_n \Lambda_n$  is finite.

Note that the event in the above definition is a 0-1 event. Thus, if  $W$  is  $\sigma$ -summable, then  $\sum_n \min_{1 \leq j \leq \sigma_n} W_n^j$  is finite with probability one. For a characterization of  $\sigma$ -summable weight distributions see Proposition 4.1. Examples are given at the end of Section 4.

We note that if  $W$  is  $\sigma$ -summable and  $\tau$ -summable, then  $W$  is  $\sigma \cup \tau$ -summable, and if  $\sigma_n \leq \tau_n$  for all  $n$ ,  $\sigma$ -summability implies  $\tau$ -summability. We also have

**LEMMA 2.2.** *Let  $\sigma$  be any increasing sequence, and let  $\tau$  be defined by  $\tau_n = \sigma_{\gamma n}$  for some constant  $\gamma$ , a positive integer. Then  $W$  is  $\sigma$ -summable iff it is  $\tau$ -summable.*

**PROOF.** Write  $\sigma = \sigma^0 \cup \sigma^1 \cup \dots \cup \sigma^{\gamma-1}$ , where  $\sigma^i := \{\sigma_{\gamma n+i} : n \in \mathbb{N}\}$ . Since  $\sigma$  is increasing, if  $W$  is  $\sigma^i$ -summable and  $i < j$ , then  $W$  is  $\sigma^j$ -summable. So if  $W$  is  $\tau = \sigma^0$ -summable, then it is  $\sigma^i$ -summable for all  $0 \leq i \leq \gamma - 1$ , and thus  $\sigma$ -summable. The other direction follows trivially since  $\tau \subseteq \sigma$ .  $\square$

The following proposition relates the condition of the Equivalence Theorem to the notion of  $\sigma$ -summability under the presence of a speed function for the Galton-Watson distribution.

**PROPOSITION 2.3.** *Let  $W$  be a weight distribution and  $Z$  an offspring distribution. Suppose that  $f : \mathbb{N} \rightarrow \mathbb{R}^+$  is a speed for  $Z$ . Then  $W \in \mathcal{W}_{\text{MS}}(Z)$  if and only if  $W$  is  $\sigma$ -summable for the sequence  $\sigma = (f(n))_{n \in \mathbb{N}}$ .*

**PROOF.** Since  $f$  is a speed for  $Z$ , the event

$$R := \{Z_{n/a} \leq f(n) \leq Z_{bn} \text{ for all } n\}$$

occurs with positive probability. Let  $\sigma^a$  be the sequence given by  $\sigma_n^a = f(an)$ , and  $\sigma^b$  the sequence defined by  $\sigma_n^b = f(\lfloor n/b \rfloor)$ . Suppose  $W$  is  $\sigma$ -summable; then by Lemma 2.2,  $W$  is  $\sigma^b$ -summable. Whenever  $R$  occurs,  $Z_n \geq \sigma_n^b$  for all  $n$ , and hence  $T_Z$  has the min-summability property almost surely. Thus,  $W \in \mathcal{W}_{\text{MS}}(T_Z)$  with positive probability, and hence  $W \in \mathcal{W}_{\text{MS}}(Z)$ .

Conversely, if  $W$  is not  $\sigma$ -summable, then again by Lemma 2.2, it is not  $\sigma^a$ -summable. Thus, even when conditioning on survival,  $W \notin \mathcal{W}_{\text{MS}}(T_Z)$  with positive probability, and hence  $W \notin \mathcal{W}_{\text{MS}}(Z)$ .  $\square$

*Definition of a speed function for plump distributions  $Z$ .* We are now in a position to partially explain the mysterious function  $h$  defined in (1.4), which recall was defined by

$$h(0) = m_0 \quad \text{and} \quad h(n+1) = F_Z^{-1}(1 - 1/h(n)).$$

It will turn out that this function defines a speed function for the offspring distribution  $Z$  in the sense of Definition 2.1.

**THEOREM 2.4.** *If the offspring distribution  $Z$  is plump, then the function  $h$  is a speed of  $Z$ .*

Although it is possible to present a proof at this stage, to avoid redundancy we postpone it until Section 3.

It will actually be convenient in our proofs to consider a slight variation on  $h$ . Let  $\alpha = (1 + \epsilon)^{-1/2}$ , and define  $f$  by

$$(2.1) \quad f(0) = \tilde{m}_0 \quad \text{and} \quad f(n+1) = F_Z^{-1}(1 - f(n)^{-\alpha}),$$

where  $\tilde{m}_0$  is the least integer such that Condition (1.3) holds with  $m_0 = \tilde{m}_0^\alpha$ , and the following inequalities hold:  $\tilde{m}_0^{1-\alpha} \geq 16(1-\alpha)^{-1} + 16$  and  $\tilde{m}_0^{\alpha^{-1}-1} \geq 4^{\lceil(\alpha^{-1}-1)^{-1}\rceil+1}$ .

The functions  $h$  and  $f$  are essentially equivalent as far as we are concerned. The following lemma demonstrates their equivalence as speeds.

**LEMMA 2.5.** *For any plump distribution  $Z$ ,  $h$  is a speed for  $Z$  if and only if  $f$  is.*

**PROOF.** Since  $h$  is increasing, for some constant  $c$  we have  $h(c) \geq \tilde{m}_0 = f(0)$ . Inductively, we then have  $f(n) \leq h(n+c)$  for all  $n$ . Since  $Z$  is plump, we have from the definition of  $f$  that

$$f(n+1) \geq f(n)^{\alpha(1+\epsilon)} = f(n)^{1/\alpha} \quad \text{for any } n.$$

Thus,

$$f(n+2) = F_Z^{-1}(1 - f(n+1)^{-\alpha}) \geq F_Z^{-1}(1 - f(n)^{-1}).$$

It follows that if  $f(n) \geq h(m)$ , then  $f(n+2) \geq h(m+1)$ . So by induction, we have  $f(2n) \geq h(n)$ .

Considering the definition of a speed for  $Z$ , we see that if one is a speed, so is the other.  $\square$

In the following lemma, we state some direct consequences of Condition (1.3) (i.e. the assumption  $Z$  is plump), and the definition of  $f$ , that will be helpful later.

LEMMA 2.6. *Let  $Z$  be a plump distribution and let  $f(n)$  be defined as in (2.1).*

(i) *For all  $n$ ,*

$$(2.2) \quad f(n+2) \geq F_Z^{-1}(1 - 1/f(n)).$$

(ii)  *$f(n+1) \geq 4^{n+1}f(n)$  for all  $n \geq 0$ . In particular,  $f(n)^{1-\alpha} \geq 16n + 16$  for all  $n \geq 1$ , and for any positive  $r$ ,  $f(n) = \Omega(r^n)$ .*

(iii) *For each  $k \geq 2$  and for all  $n$ ,*

$$(2.3) \quad f(n + 2\lceil \log k / \log(1 + \epsilon) \rceil) \geq f(n)^k.$$

PROOF. Part (i) follows immediately from the proof of Lemma 2.5. To prove Part (ii) we begin by noting that the ratio  $f(n+1)/f(n)$  is at least  $f(n)^{\alpha^{-1}-1}$ , as  $\alpha(1+\epsilon) = \alpha^{-1}$ . We therefore prove that  $f(n)^{\alpha^{-1}-1} \geq 4^{n+1}$  for all  $n$ . Let  $n_0 = \lceil (\alpha^{-1} - 1)^{-1} \rceil$ , and note that since  $\tilde{m}_0^{\alpha^{-1}-1} \geq 4^{\lceil (\alpha^{-1}-1)^{-1} \rceil + 1}$ , the inequality  $f(n)^{\alpha^{-1}-1} \geq 4^{n+1}$  holds trivially for  $n \leq n_0$ . For  $n > n_0$ , the result follows easily by induction as

$$f(n)^{\alpha^{-1}-1} \geq (4^n f(n-1))^{\alpha^{-1}-1} = 4^{(\alpha^{-1}-1)n} f(n-1)^{\alpha^{-1}-1} \geq 4f(n-1)^{\alpha^{-1}-1}.$$

To conclude the proof of Part (ii), we have to show  $f(n)^{1-\alpha} \geq 16n + 16$  for all  $n$ . For  $n \leq (1-\alpha)^{-1}$ , we trivially have

$$f(n)^{1-\alpha} \geq f(0)^{1-\alpha} = \tilde{m}_0^{1-\alpha} \geq 16(1-\alpha)^{-1} + 16.$$

For  $n \geq (1-\alpha)^{-1} + 1$ , we have  $f(n)^{1-\alpha}/f(n-1)^{1-\alpha} \geq 4$ , and the result easily follows by induction.

To prove Part (iii), we note that

$$f(n+2) = F_Z^{-1}(1 - 1/f(n)) \geq f(n)^{1+\epsilon}.$$

An inductive argument now easily yields that

$$f(n+2\ell) \geq f(n)^{(1+\epsilon)^\ell}$$

for any  $n$  and  $\ell$ . It follows that  $f(2n) \geq m_0^{(1+\epsilon)^n}$ . We conclude by setting  $\ell = \lceil \log k / \log(1 + \epsilon) \rceil$ .  $\square$

**3. Proof of the Equivalence Theorem.** In this section we prove the Equivalence Theorem. We first prove it in the second (technically weaker) form and then describe how the first form may be deduced.

Let  $Z$  be a plump offspring distribution, and let  $\epsilon$  and  $m_0$  be such that Condition (1.3) holds for the triple  $Z$ ,  $\epsilon$ , and  $m_0$ . Fix an arbitrary  $W \in \mathcal{W}_{\text{MS}}(Z)$ . We shall prove that  $W \in \mathcal{W}_{\text{EX}}(Z)$  (and the theorem will follow). We define an algorithm which selects a path in the tree in a very precise way; then using the properties of  $W$ , we prove that with positive probability this path is an exploding path. Since, conditioned on survival, the event that there is an exploding path is a 0-1 event, this is enough to prove the theorem.

The algorithm depends on a parameter  $\alpha$ , defined in the previous section:  $\alpha := (1 + \epsilon)^{-\frac{1}{2}}$ . The reason for this choice of exponent will be clarified later in the proof.

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**Algorithm FINDPATH:**

Let  $x_0$  be the root of the tree.

**For**  $n = 0, 1, 2, \dots$ ,

- Consider node  $x_n$ , which is the lowest node in the candidate exploding path we are constructing. Let  $Y_{n+1}$  denote the number of children of  $x_n$ .
  - Order the children of  $x_n$  by how many children they in turn have, from largest to smallest. Let  $X_{n+1} := \lceil (Y_{n+1})^{(1-\alpha)}/2 \rceil$ . We define the *options* from  $x_n$  to be the first  $X_{n+1}$  children of  $x_n$  in the ordering.
  - If  $X_{n+1} = 0$ , the algorithm terminates in failure. Otherwise, of the  $X_{n+1}$  choices, pick the option whose edge from  $x_n$  has the smallest weight, and set  $x_{n+1}$  to be this child.
- 

The analysis of the algorithm, and the proof that it provides with positive probability an exploding path, will be based on the following assertion.

**CLAIM 3.1.** *There exists a positive integer  $a$  such that, with positive probability,  $Z_n \leq f(an)$  and  $Y_n \geq f(n)$  hold simultaneously for all  $n \in \mathbb{N}$ , where  $f$  is the function defined in Equation (2.1).*

Indeed, given this, we may deduce immediately that with positive probability  $Z_{n/a} \leq f(n) \leq Z_n$  for all  $n \in \mathbb{N}$ , implying that  $f(n)$  is a speed of  $Z$ . Furthermore, since  $X_n$ , the number of options of  $x_{n-1}$ , is defined by



$X_n = \lceil Y_n^{(1-\alpha)}/2 \rceil$ , there is a positive probability that  $X_n \geq f(n - \gamma)$  for all  $n \in \mathbb{N}$ , where  $\gamma = 2\lceil \log(1 - \alpha)^{-1}/\log(1 + \epsilon) \rceil + 1$  (this follows from Lemma 2.6 (iii)).

We now observe that conditional on the inequality  $X_n \geq f(n - \gamma)$  holding for all  $n \in \mathbb{N}$ , the path  $x_0, x_1, x_2, \dots$  is an exploding path almost surely. The distribution of the sum of weights along the path  $x_0, x_1, x_2, \dots$ , dependent on  $X_1, X_2, X_3, \dots$ , is given by

$$\sum_{n \geq 1} \min \{ W_n^1, \dots, W_n^{X_n} \},$$

where the  $W_n^j$  are i.i.d with distribution  $W$ . Thus, conditional on the event that  $X_n \geq f(n - \gamma)$  for all  $n \in \mathbb{N}$ , this sum is stochastically smaller than  $\sum_{n \geq 1} \min \{ W_n^1, \dots, W_n^{f(n-\gamma)} \}$ . Moreover, Lemma 2.2 implies that  $W$  is  $\sigma$ -summable for the sequence  $\sigma = (f(n))_{n \in \mathbb{N}}$ , and since the contribution of any finite number of terms is finite,  $W$  is also  $\sigma$ -summable for the sequence  $\sigma = (f(n - \gamma))_{n \in \mathbb{N}}$ . This proves that  $x_0, x_1, x_2, \dots$  is an exploding path almost surely.

So it remains to prove Claim 3.1, which we will do for the choice  $a = 3 + 2\lceil \log 2/\log(1 + \epsilon) \rceil$ .

Define the two families of events  $\{A_n\}_{n \geq 1}$  and  $\{B_n\}_{n \geq 1}$  by

$$A_n := \{Y_n < f(n)\}, \quad B_n := \{Z_n > f(an)\}.$$

We are led to prove that there is a positive probability that none of the events  $A_n$  or  $B_n$  occur. Let  $C = A_1^c \cap B_1^c$ . The definition of  $f$  implies that  $Z$  assigns a positive probability to the range  $[f(1), f(a)]$ , so that  $\mathbb{P}\{C\} > 0$ . We will show below that

$$(3.1) \quad \mathbb{P}\{A_2 \mid C\} \leq 1/16 \quad \text{and} \quad \mathbb{P}\{A_{n+1} \mid A_n^c\} \leq 4^{-n-1} \quad \text{for } n \geq 2;$$

$$(3.2) \quad \mathbb{P}\{B_2 \mid C\} \leq 1/16 \quad \text{and} \quad \mathbb{P}\{B_{n+1} \mid B_n^c\} \leq 4^{-n-1} \quad \text{for } n \geq 2.$$

Assuming the above inequalities, we infer that

$$\begin{aligned} \mathbb{P}\left\{C \cap \bigcap_{n \geq 1} A_{n+1}^c\right\} &= \mathbb{P}\{C\} \prod_{n \geq 1} \mathbb{P}\left\{A_{n+1}^c \mid A_n^c, A_{n-1}^c, \dots, A_2^c, C\right\} \\ &= \mathbb{P}\{C\} \mathbb{P}\{A_2^c \mid C\} \prod_{n \geq 2} \mathbb{P}\{A_{n+1}^c \mid A_n^c\} \\ &\quad \text{(Since the sequence } Y_1, Y_2, Y_3, \dots \text{ is Markovian)} \\ &\geq \left(1 - \sum_{n \geq 1} 4^{-n-1}\right) \mathbb{P}\{C\}. \end{aligned}$$

In the same way, we obtain  $\mathbb{P}\left\{C \cap \bigcap_{n \geq 1} B_{n+1}^c\right\} \geq (1 - \sum_{n \geq 1} 4^{-n-1})\mathbb{P}\{C\}$ . Since both the events  $C \cap \bigcap_{n \geq 1} A_{n+1}^c$  and  $C \cap \bigcap_{n \geq 1} B_{n+1}^c$  are contained in  $C$ , we conclude that with positive probability none of the events  $A_n$  and  $B_n$  occur, finishing the proof of the claim.

All that remains is to prove inequalities (3.1) and (3.2). We first prove the bound on  $\mathbb{P}\{A_{n+1} | A_n^c\}$  (it will be seen that the bound on  $\mathbb{P}\{A_2 | C\}$  follows by the same proof). Call a child of  $x_n$  *good* if it has at least  $f(n+1)$  children, and write  $G_n$  for the number of good children of  $x_n$ . We note that given  $Y_n$ , the distribution of  $G_n$  is  $\text{Bin}(Y_n, p)$  where  $p$ , the probability that a given child is good, is at least  $1 - F_Z(f(n+1)) = f(n)^{-\alpha}$ . By the way the algorithm chooses the vertex  $x_{n+1}$ , we also note that  $A_{n+1}$  can occur only if  $G_n < Y_n^{1-\alpha}/2$ . Thus, conditional on  $Y_n \geq f(n)$ , if  $A_{n+1}$  occurs then

$$G_n < Y_n^{1-\alpha}/2 \leq Y_n f(n)^{-\alpha}/2 \leq \mathbb{E}\{G_n\}/2.$$

Hence

$$\begin{aligned} \mathbb{P}\{A_{n+1} | A_n^c\} &\leq \mathbb{P}\left\{G_n \leq \frac{Y_n^{1-\alpha}}{2} \mid Y_n \geq f(n)\right\} \\ &\leq \exp\left(\frac{-f(n)^{1-\alpha}}{8}\right) \\ &\leq \frac{1}{4^{n+1}}. \end{aligned} \quad (\text{By Lemma 2.6 (ii)})$$

We now prove  $\mathbb{P}\{B_{n+1} | B_n^c\} \leq 4^{-(n+1)}$  (the proof bounding  $\mathbb{P}\{B_2 | C\}$  being identical). Note that by Lemma 2.6 (iii),

$$f(an+a) \geq f(an)f(an+3).$$

Thus in order for the event  $Z_{n+1} \geq f(an+a)$  to occur, conditional on  $Z_n \leq f(an)$ , there must be some node in generation  $n$  having at least  $f(an+3)$  children. Taking  $Z(i)$  to be an independent copy of  $Z$  for each  $i$ , the probability of this is bounded as follows:

$$\begin{aligned} \mathbb{P}\{\max\{Z(1), \dots, Z(f(an))\} > f(an+3)\} &\leq f(an) \mathbb{P}\{Z > f(an+3)\} \\ &\leq f(an)(1 - F_Z(f(an+3))) \\ &\leq f(an)f(an+1)^{-1} \\ &\quad (\text{By Lemma 2.6 (i)}) \\ &\leq \frac{1}{4^{n+1}}. \end{aligned} \quad (\text{By Lemma 2.6 (ii)})$$

The proof of the Equivalence Theorem (in its second form) is complete. Note that in the process, we have also proved that  $f$  is a speed of  $Z$ ; thus by Lemma 2.5, Theorem 2.4 also follows.

*First form of the Equivalence Theorem.* One might hope that the first form of the Equivalence Theorem could be deduced from the second by some very simple reasoning, perhaps considering for each weight distribution  $W$  the set of trees  $T$  for which  $\mathcal{W}_{\text{EX}}(T) \neq \mathcal{W}_{\text{MS}}(T)$ . However, the fact that there are uncountably many possible weight distributions seems to be problematic for such a direct approach.

Taking  $T$  to be a random Galton-Watson tree with offspring distribution  $Z$  conditioned to survive, we will prove that the following chain of containments holds almost surely:

$$\mathcal{W}_{\text{MS}}(T) \subseteq \mathcal{W}_{\text{MS}}(Z) \subseteq \mathcal{W}_{\text{EX}}(T).$$

From this the Equivalence Theorem in its first form immediately follows.

That the first inclusion holds almost surely follows from the fact that the rate of growth of generation sizes of  $T$  may almost surely be bounded in terms of the speed  $f$  of  $Z$ . Specifically, taking  $a = 3 + 2\lceil \log 2 / \log(1 + \epsilon) \rceil$  as in Claim 3.1, we will show that almost surely there exists a constant  $c$  such that  $Z_n \leq f(an + c)$  for all  $n$ . For  $z \in \mathbb{N}$ , let  $r(z)$  denote the greatest  $r$  for which  $z \geq f(r)$ . If no bound of the form  $Z_n \leq f(an + c)$  holds, then there must be infinitely many  $n$  for which  $r(Z_{n+1}) > r(Z_n) + a$ . However, our proof of (3.2) demonstrates that the probability that  $Z_{n+1} \geq f(r + a)$  given that  $Z_n \leq f(r)$  is at most  $4^{-r}$ . Since  $f$  is a speed of  $Z$ , the sequence of probabilities  $4^{-r(Z_n)}$  is summable almost surely, and so this event has probability zero.

That the second inclusion holds almost surely follows from the fact that we may apply the above algorithmic approach to finding an exploding path to any rooted subtree of  $T$  which survives. For a node  $v$ , let  $T_v$  denote the subtree of its descendants. Denote by  $s(n)$  the number of nodes of generation  $n$  for which  $T_v$  is infinite. As  $T$  is conditioned on survival, the function  $s(n)$  is unbounded almost surely [4, Ch. 10–12]. Let now  $W \in \mathcal{W}_{\text{MS}}(Z)$ . The above algorithm, applied independently to each node of generation  $n$  for which  $T_v$  is infinite, has positive probability  $p > 0$  of producing an exploding path in each. Thus the probability of no exploding path is at most  $(1 - p)^s$  for all  $s$ , and so is 0.

*The set of weights of infinite rooted paths.* The following theorem characterizes the set of all possible values the weights of infinite rooted paths can

take conditioned on the survival of the Galton-Watson tree. Note that the theorem is valid in general and does not require the plumpness condition.

**THEOREM 3.2.** *Let  $Z$  be an offspring distribution and  $W$  a nonnegative weight distribution which is not a.s. zero. Then almost surely conditioned on survival the set of weights of infinite rooted paths is  $[A, \infty]$  where  $A$  is the infimum weight of infinite rooted paths.*

**PROOF.** By applying the transformation discussed in the introduction if necessary, we may assume that  $W$  has no atom at zero. Note that clearly the transformation does not change the weights of infinite rooted paths.

The theorem is clearly true if  $W \notin \mathcal{W}_{\text{EX}}(Z)$  since in this case, conditioned on survival, all infinite rooted paths have infinite weight. So in the following we assume  $W \in \mathcal{W}_{\text{EX}}(Z)$ .

By a straightforward compactness argument, it suffices to show that for any  $\epsilon' > 0$ , there exists (almost surely) an infinite path with weight in  $[a, a + \epsilon']$ , for all  $a \geq A$ .

Let  $\epsilon \leq \epsilon'/4$  be such that  $\mathbb{P}\{W \in (\epsilon, 2\epsilon)\} > 0$ ; such an  $\epsilon$  must exist since  $W \in \mathcal{W}_{\text{EX}}(Z)$  and  $W$  has no atom at zero. Define the *path-weight*  $\text{pw}(v)$  of a node  $v$  to be the sum of the edge weights on the path from  $v$  to the root. Now let

$$S_i = \{v \in T \mid \text{pw}(v) \in [i\epsilon, (i+1)\epsilon]\}.$$

The choice of  $\epsilon$  is such that if  $v \in S_i$ , then for any given child  $w$  of  $v$ ,  $w \in S_{i+1} \cup S_{i+2}$  with a constant positive probability.

Since explosion occurs, there is some least integer  $\ell$  such that  $S_\ell$  is infinite; we then have  $A \geq \ell\epsilon$ . We may explore  $S_0, S_1, \dots$  in turn, each time uncovering all of  $S_i$ , as well as all children of nodes in  $S_i$ . In the process of exploring  $S_\ell$ , each node we explore whose parent is in  $S_\ell$  will have a constant positive probability of being in  $S_{\ell+1} \cup S_{\ell+2}$ ; thus a.s. at least one of  $S_{\ell+1}$  and  $S_{\ell+2}$  is infinite too. Moreover, since explosion occurs, each such node will have a positive probability of being the root of an infinite path of length at most  $\epsilon$ . Thus  $S_{\ell+1} \cup S_{\ell+2} \cup S_{\ell+3}$  must contain an infinite path a.s. Continuing inductively, we find that a.s. for any integer  $j \geq \ell$ , one of the sets  $S_j$  or  $S_{j+1}$  should be infinite, and there is an infinite path of total weight in  $[j\epsilon, (j+4)\epsilon]$ .

Now choosing  $j$  such that  $a \in [j\epsilon, (j+1)\epsilon]$ , we infer the existence of an infinite path with length in the interval  $[a, a + 4\epsilon] \subseteq [a, a + \epsilon']$ .  $\square$

**4. Equivalent conditions for min-summability.** In the previous section, we proved an equivalence theorem between explosion and min-summability for branching processes with plump offspring distributions.

Though, the existence of such a result is certainly nice in its own, one may wonder if the property of min-summability is in any sense substantially simpler than that of explosion. The aim of this section is to answer this question in the affirmative by proving Theorem 1.3, which provides a necessary and sufficient condition for min-summability that involves a calculation based only on the distributions. We then provide some examples at the end of this section.

Let  $W$  be a random variable taking values in  $[0, \infty)$  and let  $\sigma = (\sigma_i)_{i \geq 0}$  be a sequence of positive integers. Then we have

PROPOSITION 4.1. *The nonnegative random variable  $W$  is  $\sigma$ -summable if and only if the following two conditions are satisfied:*

$$(i) \quad \sum_n (\mathbb{P}\{W > 1\})^{\sigma_n} < \infty, \quad \text{and}$$

$$(ii) \quad \sum_n \int_0^1 (\mathbb{P}\{W > t\})^{\sigma_n} dt < \infty.$$

PROOF. As in Section 2, let  $W_n^j$  be an independent copy of  $W$  for each  $n, j \in \mathbb{N}$  and let

$$\Lambda_n := \min_{1 \leq j \leq \sigma_n} W_n^j.$$

Clearly,  $\Lambda_n$  is a sequence of nonnegative and independent random variables. By Kolmogorov's three-series theorem (see, e.g., Kallenberg [24] or Petrov [29]), we have  $\sum_n \Lambda_n < \infty$  almost surely if and only if

$$\sum_n \mathbb{P}\{\Lambda_n > 1\} < \infty,$$

$$\sum_n \mathbb{E}\{\Lambda_n \mathbf{1}_{[\Lambda_n \leq 1]}\} < \infty,$$

and

$$\sum_n \text{Var}\{\Lambda_n \mathbf{1}_{[\Lambda_n \leq 1]}\} < \infty.$$

Since  $W$  is nonnegative, random variables  $\Lambda_n \mathbf{1}_{[\Lambda_n \leq 1]}$  take value in  $[0, 1]$ , and so the third condition follows from the second one. Now,  $\mathbb{P}\{\Lambda_n > 1\} = (\mathbb{P}\{W > 1\})^{\sigma_n}$ , and  $\mathbb{E}\{\Lambda_n \mathbf{1}_{[\Lambda_n \leq 1]}\} = \left(\int_0^1 (\mathbb{P}\{W > t\})^{\sigma_n} dt\right) - \mathbb{P}\{\Lambda_n > 1\}$ , thus proving the theorem.  $\square$

In the case of a random integer sequence given by the generation sizes, it is also possible to give a result analogous to Proposition 4.1 (whose proof is omitted).

PROPOSITION 4.2. *Let  $\{Z_n\}$  be a Galton-Watson process with an offspring distribution  $Z$ , satisfying  $Z \geq 1$  almost surely. Let  $\Lambda_n$  be the minimum weight of the  $n$ -th generation. We have*

$$\mathbb{P}\left\{\sum_n \Lambda_n < \infty\right\} = 1$$

*if and only if the following two conditions are satisfied*

$$(i) \quad \mathbb{P}\left\{\sum_n (\mathbb{P}\{W > 1\})^{Z_n} < \infty\right\} = 1, \quad \text{and}$$

$$(ii) \quad \mathbb{P}\left\{\sum_n \int_0^1 (\mathbb{P}\{W > t\})^{Z_n} dt < \infty\right\} = 1.$$

*Otherwise,  $\mathbb{P}\{\sum_n \Lambda_n < \infty\} = 0$ .*

The two above propositions are likely the most general form of necessary and sufficient conditions on min-summability one may hope for. However, under some extra conditions on the sequence  $\sigma$ , it is possible to unify the two Conditions of Proposition 4.1 into one single and simpler condition.

COROLLARY 4.3. *Let  $\sigma$  be a sequence of integers such that there exists  $c > 1$  with the property that for all large enough values of  $n$ ,  $\sigma_{n+1} \geq c \cdot \sigma_n$  (think of the speed function  $f$ , see Lemma 2.6). Then  $W$  is  $\sigma$ -summable if and only if  $\sum_n F_W^{-1}(\frac{1}{\sigma_n}) < \infty$ .*

PROOF. Note that, under the assumption of the corollary on the growth of  $\sigma_n$ , Condition (i) of Proposition 4.1 always holds, provided that  $\mathbb{P}\{W > 1\} < 1$ .

Let  $\sigma$  be a sequence satisfying the condition  $\sigma_{n+1} \geq c \cdot \sigma_n$  for all  $n$ . Let  $a_0 = 0$  and  $a_n = F_W^{-1}(\frac{1}{\sigma_n})$  for  $n \geq 1$ , and suppose that  $\sum_{n \geq 0} a_n < \infty$ . In this case, trivially  $\mathbb{P}\{W > 1\} < 1$ . We show that Condition (ii) of Proposition 4.1 holds. We have

$$\begin{aligned} \int_0^1 (\mathbb{P}\{W > t\})^{\sigma_n} dt &= \int_0^{a_{n-1}} (\mathbb{P}\{W > t\})^{\sigma_n} dt + \int_{a_{n-1}}^1 (\mathbb{P}\{W > t\})^{\sigma_n} dt \\ &\leq a_{n-1} + \sum_{m=1}^n a_{m-1} ((\mathbb{P}\{W > a_m\})^{\sigma_n} - (\mathbb{P}\{W > a_{m-1}\})^{\sigma_n}) \\ &\leq a_{n-1} + \sum_{m=1}^n a_{m-1} (1 - 1/\sigma_m)^{\sigma_n}. \end{aligned}$$

Thus,

$$\begin{aligned}
\sum_n \int_0^1 (\mathbb{P}\{W > t\})^{\sigma_n} dt &\leq \sum_n a_n + \sum_m a_{m-1} \sum_{n \geq m} (1 - 1/\sigma_m)^{\sigma_n} \\
&\leq \sum_n a_n + \sum_m a_{m-1} \sum_{n \geq m} (1 - 1/\sigma_m)^{e^{n-m} \sigma_m} \\
&\leq \sum_n a_n + \sum_m a_{m-1} \sum_{j=0}^{\infty} e^{-c^j} \\
&= O(1) \sum_n a_n < \infty.
\end{aligned}$$

This shows that  $W$  is  $\sigma$ -summable.

To prove the other direction, suppose that  $W$  is  $\sigma$ -summable, so that by Proposition 4.1,

$$\sum_n \int_0^1 (\mathbb{P}\{W > t\})^{\sigma_n} dt < \infty.$$

Since  $W$  is  $\sigma$ -summable, we have  $F_W(1) > 0$  and so there exists an integer  $N$  such that for  $n \geq N$ ,  $a_n \leq 1$ . Thus,

$$\begin{aligned}
\sum_n \int_0^1 (\mathbb{P}\{W > t\})^{\sigma_n} dt &\geq \sum_{n \geq N} \int_0^{a_n} (\mathbb{P}\{W > t\})^{\sigma_n} dt \\
&\geq \sum_{n \geq N} \int_0^{a_n} (1 - \mathbb{P}\{W \leq a_n\})^{\sigma_n} dt \\
&= \sum_{n \geq N} \int_0^{a_n} \left(1 - \frac{1}{\sigma_n}\right)^{\sigma_n} dt \\
&= \Omega(1) \sum_{n \geq N} a_n.
\end{aligned}$$

It follows that  $\sum_n a_n < \infty$  and the corollary follows.  $\square$

Combining the above corollary with Theorem 2.4 and Proposition 2.3, we infer a proof of Theorem 1.3.

*Examples and special cases.* Here we give a family of examples of applications of Proposition 4.1. The notations are those of Proposition 4.1. (In particular,  $\Lambda_n$  is the minimum of  $\sigma_n$  copies of the weight distribution  $W$ .)

- (i) If  $W \geq a > 0$ , then Condition (ii) of Proposition 4.1 does not hold, and so  $\sum_n \Lambda_n = \infty$ . (This also trivially follows from  $\Lambda_n \geq a$ .) This example shows that the only interesting cases occur when 0 is an accumulation point of the distribution.
- (ii) If  $W = 0$  with probability  $p > 0$ , then both the conditions of Proposition 4.1 hold if  $\sum_n (1-p)^{\sigma_n} < \infty$ . On the other hand,  $\sum_n \Lambda_n < \infty$  implies that  $\sum_n (1-p-\epsilon)^{\sigma_n} < \infty$  for every  $\epsilon \in (0, p)$ . This case is not of prime interest either. The case  $p = 0$  with 0 being an accumulation point of  $W$  is the most interesting.
- (iii) If  $W$  is uniform on  $[0, 1]$ , then the conditions of Proposition 4.1 are equivalent to

$$\sum_n \frac{1}{\sigma_n + 1} < \infty.$$

- (iv) If  $W$  is exponential, then  $\Lambda_n \stackrel{\mathcal{L}}{=} E/\sigma_n$ , where  $E$  is exponential. The sequence  $\Lambda_n$  has almost surely a finite sum if and only if

$$\sum_n \frac{1}{\sigma_n} < \infty.$$

- (v) For the sequence  $\sigma_n = n$ , assuming that there is no atom at the origin and that 0 is an accumulation point for  $W$ , it is easy to verify that  $\sum_n \Lambda_n < \infty$  almost surely if and only if

$$\int_0^1 \frac{1}{\mathbb{P}\{W > t\}} dt < \infty.$$

- (vi) For the sequence  $\sigma_n \sim c^n$ , with  $c > 1$  a positive constant, and assuming no atom at the origin, but with 0 an accumulation point for  $W$ , it is easy to verify that  $\sum_n \Lambda_n < \infty$  almost surely if and only if

$$\int_0^1 \ln \left( \frac{1}{\mathbb{P}\{W > t\}} \right) dt < \infty.$$

**5. Sharpness of the condition in the Equivalence Theorem.** The main result of this article, the Equivalence Theorem, gives a sufficient condition on a distribution  $Z$  for the equality  $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$  to occur. This condition, that for some  $\epsilon > 0$ , the inequality  $\mathbb{P}\{Z \geq m^{1+\epsilon}\} \geq 1/m$  holds for all sufficiently large  $m \in \mathbb{N}$ , demands that  $Z$  has a heavy tail, and, furthermore, that the tail is consistently heavy. This condition ensures that the generation sizes (equivalently, the speed) of the corresponding branching process are at least double exponential. Furthermore, it ensures that the



rate of growth is always at least the rate associated with double exponential functions (i.e.  $f(n+1) \geq f(n)^{1+\epsilon}$ ). It is therefore natural to ask:

- (i) Could a weaker version of our condition still imply  $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$ ?
- (ii) Could a lower bound on the speed of  $Z$  alone (e.g.,  $Z$  has a speed  $f$  which is at least double exponential) be sufficient to guarantee  $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$ ?

Theorem 5.1 answers (i) in the negative (almost completely) by showing that no substantially weaker version of our condition implies  $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$ . Theorem 5.2 answers (ii), completely, in the negative. In a sense, these results show the Equivalence Theorem to be best possible.

**THEOREM 5.1.** *Let  $g : \mathbb{N} \rightarrow \mathbb{N}$  be an increasing function satisfying  $g(m) = m^{1+o(1)}$ . Then there is a distribution  $Z$ , satisfying  $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$  for all  $m \in \mathbb{N}$ , but for which  $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$ .*

**THEOREM 5.2.** *Let  $s : \mathbb{N} \rightarrow \mathbb{N}$  be any function. Then there is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , satisfying  $f(n) \geq s(n)$  for all  $n \in \mathbb{N}$ , and a distribution  $Z$  for which  $f$  is a speed, such that  $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$ .*

There does not seem to be an obvious intuitive way to judge, for a given distribution  $Z$ , whether the equality  $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$  should hold or not. So before giving our proof of Theorem 5.1, we establish a sufficient condition for the equality to fail, see Proposition 5.4 below.

We recall that a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a speed of a distribution  $Z$  if there exist  $a, b \in \mathbb{N}$  such that with positive probability the bounds  $Z_{n/a} \leq f(n) \leq Z_{bn}$  hold for all  $n$ . We shall say that  $f$  is a *dominating* speed if we may take  $a = 1$ . We shall say that  $f$  is *swift* if, for some  $c > 1$ , the inequality  $f(n+1) > cf(n)$  holds for all  $n \geq 0$ . It will be useful (for technical reasons) to restrict our attention to swift dominating speeds. The following direct consequence of Corollary 4.3 and Proposition 2.3 will be useful in our proof of Proposition 5.4.

**LEMMA 5.3.** *Let  $Z$  be a distribution with mean greater than 1,  $f$  a swift speed of  $Z$ , and  $W$  a weight distribution for which the sum  $\sum_{n=1}^{\infty} F_W^{-1}(f(n)^{-1})$  is bounded. Then  $W \in \mathcal{W}_{\text{MS}}(Z)$ .*

**PROPOSITION 5.4.** *Let  $Z$  be any distribution with a swift dominating speed  $f$  satisfying*

$$(5.1) \quad \liminf_{n \rightarrow \infty} 2^n f(n) f(\lceil n/\omega(n) \rceil)^{-n/2} = 0,$$

*for some function  $\omega(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Then  $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$ .*

PROOF. We must prove the existence of a weight distribution  $W$  such that  $W \in \mathcal{W}_{\text{MS}}(Z)$  but  $W \notin \mathcal{W}_{\text{EX}}(Z)$ . Before defining  $W$ , we first define some sequences on which its definition will be based. From our assumption on  $f$ , there exists an increasing sequence  $n_i$  such that

$$(5.2) \quad \lim_{i \rightarrow \infty} 2^{n_i} f(n_i) f(\lceil n_i / \omega(n_i) \rceil)^{-n_i/2} = 0.$$

Let us define the sequence  $\omega_i$  by  $\omega_i = \omega(n_i)$  and the sequence  $\beta_i$  by  $\beta_i = \sqrt{\omega_i}$ . We note that  $\beta_i \rightarrow \infty$  as  $i \rightarrow \infty$ , and so we may choose a subsequence  $\beta_{i_j}$  with the property that  $\beta_{i_j} \geq 2^j$  for each  $j \geq 1$ . Finally, set  $m_i := \lceil n_i / \omega_i \rceil$ . We now define the weight distribution  $W$  to satisfy

$$\mathbb{P} \left\{ W < \frac{1}{\beta_{i_j} m_{i_j}} \right\} = \frac{1}{f(m_{i_j})} \quad \text{for all } j \geq 1,$$

by placing probability mass  $f(m_{i_j})^{-1} - \sum_{j' > j} f(m_{i_{j'}})^{-1}$  at position  $1/\beta_{i_{j+1}} m_{i_{j+1}}$  for each  $j \geq 1$ , and probability mass  $1 - \sum_{j' \geq 1} f(m_{i_{j'}})^{-1}$  at 1.

We first observe that  $W \in \mathcal{W}_{\text{MS}}(Z)$ . Indeed, this follows immediately from Lemma 5.3 and the observation that

$$\sum_{n \geq 1} F_W^{-1}(f(n)^{-1}) \leq \sum_{j \geq 1} m_{i_j} \cdot \frac{1}{\beta_{i_j} m_{i_j}} \leq \sum_{j \geq 1} \frac{1}{\beta_{i_j}} \leq \sum_{j \geq 1} \frac{1}{2^j} = 1.$$

We now observe that  $W \notin \mathcal{W}_{\text{EX}}(Z)$ . We must prove that  $\mathbb{P}\{E\} < 1$ , where  $E$  denotes the event of an infinite path of finite weight. Let  $G$  be the event that  $Z_n \leq f(n)$  for all  $n \in \mathbb{N}$ ; since  $f$  is a dominating speed of  $Z$ ,  $G$  has positive probability. Thus it suffices to prove that  $\mathbb{P}\{E \mid G\} = 0$ .

Let  $A_j$  be the event that there exists a path from the root to generation  $n_{i_j}$  of weight less than  $\beta_{i_j}/2$ . The event  $E$  may occur only if  $A_j$  occurs for all sufficiently large  $j$ , so it suffices to prove that  $\mathbb{P}\{A_j \mid G\} \rightarrow 0$  as  $j \rightarrow \infty$ .

For the event  $A_j$  to occur there must exist a path from the root to generation  $n_{i_j}$  at least half of whose edges have weight less than  $\beta_{i_j}/n_{i_j}$ . Since under event  $G$  there are at most  $f(n_{i_j})$  such paths, and for each path there are less than  $2^{n_{i_j}}$  choices for a subset of half its edges, we have

$$\mathbb{P}\{A_j \mid G\} \leq 2^{n_{i_j}} f(n_{i_j}) (\mathbb{P}\{W < \beta_{i_j}/n_{i_j}\})^{n_{i_j}/2}.$$

Since

$$\mathbb{P}\{W < \beta_{i_j}/n_{i_j}\} = \mathbb{P}\{W < 1/(\beta_{i_j} m_{i_j})\} = 1/f(m_{i_j}),$$

it follows from (5.2) that  $\mathbb{P}\{A_j \mid G\} \rightarrow 0$  as required.  $\square$

PROOF OF THEOREM 5.1. Let  $g$  be any increasing function satisfying the condition of the theorem, i.e.,  $g(m) = m^{1+o(1)}$ . We define a distribution  $Z$  satisfying  $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$  for all  $m \in \mathbb{N}$ , which has a swift dominating speed  $f$  satisfying  $\liminf_{n \rightarrow \infty} 2^n f(n) f(\lceil n^{1/2} \rceil)^{-n/2} = 0$ ; the proof is then complete by Proposition 5.4.

There is a sense in which it is difficult to achieve these two objectives simultaneously. The first asks that  $Z$  has a sufficiently heavy tail, while the second would seem to get more likely to occur if the tail of  $Z$  were less heavy. Our approach to achieving the objectives simultaneously is to define  $Z$  to have a heavy, but not at all smooth, tail. In the resulting Galton-Watson branching process the growth of generation sizes does not at all resemble a smooth fast growing function (such as a double exponential), but instead consists of a number of periods of exponential growth, each period much longer than all preceding periods, and with a multiplicative factor very much larger (in fact the lengths will be  $(2n_i)_{i \geq 1}$  and the multiplicative factors  $(m_i)_{i \geq 1}$ ; these sequences are defined below)

Define  $n_i = 10^{10^i}$  for each  $i \geq 1$ , and  $\epsilon_i = 1/10n_i = 10^{-(10^i+1)}$ . As  $g(m) = m^{1+o(1)}$ , there exists, for each  $\epsilon_i$ , a natural number  $m_i$  such that  $g(m) \leq m^{1+\epsilon_i}$  for all  $m \geq m_i^{1/2}$ . Furthermore, we may choose  $(m_i)_{i \in \mathbb{N}}$  to in addition satisfy

$$(5.3) \quad m_i \geq 16n_i^2 M_{i-1}^2 \quad \text{for all } i \geq 1,$$

where  $M_0 = 1$  and  $M_j := \prod_{i=1}^j m_i^{2n_i}$  for  $j \geq 1$ . Next define sequences  $(N_j)_{j \in \mathbb{N}}$  and  $(L_j)_{j \in \mathbb{N}}$  by

$$N_j := \sum_{i=1}^j n_i \quad \text{and} \quad L_j := m_j \prod_{i=1}^{j-1} m_i^{2n_i}.$$

As we mentioned above, we shall define the distribution  $Z$  so that the growth of generation sizes of  $T_Z$  consists of a number of periods of exponential growth, each period much longer than all preceding periods, and with multiplicative factor very much larger. (The  $j$ th period of growth will have length (approximately)  $2n_j$  and multiplicative factor  $m_j$ .) In this context  $L_j$  is approximately the generation size at the start of this  $j$ th period of growth (in fact after the first step of this period) and  $M_j$  the generation size when it ends (i.e., at the point at which we shall switch into the next, faster, period of growth). One may observe that  $L_j = m_j M_{j-1}$ ; note however that  $L_j$  is much larger than  $M_{j-1}$ , since (5.3) implies that  $m_j$  is already much larger.

Define the distribution  $Z$  by

$$\begin{aligned}\mathbb{P}\{Z \geq L_1\} &= 1; \\ \mathbb{P}\{Z \geq m^{1+\epsilon_i}\} &= \frac{1}{m}, \quad L_i^{1/(1+\epsilon_i)} < m \leq M_i, \quad i \geq 1; \\ \mathbb{P}\{Z \geq L_{i+1}\} &= \frac{1}{M_i}, \quad i \geq 1.\end{aligned}$$

It is easily verified that this distribution satisfies  $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$  for all  $m \in \mathbb{N}$ . Now define the function  $f : \mathbb{N} \rightarrow \mathbb{N}$  (which will be a speed for  $Z$ ) by

$$f(n) = L_{i+1}m_{i+1}^{2(n-N_i)-1} \quad \text{with } i \text{ chosen so that } N_i < n \leq N_{i+1}.$$

It is also quite easily verified that  $f$  satisfies (5.1), using  $\omega(n) = n^{1/2}$ . In particular, we observe that  $f(n_i) \leq L_i m_i^{2n_i}$ , and, since  $\lceil n_i^{1/2} \rceil - N_{i-1} \geq n_{i-1}$ , we have that  $f(\lceil n_i^{1/2} \rceil^{n_i/2}) \geq L_i m_i^{n_i - 1n_i}$ . It is also easily observed that  $f$  is swift. Thus, in light of Proposition 5.4, all that is required to complete the proof is to demonstrate that  $f$  is a dominating speed of  $Z$ . Though it is conceptually straightforward, the proof is rather long; we stress that it is really just a technical detail.

We prove that with positive probability the bounds  $Z_n \leq f(n) \leq Z_{4n}$  hold for all  $n \in \mathbb{N}$ . Let  $E$  be the event that  $Z_n > f(n)$  for some  $n$ , and let  $F$  be the event that  $Z_{4n} < f(n)$  for some  $n$ . Let us subdivide these events by the minimum  $n$  for which the required inequality fails. Let  $E_n$  to be the event that  $n$  is minimal such that  $Z_n > f(n)$ , and  $F_n$  the event that  $n$  is minimal such that  $Z_{4n} < f(n)$ . We will show that  $\sum_{n \geq 1} E_n \leq 1/4$  and  $\sum_{n \geq 1} F_n \leq 1/4$ , which will complete the proof.

We have stated that our example is designed to exhibit a number of periods of exponential growth. Once the number of nodes of a given generation is much larger than  $M_{i-1}$ , it is clear that, from this point on, the growth should always be at least geometric (i.e., exponential) with multiple  $m_i$ . Indeed, among  $m \gg M_{i-1}$  nodes, one expects about  $m/M_{i-1}$  to have  $L_i = m_i M_{i-1}$  children. Considering these children alone we see that the size of the next generation should be at least  $m_i$  times as large.

Our bound on the probability of the event  $F$  is therefore relatively straightforward, requiring us to formalize the above statement. The bound on the probability of  $E$  is more difficult as we are required to control all ways in which the process could grow faster.

CLAIM.  $\mathbb{P}\{E\} \leq 1/4$ .

PROOF. We shall define two sequences  $p_{i,j,k}$  and  $q_i$  of probabilities, corresponding to the probabilities of certain unlikely events (events that would cause faster than expected growth). We then prove a bound on the probability of  $E$  based on the  $p_{i,j,k}$  and  $q_i$ , specifically that this probability is at most their sum. It then suffices to bound by  $1/4$  the sum  $\sum_{i,j,k} p_{i,j,k} + \sum_i q_i$ .

For each triple  $i, j, k \in \mathbb{N}_0$  such that  $i \geq 1$ ,  $1 \leq j \leq n_i - 1$  and  $0 \leq k \leq 4j$ , we define  $p_{i,j,k}$  to be the probability that amongst  $M_{i-1}m_i^{2j}$  independent copies of  $Z$ , at least  $M_{i-1}m_i^{k/2}$  exceed  $M_{i-1}m_i^{2j+1-k/2}$ . We define  $q_1$  to be the probability that  $Z \geq m_1^2$ , and, for  $i \geq 2$ , we define  $q_i$  to be the probability that amongst  $M_{i-1}$  copies of  $Z$ , at least one of them exceeds  $M_{i-1}m_i^{3/2}$ .

We prove the bound

$$\mathbb{P}\{E\} = \sum_{n \geq 1} \mathbb{P}\{E_n\} \leq \sum_{i,j,k} p_{i,j,k} + \sum_i q_i.$$

Notice that for the event  $E_{N_{i-1}+1}$  to occur, we must have

$$Z_{N_{i-1}} \leq f(N_{i-1}) = M_{i-1} \quad \text{and} \quad Z_{N_{i-1}+1} > f(N_{i-1} + 1) = M_{i-1}m_i^2.$$

This in turn implies that at least one of the nodes in generation  $N_{i-1}$  has more than  $M_{i-1}m_i^{3/2}$  children (as  $M_{i-1} \leq m_i^{1/2}$ , see Condition (5.3)). Thus we may bound for each  $i$  the probability of the event  $E_{N_{i-1}+1}$  by  $q_i$ .

Next, for  $n$  of the form  $N_{i-1} + j + 1$  for some  $i \in \mathbb{N}$  and  $1 \leq j \leq n_i - 1$ , we note that the occurrence of  $E_n$  implies that

$$Z_{n-1} \leq M_{i-1}m_i^{2j} \quad \text{and} \quad Z_n > M_{i-1}m_i^{2j+2}.$$

It follows that for some  $0 \leq k \leq 4j$ , there are at least  $M_{i-1}m_i^{k/2}$  nodes of generation  $n - 1$  with more than  $M_{i-1}m_i^{2j+1-k/2}$  children. Indeed, if this were not the case, then we would have

$$\begin{aligned} Z_n &\leq \sum_{k=0}^{4j} (M_{i-1}m_i^{k/2})(M_{i-1}m_i^{2j+3/2-k/2}) \\ &= (4j+1)M_{i-1}^2m_i^{2j+3/2} \\ &\leq M_{i-1}m_i^{2j+2} \quad (\text{since } (4j+1)M_{i-1} \leq 4n_iM_{i-1} \leq m_i^{1/2}). \end{aligned}$$

It easily follows that  $\mathbb{P}\{E_n\} \leq \sum_{0 \leq k \leq 4j} p_{i,j,k}$ .

We now prove the bound  $\sum_{i,j,k} p_{i,j,k} + \sum_i q_i \leq 1/4$ . By the bounds (5.3) it suffices to prove for each triple  $i, j, k \in \mathbb{N}_0$  with  $i \geq 1$ ,  $1 \leq j \leq n_i - 1$  and  $0 \leq k \leq 4j$ , that

$$(5.4) \quad p_{i,j,k} \leq (m_i/e^2)^{-M_{i-1}m_i^{k/2}/2}$$

and

$$q_i \leq \frac{M_{i-1}}{m_i}.$$

The bound on  $q_i$  is trivial; since  $1/(1 + \epsilon_i) \geq 2/3$ , it follows that

$$\mathbb{P}\{Z \geq M_{i-1}m_i^{3/2}\} = (M_{i-1}m_i^{3/2})^{-1/(1+\epsilon_i)} \leq m_i^{-1}.$$

We bound the probability  $p_{i,j,k}$  (that amongst  $M_{i-1}m_i^{2j}$  independent copies of  $Z$  at least  $M_{i-1}m_i^{k/2}$  exceed  $M_{i-1}m_i^{2j+1-k/2}$ ) using a union bound. By the familiar estimate  $\binom{s}{t} \leq (es/t)^t$ , the number of choices of the set of  $M_{i-1}m_i^{k/2}$  copies is

$$\binom{M_{i-1}m_i^{2j}}{M_{i-1}m_i^{k/2}} \leq (em_i^{2j-k/2})^{M_{i-1}m_i^{k/2}}.$$

For each copy of  $Z$  we have

$$\mathbb{P}\{Z > M_{i-1}m_i^{2j+1-k/2}\} = (M_{i-1}m_i^{2j+1-k/2})^{-1/(1+\epsilon_i)} \leq m_i^{-(2j+1/2-k/2)},$$

where for the final inequality we have used that  $\epsilon_i = 1/(10n_i)$  and (since  $2j + 1/2 - k/2 \leq 2n_i$ )

$$2j + 1 - k/2 = 2j + 1/2 - k/2 + 1/2 \geq (2j + 1/2 - k/2)(1 + 1/(4n_i)).$$

Thus the probability that a given set of  $M_{i-1}m_i^{k/2}$  copies of  $Z$  all exceed  $M_{i-1}m_i^{2j+1-k/2}$  is at most

$$m_i^{-(2j+1/2-k/2)M_{i-1}m_i^{k/2}},$$

and (5.4) now follows by a union bound.  $\square$

CLAIM 5.5.  $\sum_{n \geq 1} \mathbb{P}\{F_n\} \leq 1/4$ .

PROOF. Our approach is similar to that used in the previous proof. For  $i \geq 1$  and  $2 \leq j \leq 4n_i$ , we define  $p_{i,j}$  to be the probability that from a collection of  $M_{i-1}m_i^{j/2}$  copies of  $Z$ , fewer than  $M_{i-1}m_i^{j/2-1/2}$  exceed  $m_i$ . For each  $i \geq 1$ , we define  $q_i$  to be the probability that the maximum of  $M_i m_i^{1/2}$  copies of  $Z$  is less than  $L_{i+1}$ . We prove for  $n$  of the form  $n = N_i + 1$  that

$$\mathbb{P}\{F_n\} \leq p_{i,4n_i} + q_i + p_{i+1,2} + p_{i+1,3},$$

and for  $n$  of the form  $n = N_i + k$ ,  $k = 2, \dots, n_{i+1}$ , that

$$\mathbb{P}\{F_n\} \leq p_{i+1,4k-4} + p_{i+1,4k-3} + p_{i+1,4k-2} + p_{i+1,4k-1}.$$

It will then suffice to bound by  $1/4$  the sum  $\sum_{i,j} p_{i,j} + \sum_i q_i$ . For  $n = N_i + k$ ,  $k = 2, \dots, n_{i+1}$ , if the event  $F_n$  occurs then  $Z_{4n-4} \geq f(n-1) = M_i m_{i+1}^{2k-2}$  and  $Z_{4n} < f(n) = M_i m_{i+1}^{2k}$ . The required bound now follows, as the probability for a given  $0 \leq l \leq 3$  that  $l$  is minimal such that  $Z_{4n-l} < M_i m_{i+1}^{2k-l/2}$  is at most  $p_{i+1,4k-l-1}$ . The case  $n = N_i + 1$  is similar, differing only in that we do not consider the events  $Z_{4n-l} < M_i m_{i+1}^{2k-l/2}$  for  $0 \leq l \leq 3$ , but rather the events  $Z_{4n-3} < M_i m_i^{1/2}$ ,  $Z_{4n-2} < L_{i+1}$ ,  $Z_{4n-1} < L_{i+1} m_i^{1/2}$  and  $Z_{4n} < L_{i+1} m_i$ .

Finally we prove the bound  $\sum_{i,j} p_{i,j} + \sum_i q_i < 1/4$ . It is trivial, using the inequality  $(1-p)^n \leq e^{-pn}$ , that  $q_i \leq \exp(-\sqrt{m_i})$ . To bound  $p_{i,j}$  we first note that  $\mathbb{P}\{Z > m_i\} \geq 1/M_{i-1}$ , so from a collection of  $M_{i-1} m_i^{j/2}$  copies of  $Z$  the distribution for the number exceeding  $m_i$  is  $\text{Bin}(M_{i-1} m_i^{j/2}, 1/M_{i-1})$ . Since this binomial has expected value  $m_i^{j/2} \geq 2M_{i-1} m_i^{j/2-1/2}$ , an application of Chernoff's inequality yields

$$p_{i,j} \leq \exp\left(\frac{-m_i^{j/2}}{8}\right).$$

□

The proof of Theorem 5.1 is now complete. □

The proof of Theorem 5.2 is essentially identical to the above. The only change required is that the following extra condition should be included in (5.3):

$$m_i \geq \max_{n \leq n_i} s(n) \quad i \geq 1.$$

This ensures that the inequality  $f(n) \geq s(n)$  holds for all  $n \in \mathbb{N}$ . Since the proofs that  $f$  is a speed of  $Z$  and that  $\mathcal{W}_{\text{EX}}(Z) \neq \mathcal{W}_{\text{MS}}(Z)$  are unaffected by this change, Theorem 5.2 does indeed follow.

**6. Limit theorem in the case of no explosion.** So far we only considered the appearance of the event of explosion. In this section, we consider the case of weight distributions for a heavy-tailed branching random walk for which explosion does not happen, and obtain a precise limit theorem for the minimum displacement  $M_n$  under some quite strong (smoothness) assumption on the tails of  $Z$ . To explain this, let  $Z$  be a plump random variable, and denote by  $G_Z(\cdot)$  the moment generating function of  $Z$  as before.

Note that

$$\begin{aligned}
 K_Z(s) &= 1 - G_Z(1-s) = \sum_{k=0}^{\infty} \left( \mathbb{P}\{Z = k\} - (1-s)^k \mathbb{P}\{Z = k\} \right) \\
 &= s \sum_{k=1}^{\infty} \mathbb{P}\{Z = k\} (1 + \dots + (1-s)^{k-1}) \\
 (6.1) \quad &= s \left( 1 - \mathbb{P}\{Z = 0\} + \sum_{k=1}^{\infty} (1-s)^k (1 - F_Z(k)) \right).
 \end{aligned}$$

Consider now the smoothness Condition (1.5) on  $Z$ :

$$1 - F_Z(k) = k^{-\eta} \ell(k),$$

for some function  $\ell$  which is continuous-bounded-and-non-zero at infinity. In particular, note that one can define  $\ell(\infty) \neq 0, \infty$ . Using Equation (6.1) and applying a Tauberian theorem (see for example Feller [18, XIII.5, Thm. 5]), we see that Condition (1.5) is equivalent to the condition

$$(\star) \quad K_Z(s) \sim a s^\eta \ell\left(\frac{1}{s}\right)$$

near  $s = 0$  for some  $a > 0$  (indeed,  $a = \Gamma(1 - \eta)$ ). This in particular implies that  $Z$  is plump and

$$(\star\star) \quad F_Z^{-1}\left(1 - \frac{1}{m}\right) = m^{1+\epsilon} \tilde{\ell}(m),$$

for a slowly growing function  $\tilde{\ell}$  and  $1 + \epsilon = \eta^{-1}$ . We have

**THEOREM 6.1.** *Let  $Z$  be an offspring distribution satisfying  $(\star)$ . Let  $W$  be a nonnegative weight distribution and assume that  $W \notin \mathcal{W}_{\text{EX}}(Z)$ . Conditional on the survival of the Galton-Watson process,*

$$\lim_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\left(\frac{1}{h(k)}\right)} = 1.$$

Here  $h(k) = \exp((1 + \epsilon)^k)$ , where  $\epsilon$  is as in  $(\star\star)$  and  $\eta = (1 + \epsilon)^{-1}$  as in  $(\star)$ .

The proof will essentially use the algorithm we presented in Section 3. However, we first need to obtain a more precise information on the speed of the Galton-Watson tree under Condition  $(\star)$ .



DEFINITION 6.2 (Additive speed). An increasing function  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  is an *additive speed* for a Galton-Watson offspring distribution  $Z$  if the probability of the increasing events  $E_r$  defined as

$$E_r := \left\{ h(n-r) \leq Z_n \leq h(n+r) \quad \text{for all large enough } n \right\}$$

tend to one as  $r$  goes to infinity conditional on survival.

LEMMA 6.3. *Let  $Z$  be an offspring distribution satisfying Condition  $(\star)$ . Then the function  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  defined by  $h(n) = \exp((1+\epsilon)^k)$  is an additive speed for  $Z$ .*

PROOF OF THEOREM 6.1. Since  $h(n)$  is an additive speed for  $Z$ , we obtain by Lemma 6.3 that conditional on survival,

$$\lim_{r \rightarrow \infty} \mathbb{P}\{E_r\} = 1.$$

Fix the integer  $r$  and suppose the event  $E_r$  holds. This means  $Z_n \leq h(n+r)$  for large enough  $n$ . This implies that the minimum of level  $n$  is at least  $F_W^{-1}(\frac{1}{h(n+r)})$  for all large enough  $n$ . Since by our Equivalence Theorem we have a.s.  $\sum F_W^{-1}(1/h(n)) = \infty$ , we obtain

$$\liminf_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}(\frac{1}{h(k)})} = \liminf_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}(\frac{1}{h(k+r)})} \geq 1,$$

on  $E_r$ . We infer that on the union of  $E_r$ , i.e., on the event of non-extinction, we have

$$\liminf_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}(\frac{1}{h(k)})} \geq 1.$$

We now show that on the union of  $E_r$ , we have

$$\limsup_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}(\frac{1}{h(k)})} \leq 1.$$

This will finish the proof of the theorem above.

It will be enough to show this on each  $E_r$ . In addition, we can also fix an  $n_0$  and suppose that for all  $n \geq n_0$ , we have  $Z_n \geq h(n-r)$  (and then make  $n_0$  tend to infinity). Fix a small  $\delta > 0$ . One can now apply a variant of the algorithm of Section 3, by modifying  $\alpha$  to  $(1+\epsilon)^{-\delta}$ , started at some large  $N > n_0$ , and show that w.h.p., as  $N$  goes to infinity, we have for all

$n \geq N$ ,  $X_n \geq h((1 - \delta)n)$  (this follows from a variant of the inequalities (3.1) and (3.2)). In addition, given the double exponential growth of  $h(n)$ , a union bound argument shows that we can assume with height probability that for large enough  $n$ , the weight of the  $n$ -th edge on the path constructed in the algorithm is bounded above by  $F_W^{-1}(1/h((1 - 2\delta)n))$ . Applying now the Equivalence Theorem, since both  $M_n$  and  $\sum_{k=1}^n F_W^{-1}(\frac{1}{h((1-2\delta)k)})$  tend to infinity, we obtain that

$$\limsup_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}(\frac{1}{h((1-2\delta)k)})} \leq 1.$$

Since this holds for any small enough  $\delta > 0$ , and since the function  $F_W^{-1}(1/m)$  is a decreasing function of  $m$ , a simple argument shows that

$$\limsup_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}(\frac{1}{h(k)})} = \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}(\frac{1}{h((1-2\delta)k)})} \leq 1.$$

The theorem follows.  $\square$

PROOF OF LEMMA 6.3. Under some extra conditions on  $\ell$  as in Seneta [31] or [32], a combination of the results of Darling [14] and Cohn [13] with the above mentioned results of Seneta [31, 32] ensures the existence of a limiting random variable  $V$  such that

$$(1 + \epsilon)^{-n} \log(Z_n + 1) \rightarrow V \quad \text{almost surely,}$$

for  $V$  having a strictly increasing continuous distribution  $v$ ,  $V > 0$  a.s. on the set of non-extinction of the process, and  $v(0+) = q$ , where  $q$  is the extinction probability of the Galton-Watson process. In the general case of a function  $\ell$  continuous bounded and non-zero at infinity, the above limit theorem still holds as we now briefly explain by following closely Bramson's strategy in [10]. Define  $\alpha = 1 + \epsilon = \eta^{-1}$ . The general idea in proving such a limit theorem is to prove first the convergence of the sequences  $K^{(n)}(\exp(-\alpha^n s))$  uniformly on compact sets. Here,  $K^{(n)}(\cdot) = K_Z^{(n)}(\cdot) = K_{Z_n}(\cdot)$  is the  $n$ -times composition of  $K_Z$  (and  $K_Z$  is as in Equation (6.1)). For this, define

$$H(s) := -\log K(\exp(-s)),$$

and notice that  $H^{(n)}(s) = -\log K^{(n)}(\exp(-s))$ , so that we are left to prove the convergence of the sequence  $H^{(n)}(\alpha^n s)$  as  $n$  goes to infinity, for  $s \geq 0$ .

By an abuse of the notation (from Condition  $(\star)$ ), assume that  $K_Z(s) = s^\eta \ell(\frac{1}{s})$  for a function  $\ell$  continuous bounded and non-zero at infinity, and define

$$L(s) = -\log \ell(\exp(s)).$$

By the assumptions on  $\ell$ , it follows that  $L$  is continuous at infinity and  $L(\infty) \neq \pm\infty$ , and so for each  $a > 0$ , there is an  $N_a$  such that for  $s_1$  and  $s_2$  larger than  $N_a$ , we have  $|L(s_1) - L(s_2)| \leq a$ . A simple induction shows that

$$(6.2) \quad H^{(m)}(\alpha^m s) = s + \sum_{k=1}^m \frac{1}{\alpha^{m-k}} (-1)^k L\left(H^{(k-1)}(\alpha^{m-k+1} s)\right).$$

By the definition of  $H$ , one can easily verify that  $H$  is 1-Lipschitz, i.e.,

$$\text{for any two } s_1, s_2 \geq 0, \quad |H(s_1) - H(s_2)| \leq |s_1 - s_2|.$$

We now show that the sequence  $\{H^{(n)}(\alpha^n s), n \in \mathbb{N}\}$  is Cauchy, proving the point-wise convergence. The same argument shows that the sequence is uniformly Cauchy on compact intervals of  $[0, \infty)$  concluding the proof of the uniform convergence.

Fix a large  $m \in \mathbb{N}$  and note that replacing  $s$  by  $\alpha^n s$  in (6.2), we get

$$H^{(m)}(\alpha^{n+m} s) = \alpha^n s + \sum_{k=1}^m \frac{1}{\alpha^{m-k}} (-1)^k L\left(H^{(k-1)}(\alpha^{m-k+1+n} s)\right).$$

We claim that as  $n$  goes to infinity each term  $H^{(k-1)}(\alpha^{m-k+1+n} s)$  tends to infinity. Indeed, more precisely, the rate of convergence to infinity of this term is as  $\alpha^{n+m-2k+2} s + O(1)$ ; this can be shown by a simple induction from (6.2), using the bounded continuity of  $L$  at infinity.

For two fixed  $m$  and  $M$ , we have

$$\begin{aligned} \left| H^{(m)}(\alpha^{n+m} s) - H^{(M)}(\alpha^{n+M} s) \right| &= \left| \sum_{k=1}^m \frac{1}{\alpha^{m-k}} (-1)^k L\left(H^{(k-1)}(\alpha^{m-k+1+n} s)\right) \right. \\ &\quad \left. - \sum_{k=1}^M \frac{1}{\alpha^{M-k}} (-1)^k L\left(H^{(k-1)}(\alpha^{M-k+1+n} s)\right) \right|. \end{aligned}$$

For  $n$  large enough, we can assume that each term  $L(H^{(k-1)}(\alpha^{m-k+1+n} s))$  differs from  $L(\infty)$  by an arbitrary small positive number  $a$ . It follows then

$$\begin{aligned} \left| H^{(m)}(\alpha^{n+m} s) - H^{(M)}(\alpha^{n+M} s) \right| &\leq a \left[ \sum_{k=1}^m \frac{1}{\alpha^{m-k}} + \sum_{k=1}^M \frac{1}{\alpha^{M-k}} \right] \\ &\quad + \left| \sum_{k=1}^m \frac{1}{\alpha^{m-k}} (-1)^k L(\infty) - \sum_{k=1}^M \frac{1}{\alpha^{M-k}} (-1)^k L(\infty) \right|. \end{aligned}$$

Since  $\alpha > 0$  and  $L(\infty) < \infty$ , and  $a$  can be chosen arbitrarily small, obviously the right term of the above inequality can be made arbitrarily small, provided that  $n$  is sufficiently large and the constants  $m$  and  $M$  are large enough. We conclude that for any  $a > 0$ , there exist integer constants  $N_a$  and  $M_a$  such that

$$\begin{aligned} |H^{(n+m)}(\alpha^{n+m}s) - H^{(n+M)}(\alpha^{n+M}s)| &\leq |H^{(m)}(\alpha^{n+m}s) - H^{(M)}(\alpha^{n+M}s)| \\ &\leq a \end{aligned}$$

for any  $n$  larger than  $N_a$ , provided that  $m$  and  $M$  are larger than  $M_a$ . This shows that the sequence is Cauchy. In the same way, we can easily prove that the sequence is uniformly Cauchy on compact subsets of  $[0, \infty)$ . This shows the existence of a continuous limit  $w$  for the sequence  $H^{(n)}(\alpha^n s)$ .

We now show that  $w$  is strictly increasing and  $w(\infty) = \infty$ . For this, note that for  $s_1 < s_2$ , the above arguments show that for large enough  $m$  and  $n$ , one has  $H^{(m)}(\alpha^{n+m}s_i) = \alpha^n s_i + O(1)$ . In particular for  $n$  large enough constant and for all  $m$ ,  $H^{(m)}(\alpha^{n+m}s_2) - H^{(m)}(\alpha^{n+m}s_1) > \frac{1}{2}\alpha^n(s_2 - s_1)$ . Since  $H$  is itself strictly increasing, and so  $H^{(n)}$  is, one conclude that the limit  $w$  is strictly increasing. A similar argument shows that  $w(\infty) = \infty$ .

Finally, we observe that  $w(0^+) = -\log(1 - q)$ . This follows from a simple fixed point argument: fix an  $s > 0$  and note that

$$\begin{aligned} w(0^+) &= \lim_{m \rightarrow \infty} w(\alpha^{-m}s) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} H^{(m)}H^{(n-m)}(\alpha^{n-m}s) \\ &= \lim_{m \rightarrow \infty} H^{(m)}(w(s)), \end{aligned}$$

by the continuity of  $H^{(m)}$  for each fixed  $m$ .

Since  $H^{(m)}(w(s)) = -\log K_Z^{(m)}(\exp(-w(s)))$  and  $w(s) \geq 0$ , it follows easily that for each  $s > 0$ , when  $m$  goes to infinity,  $H^{(m)}(w(s))$  tends to the unique finite fixed point of  $H$ . This is  $-\log(1 - q)$ , a consequence of the corresponding statement for  $K^{(m)}$  given that the unique fixed point of  $K_Z$  in  $(0, 1)$  is  $1 - q$ .

These then allow us to conclude the proof of the above convergence result by first proceeding as in Darling [14] to obtain the convergence in distribution, and next by applying the result of Cohn [13] to obtain the almost sure convergence.

To conclude the proof of the lemma, note that for two constants  $\delta, \Delta > 0$ ,  $\delta < \Delta$ , the event

$$E_{\delta, \Delta} := \left\{ \delta(1 + \epsilon)^n \leq \log(Z_n + 1) \leq \Delta(1 + \epsilon)^n \quad \text{for large enough } n \right\}$$

happens with a probability tending to  $1 - q$  as  $\delta \rightarrow 0$  and  $\Delta \rightarrow \infty$ . For two fixed constants  $\delta$  and  $\Delta$ , we have for  $r$  large enough,  $(1 + \epsilon)^{-r} \leq \delta$  and  $(1 + \epsilon)^r \geq \Delta$ . This shows that the event  $E_{\delta, \Delta}$  is contained in the event  $E_r$  for  $r$  sufficiently large, and the lemma follows.  $\square$

**7. Conclusion.** We have proved the equivalence of  $\mathcal{W}_{\text{EX}}(Z)$  and  $\mathcal{W}_{\text{MS}}(Z)$  for plump offspring distributions  $Z$ , and shown that the plumpness condition is essentially best possible, in terms of conditions of the form  $F_Z(1 - 1/m) \geq g(m)$ . However, this is very far from being a characterization of all offspring distributions for which explosion and min-summability are equivalent. For example, a simple adaptation of the proof of the Equivalence Theorem shows that  $\mathcal{W}_{\text{EX}}(Z) = \mathcal{W}_{\text{MS}}(Z)$  for  $Z$  defined by

$$\mathbb{P} \left\{ Z \geq m \exp \left( \exp \left( \log \log m - \sqrt{\log \log m} + \frac{1}{2} \log \log \log m \right) \right) \right\} = \frac{1}{m}.$$

The function

$$f(n) = e^{e^{\log^2 n}}$$

is a speed of  $Z$ . This illustrates that the equivalence can occur for distributions with speeds very much slower than doubly exponential. By contrast, any plump distribution has a speed that grows at least as fast as a double exponential.

We remark that the above example is extremely close to best possible. It follows from Proposition 5.4 that the equivalence cannot hold for an offspring distribution which has a speed of the form

$$f(n) = e^{e^{o(\log^2 n)}}.$$

We do not know how general the equivalence of  $\mathcal{W}_{\text{EX}}(Z)$  and  $\mathcal{W}_{\text{MS}}(Z)$  should be when  $Z$  has speed slower than doubly exponential. Obtaining a complete characterization of offspring distributions where equivalence occurs remains an interesting open question.

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