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LIMIT THEOREMS FOR WEIGHTED NON-LINEAR TRANSFORMATIONS OF GAUSSIAN STATIONARY PROCESSES WITH SINGULAR SPECTRA

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The limit Gaussian distribution of multivariate weighted functionals of non-linear transformations of Gaussian stationary processes, having multiple singular spectra, is derived, under very general conditions on the weight function. This paper is motivated by its potential applications in non-linear regression, and asymptotic inference on non-linear functionals of Gaussian stationary processes with singular spectra.

1. Introduction. During the last thirty years, a number of papers have been devoted to limit theorems for non-linear transformations of Gaussian processes and random fields. The pioneering results are those of Taqqu [29, 30] and Dobrushin and Major [8], for convergence to Gaussian and non-Gaussian distributions, under long range dependence, in terms of Hermite expansions, as well as Breuer and Major [5], Ivanov and Leonenko [17], Chambers and Slud [7], on convergence to the Gaussian distribution by using diagram formulae or graphical methods. This line of research continues to be of interest today, see Berman [4] for m-dependent approximation approach, Ho and Hsing [13] for martingale approach, Nualart and Peccati [23] (see also Peccati and Tudor [27]) for the application of Malliavin calculus, Nourdin and Peccati [20] in relation to Stein's method and exact Berry-Esseen asymptotics for functionals of Gaussian fields, Avram, Leonenko and Sakhno [2] for an extension of graphical method for random fields, to name only a few papers. The volume of Doukhan, Oppenheim and Taqqu [9] contains outstanding surveys of the field. Limit theorems for weighted functionals of stochastic processes, and for processes with seasonalities were considered by a number of authors, including Rosenblatt [28], Oppenheim *et al.* [24], Haye [11], and their references. Limit theorems for non-linear transformations of vector Gaussian processes have been obtained by Arcones [1] (see also his references).

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In this paper, our main result (Theorem 2) states the convergence to the Gaussian distribution of the multivariate weighted functionals of non-linear transformations $\psi(\xi(t))$ of Gaussian stationary processes $\xi(t)$, with multiple singularities in their spectra, having covariance function (c.f.) belonging to a parametric family defined in assumption **A2** below. Here, $\psi \in L_2(\mathbb{R}, \varphi(x)dx)$ (see Assumption **A3** in Section 2), where $\varphi(x) = e^{-\frac{x^2}{2}}/\sqrt{2\pi}$, $x \in \mathbb{R}$, denotes the standard Gaussian probability density. Specifically, under suitable conditions, the convergence to the Gaussian distribution of

$$\boldsymbol{\zeta}_T = W_T^{-1} \int_0^T \mathbf{w}(t)\psi(\xi(t))\nu(dt), \tag{1}$$

as $T \to \infty$, is obtained for certain ranges of the parameters defining the spectral singularities of ξ (see assumption **A4** in the next section). For each $T > 0$,

$$\mathbf{w}(t) = (w_1(t), \ldots, w_q(t))', \; W_T^2 = diag\left(W_{iT}^2\right)_{i=1}^q, \; W_{iT}^2 = \int_0^T w_i^2(t)\nu(dt), \; i = 1, \ldots, q, \tag{2}$$

where, to ensure a finite limit variance, the weak convergence of the family of matrix measures associated with $\mathbf{w}$ over the intervals $\{[0, T], \; T > 0\}$ is also assumed, jointly with some restrictions on the boundedness of their components in some neighborhoods of the spectral singularities of ξ (see assumptions **B1** and **B2** in Section 4). The convergence to the Gaussian distribution also requires some conditions to be assumed on the norms of components of function $\mathbf{w}$ (see condition **B3** in Section 5).

As commented, the spectral density (s.d.) f of $\xi(t)$ is assumed to display several singularities denoted as $\Xi_{noise} = \{\pm\varkappa_0, \ldots, \pm\varkappa_r\}$, with $0 \leq \varkappa_0 < \varkappa_1 < \cdots < \varkappa_r$. In the case where the weak-sense limit of the measures associated with the multivariate weight function $\mathbf{w}$ is an atomic measure, it is also assumed that its atoms $\Xi_{regr} = \{\delta_1, \ldots, \delta_n\}$ do not intersect with the singularities of f (i.e., $\delta_i \neq \pm\varkappa_j, \; i = 1, \ldots, n, \; j = 0, 1, \ldots, r$). The convergence to the Gaussian distribution then holds with standard normalization.

The nature of the limit results obtained depends on the intersection of the two spectral point sets Ξ_{noise} and Ξ_{regr}. In the discrete case, this phenomenon was discussed by Yajima [31, 32] in some other regression scheme. Otherwise, different normalizing factors must be derived, and new limiting distributions are obtained, for Hermite rank $m \geq 2$. Note that the classical non-central limit theorems (Taqqu [30], and Dobrushin and Major [8]) can be viewed as particular cases of the general setting considered here, when there is an unique singular point in the spectrum of ξ, with $\varkappa_0 = 0$, and $w(t) = 1$. In this case, the measure sequence, constructed from the weight function w, is given in terms of the Fejer kernel, which tends to the delta-measure with atom at zero. Some limiting distributions for the case when the two spectral point sets Ξ_{noise} and Ξ_{regr} are in fact overlapped, in discrete time, can be derived from the papers by Rosenblatt [28], Arcones [1], Oppenheim *et al.* [24] and Haye [11]. In continuous time, the limiting distributions for non-empty set, $\Xi_{noise} \cap \Xi_{regr}$, can be obtained from the paper of Ivanov and Leonenko [18], and the book by Ivanov and Leonenko [17]. This subject will be considered in subsequent papers.

In the derivation of the main result of this paper, the Central Limit Theorem by Peccati and Tudor [27] (see also Nualart and Peccati [23]), for a family of vectors of random variables (r.v.-s) belonging to fixed Wiener chaoses, is applied. The outline of the paper is the following. Motivating examples, as well as preliminary identities, and conditions needed in the derivation of the subsequent results are provided in Section 2. The zero-mean Gaussian random field family considered is embedded into an isonormal process family in Section 3. The conditions

needed for the weak-convergence (in particular, to an atomic measure) of the matrix-valued measures associated with the class of vectorial weight functions studied are established in Section 4. The asymptotic normality of the corresponding weighted functionals of non-linear transformations of zero-mean Gaussian stationary random processes is obtained in Section 5. Section 6 provides the final comments, and our main conjecture on the work developed.

2. Stationary processes with singular spectra. Let us consider simultaneously discrete and continuous time cases in the following development. Specifically, for a stationary process ξ defined on a complete probability space $(\Omega, \mathbb{F}, P)$, the following notation will be followed

$$\xi(t) = \xi(\omega, t) : \Omega \times \mathbb{S} \to \mathbb{R},$$

where $\mathbb{S} = \mathbb{Z}$, for discrete time $t \in \mathbb{Z}$, and $\mathbb{S} = \mathbb{R}$, for continuous time $t \in \mathbb{R}$. Such a process is assumed to be measurable and mean-square continuous in the case of continuous time (see also Assumption **A1** below).

In the definition of integrals, $\nu(dt)$ will represent a counting measure in the case of discrete time (i.e., $\nu(\{t\}) = 1$, $t \in \mathbb{Z}$), and the Lebesgue measure dt, in the case of continuous time (i.e., $\nu(dt) = dt$ if $t \in \mathbb{R}$). According to this notation, the integral

$$\int_0^T g(t)\xi(t)\nu(dt)$$

represents the sum $\sum_{t=1}^{T} \xi(t)g(t)$, for discrete time, and the Lebesgue integral $\int_0^T g(t)\xi(t)dt$, for continuous time, where $g(t)$ is a non-random (measurable for continuous time) function.

Consider now the following motivating example.

Example. Let x be defined in terms of the nonlinear regression model

$$x(t) = g(t, \theta) + \psi(\xi(t)), \quad t \in \mathbb{S}_+, \tag{3}$$

where $\mathbb{S}_+ = \mathbb{R}_+$, for continuous time and $\mathbb{S}_+ = \mathbb{N}$, for discrete time, and with $g(t, \theta) : \mathbb{S}_+ \times \Theta \to \mathbb{R}$ being a continuously differentiable function of an unknown parameter $\theta \in \Theta \subset \mathbb{R}^q$, consider $g_i(t, \theta) = (\partial/\partial\theta_i)g(t, \theta)$, $i = 1, \ldots, q$, such that

$$d_{iT}^2 = \int_0^T [g_i(t, \theta)]^2 \nu(dt) < \infty, \ T > 0, \quad i = 1, \ldots, q, \tag{4}$$

and $\psi(\xi(t))$ represents the noise, with $E\psi(\xi(t)) = 0$. The least squares estimate (LSE) $\hat{\theta}_T$ of an unknown parameter $\theta \in \Theta$, obtained from the observations $x(t)$,$t \in [0, T]$, or $t = 1, ..., T$, is any r.v. $\hat{\theta}_T \in \Theta^c$, having the property

$$Q_T(\hat{\theta}_T) = \inf_{\tau \in \Theta^c} Q_T(\tau), \quad Q_T(\tau) = \int_0^T [x(t) - g(t, \tau)]^2 \nu(dt),$$

where Θ^c is the closure of Θ. Let $\nabla g(t, \theta) = (g_1(t, \theta), ..., g_q(t, \theta))'$ be the column vector-gradient of the function $g(t, \theta)$. We denote $d_T^2(\theta) = diag\left(d_{iT}^2\right)_{i=1}^q$, where d_{iT}^2, $i = 1, \ldots, q$, are

defined by (4). In the theory of statistical estimation of unknown parameter $\theta \in \Theta \subset \mathbb{R}^q$ for the scheme (3), the asymptotic behavior, as $T \to \infty$, of the functional

$$\zeta_T = d_T^{-1}(\theta) \int_0^T \nabla g(t,\theta)\psi(\xi(t))\nu(dt), \tag{5}$$

plays a crucial role, since, under certain number of conditions, the asymptotic distributions of the normalized LSE $d_T(\theta)(\hat{\theta}_T - \theta)$, and properly normalized functional (5) coincide, as $T \to \infty$ (see Ivanov and Leonenko [17, 18]).

In this setting, an interesting case corresponds to $\xi(t)$ to be a Gaussian stationary process with s.d. $f(\lambda)$ displaying singularities at the points $\Xi_{noise} = \{\pm\, \varkappa_j,\ j = 0, 1, 2, ...r\}$, see (7) below. The following nonlinear functions:

$$g(t,\theta) = t^\beta \cos(t\vartheta + \phi), \quad \beta \geq 0,\ \vartheta \in \mathbb{R},\ \phi \in (-\pi, \pi],\ \theta = (\beta, \vartheta, \phi),$$

are of particular interest in applications because they themselves also involve various seasonalities.

Let us consider $\{\xi(t),\ t \in \mathbb{S}\}$ to be a stochastic process satisfying the following assumptions:

A1. Process ξ is a real stationary mean-square continuous Gaussian process with $E\xi(t) = 0$, $E\xi^2(t) = 1$.

A2. The c.f. of ξ is of the form:

$$B(t) = E[\xi(0)\xi(t)] = \sum_{j=0}^{r} A_j B_{\alpha_j,\varkappa_j}(t), \quad t \in \mathbb{S}, \quad r \geq 0, \tag{6}$$

where

$$B_{\alpha_j,\varkappa_j}(t) = \frac{\cos(\varkappa_j t)}{(1+t^2)^{\alpha_j/2}}, \quad 0 \leq \varkappa_0 < \varkappa_1 < \cdots < \varkappa_r, \quad 0 < \alpha_j < 1, \quad t \in \mathbb{S}, \quad j = 0, \ldots, r,$$

$$\sum_{j=0}^{r} A_j = 1, \quad A_j \geq 0, \quad j = 0, \ldots, r.$$

The c.f. $B(t)$, $t \in \mathbb{S}$, admits the following spectral decomposition:

$$B(t) = \int_\Lambda e^{i\lambda t} f(\lambda) d\lambda, \quad t \in \mathbb{R},$$

where the set $\Lambda = (-\pi, \pi]$, in the discrete case ($t \in \mathbb{Z}$), and $\Lambda = \mathbb{R}$, in the continuous case ($t \in \mathbb{R}$), and the s.d. f in the continuous time is of the form:

$$f(\lambda) = \sum_{j=0}^{r} A_j f_{\alpha_j,\varkappa_j}(\lambda), \quad \lambda \in \mathbb{R}, \tag{7}$$

where, for $j = 0, \ldots, r$,

$$f_{\alpha_j,\varkappa_j}(\lambda) = \frac{c_1(\alpha_j)}{2}\left[K_{\frac{\alpha_j-1}{2}}(|\lambda+\varkappa_j|)\,|\lambda+\varkappa_j|^{\frac{\alpha_j-1}{2}} + K_{\frac{\alpha_j-1}{2}}(|\lambda-\varkappa_j|)\,|\lambda-\varkappa_j|^{\frac{\alpha_j-1}{2}}\right], \quad \lambda \in \mathbb{R},$$

with

$$c_1(\alpha) = \frac{2^{(1-\alpha)/2}}{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)},$$

and

$$K_\nu(z) = \frac{1}{2}\int_0^\infty s^{\nu-1}\exp\left\{-\frac{1}{2}\left(s+\frac{1}{s}\right)z\right\}ds, \quad z \geq 0, \quad \nu \in \mathbb{R},$$

being the modified Bessel function of the third kind of order ν or McDonald's function. We also note that $K_{-\nu}(z) = K_\nu(z)$, and for $z \downarrow 0$ $K_\nu(z) \sim \Gamma(\nu)\,2^{\nu-1}z^{-\nu}$, $\nu > 0$.

Thus, as $\lambda \to \pm\varkappa_j$, for $j = 0, \ldots, r$,

$$f_{\alpha_j,\varkappa_j}(\lambda) = \frac{c_2(\alpha_j)}{2}\left[|\lambda+\varkappa_j|^{\alpha_j-1}(1-h_j(|\lambda+\varkappa_j|)) + |\lambda-\varkappa_j|^{\alpha_j-1}(1-h_j(|\lambda-\varkappa_j|))\right],$$

where

$$c_2(\alpha) = \frac{1}{2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}},$$

$$h_j(|\lambda|) = \frac{\Gamma\left(\frac{\alpha_j+1}{2}\right)}{\Gamma\left(\frac{3-\alpha_j}{2}\right)}\left|\frac{\lambda}{2}\right|^{1-\alpha_j} + \frac{\Gamma\left(\frac{\alpha_j+1}{2}\right)}{4\Gamma\left(\frac{3+\alpha_j}{2}\right)}\left|\frac{\lambda}{2}\right|^2 - o\left(|\lambda|^2\right), \quad \lambda \longrightarrow 0, \quad j = 0, \ldots, r.$$

Therefore, the s.d. f has $2r+2$ different singular points (see condition **A2**), when $\varkappa_0 \neq 0$.

A model with discrete time which satisfy condition **A2** can be obtained by using discretization procedure and the formula for s.d. of stationary processes with discrete time of the form:

$$\sum_{k=-\infty}^{\infty} f(\lambda + 2k\pi).$$

We will use the same notation for the s.d. in both cases corresponding to discrete and continuous time.

Similar results can be obtained for c.f.'s of the form

$$R_{\alpha_j,\varkappa_j}(t) = \frac{\cos(\varkappa_j t)}{(1+|t|^{\rho_j})^{\alpha_j}}, \quad \varkappa_j \in \mathbb{R}, \quad 0 < \alpha_j\rho_j < 1, \quad \varkappa_j \neq 0, \quad j = 0, \ldots, r$$

(see again Ivanov and Leonenko [18] for details).

It is well-known that the Hermite polynomials $H_k(x) = (-1)^k e^{\frac{x^2}{2}}\frac{d^k}{dx^k}e^{-\frac{x^2}{2}}$, $k = 0, 1, \ldots$ constitute a complete orthogonal system in the Hilbert space $L_2(\mathbb{R}, \varphi(x)dx)$ of square integrable functions with respect to the standard Gaussian density φ.

A3. Assume that the function $\psi \in L_2(\mathbb{R}, \varphi(x)dx)$, that is, $E\psi^2(\xi(0)) < \infty$, and $C_0(\psi) = E\psi(\xi(0)) = 0$.

Definition 1 *A function $\psi \in L_2(\mathbb{R}, \varphi(x)dx)$ has Hermite rank $Hrank(\psi) = m$, if either $C_1(\psi) \neq 0$ and $m = 1$, or for some $m \geq 2$, $C_1(\psi) = \cdots = C_{m-1}(\psi) = 0$, $C_m(\psi) \neq 0$.*

A4. Either **(i)** $Hrank(\psi) = 1$, $\alpha > 1/2$; or **(ii)** $Hrank(\psi) = m$, $\alpha m > 1$, where $\alpha = \min_{j=0,\ldots,r}\alpha_j$, with α_j, $j = 0, \ldots, r$, introduced in **A2**.

Under condition **A3,** function $\psi(x)$ of $Hrank(\psi) = m$ can be expanded into a Hermite series in the Hilbert space $L_2(\mathbb{R}, \varphi(x)dx)$:

$$\psi(x) = \sum_{k=m}^{\infty}\frac{C_k(\psi)}{k!}H_k(x), \tag{8}$$

or the process $\psi(\xi(t))$ admits a Hermite series expansion in the Hilbert space $L_2(\Omega, \mathbb{F}, P)$:

$$\psi(\xi(t)) = \sum_{k=m}^{\infty} \frac{C_k(\psi)}{k!} H_k(\xi(t)), \tag{9}$$

where

$$C_k(\psi) = \int_{\mathbb{R}} \psi(x) H_k(x) \varphi(x) dx, \quad k \geq 0.$$

3. Some elements of the theory of isonormal processes. In this section, we introduce basic notation, elements and results in relation to Gaussian Hilbert spaces, isonormal processes, and chaos expansions needed for our purposes (see Nualart [22]; Janson [19]; Nualart and Peccati [23]; Peccati and Tudor [27]; Peccati [25]; Nourdin, Peccati, and Réveillac [21], among others).

Definition 2 *Let H be a real separable Hilbert space. The set of r.v.'s $X = \{X(h) : h \in H\}$ is said to be an isonormal process on H if X is a centered $H-$indexed Gaussian family defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and it satisfies*

$$E[X(h)X(g)] = \langle h, g \rangle_H, \quad h, g \in H.$$

Let us now consider a real-valued centered Gaussian process ξ indexed over $\mathbb{S} = \mathbb{R}$. By $\mathcal{E}$ denote the collection of all finite linear combinations of indicator functions of the type $\mathbf{l}_{(-\infty,t]}$, with $t \in \mathbb{R}$. To embed a real-valued centered Gaussian process ξ indexed by $\mathbb{R}$ into some isonormal process X, we introduce a separable Hilbert space H defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle f, h \rangle_H := \sum_{i,j} a_i c_j E\xi(s_i)\xi(t_j), \tag{10}$$

for given functions $f = \sum_i a_i \mathbf{l}_{(-\infty,s_i]}$ and $h = \sum_j c_j \mathbf{l}_{(-\infty,t_j]}$ in H. Thus, for any function $h = \sum_i c_i \mathbf{l}_{(-\infty,t_i]} \in \mathcal{E}$, define

$$X(h) = \sum_i c_i \xi(t_i). \tag{11}$$

Additionally, for any function $h \in H$, $X(h)$ can be defined as the limit in $L_2(\Omega, \mathbb{F}, \mathbb{P})$ of $X(h_n)$ for any sequence $\{h_n\} \subset \mathcal{E}$ convergent to h in H. This sequence may be not unique, but the definition of $X(h)$ does not depend on the choice of the sequence $\{h_n\}$. From this construction, process X is an isonormal process over H defined as $X(\mathbf{l}_{(-\infty,t]}) = \xi(t)$.

When $\mathbb{S} = \mathbb{Z}$, a similar development in terms of sequences leads to the definition of an isonormal process from a Gaussian process ξ on $\mathbb{Z}$. Now, $\mathcal{E}$ denotes the set of all real-valued sequences $h = \{h_l : l \in \mathbb{Z}\}$ such that $h_l \neq 0$ only for a finite number of integers l. The real separable Hilbert space H is then introduced as the closure of the set $\mathcal{E}$ with respect to the scalar product

$$\langle f, h \rangle_H := \sum_{k,l} f_k h_l E\xi(k)\xi(l),$$

for given sequences $f = \{f_k : k \in \mathbb{Z}\}$ and $h = \{h_l : l \in \mathbb{Z}\}$. If $h \in H$, then the series $\sum_{l \in \mathbb{Z}} h_l \xi(l)$ converges in $L_2(\Omega, \mathbb{F}, \mathbb{P})$. Thus, the centered Gaussian family $\{X(h) : h \in H\}$, with

$$X(h) = \sum_{l \in \mathbb{Z}} h_l \xi(l) \tag{12}$$

is an isonormal process over H.

Let X be an isonormal process defined on H as before, i.e., from a centered Gaussian random process ξ. Let us write $\mathcal{H}_0(X) = \mathbb{R}^1$, and $\mathcal{H}_1(X)$ the closed linear subspace of the set of r.v.'s $\{X(h) : h \in H\}$ in the Hilbert space $L_2(\Omega, \mathbb{F}, \mathbb{P})$. Thus,

$$X : H \longrightarrow \mathcal{H}_1(X)$$

$$h \longrightarrow X(h).$$

For any $n \geq 2$, by $\mathcal{H}_n(X)$, the nth Wiener chaos of process X is denoted, that is, the closed subspace of $L_2(\Omega, \mathbb{F}, \mathbb{P})$ generated by the r.v.'s $H_n(Y)$, where $Y \in \mathcal{H}_1(X)$, and $E[Y^2] = 1$, with H_n denoting, as before, the nth Hermite polynomial. Let us now consider the isometry

$$I_n^X : H^{\odot n} \longrightarrow \mathcal{H}_n(X), \tag{13}$$

between the symmetric tensor product $H^{\odot n}$, equipped with the norm $\sqrt{n!}\|\cdot\|_{H^{\otimes n}}$, and the nth Wiener chaos $\mathcal{H}_n(X)$ of X. For any $h \in H^{\otimes n}$, $I_n^X(h)$ is then defined as $I_n^X(h) := I_n^X(\widetilde{h})$, with $\widetilde{h}$ denoting the symmetrization of h. For any $g \in H^{\otimes m}$ and $h \in H^{\otimes n}$,

$$E[I_m^X(g) I_n^X(h)] = \delta_{mn} m! \left\langle \widetilde{g}, \widetilde{h} \right\rangle_{H^{\otimes m}}.$$

The pth contraction of $g = g_1 \otimes \cdots \otimes g_k \in H^{\otimes k}$ and $h = h_1 \otimes \cdots \otimes h_k \in H^{\otimes k}$, designated as $g \otimes_p h$, is the element of $H^{\otimes 2(k-p)}$ given by

$$g \otimes_p h = \langle h_1, g_1 \rangle_H \ldots \langle h_p, g_p \rangle_H \, g_{p+1} \otimes \cdots \otimes g_k \otimes h_{p+1} \otimes \cdots \otimes h_k. \tag{14}$$

The definition can be extended by linearity to any element of $H^{\otimes k}$. Finally, any r.v. $F \in L_2(\Omega, \mathbb{G}, \mathbb{P})$, with σ- field $\mathbb{G}$ generated by the r.v.'s $\{X(h), h \in H\}$, admits an unique chaos decomposition $F = \sum_{k=0}^{\infty} I_k^X(h_k)$, where $h_k \in H^{\odot k}$.

From the constructions (11) and (12) of an isonormal process X from a Gaussian process ξ respectively defined over continuous and discrete time, $\mathcal{H}_n(X)$, $n \geq 1$, coincides with the n-th Wiener chaos associated with ξ, $\mathcal{H}_n(\xi)$, $n \geq 1$. Since, by definitions (11) and (12), $\mathcal{H}_1(X) = \mathcal{H}_1(\xi)$, and, as stated before, the nth Wiener chaos of process X is the closed subspace of $L_2(\Omega, \mathbb{F}, \mathbb{P})$ generated from the evaluation of nth Hermite polynomial H_n over the r.v.'s of the space $\mathcal{H}_1(X) = \mathcal{H}_1(\xi)$.

The next statement is a convenient, for our purposes, modification of Theorem 1 of Peccati and Tudor [27], see also Nualart and Peccati [23] (in the above papers all statements are formulated for positive integers $T \in \{1, 2, ...\}$, but it is easy to see that one can formulate similar results for continuous $T > 0$ as well).

Proposition 1 *Let $\{\xi(t),\ t \in \mathbb{S}\}$ be a centered Gaussian process, and X is the isonormal process constructed from it as given in (11) and (12). Consider the natural numbers: $1 \leq n_1 < n_2 < \cdots < n_d < \infty$, $d \geq 2$, and the set of r.v.'s $\pi_{T,n_j}(\xi) \in \mathcal{H}_{n_j}(\xi)$, where, for $T > 0$, $\pi_{T,n_j}(\xi) = I_{n_j}^X(f_{j,T})$, for certain $f_{j,T} \in H^{\odot n_j}$, $j = 1, \ldots, d$, such that*

$$\lim_{T \to \infty} E\pi_{T,n_j}^2(\xi) = \lim_{T \to \infty} n_j! \|f_{j,T}\|_{H^{\otimes n_j}}^2 = 1, \quad j = 1, \ldots, d. \tag{15}$$

Then, the following conditions are equivalent:

(i) For each $j = 1, \ldots, d$,

$$\lim_{T \to \infty} \|f_{j,T} \otimes_p f_{j,T}\|_{H^{\otimes 2(n_j - p)}} = 0,$$

for every $p = 1, \ldots, n_j - 1$.

(ii) For every $j = 1, \ldots, d$,

$$\lim_{T\to\infty} E\left[\left(I^X_{n_j}(f_{j,T})\right)^4\right] = 3.$$

(iii) As $T \to \infty$, *the vector* $\left(I^X_{n_1}(f_{1,T}), \ldots, I^X_{n_d}(f_{d,T})\right)$ *converges in distribution to a* d*-dimensional standard Gaussian vector* $N_d(0, \mathbb{I}_d)$.

The proof follows from Peccati and Tudor [27], and Nualart and Peccati [23], considering the fact that $\mathcal{H}_n(\xi) = \mathcal{H}_n(X)$, for any $n \geq 1$, with X being the isonormal process constructed from identity (11), in the continuous time case, and, similarly, in the discrete time case, from equation (12).

Corollary 1 *Assume that conditions (15) and (i) or (ii) of Proposition 1 are satisfied for r.v.'s*

$$\pi_{T,n_j}(\xi) = \int_0^T r_{T,j}(t) H_{n_j}(\xi(t))\nu(dt), \tag{16}$$

where, in the case of continuous time, it is also assumed that $r_{T,j}(t) \in C([0,\infty))$, *for* $T > 0$, *and* $j = 1, \ldots, d$. *Then, the vector*

$$\pi_{T,d}(\xi) = \left(\int_0^T r_{T,1}(t) H_{n_1}(\xi(t))\nu(dt), \ldots, \int_0^T r_{T,d}(t) H_{n_d}(\xi(t))\nu(dt)\right) \tag{17}$$

converges in distributions, as $T \to \infty$, *to a standard Gaussian vector* $\pi_d \sim N(0, \mathbb{I}_d)$.

Proof. In the case of continuous time, since $\xi(t) = X\left(\mathbf{l}_{(-\infty,t]}\right)$,

$$H_{n_j}(\xi(t)) = H_{n_j}\left(X\left(\mathbf{l}_{(-\infty,t]}\right)\right) = I^X_{n_j}\left(\mathbf{l}^{\otimes n_j}_{(-\infty,t]}\right),$$

where $I^X_{n_j}$ denotes the isometry introduced in (13). Therefore, for $r_{T,j}(t) \in C([0,\infty))$, $T > 0$, and, for $j = 1, \ldots, d$,

$$\pi_{T,n_j}(\xi) = \int_0^T r_{T,j}(t) H_{n_j}(\xi(t))dt = I^X_{n_j}\left(\int_0^T r_{T,j}(t)\mathbf{l}^{\otimes n_j}_{(-\infty,t]}dt\right).$$

Thus, considering in (iii) of Proposition 1

$$f_{j,T}(s_1, \ldots, s_{n_j}) = \int_0^T r_{T,j}(t)\mathbf{l}^{\otimes n_j}_{(-\infty,t]}(s_1, \ldots, s_{n_j})dt,$$

for $j = 1, \ldots, d$, we obtain the desired result.

Similarly, for the case of discrete time, we have, from (12),

$$X(\delta_{\cdot,l}) = \xi(l), \quad l \in \mathbb{Z}, \tag{18}$$

where, for each $l \in \mathbb{Z}$, $\delta_{\cdot,l}$ denotes the Kronecker delta function, i.e.,

$$\delta_{i,l} = \begin{cases} 1 & \text{if} \quad i = l \\ 0 & \text{if} \quad i \neq l, \quad i \in \mathbb{Z}. \end{cases}$$

Therefore,

$$H_{n_j}(\xi(l)) = H_{n_j}\left(X\left(\delta_{\cdot,l}\right)\right) = I^X_{n_j}\left(\delta^{\otimes n_j}_{\cdot,l}\right).$$

Proposition 1(iii) is then applied, considering

$$f_{j,T}(m_1, \ldots, m_{n_j}) = \sum_{l=1}^T r_{T,j}(l)\prod_{i=1}^j \delta_{m_i,l}, \quad m_1, \ldots, m_j \in \mathbb{Z},$$

for $j = 1, \ldots, d$. ■

4. Spectral measures of weight functions and admissible spectral densities. Let us first establish some results on weak-convergence of matrix-valued measures, given by

$$\mu_T^{jl}(d\lambda) = \frac{w_T^j(\lambda)\overline{w_T^l(\lambda)}d\lambda}{\sqrt{\int\limits_\Lambda \left|w_T^j(\lambda)\right|^2 d\lambda \int\limits_\Lambda \left|w_T^l(\lambda)\right|^2 d\lambda}}, \quad j,l = 1,\dots,q, \tag{19}$$

where

$$w_T^j(\lambda) = \int\limits_0^T e^{\mathrm{i}t\lambda} w_j(t)\nu(dt), \quad j = 1,\dots,q,$$

and the functions $w_j(t)$, $j = 1,\dots,q$, are, as before, the functions (2) involved in the definition of the random vector (1).

B1. Assume that the weak-convergence $\boldsymbol{\mu}_T \Rightarrow \boldsymbol{\mu}$, when $T \to \infty$, holds, where $\boldsymbol{\mu}_T$ is defined by (19), and $\boldsymbol{\mu}$ is a positive definite matrix measure.

The above condition means that an element μ^{jl} of the matrix-valued measure $\boldsymbol{\mu}$ is a signed measure of bounded variation, and the matrix $\boldsymbol{\mu}(A)$ is positive definite for any set $A \in \mathcal{A}$, with $\mathcal{A}$ denoting the $\sigma-$algebra of measurable subsets of $\mathbb{R}$ (see, for example, Ibragimov and Rozanov [14]).

The following definition can be found in Grenander and Rosenblatt [10], Ibragimov and Rozanov [14], and Ivanov and Leonenko [17].

Definition 3 *The non-degenerate matrix-valued measure* $\boldsymbol{\mu}(d\lambda) = \{\mu^{jl}(d\lambda)\}_{j,l=1}^q$ *is said to be the spectral measure of function* $\mathbf{w}(t)$.

Definition 4 *(Ibragimov and Rozanov [14]). The s.d.* f *is said to be* $\boldsymbol{\mu}-$ *admissible, if it is integrable, that is, all elements of the matrix*

$$\int_\Lambda f(\lambda)\boldsymbol{\mu}(d\lambda),$$

are finite, and

$$\lim_{T\to\infty} \int\limits_\Lambda f(\lambda)\boldsymbol{\mu}_T(d\lambda) = \int\limits_\Lambda f(\lambda)\boldsymbol{\mu}(d\lambda). \tag{20}$$

Let us introduce two conditions on the s.d. f that guarantee its $\boldsymbol{\mu}-$admissibility. These assumptions are related to basic conditions on the c.f. and s.d. **A2.** In the following, J denotes one of the three sets:

$$\{-r,\dots,-1,0,1,\dots,r\}; \{-r,\dots,-1,1,\dots,r\}; \{0\}.$$

We formulate the following condition for a set $J = \{-r,\dots,-1,0,1,\dots,r\}$.

I. The s.d. $f \in \mathcal{C}\left(\Lambda \setminus \{\varkappa_j, j \in J\}\right)$, with

$$\varkappa_{-j} = -\varkappa_j,\ j = 0,1,\dots,r,\ 0 \le \varkappa_0 < \varkappa_1 < \dots < \varkappa_r,$$

and

$$\lim_{\lambda\to\varkappa_j} f(\lambda)\left|\lambda - \varkappa_j\right|^{1-\alpha_j} = a_j > 0,\ \alpha_j \in (0,1),\ j \in J;\ \alpha_{-j} = \alpha_j,\ a_{-j} = a_j,\ j = 0,1,\dots,r. \tag{21}$$

We obtain from (21) that, for any $\varepsilon > 0$, and $j \in J$, there exists $\delta_j = \delta_j(\varepsilon)$, such that for $|\lambda - \varkappa_j| < \delta_j$

$$f(\lambda) < \frac{a_j + \varepsilon}{|\lambda - \varkappa_j|^{1-\alpha_j}}.$$

Then, for $|\lambda - \varkappa_j| < \delta_j$, we have the following:

$$\{\lambda : f(\lambda) > c\} \subset V_j(c) = \left\{\lambda : |\lambda - \varkappa_j| < \left(\frac{a_j + \varepsilon}{c}\right)^{\frac{1}{1-\alpha_j}}\right\}.$$

Moreover c must satisfy the inequality:

$$\left(\frac{a_j + \varepsilon}{c}\right)^{\frac{1}{1-\alpha_j}} \leq \delta_j,$$

equivalently

$$c \geq \frac{a_j + \varepsilon}{\delta_j^{1-\alpha_j}(\varepsilon)} = c_j(\varepsilon). \tag{22}$$

II. Let $\varepsilon_0 > 0$ be fixed. There exists $c_0 = \max_{j \in J} c_j(\varepsilon_0)$, such that for $c \geq c_0$

$$\{\lambda : f(\lambda) > c\} \subset \cup_{j \in J} V_j(c), \tag{23}$$

where $c_j(\varepsilon)$ are defined by (22).

It is easy to see that for sufficiently large c (say, $c \geq c_0$), the neighborhoods $V_j(c), j \in J$, in (23), are non overlapping, and

$$|V_j(c)| \downarrow 0,$$

as $c \to \infty$.

Note that the function (7) satisfies conditions **I** and **II.**

B2. For T sufficiently large (say, $T \geq T_0$)

$$W_{iT}^{-1} \max_{\lambda \in V_j(c_0)} \left|w_T^i(\lambda)\right| \leq k_{ij} < \infty, \quad j \in J, \quad i = 1, \ldots, q. \tag{24}$$

In condition **B2**, one can assume that (24) holds only for $j = 0, 1, \ldots, r$, since $V_{-j}(c_0) = -V_j(c_0)$, $j = 0, 1, \ldots, r$.

Theorem 1 *Assume that conditions **B1, B2,** as well as **I, II** are satisfied, and the s.d. f is integrable with respect to the spectral measure $\boldsymbol{\mu}$, then the s.d. f is $\boldsymbol{\mu}-$ admissible.*

Proof. For $c \geq c_0$, we consider

$$f^c(\lambda) = f(\lambda) 1_{\langle f(\lambda) < c \rangle}(\lambda) + c 1_{\langle f(\lambda) \geq c \rangle}(\lambda).$$

Then, for $k, l = 1, \ldots, q$,

$$\begin{aligned}
&\left|\int_\Lambda f(\lambda)\mu_T^{k,l}(d\lambda) - \int_\Lambda f(\lambda)\mu^{k,l}(d\lambda)\right| \\
&\leq \left|\int_\Lambda f(\lambda)\mu_T^{k,l}(d\lambda) - \int_\Lambda f^c(\lambda)\mu_T^{k,l}(d\lambda)\right| + \\
&+ \left|\int_\Lambda f^c(\lambda)\mu_T^{k,l}(d\lambda) - \int_\Lambda f^c(\lambda)\mu^{k,l}(d\lambda)\right| \\
&+ \left|\int_\Lambda f^c(\lambda)\mu^{k,l}(d\lambda) - \int_\Lambda f(\lambda)\mu^{k,l}(d\lambda)\right| = I_1^{k,l}(T,c) + I_2^{k,l}(T,c) + I_3^{k,l}(c).
\end{aligned} \tag{25}$$

By assumption **B1**, for any complex numbers $z = (z_1, \dots, z_q)$, the function

$$M_z(A) = \sum_{k,l=1}^{q} \mu^{k,l}(A) z_k \bar{z}_l \geq 0, \quad A \in \mathcal{A},$$

is a measure. Thus, by Lebesgue's monotone convergence theorem,

$$\int_\Lambda f^c(\lambda) M_z(d\lambda) \underset{c\to\infty}{\longrightarrow} \int_\Lambda f(\lambda) M_z(d\lambda). \tag{26}$$

Note that the diagonal elements $\mu^{k,k}$ and $\mu^{l,l}$ are measures, thus, if only z_k and z_l are non zero among $z = (z_1, \dots, z_q)$, we obtain from (26), that

$$\int_\Lambda (f(\lambda) - f^c(\lambda)) \left(\mu^{k,l}(d\lambda) z_k \bar{z}_l + \mu^{l,k}(d\lambda) z_l \bar{z}_k \right) \longrightarrow 0, \quad c \to \infty. \tag{27}$$

Note that $\mu^{l,k} = \overline{\mu^{k,l}}$, and choosing, for instance, $z_k = z_l = 1$, we have from (27),

$$\int_\Lambda (f(\lambda) - f^c(\lambda)) \operatorname{Re}(\mu^{k,l})(d\lambda) \longrightarrow 0, \quad c \to \infty.$$

If we choose $z_k = 1$, $z_l = -\mathrm{i}$, then

$$\int_\Lambda (f(\lambda) - f^c(\lambda)) \operatorname{Im}(\mu^{k,l})(d\lambda) \longrightarrow 0, \quad c \to \infty.$$

Thus

$$\lim_{c\to\infty} I_3^{k,l}(c) = 0.$$

For a fixed c, we obtain, from condition **B1**, that

$$\lim_{T\to\infty} I_2^{k,l}(T, c) = 0.$$

On the other hand, under the conditions assumed in this theorem, for $T \geq T_0$,

$$I_1^{k,l}(T, c) \leq \frac{1}{2\pi} \int\limits_{\{\lambda : f(\lambda) > c\}} (f(\lambda) - c) \frac{\left|w_T^k(\lambda)\right| \left|w_T^l(\lambda)\right|}{W_{k,T} W_{l,T}} d\lambda \leq \frac{1}{2\pi} \sum_{j\in J} k_{j,k} k_{j,l} \int\limits_{V_j(c)} f(\lambda) d\lambda \to 0,$$

when $c \to \infty$. Thus, for any $\varepsilon > 0$ and $T \geq T_0$, one can choose $c_1 = c_1(\varepsilon) \geq c_0$, such that for $c > c_1$, we have $I_1^{k,l}(T, c) < \varepsilon/3$. Then, once can take $c_2 = c_2(\varepsilon) \geq c_0$, such that for $c > c_2$, we have $I_3^{k,l}(c) < \varepsilon/3$.

Let us now fix $c = \max(c_1, c_2)$, then, there exists $T_1 = T_1(\varepsilon) > T_0$, such that for $T > T_0$, $I_2^{k,l}(T, c) < \varepsilon/3$, and the left hand side of (25) is less than ε. ■

5. Central limit theorem for weighted functionals. This section provides the asymptotic normality as $T \to \infty$ of the vector (1), i.e., we will prove that the vector $\boldsymbol{\zeta}_T$ converges in distribution ($\Longrightarrow$) to some Gaussian vector $\boldsymbol{\zeta}$. Thus, for any $z \in \mathbb{R}^q$, we prove that $\langle \boldsymbol{\zeta}_T, z \rangle \Longrightarrow \langle \boldsymbol{\zeta}, z \rangle$, as $T \to \infty$. Denoting, for $z = (z_1, ..., z_q)$,

$$\sum_{i=1}^{q} z_i W_{iT}^{-1} w_i(t) = R_T(t, z) = R_T(t),$$

from (9), we have

$$\langle \boldsymbol{\zeta}_T, z\rangle = \int_0^T \psi(\xi(t))R_T(t)\nu(dt) = \sum_{j=m}^{\infty} \frac{C_j(\psi)}{j!} \int_0^T R_T(t)H_j(\xi(t))\nu(dt).$$

In the derivation of the proof of our main result, the following additional conditions are required:
B3. For $T > T_0$,

$$W_{i,T}^{-1} \sup |w_i(t)| \leq k_i T^{-1/2}, \quad i = 1, \ldots, q, \tag{28}$$

where the supremum is taken over t in the interval $[0, T]$, in the case of continuous time, and over t in the set $\{1, \ldots, T\}$, in the case of discrete time.

Let $f^{(*1)}(\lambda) = f(\lambda)$, and for $j \geq 2$,

$$f^{*(j)}(\lambda) = \int_{\Lambda^{j-1}} f(\lambda - \lambda_2 - \cdots - \lambda_j) \prod_{i=2}^{j} f(\lambda_i) d\lambda_2 \ldots d\lambda_j,$$

the jth convolution of the s.d. $f(\lambda)$.

C. The matrix integrals

$$\int_{\Lambda} f^{*(j)}(\lambda)\boldsymbol{\mu}(d\lambda), \quad j \geq 1,$$

are positive definite.

We now proceed the formulation of our main result.

Theorem 2 *Suppose that conditions **A1-A4**, **B1-B3** and **C** are fulfilled. Then, the r.v. $\boldsymbol{\zeta}_T$ in (1) converges in distribution, as $T \longrightarrow \infty$, to the Gaussian r.v. $\boldsymbol{\zeta}$ with zero mean and covariance matrix*

$$\Xi = 2\pi \sum_{j=m}^{\infty} \frac{C_j^2(\psi)}{j!} \int_{\Lambda} f^{*(j)}(\lambda)\boldsymbol{\mu}(d\lambda), \tag{29}$$

where $\boldsymbol{\mu}$ is the weak-sense limit of the family of matrix-valued measures introduced in (19) and associated with the weight function $\mathbf{w}(t)$ in (2), given from functional (1).

In the proof of the above theorem, the following identities will be applied jointly with Lemma 1 formulated below. Specifically, from the orthogonality of Hermite polynomials, we obtain

$$E\left\langle \boldsymbol{\zeta}_T, z\right\rangle^2 = \sum_{j=m}^{\infty} \left[\frac{C_j(\psi)}{j!}\right]^2 \sigma_T^2(j, z) = \sum_{j=m}^{\infty} \frac{C_j^2(\psi)}{j!} \int_0^T \int_0^T R_T(t)R_T(s)B^j(t-s)\nu(dt)\nu(ds). \tag{30}$$

We will prove the asymptotic normality of (1) under condition **A4**(i). The proof under condition **A4**(ii) is even simpler.

By conditions **A2** and **A4**(i), for $j \geq 2$, all the convolutions $f^{*(j)}$ are bounded and continuous functions, and by **B1**

$$\begin{aligned}
\sigma_T^2(j,z) &= j! \int_0^T \int_0^T B^j(t-s) R_T(t) R_T(s) \nu(dt)\nu(ds) \\
&= \sum_{k,l=1}^{q} \left(j! \int_0^T \int_0^T B^j(t-s) \frac{w_k(t)}{W_{k,T}} \frac{w_l(s)}{W_{l,T}} \nu(dt)\nu(ds) \right) z_k z_l \\
&= 2\pi j! \int_\Lambda f^{*(j)}(\lambda) \left(\sum_{k,l=1}^{q} \mu_T^{k,l}(\lambda) z_k z_l \right) d\lambda \\
&\underset{T\to\infty}{\longrightarrow} 2\pi j! \int_\Lambda f^{*(j)}(\lambda) m_z(d\lambda) = \sigma^2(j,z), \quad j \geq 2,
\end{aligned} \tag{31}$$

where $m_z(d\lambda) = \sum_{k,l=1}^{q} \mu^{k,l}(d\lambda) z_k z_l$.

Under condition **A2**, from Theorem 1, we obtain, for $j = 1$,

$$\lim_{T\to\infty} \sigma_T^2(1,z) = 2\pi \int_\Lambda f(\lambda) m_z(d\lambda) = \sigma^2(1,z). \tag{32}$$

Thus,

$$\lim_{T\to\infty} E\left\langle \zeta_T, z \right\rangle^2 = \sum_{j=1}^{\infty} \left[\frac{C_j(\psi)}{j!} \right]^2 \sigma^2(j,z) = \sigma^2(z). \tag{33}$$

In Lemma 1 below, we will consider the following decomposition:

$$\tau_T = \langle \zeta_T, z \rangle = \tau_T(d) + \tau_T'(d) = \left(\sum_{j=1}^{d} + \sum_{j=d+1}^{\infty} \right) \frac{C_j(\psi)}{j!} \int_0^T R_T(t) H_j(\xi(t)) \nu(dt). \tag{34}$$

Lemma 1 *Suppose that conditions **A1-A4** and **B1-B3** hold. If for any $d \geq 1$, as $T \to \infty$, $\tau_T(d) \Rightarrow \tau_d \sim N(0, \sigma_d^2(z))$, where*

$$\sigma_d^2(z) = \sum_{j=1}^{d} \left[\frac{C_j(\psi)}{j!} \right]^2 \sigma^2(j,z), \tag{35}$$

then $\tau_T \Rightarrow \tau \sim N(0, \sigma^2(z))$.

Proof. Note that $E[(\tau_T'(d))^2] \to 0,\ d \to \infty$, uniformly in T. Really, by condition **B3**,

$$|R_T(t)| = \left| \sum_{i=1}^{q} z_i w_i(t) W_{i,T}^{-1} \right| \leq T^{-1/2} \|z\| \left\| \tilde{k} \right\|, \ \tilde{k} = (k_1, \ldots, k_q). \tag{36}$$

Then, under **A4**(i),

$$\begin{aligned}
E[(\tau_T'(d))^2] &= \sum_{j=d+1}^{\infty} \frac{C_j^2(\psi)}{j!} \int_0^T \int_0^T B^j(t-s) R_T(t) R_T(s) \nu(dt)\nu(ds) \\
&\leq T^{-1} \|z\|^2 \left\| \tilde{k} \right\|^2 \int_0^T \int_0^T B^2(t-s) \nu(dt)\nu(ds) \sum_{j=d+1}^{\infty} \frac{C_j^2(\psi)}{j!} \\
&\leq \|z\|^2 \left\| \tilde{k} \right\|^2 \int_{\mathbb{R}} B^2(t) \nu(dt) \sum_{j=d+1}^{\infty} \frac{C_j^2(\psi)}{j!} = \beta(d) \longrightarrow 0, \quad d \to \infty,
\end{aligned}$$

since by Parseval identity

$$\sum_{j=1}^{\infty} \frac{C_j^2(\psi)}{j!} = E\psi^2(\xi(0)) < \infty.$$

Thus, for any $\varepsilon > 0$, uniformly in T,

$$P\left\{\left|\tau_T'(d)\right| > \varepsilon\right\} \leq \frac{\beta(d)}{\varepsilon^2} \longrightarrow 0, \quad d \to \infty.$$

For any $\varepsilon > 0$, and $d \geq 1$, we then obtain

$$\overline{\lim_{T\to\infty}} P\left\{\tau_T \leq x\right\} \leq \Phi_d(x+\varepsilon) + \frac{\beta(d)}{\varepsilon^2}, \tag{37}$$

where Φ_d is the distribution function of a Gaussian r.v. with zero mean and variance $\sigma_d^2(z)$.

Also, for any $\varepsilon > 0$, and $d \geq 1$, as $T \to \infty$, the following inequality holds:

$$\underline{\lim} P\left\{\tau_T \leq x\right\} \geq \Phi_d(x-\varepsilon) - \frac{\beta(d)}{\varepsilon^2}. \tag{38}$$

If $d \to \infty$, we obtain, from equations (37) and (38) that, as $T \to \infty$,

$$\Phi_\infty(x-\varepsilon) \leq \underline{\lim} P\left\{\tau_T \leq x\right\} \leq \overline{\lim} P\left\{\tau_T \leq x\right\} \leq \Phi_\infty(x+\varepsilon),$$

where Φ_∞ is the distribution function of a Gaussian r.v. with zero mean and the variance $\sigma^2(z)$ given by (33). Thus, if $\varepsilon \to 0$, $\lim_{T\to\infty} P\left\{\tau_T \leq x\right\} = \Phi_\infty(x)$, $x \in \mathbb{R}$. ■

Now we are in position to derive the proof of Theorem 2. In such a proof, we will check condition (i) of Proposition 1, but the proof can also be developed from the verification of condition (ii) in Proposition 1, using diagram formula. We place this proof into Appendix, due to its methodological interest in relation to the approach it presents for the analysis of non-regular diagrams, providing the classification of their levels into recipients and donors.

Proof of Theorem 2. From Lemma 1, it is sufficient to show the asymptotic normality of the r.v.'s $\tau_T(d)$. Consider then the r.v.'s

$$\pi_{T,d}(\xi) = \left(\int_0^T r_{T,1}(t) H_1(\xi(t))\nu(dt), \ldots, \int_0^T r_{T,d}(t) H_d(\xi(t))\nu(dt)\right)', \tag{39}$$

where

$$r_{T,j}(t) = \frac{R_T(t)}{\sigma(j,z)}, \quad j = 1, \ldots, d. \tag{40}$$

The proof will follow from the application of Corollary 1, after checking condition (i) of Proposition 1 for the random vector $\pi_{T,d}(\xi)$ defined by (39) and (40). From Theorem 1 and equation (31),

$$E\left[\int_0^T r_{T,j}(t) H_j(\xi(t))\nu(dt)\right]^2 = \frac{\sigma_T^2(j,z)}{\sigma^2(j,z)} \longrightarrow 1, \quad T \to \infty, \quad j = 1, \ldots, d. \tag{41}$$

Now, $\pi_{T,d}(\xi) \Longrightarrow \pi_d \sim \mathcal{N}\left(0, \mathbb{I}_d\right)$, $T \to \infty$, if and only if

$$\lim_{T\to\infty} \|f_{j,T} \otimes_p f_{j,T}\|_{H^{\otimes 2(j-p)}} = 0,$$

for $p = 1, \dots, j-1$, $2 \le j \le d$, where

$$f_{j,T}(s_1, \dots s_j) = \int_0^T R_T(t) \prod_{i=1}^{j} \mathbf{1}_{(-\infty,t]}(s_i) dt.$$

We first check the convergence to zero of contractions in the continuous time case. The pth contraction is computed by applying formula (14) with $k = j$ as follows

$$\begin{aligned} f_{j,T} \otimes_p f_{j,T}(x_1, \dots, x_{2j-2p}) &= \int_0^T \int_0^T R_T(t) R_T(s) B(t-s) \times \underset{p}{\dots} \times B(t-s) \\ &\qquad \times \prod_{i=p+1}^{j} \mathbf{1}_{(-\infty,t]}(x_i) \prod_{l=p+1}^{j} \mathbf{1}_{(-\infty,s]}(x_l) ds dt \\ &= \int_0^T \int_0^T R_T(t) R_T(s) B^p(t-s) \prod_{i=1}^{j-p} \mathbf{1}_{(-\infty,t]}(x_i) \\ &\qquad \times \prod_{l=j-p+1}^{2j-2p} \mathbf{1}_{(-\infty,s]}(x_l) ds dt. \end{aligned} \tag{42}$$

The norm of the pth contraction (42) in the space $H^{\otimes 2(j-p)}$ is then given by:

$$\begin{aligned} \|f_{j,T} \otimes_p f_{j,T}\|^2_{H^{\otimes 2(j-p)}} &= \int_0^T \int_0^T \int_0^T \int_0^T R_T(t_1) R_T(s_1) R_T(t_2) R_T(s_2) \\ &\qquad \times B^{j-p}(t_1 - t_2) B^{j-p}(s_1 - s_2) B^p(t_1 - s_1) B^p(t_2 - s_2) ds_1 ds_2 dt_1 dt_2. \end{aligned}$$

By condition **B3**, for $2 \le j \le d$, and $p = 1, \dots, j-1$,

$$\begin{aligned} \|f_{j,T} \otimes_p f_{j,T}\|^2_{H^{\otimes 2(j-p)}} &\le \frac{\|z\|^4 \left\|\tilde{k}\right\|^4}{T^2} \int_0^T \int_0^T \int_0^T \int_0^T |B^p(t_1 - s_1) B^p(t_2 - s_2) B^{j-p}(t_1 - t_2)| \\ &\qquad \times |B^{j-p}(s_1 - s_2)| ds_1 ds_2 dt_1 dt_2 \\ &\le \frac{\|z\|^4 \left\|\tilde{k}\right\|^4}{T^2} \int_0^T \int_0^T \int_0^T \int_0^T |B(t_1 - s_1) B(t_2 - s_2) B(t_1 - t_2) B(s_1 - s_2)| \\ &\qquad \times dt_1 dt_2 ds_1 ds_2 \\ &\le \frac{\|z\|^4 \left\|\tilde{k}\right\|^4}{T^2} \int_0^T \int_0^T \int_0^T \frac{1}{2} \int_0^T \left[B^2(t_1 - s_1) + B^2(t_1 - t_2)\right] dt_1 \\ &\qquad \times |B(t_2 - s_2) B(s_1 - s_2)| ds_1 ds_2 dt_2 \\ &\le 4 \|z\|^4 \left\|\tilde{k}\right\|^4 \left[\int_0^\infty B^2(t_1) dt_1\right] \left[\int_0^T |B(t_2)| dt_2\right] \\ &\qquad \times T^{-2} \int_0^T \int_0^T |B(s_1 - s_2)| ds_1 ds_2. \end{aligned} \tag{43}$$

In (43), as $T \to \infty$,

$$\begin{aligned} \int_0^T |B(t_2)| dt_2 &= \mathcal{O}(T^{1-\alpha}) \\ T^{-2} \int_0^T \int_0^T |B(s_1 - s_2)| ds_1 ds_2 &= \mathcal{O}(T^{-\alpha}). \end{aligned} \tag{44}$$

From condition **A4**, in the case considered of Hermite rank $m = 1$, we have $\alpha > 1/2$, and therefore, from (43) and (44), we obtain, for $j \geq 2$, $p = 1, \ldots, j-1$,

$$\lim_{T\to\infty} \|f_{j,T} \otimes_p f_{j,T}\|_{H^{\otimes 2(j-p)}} = 0. \tag{45}$$

The proof in the discrete time case can be similarly derived in terms of definition (12) of isonormal process X, considering the counting measure $\nu(\cdot)$. Specifically, from (18), for $T > 0$, and $2 \leq j \leq d$, we consider the sequence of kernels,

$$f_{j,T}(m_1, \ldots, m_j) = \sum_{l=1}^{T} R_T(l) \prod_{i=1}^{j} \delta_{m_i,l}, \quad m_1, \ldots, m_j \in \mathbb{Z}.$$

For $p = 1, \ldots, j-1$, the pth self-contraction of this kernel is given by

$$f_{j,T} \otimes_p f_{j,T}(m_1, \ldots, m_{2j-2p}) = \sum_{q=1}^{T}\sum_{l=1}^{T} R_T(q)R_T(l)B^p(q-l) \prod_{i=1}^{j-p} \delta_{m_i,q} \prod_{i=j-p+1}^{2j-2p} \delta_{m_i,l}.$$

Therefore, since

$$\begin{aligned} \|f_{j,T} \otimes_p f_{j,T}\|^2_{H^{\otimes 2(j-p)}} &= \sum_{q=1}^{T}\sum_{k=1}^{T}\sum_{l=1}^{T}\sum_{i=1}^{T} R_T(q)R_T(l)R_T(k)R_T(i) \\ &\quad \times B^{j-p}(q-k)B^{j-p}(l-i)B^p(q-l)B^p(k-i), \end{aligned}$$

in a similar way to the continuous time case, we obtain

$$\begin{aligned} \|f_{j,T} \otimes_p f_{j,T}\|^2_{H^{\otimes 2(j-p)}} &\leq \frac{\|z\|^4 \left\|\tilde{k}\right\|^4}{T^2} \sum_{q=1}^{T}\sum_{k=1}^{T}\sum_{l=1}^{T}\sum_{i=1}^{T} |B^p(q-l)B^p(k-i)B^{j-p}(q-k)| \\ &\qquad \times |B^{j-p}(l-i)| \\ &\leq \frac{\|z\|^4 \left\|\tilde{k}\right\|^4}{T^2} \sum_{q=1}^{T}\sum_{k=1}^{T}\sum_{l=1}^{T}\sum_{i=1}^{T} |B(q-l)B(k-i)B(q-k)B(l-i)| \\ &\leq 4\,\|z\|^4 \left\|\tilde{k}\right\|^4 \left[\sum_{q=1}^{\infty} B^2(q)\right] \left[\sum_{k=1}^{T} |B(k)|\right] \\ &\qquad \times T^{-2} \sum_{l=1}^{T}\sum_{i=1}^{T} |B(l-i)|. \end{aligned} \tag{46}$$

Thus, as $T \to \infty$,

$$\begin{aligned} \sum_{k=1}^{T} |B(k)| &= \mathcal{O}(T^{1-\alpha}) \\ T^{-2} \sum_{l=1}^{T}\sum_{i=1}^{T} |B(l-i)| &= \mathcal{O}(T^{-\alpha}). \end{aligned} \tag{47}$$

Again, from condition **A4**, and equations (46) and (47), Proposition (i) holds and the convergence to the Gaussian distribution follows.

■

Example(continuation). Consider now model (3) with nonlinear regression function

$$g(t,\theta)=\sum_{k=1}^{N}\left(A_k\cos\varphi_k t+B_k\sin\varphi_k t\right), \tag{48}$$

where $\theta=(A_1,B_1,\varphi_1,\dots,A_N,B_N,\varphi_N)$, $C_k^2=A_k^2+B_k^2>0$, $k=1,\dots,N$, $0<\varphi_1<\dots<\varphi_N<\infty$. In this case, $q=3N$, function $g(t,\theta)$ then has a block-diagonal measure $\mu(d\lambda)$ (see, for example, Ivanov [15]) with blocks

$$\begin{pmatrix} \kappa_k & \mathrm{i}\rho_k & \bar\beta_k \\ -\mathrm{i}\rho_k & \kappa_k & \bar\gamma_k \\ \beta_k & \gamma_k & \kappa_k \end{pmatrix}, \quad k=1,\dots,N,$$

where

$$\beta_k=\frac{\sqrt3}{2C_k}\left(B_k\kappa_k+\mathrm{i}A_k\rho_k\right), \quad \gamma_k=\frac{\sqrt3}{2C_k}\left(-A_k\kappa_k+\mathrm{i}B_k\rho_k\right).$$

Here, the measure $\kappa_k=\kappa_k(d\lambda)$ and the signed measure $\rho_k=\rho_k(d\lambda)$ are located at the points $\pm\varphi_k$, and $\kappa_k\left(\{\pm\varphi_k\}\right)=\frac12$, $\quad\rho_k\left(\{\pm\varphi_k\}\right)=\pm\frac12$. We then have

$$g_{3k-2}(t,\theta)=\frac{\partial}{\partial A_k}g(t,\theta)=\cos\varphi_k t, \quad g_{3k-1}(t,\theta)=\frac{\partial}{\partial B_k}g(t,\theta)=\sin\varphi_k t,$$

$$g_{3k}(t,\theta)=\frac{\partial}{\partial\varphi_k}g(t,\theta)=-A_k t\sin\varphi_k t+B_k t\cos\varphi_k t, \quad k=1,\dots,N.$$

It is easy to see that if the s.d. f satisfies **I**, and $\varkappa_j\neq\varphi_k$, $j=0,1,\dots,r$, $k=1,\dots,N$, one can find a neighborhood $V_j(c_0)$ of the point $\varkappa_j$, for $j=0,1,\dots,r$, which does not contain the points φ_k, $k=1,\dots,N$. Thus, for $T>T_0$, the following condition holds:

$$W_{iT}^{-1}\max_{\lambda\in V_j(c_0)}\left|w_T^i(\lambda)\right|\le k_{ij}T^{-1/2}, \quad j\in J; \quad i=3k-2,3k-1,3k; \quad k=1,\dots,N.$$

In relation to the considered function $\mathbf{w}(t)=\nabla g(t,\theta)$, the measure $\mu_T^{jl}(d\lambda)=\mu_T^{jl}(d\lambda,\theta)$ approximates, in the weak sense, the spectral measure $\boldsymbol{\mu}(d\lambda)=\{\mu^{jl}(d\lambda)\}_{j,l=1}^q$ of the nonlinear regression function $g(t,\theta)$ (see (3)), where

$$\mu_T^{jl}(d\lambda,\theta)=\frac{g_T^j(\lambda,\theta)\overline{g_T^l(\lambda,\theta)}d\lambda}{\sqrt{\int\limits_\Lambda\left|g_T^j(\lambda,\theta)\right|^2d\lambda\int\limits_\Lambda\left|g_T^l(\lambda,\theta)\right|^2d\lambda}}, \quad j,l=1,\dots,q,$$

$$g_T^j(\lambda,\theta)=\int\limits_0^T e^{\mathrm{i}t\lambda}g_j(t,\theta)\nu(dt), \quad j=1,\dots,q,$$

and $g_j(t,\theta)$ defines the jth-component of $\mathbf{w}(t)=\nabla g(t,\theta)$, for $j=1,\dots,q$.

If the s.d. $f(\lambda)$ satisfies condition **II,** then f is $\boldsymbol{\mu}-$admissible, and the block-diagonal matrix $\int_\Lambda f(\lambda)\boldsymbol{\mu}(d\lambda)$ consists of the blocks:

$$f(\varphi_k)\begin{pmatrix} 1 & 0 & \frac{\sqrt3}{2}\frac{B_k}{C_k} \\ 0 & 1 & -\frac{\sqrt3}{2}\frac{A_k}{C_k} \\ \frac{\sqrt3}{2}\frac{B_k}{C_k} & -\frac{\sqrt3}{2}\frac{A_k}{C_k} & 1 \end{pmatrix}, \quad k=1,\dots,N.$$

It is easy to see that, for the function $g(t,\theta)$ given by (48), the matrix Ξ is a block-diagonal with blocks of the form

$$\Xi_k = 2\pi \sum_{j=m}^{\infty} \frac{C_j^2(\psi)}{j!} f^{*(j)}(\varphi_k) \begin{pmatrix} 1 & 0 & B_k \frac{\sqrt{3}}{2C_k} \\ 0 & 1 & -A_k \frac{\sqrt{3}}{2C_k} \\ B_k \frac{\sqrt{3}}{2C_k} & -A_k \frac{\sqrt{3}}{2C_k} & 1 \end{pmatrix}, \quad k = 1, \ldots, N.$$

6. Final Comments. This paper addresses the problem of Gaussian limit theory of weighted functionals of nonlinear transformations of Gaussian stationary random processes ξ having multiple singularities in their spectra. The general case where the Fourier transform of the weight function also displays multiple singularities in the limit, which do not coincide with the singularities of the spectral density of ξ, is also covered here. This subject has several applications in asymptotic statistical inference. We are especially motivated by its application in the limit theory of non-linear regression problems with regression function and errors having multiple singularities in their spectra. This actually constitutes an active research area, due to the existence of several open problems and applications. Note that, although here we have considered the parameter range

$$\alpha = \min_{j=0,1,\ldots,r} \alpha_j > 1/2,$$

which, in particular, allows to consider long-range dependence models. Our conjecture is that the Gaussian limit results hold for $\alpha_j \in (0,1)$, $j = 0, 1, \ldots, r$. The proof of this conjecture will lead to a general scenario where most of the limit results derived for random fields with singular spectra (see Taqqu [29, 30]; Dobrushin and Major [8]; Nualart and Peccati [23]; and the references therein) can be obtained as particular cases.

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Appendix. Proof of Theorem 2 based on diagram formula. As before, we will prove this result for Hermite rank $m = 1$. To show the asymptotic normality of the r.v.'s $\tau_T(d)$, consider the r.v.'s $\pi_{T,d}(\xi)$ and $r_{T,j}(t)$, $j = 1, \ldots, d$, defined by (39) and (40).

We will check condition (ii) of Proposition 1. Then, from Corollary 1, $\pi_{T,d}(\xi) \Rightarrow \pi_d \sim N(0, \mathbb{I}_d)$, that is, $\tau_T(d) \Rightarrow \tau_d \sim N(0, \sigma_d^2(z))$, as $T \to \infty$.

We apply diagram technique for proving condition (ii) of Proposition 1. Let us first introduce some definitions.

A graph $\Gamma = \Gamma(l_1, \ldots, l_p)$ with $l_1 + \cdots + l_p$ vertices is called a diagram of order $(l_1, \ldots, l_p)$ if:

a) the set of vertices V of the graph Γ is of the form $V = \cup_{j=1}^{p} W_j$, where $W_j = \{(j,l) : 1 \le l \le l_j\}$ is the jth level of the graph Γ, $1 \le j \le p$ (if $l_j = 0$, assume $W_j = \emptyset$);

b) each vertex is of degree 1;

c) if $((j_1, l_1),\ (j_2, l_2)) \in \Gamma$ then $j_1 \ne j_2$, that is, the edges of the graph Γ may connect only different levels.

Let $L = L(l_1, \ldots, l_p)$ be a set of diagrams Γ of order $(l_1, \ldots, l_p)$. Denote by Z_Γ the set of edges of a graph $\Gamma \in L$. For the edge $\varpi = ((j_1, l_1),\ (j_2, l_2)) \in Z_\Gamma$, $j_1 < j_2$, we set

$d_1(\varpi) = j_1$, $d_2(\varpi) = j_2$. We call a diagram Γ regular if its levels can be splitted into pairs in such a manner that no edge connects the levels belonging to different pairs. We denote by L^* the set of regular diagrams $L^* \subseteq L(l_1, \dots, l_p)$. If p is odd, then $L^* = \emptyset$.

The following lemma provides the Diagram Formula (see Taqqu [30], Lemma 3.2 or Doukhan, Oppenheim and Taqqu [9], p.74, or Peccati and Taqqu [26]).

Lemma 2 *Let $(\xi_1, \dots, \xi_p)$, $p \geq 2$, be a Gaussian vector with $E\xi_j = 0$, $E\xi_j^2 = 1$, $E\xi_i\xi_j = B(i,j)$, $i,j = 1, \dots, p$, and let $H_{l_1}(u), \dots, H_{l_p}(u)$ be the Hermite polynomials. Then,*

$$E\left\{\prod_{j=1}^{p} H_{l_j}(\xi_j)\right\} = \sum_{\Gamma \in L} \prod_{\varpi \in Z_\Gamma} B(d_1(\varpi), d_2(\varpi)). \tag{49}$$

From (49), we obtain, for $p = 4$, $l_1 = l_2 = l_3 = l_4 = j$, $\Gamma = \Gamma(j,j,j,j)$, and $(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi(t_1), \xi(t_2), \xi(t_3), \xi(t_4))$:

$$E\pi_{T,j}^4(\xi) = \int_0^T \int_0^T \int_0^T \int_0^T \prod_{i=1}^{4} r_{T,j}(t_i) E\left[\prod_{i=1}^{4} H_j(\xi(t_i))\right] \nu(dt_1)\nu(dt_2)\nu(dt_3)\nu(dt_4). \tag{50}$$

We then have

$$E\pi_{T,1}^4(\xi) = \frac{3}{\sigma^4(1,z)} \left[\int_0^T \int_0^T B(t_1 - t_2) R_T(t_1) R_T(t_2) \nu(dt_1)\nu(dt_2)\right]^2 =$$

$$= 3\frac{\sigma_T^4(1,z)}{\sigma^4(1,z)} \longrightarrow 3, \quad T \to \infty. \tag{51}$$

For $j \geq 2$, the sum in (49) is splitted into two sums corresponding to regular and non-regular diagrams:

$$\sum_{\Gamma \in L} \dots = \sum_{\Gamma \in L^*} \dots + \sum_{\Gamma \in L \setminus L^*} \dots$$

and the right-hand side of (50) is spitted into two these parts, as well.

Analysis of the regular diagrams:

We have

$$\sum^{*}(T) = \sum_{\Gamma \in L^*} F_\Gamma(T), \tag{52}$$

where

$$F_\Gamma(T) = \int_0^T \int_0^T \int_0^T \int_0^T \prod_{i=1}^{4} r_{T,j}(t_i) \prod_{\varpi \in Z_\Gamma} B(t_{d_1(\varpi)} - t_{d_2(\varpi)}) \nu(dt_1)\nu(dt_2)\nu(dt_3)\nu(dt_4). \tag{53}$$

Each regular diagram $\Gamma \in L^*$ consists of 4 levels of cardinality j. There are only 3 subdivisions of the 4 levels into pairs, and in each pair the vertices can be connected by $j!$ ways. Thus, there is only

$$|L^*| = 3(j!)^2$$

regular diagrams, and, in this case, the sum (52) is subdivided into product of pairs of integrals:

(54)
$$\sum^{*}(T)=\frac{3(j!)^2}{\sigma^4(j,z)}\left(\int_0^T\int_0^T B^j(t_1-t_2)R_T(t_1)R_T(t_2)\nu(dt_1)\nu(dt_2)\right)^2=3\frac{\sigma_T^4(j,z)}{\sigma^4(j,z)}\longrightarrow 3,\quad T\to\infty.$$

Analysis of the nonregular diagrams:

First, we consider

$$\sum(T)=\sum_{\Gamma\in L\backslash L^*}F_\Gamma(T), \tag{55}$$

where F_Γ is defined as in (53). We now prove that $\lim_{T\to\infty}\sum(T)=0$. Then, the assertion of the theorem will follow from (51) and (54).

From (36),

(56)
$$|F_\Gamma(T)|\leq\frac{\|z\|^4\left\|\tilde{k}\right\|^4}{\sigma^4(j,z)}T^{-2}\int_0^T\int_0^T\int_0^T\int_0^T\prod_{i=1}^4\prod_{\varpi\in Z_\Gamma,d_1(\varpi)=i}\left|B(t_i-t_{d_2(\varpi)})\right|\nu(dt_1)\nu(dt_2)\nu(dt_3)\nu(dt_4).$$

Let $q_\Gamma(i)$ be the number of edges $\varpi\in Z_\Gamma$, such that $d_1(\varpi)=i$. Then, for $q_\Gamma(i)\geq 1$,

(57)
$$\int_0^T\prod_{\varpi\in Z_\Gamma,d_1(\varpi)=i}\left|B(t_i-t_{d_2(\varpi)})\right|\nu(dt_i)\leq\frac{1}{q_\Gamma(i)}\sum_{\varpi\in Z_\Gamma,d_1(\varpi)=i}\int_0^T\left|B(t_i-t_{d_2(\varpi)})\right|^{q_\Gamma(i)}\nu(dt_i)\leq$$
$$\leq 2\int_0^T|B(t_i)|^{q_\Gamma(i)}\nu(dt_i).$$

If $q_\Gamma(i)=0$, the integrals regarded to these variables (t_4, and possibly t_3), in the left-hand side of (57), give a contribution in the form of a multiplier of T in the estimate (56).

Definition 5 . *The level i of a nonregular diagram $\Gamma\in L\backslash L^*$ is said to be a donor, if $q_\Gamma(i)\geq 1$, and a strongly donor, if $q_\Gamma(i)=j$. The level i of a nonregular diagram $\Gamma\in L\backslash L^*$ is said to be a recipient, if it is not donor, that is $q_\Gamma(i)=0$.*

Let ρ_{sd} be a number of strongly donor levels, and ρ_r is a number of recipient levels. Obviously, the level 1 is a strongly donor one, while the level 4 is a recipient one. If $\rho_{sd}=1$, then $\rho_r=1$, while if $\rho_{sd}=2$, then $\rho_r=2$.

Formulae (56) and (57) then imply

$$|F_\Gamma(T)|\leq 2^{4-\rho_r}\frac{\|z\|^4\left\|\tilde{k}\right\|^4}{\sigma^4(j,z)}T^{-2}\prod_{i=1}^4\int_0^T|B(t)|^{q_\Gamma(i)}\nu(dt). \tag{58}$$

Since $j\geq 2$, and $\alpha>1/2$, for a strongly donor level i with $q_\Gamma(i)=j$

$$\int_0^T|B(t)|^j\nu(dt)\leq\int_0^\infty[B(t)]^2\nu(dt)<\infty. \tag{59}$$

Thus, for the recipient levels ($q_\Gamma(i) = 0$) and the strongly donor levels ($q_\Gamma(i) = j$) we obtain

$$\int_0^T |B(t)|^{q_\Gamma(i)}\,\nu(dt) \le C_0 T^{1-z(i)}, \tag{60}$$

where

$$z(i) = \frac{q_\Gamma(i)}{j}, \quad C_0 = \max\left(1, \int_0^\infty B^2(t)\nu(dt)\right).$$

Let now $0 < q_\Gamma(i) < j$, that is, a level i is a donor one, but not strongly donor, then

$$\int_0^T |B(t)|^{q_\Gamma(i)}\,\nu(dt) = \left[\int_0^1 + \int_1^T\right] |B(t)|^{q_\Gamma(i)}\,\nu(dt) \le$$

$$\le 1 + \frac{T^{1-\alpha q_\Gamma(i)} - 1}{1 - \alpha q_\Gamma(i)} = \frac{\alpha q_\Gamma(i)}{\alpha q_\Gamma(i) - 1} + \frac{T^{1-\alpha q_\Gamma(i)}}{1 - \alpha q_\Gamma(i)} = o\left(T^{1-z(i)}\right), \tag{61}$$

since $\alpha q_\Gamma(i) = \alpha j z(i)$, and $\alpha j > 1$. We will show that

$$\mu = 2 - \sum_{i=1}^4 z(i) = 0.$$

Indeed,

$$\sum_{i=1}^4 z(i) = 1 + \frac{q_\Gamma(2) + q_\Gamma(3)}{j},$$

and $q_\Gamma(2) + q_\Gamma(3) = j$, since $|Z_\Gamma| = 2j$.

Formulae (60), (61) and (57) together with (58) then imply that

$$|F_\Gamma(T)| = O(1), \quad T \to \infty, \tag{62}$$

when $\rho_{sd} = \rho_r = 2$, and

$$|F_\Gamma(T)| = o(1), \quad T \to \infty, \tag{63}$$

when $0 < q_\Gamma(i) < j$, for i=2,3 ($\rho_{sd} = \rho_r = 1$).

The estimate (62) is not exact. Thus, let us consider again the case of nonregular diagram Γ, which has 2 strongly donor levels and the rest 2 levels are recipient ones. The recipient level 3 takes edges from the strongly donor levels 1 and 2, while the level 2 does not supply the level 3 in full. Let us permutate the levels 2 and 3, and denote this permutation by π, that is $\pi(2) = 3, \pi(3) = 2$, and, from the level $\pi(3)$ to the level $\pi(2)$, there are less than j edges. Moreover, from the level $\pi(3)$ there is no edges down, except the edges which connect $\pi(3)$ with $\pi(2)$, since the level $\pi(3)$ took all edges from the top, i.e.,

$$q_{\pi\Gamma}(\pi(3)) = q_{\pi\Gamma}(2) < j,$$

where $\pi\Gamma$ is a nonregular diagram, taken from Γ by permutating the levels 2 and 3. Note that this permutation does not change the value of integral defining $F_\Gamma(T)$ in (55), since it is equivalent to the renaming of the variables t_2 and t_3. From (63), we then obtain

$$|F_\Gamma(T)| = |F_{\pi\Gamma}(T)| \to 0, \quad T \to \infty. \tag{64}$$

The assertion of this theorem then follows from equations (62), (63) and (64).

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