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A MULTIVARIATE GNEDENKO LAW OF LARGE NUMBERS

By Daniel Fresen

University of Missouri

In memory of Nigel J. Kalton

We show that the convex hull of a large i.i.d. sample from an absolutely continuous log-concave distribution approximates a predetermined body in the logarithmic Hausdorff distance and in the Banach-Mazur distance. For log-concave distributions that decay super-exponentially, we also have approximation in the Hausdorff distance. These results are multivariate versions of the Gnedenko law of large numbers, which guarantees concentration of the maximum and minimum in the one dimensional case.

We provide quantitative bounds in terms of the number of points and the dimension of the ambient space.

1. Introduction. The Gnedenko law of large numbers [11] states that if F is the cumulative distribution of a probability measure μ on $\mathbb{R}$ such that for all $\varepsilon > 0$

$$\lim_{t\to\infty} \frac{F(t+\varepsilon) - F(t)}{1 - F(t+\varepsilon)} = \infty \tag{1}$$

then there are functions δ, T and $\mathcal{P}$ defined on $\mathbb{N}$ with

$$\lim_{n\to\infty} \delta_n = 0 \tag{2}$$

$$\lim_{n\to\infty} \mathcal{P}_n = 1 \tag{3}$$

such that for any $n \in \mathbb{N}$ and any i.i.d. sample $(\gamma_i)_1^n$ from μ, with probability $\mathcal{P}_n$ we have

$$|\max\{\gamma_i\}_1^n - T_n| < \delta_n$$

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We define $0/0 = \infty$ to allow for the trivial case when μ has bounded support. The condition (1) implies super-exponential decay of the tail probabilities $1 - F(t)$, i.e. for all $c > 0$,

$$\lim_{t\to\infty} e^{ct}(1 - F(t)) = 0$$

The converse is almost true and can be achieved if we impose some sort of regularity on F. One such regularity condition is log-concavity (see Section 3). Of course all of this can be re-worded in multiplicative form. Provided $1-F(t)$ is regular enough and decays super-polynomially, i.e. for any $m \in \mathbb{N}$,

$$\lim_{t\to\infty} t^m(1 - F(t)) = 0$$

then (2) and (3) hold, and with probability $\mathcal{P}_n$,

$$\left|\frac{\max\{\gamma_i\}_1^n}{T_n} - 1\right| \leq \delta_n$$

Note that rapid decay of the left hand tail provides concentration of $\min\{\gamma_i\}_1^n$, and that $[\min\{\gamma_i\}_1^n, \max\{\gamma_i\}_1^n] = conv\{\gamma_i\}_1^n$.

In this paper we extend the Gnedenko law of large numbers to the multivariate setting. We consider a large collection of i.i.d. random vectors $\{x_i\}_1^n$ in $\mathbb{R}^d$ that follow a log-concave distribution μ with density function f. The object of interest is the convex hull $P_n = conv\{x_i\}_1^n$, which is called a random polytope. It is shown that with high probability, P_n approximates a deterministic body $F_{1/n}$ called the floating body, which is what remains of $\mathbb{R}^d$ after deleting all open half-spaces $\mathfrak{H}$ such that $\mu(\mathfrak{H}) < 1/n$. As in the one dimensional case, the way in which P_n approximates $F_{1/n}$ depends on how rapidly μ decays. Of primary interest is a quantitative analysis in terms of the number of points, and in this regard our results are essentially optimal (see Section 8).

The fact that the floating body can be used in order to model random polytopes is well known in the setting where μ is the uniform distribution on a convex body (see for example [4] and [6]). Our main contribution is to study this approximation in the more general setting of log-concave measures. Unlike the former case, the objects that we study can have many different shapes as $n \to \infty$ and are not limited to lie within a bounded region of space.

The notion of a multivariate Gnedenko law of large numbers has also been considered by Goodman [12] in the setting of Gaussian measures on seperable Banach spaces. In his paper he shows that with probability 1, the Hausdorff distance between the sample $\{x_i\}_1^n$ and the ellipsoid $\sqrt{2\log n}\,\mathcal{E}$

converges to zero as $n \to \infty$, where $\mathcal{E}$ is the unit ball of the reproducing kernel Hilbert space associated to μ.

2. Main Results. Let $d \geq 1$, $n \geq d+1$ and let μ be a log-concave probability measure on $\mathbb{R}^d$ with a density function $f = d\mu/dx$. This means that f is of the form $f(x) = \exp(-g(x))$ where g is convex. Let $(x_i)_1^n$ denote a sequence of i.i.d. random vectors in $\mathbb{R}^d$ with distribution μ, and consider the random polytope $P_n = conv\{x_i\}_1^n$. For any $x \in \mathbb{R}^d$, define

$$\widetilde{f}(x) = \inf_{\mathfrak{H}} \mu(\mathfrak{H})$$

where $\mathfrak{H}$ runs through the collection of all open half-spaces that contain x. For any $\delta > 0$, the *floating body* is defined as

$$F_\delta = \{x \in \mathbb{R}^d : \widetilde{f}(x) \geq \delta\} \tag{4}$$

Note that F_δ is the intersection of all closed half-spaces $\mathfrak{H}$ such that $\mu(\mathfrak{H}) \geq 1/n$, and is therefore convex. If $\mathfrak{H}$ is any open half-space that contains the centroid of μ, then $\mu(\mathfrak{H}) \geq e^{-1}$ (see Lemma 5.12 in [16] or Lemma 3.3 in [7]) hence F_δ is non-empty provided that $\delta \leq e^{-1}$. Such a floating body was defined by Schütt and Werner [23] in the case where μ is the uniform distribution on a convex body. We define the *logarithmic Hausdorff distance* between convex bodies $K, L \subset \mathbb{R}^d$ as,

$$d_{\mathfrak{L}}(K,L) = \inf\{\lambda \geq 1 : \exists x \in int(K \cap L),\ \lambda^{-1}(L-x)+x \subset K \subset \lambda(L-x)+x\}$$

where we use the convention that $\inf(\emptyset) = \infty$. The main result of the paper is as follows:

Theorem 1 *There exist universal constants $c, c', \widetilde{c} > 0$ with the following property. Let $q \geq 1$, $d \in \mathbb{N}$ and $n \geq c \exp\exp(5d) + c'q^3$. Let μ be a probability measure on $\mathbb{R}^d$ with a log-concave density function, $(x_i)_1^n$ an i.i.d. sample from μ, $P_n = conv\{x_i\}_1^n$ and $F_{1/n}$ the floating body as in (4). With probability at least $1 - 3^{d+3}(\log n)^{-q}$,*

$$d_{\mathfrak{L}}(P_n, F_{1/n}) \leq 1 + \widetilde{c}d(d+q)\frac{\log\log n}{\log n} \tag{5}$$

The strategy of the proof is to use quantitative bounds in the one dimensional case to analyze the Minkowski functional of P_n in different directions. The idea is simple, however there are some subtle complications. The lack of symmetry is a complicating factor, and the fact that the half-spaces of mass $1/n$ do not necessarily touch $F_{1/n}$ adds to the intricacy of the proof.

We define f to be p-log-concave if it is of the form $f(x) = c\exp(-g(x)^p)$ where g is a non-negative convex function and $c > 0$.

Theorem 2 *For all $q > 0$, $p > 1$ and $d \in \mathbb{N}$, and any probability measure μ on $\mathbb{R}^d$ with a non-vanishing p-log-concave density function, there exist $c, \widetilde{c} > 0$ such that for all $n \in \mathbb{N}$ with $n \geq d+2$, if $(x_i)_1^n$ is an i.i.d. sample from μ, $P_n = conv\{x_i\}_1^n$ and $F_{1/n}$ is the floating body as in (4), then with probability at least $1 - \widetilde{c}(\log n)^{-q}$ we have*

$$d_{\mathcal{H}}(P_n, F_{1/n}) \leq c \frac{\log\log n}{(\log n)^{1-\frac{1}{p}}} \tag{6}$$

Theorem 2 can easily be extended to a much larger class of log-concave distributions. Using Theorem 1, any bound on the growth rate of $diam(F_{1/n})$ automatically transfers to a bound on $d_{\mathcal{H}}(P_n, F_{1/n})$.

Our prototypical example is the class of distributions introduced by Schechtman and Zinn [22] of the form $f(x) = c_p^d \exp(-||x||_p^p)$, where $1 \leq p < \infty$ and $c_p = p/(2\Gamma(p^{-1}))$. For these distributions, $P_n \approx (\log(c_p^d n))^{1/p} B_p^d$. Of particular interest is the Gaussian distribution, where $p = 2$. In this case (actually for the standard Gaussian distribution), Bárány and Vu [5] obtained a similar approximation (see Remark 9.6 in their paper) and showed that there exist two radii R and r, both functions of n and d, such that for any fixed $d \geq 2$ both $r, R = (2\log n)^{1/2}(1 + o(1))$ as $n \to \infty$, and with 'high probability' $rB_2^d \subset P_n \subset RB_2^d$. Their sandwiching result served as a key step in their proof of the central limit theorem for Gaussian polytopes (asymptotic normality of various functionals such as the volume and the number of faces).

In the setting where μ is the uniform distribution on a convex body, the floating body is usually denoted by K_δ. In this context it is trivial that $\lim_{n\to\infty} d_{\mathcal{H}}(P_n, K) = 0$ (almost surely) and the phenomenon of interest is the rate at which P_n approached the boundary of K. Bárány and Larman [4] proved that for $n \geq n_0(d)$,

$$c' \mathrm{vol}_d(K \backslash K_{1/n}) \leq \mathbb{E}\mathrm{vol}_d(K \backslash P_n) \leq c''(d) \mathrm{vol}_d(K \backslash K_{1/n})$$

The reader may be interested to contrast our results with the results in [9]. The results presented here require a very large sample size and guarantee a precise approximation, somewhat in the spirit of the 'almost-isometric' theory of convex bodies. On the other hand, the results presented in [9] describe a type of approximation in the spirit of the 'isomorphic' theory, and are most interesting specifically in high dimensional spaces.

We also study two other deterministic bodies that serve as approximants to the random body. Define

$$f^\sharp(x) = \inf_{\mathcal{H}} \int_{\mathcal{H}} f(y) d_{\mathcal{H}}(y)$$

where $\mathcal{H}$ runs through the collection of all hyperplanes that contain x, and $d_{\mathcal{H}}$ stands for Lebesgue measure on $\mathcal{H}$. For any $\delta > 0$, define the bodies

$$D_\delta = Cl\{x \in \mathbb{R}^d : f(x) \geq \delta\}$$

$$R_\delta = Cl\{x \in \mathbb{R}^d : f^\sharp(x) \geq \delta\}$$

where $Cl(E)$ denotes the closure of a set E. By log-concavity of f, both D_δ and R_δ are convex.

Theorem 3 *Let $d \in \mathbb{N}$ and let μ be a probability measure on $\mathbb{R}^d$ with a non-vanishing log-concave density function. Then we have*

$$\lim_{\delta \to 0} d_{\mathfrak{L}}(F_\delta, D_\delta) = 1 \tag{7}$$

$$\lim_{\delta \to 0} d_{\mathfrak{L}}(F_\delta, R_\delta) = 1 \tag{8}$$

Similar results hold in the Hausdorff distance for log-concave distributions that decay super-exponentially.

Let $X \in \mathbb{R}^d$ be a random vector with distribution μ. The random variable $-\log f(X)$ is a sort of differential information content (see [8]). The differential entropy of μ is defined as

$$\begin{aligned} h(\mu) &= -\mathbb{E} \log f(X) \\ &= -\int_{\mathbb{R}^d} f(x) \log f(x) dx \end{aligned}$$

and the entropy power defined as $N(\mu) = \exp(2d^{-1}h(\mu))$. Note that the distribution of $-\log f(X)$ can be expressed in terms of the function $\delta \mapsto \mu(D_\delta)$,

$$\mathbb{P}\{-\log f(X) \leq t\} = \mu(D_\delta) \quad : \quad \delta = e^{-t}$$

Because of the rapid decay of f, the body D_δ acts as an essential support for the measure μ. For $\delta = e^{-d}$, this was studied by Klartag and Milman [14] (see Lemma 2.2 and Corollary 2.4 in their paper). Bobkov and Madiman later provided a more precise description. In [8] they show that the variance of $-\log f(X)$ is at most Cd, where $C > 0$ is a universal constant, and that in high dimensional spaces, $f(X)^{2/d}$ is strongly concentrated around $N(\mu)$. Theorem 1.1 in their paper can be written as

$$\mu\{x \in \mathbb{R}^d : N(\mu)^{d/2}\delta \leq f(x) \leq N(\mu)^{d/2}\delta^{-1}\} \geq 1 - 2\delta^{p(d)}$$

provided $\delta \in (0,1)$, where $p(d) = 16^{-1}d^{-1/2}$. In Lemma 16 we show that if μ is isotropic and has a continuous density function, then for all $\delta < \exp(-11d\log d - 7)$,

$$\mu\{x \in \mathbb{R}^d : f(x) \geq \delta\} \geq 1 - \alpha_d \delta (\log \delta^{-1})^d \tag{9}$$

where $\alpha_d = \exp(d^2(2\log d + c_1))$. In a fixed dimension, inequality (9) displays the natural quantitative behaviour of $\mu(D_\delta)$ as $\delta \to 0$ and is sharp up to a factor of $\log \delta^{-1}$.

Let $\mathcal{K}_d$ denote the collection of all convex bodies in $\mathbb{R}^d$. For all $K, L \in \mathcal{K}_d$, define

$$d_{BM}(K,L) = \inf\{\lambda \geq 1 : \exists x \in \mathbb{R}^d,\ \exists T,\ K \subset TL \subset \lambda(K-x)+x\} \tag{10}$$

where T represents any affine transformation of $\mathbb{R}^d$. This is a modification of the classical Banach-Mazur distance between normed spaces (origin symmetric bodies).

Theorem 4 *For all $d \in \mathbb{N}$, there exists a probability measure μ on $\mathbb{R}^d$ with the following universality property. Let $(x_i)_1^\infty$ be an i.i.d. sample from μ, and for each $n \in \mathbb{N}$ with $n \geq d+1$ let $P_n = conv\{x_i\}_1^n$. Then with probability 1, the sequence $(P_n)_{d+1}^\infty$ is dense in $\mathcal{K}_d$ with respect to d_{BM}.*

Throughout the paper we will make use of variables c, $\widetilde{c}$, c_1, c_2, n_0, m etc. At times they represent universal constants and at other times they depend on parameters such as the dimension d or the measure μ. Such dependence will always be clear from the context, and will either be indicated explicitly as c_d, $c(d)$, $n_0(d)$ etc., or implicitly as in Theorem 2, where c and $\widetilde{c}$ depend on q, p, d and μ. Half-spaces shall be indexed as $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle x, \theta\rangle \geq t\}$ and hyperplanes as $\mathcal{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle x, \theta\rangle = t\}$, where $\theta \in S^{d-1}$ and $t \in \mathbb{R}$.

3. Background. Most of the material in this section is discussed in [2], [3], [17] and [18]. We denote the standard Euclidean norm on $\mathbb{R}^d$ by $||\cdot||_2$. For any $\varepsilon > 0$, an ε-net in S^{d-1} is a subset $\mathcal{N}$ such that for any distinct $\omega_1, \omega_2 \subset \mathcal{N}$, $||\omega_1 - \omega_2||_2 > \varepsilon$, and for all $\theta \in S^{d-1}$ there exists $\omega \in \mathcal{N}$ such that $||\theta - \omega||_2 \leq \varepsilon$. Such a subset can easily be constructed using induction. By a standard volumetric argument we have

$$|\mathcal{N}| \leq \left(\frac{3}{\varepsilon}\right)^d \tag{11}$$

By induction, any $\theta \in S^{d-1}$ can be expressed as a series

$$\theta = \omega_0 + \sum_{i=1}^{\infty} \varepsilon_i \omega_i \tag{12}$$

where each $\omega_i \in \mathcal{N}$ and $0 \leq \varepsilon_i \leq \varepsilon^i$. To see this, express $\theta = \omega_0 + r_0$, where $\omega_0 \in \mathcal{N}$ and $||r_0||_2 \leq \varepsilon$. Then express $||r_0||^{-1} r_0 \in S^{d-1}$ in a similar fashion and iterate this procedure.

Define the functional

$$||x||_{\mathcal{N}} = \max\{\langle x, \omega\rangle : \omega \in \mathcal{N}\}$$

As an easy consequence of the Cauchy-Schwarz inequality, provided $\varepsilon \in (0,1)$ we have

$$(1-\varepsilon)||x||_2 \leq ||x||_{\mathcal{N}} \leq ||x||_2 \tag{13}$$

which implies that

$$B_2^d \subset B_{\mathcal{N}} \subset (1-\varepsilon)^{-1} B_2^d \tag{14}$$

where $B_{\mathcal{N}} = \{x : ||x||_{\mathcal{N}} \leq 1\}$. The body $B_{\mathcal{N}}$ is what remains if one deletes all half-spaces that are tangent to B_2^d at points in $\mathcal{N}$.

A *convex body* is a compact convex subset of Euclidean space with nonempty interior. For a convex body $K \subset \mathbb{R}^d$ that contains the origin as an interior point, its *Minkowski functional* is defined as

$$||x||_K = \inf\{\lambda > 0 : x \in \lambda K\}$$

for all $x \in \mathbb{R}^d$. By convexity of K, one can easily show that $||\cdot||_K$ obeys the triangle inequality. The dual Minkowski functional is defined as

$$||y||_{K^\circ} = \sup\{\langle x, y\rangle : x \in K\}$$

for all $y \in \mathbb{R}^d$, and the *polar* of K is

$$K^\circ = \{y \in \mathbb{R}^d : ||y||_{K^\circ} \leq 1\}$$

By the Hahn-Banach theorem, $K^{\circ\circ} = K$.

The *Hausdorff distance* $d_{\mathcal{H}}$ between K and L is defined as

$$d_{\mathcal{H}}(K, L) = \max\{\max_{k \in K} d(k, L); \max_{l \in L} d(K, l)\}$$

By convexity this reduces to

$$\begin{aligned} d_{\mathcal{H}}(K, L) &= \sup_{\theta \in S^{d-1}} |\sup_{k \in K} \langle k, \theta\rangle - \sup_{l \in L} \langle l, \theta\rangle| \\ &= \sup_{\theta \in S^{d-1}} |(||\theta||_{K^\circ} - ||\theta||_{L^\circ})| \end{aligned}$$

We define the *logarithmic Hausdorff distance* between K and L about a point $x \in int(K \cap L)$ as

$$d_{\mathfrak{L}}(K, L, x) = \inf\{\lambda \geq 1 : \lambda^{-1}(L - x) + x \subset K \subset \lambda(L - x) + x\}$$

provided $int(K \cap L) \neq \emptyset$, and

$$d_{\mathfrak{L}}(K, L) = \inf\{d_{\mathfrak{L}}(K, L, x) : x \in int(K \cap L)\}$$

Note that

$$\log d_{\mathfrak{L}}(K, L, 0) = \sup_{\theta \in S^{d-1}} |\log ||\theta||_K - \log ||\theta||_L|$$

The following relations follow from the definitions above,

$$\begin{aligned} d_{\mathfrak{L}}(K, L, 0) &= d_{\mathfrak{L}}(K^\circ, L^\circ, 0) \\ d_{\mathfrak{L}}(TK, TL) &= d_{\mathfrak{L}}(K, L) \end{aligned} \tag{15}$$

where T is any invertible affine transformation. In addition one can check that,

$$\begin{aligned} d_{BM}(K, L) &\leq d_{\mathfrak{L}}(K, L)^2 \\ d_{\mathcal{H}}(K, L) &\leq diam(K)(d_{\mathfrak{L}}(K, L) - 1) \end{aligned} \tag{16}$$

hence all of our bounds in terms of $d_{\mathfrak{L}}$ apply equally well to d_{BM}. For large bodies, $d_{\mathcal{H}}$ is more sensative than $d_{\mathfrak{L}}$. More precisely, if $r > 1$ and $rB_2^d + x \subset K$ for some $x \in \mathbb{R}^d$, and if $d_{\mathcal{H}}(K, L) \leq 1/2$, then

$$d_{\mathfrak{L}}(K, L) \leq 1 + 2r^{-1} d_{\mathcal{H}}(K, L) \tag{17}$$

By a simple compactness argument, there is an ellipsoid of maximal volume $\mathcal{E}_K \subset K$. This ellipsoid is called the *John ellipsoid* [3] associated to K. It can be shown that $\mathcal{E}_k$ is unique and has the property that $K \subset d(\mathcal{E}_k - x) + x$, where x is the center of $\mathcal{E}_k$. In particular, $d_{\mathfrak{L}}(\mathcal{E}_k, K) \leq d$.

In [10] it is shown that provided $\lambda < 8^{-d}$, we have

$$d_{\mathfrak{L}}(K, K_\lambda, x) \leq 1 + 8\lambda^{1/d} \tag{18}$$

where x is the centroid of K and K_δ is the floating body inside K.

The *cone measure* on ∂K is defined as

$$\mu_K(E) = \mathrm{vol}_d(\{r\theta : \theta \in E,\ r \in [0, 1]\})$$

for all measurable $E \subset \partial K$. The significance of the cone measure is that it leads to a natural polar integration formula (see [19]); for all $f \in L_1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(x)dx = d \int_0^\infty \int_{\partial K} r^{d-1} f(r\theta) d\mu_K(\theta) dr \tag{19}$$

A probability measure μ is called *isotropic* if its centroid lies at the origin and its covariance matrix is the $d \times d$ identity matrix.

A function $f : \mathbb{R}^d \to [0, \infty)$ is called *log-concave* (see [14]) if

$$f(\lambda x + (1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^d$ and all $\lambda \in (0,1)$. Any such function can be written in the form $f(x) = e^{-g(x)}$ where $g : \mathbb{R}^d \to (-\infty, \infty]$ is convex. If f is the density of a probability measure μ, then it must decay exponentially to zero. In this case g lies above a cone, i.e.

$$g(x) \geq m||x||_2 - c \tag{20}$$

with $m, c > 0$. As a consequence of the Prékopa-Leindler inequality [2], if x is a random vector with log-concave density and y is any fixed vector, then $\langle x, y \rangle$ has a log-concave density in $\mathbb{R}$. Log-concave functions are very rigid. One such example of this rigidity (see Lemma 5.12 in [16]) is the fact that if $\mathfrak{H}$ is any half-space containing the centroid of μ, then $\mu(\mathfrak{H}) \geq e^{-1}$. Another example (see Theorem 5.14 in [16]) is that if μ is isotropic, then

$$2^{-7d} \leq f(0) \leq d(20d)^{d/2} \tag{21}$$

$$(4\pi e)^{-d/2} \leq ||f||_{\infty} \leq 2^{8d} d^{d/2} \tag{22}$$

and if $||x||_2 \leq 1/9$ then,

$$2^{-8d} \leq f(x) \leq d2^d (20d)^{d/2} \tag{23}$$

Let $1 \leq p < \infty$. If $g : \mathbb{R}^d \to [0, \infty]$ is convex and $\lim_{x \to \infty} g(x) = \infty$, then the probability measure with density given by

$$f(x) = ce^{-g(x)^p}$$

will be called p-log-concave. This is a natural generalization of the normal distribution. If f is p-log-concave, then it is also p'-log-concave for all $1 \leq p' \leq p$.

Let $\mathbf{H}_d$ denote the collection of all $d-1$ dimensional affine subspaces (hyperplanes) of $\mathbb{R}^d$. The Radon transform of a log-concave function $f : \mathbb{R}^d \to [0, \infty)$ is the function $Rf : \mathbf{H}_d \to [0, \infty)$ defined by

$$Rf(\mathcal{H}) = \int_{\mathcal{H}} f(y) d_{\mathcal{H}}(y)$$

where $d_{\mathcal{H}}$ is Lebesgue measure on $\mathcal{H}$. The Radon transform is closely related to the Fourier transform. See [15] for a discussion of these operators and their connections to convex geometry.

4. The one dimensional case. Let f be a non-vanishing log-concave probability density function on $\mathbb{R}$ associated to a probability measure μ. In particular, $f(t) = e^{-g(t)}$ where $g : \mathbb{R} \to \mathbb{R}$ is convex. For $t \in \mathbb{R}$, define

$$\begin{aligned} J(t) &= \int_{-\infty}^{t} f(s)ds \\ u(t) &= -\log(1 - J(t)) \end{aligned}$$

The cumulative distribution function J is a strictly increasing bijection between $\mathbb{R}$ and $(0, 1)$. The following lemma is a standard result (see e.g. Theorem 5.1 in [16] for the statement, and the references given there). However we include a short proof here for completeness.

Lemma 5 *u is convex*

Proof. Assume momentarily that $g \in C^2(\mathbb{R})$. For $t \in (0, 1)$ define

$$\psi(t) = f(J^{-1}(1 - t))$$

Note that

$$\psi''(t) = \frac{-g''(J^{-1}(1 - t))}{\psi(t)} \leq 0$$

Hence ψ is concave. In addition, $\lim_{t\to 0} \psi(t) = \lim_{t\to 1} \psi(t) = 0$. Hence, the function $\kappa(t) = \psi(t)/t$ is non-increasing on $(0, 1)$ and the function $f(t)/(1 - J(t)) = \kappa(1 - J(t))$ is non-decreasing on $\mathbb{R}$. Since $u'(t) = f(t)/(1 - J(t))$, u is convex.

If $g \notin C^2(\mathbb{R})$, then the result follows by approximation (convolve μ with a Gaussian). ■

Lemma 6 *If $T \geq 1$ and $x > 2T \log T$, then $(\log x)/x < T^{-1}$.*

Proof. Since the function $y = e^{-1}x$ is tangent to the strictly concave function $y = \log x$, the function $y = (\log x)/x$ has a global maximum of e^{-1} and is decreasing on $[e, \infty)$. We now consider two cases. In case 1, $T < e$ and therefore $(\log x)/x \leq e^{-1} < T^{-1}$. In case 2, $T \geq e$. Since $(\log T)/T < 2^{-1}$, it follows that $\log(2 \log T) < \log T$. For $x' = 2T \log T$,

$$\frac{\log x'}{x'} = \frac{\log T + \log(2 \log T)}{2T \log T} < \frac{1}{T}$$

Since $x > x' > e$, $(\log x)/x < (\log x')/x'$ and the result follows. ■

The following lemma is a quantitative version of the Gnedenko law of large numbers for log-concave probability measures on $\mathbb{R}$.

Lemma 7 *Let $q \geq 1$ and $n \geq 120q^2(2+\log q)^2$. Let μ be a probability measure on $\mathbb{R}$ with a non-vanishing log-concave density function and cumulative distribution function J, and let $(\gamma_i)_1^n$ be an i.i.d. sample from μ. With probability at least $1-2(\log n)^{-q}$,*

$$\frac{\left|\gamma_{(n)} - J^{-1}(1-1/n)\right|}{J^{-1}(1-1/n) - \mathbb{E}\mu} \leq 6q\frac{\log\log n}{\log n} \tag{24}$$

where $\gamma_{(n)} = \max\{\gamma_i\}_1^n$ and $\mathbb{E}\mu$ denotes the centroid of μ.

Proof. We shall implicitly make use of Lemma 6 several times throughout the proof. Let $a = (\log n)^{-q}$ and $b = q\log n$. It follows that $0 < a < b < ne^{-1}$. Set $s = J^{-1}(1-b/n)$ and $t = J^{-1}(1-a/n)$. As mentioned in the preliminaries (see also Lemma 3.3 in [7]), $1 - J(\mathbb{E}\mu) \geq e^{-1}$, hence $u(\mathbb{E}\mu) \leq 1$. Since $b/n < e^{-1}$, we have $\mathbb{E}\mu < s < t$. By convexity of u we have the inequality $(s-\mathbb{E}\mu)^{-1}(u(s)-u(\mathbb{E}\mu)) \leq (t-s)^{-1}(u(t)-u(s))$ which can be rewritten as

$$\frac{J^{-1}(1-a/n) - J^{-1}(1-b/n)}{J^{-1}(1-b/n) - \mathbb{E}\mu} \leq \frac{\log b - \log a}{\log n - \log b - 1} \tag{25}$$

Since $2qe\log\sqrt{n} \leq \sqrt{n}$, it follows that $\log(qe\log n) \leq \frac{1}{2}\log n$ which implies that

$$\frac{\log b - \log a}{\log n - \log b - 1} \leq \frac{3q\log\log n}{\log n - \log(qe\log n)} \leq 6q\frac{\log\log n}{\log n}$$

By independence,

$$\begin{aligned}
\mathbb{P}\{J^{-1}(1-b/n) &\leq \gamma_{(n)} \leq J^{-1}(1-a/n)\} \\
&= \left(1-\frac{a}{n}\right)^n - \left(1-\frac{b}{n}\right)^n \\
&\geq 1 - a - e^{-b} \\
&\geq 1 - 2(\log n)^{-q}
\end{aligned}$$

If the event $\{J^{-1}(1-b/n) \leq \gamma_{(n)} \leq J^{-1}(1-a/n)\}$ occurs, then the event defined by inequality (24) also occurs. ■

Although Lemma 7 applies to the multiplicative version of the Gnedenko law of large numbers, it also recovers the additive version as long as

$$J^{-1}(1-1/n) = o\left(\frac{\log n}{\log\log n}\right) \tag{26}$$

If, in the proof, we take $a^{-1} = b = \log_{(m)} n$ (the mth iterate of the logarithm), then the probability bound becomes $1 - 2(\log_{(m)} n)^{-1}$, and the right hand side of (24) becomes

$$\frac{4\log_{(m+1)} n}{\log n}$$

provided $n > n_0(m)$.

5. Main Proofs. Since Lebesgue measure depends on the underlying Euclidean structure of $\mathbb{R}^d$, so does the definition of $f = d\mu/dx$, and therefore also the definition of $D_\delta = Cl\{x : f(x) \geq \delta\}$. A natural variation of the body D_δ which does not depend on Euclidean structure is the body

$$D_\delta^\natural = Cl\{x \in \mathbb{R}^d : f(x) \geq \tau_d^{-1} 9^d |\det cov(\mu)|^{-1/2}\delta\}$$

where the quantity

$$\tau_d = \mathrm{vol}_{d-1}(B_2^{d-1}) \int_{1/2}^1 (1-t^2)^{(d-1)/2} dt \tag{27}$$

represents the volume of the set $\{x \in \mathbb{R}^d : ||x||_2 \leq 1,\, x_1 \geq 1/2\}$. Associated to $D_\delta^\natural$ are three ellipsoids that play a central role in our proof. The John ellipsoid of $D_\delta^\natural$ is denoted $\mathcal{E}_{D_\delta^\natural}$ and the centroid of $\mathcal{E}_{D_\delta^\natural}$ will be denoted $\mathcal{O}_\delta$. We also consider

$$\mathcal{E}_\delta^\sharp = 3d(\mathcal{E}_{D_\delta^\natural} - \mathcal{O}_\delta) + \mathcal{O}_\delta \tag{28}$$

and

$$\mathcal{E}_\delta^\flat = \frac{1}{2}(\mathcal{E}_{D_\delta^\natural} - \mathcal{O}_\delta) + \mathcal{O}_\delta \tag{29}$$

The advantage of using $D_\delta^\natural$ is that we may place μ in different positions at various stages of our analysis. We first position μ to be isotropic and then position it so that $\mathcal{E}_{D_\delta^\natural} = B_2^d$. We include the proofs of Lemma 8 and Lemma 9 in Section 6.

Lemma 8 *There exists a universal constant $c > 0$ with the following property. Let $d \in \mathbb{N}$, let μ be a log-concave probability measure with a continuous density function f, and let $\delta < c\exp(-5d^2 \log d)$. Let $\mathfrak{H}$ be a half-space (either open or closed) with $\mu(\mathfrak{H}) = \delta$ and let $\mathcal{E}_\delta^\sharp$ and $\mathcal{E}_\delta^\flat$ be defined by (28) and (29) respectively. Then*

$$\mathfrak{H} \cap \mathcal{E}_\delta^\sharp \neq \emptyset \tag{30}$$

$$\mathfrak{H} \cap \mathcal{E}_\delta^\flat = \emptyset \tag{31}$$

Consequently,

$$\mathcal{E}_\delta^\flat \subset F_\delta \subset \mathcal{E}_\delta^\sharp$$

We shall use the Euclidean structure corresponding to $\mathcal{E}_{D_\delta^\natural}$ in order to compare $F_{1/n}$ and P_n. The following lemma together with Lemma 8 allows us to do so.

Lemma 9 *Let $d \in \mathbb{N}$ and let K and L be convex bodies in $\mathbb{R}^d$ such that $rB_2^d \subset K \subset RB_2^d$ for some $r, R > 0$. Let $0 < \rho < 1/2$ and $0 < \varepsilon < (16R/r)^{-1}$, and let $\mathcal{N}$ be an ε-net in S^{d-1}. Suppose that for each $\omega \in \mathcal{N}$,*

$$(1-\rho)||\omega||_L \leq ||\omega||_K \leq (1+\rho)||\omega||_L \tag{32}$$

Then for all $x \in \mathbb{R}^d$ we have

$$(1+2\rho+28Rr^{-1}\varepsilon)^{-1}||x||_L \leq ||x||_K \leq (1+2\rho+28Rr^{-1}\varepsilon)||x||_L \tag{33}$$

In particular,

$$d_{\mathfrak{L}}(K,L) \leq d_{\mathfrak{L}}(K,L,0) \leq 1+2\rho+28Rr^{-1}\varepsilon \tag{34}$$

Proof of Theorem 1 . By convolving μ with a Gaussian measure of the form

$$\phi_{\lambda,d}(x) = \lambda^{-d}\phi_d(\lambda^{-1}x)$$

where $\phi_d(x) = (2\pi)^{-d}\exp(-2^{-1}||x||_2^2)$ is the standard normal density function, and taking $\lambda \to 0$, we may assume that the density of μ is continuous and non-vanishing. This is possible because the bounds in the theorem do not depend on μ. The condition $n \geq c\exp\exp(5d) + c'q^3$ insures that the probability bound is non-trivial. It is also sufficiently large so that we may use Lemma 8 with $\delta = 1/n$ and Lemma 7 with $q' = d+q$. Let $\varepsilon = (\log n)^{-1}$. By applying a suitable affine transformation, we may assume that $\mathcal{E}_{D_{1/n}^\natural} = B_2^d$. By Lemma 8, if $\mathfrak{H}_{\theta,t}$ is a half-space with $\mu(\mathfrak{H}_{\theta,t}) = 1/n$, then

$$1/2 \leq t \leq 3d \tag{35}$$

This implies that $1/2B_2^d \subset F_{1/n} \subset 3dB_2^d$. For each $\theta \in S^{d-1}$, the function $f_\theta(t) = -\frac{d}{dt}\mu(\mathfrak{H}_{\theta,t})$ is the density of a log-concave probability measure μ_θ on $\mathbb{R}$ with cumulative distribution function $J_\theta(t) = 1-\mu(\mathfrak{H}_{\theta,t})$. Furthermore, the sequence $(\langle\theta, x_i\rangle)_{i=1}^n$ is an i.i.d. sample from this distribution. Recalling the definition of the dual Minkowski functional, for any $y \in \mathbb{R}^d$

$$\begin{aligned} ||y||_{P_n^\circ} &= \sup\{\langle x, y\rangle : x \in P_n\} \\ &= \max_{i=1\ldots n}\langle x_i, y\rangle \end{aligned}$$

We use this notation even when $0 \notin P_n$. Let $\mathcal{N}$ denote a generic ε-net in S^{d-1} and consider the function

$$\widetilde{f}_{\mathcal{N}}(x) = \inf\{\mu(\mathfrak{H}_{\omega,t}) : \omega \in \mathcal{N},\ t = \langle \omega, x\rangle\}$$

For all $\delta > 0$, define the discrete floating body

$$F_\delta^{\mathcal{N}} = \{x \in \mathbb{R}^d : \widetilde{f}_{\mathcal{N}}(x) \geq \delta\}$$

Note that $\widetilde{f}(x) = \inf_{\mathcal{N}} \widetilde{f}_{\mathcal{N}}(x)$ and $F_\delta = \cap_{\mathcal{N}} F_\delta^{\mathcal{N}}$, where $\mathcal{N}$ runs through the collection of all ε-nets in S^{d-1}. By (35), $\frac{1}{2}B_2^d \subset F_{1/n}^{\mathcal{N}} \subset 3dB_{\mathcal{N}}$ and by (13) we have $1/2B_2^d \subset F_{1/n}^{\mathcal{N}} \subset 4dB_2^d$ which implies that $(4d)^{-1}B_2^d \subset (F_{1/n}^{\mathcal{N}})^\circ \subset 2B_2^d$. For each $\theta \in S^{d-1}$, we have

$$\begin{aligned} \mathbb{E}\mu_\theta &\geq J_\theta^{-1}(e^{-1}) \\ &\geq J_\theta^{-1}(1/n) \end{aligned}$$

Combining this and (24), with probability at least $1 - 2(\log n)^{-d-q}$ we have that,

$$\frac{\left|||\theta||_{P_n^\circ} - J_\theta^{-1}(1-1/n)\right|}{J_\theta^{-1}(1-1/n) - J_\theta^{-1}(1/n)} \leq 6(d+q)\frac{\log\log n}{\log n}$$

Since both $-J_\theta^{-1}(1/n)$ and $J_\theta^{-1}(1-1/n)$ lie in the interval $[1/2, 3d]$, both have roughly the same order of magnitude and we have

$$(1-\rho)J_\theta^{-1}(1-1/n) \leq ||\theta||_{P_n^\circ} \leq (1+\rho)J_\theta^{-1}(1-1/n)$$

where

$$\rho = 42d(d+q)\frac{\log\log n}{\log n}$$

With probability at least $1 - \varepsilon^{-d}3^{d+1}(\log n)^{-d-q} = 1 - 3^{d+1}(\log n)^{-q}$, this happens for all $\omega \in \mathcal{N}$. Hence,

$$(1+\rho)^{-1}P_n \subset F_{1/n}^{\mathcal{N}}$$

which implies that

$$(1-\rho)||\theta||_{P_n^\circ} \leq ||\theta||_{(F_{1/n}^{\mathcal{N}})^\circ}$$

for all $\theta \in S^{d-1}$. On the other hand, for all $\omega \in \mathcal{N}$ we have

$$\begin{aligned} ||\omega||_{P_n^\circ} &\geq (1-\rho)J_\omega^{-1}(1-1/n) \\ &\geq (1-\rho)||\omega||_{(F_{1/n}^{\mathcal{N}})^\circ} \end{aligned}$$

By (33),

$$(1+4\rho+224d\varepsilon)^{-1}||x||_{P_n^\circ} \leq ||x||_{(F_{1/n}^{\mathcal{N}})^\circ} \leq (1+4\rho+224d\varepsilon)||x||_{P_n^\circ} \tag{36}$$

for all $x \in \mathbb{R}^d$. Let $\mathcal{M}$ be any other ε-net in S^{d-1}. By the calculations above, with probability at least $1-3^{d+1}(\log n)^{-q}$,

$$(1+4\rho+224d\varepsilon)^{-1}||x||_{P_n^\circ} \leq ||x||_{(F_{1/n}^{\mathcal{M}})^\circ} \leq (1+4\rho+224d\varepsilon)||x||_{P_n^\circ} \tag{37}$$

for all $x \in \mathbb{R}^d$. By the union bound, with probability at least $1-3^{d+2}(\log n)^{-q} > 0$, both (36) and (37) hold. Since both $F_{1/n}^{\mathcal{N}}$ and $F_{1/n}^{\mathcal{M}}$ are deterministic bodies, the only way that this can be true is if

$$(1+4\rho+224d\varepsilon)^{-2}F_{1/n}^{\mathcal{N}} \subset F_{1/n}^{\mathcal{M}} \subset (1+4\rho+224d\varepsilon)^2 F_{1/n}^{\mathcal{N}}$$

Since $F_{1/n} = \cap_{\mathcal{M}} F_{1/n}^{\mathcal{M}}$, where the intersection is taken over all ε-nets in S^{d-1}, we have

$$(1+4\rho+224d\varepsilon)^{-2}F_{1/n}^{\mathcal{N}} \subset F_{1/n} \subset (1+4\rho+224d\varepsilon)^2 F_{1/n}^{\mathcal{N}}$$

Combining this with the polar of (36) gives that with probability at least $1-3^{d+3}(\log n)^{-q}$ we have

$$(1+4\rho+224d\varepsilon)^{-3}P_n \subset F_{1/n} \subset (1+4\rho+224d\varepsilon)^3 P_n$$

from which the result follows by the inequality $(1+\varepsilon')^3 \leq 1+12\varepsilon'$ valid if $0 \leq \varepsilon' \leq 1$. ■

Lemma 10 *Let $g : \mathbb{R}^d \to [0,\infty)$ be convex with $\lim_{x\to\infty} g(x) = \infty$, let $K \subset \mathbb{R}^d$ be a convex body containing 0 in its interior, and let $p > 1$. Then there exist $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}^d$,*

$$g(x)^p \geq c_1||x||_K^p - c_2 \tag{38}$$

Proof. We leave the easy proof of this to the reader. ■

Lemma 11 *Let $p > 1$, $d \in \mathbb{N}$ and let μ be a p-log-concave probability measure on $\mathbb{R}^d$. Then there exist $c_1, c_2, t_0 > 0$ such that for all $\theta \in S^{d-1}$ and all $t \geq t_0$,*

$$\mu(\mathfrak{H}_{\theta,t}) \leq c_1 t^{1-p} e^{-c_2 t^p} \tag{39}$$

where $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle x, \theta\rangle \geq t\}$.

Proof. For all $t \geq 1$ we have

$$\begin{aligned} e^{-t^p} &\leq -\frac{d}{dt}\left(p^{-1}t^{1-p}e^{-t^p}\right) \\ &= p^{-1}(p-1)t^{-p}e^{-t^p} + e^{-t^p} \\ &\leq p^{-1}(2p-1)e^{-t^p} \end{aligned}$$

Hence, by the fundamental theorem of calculus,

$$(2p-1)^{-1}t^{1-p}e^{-t^p} \leq \int_t^\infty e^{-s^p}ds \leq p^{-1}t^{1-p}e^{-t^p} \tag{40}$$

Since the image of a p-log-concave probability measure under an orthogonal transformation is p-log-concave, we may assume without loss of generality that $\theta = e_1 = (1,0,0,0\ldots)$. By (38), there exist $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}^d$,

$$f(x) \leq c_1 e^{-c_2||x||_p^p}$$

where $||x||_p^p = \sum_{i=1}^d |x_i|^p$. Hence,

$$\begin{aligned} \mu(\mathfrak{H}_{\theta,t}) &\leq \int_{\mathfrak{H}_{\theta,t}} c_1 e^{-c_2||x||_p^p}dx \\ &= \int_t^\infty c_3 e^{-c_2 s^p}ds \end{aligned} \tag{41}$$

The result now follows from a change of variables, (41) and (40). ■

Proof of Theorem 2. Let c_1, c_2 and t_0 be the constants appearing in Lemma 11. Let $n_0 > c_1 + \exp(2^{-1}c_2 t_0^p)$. Without loss of generality, $t_0 > 1$ and $n > n_0$. Set $\alpha = (2c_2^{-1}\log n)^{1/p}$ and consider any $x \in \mathbb{R}^d$ with $||x||_2 > \alpha$. Let $\theta = ||x||_2^{-1}x$ and $t = (\alpha + ||x||_2)/2$. Since $t > \alpha > t_0$ and $n > c_1$, Lemma 11 implies that

$$\mu(\mathfrak{H}_{\theta,t}) < c_1 n^{-2} < n^{-1}$$

Since $||x||_2 > t$, $x \in int(\mathfrak{H}_{\theta,t})$. By definition of the floating body, $x \notin F_{1/n}$. Since this is true for all such x, $diam(F_{1/n}) \leq 2\alpha = c_4(\log n)^{1/p}$. The result now follows from Theorem 1 and the relation (16) between the Hausdorff and the logarithmic Hausdorff distances. ■

6. Technical Lemmas. This section contains some technical results on the rigidity of log-concave functions that enable us to obtain a lower bound on the sample size.

Lemma 12 *There exist universal constants $c_1, c_2 > 0$ such that for all $d \in \mathbb{N}$,*

$$c_1^d d^{-d/2} \leq \mathrm{vol}_d(B_2^d) \leq c_2^d d^{-d/2}$$

Proof. This follows from Stirling's formula and the expression $\mathrm{vol}_d(B_2^d) = \pi^{d/2}(\Gamma(1+d/2))^{-1}$ (see Corollary 2.20 in [15] or p.11 in [20]). ■

Lemma 13 *There exists a universal constant $c > 0$ with the following property. Let $d \in \mathbb{N}$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^d$ with density function f. For all $x \in \mathbb{R}^d$,*

$$f(x) \leq e^{-\alpha_d ||x||_2 + \beta_d}$$

where $\alpha_d = c^d d^{-d/2}$ and $\beta_d = 11d\log(d) + 7$.

Proof. We first consider the case $d \geq 2$. The volume of a cone in $\mathbb{R}^d$ with height h and base radius r is $d^{-1}r^{d-1}h\mathrm{vol}_{d-1}(B_2^{d-1})$. For any $x \in \mathbb{R}^d$, let A_x be the cone with vertex x and base $(1/9)B_2^d \cap x^\perp$. Then $\mathrm{vol}_d(A_x) = d^{-1}9^{-d+1}||x||_2\mathrm{vol}_{d-1}(B_2^{d-1}) > e^{-4d+3}||x||_2\mathrm{vol}_{d-1}(B_2^{d-1})$. By log-concavity of f and inequality (23), for all $y \in A_x$,

$$f(y) \geq \min\{f(x), 2^{-8d}\} \tag{42}$$

If $f(x) \geq 2^{-8d}$, then

$$1 \geq \int_{A_x} f(y)dy \geq 2^{-8d}\mathrm{vol}_d(A_x) > e^{-10d+3}||x||_2\mathrm{vol}_{d-1}(B_2^{d-1})$$

and it follows that

$$||x||_2 < \frac{e^{10d-3}}{\mathrm{vol}_{d-1}(B_2^{d-1})}$$

Hence, if $||x||_2 \geq e^{10d-3}/\mathrm{vol}_{d-1}(B_2^{d-1})$ then $f(x) < 2^{-8d}$ and (42) becomes $f(y) \geq f(x)$. Then

$$1 \geq \int_{A_x} f(y)dy \geq f(x)\mathrm{vol}_d(A_x) > f(x)e^{-4d+3}||x||_2\mathrm{vol}_{d-1}(B_2^{d-1})$$

which implies that

$$f(x) < \frac{e^{4d-3}}{\mathrm{vol}_{d-1}(B_2^{d-1})}||x||_2^{-1}$$

If $\widetilde{x} \in \mathbb{R}^d$ obeys $||\widetilde{x}||_2 = e^{10d-3}/\mathrm{vol}_{d-1}(B_2^{d-1})$, then

$$f(\widetilde{x}) < e^{-6d} \tag{43}$$

For any $x \in \mathbb{R}^d$ with $||x||_2 \geq e^{10d-3}/\mathrm{vol}_{d-1}(B_2^{d-1})$, we have the convex combination

$$\frac{e^{10d-3}}{\mathrm{vol}_{d-1}(B_2^{d-1})}\frac{x}{||x||_2} = \frac{e^{10d-3}}{||x||_2\mathrm{vol}_{d-1}(B_2^{d-1})}x + \left(1 - \frac{e^{10d-3}}{||x||_2\mathrm{vol}_{d-1}(B_2^{d-1})}\right)0$$

Set

$$\widetilde{x} = \frac{e^{10d-3}}{\mathrm{vol}_{d-1}(B_2^{d-1})} \frac{x}{||x||_2}$$

Using concavity of $\log f$ and inequality (43),

$$-6d \geq \left(\frac{e^{10d-3}}{||x||_2 \mathrm{vol}_{d-1}(B_2^{d-1})} \right) \log f(x) + \left(1 - \frac{e^{10d-3}}{||x||_2 \mathrm{vol}_{d-1}(B_2^{d-1})} \right) \log f(0)$$

After some simplification, and using inequality (21), we get

$$f(x) \leq \exp\left(-de^{-10d+3} \mathrm{vol}_{d-1}(B_2^{d-1}) ||x||_2 - 7d \log 2 \right)$$

If, on the other hand, $||x||_2 < e^{10d-3}/\mathrm{vol}_{d-1}(B_2^{d-1})$, then by (22)

$$\begin{aligned} f(x) &\leq ||f||_\infty \leq d^{d/2} 2^{8d} \\ &\leq \exp(-de^{-10d+3} \mathrm{vol}_{d-1}(B_2^{d-1}) ||x||_2 + 11d \log d) \end{aligned}$$

The case $d = 1$ is simpler and we leave the details to the reader. First show that $f(2^8) \leq 2^{-8}$ and then proceed as in the case $d \geq 2$ to obtain $f(x) \leq \exp(-2^{-9}|x| + 7)$ for all $x \in \mathbb{R}$. The result now follows from Lemma 12. ■

Corollary 14 *There exist universal constants $c_1, c_2 > 0$ with the following property. Let $d \in \mathbb{N}$ and let μ be an absolutely continuous isotropic log-concave probability measure. For all $\delta < e^{-11d \log d - 7}$,*

$$D_\delta \subset c_1^d d^{d/2} (\log \delta^{-1}) B_2^d$$

In particular, $\mathrm{vol}_d(D_\delta) \leq c_2 \exp(d^2 \log d)(\log \delta^{-1})^d$.

Proof. By (21), $D_\delta \neq \emptyset$. By the bounds on δ, it follows that $11d \log d + 7 \leq \log \delta^{-1}$. The result now follows from lemmas 13 and 12. ■

Lemma 15 *There exists a universal constant $c > 0$ with the following property. Let $d \in \mathbb{N}$ and let μ be an isotropic log-concave probability measure with density f. Let $r > 1$ and $x \in \mathbb{R}^d$. If $f(x) < 2^{-8d}$ then*

$$f(rx) \leq f(x) \exp(-c^d d^{-d/2} (r-1) ||x||_2) \tag{44}$$

Proof. Let $g = -\log f$. By Lemma 13 and Lemma 12, there exists a universal constant $c_2 > 0$ such that $f(\widetilde{x}) \leq e^{-6d}$ for all $\widetilde{x}$ with $||\widetilde{x}||_2 \geq c_2^d d^{d/2}$, see in particular (43). Let $x \in \mathbb{R}^d$ be the point specified in the statement of the lemma. We consider two cases. In the first case $||x||_2 \geq c_2^d d^{d/2}$. Let

$\widetilde{x} = c_2^d d^{d/2} ||x||_2^{-1} x$. By inequality (21), $f(0) \geq 2^{-7d}$. By convexity of g and the definition of c_2,

$$\frac{g(rx) - g(x)}{(r-1)||x||_2} \geq \frac{g(\widetilde{x}) - g(0)}{||\widetilde{x}||} = ||\widetilde{x}||_2^{-1} \ln \frac{f(0)}{f(\widetilde{x})} \geq c_2^{-d} d^{1-d/2}$$

In the second case, $||x||_2 < c_2^d d^{d/2}$. Recall that, by hypothesis, $f(x) < 2^{-8d}$. Therefore,

$$\frac{g(rx) - g(x)}{(r-1)||x||_2} \geq \frac{g(x) - g(0)}{||x||} \geq ||x||_2^{-1} \ln \frac{f(0)}{f(x)} \geq \ln(2) c_2^{-d} d^{1-d/2}$$

from which the result follows with $c = (2c_2)^{-1}$. ■

Lemma 16 *There exists a universal constant $c_1 > 0$ with the following property. Let $d \in \mathbb{N}$ and let μ be an isotropic log-concave probability measure with a continuous density function f. For all $\delta < e^{-11d \log d - 7}$,*

$$\mu(\mathbb{R}^d \backslash D_\delta) \leq \alpha_d \delta (\log \delta^{-1})^d \tag{45}$$

where $\alpha_d = c_1 \exp(3d^2 \log d)$.

Proof. Since f is continuous, for all $\theta \in \partial D_\delta$ we have $f(\theta) = \delta$. By the polar integration formula (19) and inequality (44),

$$\begin{aligned}
\mu(\mathbb{R}^d \backslash D_\delta) &= \int_{\mathbb{R}^d \backslash D_\delta} f(x) dx \\
&= d \int_1^\infty \int_{\partial D_\delta} r^{d-1} f(r\theta) d\mu_{D_\delta}(\theta) dr \\
&\leq d \int_1^\infty \int_{\partial D_\delta} r^{d-1} \delta \exp(-c^d d^{-d/2} (r-1) ||\theta||_2) d\mu_{D_\delta}(\theta) dr
\end{aligned}$$

By (23) and the fact that $\delta < 2^{-8d}$, we have $1/9 B_2^d \subset D_\delta$. By Corollary 14,

$$\begin{aligned}
\mu(\mathbb{R}^d \backslash D_\delta) &\leq d \int_1^\infty \int_{\partial D_\delta} r^{d-1} \delta \exp(-c^d d^{-d/2} (r-1) 9^{-1}) d\mu_{D_\delta}(\theta) dr \\
&= \delta \mathrm{vol}_d(D_\delta) d \int_1^\infty r^{d-1} \exp(-c_2^d d^{-d/2} (r-1)) dr \\
&\leq \beta_d \delta (\log \delta^{-1})^d d \exp(d^2 \log d + c_3))
\end{aligned}$$

where

$$\beta_d = \int_1^\infty r^{d-1} \exp(-c_2^d d^{-d/2} (r-1)) dr$$

Set $\omega_d = c_2^d d^{-d/2}$ and $t = \omega_d r$. Recall the definition of the gamma function $\Gamma(z) = \int_0^\infty e^{-r} r^{z-1} dr$. By a change of variables and Stirling's formula,

$$\begin{aligned} \beta_d &\leq \exp(\omega_d) \int_0^\infty r^{d-1} \exp(-\omega_d r) dr \\ &\leq c_4 \omega_d^{-d} \int_0^\infty t^{d-1} e^{-t} dt \\ &\leq \exp(2^{-1} d^2 \log d + c_5 d^2) \end{aligned}$$

from which the result follows. ■

Lemma 16 is optimal in δ up to a factor $\log \delta^{-1}$ as can be seen from the example $f(x) = 2^{-d} \exp(-||x||_2)$, in which case $\mu(\mathbb{R}^d \backslash D_\delta) \geq c_d \delta (\log \delta^{-1})^{d-1}$ for $\delta < \delta_0(d)$. To see this, apply the polar integration formula just as in the proof of Lemma 16.

Lemma 17 *There exists a universal constant $c > 0$ such that for all $d \in \mathbb{N}$, if $t > d^{5d}$ then $\sqrt{t} \geq c(\log t)^d$.*

Note that the inequality fails for $t = d^{2d}$.

Proof. There exists $d_0 \in \mathbb{N}$ such that for all $d > d_0$, $(2d)^{4d} < d^{5d}$. Consider any such d. Set $T = 2d$ and $x = \log t$. Since $(2d)^{4d} < d^{5d} < t$, it follows that $2T \log T < x$. By Lemma 6, $(\log x)/x < T^{-1}$, or equivalently

$$\frac{\log \log t}{\log t} < \frac{1}{2d}$$

which is in turn equivalent to $\sqrt{t} \geq (\log t)^d$. By elementary analysis, the number

$$c' = \inf\{t^{1/2} (\log t)^{-d} : d \leq d_0, t > d^{5d}\}$$

is strictly positive and the result follows with $c = \min\{c', 1\}$. ■

Lemma 18 *There exists a universal constant $\widetilde{c} > 0$ with the following property. Let $d \in \mathbb{N}$ and let μ be an isotropic log-concave probability measure with a continuous density function f. For all $\delta < \widetilde{c} \exp(-5d^2 \log d)$,*

$$\mu(\mathbb{R}^d \backslash 2D_{\tau^{-1} 9^d \delta}) < \delta$$

where $\tau = \tau_d = \mathrm{vol}_{d-1}(B_2^{d-1}) \int_{1/2}^1 (1-t^2)^{(d-1)/2} dt$.

Proof. Consider the quantity $\alpha_d = \exp(d^2(2 \log d + c_1))$. By concavity, $1 - t^2 \geq 3(1-t)/2$ for all $1/2 \leq t \leq 3/4$. By a change of variables and Lemma 12, one sees that $\tau > c_2^d d^{-d/2}$. Let $\kappa = \tau^{-1} 9^d \delta$. Consider any $y \in \partial(2D_\kappa)$.

Then $x = y/2 \in \partial D_\kappa$ and we have the convex combination $x = \frac{1}{2}0 + \frac{1}{2}y$. By log-concavity, $f(x) \geq f(0)^{1/2} f(y)^{1/2}$ and by inequality (21),

$$f(y) \leq \frac{f(x)^2}{f(0)} < 2^{8d}\kappa^2$$

and $y \notin D_\varepsilon$ with $\varepsilon = 2^{8d}\kappa^2$. Since this is true for all $y \in \partial(2D_\kappa)$, $D_\varepsilon \subset 2D_\kappa$. For a sufficiently small choice of $\widetilde{c}$ (chosen independently of d), $\varepsilon < e^{-11d\log d - 7}$. By Lemma 16,

$$\begin{aligned}
\mu(\mathbb{R}^d \backslash 2D_\kappa) &\leq \mu(\mathbb{R}^d \backslash D_\varepsilon) \leq \alpha_d \varepsilon (\log \varepsilon^{-1})^d \\
&\leq e^{10d}\alpha_d \tau^{-2}\delta^2 (2\log\delta^{-1} - \log(c_3^d \tau^{-2}))^d \\
&\leq e^{10d}\alpha_d \tau^{-2}\delta^2 (\log\delta^{-1})^d \\
&= \delta(ce^{10d}\alpha_d\tau^{-2}\delta^{1/2})c^{-1}\delta^{1/2}(\log\delta^{-1})^d
\end{aligned}$$

where c is the constant from Lemma 17. By the bound imposed on δ, $ce^{10d}\alpha_d\tau^{-2}\delta^{1/2} < 1$. The result now follows from Lemma 17. ■

Recall that $\mathcal{E}_K$ denotes the John ellipsoid of a convex body K and that $\mathcal{E}_K \subset K \subset d(\mathcal{E}_K - x_0) + x_0$, where x_0 is the center of $\mathcal{E}_K$.

Lemma 19 *Let $K \subset \mathbb{R}^d$ be a convex body with $0 \in K$. Then $2K \subset 3d(\mathcal{E}_K - x_0) + x_0$.*

Proof. By applying a suitable linear transformation, we may assume that $\mathcal{E}_K = B_2^d + x_0$. Take any $x \in K$. Since $\max\{||x_0 - x||_2, ||x_0||_2\} \leq d$, it follows that $||x||_2 \leq ||x_0||_2 + ||x - x_0||_2 \leq 2d$ and that $||x_0 - 2x||_2 \leq ||x_0 - x||_2 + ||x - 2x||_2 \leq 3d$. ■

Lemma 20 *Let $\mathcal{E}$ be an ellipsoid with centroid $\mathcal{O}$ and let $\mathfrak{H}$ be a hyperplane with $\mathrm{vol}_d(\mathfrak{H} \cap \mathcal{E}) \times \mathrm{vol}_d(B_2^d) < \tau_d \mathrm{vol}_d(\mathcal{E})$. Then $\mathfrak{H}$ and $\frac{1}{2}(\mathcal{E} - \mathcal{O}) + \mathcal{O}$ are disjoint.*

Proof. The truth of the lemma is invariant under affine transformations of $\mathcal{E}$ and we may therefore assume that $\mathcal{E} = B_2^d$. The result now follows from the definition of τ_d (see equation (27) and the fact that $\tau_d = \mathrm{vol}_d\{x \in \mathbb{R}^d : ||x||_2 \leq 1,\, x_1 \geq 1/2\}$. ■

Proof of Lemma 8. Consider $\tau = \tau_d$ defined by (27). We may assume that μ is in isotropic position, which implies that $D_\delta^\natural = D_{\tau^{-1}9^d\delta}$. Lemma 18 and Lemma 19 together imply that $\mathfrak{H} \cap \mathcal{E}_\delta^\sharp \neq \emptyset$. For each $x \in D_\delta^\natural$, $f(x) \geq \tau^{-1}9^d\delta$. Therefore $\delta \geq \mu(\mathfrak{H} \cap \mathcal{E}_{D_\delta^\natural}) \geq \tau^{-1}9^d\delta\mathrm{vol}_d(\mathfrak{H} \cap \mathcal{E}_{D_\delta^\natural})$ which implies that $\mathrm{vol}_d(\mathfrak{H} \cap \mathcal{E}_{D_\delta^\natural}) \leq \tau 9^{-d}$. Since the density function f is continuous and $\tau^{-1}9^d\delta < 2^{-8d}$, inequality (21) implies that $(9^{-1} + \kappa)B_2^d \subset D_\delta^\natural$ for some

$\kappa > 0$. Since $\mathcal{E}_{D_\delta^\natural}$ is the ellipsoid of maximal volume inside $D_\delta^\natural$, we have $\mathrm{vol}_d(\mathcal{E}_{D_\delta^\natural}) > 9^{-d}\mathrm{vol}_d(B_2^d)$. From the definition of $\mathcal{E}_\delta^\flat$ and Lemma 20, we see that $\mathfrak{H} \cap \mathcal{E}_\delta^\flat = \emptyset$. Finally, the claim that $\mathcal{E}_\delta^\flat \subset F_\delta$ follows from the definition of F_δ while the claim that $F_\delta \subset \mathcal{E}_\delta^\sharp$ follows from the Hahn-Banach theorem (any $x \notin \mathcal{E}_\delta^\sharp$ lies in an open half-space $\mathfrak{H}$ with $\mathfrak{H} \cap \mathcal{E}_\delta^\sharp = \emptyset$ and therefore $\mu(\mathfrak{H}) < \delta$ and $x \notin F_\delta$). ■

Proof of Lemma 9. Note that $1+\rho \leq (1-\rho)^{-1} \leq 1+2\rho$ and $1-\rho \leq (1+\rho)^{-1} \leq 1-\rho/2$, and the same inequalities hold for ε. Since $rB_2^d \subset K \subset RB_2^d$, we have that

$$R^{-1}||x||_2 \leq ||x||_K \leq r^{-1}||x||_2$$

for all $x \in \mathbb{R}^d$. Combining this with (32) gives

$$R^{-1}(1+\rho)^{-1} \leq ||\omega||_L \leq r^{-1}(1-\rho)^{-1}$$

for all $\omega \in \mathcal{N}$. Consider any $\theta \in S^{d-1}$. By the series representation (12) and the triangle inequality,

$$||\theta||_L \leq r^{-1}(1-\rho)^{-1}(1-\varepsilon)^{-1}$$

Hence $||x||_L \leq r^{-1}(1-\rho)^{-1}(1-\varepsilon)^{-1}||x||_2$ for all $x \in \mathbb{R}^d$. Using the triangle inequality in a bit of a different way,

$$\begin{aligned}
||\theta||_L &\geq ||\omega_0||_L - \sum_{i=1}^{\infty} \varepsilon_i ||\omega_i||_L \\
&\geq R^{-1}(1+\rho)^{-1} - r^{-1}\varepsilon(1-\varepsilon)^{-1}(1-\rho)^{-1} \\
&\geq R^{-1}/2 - 4r^{-1}\varepsilon \\
&= R^{-1}(1-8Rr^{-1}\varepsilon)/2 \\
&\geq (4R)^{-1}
\end{aligned}$$

which holds since $8Rr^{-1}\varepsilon \leq 1/2$. Thus,

$$\begin{aligned}
||\theta||_L &\leq ||\omega_0||_L + ||\theta - \omega_0||_L \\
&\leq (1-\rho)^{-1}||\omega_0||_K + r^{-1}(1-\rho)^{-1}(1-\varepsilon)^{-1}\varepsilon \\
&\leq (1-\rho)^{-1}(||\theta||_K + ||\omega_0 - \theta||_K) + r^{-1}(1-\rho)^{-1}(1-\varepsilon)^{-1}\varepsilon \\
&\leq (1-\rho)^{-1}||\theta||_K + r^{-1}(1-\rho)^{-1}\varepsilon(1+(1-\varepsilon)^{-1}) \\
&\leq (1-\rho)^{-1}||\theta||_K + Rr^{-1}(1-\rho)^{-1}\varepsilon(1+(1-\varepsilon)^{-1})||\theta||_K \\
&\leq (1+2\rho)(1+3Rr^{-1}\varepsilon)||\theta||_K \\
&\leq (1+2\rho+6Rr^{-1}\varepsilon)||\theta||_K
\end{aligned}$$

where ω_0 is the element of $\mathcal{N}$ that minimizes $||\theta - \omega_0||_2$. On the other hand,

$$\begin{aligned}
||\theta||_K &\leq ||\omega_0||_K + ||\theta - \omega_0||_K \\
&\leq (1+\rho)||\omega_0||_L + r^{-1}\varepsilon \\
&\leq (1+\rho)(||\theta||_L + ||\omega_0 - \theta||_L) + r^{-1}\varepsilon \\
&\leq (1+\rho)||\theta||_L + r^{-1}(1+\rho)(1-\rho)^{-1}(1-\varepsilon)^{-1}\varepsilon + r^{-1}\varepsilon \\
&\leq (1+\rho)||\theta||_L + 7r^{-1}\varepsilon \cdot 4R||\theta||_L \\
&\leq (1+\rho+28Rr^{-1}\varepsilon)||\theta||_L
\end{aligned}$$

The result follows by positive homogeneity. ■

7. Proof of Theorem 3. Fix μ and d as in the statement of Theorem 3. Let f be the density of μ and let $g = -\log f$. All variables in this section depend on both d and μ.

Lemma 21 *There exist $c, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\mu(\mathbb{R}^d \backslash D_\varepsilon) < c\varepsilon \left(\log \varepsilon^{-1}\right)^d \tag{46}$$

Proof. Since μ has a log-concave density, it necessarily has a nonsingular covariance matrix, and there exists an affine map T such that $\mu' = T\mu$ is isotropic. The density of μ' is

$$\widetilde{f}(x) = \det(T^{-1}) f(T^{-1}x)$$

and $D_\varepsilon = T^{-1}\widetilde{D}_{\widetilde{\varepsilon}}$, where $\widetilde{\varepsilon} = \varepsilon \det T^{-1}$ and $\widetilde{D}_{\widetilde{\varepsilon}} = \{x : \widetilde{f}(x) \geq \widetilde{\varepsilon}\}$. Since μ' is isotropic, we may use Lemma (16), which gives

$$\begin{aligned}
\mu(\mathbb{R}^d \backslash D_\varepsilon) &= \mu'(\mathbb{R}^d \backslash \widetilde{D}_{\widetilde{\varepsilon}}) \\
&\leq c'\widetilde{\varepsilon}(\log \widetilde{\varepsilon}^{-1})^d \\
&\leq c\varepsilon \left(\log \varepsilon^{-1}\right)^d
\end{aligned}$$

■

Lemma 22 *For any $x \in \mathbb{R}^d$ there exist $c', \delta_0 > 0$ and a function $p : (0, \delta_0) \to (0, \infty)$ such that for all $\delta \in (0, \delta_0)$,*

$$p(\delta) \leq c' \frac{\log\log \delta^{-1}}{\log \delta^{-1}} \tag{47}$$

and

$$F_\delta \subset (1+p)(D_\delta - x) + x \tag{48}$$

Proof. Let $c > 0$ be the constant in (46). A brief analysis of the function $t \mapsto ct\left(\log t^{-1}\right)^d$ shows that there exists $\delta_0 > 0$ and a function $\varepsilon = \varepsilon(\delta)$ defined implicitly for all $\delta \in (0, \delta_0)$ by the equation $\delta = c\varepsilon\left(\log \varepsilon^{-1}\right)^d$. We can take δ_0 small enough to ensure that $\varepsilon < \delta$ and that $\log \delta^{-1} < \log \varepsilon^{-1} < 2 \log \delta^{-1}$. If we define

$$p(\delta) = 3\frac{\log \varepsilon^{-1} - \log \delta^{-1}}{\log \delta^{-1}}$$

then $\delta^{1+p/2} < \varepsilon$ and (47) holds. Since D_ε is both compact and convex, for any point $y \in D_\varepsilon$ there exists (by the Hahn-Banach theorem), a closed halfspace $\mathfrak{H}$ with $y \in \mathfrak{H}$ and $\mathfrak{H} \cap D_\varepsilon = \emptyset$. Since $\mathfrak{H} \subset \mathbb{R}^d \backslash D_\varepsilon$, (46) implies that $\mu(\mathfrak{H}) < \delta$ and by definition of F_δ, $y \notin F_\delta$. This goes to show that $F_\delta \subset D_\varepsilon$. Let $x \in \mathbb{R}^d$. For any $\theta \in S^{d-1}$ consider the function $f_\theta(t) = f(x + t\theta) = e^{-g_\theta(t)}$, $t \in \mathbb{R}$. This notation differs slightly from that in the proof of theorem 1. If ε is small enough then for all $\theta \in S^{d-1}$ there is a unique $v > 0$ such that $f_\theta(v) = \varepsilon$; we denote this number by $f_\theta^{-1}(\varepsilon)$. We may assume that $\delta_0 < \min\{1, f(x)^2\}$. Note that $1 < \delta/\varepsilon < \delta^{-p/2}$ and $\log \delta^{-1} + \log f(x) \geq 1/2 \log \delta^{-1}$. By convexity of g_θ, for any $0 < s < v$ we have $s^{-1}(g_\theta(s) - g_\theta(0)) \leq (v - s)^{-1}(g_\theta(v) - g_\theta(s))$. Taking $v = f_\theta^{-1}(\varepsilon)$ and $s = f_\theta^{-1}(\delta)$, this becomes

$$\begin{aligned} \frac{f_\theta^{-1}(\varepsilon) - f_\theta^{-1}(\delta)}{f_\theta^{-1}(\delta)} &\leq \frac{\log \varepsilon^{-1} - \log \delta^{-1}}{\log \delta^{-1} + \log f(x)} \\ &< p \end{aligned} \tag{49}$$

Inequality (49) reduces to $f_\theta^{-1}(\varepsilon) \leq (1 + p) f_\theta^{-1}(\delta)$. Since this holds for any $\theta \in S^{d-1}$, $D_\varepsilon \subset (1 + p)(D_\delta - x) + x$ and (48) follows. ■

Lemma 23 *There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have the relation*

$$(1 + 8\lambda^{1/d})^{-1}(D_\delta - x') + x' \subset F_\delta \tag{50}$$

where $\lambda = \mathrm{vol}_d(D_\delta)^{-1}$ and x' is the centroid of D_δ.

Proof. Let δ_0 be such that $\mathrm{vol}_d(D_{\delta_0}) > 8^d$. We use the notation $(D_\delta)_\lambda$ for the convex floating body with parameter $\lambda > 0$ corresponding to the uniform probability measure on D_δ. If $\mathfrak{H}$ is any half-space with $\mu(\mathfrak{H}) < \delta$, then $\mathrm{vol}_d(\mathfrak{H} \cap D_\delta) < 1$. Hence $(D_\delta)_\lambda \subset F_\delta$, where $\lambda = \mathrm{vol}_d(D_\delta)^{-1}$. The result now follows from inequality (18). ■

Lemma 24 *Let $K, L \subset \mathbb{R}^d$ be convex bodies such that there exist $x, x' \in int(K \cap L)$ and $0 < r < (8d)^{-1}$ for which*

$$(1 + r)^{-1}(K - x) + x \subset L \subset (1 + r)(K - x') + x' \tag{51}$$

Then

$$d_{\mathfrak{L}}(K,L) \leq 1 + 8dr \tag{52}$$

Proof. Since the statement of the lemma is invariant under affine transformations of K and L, we may assume without loss of generality that the John ellipsoid of K is B_2^d. Hence $B_2^d \subset K \subset dB_2^d$ and $||x||_2, ||x'||_2 \leq d$. Note also that $L \subset 3dB_2^d$. Using these facts and manipulating (51) in the obvious way, we see that both of the following relations hold

$$\begin{aligned} L &\subset K + 2drB_2^d \\ K &\subset L + 4drB_2^d \end{aligned}$$

By definition of the Hausdorff distance, $d_{\mathcal{H}}(K,L) \leq 4dr$. Since $B_2^d \subset K$, $d_{\mathfrak{L}}(K,L) \leq (1-4dr)^{-1} \leq 1 + 8dr$. ■

Proof of equation (7). Since $\lim_{\delta\to 0} p(\delta) = \lim_{\delta \to 0} \lambda(\delta) = 0$, equation (7) now follows from (48), (50) and (52). ■

Remark 25 *There is no lower bound on the growth rate of* $\mathrm{vol}_d(D_\delta)$*, indeed the function could grow arbitrarily slowly. However in the case of the Schechtman-Zinn distributions,* $\mathrm{vol}_d(D_\delta) = (\log(c_p^d/\delta))^{d/p}\mathrm{vol}_d(B_p^d)$ *and we leave it to the reader to combine this with (48), (47) and (50) to obtain a quantitative upper bound on* $d_{\mathfrak{L}}(F_\delta, D_\delta)$*.*

Proof of equation (8). Let $\varepsilon > 0$ be given. Using the notation from the proof of theorem 1, for any $\theta \in S^{d-1}$ we define

$$f_\theta(t) = -\frac{d}{dt}\mu(\mathfrak{H}_{\theta,t})$$

This function is the density of a log-concave probability measure on $\mathbb{R}$ with cumulative distribution function $J_\theta(t) = 1 - \mu(\mathfrak{H}_{\theta,t})$. By Fubini's theorem we have

$$f_\theta(t) = Rf(\mathcal{H}_{\theta,t})$$

Define $\alpha = \inf\{f_\theta(0) : \theta \in S^{d-1}\}$. By (20) there exists $t_0 > 0$ such that if $\beta = \sup\{f_\theta(t_0) : \theta \in S^{d-1}\}$, then $\beta < \alpha$. Since f is non-vanishing, S^{d-1} is compact and the function $\theta \mapsto f_\theta(t)$ is continuous, $\beta > 0$. Define $g_\theta(t) = -\log f_\theta(t)$ and let $\lambda = t_0^{-1}(\log\alpha - \log\beta)$ and $\Delta = \max\{1, \lambda^{-1}\log\lambda^{-1}\}$. By definition of α, β and λ, for all $\theta \in S^{d-1}$ we have $t_0^{-1}(g_\theta(t_0) - g_\theta(0)) \geq \lambda$. By convexity of g_θ, if $u > v \geq t_0$ then $g_\theta(u) \geq g_\theta(v) + \lambda(u-v)$, which translates into $f_\theta(u) \leq f_\theta(v)e^{-\lambda(u-v)}$. Let $\delta_0 < \inf\{f_\theta(t_0+1) : \theta \in S^{d-1}\}$ be such that $\Delta\varepsilon^{-1}B_2^d \subset F_{\delta_0}$. Consider any $\delta < \delta_0$ and momentarily fix $\theta \in S^{d-1}$. Let

$s = J_\theta^{-1}(\delta)$ and denote by $t = f_\theta^{-1}(\delta)$ the unique positive number such that $f_\theta(t) = \delta$. Consider the hyperplane $\mathcal{H}_{\theta,t}$ and the half-space $\mathfrak{H}_{\theta,s}$. Note that

$$\mu(\mathfrak{H}_{\theta,s}) = Rf(\mathcal{H}_{\theta,t}) = \delta$$

By log-concavity we have $f_\theta(u) \geq \delta_0$ for all $0 < u < t_0 + 1$, hence $t > t_0 + 1$. By the fundamental theorem of calculus and the fact that $f_\theta(u) \geq \delta$ for all $u \in [t-1, t]$ we have

$$\begin{aligned} \mu(\mathfrak{H}_{\theta,t-1}) &> \mu\{x \in \mathbb{R}^d : t-1 \leq \langle \theta, x \rangle \leq t\} \\ &= \int_{t-1}^{t} f_\theta(u)du \\ &\geq \delta \end{aligned}$$

hence $\mathfrak{H}_{\theta,s} \subset \mathfrak{H}_{\theta,t-1}$ which implies that $s > t-1 > t_0$. Thus, if $s \leq t$ then $|s-t| \leq 1$. If $s > t$ then

$$\begin{aligned} \delta &= \int_s^\infty f_\theta(u)du \\ &\leq f_\theta(s) \int_s^\infty e^{-\lambda(u-s)}du \\ &\leq \delta e^{-\lambda(s-t)}\lambda^{-1} \end{aligned}$$

from which it follows that $s - t \leq \lambda^{-1}\log\lambda^{-1}$. Either way, $|s-t| \leq \max\{1, \lambda^{-1}\log\lambda^{-1}\} = \Delta$. Since $\Delta\varepsilon^{-1}B_2^d \subset F_{\delta_0}$, it follows that $(1-\varepsilon)s \leq t \leq (1+\varepsilon)s$. Since this holds for all $\theta \in S^{d-1}$ we have

$$(1-\varepsilon)F_\delta \leq R_\delta \leq (1+\varepsilon)F_\delta$$

■

8. Optimality. Let Φ denote the cumulative standard normal distribution on $\mathbb{R}$,

$$\Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^{t} e^{-\frac{1}{2}s^2}ds$$

By (40) there exists $c > 0$ such that for all $n \geq 3$,

$$\Phi^{-1}(1-1/n) \geq c(\log n)^{1/2} \tag{53}$$

Lemma 26 *For all $q > 0$ and all $d \in \mathbb{N}$, there exists $c, \widetilde{c} > 0$ such that for all $n \geq d+1$, if $(x_i)_1^n$ is an i.i.d. sample from the standard normal distribution*

on $\mathbb{R}^d$ *and* $P_n = conv\{x_i\}_1^n$*, then with probability at least* $1-\widetilde{c}(\log n)^{-q(d-1)/2}$ *both of the following events occur,*

$$d_{\mathcal{H}}(P_n, F_{1/n}) \geq c(\log n)^{-\frac{1}{2}-q} \tag{54}$$

$$d_{\mathfrak{L}}(P_n, F_{1/n}) \geq 1 + c(\log n)^{-1-q} \tag{55}$$

Proof. A standard result in approximation theory ([13] p. 326) is that for any polytope $K_m \subset \mathbb{R}^d$ with at most m vertices,

$$d_{\mathcal{H}}(K_m, B_2^d) > c_d \left(\frac{1}{m}\right)^{\frac{2}{d-1}} \tag{56}$$

Since $F_{1/n} = \Phi^{-1}(1-1/n)B_2^d$, inequality (53) implies that

$$d_{\mathcal{H}}(K_m, F_{1/n}) > c_d(\log n)^{1/2} \left(\frac{1}{m}\right)^{\frac{2}{d-1}}$$

By a result of Raynaud [21], the number of vertices of P_n, denoted by $f_0(P_n)$, obeys the inequality $\mathbb{E}f_0(P_n) < \widetilde{c}_d(\log n)^{(d-1)/2}$. By Chebyshev's inequality we have

$$\mathbb{P}\{f_0(P_n) > (\log n)^{\frac{(d-1)(q+1)}{2}}\} \leq \frac{\mathbb{E}f_0(P_n)}{(\log n)^{\frac{(d-1)(q+1)}{2}}} < \widetilde{c}_d(\log n)^{-\frac{q(d-1)}{2}}$$

and if the complement of this event occurs, then so does (54). By (56) and (16) we get

$$d_{\mathfrak{L}}(K_m, B_2^d) > 1 + c_d \left(\frac{1}{m}\right)^{\frac{2}{d-1}}$$

Since $d_{\mathfrak{L}}$ is preserved by invertible affine transformations (as per (15)), the same inequality holds for all Euclidean balls. This gives (55). ■

We can choose q to be arbitrarily small, in which case (54) and (55) complement Theorem 1 and Theorem 2.

9. Proof of Theorem 4. If Ω is a convex subset of a real vector space and $\mathcal{K}_d$ is the collection of all convex bodies in $\mathbb{R}^d$, then we define a function $\kappa : \Omega \to \mathcal{K}_d$ to be concave if for all $x, y \in \Omega$ and all $\lambda \in (0,1)$ we have

$$\lambda\kappa(x) + (1-\lambda)\kappa(y) \subset \kappa(\lambda x + (1-\lambda)y)$$

If Ω has an ordering then we define κ to be non-decreasing if for all $x, y \in \Omega$ with $x \leq y$ we have $\kappa(x) \subset \kappa(y)$.

Lemma 27 *If $\kappa : [0,\infty) \to \mathcal{K}_d$ is concave, non-decreasing and $\cup_{t\in[0,\infty)}\kappa(t) = \mathbb{R}^d$, then the function $g : \mathbb{R}^d \to [0,\infty)$ defined by*

$$g(x) = \inf\{t \geq 0 : x \in \kappa(t)\} \tag{57}$$

is convex. Furthermore, κ is continuous with respect to the Hausdorff distance and for all $t > 0$

$$\kappa(t) = \{x \in \mathbb{R}^d : g(x) \leq t\} \tag{58}$$

Proof. By hypothesis, $\kappa(0) \neq \emptyset$. If $0 \notin \kappa(0)$, then we define $\kappa^\sharp(t) = \kappa(t) - x_0$, where $x_0 \in \kappa(0)$. The function $\kappa^\sharp$ enjoys all of the properties that κ does, and the function

$$g^\sharp(x) = \inf\{t \geq 0 : x \in \kappa^\sharp(t)\}$$

is related to g by the equation $g^\sharp(x) = g(x + x_0)$. Note that $0 \in \kappa^\sharp(0)$. If the lemma holds for the functions $\kappa^\sharp$ and $g^\sharp$, it will necessarily hold for κ and g. We may therefore, without loss of generality, assume that $0 \in \kappa(0)$. For any $0 < \varepsilon < t$ we have the convex combination

$$t = \frac{\varepsilon}{t+\varepsilon}0 + \frac{t}{t+\varepsilon}(t+\varepsilon)$$

Exploiting the concavity of κ, this leads to

$$\kappa(t+\varepsilon) \subset \frac{t+\varepsilon}{t}\kappa(t)$$

Similarly,

$$\frac{t-\varepsilon}{t}\kappa(t) \subset \kappa(t-\varepsilon)$$

Hence κ is continuous with respect to the Hausdorff distance. By definition of g, $\kappa(t) \subset \{x \in \mathbb{R}^d : g(x) \leq t\}$. Since $\kappa(t)$ is a closed set, if $x \notin \kappa(t)$ then $d(x, \kappa(t)) > 0$ and by continuity of κ, $g(x) > t$. This implies (58). Consider any $x, y \in \mathbb{R}^d$ and $\lambda \in (0,1)$. Let $t = g(x)$ and $s = g(y)$. By (58), $x \in \kappa(t)$ and $y \in \kappa(s)$. Therefore

$$\begin{aligned} \lambda x + (1-\lambda)y &\in \lambda\kappa(t) + (1-\lambda)\kappa(s) \\ &\subset \kappa(\lambda t + (1-\lambda)s) \end{aligned}$$

This implies that $g(\lambda x + (1-\lambda)y) \leq \lambda t + (1-\lambda)s$ which shows that g is convex. ■

Note that the function g is a generalization of the Minkowski functional of a convex body K, in which case $\kappa(t) = tK$. Including $\{0\}$ as an honorary

member of $\mathcal{K}_d$ does no harm to the preceding lemma. If $(K_n)_{n=1}^{\infty}$ is a sequence of convex bodies then we define the corresponding Minkowski series as,

$$\sum_{n=1}^{\infty} K_n = \left\{ \sum_{n=1}^{\infty} x_n : \forall n,\, x_n \in K_n \right\}$$

where we take $\sum x_n$ to have meaning only if it converges. We leave the easy proof of the following lemma to the reader.

Lemma 28 *For each $n \in \mathbb{N}$, let $\alpha_n : [0,\infty) \to [0,\infty)$ be a concave function and let K_n be a convex body with $0 \in K_n$. Provided that*

$$\sum_{n=1}^{\infty} \alpha_n(t) diam(K_n) < \infty$$

for all $t \geq 0$, then the function $\kappa : [0,\infty) \to \mathcal{K}_d$ defined by

$$\kappa(t) = \sum_{n=1}^{\infty} \alpha_n(t) K_n$$

is concave.

The space $\mathcal{K}_d$ is separable with respect to d_{BM} and we shall use a dence sequence $(K_n)_{n=1}^{\infty}$ that is dense in $\mathcal{K}_d$. Since d_{BM} is blind to affine transformations we can assume that the John ellipsoid of each K_n is B_2^d. As coefficients, we shall use the functions

$$\alpha_n(t) = \begin{cases} 2^{-n^2} t & : \quad 0 \leq t \leq 2^{2n^2} \\ 2^{n^2} & : \quad 2^{2n^2} < t < \infty \end{cases}$$

Note that for large values of n, the dominant coefficient at the value $t = 2^{2n^2}$ is α_n. In fact $\sum_{j \neq n} \alpha_j(2^{2n^2})$ is much smaller than $\alpha_n(2^{2n^2})$,

$$\begin{aligned} \sum_{j \neq n} \alpha_j(2^{2n^2}) &= \sum_{j=1}^{n-1} 2^{j^2} + 2^{2n^2} \sum_{j=n+1}^{\infty} 2^{-j^2} \\ &\leq \sum_{j=1}^{n-1} 2^{nj} + 2^{2n^2} \sum_{j=n+1}^{\infty} 2^{-nj} \\ &\leq 2^{n^2-n+2} \\ &= 2^{-n+2} \alpha_n(2^{2n^2}) \end{aligned}$$

Hence,

$$d_{BM}(\kappa(2^{2n^2}), K_n) \leq 1 + 2^{-n+2} d$$

Thus the sequence $(\kappa(n))_{n=1}^{\infty}$ is dense in $\mathcal{K}_d$. Since each coefficient α_n is non-decreasing and concave, κ is concave and the function g as defined by (57) is convex. Clearly, $\lim_{x\to\infty} g(x) = \infty$. For some $c > 0$, the function

$$f(x) = 2^{-g(cx)}$$

is the density of a log-concave probability measure μ on $\mathbb{R}^d$. For each $n \in \mathbb{N}$, $D_{2^{-n}} = \{x \in \mathbb{R}^d : f(x) \geq 2^{-n}\} = \{x \in \mathbb{R}^d : g(cx) \leq n\} = c^{-1}\kappa(n)$. Hence the sequence $(D_{1/n})_{n=1}^{\infty}$ is dense in $\mathcal{K}_d$. By (7), the sequence $(F_{1/n})_{n=3}^{\infty}$ is also dense in $\mathcal{K}_d$.

We now use Theorem 1 with $q = 1$. Let $\widetilde{\mathcal{K}}_d$ denote a countably dense subset of $\mathcal{K}_d$ and let $K \in \widetilde{\mathcal{K}}_d$. Note that there exists an increasing sequence of natural numbers $(k_n)_1^{\infty}$ such that $\lim_{n\to\infty} d_{BM}(F_{1/k_n}, K) = 1$ and $\sum_{n=1}^{\infty} \widetilde{c}(\log k_n)^{-1} < \varepsilon$. By (5),

$$\lim_{n\to\infty} d_{BM}(P_{k_n}, K) = 1$$

with probability at least $1-\varepsilon$. Since this holds for all $\varepsilon > 0$, $K \in cl_{BM}\{P_n : n \in \mathbb{N},\ n \geq d+1\}$ almost surely, where cl_{BM} denotes closure in $\mathcal{K}_d$ with respect to d_{BM}. Since this holds for all $K \in \widetilde{\mathcal{K}}_d$ and $\widetilde{\mathcal{K}}_d$ is countable, $\widetilde{\mathcal{K}}_d \subset cl_{BM}\{P_n : n \in \mathbb{N},\ n \geq d+1\}$ almost surely. The result now follows since $\widetilde{\mathcal{K}}_d$ is dense in $\mathcal{K}_d$.

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Department of Mathematics
University of Missouri
Columbia, Missouri 65211
USA

E-mail: djfb6b@mail.missouri.edu