Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials

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January 17, 2011

[1]

Abstract

We investigate the construction of diffusions consisting of infinitely numerous Brownian particles moving in $\mathbb{R}^d$ and interacting via logarithmic functions (two dimensional Coulomb potentials). These potentials are very strong and act over a long range in nature. The associated equilibrium states are no longer Gibbs measures.

We present general results for the construction of such diffusions and, as applications thereof, construct two typical interacting Brownian motions with logarithmic interaction potentials, namely the Dyson model in infinite dimensions and Ginibre interacting Brownian motions. The former is a particle system in $\mathbb{R}$ while the latter is in $\mathbb{R}^2$. Both models are translation and rotation invariant in space, and as such, are prototypes of dimensions $d = 1, 2$, respectively. The equilibrium states of the former diffusion model are determinantal or Pfaffian random point fields with sine kernels. They appear in the thermodynamical limits of the spectrum of the ensembles of Gaussian random matrices such as GOE, GUE and GSE. The equilibrium states of the latter diffusion model are the thermodynamical limits of the spectrum of the ensemble of complex non-Hermitian Gaussian random matrices known as the Ginibre ensemble.

1 Introduction

Interacting Brownian motions (IBMs) in infinite dimensions are diffusions $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{Z}}$ consisting of infinitely many particles moving in $\mathbb{R}^d$ with the effect of the external force coming from a self potential $\Phi:\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ and that of the mutual interaction coming from an interacting potential $\Psi:\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ such that $\Psi(x,y) = \Psi(y,x)$.

Intuitively, an IBM is described by the infinitely dimensional stochastic differential equation (SDE) of the form

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\in\mathbb{Z},j\neq i}\nabla\Psi(X_t^i, X_t^j)dt \quad (i \in \mathbb{Z}). \tag{1.1}$$

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Key words: Interacting Brownian particles, Random matrices, Coulomb potentials, Infinitely many particle systems, Diffusions

The state space of the process $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{Z}}$ is $(\mathbb{R}^d)^{\mathbb{Z}}$ by construction. Let X be the configuration-valued process given by

$$\mathsf{X}_t = \sum_{i\in\mathbb{Z}} \delta_{X_t^i}. \tag{1.2}$$

Here δ_a denotes the delta measure at a and a configuration is a Radon measure consisting of a sum of delta measures. We call $\mathbf{X}$ the labeled dynamics and X the unlabeled dynamics.

The SDE (1.1) was initiated by Lang [9], [10]. He studied the case $\Phi = 0$, and $\Psi(x, y) = \Psi(x - y)$, where Ψ is of $C_0^3(\mathbb{R}^d)$, superstable and regular according to Ruelle [20]. With the last two assumptions, the corresponding unlabeled dynamics X has Gibbsian equilibrium states. See [21], [4], and [25] for other works concerning the SDE (1.1).

In [13] the unlabeled diffusion was constructed using the Dirichlet form. The advantage of this method is that it gives a general and simple proof of construction, and more significantly it allows us to apply singular interaction potentials, which are particularly of interest, such as the Lennard-Jones 6-12 potential and hard core potential. We note that all these potentials were excluded in the SDE approach. See [28], [1] [26], and [27] for other works on applying the Dirichlet form approach to IBMs.

We remark that in all these works, except some parts of [13], the equilibrium states are supposed to be Gibbs measures with Ruelle's class interaction potentials Ψ. Thus, the equilibrium states are described by the Dobrushin-Lanford-Ruelle (DLR) equations (see (2.11)), the usage of which plays a pivotal role in the previous works.

The purpose of this paper is to construct unlabeled IBMs in infinite dimensions with the logarithmic interaction potentials

$$\Psi(x, y) = -\beta \log |x - y|. \tag{1.3}$$

We present a sequence of general theorems to construct IBMs and apply these to logarithmic potentials. We remark that the equilibrium states are not Gibbs measures because the logarithmic interaction potentials are unbounded at infinity.

The above potential Ψ in (1.3) is known to be the two-dimensional Coulomb potential. In practice, such systems are regarded as one-component plasma consisting of equally charged particles. To prevent the particles all repelling to explode, a neutralizing background charge is imposed. The self potential Φ denotes this particle-background interaction (see [2]).

We study two typical examples, namely Dyson's model (Section 2.1) and Ginibre IBMs (Section 2.2). In the first example, we take $d = 1$, $\Phi = 0$, and $\Psi(x, y) = -\beta \log |x - y|$ $(\beta = 1, 2, 4)$, while in the second $d = 2$, $\Phi(z) = |z|^2$, and $\Psi(x, y) = -2 \log |x - y|$.

For the special values $\beta = 1, 2, 4$ and particular self potentials Φ, the associated equilibrium states are limits of the spectrum of random matrices. Recently, much intensive research has been carried out on random point fields related to random matrices. Our purpose in this paper is a rather more dynamical one; that is, we construct diffusions, the equilibrium states of which are these random point fields related to random matrices.

The labeled dynamics of the Dyson model in infinite dimensions is represented by the following SDE.

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{R\to\infty} \sum_{|X_t^j|\le R, j\in\mathbb{Z}, j\neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{Z}). \tag{1.4}$$

Here $\beta = 1, 2, 4$, corresponding to the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble (GUE) and the Gaussian symplectic ensemble (GSE), respectively. The invariant probability measures $\mu_{\mathrm{dys},\beta}$ of the (unlabeled) Dyson models are translation invariant. Hence, if the distribution of X_0 equals $\mu_{\mathrm{dys},\beta}$, then for all t

$$\sum_{j\in\mathbb{Z}, j\neq i} \frac{1}{|X_t^i - X_t^j|} dt = \infty \quad \text{a.s..} \tag{1.5}$$

This means that only conditional convergence is possible in the summation of the drift term in (1.4), which is the cause of the difficulty in dealing with the Dyson model. It is well known that the equilibrium states are the thermodynamic limits of the distribution of the spectrum of Gaussian random matrices at the bulk [23], [2], [12].

The labeled dynamics of Ginibre IBMs is represented by the following SDE. For convenience, we regard S as $\mathbb{C}$ rather than $\mathbb{R}^2$.

$$dZ_t^i = dB_t^i - Z_t^i dt + \lim_{R\to\infty} \sum_{|Z_t^j|\le R,\ j\in\mathbb{Z},\ j\neq i} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \quad (i \in \mathbb{Z}). \tag{1.6}$$

Here $Z_t^i = X_t^i + \mathrm{i} Y_t^i \in \mathbb{C}$, where $\mathrm{i} = \sqrt{-1}$, and $\{B_t^i\}_{i\in\mathbb{Z}}$ are independent complex Brownian motions. That is, $B_t^i = B_t^{i,\mathrm{Re}} + \mathrm{i} B_t^{i,\mathrm{Im}}$, where $\{B_t^{i,\mathrm{Re}}, B_t^{i,\mathrm{Im}}\}_{i\in\mathbb{Z}}$ is a system of independent one-dimensional Brownian motions. The stationary measure μ_{gin} of the unlabeled dynamics is the thermodynamic limit of the distribution of the spectrum of random Gaussian matrices called the Ginibre ensemble (cf. [23]). μ_{gin} is a random point field with logarithmic interaction potential and is known to be translation invariant. If Ginibre IBMs $\mathsf{Z} = \{\mathsf{Z}_t\} = \{\sum_i \delta_{Z_t^i}\}$ start from the stationary measure μ_{gin}, then Z is also translation invariant in space. Moreover, Ginibre IBMs Z satisfy the SDE of the translation invariant form:

$$dZ_t^i = dB_t^i + \lim_{R\to\infty} \sum_{|Z_t^i - Z_t^j|\le R,\ j\in\mathbb{Z},\ j\neq i} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \quad (i \in \mathbb{Z}). \tag{1.7}$$

This variety of SDE representations of Ginibre IBMs is a result of the strength of the interaction potential.

A diffusion (X, P) is a family of probability measures $\mathsf{P} = \{\mathsf{P}_{\mathsf{x}}\}$ with continuous sample path $\mathsf{X} = \{\mathsf{X}_t\}$ starting at each point x of the state space with a strong Markov property (see [3]). We emphasize that we construct not only a Markov semi-group or a stationary Markov process but also a diffusion in the above sense, and also that, to apply stochastic analysis effectively, we require the construction of diffusions.

In [16], we give another general result for the SDE representation of unlabeled diffusions constructed in this paper. The SDEs (1.4), (1.6), and (1.7) of the labeled dynamics are solved there using the main results Theorem 2.2 and Theorem 2.3 in the present paper. These SDEs provide a clear trajectory level description of the diffusions obtained in the present paper. We also note that in [16] the fully labeled dynamics $\mathbf{X}_t$ is a diffusion on $\mathbb{R}^{\mathbb{Z}}$ (Dyson's model) and $(\mathbb{R}^2)^{\mathbb{Z}}$ (Ginibre IBMs).

Because of the long range nature of the logarithmic interaction, the diffusion has not yet been constructed. The only exception is the Dyson model with $\beta = 2$. In [24] Spohn proved the closability of the Dirichlet form associated with (1.1) for this model. This implies the construction of the unlabeled dynamics (1.2) in the sense of an L^2-Markovian semigroup.

An associated diffusion was constructed in [13] by combining Spohn's result with the result from [13, Theorem 0.1] for the quasi-regularity of Dirichlet forms.

In one space dimension, some explicit computations of space-time correlation functions of infinite particle systems related to random matrices have been obtained. Indeed, Katori and Tanemura [8] recently studied the thermodynamic limit of the space-time correlation functions related to the Dyson model and Airy process. Their limit space-time correlation functions define a stochastic process starting from a limited set of initial distributions. However, the Markov (semi-group) property of the process has not yet been proved. They also proved that, if their process is Markovian, the associated Dirichlet form is the same as the one obtained in this paper and their processes coincide with the processes constructed here. It is an interesting open problem to prove the Markov property of their processes and identify these two processes. We also refer to [5], [6], [7], and [18] for stochastic processes of one dimensional infinite particle systems related to random matrices.

As for two dimensional infinite systems with logarithmic interactions, the construction of stochastic processes based on the explicit computation of space-time correlation functions has not been done. Techniques useful in one dimension, such as applying the Karlin-McGregor formula, are no longer valid in two dimensions.

Let us briefly explain the main idea. We introduce the notion of quasi-Gibbs measures as a substitution for Gibbs measures. These measures satisfy inequality (2.8) involving a (finite volume) Hamiltonian. Inequality (2.8) is sufficient for the closability of the Dirichlet forms and the construction of the diffusions.

To obtain the above-mentioned inequality we control the difference of the infinite volume Hamiltonians in stead of the Hamiltonian itself. The key point of the control is the usage of the geometric property of the random point fields behind the dynamics. Indeed, although the difference still diverges for Poisson random fields and Gibbs measures with translation invariance, it becomes finite for random point fields such as Dyson random point fields and Ginibre random point fields. For these random point fields the fluctuations of particles are extremely suppressed because the logarithmic potentials are quite strong. This cancels the sum of the difference of the infinite-volume Hamiltonians.

The organization of the paper is as follows. In Section 2, we describe the set up and state the main results (Theorem 2.2, Theorem 2.3). We first introduce the notion of quasi-Gibbs measures and give a general result (Lemma 2.1) concerning the closability of bilinear forms. As applications, we then construct the diffusions of the Dyson model and the Ginibre IBMs cited above in Theorem 2.2 and Theorem 2.3, respectively. Section 3 is devoted to preparation from the Dirichlet form theory and the proof of Lemma 2.1. The most crucial assumption of Lemma 2.1 is the quasi-Gibbs property. In Section 4, we introduce Theorem 4.1, which gives a pair of sufficient conditions (A.4) and (A.5) for the quasi-Gibbs property. We also explain the strategy of the proof of Theorem 4.1. In Section 5, we prove Theorem 4.1. In Section 6, we prove Theorem 6.2, which allows us to deduce (A.5) from the new condition (A.6). In Section 7, we give a sufficient condition of (A.6), directly used in the proof of Theorem 2.2 and Theorem 2.3. In Section 8, we give a representation of the L^2-norm of linear statistics in terms of Fourier series when random fields are periodic, which is a preparation of the proof of Theorem 2.2. In Section 9, we prove Theorem 2.2. In Section 10, we prove Theorem 2.3. In Appendix 11.1 we prove Lemma 3.4 and Lemma 3.5, in Appendix 11.2 we prove Lemma 11.1.

2 Set up and main results

Let S be a closed set in $\mathbb{R}^d$ such that $0 \in S$ and $\overline{S^{\text{int}}} = S$, where S^{int} means the interia of S. Let $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\, ;\, \mathsf{s}(K) < \infty$ for any compact set $K\}$, where $\{s_i\}$ is a sequence in S. Then S is the set of configurations on S by definition. We endow S with vague topology, under which S is a Polish space.

Let μ be a probability measure on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$. We construct μ-reversible diffusions (X, P) with state space S using the Dirichlet form theory. Hence, we begin by introducing Dirichlet forms in the following.

For a subset $A \subset S$, we define the map $\pi_A : \mathsf{S} \to \mathsf{S}$ by $\pi_A(\mathsf{s}) = \mathsf{s}(A \cap \cdot)$. We say a function $f : \mathsf{S} \to \mathbb{R}$ is local if f is $\sigma[\pi_A]$-measurable for some bounded Borel set A. We say f is smooth if $\tilde{f}$ is smooth, where $\tilde{f}((s_i))$ is the permutation invariant function in (s_i) such that $f(\mathsf{s}) = \tilde{f}((s_i))$ for $\mathsf{s} = \sum_i \delta_{s_i}$.

Let $\mathsf{S} \bullet S = \{(\mathsf{s}, s) \in \mathsf{S} \times S\, ;\, \mathsf{s}(\{s\}) \geq 1\}$. Let $a = (a_{kl}) : \mathsf{S} \bullet S \to \mathbb{R}^{d^2}$ be such that $a_{kl} = a_{lk}$ and $(a_{kl}(\mathsf{s}, s))$ is nonnegative definite. Set

$$(2.1) \qquad \mathbb{D}^a[f, g](\mathsf{s}) = \frac{1}{2} \sum_i \sum_{k,l=1}^{d} a_{kl}(\mathsf{s}, s_i) \frac{\partial \widetilde{f}}{\partial s_{ik}} \cdot \frac{\partial \widetilde{g}}{\partial s_{il}}.$$

Here $s_i = (s_{i1}, \ldots, s_{id}) \in S$ and $\mathsf{s} = \sum_i \delta_{s_i}$. For given f and g, it is easy to see that the right-hand side depends only on s. Therefore, the square field $\mathbb{D}^a[f, g]$ is well defined. We assume $\mathbb{D}^a[f, g] : \mathsf{S} \to \mathbb{R}$ is $\mathcal{B}(\mathsf{S})$-measurable for each of the local, smooth functions f and g.

For a and μ, we consider the bilinear form $(\mathcal{E}^{a,\mu}, \mathcal{D}_\infty^{a,\mu})$ defined by

$$(2.2) \qquad \mathcal{E}^{a,\mu}(f, g) = \int_{\mathsf{S}} \mathbb{D}^a[f, g] d\mu,$$

$$\mathcal{D}_\infty^{a,\mu} = \{f \in L^2(\mathsf{S}, \mu); f \text{ is local and smooth, } \ \mathcal{E}^{a,\mu}(f, f) < \infty\}.$$

When $a_{kl} = \delta_{kl}$ (δ_{kl} is the Kronecker delta), we write $\mathbb{D}^a = \mathbb{D}$, $\mathcal{E}^{a,\mu} = \mathcal{E}^\mu$, and $\mathcal{D}_\infty^{a,\mu} = \mathcal{D}_\infty^\mu$.

All examples in this paper satisfy $a_{kl} = \delta_{kl}$. We, however, state the assumption in a general framework. We assume the coefficients $\{a_{kl}\}$ satisfy the following,

(A.0) There exists a nonnegative, bounded, lower semicontinuous function $a_0 : \mathsf{S} \bullet S \to [0, \infty)$ and a constant $c_1 \geq 1$ such that

$$(2.3) \qquad c_1^{-1} a_0(\mathsf{s}, s)|x|^2 \leq \sum_{k,l=1}^{d} a_{kl}(\mathsf{s}, s) x_k x_l \leq c_1 a_0(\mathsf{s}, s)|x|^2$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $(\mathsf{s}, s) \in \mathsf{S} \bullet S$.

We call a function ρ^n the n-correlation function of μ with respect to (w.r.t.) the Lebesgue measure if $\rho^n : S^n \to \mathbb{R}$ is a permutation invariant function such that

$$(2.4) \qquad \int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\mu$$

for any sequence of disjoint bounded measurable subsets $A_1, \ldots, A_m \subset S$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. It is well known [23] that under a mild condition, the correlation functions $\{\rho^n\}_{n \in \mathbb{N}}$ determine the measure μ.

We assume μ satisfies the following.

(A.1) The measure μ has a locally bounded, n-correlation function ρ^n for each $n \in \mathbb{N}$.

We introduce a Hamiltonian on a bounded Borel set A as follows. For Borel measurable functions $\Phi : S \to \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \to \mathbb{R} \cup \{\infty\}$ with $\Psi(x,y) = \Psi(y,x)$, let

$$\mathcal{H}_A^{\Phi,\Psi}(\mathsf{x}) = \sum_{x_i \in A} \Phi(x_i) + \sum_{x_i,x_j \in A, i<j} \Psi(x_i,x_j), \quad \text{where } \mathsf{x} = \sum_i \delta_{x_i}. \tag{2.5}$$

We assume $\Phi < \infty$ a.e. to avoid triviality.

For two measures ν_1, ν_2 on a measurable space $(\Omega, \mathcal{B})$ we write $\nu_1 \le \nu_2$ if $\nu_1(A) \le \nu_2(A)$ for all $A \in \mathcal{B}$. We say a sequence of finite Radon measures $\{\nu^N\}$ on a Polish space Ω converge weakly to a finite Radon measure ν if $\lim_{N\to\infty} \int f d\nu^N = \int f d\nu$ for all $f \in C_b(\Omega)$.

Throughout this paper, $\{b_r\}$ denotes an increasing sequence of natural numbers. We set

$$S_r = \{s \in S\,;\, |s| < b_r\}, \quad \mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}; \mathsf{s}(S_r) = m\}. \tag{2.6}$$

Definition 2.1. A probability measure μ is said to be a (Φ,Ψ)-quasi Gibbs measure if there exists an increasing sequence $\{b_r\}$ of natural numbers and measures $\{\mu_{r,k}^m\}$ such that, for each $r, m \in \mathbb{N}$, $\mu_{r,k}^m$ and $\mu_r^m := \mu(\cdot \cap \mathsf{S}_r^m)$ satisfy

$$\mu_{r,k}^m \le \mu_{r,k+1}^m \text{ for all } k, \quad \lim_{k\to\infty} \mu_{r,k}^m = \mu_r^m \quad \text{weakly}, \tag{2.7}$$

and that, for all $r, m, k \in \mathbb{N}$ and for $\mu_{r,k}^m$-a.e. $\mathsf{s} \in \mathsf{S}$,

$$c_2^{-1} e^{-\mathcal{H}_r(\mathsf{x})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda(d\mathsf{x}) \le \mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) \le c_2 e^{-\mathcal{H}_r(\mathsf{x})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda(d\mathsf{x}). \tag{2.8}$$

Here $\mathcal{H}_r(\mathsf{x}) = \mathcal{H}_{S_r}^{\Phi,\Psi}(\mathsf{x})$, $c_2 = c_2(r,m,k,\pi_{S_r^c}(\mathsf{s}))$ is a positive constant, Λ is the Poisson random point field whose intensity is the Lebesgue measure on S, and $\mu_{r,k,\mathsf{s}}^m$ is the conditional probability measure of $\mu_{r,k}^m$ defined by

$$\mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) = \mu_{r,k}^m(\pi_{S_r} \in d\mathsf{x} |\ \pi_{S_r^c}(\mathsf{s})). \tag{2.9}$$

We call Φ (resp. Ψ) a free (interaction) potential. When Ψ is an interaction potential, we implicitly assume that $\Psi(x,y) = \Psi(y,x)$. Our second assumption is as follows.

(A.2) μ is a (Φ,Ψ)-quasi Gibbs measure.

Remark 2.1. (1) By definition, $\mu_{r,k}^m((\mathsf{S}_r^m)^c) = 0$. Since $\mu_{r,k,\mathsf{s}}^m$ is $\sigma[\pi_{S_r^c}]$-measurable in s, we have the disintegration of the measure $\mu_{r,k}^m$

$$\mu_{r,k}^m \circ \pi_{S_r}^{-1}(d\mathsf{x}) = \int_{\mathsf{S}} \mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) \mu_{r,k}^m(d\mathsf{s}). \tag{2.10}$$

(2) Let $\mu_{r,\mathsf{s}}^m(d\mathsf{x}) = \mu_r^m(\pi_{S_r}(\mathsf{s}) \in d\mathsf{x} |\ \pi_{S_r^c}(\mathsf{s}))$. Recall that a probability measure μ is said to be a (Φ,Ψ)-canonical Gibbs measure if μ satisfies the DLR equation (2.11), that is, for each $r, m \in \mathbb{N}$, the conditional probability $\mu_{r,\mathsf{s}}^m$ satisfies

$$\mu_{r,\mathsf{s}}^m(d\mathsf{x}) = \frac{1}{c_3} e^{-\mathcal{H}_r(\mathsf{x}) - \Psi_r(\mathsf{x},\mathsf{s})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda(d\mathsf{x}) \quad \text{for } \mu_r^m\text{-a.e. } \mathsf{s}. \tag{2.11}$$

Here $0 < c_3 < \infty$ is the normalization and, for $\mathsf{x} = \sum_i \delta_{x_i}$ and $\mathsf{s} = \sum_j \delta_{s_j}$, we set

$$\Psi_r(\mathsf{x},\mathsf{s}) = \sum_{x_i \in S_r, s_j \in S_r^c} \Psi(x_i, s_j). \tag{2.12}$$

We remark that (Φ,Ψ)-canonical Gibbs measures are (Φ,Ψ)-quasi Gibbs measures. The converse is, however, not true. When $\Psi(x,y) = -\beta \log|x-y|$ and μ are translation invariant, μ are not (Φ,Ψ)-canonical Gibbs measures. This is because the DLR equation does not make sense. Indeed, $|\Psi_r(\mathsf{x},\mathsf{s})| = \infty$ for μ-a.s. s. The point is that one can expect a cancellation between c_3 and $e^{-\Psi_r(\mathsf{x},\mathsf{s})}$ even if $|\Psi_r(\mathsf{x},\mathsf{s})| = \infty$.

(A.3) There exist upper semicontinuous functions $\Phi_0, \Psi_0 : S \to \mathbb{R}\cup\{\infty\}$ and positive constants c_4 and c_5 such that

$$c_4^{-1}\Phi_0(s) \le \Phi(s) \le c_4\Phi_0(s) \tag{2.13}$$

$$c_5^{-1}\Psi_0(s-t) \le \Psi(s,t) \le c_5\Psi_0(s-t), \quad \Psi_0(s) = \Psi_0(-s) \quad (\forall s). \tag{2.14}$$

Moreover, Φ_0 and Ψ_0 are locally bounded from below and $\mathsf{\Gamma} := \{s\,;\,\Psi_0(s) = \infty\}$ is a compact set.

We use the following result obtained in [13] and [14].

Lemma 2.1 ([13], [14]). *Assume* (A.0)–(A.3). *Then* $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}_\infty, L^2(\mathsf{S},\mu))$ *is closable, and its closure* $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$ *is a local, quasi-regular Dirichlet space.*

See Section 3 for the definition of "a local, quasi-regular Dirichlet space" and necessary notions of the Dirichlet form theory. Combining Lemma 2.1 with the Dirichlet form theory developed in [3] and [11], we obtain the following.

Corollary 2.1 . Assume (A.0)–(A.3). Then there exists a diffusion (X,P) associated with $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$. Moreover, the diffusion (X,P) is μ-reversible.

We say a diffusion (X,P) is associated with the Dirichlet space $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$ if $E_{\mathsf{x}}[f(\mathsf{X}_t)] = T_t f(\mathsf{x})$ μ-a.e. x for all $f \in L^2(\mathsf{S},\mu)$. Here T_t is the L^2-semi group associated with the Dirichlet space $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$. Moreover, (X,P) is called μ-reversible if (X,P) is μ-symmetric and μ is an invariant probability measure of (X,P).

2.1 The Dyson model in infinite dimensions (Dyson IBMs).

Let $S = \mathbb{R}$. Let $\mu_{\mathrm{dys},\beta}$ $(\beta = 1,2,4)$ be the probability measure on S whose n-correlation function ρ^n is given by

$$\rho^n(x_1,\dots,x_n) = \det[\mathsf{K}_{\mathrm{dys},\beta}(x_i,x_j)]_{1\le i,j\le n}. \tag{2.15}$$

Here for $\beta = 2$, we take $\mathsf{K}_{\mathrm{dys},2}(x,y) = \sin(\pi(x-y))/\pi(x-y)$. $\mathsf{K}_{\mathrm{dys},2}$ is called the sine kernel. We remark that $\mathsf{K}_{\mathrm{dys},2}(x,y) = \frac{1}{2\pi}\int_{|k|\le\pi} e^{\mathrm{i}k(x-y)}\,dk$ and $0 \le \mathsf{K}_{\mathrm{dys},2} \le \mathrm{Id}$ as an operator on $L^2(\mathbb{R})$. It is known that $\mathsf{K}_{\mathrm{dys},2}$ generates a determinantal random point field [23]. The definition of $\mathsf{K}_{\mathrm{dys},\beta}$ for $\beta = 1,4$ is given by (9.5) and (9.7). We use quaternions to denote the kernel $\mathsf{K}_{\mathrm{dys},\beta}$ for $\beta = 1,4$. The precise meaning of the determinant of (2.15) for $\beta = 1,4$ is given by (9.3).

Theorem 2.2. *Let* $\Phi(x) = 0$ *and* $\Psi(x,y) = -\beta\log|x-y|$. *Then* $\mu_{\mathrm{dys},\beta}$ *is a quasi-Gibbs measure with potentials* (Φ,Ψ).

From Corollary 2.1 and Theorem 2.2 we obtain

Corollary 2.2 . Let $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}, L^2(\mathsf{S},\mu))$ be the Dirichlet space in Lemma 2.1 with $a = (\delta_{kl})$ and $\mu = \mu_{\mathrm{dys},\beta}$. Then there exists a μ-reversible diffusion (X,P) associated with $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}, L^2(\mathsf{S},\mu))$.

Remark 2.2. (1) We write $\mathsf{X}_t = \sum_{i\in\mathbb{Z}}\delta_{X_t^i}$. Here $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{Z}}$ is the associated labeled dynamics. It is known [15] that particles X_t^i never collide with each other. Moreover, in [16], we prove that the associated labeled dynamics $(X_t^i)_{i\in\mathbb{Z}}$ is a solution of the SDE

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{R\to\infty}\sum_{|X_t^j|\le R, j\in\mathbb{Z}, j\ne i}\frac{1}{X_t^i - X_t^j}dt \quad (i\in\mathbb{Z}) \tag{2.16}$$

with $(X_0^i) = (x_i)$ for $\mu_{\mathrm{dys},\beta}$-a.s. $\mathsf{x} = \sum_i \delta_{x_i}$.
(2) We remark that $\mu_{\mathrm{dys},\beta}$ is translation invariant. The dynamics X_t inherits the translation invariance from the equilibrium state $\mu_{\mathrm{dys},\beta}$. Indeed, if X_t starts from the distribution $\mu_{\mathrm{dys},\beta}$, then the distribution of X_t becomes translation invariant in time and space.
(3) One can easily see that $\rho^1(x) = 1$. By scaling in space, we can treat $\mu_{\mathrm{dys},\beta}$ with intensity $\rho^1(x) = \bar{\rho}$ for any $0 < \bar{\rho} < \infty$.

2.2 Ginibre interacting Brownian motions.

Next we proceed with the Ginibre IBMs. For this purpose, we first introduce a Ginibre random point field, which is a stationary probability measure for a Ginibre IBM.

Let the state space S of particles be $\mathbb{C}$. Let

$$\mathsf{K}_{\mathrm{gin}}(z_1, z_2) = \frac{1}{\pi} \exp(-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} + z_1 \cdot \bar{z}_2). \tag{2.17}$$

Here $z_1, z_2 \in \mathbb{C}$ and $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Let μ_{gin} be the probability measure whose n-correlation ρ^n_{gin} is given by

$$\rho^n_{\mathrm{gin}}(z_1, \ldots, z_n) = \det[\mathsf{K}_{\mathrm{gin}}(z_i, z_j)]_{1 \le i,j \le n}. \tag{2.18}$$

We call μ_{gin} the Ginibre random point field. It is well known [12] that μ_{gin} is the thermodynamic limit of the distribution of the spectrum of the random Gaussian matrix called the Ginibre ensemble (cf. [23]), which is the ensemble of complex non-Hermitian random $N \times N$ matrices whose $2N^2$ parameters are independent Gaussian random variables with mean zero and variance $1/2$.

Theorem 2.3. *Let $\Phi(z) = |z|^2$ and $\Psi(z_1, z_2) = -2 \log |z_1 - z_2|$. Then μ_{gin} is a quasi-Gibbs measure with potential (Φ, Ψ).*

From Corollary 2.1 and Theorem 2.3, we obtain

Corollary 2.3 . Let $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathsf{S}, \mu))$ be the Dirichlet space in Lemma 2.1 with $a = (\delta_{kl})$ and $\mu = \mu_{\mathrm{gin}}$. Then there exists a μ-reversible diffusion (Z, P) associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathsf{S}, \mu))$.

We write $\mathsf{Z}_t = \sum_{i \in \mathbb{Z}} \delta_{Z_t^i}$. In [16], we prove that the associated labeled dynamics $(Z_t^i)_{i \in \mathbb{Z}}$ is a solution of the SDE

$$dZ_t^i = dB_t^i - Z_t^i dt + \lim_{R \to \infty} \sum_{|Z_t^j| \le R,\ j \in \mathbb{Z},\ j \ne i} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \quad (i \in \mathbb{Z}). \tag{2.19}$$

Here $Z_t^i \in \mathbb{C}$ and $\{B_t^i\}_{i \in \mathbb{Z}}$ are independent complex Brownian motions.

We remark that the kernel $\mathsf{K}_{\mathrm{gin}}$ is *not* translation invariant. The measure μ_{gin} is, however, rotation and translation invariant. Such invariance is inherited by the unlabeled diffusion $\mathsf{Z}_t = \sum_{i \in \mathbb{Z}} \delta_{Z_t^i}$. This may be surprising because SDE (2.19) is not translation invariant at first glance. In [16], we prove that $(Z_t^i)_{i \in \mathbb{Z}}$ satisfies the following SDE

$$dZ_t^i = dB_t^i + \lim_{R \to \infty} \sum_{|Z_t^i - Z_t^j| \le R,\ j \in \mathbb{Z},\ j \ne i} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \quad (i \in \mathbb{Z}) \tag{2.20}$$

if Z_t starts from the distribution μ_{gin}. The passage from (2.19) to (2.20) is a result of the cancellation between the repulsion of the mutual interaction of the particles and the neutralizing background charge.

3 Preliminaries from the Dirichlet form theory.

In this section, we prepare some results from the Dirichlet form theory and give a proof of Lemma 2.1. The proof of Lemma 2.1 is essentially the same as that in [13] and [14] although the notion of quasi-Gibbs measures was not introduced in those papers and the statement was different to Lemma 2.1. For the reader's convenience, we present the proof here.

We begin by recalling the definition of Dirichlet forms and related notions according to [3] and [11]. Let X be a Polish space and m be a σ-finite Borel measure on X whose topological support equals X. Let $\mathcal{F}$ be a dense subspace of $L^2(X,m)$ and $\mathcal{E}$ be a non-negative bilinear form defined on $\mathcal{F}$. We call $(\mathcal{E},\mathcal{F})$ a Dirichlet form on $L^2(X,m)$ if $(\mathcal{E},\mathcal{F})$ is closed and Markovian. Here we say $(\mathcal{E},\mathcal{F})$ is Markovian if $\bar{u} := \min\{\max\{u,0\},1\} \in \mathcal{F}$ and $\mathcal{E}(\bar{u},\bar{u}) \le \mathcal{E}(u,u)$ for any $u \in \mathcal{F}$. The triplet $(\mathcal{E},\mathcal{F},L^2(X,m))$ is called a Dirichlet space. We say $(\mathcal{E},\mathcal{F},L^2(X,m))$ is local if $\mathcal{E}(u,v)=0$ for any $u,v \in \mathcal{F}$ with disjoint compact supports. Here a support of $u \in \mathcal{F}$ is the topological support of the signed measure udm (see [3]).

For a given Dirichlet space, there exists an L^2-Markovian semi-group associated with the Dirichlet space. If the Dirichlet space satisfies the quasi-regularity explained below, then there exists a Hunt process associated with the Dirichlet space. Moreover, if the Dirichlet form is local, then the Hunt process becomes a diffusion; that is, a strong Markov process with continuous sample paths.

We say a Dirichlet space $(\mathcal{E},\mathcal{F},L^2(X,m))$ is quasi-regular if

(Q.1) There exists an increasing sequence of compact sets $\{K_n\}$ such that $\cup_n \mathcal{F}(K_n)$ is dense in $\mathcal{F}$ w.r.t. $\mathcal{E}_1^{1/2}$-norm. Here $\mathcal{F}(K_n) = \{f \in \mathcal{F}\, ;\, f=0 \ m\text{-a.e. on } K_n^c\}$, and $\mathcal{E}_1^{1/2}(f) = \mathcal{E}(f,f)^{1/2} + \|f\|_{L^2(E,m)}$.

(Q.2) There exists a $\mathcal{E}_1^{1/2}$-dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi continuous m-version.

(Q.3) There exists a countable set $\{u_n\}_{n\in\mathbb{N}}$ having $\mathcal{E}$-quasi continuous m-version $\tilde{u}_n$, and an exceptional set $\mathcal{N}$ such that $\{\tilde{u}_n\}_{n\in N}$ separates the points of $E\backslash\mathcal{N}$.

Lemma 3.1. (1) *Assume* (A.1). *Let* $(\mathcal{E}^{\mu},\mathcal{D}_{\infty}^{\mu})$ *be as in* (2.2) *with* $a_{kl}=\delta_{kl}$. *Assume* $(\mathcal{E}^{\mu},\mathcal{D}_{\infty}^{\mu})$ *is closable on* $L^2(\mathsf{S},\mu)$. *Then its closure* $(\mathcal{E}^{\mu},\mathcal{D}^{\mu})$ *on* $L^2(\mathsf{S},\mu)$ *is a local, quasi-regular Dirichlet form.*

(2) *In addition, assume* (A.0) *and that* $(\mathcal{E}^{a,\mu},\mathcal{D}_{\infty}^{a,\mu})$ *is closable* $L^2(\mathsf{S},\mu)$. *Then its closure* $(\mathcal{E}^{a,\mu},\mathcal{D}_{\infty}^{a,\mu})$ *on* $L^2(\mathsf{S},\mu)$ *is a local, quasi-regular Dirichlet form.*

Proof. (1) follows from [13, Theorem 1], in which we suppose that the density functions are locally bounded and $\sum_{m=1}^{\infty} m\mu(\mathsf{S}_r^m) < \infty$. We remark that these assumptions follow immediately from (A.1). We have thus obtained (1).

Let $c_6 = c_1 \sup|a_0(\mathsf{s},s)|$. Then by (A.0), we see that $c_6 < \infty$ and

$$\mathcal{D}^{a,\mu} \supset \mathcal{D}^{\mu}, \quad \mathcal{E}^{a,\mu}(f,f) \le c_6 \mathcal{E}^{\mu}(f,f) \quad \text{ for all } f \in \mathcal{D}^{\mu}.$$

Hence, (2) follows from (1). □

We now proceed with the proof of closability. Let μ_r^m be as in Definition 2.1. We remark that $\sum_{m=0}^{\infty} \mu_r^m = \mu$ by construction. Let $\mathcal{E}_r^{m,a,\mu}$ be the bilinear form defined by

$$\mathcal{E}_r^{m,a,\mu}(f,g) = \int \mathbb{D}^a[f,g] d\mu_r^m. \tag{3.1}$$

Then we have $\mathcal{E}^{a,\mu} = \sum_{m=1}^{\infty} \mathcal{E}_r^{m,a,\mu}$ for each $r \in \mathbb{N}$, where $\mathcal{E}^{a,\mu}$ is the bilinear form given by (2.2). We now quote a result from [13].

Lemma 3.2 (Theorem 2 in [13]). *Assume* $(\mathcal{E}_r^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu})$ *is closable on* $L^2(\mathsf{S},\mu)$ *for all* $r, m \in \mathbb{N}$*. Then* $(\mathcal{E}^{a,\mu}, \mathcal{D}_\infty^{a,\mu})$ *is closable on* $L^2(\mathsf{S},\mu)$*.*

Proof. When $b_r = r$ and the coefficient is the unit matrix, Lemma 3.2 was proved in Theorem 2 in [13]. The generalization to the present case is trivial. □

Let $\mu_{r,k}^m$ as in Definition 2.1. Define the bilinear form $\mathcal{E}_{r,k}^{m,a,\mu}$ by

$$\mathcal{E}_{r,k}^{m,a,\mu}(f,g) = \int \mathbb{D}^a[f,g] d\mu_{r,k}^m. \tag{3.2}$$

Lemma 3.3. *Assume* $(\mathcal{E}_{r,k}^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu})$ *is closable on* $L^2(\mathsf{S},\mu_{r,k}^m)$ *for all* k*. Then* $(\mathcal{E}_r^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu})$ *is closable on* $L^2(\mathsf{S},\mu)$*.*

Proof. By (2.7), we have $\mu_{r,k}^m \le \mu$. This implies $(\mathcal{E}_{r,k}^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu})$ is closable not only on $L^2(\mathsf{S},\mu_{r,k}^m)$ but also on $L^2(\mathsf{S},\mu)$. By (2.7), the forms $\{(\mathcal{E}_{r,k}^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu})\}$ are nondecreasing in k and converge to $(\mathcal{E}_r^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu})$ as $k \to \infty$. Hence, $(\mathcal{E}_r^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu})$ is closable on $L^2(\mathsf{S},\mu)$ according to the monotone convergence theorem of closable bilinear forms. □

Let $\mu_{r,k,\mathsf{s}}^m$ be as in (2.9). Let $\mathcal{E}_{r,k,\mathsf{s}}^{m,a,\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}^a[f,g] d\mu_{r,k,\mathsf{s}}^m$. By (2.10) and (3.1)

$$\mathcal{E}_{r,k}^{m,a,\mu}(f,g) = \int_{\mathsf{S}} \mathcal{E}_{r,k,\mathsf{s}}^{m,a,\mu}(f,g) \mu_{r,k}^m(d\mathsf{s}), \tag{3.3}$$

$$\|f\|_{L^2(\mathsf{S}_r^m,\mu_{r,k}^m)}^2 = \int_{\mathsf{S}} \|f\|_{L^2(\mathsf{S}_r^m,\mu_{r,k,\mathsf{s}}^m)}^2 \mu_{r,k}^m(d\mathsf{s}). \tag{3.4}$$

Lemma 3.4. *Assume* $(\mathcal{E}_{r,k,\mathsf{s}}^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu}, L^2(\mathsf{S}_r^m, \mu_{r,k,\mathsf{s}}^m))$ *is closable for* $\mu_{r,k}^m$*-a.s.* s*. Then* $(\mathcal{E}_{r,k}^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu}, L^2(\mathsf{S},\mu_{r,k}^m))$ *is closable.*

Lemma 3.5. *Assume* (A.0), (A.2)*, and* (A.3)*. Then* $(\mathcal{E}_{r,k,\mathsf{s}}^{m,a,\mu}, \mathcal{D}_\infty^{a,\mu}, L^2(\mathsf{S}_r^m, \mu_{r,k,\mathsf{s}}^m))$ *is closable for* $\mu_{r,k}^m$*-a.s.* s*.*

Although the proof of Lemma 3.4 is the same as that of Theorem 4 in [13], we present it in the Appendix (Section 11.1) for the reader's convenience. We also give the proof of Lemma 3.5 in Section 11.1. We are now ready to prove the closability of $(\mathcal{E}^{a,\mu}, \mathcal{D}_\infty^{a,\mu}, L^2(\mathsf{S},\mu))$.

Lemma 3.6. *Assume* (A.0)*,* (A.2)*, and* (A.3)*. Then* $(\mathcal{E}^{a,\mu}, \mathcal{D}_\infty^{a,\mu}, L^2(\mathsf{S},\mu))$ *is closable.*

Proof. By Lemma 3.2–3.5, we conclude Lemma 3.6. □

Proof of Lemma 2.1. Lemma 2.1 follows immediately from Lemma 3.1 and Lemma 3.6. □

Proof of Corollary 2.1. By Lemma 2.1 and [3, Theorems 4.5.1], there exists a μ-symmetric diffusion whose Dirichlet space is $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$. Since $1 \in \mathcal{D}^{a,\mu}$, the diffusion is conservative, which completes the proof. □

4 A sufficient condition of the quasi-Gibbs property.

The most crucial assumption in Lemma 2.1 is that of the quasi-Gibbs property (A.2). In this section, we introduce assumptions (A.4) and (A.5) below to obtain a sufficient condition of (A.2). These conditions guarantee that μ has a good finite-particle approximation $\{\mu^N\}_{N\in\mathbb{N}}$ that enables us to prove the quasi-Gibbs property. We set $\tilde{S}_r = \{x \in S\,; |x| < r\}$ and $\tilde{S}_r^n = \prod_{m=1}^n \{|x_m| < r\}$.

(A.4) There exists a sequence of probability measures $\{\mu^N\}_{N\in\mathbb{N}}$ on S satisfying the following.
(1) The n-correlation functions ρ_N^n of μ^N satisfy

$$\lim_{N\to\infty} \rho_N^n(x_1,\ldots,x_n) = \rho^n(x_1,\ldots,x_n) \quad \text{a.e.} \quad \text{for all } n \in \mathbb{N}, \tag{4.1}$$

$$\sup\{\rho_N^n(x_1,\ldots,x_n); N\in\mathbb{N},\, (x_1,\ldots,x_n)\in\tilde{S}_r^n\} \le \{c_7 n^\delta\}^n \quad \text{for all } n, r \in \mathbb{N}, \tag{4.2}$$

where $c_7 = c_7(r) > 0$ and $\delta = \delta(r) < 1$ are constants depending on $r \in \mathbb{N}$.
(2) $\mu^N(\mathsf{s}(S) \le n_N) = 1$ for some $n_N \in \mathbb{N}$.
(3) μ^N is a (Φ^N, Ψ^N)-canonical Gibbs measure.
(4) The potentials $\Phi^N : S \to \mathbb{R}\cup\{\infty\}$ and $\Psi^N : S\times S \to \mathbb{R}\cup\{\infty\}$ satisfy the following.

$$\lim_{N\to\infty} \Phi^N(x) = \Phi(x) \quad \text{for a.e. } x, \quad \inf_{N\in\mathbb{N}} \inf_{x\in S} \Phi^N(x) > -\infty, \tag{4.3}$$

$$\lim_{N\to\infty} \Psi^N = \Psi \text{ compact uniformly in } C^1(S\times S\backslash\{x=y\}), \tag{4.4}$$

$$\inf_{N\in\mathbb{N}} \inf_{x,y\in S_r} \Psi^N(x,y) > -\infty \quad \text{for all } r \in \mathbb{N}.$$

Remark 4.1. (1) By (4.1) and (4.2), we see that $\lim_{N\to\infty} \mu^N = \mu$ weakly in S (see Lemma 11.1). By $\mu^N(\mathsf{s}(S) \le n_N) = 1$, the DLR equation (2.11) makes sense even if Ψ^N is a logarithmic function. (4.4) implies the core Γ in (A.3) becomes $\Gamma = \{0\}$ or $\emptyset$.

(2) By assumption, for each $r \in \mathbb{N}$, $\Psi^N \in C^1(\tilde{S}_r \times \tilde{S}_r \backslash \{x=y\})$ for all sufficiently large N, and $\Psi \in C^1(S\times S\backslash\{x=y\})$. We note that Ψ^N is not necessarily in $C^1(S\times S\backslash\{x=y\})$.

The difficulty in treating the logarithmic interaction is the unboundedness at infinity. Indeed, the DLR equation does not make sense for infinite volume. The key issue in overcoming this difficulty is the fact that the logarithmic functions have small variations at infinity. With this property, we can control the difference of interactions rather than the interactions themselves. Bearing this in mind, we introduce the set $\mathsf{H}_{r,k}$ in (4.6) and the assumption (A.5) below.

For $\{S_r\}$ in (2.6), we set $S_{rs} = S_s\backslash S_r$ and $S_{r\infty} = S_r^c$. For $r < s \le t < u \le \infty$, we set

$$\Psi_{rs,tu}^N(\mathsf{x},\mathsf{y}) = \sum_{x_i\in S_{rs},\ y_j\in S_{tu}} \Psi^N(x_i,y_j) \qquad (\mathsf{x} = \sum \delta_{x_i},\ \mathsf{y} = \sum \delta_{y_j} \in \mathsf{S}). \tag{4.5}$$

We write $\Psi_{r,st}^N = \Psi_{0r,st}^N$ and $\Psi_{r,rs}^N(\mathsf{x},\mathsf{y}) = \Psi_{r,rs}^N(x,\mathsf{y})$ if $\mathsf{x} = \delta_x$. We define $\mathsf{H}_{r,k}$ by

$$\mathsf{H}_{r,k} = \{\mathsf{y}\in\mathsf{S}\,; \sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} \sup_{x\neq w\in S_r} \frac{|\Psi_{r,rs}^N(x,\mathsf{y}) - \Psi_{r,rs}^N(w,\mathsf{y})|}{|x-w|} \le k\}. \tag{4.6}$$

The following is a tightness condition on $\{\mu^N\}$ according to the interaction Ψ^N.

(A.5) The measures $\{\mu^N\}$ satisfy the following.

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mu^N(\mathsf{H}_{r,k}^c) = 0 \quad \text{for all } r \in \mathbb{N}. \tag{4.7}$$

Theorem 4.1. *Assume* (A.4) *and* (A.5). *Then* μ *is a* (Φ, Ψ)*-quasi Gibbs measure.*

We will prove Theorem 4.1 in Section 5.

Corollary 4.1 . Assume (A.0), (A.1), and (A.3)–(A.5). Then we have the following.
(1) $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}_{\infty}, L^2(\mathsf{S},\mu))$ is closable, and its closure $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$ is a local, quasi-regular Dirichlet space.
(2) There exists a μ-reversible diffusion (X, P) associated with $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mathsf{S},\mu))$.

Let S_r^m be as in (2.6). Using the set $\mathsf{H}_{r,k}$, we introduce cut-off measures $\mu_{r,k}^{N,m}$:

$$\mu_{r,k}^{N,m} = \mu^N(\cdot \cap \mathsf{S}_r^m \cap \mathsf{H}_{r,k}). \tag{4.8}$$

We will prove Theorem 4.1 along this sequence $\{\mu_{r,k}^{N,m}\}$. For this, we first note the following.

Lemma 4.2. *There exists a weak convergent subsequence of* $\{\mu_{r,k}^{N,m}\}$*, denoted by the same symbol, with limit measures* $\{\mu_{r,k}^{m}\}$ *satisfying* (2.7) *for all* r, k, m.

Proof. Recall that $\{\mu^N\}$ is a weak convergent sequence. This combined with $\mu_{r,k}^{N,m} \le \mu^N$ shows that $\{\mu_{r,k}^{N,m}\}$ is relatively compact for each $r, k, m \in \mathbb{N}$. Hence, we can choose a convergent subsequence $\{\mu_{r,k}^{n_N(r,k),m}\}$ from any subsequence of $\{\mu_{r,k}^{N,m}\}$ for each r, k, m. Then by diagonal argument, we obtain a weak convergent subsequence with limit $\{\mu_{r,k}^{m}\}$.

Since $\mathsf{H}_{r,k} \subset \mathsf{H}_{r,k+1}$, we have $\mu_{r,k}^{N,m} \le \mu_{r,k+1}^{N,m}$ by (4.8). This allows us to deduce $\mu_{r,k}^{m} \le \mu_{r,k+1}^{m}$, which is the first claim of (2.7). Because of the weak convergence, we see that for $f \in C_b(\mathsf{S})$

$$\begin{aligned} |\int f d\mu_{r,k}^m - \int f d\mu_r^m| \le & \lim_{N\to\infty} \{|\int f d\mu_{r,k}^m - \int f d\mu_{r,k}^{N,m}| + |\int f d\mu_r^{N,m} - \int f d\mu_r^m|\} \\ & + \limsup_{N\to\infty} |\int f d\mu_{r,k}^{N,m} - \int f d\mu_r^{N,m}| \\ = & \limsup_{N\to\infty} |\int f d\mu_{r,k}^{N,m} - \int f d\mu_r^{N,m}| \le \{\sup_{\mathsf{s}} |f(\mathsf{s})|\} \cdot \limsup_{N\to\infty} \mu_r^{N,m}(\{\mathsf{H}_{r,k}\}^c). \end{aligned}$$

By (4.7) we deduce that the right-hand side converges to zero as $k \to \infty$, which is the second claim of (2.7). We thus see that the limit measures $\{\mu_{r,k}^{m}\}$ satisfy (2.7). □

Let $\mu_{r,k,\mathsf{s},rs}^{N,m}$ denote the conditional probability of $\mu_{r,k}^{N,m}$ defined by

$$\mu_{r,k,\mathsf{s},rs}^{N,m}(d\mathsf{x}) = \mu_{r,k}^{N,m}(\pi_{S_r} \in d\mathsf{x} |\, \pi_{S_{rs}}(\mathsf{s})).$$

We note that, although $\mu_{r,k}^{N,m}$ is not necessarily a probability measure, we take the normalizing in such a way that the conditional measure $\mu_{r,k,\mathsf{s},rs}^{N,m}$ to be a probability measure. As a result, we have $\mu_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{S}) = 1$ and

$$\mu_{r,k}^{N,m} \circ \pi_{S_r}^{-1}(d\mathsf{x}) = \int_{\mathsf{S}} \mu_{r,k,\mathsf{s},rs}^{N,m}(d\mathsf{x})\, \mu_{r,k}^{N,m} \circ \pi_{S_{rs}}^{-1}(d\mathsf{s}). \tag{4.9}$$

Recall that by (A.4), μ^N is a (Φ^N, Ψ^N)-canonical Gibbs measure. Then μ^N satisfies the DLR equation (2.11). Hence, $\mu_{r,k,\mathsf{s},rs}^{N,m}$ is absolutely continuous w.r.t. $e^{-\mathcal{H}_r^N(\mathsf{x})}\Lambda(d\mathsf{x})$. Therefore, we denote its density by $\sigma_{r,k,\mathsf{s},rs}^{N,m}$. Then by definition, we have for $\mu_{r,k}^{N,m}$-a.e. s

$$\sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}) e^{-\mathcal{H}_r^N(\mathsf{x})}\Lambda(d\mathsf{x}) = \mu_{r,k,\mathsf{s},rs}^{N,m}(d\mathsf{x}), \qquad \text{where } \mathcal{H}_r^N = \mathcal{H}_{S_r}^{\Phi^N,\Psi^N}. \tag{4.10}$$

The quasi-Gibbs property consists of two conditions: (2.7) and (2.8). We have already proved (2.7) by Lemma 4.2. Therefore, it only remains to prove (2.8). This task is the most difficult part of the proof, and it is carried out in the next section. In the rest of this section, we explain the strategy of the proof of (2.8).

By taking the representation (4.9) into account, the proof consists of two kinds of limit procedures: (4.11) $N \to \infty$ and then (4.12) $s \to \infty$, which involve the following convergence.

$$\lim_{N\to\infty} \mu^{N,m}_{r,k,\mathsf{s},rs} = \mu^{m}_{r,k,\mathsf{s},rs}, \qquad \lim_{N\to\infty} \mu^{N,m}_{r,k} \circ \pi^{-1}_{S_{rs}} = \mu^{m}_{r,k} \circ \pi^{-1}_{S_{rs}}, \tag{4.11}$$

$$\lim_{s\to\infty} \mu^{m}_{r,k,\mathsf{s},rs} = \mu^{m}_{r,k,\mathsf{s}}. \tag{4.12}$$

Note that two of these are the convergence of the *conditional* measures. Comparing with the weak convergence of $\{\mu^{N,m}_{r,k}\}$ in Lemma 4.2, it is noted that the convergence of conditional measures is much more delicate. It involves a kind of strong convergence of the conditioned variable s.

In each step, we prove the bounds of the densities being uniform in N, s ((5.6) and (5.17)) and the related quantities as well as the convergence of measures as above. The uniformity of the bounds is the crucial point of the proof. We emphasize that we can carry out the proof because we treat the cut-off measures $\{\mu^{N,m}_{r,k}\}$ defined by (4.8). This cut-off is done by the set $\mathsf{H}_{r,k}$. Therefore, the assumption (A.5) plays a significant role in the proof of Theorem 4.1.

The first step consists of three lemmas. Recall the expressions (4.9) and (4.10). We prove the uniform bounds of $\int_{\mathsf{S}^m_r} e^{-\mathcal{H}^N_r(\mathsf{x})}\Lambda(d\mathsf{x})$ (Lemma 5.1) and $\sigma^{N,m}_{r,k,\mathsf{s},rs}$ (Lemma 5.2). We then prove the weak convergence $\lim_{N\to\infty} \mu^{N,m}_{r,k} \circ \pi^{-1}_{S_{rs}} = \mu^{m}_{r,k} \circ \pi^{-1}_{S_{rs}}$ and the L^1 convergence of their densities (Lemma 5.3).

The second step consists of two lemmas. In Lemma 5.4, we prove the absolute continuity of the measures $\mu^{m}_{r,k,\mathsf{s},rs}$ and the uniform bound (5.17) of their densities $\sigma^{m}_{r,k,\mathsf{s},rs}(\mathsf{x})$. Finally, in Lemma 5.5 we prove the convergence of $\sigma^{m}_{r,k,\mathsf{s},rs}(\mathsf{x})$ as $s \to \infty$ using martingale convergence theorems to complete the proof of the quasi-Gibbs property.

5 Proof of Theorem 4.1

In this section, we prove (2.8) to complete the proof of Theorem 4.1. We fix $r, m \in \mathbb{N}$ throughout this section. We divide this section into two parts. In Section 5.1, we prove the first step (4.11), and in Section 5.2, we prove the second step (4.12).

5.1 Proof of the first step.

Lemma 5.1. *Set* $c_8(n) = \sup_{n\le N\in\mathbb{N}} \max\{\int_{\mathsf{S}^m_r} e^{-\mathcal{H}^N_r(\mathsf{x})}\Lambda(d\mathsf{x}),\ [\int_{\mathsf{S}^m_r} e^{-\mathcal{H}^N_r(\mathsf{x})}\Lambda(d\mathsf{x})]^{-1}\}$.
Then there exists an N_0 *such that* $c_8(N_0) < \infty$.

Proof. By (A.4), we see that $\sup\{e^{-\mathcal{H}^N_r(\mathsf{x})}; N \in \mathbb{N}, \mathsf{x} \in \mathsf{S}^m_r\} < \infty$. Hence, by (4.3), (4.4), and the bounded convergence theorem, we deduce that

$$\lim_{N\to\infty} \int_{\mathsf{S}^m_r} e^{-\mathcal{H}^N_r(\mathsf{x})}\Lambda(d\mathsf{x}) = \int_{\mathsf{S}^m_r} e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x}) < \infty.$$

Recall that $\Phi(x) < \infty$ a.e. by assumption (see the line after (2.5)) and $\Psi(x, y) < \infty$ a.e. by the first assumption of (4.4). Therefore, $\mathcal{H}_r(\mathsf{x}) < \infty$ a.e.. Hence, $\int_{\mathsf{S}^m_r} e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x}) > 0$. Combining these completes the proof. ∎

We next consider a decomposition of $\sigma^{N,m}_{r,k,\mathsf{s},rs}$ in (4.10). By the DLR equation and (4.8), we deduce that for $\mu^{N,m}_{r,k}$-a.e. s, the density $\sigma^{N,m}_{r,k,\mathsf{s},rs}$ is expressed in such a way that

$$\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x}) = e^{-\Psi^N_{r,rs}(\mathsf{x},\mathsf{s})}\tau^N_{r,rs}(\mathsf{x},\mathsf{s})/c^N_9(\mathsf{s}). \tag{5.1}$$

Here $\Psi^N_{r,rs}$ is given by (4.5). We define $\tau^N_{r,rs}(\mathsf{x},\mathsf{s})$ and $c^N_9(\mathsf{s})$ by

$$\tau^N_{r,rs}(\mathsf{x},\mathsf{s}) = 1_{\mathsf{S}^m_r}(\mathsf{x})\int_{\mathsf{S}} 1_{\mathsf{H}_{r,k}}(\pi_{S_{rs}}(\mathsf{s})+\mathsf{z})e^{-\Psi^N_{r,s\infty}(\mathsf{x},\mathsf{z})-\Psi^N_{rs,s\infty}(\mathsf{s},\mathsf{z})}\mu^{N,m}_{r,k}\circ\pi^{-1}_{S_{s\infty}}(d\mathsf{z}), \tag{5.2}$$

$$c^N_9(\mathsf{s}) = \int_{\mathsf{S}} e^{-\Psi^N_{r,rs}(\mathsf{x},\mathsf{s})}\tau^N_{r,rs}(\mathsf{x},\mathsf{s})e^{-\mathcal{H}^N_r(\mathsf{x})}\Lambda(d\mathsf{x}). \tag{5.3}$$

We remark that, since $\mu^N(\mathsf{s}(S)\le n_N)=1$, $\Psi^N_{r,s\infty}$ and $\Psi^N_{rs,s\infty}$ are well defined for μ^N-a.s. s.

Set $c_{10}(k) = mk\cdot\mathrm{diam}\,(S_r)$. Then from (4.5) and (4.6), we deduce that

$$\sup_{N\in\mathbb{N}}\ \sup_{r\le s<t\in\mathbb{N}}\ \sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}^m_r}\ \sup_{\mathsf{s}\in\mathsf{H}_{r,k}} |\Psi^N_{r,st}(\mathsf{x},\mathsf{s})-\Psi^N_{r,st}(\mathsf{x}',\mathsf{s})| \le c_{10} \quad \text{for each } k\in\mathbb{N}. \tag{5.4}$$

Let $\mathsf{S}^n_{rs} = \{\mathsf{x}\in\mathsf{S};\, \mathsf{x}(S_{rs})=n\}$. Then from (4.6) and $S_{rs}\subset S_s$, we deduce that

$$\sup_{N\in\mathbb{N}}\ \sup_{r\le s<t\in\mathbb{N}}\ \sup_{\mathsf{y},\mathsf{y}'\in\mathsf{S}^n_{rs}}\ \sup_{\mathsf{s}\in\mathsf{H}_{s,l}}\Big\{\frac{|\Psi^N_{rs,st}(\mathsf{y},\mathsf{s})-\Psi^N_{rs,st}(\mathsf{y}',\mathsf{s})|}{d_{\mathsf{S}^n_{rs}}(\mathsf{y},\mathsf{y}')}\Big\} \le l \quad \text{for each } n,l\in\mathbb{N}. \tag{5.5}$$

Here for $\mathsf{s},\mathsf{t}\in\mathsf{S}^n_{rs}$, we set $d_{\mathsf{S}^n_{rs}}(\mathsf{s},\mathsf{t}) = \min\sum_{i=1}^n |s_i-t_i|$, where the minimum is taken over the labeling such that $\pi_{S_{rs}}(\mathsf{s}) = \sum_{i=1}^n\delta_{s_i}$ and $\pi_{S_{rs}}(\mathsf{t}) = \sum_{i=1}^n\delta_{t_i}$. Moreover, we used the inequality $\{a_1+\cdots+a_n\}/\{b_1+\cdots+b_n\}\le\max\{a_m/b_m; m=1,\ldots,n\}$ for $a_i\ge 0$ and $b_j>0$.

Lemma 5.2. *Let $c_{11}=e^{2c_{10}}c_8(N_0)$. Then for $\mu^{N,m}_{r,k}$-a.e. s, it holds that*

$$c^{-1}_{11}\le\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x})\le c_{11} \quad \textit{for all } \mathsf{x}\in\mathsf{S}^m_r,\ r<s\in\mathbb{N},\ \textit{and } N_0\le N\in\mathbb{N}. \tag{5.6}$$

Proof. By (5.1) and (5.4), we see that

$$\frac{\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x})}{\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x}')} = e^{-\Psi^N_{r,rs}(\mathsf{x},\mathsf{s})+\Psi^N_{r,rs}(\mathsf{x}',\mathsf{s})}\frac{\tau^N_{r,rs}(\mathsf{x},\mathsf{s})}{\tau^N_{r,rs}(\mathsf{x}',\mathsf{s})} \le e^{c_{10}}\frac{\tau^N_{r,rs}(\mathsf{x},\mathsf{s})}{\tau^N_{r,rs}(\mathsf{x}',\mathsf{s})}. \tag{5.7}$$

By (5.2), we have for $\mu^{N,m}_{r,k}$-a.e. s

$$\begin{aligned}
&\sup_{N\in\mathbb{N}}\ \sup_{r<s\in\mathbb{N}}\ \sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}^m_r}\{\tau^N_{r,rs}(\mathsf{x},\mathsf{s})/\tau^N_{r,rs}(\mathsf{x}',\mathsf{s})\} \\
&=\sup_{N\in\mathbb{N}}\ \sup_{r<s\in\mathbb{N}}\ \sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}^m_r}\Big\{\frac{\int_{\mathsf{S}}1_{\mathsf{H}_{r,k}}(\pi_{S_{rs}}(\mathsf{s})+\mathsf{z})e^{-\Psi^N_{r,s\infty}(\mathsf{x},\mathsf{z})-\Psi^N_{rs,s\infty}(\mathsf{s},\mathsf{z})}\mu^{N,m}_{r,k}\circ\pi^{-1}_{S_{s\infty}}(d\mathsf{z})}{\int_{\mathsf{S}}1_{\mathsf{H}_{r,k}}(\pi_{S_{rs}}(\mathsf{s})+\mathsf{z})e^{-\Psi^N_{r,s\infty}(\mathsf{x}',\mathsf{z})-\Psi^N_{rs,s\infty}(\mathsf{s},\mathsf{z})}\mu^{N,m}_{r,k}\circ\pi^{-1}_{S_{s\infty}}(d\mathsf{z})}\Big\} \\
&=\sup_{N\in\mathbb{N}}\ \sup_{r<s<t\in\mathbb{N}}\ \sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}^m_r}\Big\{\frac{\int_{\mathsf{S}}1_{\mathsf{H}_{r,k}}(\pi_{S_{rs}}(\mathsf{s})+\mathsf{z})e^{-\Psi^N_{r,st}(\mathsf{x},\mathsf{z})-\Psi^N_{rs,st}(\mathsf{s},\mathsf{z})}\mu^{N,m}_{r,k}\circ\pi^{-1}_{S_{s\infty}}(d\mathsf{z})}{\int_{\mathsf{S}}1_{\mathsf{H}_{r,k}}(\pi_{S_{rs}}(\mathsf{s})+\mathsf{z})e^{-\Psi^N_{r,st}(\mathsf{x}',\mathsf{z})-\Psi^N_{rs,st}(\mathsf{s},\mathsf{z})}\mu^{N,m}_{r,k}\circ\pi^{-1}_{S_{s\infty}}(d\mathsf{z})}\Big\} \\
&\le e^{c_{10}} \qquad \text{by (5.4).}
\end{aligned} \tag{5.8}$$

Here we used $\mu^N(\mathsf{s}(S)\le n_N)=1$ for the third line. By (5.7) and (5.8), we deduce that

$$\sup_{N\in\mathbb{N}}\ \sup_{r<s\in\mathbb{N}}\ \sup_{\mathsf{x},\mathsf{x}'\in\mathsf{S}^m_r}\{\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x})/\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x}')\}\le e^{2c_{10}} \quad \text{for } \mu^{N,m}_{r,k}\text{-a.e. } \mathsf{s}.$$

Hence for $\mu^{N,m}_{r,k}$-a.e. s, we see that for all $\mathsf{x},\mathsf{x}'\in\mathsf{S}^m_r$, $r<s\in\mathbb{N}$, and $N\in\mathbb{N}$,

$$e^{-2c_{10}}\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x}')\le\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x})\le e^{2c_{10}}\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{x}'). \tag{5.9}$$

Multiply (5.9) by $1_{\mathsf{S}_r^m}(\mathsf{x}')e^{-\mathcal{H}_r^N(\mathsf{x}')}$ and integrate w.r.t. $\Lambda(d\mathsf{x}')$. Note that by (4.10) we have $\int_{\mathsf{S}_r^m} \sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}')e^{-\mathcal{H}_r^N(\mathsf{x}')}\Lambda(d\mathsf{x}') = 1$. Then we deduce that for $\mu_{r,k}^{N,m}$-a.e. s,

$$e^{-2c_{10}} \le \sigma_{r,k,\mathsf{s},rs}^{N,m}(\mathsf{x}) \int_{\mathsf{S}_r^m} e^{-\mathcal{H}_r^N(\mathsf{x}')}\Lambda(d\mathsf{x}') \le e^{2c_{10}} \quad \text{for all } \mathsf{x} \in \mathsf{S}_r^m.$$

This combined with Lemma 5.1 yields (5.6). □

Let $\mathcal{H}_{rs}^N = \mathcal{H}_{S_{rs}}^{\Phi^N,\Psi^N}$ and $\mathcal{H}_{rs} = \mathcal{H}_{S_{rs}}^{\Phi,\Psi}$. By (4.1) and (4.2), we see that $\mu_{r,k}^{N,m} \circ \pi_{S_{rs}}^{-1}$ and $\mu_{r,k}^{m} \circ \pi_{S_{rs}}^{-1}$ are absolutely continuous w.r.t. $e^{-\mathcal{H}_{rs}^N}\Lambda$ and $e^{-\mathcal{H}_{rs}}\Lambda$, respectively. Hence, we denote by Δ^N and Δ their Radon-Nikodym densities, respectively.

Lemma 5.3. (1) *$\mu_{r,k}^{N,m} \circ \pi_{S_{rs}}^{-1}$ converges weakly to $\mu_{r,k}^{m} \circ \pi_{S_{rs}}^{-1}$ as $N \to \infty$.*
(2) *$\Delta^N e^{-\mathcal{H}_{rs}^N}$ converges to $\Delta e^{-\mathcal{H}_{rs}}$ in $L^1(\mathsf{S},\Lambda)$ as $N \to \infty$.*

Proof. Let E be the discontinuity points of $\pi_{S_{rs}}$. Namely

$$E = \{\mathsf{s} \in \mathsf{S}\,;\ \lim_{n\to\infty} \pi_{S_{rs}}(\mathsf{s}_n) \neq \pi_{S_{rs}}(\mathsf{s}) \text{ for some } \{\mathsf{s}_n\} \text{ such that } \lim_{n\to\infty} \mathsf{s}_n = \mathsf{s}\}.$$

Then by (A.1), we deduce that $\mu_{r,k}^m(E) \le \mu(E) = 0$. Since $\mu_{r,k}^{N,m}$ converge weakly to $\mu_{r,k}^m$ by Lemma 4.2 and the discontinuity points of $\pi_{S_{rs}}^{-1}$ are $\mu_{r,k}^m$-measure zero, we obtain (1).

We proceed with (2). It only remains to prove that $\{\Delta^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$ is relatively compact in $L^1(\mathsf{S},\Lambda)$. Indeed, if this property holds, then their limit points are unique and equal to $\Delta e^{-\mathcal{H}_{rs}}$ by (1).

Recall that $\mathsf{S}_{rs}^n = \{\mathsf{x} \in \mathsf{S};\, \mathsf{x}(S_{rs}) = n\}$ and note that $\Delta^N e^{-\mathcal{H}_{rs}^N} = \Delta^N e^{-\mathcal{H}_{rs}^N} \sum_{n=0}^\infty 1_{\mathsf{S}_{rs}^n}$. By (1), we deduce that for each $\epsilon > 0$ there exists an n_0 such that

$$\sup_{N\in\mathbb{N}} \mu_{r,k}^{N,m}(\sum_{n=n_0}^\infty \mathsf{S}_{rs}^n) < \epsilon, \tag{5.10}$$

which is equivalent to

$$\sup_{N\in\mathbb{N}} \|\Delta^N e^{-\mathcal{H}_{rs}^N} \sum_{n=n_0}^\infty 1_{\mathsf{S}_{rs}^n}\|_{L^1(\mathsf{S},\Lambda)} < \epsilon. \tag{5.11}$$

According to (5.11), the relative compactness of $\{\Delta^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$ in $L^1(\mathsf{S},\Lambda)$ follows from that of $\{\Delta^N e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{S}_{rs}^n}\}_{N\in\mathbb{N}}$ for each $n \in \mathbb{N}$. Hence, we fix $n \in \mathbb{N}$ in the rest of the proof.

We set $\mu_l^N = \mu_{r,k}^{N,m}(\cdot \cap \mathsf{S}_{rs}^n \cap \mathsf{H}_{s,l})$, where $\mathsf{H}_{s,l}$ is as in (4.6). Let Δ_l^N be the Radon-Nikodym density of $\mu_l^N \circ \pi_{S_{rs}}^{-1}$ w.r.t. $e^{-\mathcal{H}_{rs}^N}\Lambda$. Since $\mu_l^N \le \mu_{r,k}^{N,m}$, we see that $\Delta_l^N e^{-\mathcal{H}_{rs}^N} \le \Delta^N e^{-\mathcal{H}_{rs}^N}$. Combining this with (4.7) yields

$$\lim_{l\to\infty} \limsup_{N\in\mathbb{N}} \|\Delta^N e^{-\mathcal{H}_{rs}^N} - \Delta_l^N e^{-\mathcal{H}_{rs}^N}\|_{L^1(\mathsf{S},\Lambda)} \le \lim_{l\to\infty} \limsup_{N\in\mathbb{N}} \mu_{r,k}^{N,m}(\mathsf{H}_{s,l}^c) = 0. \tag{5.12}$$

According to (5.12), it only remains to prove the relative compactness of $\{\Delta_l^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$ in $L^1(\mathsf{S},\Lambda)$ for each $l \in \mathbb{N}$. Hence, we fix $l \in \mathbb{N}$ in the rest of the proof.

For $q \in \mathbb{N}$ we set $B_r^q = \{0 < |s - S_r| < 1/q\}$. Let

$$\mathsf{A}_q = \{\mathsf{s} \in \mathsf{S}_{rs}^n \cap \mathsf{H}_{s,l}\,;\ \mathsf{s}(B_r^q) = 0\}. \tag{5.13}$$

By definition, A_q is the subset of $\mathsf{S}_{rs}^n \cap \mathsf{H}_{s,l}$ with no particles in B_r^q, where B_r^q is the intersection of S_r^c and the $1/q$-neighborhood of S_r. Then the relative compactness of $\{\Delta_l^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$

follows from that of $\{\Delta_l^N e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{A}_q}\}_{N\in\mathbb{N}}$ for all sufficiently large $q \in \mathbb{N}$. Indeed, by (4.1)–(4.4), for each $\epsilon > 0$ there exists a $q_0 \in \mathbb{N}$ such that, for all $q \geq q_0$,

$$\sup_{N\in\mathbb{N}} \|\Delta_l^N e^{-\mathcal{H}_{rs}^N} - \Delta_l^N e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{A}_q}\|_{L^1(\mathsf{S},\Lambda)} \leq \sup_{N\in\mathbb{N}} \mu_{r,k}^{N,m}((\mathsf{A}_q)^c) \leq \sup_{N\in\mathbb{N}} \int_{B_r^q} \rho_N^1(x)dx \leq \epsilon.$$

Let $c_{12}(q)$ be the constant defined by

$$c_{12}(q) = \sup_{N\in\mathbb{N}} \sup_{\mathsf{x}\in\mathsf{S}_r^m} \sup\{\frac{|\Psi_{r,rs}^N(\mathsf{x},\mathsf{y}) - \Psi_{r,rs}^N(\mathsf{x},\mathsf{y}')|}{d_{\mathsf{S}_{rs}^n}(\mathsf{y},\mathsf{y}')};\ \mathsf{y} \neq \mathsf{y}' \in \mathsf{A}_q\}. \tag{5.14}$$

Then we have $c_{12}(q) < \infty$. Note that $\pi_{S_{rs}^c} = \pi_{S_r} + \pi_{S_{s\infty}}$. Hence we write $\pi_{S_{rs}^c}(\mathsf{s}) = \mathsf{x} + \mathsf{z}$, where $\mathsf{x} \in \pi_{S_r}(\mathsf{S})$ and $\mathsf{z} \in \pi_{S_{s\infty}}(\mathsf{S})$. With this notation, $\Delta_l^N(\mathsf{y})$ can be written as

$$\Delta_l^N(\mathsf{y}) = \text{const.} \int_{\mathsf{S}} 1_{\mathsf{H}_{r,k}\cap\mathsf{H}_{s,l}}(\mathsf{x} + \pi_{S_{rs}}(\mathsf{y}) + \mathsf{z}) e^{-\Psi_{r,rs}^N(\mathsf{x},\mathsf{y}) - \Psi_{rs,s\infty}^N(\mathsf{y},\mathsf{z})} \mu_l^N \circ \pi_{S_{rs}^c}^{-1}(d\mathsf{x}d\mathsf{z}).$$

Then applying (5.14) and (5.5) to $\Psi_{r,rs}^N(\mathsf{x},\mathsf{y})$ and $\Psi_{rs,s\infty}^N(\mathsf{y},\mathsf{z})$ respectively, we deduce that

$$\sup_{N\in\mathbb{N}} \sup\{\frac{\Delta_l^N(\mathsf{y})}{\Delta_l^N(\mathsf{y}')};\ \mathsf{y},\mathsf{y}' \in \mathsf{A}_q\} \leq e^{(c_{12}(q)+l)d_{\mathsf{S}_{rs}^n}(\mathsf{y},\mathsf{y}')}. \tag{5.15}$$

Taking the logarithm of (5.15) and interchanging the role of y and y', we deduce that

$$\sup_{N\in\mathbb{N}} \sup\{|\log \Delta_l^N(\mathsf{y}) - \log \Delta_l^N(\mathsf{y}')|;\ \mathsf{y},\mathsf{y}' \in \mathsf{A}_q\} \leq (c_{12}(q) + l) d_{\mathsf{S}_{rs}^n}(\mathsf{y},\mathsf{y}'). \tag{5.16}$$

We deduce from (5.15) and (5.16) that $\{\Delta_l^N(\mathsf{y})\}_{N\in\mathbb{N}}$ is equi-continuous in y on A_q for each $q \in \mathbb{N}$. From the definition of Δ_l^N, we see that $\sup_{N\in\mathbb{N}} \|\Delta_l^N e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{A}_q}\|_{L^1(\mathsf{S},\Lambda)} < \infty$. We deduce from (4.3) and (4.4) that $\lim_{N\to\infty} e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{A}_q} = e^{-\mathcal{H}_{rs}} 1_{\mathsf{A}_q}$ in $L^1(\mathsf{S},\Lambda)$, and that $\|e^{-\mathcal{H}_{rs}} 1_{\mathsf{A}_q}\|_{L^1(\mathsf{S},\Lambda)} > 0$, which implies $\liminf_{N\to\infty} \|e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{A}_q}\|_{L^1(\mathsf{S},\Lambda)} > 0$. These allow us to deduce that

$$\limsup_{N\to\infty} \|\Delta_l^N 1_{\mathsf{A}_q}\|_{L^\infty(\mathsf{S},\Lambda)} < \infty$$

We therefore apply the Ascoli-Arzelá theorem to $\Delta_l^N 1_{\mathsf{A}_q}$ to deduce that $\{\Delta_l^N 1_{\mathsf{A}_q}\}$ is relatively compact in $C_b(\mathsf{A}_q)$ with uniform norm. Because $\{e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{A}_q}\}_{N\in\mathbb{N}}$ is uniformly bounded, we conclude that $\{\Delta_l^N e^{-\mathcal{H}_{rs}^N} 1_{\mathsf{A}_q}\}_{N\in\mathbb{N}}$ is relatively compact in $L^1(\mathsf{S},\Lambda)$ for each q. We therefore deduce that $\{\Delta^N e^{-\mathcal{H}_{rs}^N}\}_{N\in\mathbb{N}}$ is relatively compact in $L^1(\mathsf{S},\Lambda)$. Therefore, we complete the proof. □

5.2 Proof of the second step.

Lemma 5.4. *Let* $\mu_{r,k,\mathsf{s},rs}^m = \mu_{r,k}^m(\pi_{S_r}(\mathsf{s}) \in d\mathsf{x}|\ \pi_{S_{rs}}(\mathsf{s}))$. *Then we have the following.*
(1) $\mu_{r,k,\mathsf{s},rs}^m$ *is absolutely continuous w.r.t.* $e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})$ *for* $\mu_{r,k}^m$*-a.e.* s.
(2) *For each* $r, m, k \in \mathbb{N}$, *the Radon-Nikodym densities* $\sigma_{r,k,\mathsf{s},rs}^m$ *of* $\mu_{r,k,\mathsf{s},rs}^m$ *in* (1) *satisfy for* $\mu_{r,k}^m$*-a.e.* s *and all* $s \in \mathbb{N}$ *such that* $r < s$

$$c_{11}^{-1} \leq \sigma_{r,k,\mathsf{s},rs}^m(\mathsf{x}) \leq c_{11} \quad \textit{for } \mu_{r,k,\mathsf{s},rs}^m\textit{-a.e. } \mathsf{x}. \tag{5.17}$$

Proof. Similar to the case of Lemma 5.3 (1), we see that $\mu_{r,k}^{N,m} \circ (\pi_{S_r}, \pi_{S_{rs}})^{-1}$ converge weakly to $\mu_{r,k}^m \circ (\pi_{S_r}, \pi_{S_{rs}})^{-1}$ as $N \to \infty$. Hence, for $\mathsf{f}, \mathsf{g} \in C_b(\mathsf{S})$, we have

$$\int_{\mathsf{S}} \mathsf{f}(\pi_{S_r}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}}(\mathsf{s}))d\mu_{r,k}^m = \lim_{N\to\infty} \int_{\mathsf{S}} \mathsf{f}(\pi_{S_r}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}}(\mathsf{s}))d\mu_{r,k}^{N,m}. \tag{5.18}$$

By Lemma 5.2 and the diagonal argument, there exist subsequences of $\{\sigma^{N,m}_{r,k,\mathsf{s},rs}\}_N$, denoted by the same symbol, with a limit $\sigma^m_{r,k,\mathsf{s},rs}$ such that for all $k,m,r<s\in\mathbb{N}$,

$$\lim_{N\to\infty}\sigma^{N,m}_{r,k,\mathsf{s},rs}(\pi_{S_r}(\mathsf{s}))=\sigma^m_{r,k,\mathsf{s},rs}(\pi_{S_r}(\mathsf{s}))\quad *\text{-weakly in } L^\infty(\mathsf{S},\Lambda). \tag{5.19}$$

Here $\sigma^m_{r,k,\mathsf{s},rs}$ is a function such that $\sigma^m_{r,k,\mathsf{s},rs}(\mathsf{x})=\sigma^m_{r,k,\pi_{S_{rs}}(\mathsf{s}),rs}(\pi_{S_r}(\mathsf{x}))$. Let

$$\mathsf{F}^N(\mathsf{s})=\mathsf{f}(\pi_{S_r}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}}(\mathsf{s}))\Delta^N(\mathsf{s})e^{-\mathcal{H}^N_r(\mathsf{s})}, \tag{5.20}$$

$$\mathsf{F}(\mathsf{s})=\mathsf{f}(\pi_{S_r}(\mathsf{s}))\mathsf{g}(\pi_{S_{rs}}(\mathsf{s}))\Delta(\mathsf{s})e^{-\mathcal{H}_r(\mathsf{s})}. \tag{5.21}$$

Then by Lemma 5.3 (2), we see that F^N converge to F in $L^1(\mathsf{S},\Lambda)$. This combined with (5.19) implies

$$\lim_{N\to\infty}\int_{\mathsf{S}}\mathsf{F}^N(\mathsf{s})\sigma^{N,m}_{r,k,\mathsf{s},rs}(\mathsf{s})d\Lambda=\int_{\mathsf{S}}\mathsf{F}(\mathsf{s})\sigma^m_{r,k,\mathsf{s},rs}(\mathsf{s})d\Lambda. \tag{5.22}$$

By (5.18), (5.22) and $\Delta(\mathsf{y})e^{-\mathcal{H}_r(\mathsf{y})}\Lambda(d\mathsf{y})=\mu^m_{r,k}\circ\pi^{-1}_{S_{rs}}(d\mathsf{y})$, we obtain

$$\int_{\mathsf{S}}\mathsf{f}(\mathsf{x})\mathsf{g}(\mathsf{y})d\mu^m_{r,k}=\int_{\mathsf{S}}\mathsf{f}(\mathsf{x})\mathsf{g}(\mathsf{y})\sigma^m_{r,k,\mathsf{s},rs}(\mathsf{x})e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})\mu^m_{r,k}\circ\pi^{-1}_{S_{rs}}(d\mathsf{y}),$$

where $\mathsf{x}=\pi_{S_r}(\mathsf{s})$ and $\mathsf{y}=\pi_{S_{rs}}(\mathsf{s})$. Hence, we obtain (1) with density $\sigma^m_{r,k,\mathsf{s},rs}$.

By (5.6) and (5.19), we see that $\sigma^m_{r,k,\mathsf{s},rs}$ satisfies (5.17), which implies (2). □

Lemma 5.5. *Let $\mu^m_{r,k,\mathsf{s}}(d\mathsf{x})$ be as in (2.9). Let $\sigma^m_{r,k,\mathsf{s},rs}$ be as in Lemma 5.4. Then the following limit exists.*

$$\sigma^m_{r,k,\mathsf{s}}(\mathsf{x}):=\lim_{s\to\infty}\sigma^m_{r,k,\mathsf{s},rs}(\mathsf{x})\quad\text{for } \mu^m_{r,k,\mathsf{s}}\text{-a.s. } \mathsf{x}, \text{ for } \mu^m_{r,k}\text{-a.s. } \mathsf{s}. \tag{5.23}$$

Moreover, $\sigma^m_{r,k,\mathsf{s}}$ satisfies for $\mu^m_{r,k}$-a.e. s

$$c_{11}^{-1}\le\sigma^m_{r,k,\mathsf{s}}(\mathsf{x})\le c_{11}\quad\text{for } \mu^m_{r,k,\mathsf{s}}\text{-a.e. } \mathsf{x} \tag{5.24}$$

$$\sigma^m_{r,k,\mathsf{s}}(\mathsf{x})e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})=\mu^m_{r,k,\mathsf{s}}(d\mathsf{x}). \tag{5.25}$$

Proof. Define $M_s:\mathsf{S}\to\mathbb{R}$ by $M_s(\mathsf{s})=\sigma^m_{r,k,\mathsf{s},rs}(\mathsf{x})$, where $\mathsf{x}=\pi_{S_r}(\mathsf{s})$. Recall that $\sigma^m_{r,k,\mathsf{s},rs}$ is the Radon-Nikodym density of $\mu^m_{r,k,\mathsf{s},rs}$ w.r.t. $e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})$ and that $\mu^m_{r,k,\mathsf{s},rs}=\mu^m_{r,k,\pi_{S_{rs}}(\mathsf{s}),rs}$ by construction. Hence,

$$M_s(\mathsf{s})e^{-\mathcal{H}_r(\mathsf{x})}\Lambda(d\mathsf{x})=\mu^m_{r,k,\pi_{S_{rs}}(\mathsf{s}),rs}(d\mathsf{x}). \tag{5.26}$$

Let $\mathcal{F}_s=\sigma[\pi_{S_r},\pi_{S_{rs}}]$, where $r<s\le\infty$. Then by (5.26), we see that $\{M_s\}_{s\in[r,\infty)}$ is an $(\mathcal{F}_s)$-martingale, which implies $M_\infty(\mathsf{s}):=\lim_{s\to\infty}M_s(\mathsf{s})$ exists for $\mu^m_{r,k}$-a.e. s. Since

$$M_s(\mathsf{s})=\sigma^m_{r,k,\pi_{S_{rs}}(\mathsf{s}),rs}(\mathsf{x}),\quad\text{where } \mathsf{x}=\pi_{S_r}(\mathsf{s}),$$

we write $M_\infty(\mathsf{s})=\sigma^m_{r,k,\mathsf{s}}(\mathsf{x})$. By construction, $\sigma^m_{r,k,\mathsf{s}}(\mathsf{x})=\sigma^m_{r,k,\pi_{S_{r\infty}}(\mathsf{s})}(\mathsf{x})=\sigma^m_{r,k,\pi_{S^c_r}(\mathsf{s})}(\mathsf{x})$ and, for $\mu^m_{r,k}$-a.s. s, we can regard $\sigma^m_{r,k,\mathsf{s}}(\mathsf{x})$ as a $\sigma[\pi_{S_r}]$-measurable function in x. Hence, through disintegration (2.10), we obtain (5.23).

We immediately obtain (5.24) from (5.17) and (5.23).

We see that $\{M_s\}_{s\in[r,\infty)}$ is uniformly integrable by (5.17). Hence, by (5.23), we see that $M_s(\mathsf{s})$ converges to $M_\infty(\mathsf{s})=\sigma^m_{r,k,\mathsf{s}}(\mathsf{x})$ strongly in $L^1(\mathsf{S}^m_r,\mu^m_{r,k,\mathsf{s}})$, which combined with (5.26) and the definition $M_s(\mathsf{s})=\sigma^m_{r,k,\mathsf{s},rs}(\mathsf{x})$ yields (5.25). □

Proof of Theorem 4.1. By Lemma 4.2, we see that $\{\mu^m_{r,k}\}$ satisfies (2.7). Moreover, by (5.24) and (5.25) we deduce that $\mu^m_{r,k,\mathsf{s}}$ satisfies (2.8), which completes the proof of Theorem 4.1. □

6 A sufficient condition of (A.5)

In this section, we give a sufficient condition of (A.5) when Ψ is a logarithmic function and $d=1,2$. When $d=2$, we regard $\mathbb{R}^2$ as $\mathbb{C}$. We assume

$$\Psi(x,y) = -\beta \log |x-y| \qquad (\beta \in \mathbb{R}). \tag{6.1}$$

We take Ψ^N in two different ways. In the first case we assume $d=1,2$ and $\Psi^N=\Psi$ for all N, while in the second case Ψ^N depend on N. To unify these two cases, we introduce

$$\Psi^N(x,y) = -\beta \log |\varpi_N(x) - \varpi_N(y)|. \tag{6.2}$$

We set for the first case $d=1,2$ and

$$\varpi_N(x) = x. \tag{6.3}$$

Next we let $\mathbb{I}_N = (-N,N)$ and $n_N = 2^{4N}$. For the second case, we set $d=1$ and define the map $\varpi_N : S \to \mathbb{C}$ by

$$\varpi_N(x) = \begin{cases} \mathrm{i}\frac{n_N}{2\pi}(1-e^{2\pi \mathrm{i}x/n_N}), & \text{for } x \in \mathbb{I}_N \\ x & \text{for } x \in \mathbb{I}_{N+1}^c \\ \text{linear interpolation} & \text{for } x \in \mathbb{I}_{N+1} \backslash \mathbb{I}_N \end{cases}. \tag{6.4}$$

By construction, we have $\varpi_N(0)=0$, $\Re[\varpi_N(x)] = -\Re[\varpi_N(-x)]$, and $\Im[\varpi_N(x)] = \Im[\varpi_N(-x)]$. Here $\Re[\,\cdot\,]$ and $\Im[\,\cdot\,]$ denote the real and imaginary part of $\cdot$, respectively. It is easy to see that $|\varpi_N(x)| < |\varpi_N(y)|$ for $|x|<|y|$. We note that ϖ_N maps $\mathbb{I}_N$ into a subset of the circle in $\mathbb{C}$ centered at $\mathrm{i}\frac{n_N}{2\pi}$ with radius $\frac{n_N}{2\pi}$. We take $n_N = 2^{4N}$ such that it is large compared with N, which converges the trajectory of $\varpi_N(\mathbb{R})$ to the real axis rapidly as $N \to \infty$.

In the former case, (6.3) is used for the Ginibre random point field (Theorem 2.3). We will use this choice to prove the quasi-Gibbs property of the Bessel random point field in a forth coming paper. In the latter case, (6.4) is used for Dyson's model (Theorem 2.2), where we use circular ensembles, and thus, the above choice of ϖ_N is suitable.

The argument in this section may be generalized to higher dimensions $d \ge 3$. We restrict ourselves to the case $d=1,2$. As a result, we obtain a rather simple expression of the Tayler expansion of $\Psi(\varpi_N(x), \varpi_N(y))$. We remark that $z/|z|^2 = 1/\bar{z} \in \mathbb{C}$.

Lemma 6.1. *Assume* (6.1). *Let $x,y \in \mathbb{R}$ such that $|\varpi_N(x)| < |\varpi_N(y)|$. Then*

$$\Psi(\varpi_N(x), \varpi_N(y)) - \Psi(0, \varpi_N(y)) = \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} \Re[(\frac{\bar{\varpi}_N(x)}{\bar{\varpi}_N(y)})^{\ell}]. \tag{6.5}$$

Here $\bar{\varpi}_N$ denotes the complex conjugate of ϖ_N.

Proof. Let $r = |\varpi_N(x)|/|\varpi_N(y)|$ and $\theta = \angle(\varpi_N(x), \varpi_N(y))$. Then we see that

$$\begin{aligned} \Psi(\varpi_N(x), \varpi_N(y)) - \Psi(0, \varpi_N(y)) &= -\frac{\beta}{2} \log |\frac{\varpi_N(x)}{|\varpi_N(y)|} - \frac{\varpi_N(y)}{|\varpi_N(y)|}|^2 \\ &= -\frac{\beta}{2} \log \left(1 + r^2 - 2r\cos\theta\right) = -\frac{\beta}{2}\{\log(1-re^{\mathrm{i}\theta}) + \log(1-re^{-\mathrm{i}\theta})\}. \end{aligned}$$

Hence, (6.5) follows from the Tayler expansion. ∎

Remark 6.1. When (6.3) holds, Lemma 6.1 implies that for $0 < |x| < |y|$

$$\Psi(x,y) - \Psi(0,y) = \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} (\frac{x}{y})^{\ell} \qquad \text{if } S = \mathbb{R} \tag{6.6}$$

$$\Psi(x,y) - \Psi(0,y) = \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} \Re[(\frac{\bar{x}}{\bar{y}})^{\ell}] \qquad \text{if } S = \mathbb{C}\ . \tag{6.7}$$

Let $S_{rs} = S_s \backslash S_r = \{y \in S\,;\, b_r \le |y| < b_s\}$ as before, where S_r and b_r are given by (2.6). We set $\Psi_{rs}^N(x,\mathsf{y}) = \sum_{y_i \in S_{rs}} \Psi^N(x, y_i)$, where $\mathsf{y} = \sum_i \delta_{y_i}$. By (6.5),

$$\begin{aligned}\Psi_{rs}^N(x,\mathsf{y}) - \Psi_{rs}^N(w,\mathsf{y}) &= \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{y_i \in S_{rs}} \Re[\frac{\bar{\varpi}_N(x)^{\ell} - \bar{\varpi}_N(w)^{\ell}}{\bar{\varpi}_N(y_i)^{\ell}}] \\ &= \beta \sum_{\ell=1}^{\infty} \frac{1}{\ell} \Re[(\bar{\varpi}_N(x)^{\ell} - \bar{\varpi}_N(w)^{\ell}) \cdot \sum_{y_i \in S_{rs}} \frac{1}{\bar{\varpi}_N(y_i)^{\ell}}].\end{aligned}$$

Then, since $|\Re[ab]| \le |a||b|$ and $|\bar{a}^{\ell} - \bar{b}^{\ell}| = |a^{\ell} - b^{\ell}|$, we have

$$\frac{|\Psi_{rs}^N(x,\mathsf{y}) - \Psi_{rs}^N(w,\mathsf{y})|}{|x-w|} \le |\beta| \sum_{\ell=1}^{\infty} \frac{|\varpi_N(x)^{\ell} - \varpi_N(w)^{\ell}|}{\ell |x-w|} \cdot |\sum_{y_i \in S_{rs}} \frac{1}{\bar{\varpi}_N(y_i)^{\ell}}|. \tag{6.8}$$

Our purpose is to estimate $|\Psi_{rs}^N(x,\mathsf{y}) - \Psi_{rs}^N(w,\mathsf{y})|/|x-w|$ for $x \ne w \in S_r$. Hence, by (6.8), the main task is to control the term of the form $|\sum_{y_i \in S_{rs}} 1/\bar{\varpi}_N(y_i)^{\ell}|$. Taking this into account, we set for $r, \ell, k \in \mathbb{N}$

$$\mathsf{U}_{r,\ell,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \sup_{N \in \mathbb{N}} \sup_{r<s \in \mathbb{N}} |\sum_{y_i \in S_{rs}} \frac{1}{\bar{\varpi}_N(y_i)^{\ell}}| \le k\}, \tag{6.9}$$

$$\bar{\mathsf{U}}_{r,\ell,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \sup_{N \in \mathbb{N}} \{\sum_{y_i \in S_{r\infty}} \frac{1}{|\varpi_N(y_i)|^{\ell} - |\varpi_N(b_r)|^{\ell}}\} \le k\}. \tag{6.10}$$

Remark 6.2. When (6.3) holds, the definitions (6.9) and (6.10) become much simpler:

$$\mathsf{U}_{r,\ell,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \sup_{r<s \in \mathbb{N}} |\sum_{y_i \in S_{rs}} \frac{1}{\bar{y}_i^{\ell}}| \le k\}, \tag{6.11}$$

$$\bar{\mathsf{U}}_{r,\ell,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \{\sum_{y_i \in S_{r\infty}} \frac{1}{|y_i|^{\ell} - b_r^{\ell}}\} \le k\}. \tag{6.12}$$

(A.6) For each $r \in \mathbb{N}$, there exists an $\ell_0 \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mu^N(\bar{\mathsf{U}}_{r,\ell_0,k}^c) = 0, \tag{6.13}$$

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mu^N(\mathsf{U}_{r,\ell,k}^c) = 0 \qquad \text{for all } 1 \le \ell < \ell_0. \tag{6.14}$$

When $\ell_0 = 1$, according to our interpretation, (6.14) always holds by convention.

We now state the main theorem of this section.

Theorem 6.2. *Assume* (6.1) *and* (6.2)*. Suppose* (6.3) *or* (6.4)*. Then* (A.6) *implies* (A.5).

Proof. Let c_{13} and c_{14} be the constants defined by

$$\begin{aligned} c_{13} &= |\beta| \cdot \sup_{N\in\mathbb{N}} \max_{1\le \ell < \ell_0} \sup_{x \ne w \in S_r} \frac{|\varpi_N(x)^{\ell} - \varpi_N(w)^{\ell}|}{\ell |x-w|}, \\ c_{14} &= |\beta| \cdot \sup_{N\in\mathbb{N}} \sup_{\ell_0 \le \ell} \sup_{x \ne w \in S_r} \frac{|\varpi_N(x)^{\ell} - \varpi_N(w)^{\ell}|}{|\varpi_N(b_r)|^{\ell} \ell |x-w|}. \end{aligned} \tag{6.15}$$

Then c_{13} and c_{14} are finite. Indeed, $c_{13} < \infty$ is clear. Note that the Lipschitz norm of $\{\varpi_N\}$ on $\mathbb{R}$ is uniformly bounded in $N \in \mathbb{N}$. Moreover, $|\varpi_N(x)|/|\varpi_N(b_r)| < 1$ on S_r. Hence the Lipschitz norm of the function $\varpi_N(x)^\ell/\varpi_N(b_r)^\ell \ell$ on S_r is uniformly bounded in ℓ, $N \in \mathbb{N}$. This implies $c_{14} < \infty$.

By (6.8) and $c_{13}, c_{14} < \infty$ we have

$$\frac{|\Psi_{rs}^N(x,\mathsf{y}) - \Psi_{rs}^N(w,\mathsf{y})|}{|x-w|} \le c_{13} \sum_{\ell=1}^{\ell_0-1} \Big| \sum_{y_i \in S_{rs}} \frac{1}{\bar{\varpi}_N(y_i)^\ell} \Big| + c_{14} \sum_{\ell=\ell_0}^{\infty} \sum_{y_i \in S_{rs}} \frac{|\varpi_N(b_r)|^\ell}{|\varpi_N(y_i)|^\ell}$$
$$= c_{13} \sum_{\ell=1}^{\ell_0-1} \Big| \sum_{y_i \in S_{rs}} \frac{1}{\bar{\varpi}_N(y_i)^\ell} \Big| + c_{14} \sum_{y_i \in S_{rs}} \frac{|\varpi_N(b_r)|^{\ell_0}}{|\varpi_N(y_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0}}. \tag{6.16}$$

Here we used the formula $\sum_{\ell=\ell_0}^\infty a^\ell/b^\ell = a^{\ell_0}/(b^{\ell_0} - a^{\ell_0})$ valid for $0 < a \le b$. If $a = b$, then we interpret $\sum_{\ell=\ell_0}^\infty a^\ell/b^\ell = \infty$. Set $c_{15} = c_{14} \sup_{N\in\mathbb{N}} |\varpi_N(b_r)|^{\ell_0}$. By (6.16), we see that

$$\sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} \sup_{x\neq w \in S_r} \frac{|\Psi_{rs}^N(x,\mathsf{y}) - \Psi_{rs}^N(w,\mathsf{y})|}{|x-w|}$$
$$\le c_{13} \sum_{\ell=1}^{\ell_0-1} \{\sup_{N\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} \Big| \sum_{y_i\in S_{rs}} \frac{1}{\bar{\varpi}_N(y_i)^\ell} \Big| \} + c_{15} \{\sup_{N\in\mathbb{N}} \sum_{y_i \in S_{r\infty}} \frac{1}{|\varpi_N(y_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0}} \}.$$

Combining this with (4.6), (6.9), and (6.10), we deduce that

$$\mathsf{H}_{r,k} \supset \Big\{ \bigcap_{\ell=1}^{\ell_0-1} \mathsf{U}_{r,\ell,k/(\ell_0 c_{13})} \Big\} \bigcap \bar{\mathsf{U}}_{r,\ell_0,k/(\ell_0 c_{15})}.$$

Hence, we obtain

$$\mu^N(\mathsf{H}_{r,k}^c) \le \Big\{ \sum_{\ell=1}^{\ell_0-1} \mu^N(\mathsf{U}_{r,\ell,k/(\ell_0 c_{13})}^c) \Big\} + \mu^N(\bar{\mathsf{U}}_{r,\ell_0,k/(\ell_0 c_{15})}^c). \tag{6.17}$$

This together with (A.6) implies (4.7), which completes the proof. □

7 Sufficient conditions of (A.6)

In this section, we give sufficient conditions of (A.6). These conditions are used in the proof of Theorem 2.2 and Theorem 2.3. We begin with (6.13), the first condition of (A.6).

Lemma 7.1. *Assume* (A.4)*,* (6.1)*, and* (6.2)*. Assume* (6.3) *or* (6.4)*. Then* (6.13) *follows from* (7.1) *below.*

$$\sup_{N\in\mathbb{N}} \{ \int_{1\le|x|<\infty} \{ \sup_{M\in\mathbb{N}} \frac{1}{|\varpi_M(x)|^{\ell_0}} \} \rho_N^1(x) dx \} < \infty. \tag{7.1}$$

In particular, if (6.3) *is satisfied, then* (6.13) *follows from a simpler condition* (7.2)*:*

$$\sup_{N\in\mathbb{N}} \{ \int_{1\le|x|<\infty} \frac{1}{|x|^{\ell_0}} \rho_N^1(x) dx \} < \infty. \tag{7.2}$$

Proof. Let b_r be as in (2.6). We divide the set $S_{r\infty} = \{b_r \le |x| < \infty\}$ in (6.10) into two parts $S_{r(r+1)} = \{b_r \le |x| < b_{r+1}\}$ and $S_{(r+1)\infty} = \{b_{r+1} \le |x| < \infty\}$. Let

$$\mathsf{V}_{1,k} = \{\mathsf{x} \in \mathsf{S}\,;\, \{\sup_{N\in\mathbb{N}} \sum_{x_i \in S_{r(r+1)}} \frac{1}{|\varpi_N(x_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0}} \} \le \frac{k}{2} \}$$

$$\mathsf{V}_{2,k} = \{\mathsf{x} \in \mathsf{S}\,;\, \{\sup_{N\in\mathbb{N}} \sum_{x_i \in S_{(r+1)\infty}} \frac{1}{|\varpi_N(x_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0}} \} \le \frac{k}{2} \}, \quad \text{where } \mathsf{x} = \sum_i \delta_{x_i}.$$

Then clearly $\bar{\mathsf{U}}_{r,\ell_0,k} \supset \mathsf{V}_{1,k} \cap \mathsf{V}_{2,k}$. To estimate $\mathsf{V}_{1,k}$, we observe that for $\mathsf{x} = \sum_i \delta_{x_i}$

$$\sum_{x_i \in S_{r(r+1)}} \frac{1}{|\varpi_N(x_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0}} \le \{ \sup_{x_i \in S_{r(r+1)}} \frac{1}{|\varpi_N(x_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0}} \} \cdot \mathsf{x}(S_{r(r+1)}).$$

Here $\mathsf{x}(S_{r(r+1)})$ is the number of points x_i in $S_{r(r+1)}$. Taking this into account, we set

$$\mathsf{V}_{3,k} = \{ \mathsf{x} \in \mathsf{S}\, ; \sup_{N \in \mathbb{N}} \sup_{x_i \in S_{r(r+1)}} \frac{1}{|\varpi_N(x_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0}} \le \sqrt{k/2} \},$$
$$\mathsf{V}_{4,k} = \{ \mathsf{x} \in \mathsf{S};\, \mathsf{x}(S_{r(r+1)}) \le \sqrt{k/2} \}.$$

Then we have $\mathsf{V}_{1,k} \supset \mathsf{V}_{3,k} \bigcap \mathsf{V}_{4,k}$. We therefore obtain $\bar{\mathsf{U}}_{r,\ell_0,k} \supset \mathsf{V}_{2,k} \cap \mathsf{V}_{3,k} \cap \mathsf{V}_{4,k}$ by combining these two inclusions. Hence we deduce (6.13) from

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mu^N(\mathsf{V}_{l,k}^c) = 0 \quad \text{for all } l = 2,3,4. \tag{7.3}$$

We will check (7.3) for each $l = 2, 3, 4$.

As for (7.3) with $l = 2$, according to the Chebyshev inequality, we have

$$\begin{aligned} \mu^N(\mathsf{V}_{2,k}^c) &\le \frac{2}{k} E^{\mu^N}[\sup_{M\in\mathbb{N}} \sum_{x_i \in S_{(r+1)\infty}} \frac{1}{|\varpi_M(x_i)|^{\ell_0} - |\varpi_M(b_r)|^{\ell_0}}] \\ &= \frac{2}{k} \int_{S_{(r+1)\infty}} \sup_{M\in\mathbb{N}} \{ \frac{1}{|\varpi_M(x)|^{\ell_0} - |\varpi_M(b_r)|^{\ell_0}} \} \rho_N^1(x)dx \\ &= \frac{2}{k} \int_{S_{(r+1)\infty}} \sup_{M\in\mathbb{N}} \{ \frac{|\varpi_M(x)|^{\ell_0}}{|\varpi_M(x)|^{\ell_0} - |\varpi_M(b_r)|^{\ell_0}} \frac{1}{|\varpi_M(x)|^{\ell_0}} \} \rho_N^1(x)dx \\ &\le \frac{2}{k} \sup_{M\in\mathbb{N}} \{ \frac{|\varpi_M(b_{r+1})|^{\ell_0}}{|\varpi_M(b_{r+1})|^{\ell_0} - |\varpi_M(b_r)|^{\ell_0}} \} \cdot \int_{S_{(r+1)\infty}} \sup_{M\in\mathbb{N}} \{ \frac{1}{|\varpi_M(x)|^{\ell_0}} \} \rho_N^1(x)dx. \end{aligned} \tag{7.4}$$

Here we used the fact that $|\varpi_M(x)| < |\varpi_M(y)|$ for $|x| < |y|$, which implies

$$\sup_{x\in S_{(r+1)\infty}} \frac{|\varpi_M(x)|^{\ell_0}}{|\varpi_M(x)|^{\ell_0} - |\varpi_M(b_r)|^{\ell_0}} \le \frac{|\varpi_M(b_{r+1})|^{\ell_0}}{|\varpi_M(b_{r+1})|^{\ell_0} - |\varpi_M(b_r)|^{\ell_0}}$$

By (7.1) and (7.4), we obtain (7.3) with $l = 2$.

We next consider (7.3) with $l = 3$. Let

$$U_k = \bigcup_{N\in\mathbb{N}} \{ x \in S_{r(r+1)};\, |\varpi_N(b_r)|^{\ell_0} \le |x|^{\ell_0} < |\varpi_N(b_r)|^{\ell_0} + \sqrt{2/k} \}.$$

It is not difficult to see that U_k is nonincreasing and $\lim_{k\to\infty} U_k = \emptyset$. We note that

$$\begin{aligned} \mathsf{V}_{3,k}^c &= \{ \mathsf{x} \in \mathsf{S};\, \inf_{N\in\mathbb{N}} \inf_{x_i \in S_{r(r+1)}} \{ |\varpi_N(x_i)|^{\ell_0} - |\varpi_N(b_r)|^{\ell_0} \} < \sqrt{2/k} \} \\ &= \{ \mathsf{x} \in \mathsf{S};\ 1 \le \mathsf{x}(U_k) \}. \end{aligned} \tag{7.5}$$

Here we use a convention such that $\inf \emptyset = \infty$; that is, we interpret $\mathsf{x} \notin \mathsf{V}_{3,k}^c$ when $\mathsf{x}(S_{r(r+1)}) = 0$. Let $c_{16} = \sup\{ \rho_N^1(x); N \in \mathbb{N}, x \in S_{r(r+1)} \}$. Then by (4.2), we have $c_{16} < \infty$. From the second equality in (7.5) and the Chebyshev inequality we obtain

$$\mu^N(\mathsf{V}_{3,k}^c) \le E^{\mu^N}[\mathsf{x}(U_k)] = \int_{U_k} \rho_N^1(x)dx \le c_{16} \int_{U_k} dx. \tag{7.6}$$

Hence, we deduce (7.3) with $l = 3$ from (7.6) and $\lim_{k\to\infty} U_k = \emptyset$.

We finally consider (7.3) with $l = 4$. From the Chebyshev inequality we obtain

$$\mu^N(\mathsf{V}_{4,k}^c) \le \sqrt{\frac{2}{k}} E^{\mu^N}[\mathsf{x}(S_{r(r+1)})] = \sqrt{\frac{2}{k}} \int_{S_{r(r+1)}} \rho_N^1(x)dx \le \sqrt{\frac{2}{k}} c_{16} \int_{S_{r(r+1)}} dx.$$

This deduces (7.3) with $l = 4$ immediately. □

We proceed with (6.14), the second condition of (A.6).

Let $\tilde{S}_r = \{s \in S;\ |s| < r\}$ and $\tilde{S}_{rs} = \tilde{S}_s \backslash \tilde{S}_r$. Let $\mathsf{v}_{\ell,rs}^N : \mathsf{S} \to \mathbb{C}$ such that

$$\mathsf{v}_{\ell,rs}^N(\mathsf{x}) = \sum_{x_i \in \tilde{S}_{rs}} \frac{1}{\bar{\varpi}_N(x_i)^\ell} \quad \text{for } 1 \le r < s \le \infty. \tag{7.7}$$

Here we write $\mathsf{x} = \sum_i \delta_{x_i}$ as usual. Note that the sum in (7.7) makes sense for μ^N-a.s. x even if $s = \infty$. Indeed, by (2) of (A.4), the total number of particles has the deterministic bound n_N under μ^N. Hence, $\mathsf{v}_{\ell,rs}^N(\mathsf{x})$ is well defined and finite for μ^N-a.s. x for all $N \in \mathbb{N}$.

Lemma 7.2. *Under the same assumptions as Lemma 7.1,* (6.14) *follows from* (7.8) *below.*

$$\lim_{r\to\infty} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{v}_{\ell,r\infty}^M| \|_{L^1(\mathsf{S},\mu^N)} = 0 \quad \textit{for all } 1 \le \ell < \ell_0. \tag{7.8}$$

Proof. By (7.8), we can and do choose $\{b_r\}$ and $c_{17} > 0$ in such a way that

$$\sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{v}_{\ell,b_r\infty}^M| \|_{L^1(\mathsf{S},\mu^N)} \le c_{17} 3^{-r} \quad \text{for all } r \in \mathbb{N}. \tag{7.9}$$

We note that $\sum_{y_i \in S_{rs}} 1/\bar{\varpi}_N(y_i)^\ell = \mathsf{v}_{\ell,b_r\infty}^N(\mathsf{x}) - \mathsf{v}_{\ell,b_s\infty}^N(\mathsf{x})$. Then by (6.9), we see that

$$\begin{aligned} \mu^N(\{\mathsf{U}_{r,\ell,k}\}^c) =& \mu^N(\sup_{M\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} |\mathsf{v}_{\ell,b_r\infty}^M - \mathsf{v}_{\ell,b_s\infty}^M| > k) \\ \le& \mu^N(\sup_{M\in\mathbb{N}} |\mathsf{v}_{\ell,b_r\infty}^M| > k/2) + \mu^N(\sup_{M\in\mathbb{N}} \sup_{r<s\in\mathbb{N}} |\mathsf{v}_{\ell,b_s\infty}^M| > k/2) \\ \le& \mu^N(\sup_{M\in\mathbb{N}} |\mathsf{v}_{\ell,b_r\infty}^M| > k/2) + \sum_{s=r+1}^{\infty} \mu^N(\sup_{M\in\mathbb{N}} |\mathsf{v}_{\ell,b_s\infty}^M| > k/2) \\ \le& \frac{2}{k} \cdot \{\sum_{s=r}^{\infty} \| \sup_{M\in\mathbb{N}} |\mathsf{v}_{\ell,b_s\infty}^M| \|_{L^1(\mathsf{S},\mu^N)}\}. \end{aligned} \tag{7.10}$$

Here we used Chebyshev's inequality in the last line. By (7.9) and (7.10) we have

$$\sup_{N\in\mathbb{N}} \mu^N(\{\mathsf{U}_{r,\ell,k}\}^c) \le \frac{2}{k} \cdot \frac{c_{17}3^{-r}}{1-3^{-1}}.$$

Hence, $\lim_{k\to\infty} \sup_{N\in\mathbb{N}} \mu^N(\{\mathsf{U}_{r,\ell,k}\}^c) = 0$, which implies (6.14). □

We refine Lemma 7.2 in Lemma 7.3, used in the proof of Theorems 2.2 and 2.3 directly.

Lemma 7.3. *Let $u_{\ell,r}^N : S \to \mathbb{C}$ such that $u_{\ell,r}^N(x) = 1_{\tilde{S}_{1r}}(x) \lceil |\varpi_N(x)| \rceil^\ell / \bar{\varpi}_N(x)^\ell$. Here $\lceil\cdot\rceil$ is the minimal integer greater than or equal to $\cdot$. Let $\mathsf{u}_{\ell,r}^N : \mathsf{S} \to \mathbb{C}$ such that $\mathsf{u}_{\ell,r}^N(\mathsf{x}) = \sum_i u_{\ell,r}^N(x_i)$, where $\mathsf{x} = \sum_i \delta_{x_i}$. Suppose there exists a positive constant c_{18} such that*

$$\sup_{r\in\mathbb{N}} r^{c_{18}-\ell} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{u}_{\ell,r}^M| \|_{L^1(\mathsf{S},\mu^N)} < \infty \quad \textit{for all } 1 \le \ell < \ell_0. \tag{7.11}$$

In addition, assume the same conditions as for Lemma 7.1. Then (6.14) *holds.*

Proof. Let $1 \le \ell < \ell_0$ be fixed. Define $\mathsf{w}_r^j : \mathsf{S} \to \mathbb{C}$ by $\mathsf{w}_r^j(\mathsf{x}) = \sum_{x_i \in \tilde{S}_{1r}} \lceil |\varpi_M(x_i)| \rceil^j / \bar{\varpi}_M(x_i)^\ell$. Although w_r^j depends on $M \in \mathbb{N}$, we omit M from the notation for simplicity. Let $\mathsf{v}_{\ell,1r}^M$ be as in (7.7). Then $\mathsf{w}_r^0 = \mathsf{v}_{\ell,1r}^M$ and $\mathsf{w}_r^\ell = \mathsf{u}_{\ell,r}^M$ by definition. Moreover, we easily deduce that

$$\mathsf{w}_r^j = \sum_{q=2}^r q(\mathsf{w}_q^{j-1} - \mathsf{w}_{q-1}^{j-1}) \quad \text{for } r \ge 2, \qquad \mathsf{w}_1^j = 0.$$

Hence, through a straightforward calculation, we have

$$\mathsf{w}_r^{j-1} = \frac{\mathsf{w}_r^j}{r} + \sum_{q=2}^{r-1} \frac{\mathsf{w}_q^j}{q(q+1)} \quad \text{for } r \ge 3, \qquad \mathsf{w}_2^{j-1} = \frac{1}{2}\mathsf{w}_2^j. \tag{7.12}$$

By (2) of (A.4), we see that $\lim_{r\to\infty} r^{-1} \|\mathsf{w}_r^j\|_{L^1(\mathsf{S},\mu^N)} = 0$ and that $\mathsf{w}_\infty^j := \lim_{r\to\infty} \mathsf{w}_r^j$ exists in $L^1(\mathsf{S},\mu^N)$. Hence, by taking $r \to \infty$ in (7.12), we obtain

$$\mathsf{w}_\infty^{j-1} = \sum_{q=2}^\infty \frac{\mathsf{w}_q^j}{q(q+1)} \quad \text{in } L^1(\mathsf{S},\mu^N).$$

Subtracting (7.12) from this yields

$$\mathsf{w}_\infty^{j-1} - \mathsf{w}_r^{j-1} = -\frac{\mathsf{w}_r^j}{r} + \sum_{q=r}^\infty \frac{\mathsf{w}_q^j}{q(q+1)} \quad \text{in } L^1(\mathsf{S},\mu^N). \tag{7.13}$$

Take the supremum of the modulus of each terms of (7.12) and (7.13) w.r.t. $M \in \mathbb{N}$. Apply Minkowski's inequality to the right-hand sides of (7.12) and (7.13). Then by taking the supremum w.r.t. $N \in \mathbb{N}$, we obtain

$$\sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_r^{j-1}| \|_{L^1(\mathsf{S},\mu^N)} \le \frac{1}{r} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_r^j| \|_{L^1(\mathsf{S},\mu^N)} + \sum_{q=2}^{r-1} \frac{1}{q(q+1)} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_q^j| \|_{L^1(\mathsf{S},\mu^N)}, \tag{7.14}$$

$$\sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_\infty^{j-1} - \mathsf{w}_r^{j-1}| \|_{L^1(\mathsf{S},\mu^N)} \le \frac{1}{r} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_r^j| \|_{L^1(\mathsf{S},\mu^N)} + \sum_{q=r}^{\infty} \frac{1}{q(q+1)} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_q^j| \|_{L^1(\mathsf{S},\mu^N)}. \tag{7.15}$$

For each $j = 1, \dots, \ell$, there exists a positive constant $c_{19} = c_{19}(j)$ such that

$$\sup_{r\in\mathbb{N}} r^{c_{19}-j} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_r^j| \, \|_{L^1(\mathsf{S},\mu^N)} < \infty. \tag{7.16}$$

Indeed, when $j = \ell$, (7.16) holds by (7.11) because $\mathsf{w}_r^\ell = \mathsf{u}_{\ell,r}^M$. Suppose (7.16) holds for some $2 \le j \le \ell$ with a positive constant $c_{19}(j)$. Then by (7.14), we have (7.16) for $j-1$ with a positive constant $c_{19}(j-1)$. Therefore, through induction, (7.16) holds for all $j = 1, \dots, \ell$.

Combining (7.15) and (7.16), we easily deduce that for each $j = 1, \dots, \ell$,

$$\lim_{r\to\infty} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{w}_\infty^{j-1} - \mathsf{w}_r^{j-1}| \|_{L^1(\mathsf{S},\mu^N)} = 0. \tag{7.17}$$

Recalling $\mathsf{w}_r^0 = \mathsf{v}_{\ell,1r}^M$, we have $\mathsf{w}_\infty^0 - \mathsf{w}_r^0 = \mathsf{v}_{\ell,r\infty}^M$. Hence, by taking $j = 1$ in (7.17), we obtain

$$\lim_{r\to\infty} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{v}_{\ell,r\infty}^M| \|_{L^1(\mathsf{S},\mu^N)} = 0.$$

This allows us to deduce (7.8) in Lemma 7.2. We therefore obtain (6.14) by Lemma 7.2. $\square$

8 Translation invariant periodic measures.

In this section, we make preparations for a proof of Theorem 2.2.

Let $S = \mathbb{R}^d$. Let $\tau_x : \mathsf{S} \to \mathsf{S}$ be the translation defined by $\tau_x(\mathsf{s}) = \sum_i \delta_{x+s_i}$ for $\mathsf{s} = \sum_i \delta_{s_i}$. We say that a measure ν on S is translation invariant if $\nu \circ \tau_x^{-1} = \nu$ for all $x \in \mathbb{R}^d$. We say that ν is L-periodic if $\nu(\tau_{L\mathbf{e}_i}(\mathsf{s}) = \mathsf{s}) = 1$ for all $i = 1, \dots, d$. Moreover, we say that ν is concentrated on A if $\nu(\mathsf{s}(A^c) > 0) = 0$. A measure ν concentrated on $(-L/2, L/2]^d$ can be extended naturally to the L-periodic measure $\bar{\nu}$ on the configuration space on $\mathbb{R}^d$. We refer to this measure $\bar{\nu}$ as the L-periodic extension of ν.

Let $\mathbb{T}_N = (-n_N/2, n_N/2]^d$. We assume that ν is concentrated on $\mathbb{T}_N$ and that ν has a periodic extension that is translation invariant. Let ρ_N^n be the n-correlation function of ν. Then $\rho_N^n(x) = 0$ for $x \notin (\mathbb{T}_N)^{n_N}$ by assumption. Let $\mathcal{T}_N$ be the two-level cluster function of ν:

$$\mathcal{T}_N(x, y) = \rho_N^1(x)\rho_N^1(y) - \rho_N^2(x, y). \tag{8.1}$$

Then $\mathcal{T}_N(x, y) = 0$ if $(x, y) \notin (\mathbb{T}_N)^2$. If $(x, y) \in (\mathbb{T}_N)^2$, $\mathcal{T}_N(x, y)$ depends only on $x - y$ modulo $N\mathbf{e}_i$ $(i = 1, \dots, d)$, where $\mathbf{e}_i$ is the ith unit vector. Therefore, let $\mathcal{T}_N : \mathbb{R}^d \to \mathbb{R}$ be the n_N-periodic function such that $\mathcal{T}_N(x) = \mathcal{T}_N(x, 0)$ for $x \in \mathbb{T}_N$. We set

$$\mathfrak{m}_N(\xi) = \rho_N^1(0) - \mathcal{F}_N(\mathcal{T}_N)(\xi). \tag{8.2}$$

Here $\mathcal{F}_N(f)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi\sqrt{-1}\xi\cdot x} f 1_{\mathbb{T}_N}(x)dx$ denotes the Fourier transform of $f 1_{\mathbb{T}_N}$.

Lemma 8.1. *Assume that ν is concentrated on $\mathbb{T}_N$ and that ν has a periodic extension that is translation invariant. Let $h : \mathbb{R}^d \to \mathbb{R}$ be real valued. Set $\mathsf{h}_N(\mathsf{s}) = \sum_{s_i \in \mathbb{T}_N} h(s_i)$, where $\mathsf{s} = \sum_i \delta_{s_i}$. Then*

$$\|\mathsf{h}_N\|^2_{L^2(\mathsf{S},\nu)} = (\rho_N^1(0) \int_{\mathbb{T}_N} h(x)dx)^2 + \frac{1}{(n_N)^d} \sum_{\xi \in \mathbb{T}_N \cap (\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)|^2(\xi)\mathfrak{m}_N(\xi). \tag{8.3}$$

Proof. From $\rho_N^1(x) = \rho_N^1(0)1_{\mathbb{T}_N}(x)$, we see that

$$\int_{\mathsf{S}} \mathsf{h}_N d\nu = \int_{\mathbb{T}_N} h(x)\rho_N^1(x)dx = \rho_N^1(0) \int_{\mathbb{T}_N} h(x)dx. \tag{8.4}$$

Let $\mathrm{Var}^\nu[\mathsf{h}_N]$ be the variance of h_N w.r.t. ν. By (8.1) and the general property of correlation functions, we see that

$$\begin{aligned} \mathrm{Var}^\nu[\mathsf{h}_N] &= \int_{\mathbb{R}^d} h^2(x)\rho_N^1(x)dx - \int_{\mathbb{R}^d \times \mathbb{R}^d} h(x)h(y)\mathcal{T}_N(x, y)dxdy \\ &= \rho_N^1(0) \int_{\mathbb{T}_N} h^2(x)dx - \int_{\mathbb{T}_N \times \mathbb{T}_N} h(x)h(y)\mathcal{T}_N(x - y)dxdy. \end{aligned}$$

We used $\rho_N^1(x) = \rho_N^1(0)1_{\mathbb{T}_N}(x)$ and $\mathcal{T}_N(x, y) = 1_{\mathbb{T}_N}(x)1_{\mathbb{T}_N}(y)\mathcal{T}_N(x - y)$ in the second line. By a direct calculation of the Fourier series, we see that

$$\int_{\mathbb{T}_N} h^2(x)dx = \frac{1}{(n_N)^d} \sum_{\xi \in \mathbb{T}_N \cap (\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)(\xi)|^2$$

and

$$\int_{\mathbb{T}_N\times\mathbb{T}_N} h(x)h(y)\mathcal{T}_N(x-y)dxdy = \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} \mathcal{F}_N(h)(\xi)\overline{\mathcal{F}_N(h*\mathcal{T}_N)(\xi)}$$
$$= \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)(\xi)|^2\overline{\mathcal{F}_N(\mathcal{T}_N)(\xi)}$$
$$= \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)(\xi)|^2\mathcal{F}_N(\mathcal{T}_N)(\xi).$$

Here we used the fact that $\mathcal{F}_N(\mathcal{T}_N)$ is real valued because $\mathcal{T}_N(x)=\mathcal{T}_N(-x)$. Combining these with (8.2) yields

$$\mathrm{Var}^\nu[\mathsf{h}_N] = \frac{1}{(n_N)^d}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}^d/n_N)} |\mathcal{F}_N(h)|^2(\xi)\mathfrak{m}_N(\xi). \tag{8.5}$$

(8.3) follows from (8.4) and (8.5) immediately. □

9 Proof of Theorems 2.2.

In this section, we prove Theorem 2.2 using the previous results. We begin by defining $\mathsf{K}_{\mathrm{dys},\beta}$ for $\beta=1,4$. Let $\mathrm{i}=\sqrt{-1}$ as before. To define $\mathsf{K}_{\mathrm{dys},\beta}$, we recall the standard quaternion notation for 2×2 matrices (see [12, Ch. 2.4]):

$$\mathbf{1}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \mathbf{e}_1=\begin{bmatrix}\mathrm{i}&0\\0&-\mathrm{i}\end{bmatrix},\quad \mathbf{e}_2=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\quad \mathbf{e}_3=\begin{bmatrix}0&\mathrm{i}\\\mathrm{i}&0\end{bmatrix}.$$

A quaternion q is represented as $q=q^{(0)}\mathbf{1}+q^{(1)}\mathbf{e}_1+q^{(2)}\mathbf{e}_2+q^{(3)}\mathbf{e}_3$. Here the $q^{(i)}$ are complex numbers. There is identification between the 2×2 complex matrices and the quaternions given by

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}=\frac{1}{2}(a+d)\mathbf{1}-\frac{\mathrm{i}}{2}(a-d)\mathbf{e}_1+\frac{1}{2}(b-c)\mathbf{e}_2-\frac{\mathrm{i}}{2}(b+c)\mathbf{e}_3 \tag{9.1}$$

or equivalently

$$\begin{bmatrix}q^{(0)}+\mathrm{i}q^{(1)}&q^{(2)}+\mathrm{i}q^{(3)}\\-q^{(2)}+\mathrm{i}q^{(3)}&q^{(0)}-\mathrm{i}q^{(1)}\end{bmatrix}=q^{(0)}\mathbf{1}+q^{(1)}\mathbf{e}_1+q^{(2)}\mathbf{e}_2+q^{(3)}\mathbf{e}_3. \tag{9.2}$$

We denote by $\Theta(q^{(0)}\mathbf{1}+q^{(1)}\mathbf{e}_1+q^{(2)}\mathbf{e}_2+q^{(3)}\mathbf{e}_3)$ the 2×2 complex matrix defined by the left-hand side of (9.2). By definition, $\Theta^{-1}(\begin{bmatrix}a&b\\c&d\end{bmatrix})$ is the quaternion on the right-hand side of (9.1). We also remark that these relations can be naturally extended to the ones between $(2N)\times(2N)$ complex matrices and $N\times N$ quaternion matrices.

For a quaternion $q=q^{(0)}\mathbf{1}+q^{(1)}\mathbf{e}_1+q^{(2)}\mathbf{e}_2+q^{(3)}\mathbf{e}_3$, we call $q^{(0)}$ the scalar part of q. A quaternion is called scalar if $q^{(i)}=0$ for $i=1,2,3$. We often identify a scalar quaternion $q=q^{(0)}\mathbf{1}$ with the complex number $q^{(0)}$ by the obvious correspondence.

Let $\bar{q}=q^{(0)}\mathbf{1}-\{q^{(1)}\mathbf{e}_1+q^{(2)}\mathbf{e}_2+q^{(3)}\mathbf{e}_3\}$. A quaternion matrix $A=[a_{ij}]$ is called self-dual if $a_{ij}=\bar{a}_{ji}$ for all i,j. For a self-dual $n\times n$ quaternion matrix $A=[a_{ij}]$, we set

$$\det A=\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sign}[\sigma]\prod_{i=1}^{L(\sigma)}[a_{\sigma_i(1)\sigma_i(2)}a_{\sigma_i(2)\sigma_i(3)}\cdots a_{\sigma_i(\ell-1)\sigma_i(\ell)}a_{\sigma_i(\ell)\sigma_i(1)}]^{(0)}. \tag{9.3}$$

Here $\sigma = \sigma_1 \cdots \sigma_{L(\sigma)}$ is a decomposition of σ to products of the cyclic permutations $\{\sigma_i\}$ with disjoint indices. We write $\sigma_i = (\sigma_i(1), \sigma_i(2), \dots, \sigma_i(\ell))$, where ℓ is the length of the cyclic permutation σ_i. The decomposition is unique up to the order of $\{\sigma_i\}$. As before, $[\cdot]^{(0)}$ means the scalar part of the quaternion $\cdot$. It is known that the right-hand side is well defined. See Section 5.1 in [12] for details.

For a self-dual $N \times N$ quaternion matrix $A = [a_{ij}]$, it holds that [12, (5.1.15)]

$$\det \Theta(A) = (\det A)^2. \tag{9.4}$$

Here $\Theta(A)$ is the $(2N) \times (2N)$ complex matrix given by the relation (9.1). We note that the determinant on the left-hand side of (9.4) is of the $(2N) \times (2N)$ matrix with complex elements, while that on the left-hand side of (9.4) is of the $N \times N$ matrix with quaternion elements.

We are now ready to introduce $\mathsf{K}_{\mathrm{dys},\beta}$. Let $S(x) = \sin(\pi x)/\pi x$, $D(x) = \frac{dS}{dx}(x)$ and $I(x) = \int_0^x S(y)dx$. Let $\varepsilon(t) = -1/2$ $(t < 0)$, $\varepsilon(t) = 0$ $(t = 0)$, and $\varepsilon(t) = 1/2$ $(t < 0)$.

$$\mathsf{K}_{\mathrm{dys},1}(x,y) = \Theta^{-1}(\begin{bmatrix} S(x-y) & D(x-y) \\ I(x-y) - \varepsilon(x-y) & S(x-y) \end{bmatrix}) \tag{9.5}$$

$$\mathsf{K}_{\mathrm{dys},2}(x,y) = S(x-y) \tag{9.6}$$

$$\mathsf{K}_{\mathrm{dys},4}(x,y) = \Theta^{-1}(\begin{bmatrix} S(2(x-y)) & D(2(x-y)) \\ I(2(x-y)) & S(2(x-y)) \end{bmatrix}). \tag{9.7}$$

We thus clarify the meaning of (2.15). It is known that the matrices $[\mathsf{K}_{\mathrm{dys},\beta}(x_i, x_j)]_{1 \le i,j \le n}$ are self-dual $(\beta = 1, 4)$, and that there exist unique random point fields $\mu_{\mathrm{dys},\beta}$ $(\beta = 1, 2, 4)$ whose correlation functions $\{\rho^n\}$ are given by (2.15) (see [12, Chs. 5-8]).

Lemma 9.1. $\mu_{\mathrm{dys},\beta}$ $(\beta = 1, 2, 4)$ *satisfy* (A.1).

Proof. Since the correlation functions $\{\rho^n\}$ have the expression (2.15) and the kernels $\mathsf{K}_{\mathrm{dys},\beta}$ are bounded, we see that $\{\rho^n\}$ satisfy (A.1). □

To prove the quasi-Gibbs property of $\mu_{\mathrm{dys},\beta}$, it is sufficient to check (A.4) and (A.5) by Theorem 4.1. Therefore, the problem is to construct a finite-particle approximation $\{\mu^N\}$ fulfilling the assumptions in (A.4) and (A.5). We will take $\{\mu^N\}$, whose potentials satisfy (6.2) and (6.4) for $\beta = 1, 2, 4$. Hence, we assume $n_N = 2^{4N}$ and $\mathbb{I}_N = (-N, N)$ as in (6.4). We take $\Phi(x) = 0$ and $\Phi^N(x) = -\log 1_{\mathbb{I}_N}(x)$. Ψ and Ψ^N are the same as in (6.1), (6.2), and (6.4).

To introduce the finite-particle approximation $\{\mu^N\}$ we first recall some facts about circular ensembles $\{\nu^N\}$. Let $\check{\nu}^N$ denote the probability measure on $\mathbb{R}^{n_N}$ defined by

$$d\check{\nu}^N = \frac{1}{Z} \prod_{i=1}^{n_N} 1_{\mathbb{T}_N}(x_i) \prod_{i,j=1, i<j}^{n_N} |e^{2\pi \mathrm{i} x_i/n_N} - e^{2\pi \mathrm{i} x_j/n_N}|^\beta \, dx_1 \cdots dx_{n_N}, \tag{9.8}$$

where Z is the normalization and $\mathbb{T}_N = (-n_N/2, n_N/2]$. It is well known [12], [2] that the distribution of $(e^{2\pi \mathrm{i} x_i/n_N})_{1 \le i \le n_N}$ under $\check{\nu}^N$ is equal to the distributions of the spectra of the circular orthogonal, unitary and symplectic ensembles for $\beta = 1$, 2 and 4, respectively.

Let ι be a map such that $\iota((x_i)) = \sum_i \delta_{x_i}$. Set $\nu^N = \check{\nu}^N \circ \iota^{-1}$ and let ϱ_N^n denote the n-correlation function of ν^N. Then by (9.8), we see that $\varrho_N^n = 0$ for $n > n_N$ and

$$\varrho_{n_N}^{n_N}(x_1, \dots, x_{n_N}) = \frac{n_N!}{Z} \prod_{i,j=1, i<j}^{n_N} 1_{\mathbb{T}_N}(x_i) |e^{2\pi \mathrm{i} x_i/n_N} - e^{2\pi \mathrm{i} x_j/n_N}|^\beta 1_{\mathbb{T}_N}(x_j). \tag{9.9}$$

For each $n \in \mathbb{N}$, the n-correlation function ϱ_N^n can be written as (see [12, (11.1.10)])

$$\varrho_N^n(x_1,\dots,x_n) = \det[1_{\mathbb{T}_N}(x_i)\mathsf{K}_{\mathrm{dys},\beta}^N(x_i-x_j)1_{\mathbb{T}_N}(x_j)]_{1\le i,j\le n}, \tag{9.10}$$

where $\mathsf{K}_{\mathrm{dys},\beta}^N$ is given by (9.5)–(9.7) with the replacement of $S(x)$, $D(x)$, and $I(x)$ by $S_N(x)$, $D_N(x)$, and $I_N(x)$, respectively. Here S_N is defined as

$$S_N(x) = \frac{1}{n_N}\frac{\sin(\pi x)}{\sin(\pi x/n_N)}. \tag{9.11}$$

Moreover, we set $D_N(x) = dS_N(x)/dx$ and $I_N(x) = \int_0^x S_N(y)dy$. One can easily deduce (9.10) and (9.11) from the results in [12, Ch. 11] combined with the scaling $\theta \mapsto 2\pi x/n_N$. Indeed, these follow from (11.1.5), (11.1.6), (11.3.16), (11.3.22), (11.3.23), (11.5.6), and (11.5.13) in [12]. [2]

We are now ready to introduce the finite-particle approximation $\{\mu^N\}$.

Lemma 9.2. *Let* $\mu^N = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$. *Then* $\mu_{\mathrm{dys},\beta}$ *satisfy* (A.4) *with* μ^N. *Here we take* $\Phi^N(x) = -\log 1_{\mathbb{I}_N}(x)$ *and* Ψ^N *is given by* (6.2) *and* (6.4).

Proof. Let ρ_N^n be the n-correlation function of μ^N. Then by $\mu^N = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$, we have $\rho_N^n(x_1,\dots,x_n) = \varrho_N^n(x_1,\dots,x_n)$ on $\mathbb{I}_N^n$. Hence, by (9.10), we see that ρ_N^n satisfy

$$\rho_N^n(x_1,\dots,x_n) = \det[1_{\mathbb{I}_N}(x_i)\mathsf{K}_{\mathrm{dys},\beta}^N(x_i-x_j)1_{\mathbb{I}_N}(x_j)]_{1\le i,j\le n}. \tag{9.12}$$

By (9.11) and (9.5)–(9.7), we deduce that $1_{\mathbb{I}_N}(x)\mathsf{K}_{\mathrm{dys},\beta}^N(x-y)1_{\mathbb{I}_N}(y)$ converge compact uniformly to $\mathsf{K}_{\mathrm{dys},\beta}^N(x-y)$. This combined with (9.12) yields (4.1).

Let $k_i^{N,n}(\mathbf{x}_n)$ be the norm of the ith row vector of $[1_{\mathbb{I}_N}(x_i)\mathsf{K}_{\mathrm{dys},2}^N(x_i-x_j)1_{\mathbb{I}_N}(x_j)]_{1\le i,j\le n}$, where $\mathbf{x}_n = (x_1,\dots,x_n)$. Then there exists a constant c_{20} such that $|k_i^{N,n}(\mathbf{x}_n)| \le c_{20}n^{1/2}$ because the kernels $\mathsf{K}_{\mathrm{dys},2}^N$ are uniformly bounded. Hence, we have

$$|\det[1_{\mathbb{I}_N}(x_i)\mathsf{K}_{\mathrm{dys},2}^N(x_i-x_j)1_{\mathbb{I}_N}(x_j)]_{1\le i,j\le n}| \le k_i^{N,n}(\mathbf{x}_n)^n \le c_{20}^n n^{n/2}. \tag{9.13}$$

This combined with (9.12) yields (4.2) with $\beta = 2$. We can prove (4.2) for $\beta = 1,4$ similarly using the identity (9.4) on the quaternion determinant. We thus obtain (1) of (A.4).

(2) of (A.4) is clear because $\mu^N = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$ and ν^N consists of n_N particles.

Let $\hat{\Phi}^N(x) = -\log 1_{\mathbb{T}_N}(x)$ and $\hat{\Psi}^N(x,y) = -\beta \log|e^{2\pi \mathrm{i}x/n_N} - e^{2\pi \mathrm{i}y/n_N}|$. Then by (9.8), we see that ν^N are $(\hat{\Phi}^N, \hat{\Psi}^N)$-canonical Gibbs measures. Clearly, $\hat{\Psi}^N(x,y) = \Psi^N(x,y)$ for $x,y \in \mathbb{I}_N$. Hence, μ^N are (Φ^N, Ψ^N)-canonical Gibbs measures because $\mu^N = \nu^N \circ \pi_{\mathbb{I}_N}^{-1}$.

(4) of (A.4) is obvious through construction. □

We next proceed with the proof of (A.5). For this, it is sufficient to prove (A.6) by Theorem 6.2. We note that (A.6) consists of two conditions: (6.13) and (6.14). We prove (6.13) in the next four lemmas.

Let $\mathbb{I}_N = (-N,N)$ and $n_N = 2^{4N}$ as before. By (6.4), we easily see the following.

$$\begin{aligned} \varpi_N(x) &= \frac{n_N}{2\pi}\sin\frac{2\pi x}{n_N} + \mathrm{i}\frac{n_N}{2\pi}(1-\cos\frac{2\pi x}{n_N}) \\ &= \frac{n_N}{\pi}\sin\frac{\pi x}{n_N}\cos\frac{\pi x}{n_N} + \mathrm{i}\frac{n_N}{\pi}\sin^2\frac{\pi x}{n_N} \qquad \text{for } x \in \mathbb{I}_N, \end{aligned} \tag{9.14}$$

$$|\varpi_N(x)| = \frac{n_N}{\pi}|\sin\frac{\pi x}{n_N}| \qquad \text{for } x \in \mathbb{I}_N. \tag{9.15}$$

[2] IS_{2N} in (11.1.6) of [12] should be I_{2N}.

Hence, by (9.14) and (9.15), we have

$$\frac{\varpi_N(x)}{|\varpi_N(x)|} = \frac{\sin\frac{\pi x}{n_N}\cos\frac{\pi x}{n_N}}{|\sin\frac{\pi x}{n_N}|} + \mathrm{i}|\sin\frac{\pi x}{n_N}| \qquad \text{for } x \in \mathbb{I}_N. \tag{9.16}$$

Lemma 9.3. *Let ϖ_N be as in* (6.4). *Let $\tilde{S}_{1\infty} = \{1 \le |x| < \infty\}$. Then the following holds.*

$$\sup_{N\in\mathbb{N}}\sup_{x\in\tilde{S}_{1\infty}} \frac{|x|}{|\varpi_N(x)|} < \infty, \qquad \sup_{N\in\mathbb{N}}\sup_{x\in\tilde{S}_{1\infty}} \frac{\lceil|\varpi_N(x)|\rceil}{|\varpi_N(x)|} < \infty, \tag{9.17}$$

$$\sup_{N\in\mathbb{N}}\sup_{x\in\tilde{S}_{1\infty}} |\bar{\varpi}_N(x) - x| < \infty, \quad \sup_{N\in\mathbb{N}}\sup_{x\in\tilde{S}_{1\infty}} |\lceil|\bar{\varpi}_N(x)|\rceil - |x|| < \infty. \tag{9.18}$$

Proof. Note that, if $x \in \mathbb{I}_N$, then $|\varpi_N(x)|$ is the length of the segment between the origin and $\varpi_N(x)$, and that $|x|$ is the length of the arc connecting these two points on the circle centered at $\mathrm{i}n_N/2\pi$ with radius $n_N/2\pi$. This implies $|\varpi_N(x)| < |x|$ for $x \in \tilde{S}_{1\infty} \cap \mathbb{I}_N$. By definition, ϖ_N is linear on $\mathbb{I}_{N+1}\backslash\mathbb{I}_N$ and $\varpi_N(x) = x$ on $\mathbb{I}_{N+1}^c$. Hence, the maximum of $\frac{|x|}{|\varpi_N(x)|}$ over $\tilde{S}_{1\infty}$ is attained at $x = \pm 1$. Therefore, we have

$$\sup_{N\in\mathbb{N}}\sup_{x\in\tilde{S}_{1\infty}} \frac{|x|}{|\varpi_N(x)|} = \sup_{N\in\mathbb{N}} \frac{1}{|\varpi_N(1)|} = \sup_{N\in\mathbb{N}} \frac{\pi}{n_N}\frac{1}{\sin\frac{\pi}{n_N}} = \frac{\pi}{2^4}\frac{1}{\sin\frac{\pi}{2^4}} < \infty.$$

The second inequality in (9.17) follows from $\lceil|\varpi_N(x)|\rceil < |\varpi_N(x)|+1$ and the first inequality. We thus obtain (9.17)

Direct calculation shows that there exists a constant c_{21} independent of N such that

$$\sup_{x\in\mathbb{R}} |\{\varpi_N(x) - x\}'| = |\varpi_N'(N) - 1| \le c_{21}N2^{-4N}. \tag{9.19}$$

Since $\varpi_N(0) = 0$ and $\varpi_N(x) = x$ for $|x| \ge N+1$, (9.19) yields

$$\sup_{x\in\mathbb{R}} |\varpi_N(x) - x| \le c_{21}N(N+1)2^{-4N}. \tag{9.20}$$

Because $|\varpi_N(x) - x| = |\bar{\varpi}_N(x) - x|$, (9.20) allows us to deduce the first inequality in (9.18). The second is clear from the first. □

We set $c_{22} = \sup_{x\in\tilde{S}_{1r}, N\in\mathbb{N}} \lceil|\varpi_N(x)|\rceil/|\varpi_N(x)| < \infty$.

Lemma 9.4. *Let $u_r^N : \mathbb{R} \to \mathbb{C}$ be such that $u_r^N(x) = 1_{\tilde{S}_{1r}}(x)\lceil|\varpi_N(x)|\rceil/\bar{\varpi}_N(x)$. Then*

$$\sup_{N\in\mathbb{N}} \int_{\mathbb{R}} |u_r^N|^2 dx \le 2c_{22}^2 r, \tag{9.21}$$

$$\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}} |\int_{\mathbb{R}} 1_{\mathbb{I}_N} u_r^N dx| < \infty. \tag{9.22}$$

Proof. From $|u_r^N| \le c_{22}1_{\tilde{S}_{1r}}$ and $\tilde{S}_{1r} \subset (-r, r)$, (9.21) is obvious.

Through construction, we deduce that $|\varpi_N(x)|$ and $\lceil|\varpi_N(x)|\rceil$ are even functions. Moreover, $\Re[\varpi_N(x)]$, the real part of $\varpi_N(x)$, is an odd function in $x \in \mathbb{R}$. Hence, so is $\Re[1/\bar{\varpi}_N(x)] = \Re[\varpi_N(x)/|\varpi_N(x)|^2]$. Collecting these, we see that $\Re[1_{\mathbb{I}_N}u_r^N] = 1_{\mathbb{I}_N}1_{\tilde{S}_{1r}}\lceil|\varpi_N|\rceil\Re[1/\bar{\varpi}_N]$ becomes an odd function. Hence, we have $\int_{\mathbb{R}} \Re[1_{\mathbb{I}_N}u_r^N]dx = 0$. Therefore, it only remains to estimate $\Im[1_{\mathbb{I}_N}u_r^N]$, the imaginary part of $1_{\mathbb{I}_N}u_r^N$.

Note that $u_r^N(x) = 1_{\tilde{S}_{1r}}(x)\lceil|\varpi_N(x)|\rceil\varpi_N(x)/|\varpi_N(x)|^2$. We easily see that $\Im[u_r^N] \ge 0$ and $\Im[\varpi_N(x)/|\varpi_N(x)|]$ takes its maximum at $x = \pm N$ according to (9.14). Then by (9.16), we have

$$\sup_{x\in\mathbb{R}} |\Im[1_{\mathbb{I}_N}u_r^N(x)]| \le c_{22}\sin\frac{\pi N}{n_N} \le c_{22}\frac{\pi N}{2^{4N}}.$$

Clearly, $\Im[1_{\mathbb{I}_N} u_r^N] = 1_{\mathbb{I}_N}\Im[u_r^N] = 0$ for $x \notin \mathbb{I}_N$. Therefore, we deduce that

$$\sup_{r\in\mathbb{N}}\sup_{N\in\mathbb{N}}\int_{\mathbb{R}} |\Im[1_{\mathbb{I}_N} u_r^N(x)]|dx \le \sup_{N\in\mathbb{N}} 2Nc_{22}\frac{\pi N}{2^{4N}} < \infty.$$

This implies (9.22). □

Lemma 9.5. *Let* u_r^N *be as in Lemma 9.4. Set* $\mathsf{u}_r^N(\mathsf{x}) = \sum_i u_r^N(x_i)$*. Then*

$$\lim_{r\to\infty} r^{-3/4}\sup_{N\in\mathbb{N}} \|\mathsf{u}_r^N\|_{L^2(\mathsf{S},\mu^N)} = 0. \tag{9.23}$$

Proof. We set $\hat{\mathsf{u}}_r^N(\mathsf{x}) = \sum_i 1_{\mathbb{I}_N}(x_i)u_r^N(x_i)$. Then $\|\mathsf{u}_r^N\|_{L^2(\mathsf{S},\mu^N)} = \|\hat{\mathsf{u}}_r^N\|_{L^2(\mathsf{S},\nu^N)}$ because $\mu^N = \nu^N\circ\pi_{\mathbb{I}_N}^{-1}$. Hence, (9.23) follows from

$$\lim_{r\to\infty} r^{-3/4}\sup_{N\in\mathbb{N}} \|\hat{\mathsf{u}}_r^N\|_{L^2(\mathsf{S},\nu^N)} = 0. \tag{9.24}$$

We note that $\varrho_N^1(0) = 1$ according to (9.10). We write $1_{\mathbb{I}_N}u_r^N = \hat{u}_{r,1}^N + \mathrm{i}\hat{u}_{r,2}^N$, where $\hat{u}_{r,m}^N$ $(m = 1,2)$ are real valued. We denote by $\mathcal{F}_N$ the Fourier transform defined before Lemma 8.1. Let $\mathfrak{m}_N(\xi)$ be as in (8.2). Applying Lemma 8.1 to $\hat{\mathsf{u}}_r^N$ and using $\varrho_N^1(0) = 1$, we have

$$\|\hat{\mathsf{u}}_r^N\|_{L^2(\mathsf{S},\nu^N)}^2 = \sum_{m=1}^2 [(\int_{\mathbb{T}_N} \hat{u}_{r,m}^N dx)^2 + \frac{1}{n_N}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)} |\mathcal{F}_N(\hat{u}_{r,m}^N)|^2(\xi)\mathfrak{m}_N(\xi)]. \tag{9.25}$$

From (9.22), we deduce that $\lim_{r\to\infty} r^{-3/4}\sup_{N\in\mathbb{N}}|\int_{\mathbb{T}_N}\hat{u}_{r,m}^N dx| = 0$ $(m = 1,2)$. Hence, it only remains for (9.23) to prove

$$\lim_{r\to\infty} r^{-3/2}\sup_{N\in\mathbb{N}}\frac{1}{n_N}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)} |\mathcal{F}_N(\hat{u}_{r,m}^N)|^2(\xi)\mathfrak{m}_N(\xi) = 0 \qquad (m = 1,2). \tag{9.26}$$

Let $P_N = \{-\frac{n_N+1}{2} + p\,; 1 \le p \le n_N,\ p \in \mathbb{N}\}$. Then by an elementary calculation of the triangle series we have an expansion of $S_N(x)$ such that

$$S_N(x) = \frac{1}{n_N}\sum_{p\in P_N} e^{2\pi xp\mathrm{i}/n_N}. \tag{9.27}$$

This together with $D_N(x) = dS_N(x)/dx$ and $I_N(x) = \int_0^x S_N(y)dy$ yields

$$D_N(x) = \frac{2\pi\mathrm{i}}{n_N^2}\sum_{p\in P_N} pe^{2\pi xp\mathrm{i}/n_N} \tag{9.28}$$

$$I_N(x) = \frac{1}{2\pi\mathrm{i}}\sum_{p\in P_N}\frac{1}{p}\left(e^{2\pi xp\mathrm{i}/n_N} - 1\right) = \frac{1}{2\pi\mathrm{i}}\sum_{p\in P_N}\frac{1}{p}\,e^{2\pi xp\mathrm{i}/n_N}. \tag{9.29}$$

For (9.29) we use $0 \notin P_N$, which follows from $n_N/2 \in \mathbb{N}$.

Let $\mathcal{T}_N$ be the two-cluster function of ν^N defined by (8.1). Let $\mathcal{T}_\beta^N(x)$ be the n_N-periodic function such that $\mathcal{T}_\beta^N(x) = \mathcal{T}_N(x,0)$ for $x \in \mathbb{T}_N$. Then through construction, (see (9.3), (9.10))

$$\mathcal{T}_\beta^N(x) = [\mathsf{K}_{\mathrm{dys},\beta}^N(x)\mathsf{K}_{\mathrm{dys},\beta}^N(-x)]^{(0)} \quad \text{for } x \in \mathbb{T}_N. \tag{9.30}$$

Let $P_{N,1} = P_{N,2} = P_N$ and $P_{N,4} = \{p + \frac{1}{2}\,;\, p \in \mathbb{N}, -n_N \le N < n_N\}$. Then (9.27)–(9.30) combined with the definition of $\mathsf{K}^N_{\mathrm{dys},\beta}$ yield

$$\mathcal{T}^N_2(x) = |\mathsf{K}^N_{\mathrm{dys},2}(x)|^2 = \frac{1}{n_N^2}|\sum_{p\in P_{N,2}} e^{2\pi x p\mathrm{i}/n_N}|^2 \tag{9.31}$$

$$\mathcal{T}^N_1(x) = \frac{1}{n_N^2}|\sum_{p\in P_{N,1}} e^{2\pi x p\mathrm{i}/n_N}|^2 - \frac{1}{n_N^2}\sum_{p,q\in P_{N,1}} \frac{p}{q} e^{2\pi x(p+q)\mathrm{i}/n_N} \tag{9.32}$$

$$\mathcal{T}^N_4(x) = \frac{1}{n_N^2}|\sum_{p\in P_{N,4}} e^{4\pi x p\mathrm{i}/n_N}|^2 - \frac{1}{n_N^2}\sum_{p,q\in P_{N,4}} \frac{p}{q} e^{4\pi x(p+q)\mathrm{i}/n_N}. \tag{9.33}$$

For the reader's convenience, we provide more details of the proof of (9.32) and (9.33) as an Appendix (see Section 11.3).

We now consider the Fourier series $\mathcal{F}_N(\mathcal{T}^N_\beta)(\xi) = \int_{\mathbb{T}_N} e^{-2\pi\mathrm{i}\xi\cdot x}\mathcal{T}^N_\beta(x)dx$. By (9.31)–(9.33), we obtain $\sup_{N\in\mathbb{N}}\sup_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)}|\mathcal{F}_N(\mathcal{T}^N_\beta)(\xi)| < \infty$. Hence, $\mathfrak{m}_N$ defined by (8.2) for ν^N satisfies

$$c_{23} := \sup_{N\in\mathbb{N}}\sup_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)} |\mathfrak{m}_N(\xi)| < \infty.$$

From the isometry of the Fourier series and (9.21), we have

$$\sup_{N\in\mathbb{N}}\{\frac{1}{n_N}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)}|\mathcal{F}_N(\hat{u}^N_r)|^2(\xi)\} = \sup_{N\in\mathbb{N}}\{\int_{\mathbb{T}_N}|\hat{u}^N_{r,m}|^2dx\} \le 2c_{22}^2 r \quad (m=1,2).$$

Combining these two equations, we obtain

$$\sup_{N\in\mathbb{N}}\{\frac{1}{n_N}\sum_{\xi\in\mathbb{T}_N\cap(\mathbb{Z}/n_N)}|\mathcal{F}_N(\hat{u}^N_{r,m})|^2(\xi)\mathfrak{m}_N(\xi)\} \le c_{23}2c_{22}^2 r \quad (m=1,2), \tag{9.34}$$

which yields (9.26). We thus complete the proof. □

Lemma 9.6. *Let $u^N_r = 1_{\tilde{S}_{1r}}\lceil|\varpi_N|\rceil/\bar{\varpi}_N$ be as in Lemma 9.4. Then*

$$\lim_{r\to\infty} r^{-3/4}\sup_{N\in\mathbb{N}}\int_{\mathbb{R}}\{\sup_{M\in\mathbb{N}}|u^M_r - u^N_r|\}\rho^1_N dx = 0. \tag{9.35}$$

Proof. Through straightforward calculation, we have

$$\begin{aligned} u^M_r - u^N_r &= 1_{\tilde{S}_{1r}}\{\frac{\lceil|\varpi_M|\rceil}{\bar{\varpi}_M} - \frac{\lceil|\varpi_N|\rceil}{\bar{\varpi}_N}\} \\ &= 1_{\tilde{S}_{1r}}\{\frac{\lceil|\varpi_M|\rceil}{\bar{\varpi}_M\bar{\varpi}_N}(\bar{\varpi}_N - \bar{\varpi}_M) + \frac{1}{\bar{\varpi}_N}(\lceil|\varpi_M|\rceil - \lceil|\varpi_N|\rceil)\} \\ &= 1_{\tilde{S}_{1r}}\{\frac{\lceil|\varpi_M|\rceil}{\bar{\varpi}_M\bar{\varpi}_N}(\bar{\varpi}_N - x + x - \bar{\varpi}_M) + \frac{1}{\bar{\varpi}_N}(\lceil|\varpi_M|\rceil - |x| + |x| - \lceil|\varpi_N|\rceil)\}. \end{aligned} \tag{9.36}$$

By applying (9.17) and (9.18) to the last line of (9.36), there exists a constant c_{24} such that

$$|u^M_r(x) - u^N_r(x)| \le 1_{\tilde{S}_{1r}}(x)c_{24}\frac{1}{|x|} \quad \text{for all } x\in\tilde{S}_{1r},\ M,N\in\mathbb{N}. \tag{9.37}$$

By definition, $u^M_r(x) = 0$ on $\tilde{S}^c_{1r}$. Hence, by (9.37) and $\rho^1_N(x)\le 1$, we obtain

$$\sup_{N\in\mathbb{N}}\int_{\mathbb{R}}\{\sup_{M\in\mathbb{N}}|u^M_r - u^N_r|\}\rho^1_N dx \le c_{24}\int_{\tilde{S}_{1r}}\frac{1}{|x|}dx = c_{24}2\log r.$$

This deduces (9.35). □

Lemma 9.7. *Let u_r^N be as in Lemma 9.5. Then*

$$\lim_{r\to\infty} r^{-3/4} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{u}_r^M| \|_{L^1(\mathsf{S},\mu^N)} = 0. \tag{9.38}$$

Proof. We note that $\sup_{M\in\mathbb{N}} |\mathsf{u}_r^M| \le \{\sup_{M\in\mathbb{N}} |\mathsf{u}_r^M - \mathsf{u}_r^N|\} + |\mathsf{u}_r^N|$. Hence,

$$\sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{u}_r^M| \|_{L^1(\mathsf{S},\mu^N)} \le \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{u}_r^M - \mathsf{u}_r^N| \|_{L^1(\mathsf{S},\mu^N)} + \sup_{N\in\mathbb{N}} \|\mathsf{u}_r^N\|_{L^1(\mathsf{S},\mu^N)}.$$

By Lemma 9.5 and Hölder's inequality, we have $\lim_{r\to\infty} r^{-3/4} \sup_{N\in\mathbb{N}} \|\mathsf{u}_r^N\|_{L^1(\mathsf{S},\mu^N)} = 0$. Hence, it only remains to prove

$$\lim_{r\to\infty} r^{-3/4} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} |\mathsf{u}_r^M - \mathsf{u}_r^N| \|_{L^1(\mathsf{S},\mu^N)} = 0. \tag{9.39}$$

We write $\mathsf{x} = \sum_i \delta_{x_i}$. It is then obvious that

$$\sup_{M\in\mathbb{N}} |\mathsf{u}_r^M(\mathsf{x}) - \mathsf{u}_r^N(\mathsf{x})| = \sup_{M\in\mathbb{N}} |\sum_i \{u_r^M(x_i) - u_r^N(x_i)\}| \le \sum_i \sup_{M\in\mathbb{N}} |u_r^M(x_i) - u_r^N(x_i)|.$$

Taking the expectation of both sides w.r.t. μ^N, we deduce that

$$\| \sup_{M\in\mathbb{N}} |\mathsf{u}_r^M - \mathsf{u}_r^N| \|_{L^1(\mathsf{S},\mu^N)} \le \int_{\mathbb{R}} \{ \sup_{M\in\mathbb{N}} |u_r^M - u_r^N| \} \rho_N^1 dx. \tag{9.40}$$

Combining (9.40) with (9.35), we obtain (9.39), which completes the proof of Lemma 9.7. □

Proof of Theorem 2.2. According to Theorem 4.1, it is enough for (A.2) to check (A.4) and (A.5). We have already checked (A.4) by Lemma 9.2. By Theorem 6.2, it is sufficient for (A.5) to prove (A.6). (A.6) consists of two conditions: (6.13) and (6.14).

According to (9.17), there exists a constant c_{25} such that $1/|\varpi_M(x)|^2 \le c_{25}/x^2$ for all $M \in \mathbb{N}$ and $x \in \tilde{S}_{1r}$. This combined with $\rho_N^1(x) \le 1$ yields

$$\sup_{N\in\mathbb{N}} \int_{\tilde{S}_{1r}} \{ \sup_{M\in\mathbb{N}} \frac{1}{|\varpi_M(x)|^2} \} \rho_N^1(x) dx \le c_{25} \sup_{N\in\mathbb{N}} \int_{\tilde{S}_{1r}} \frac{1}{x^2} dx < \infty. \tag{9.41}$$

Hence, (7.1) is satisfied with $\ell_0 = 2$, and thus, we conclude (6.13) by Lemma 7.1. By Lemma 9.7, we have (7.11) with $c_{18} = 1/4$ and $\ell_0 = 2$, which yields (6.14) by Lemma 7.3 □

10 Proof of Theorem 2.3

In this section, we prove the quasi-Gibbs property (A.2) of the Ginibre random point field μ_{gin} (Theorem 2.3). Therefore, we set $S = \mathbb{C}$, $\Phi(z) = |z|^2$ and $\Psi(z_1, z_2) = -2\log|z_1 - z_2|$. From Theorem 4.1 and Theorem 6.2, we deduce (A.2) from (A.4) and (A.6). Therefore, our task is to check these two assumptions. We begin with the finite-particle approximation μ_{gin}^N.

Let μ_{gin}^N be the determinantal random point field with kernel $\mathsf{K}_{\text{gin}}^N$ given by

$$\mathsf{K}_{\text{gin}}^N(z_1, z_2) = \frac{1}{\pi} \exp\{-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2}\}\{\sum_{k=0}^{N-1} \frac{1}{k!}(z_1 \cdot \bar{z}_2)^k\}. \tag{10.1}$$

Then, by definition, its n-point correlation function $\rho_{\text{gin}}^{N,n}$ is given by

$$\rho_{\text{gin}}^{N,n}(z_1, \dots, z_n) = \det[\mathsf{K}_{\text{gin}}^N(z_i, z_j)]_{1\le i,j\le n}. \tag{10.2}$$

It is well known (see for example p. 943 in [23]) that

$$\mu_{\text{gin}}^N(\mathsf{s}(\mathbb{C})=N)=1. \tag{10.3}$$

Let $\check{\mu}_{\text{gin}}^N$ be the probability measure on $\mathbb{C}^N$ associated with μ_{gin}^N. By definition, $\check{\mu}_{\text{gin}}^N$ is the symmetric measure satisfying $d\mu_{\text{gin}}^N=\check{\mu}_{\text{gin}}^N\circ\iota^{-1}$, where $\iota((z_1,\dots,z_n))=\sum_{i=1}^N\delta_{z_i}$. It is well known (see for example p. 943 in [23]) that

$$\check{\mu}_{\text{gin}}^N=\frac{1}{Z}e^{-\sum_{i=1}^N|z_i|^2}\prod_{1\le i<j\le N}|z_i-z_j|^2dz_1\cdots dz_N. \tag{10.4}$$

Lemma 10.1. $\{\mu_{\text{gin}}^N\}_{N\in\mathbb{N}}$ *satisfy* (A.4).

Proof. It is clear that the kernels $\mathsf{K}_{\text{gin}}^N$ converge to K_{gin} compact uniformly as $N\to\infty$. Hence, (4.1) follows from (10.2). Let $k_i^{N,n}(z_1,\dots,z_n)$ be the norm of the ith row vector of the matrix $[\mathsf{K}_{\text{gin}}^N(z_i,z_j)]_{1\le i,j\le n}$. We see that $k_i^{N,n}(z_1,\dots,z_n)\le n^{1/2}/\pi$ because $|\mathsf{K}_{\text{gin}}^N(z_1,z_2)|\le 1/\pi$ by (10.1). Hence, we obtain

$$|\det[\mathsf{K}_{\text{gin}}^N(z_i,z_j)]_{1\le i,j\le n}|\le\prod_{i=1}^n k_i^{N,n}(z_1,\dots,z_n)\le\frac{n^{n/2}}{\pi^n}. \tag{10.5}$$

Therefore, we deduce (4.2) from (10.2) and (10.5). We thus see that (1) of (A.4) is satisfied.

By (10.3), we see that (2) of (A.4) is satisfied with $n_N=N$.

By (10.3) and (10.4), we see that μ_{gin}^N is a $(|z|^2,-2\log|z|)$-canonical Gibbs measure. Therefore, (3) of (A.4) holds with $\Phi^N(z)=|z|^2$ and $\Psi(z)=-2\log|z|$.

(4) of (A.4) is obvious with the above choice of Φ^N and Ψ, which completes the proof. ☐

We proceed with (A.6). For this, we prepare Lemma 10.2. We denote $\langle\mathsf{s},f\rangle=\sum_i f(s_i)$ for $\mathsf{s}=\sum_i\delta_{s_i}$. We set $\tilde{S}_r=\{z\in\mathbb{C};\,|z|<r\}$. Let $\arg z$ be the angle of $z\in\mathbb{C}$; that is, $z=|z|e^{\mathrm{i}\arg z}$. We write $f(r)=O(g(r))$ as $r\to\infty$ if $\limsup_{r\to\infty}|f(r)|/|g(r)|<\infty$.

Lemma 10.2. *Let* $h_r(z)=1_{\tilde{S}_r}(z)e^{\mathrm{i}\ell\arg z}$*, where* $\ell\in\mathbb{Z}$*. Let* $f:\mathbb{C}\to\mathbb{C}$ *be a bounded, measurable function such that* $\sup_{|z|=r}|f(z)-z_0|=O(r^{-1})$ *as* $r\to\infty$ *for some* $z_0\in\mathbb{C}$*. We then have*

$$\sup_N \mathrm{Var}^{\mu_{\text{gin}}^N}(\langle\mathsf{s},h_rf\rangle)=O(r)\ \textit{as}\ r\to\infty.\ . \tag{10.6}$$

We remark that, if we replace μ_{gin}^N by the Poisson random point field whose intensity is the Lebesgue measure, then the right-hand side of (10.6) becomes $O(r^2)$. Therefore Lemma 10.2 implies the fluctuation of $\{\mu_{\text{gin}}^N\}$ is uniformly small compared with that of the Poisson random point field. Indeed, Lemma 10.2 is the key to the proof of the quasi-Gibbs property. Shirai [22] initiated this kind of small fluctuation property for the Ginibre random point field μ_{gin} with $f=1$. In [17] Shirai's result was generalized to functions f as above. Lemma 10.2 is its N-particle version, which will be proved in Section 10.1 below.

Proof of Theorem 2.3. Applying Theorems 4.1 and 6.2, we deduce Theorem 2.3 from (A.4) and (A.6). We note that (A.4) follows from Lemma 10.1. Therefore, it only remains to prove the two assumptions (6.13) and (6.14) of (A.6). We check (6.13) and (6.14) for $\ell_0=3$.

By (10.1) and (10.2), we have $\rho_{\text{gin}}^{N,1}(z)\le\rho_{\text{gin}}^1(z)=1/\pi$. Therefore, we have

$$\int_{|z|\ge1}\frac{1}{|z|^3}\rho_{\text{gin}}^{N,1}(z)dz\le\frac{1}{\pi}\int_{|z|\ge1}\frac{1}{|z|^3}dz<\infty. \tag{10.7}$$

This implies (7.2). Hence, by Lemma 7.1, we obtain (6.13) with $\ell_0 = 3$.

We finally prove (6.14). Let $u_{\ell,r}^N$ and $\mathsf{u}_{\ell,r}^N$ be as in Lemma 7.3. It is then easy to see that

$$u_{\ell,r}^N(z) = (\frac{\lceil |z| \rceil}{|z|})^\ell 1_{\tilde{S}_{1r}}(z) e^{\mathrm{i}\ell \arg z}.$$

Hence, $u_{\ell,r}^N$ satisfies the assumption of Lemma 10.2 with $z_0 = 1$ and $f(z) = (\frac{\lceil |z| \rceil}{|z|})^\ell$. Therefore, by Lemma 10.2, we obtain

$$\lim_{r\to\infty} r^{2c_{19}-2\ell} \sup_N \mathrm{Var}^{\mu_{\mathrm{gin}}^N}[\mathsf{u}_{\ell,r}^N] = 0 \quad \text{with } c_{19} = 1/4, \text{ say, for } \ell = 1, 2. \tag{10.8}$$

Since $E^{\mu_{\mathrm{gin}}^N}[\mathsf{u}_{\ell,r}^N] = 0$, (10.8) implies $\lim_{r\to\infty} r^{2c_{19}-2\ell} \sup_{N\in\mathbb{N}} E^{\mu_{\mathrm{gin}}^N}[|\mathsf{u}_{\ell,r}^N|^2] = 0$, which allows us to deduce (7.11). Hence, by Lemma 7.3, we obtain (6.14) with $\ell_0 = 3$. We thus complete the proof. □

10.1 Proof of Lemma 10.2

The purpose of this subsection is to prove Lemma 10.2.

Let $\mathsf{g}(dz) = \frac{1}{\pi}\exp\{-|z|^2\}dz$ be the standard complex Gaussian measure. Let $\{\rho_N^n\}_{n\in\mathbb{N}}$ be the correlation function of μ_{gin}^N w.r.t. g. Then $\{\rho_N^n\}_{n\in\mathbb{N}}$ is given by

$$\rho_N^n(z_1, \dots, z_n) = \det[K_N(z_i, z_j)]_{i,j=1,\dots,n}, \tag{10.9}$$

where $K_N(z_1, z_2) = \sum_{k=0}^{N-1} \{z_1\bar{z}_2\}^k/k!$. We note that $\rho_N^n = \rho_{\mathrm{gin}}^{N,n}\pi^n e^{|z_1|^2+\cdots+|z_n|^2}$ and $K_N(w, z) = \pi e^{|w|^2/2}\mathsf{K}_{\mathrm{gin}}^N(w, z)e^{|z|^2/2}$ by construction. Let

$$K(z_1, z_2) = \sum_{k=0}^{\infty} \frac{\{z_1\bar{z}_2\}^k}{k!}, \quad K_N^*(z_1, z_2) = \sum_{k=N}^{\infty} \frac{\{z_1\bar{z}_2\}^k}{k!}. \tag{10.10}$$

Then $K = K_N + K_N^*$ by definition. Let

$$M_r^N = \int h_r(w)\overline{h_r(z)}\{|K(w, z)|^2 - |K_N(w, z)|^2 - |K_N^*(w, z)|^2\}\mathsf{g}(dw)\mathsf{g}(dz).$$

Lemma 10.3. *Let* $e_N^s = \sum_{k=0}^N s^k/k!$. *Then* $|M_r^N| \le 2\{1 - e^{-r^2}e_{N-1}^{r^2}\}\{1 - e^{-r^2}e_N^{r^2}\}$.

Proof. From $|K|^2 = |K_N|^2 + |K_N^*|^2 + K_N\overline{K}_N^* + K_N^*\overline{K}_N$ we have

$$\begin{aligned}
|M_r^N| &= |\int h_r(w)\overline{h_r}(z)\{K_N\overline{K}_N^* + K_N^*\overline{K}_N\}\mathsf{g}(dw)\mathsf{g}(dz)| \\
&= \frac{2}{(N-1)!N!}\{\int_{\tilde{S}_r} |w|^{2N-1}\mathsf{g}(dw)\}^2 \\
&\le 2\{\frac{1}{(N-1)!}\int_{\tilde{S}_r} |w|^{2N-2}\mathsf{g}(dw)\}\{\frac{1}{N!}\int_{\tilde{S}_r} |w|^{2N}\mathsf{g}(dw)\} \\
&= 2\{1 - e^{-r^2}e_{N-1}^{r^2}\}\{1 - e^{-r^2}e_N^{r^2}\}.
\end{aligned}$$

This completes the proof. □

The kernel K_N^* also generates the determinantal random point field denoted by μ_{gin}^{N*}.

Lemma 10.4. (1) *Let f be a bounded measurable function with compact support. Then*

$$\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, f\rangle) \le \frac{2}{\pi}\int_{\mathbb{C}} |f(z)|^2 dz. \tag{10.11}$$

(2) (10.11) *also hold for* μ_{gin}^{N*} *and* μ_{gin}.

Proof. Since $K_N(w,z)$ consists of a sum of pairs of orthonormal functions w.r.t. $\mathsf{g}(dz)$, we have the equality

$$K_N(z,z) = \int_{\mathbb{C}} |K_N(z,w)|^2 \mathsf{g}(dw). \tag{10.12}$$

By the standard calculation of correlation functions, we have

$$\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, f\rangle) = \int_{\mathbb{C}} |f(z)|^2 K_N(z,z)\mathsf{g}(dz) - \int_{\mathbb{C}^2} f(w)\overline{g(z)}|K_N(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz).$$

Combining these two equalities, and then using the inequalities $|a-b|^2 \le 2(|a|^2+|b|^2)$ and $|K_N(w,z)|^2 \le K_N(w,w)K_N(z,z)$, we obtain

$$\begin{aligned} \mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, f\rangle) &= \frac{1}{2}\int_{\mathbb{C}^2} |f(w)-f(z)|^2 |K_N(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz) \\ &\le \int_{\mathbb{C}^2} \{|f(w)|^2 + |f(z)|^2\} K_N(w,w)K_N(z,z)\mathsf{g}(dw)\mathsf{g}(dz). \end{aligned} \tag{10.13}$$

This, combined with the estimates $0 \le K_N(z,z)(1/\pi)e^{-|z|^2} \le 1/\pi$, allows us to conclude (10.11). The proof of (2) is the same as that of (1). □

Lemma 10.5 (Theorem 1.3 in [17]). $\sup_{1\le r} r^{-1}\mathrm{Var}^{\mu_{\mathrm{gin}}}[\langle \mathsf{s}, h_r f\rangle] < \infty$.

Proof. This lemma is a special case of Theorem 1.3 in [17]. □

Lemma 10.6. $\sup_{1\le N}\sup_{1\le r} \frac{1}{r}\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) < \infty$.

Proof. By $K = K_N + K_N^*$ and Lemma 10.4, we have

$$\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) = \mathrm{Var}^{\mu_{\mathrm{gin}}}(\langle \mathsf{s}, h_r\rangle) - M_r^N - \mathrm{Var}^{\mu_{\mathrm{gin}}^{N*}}(\langle \mathsf{s}, h_r\rangle).$$

By Lemma 10.3, we have $|M_r^N| \le 2\{1 - e^{-r^2}e_{N-1}^{r^2}\}\{1-e^{-r^2}e_N^{r^2}\}$. These, together with Lemma 10.5, complete the proof. □

Proof of Lemma 10.2. By $h_r f = z_0 h_r + h_r(f-z_0)$ and (10.11), we have

$$\begin{aligned} \mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r f\rangle) &\le 2z_0^2 \mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) + 2\mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r(f-z_0)\rangle) \\ &\le 2z_0^2 \mathrm{Var}^{\mu_{\mathrm{gin}}^N}(\langle \mathsf{s}, h_r\rangle) + \frac{4}{\pi}\int_{\tilde{S}_r} |h_r(f-z_0)|^2 dz. \end{aligned}$$

Hence, from Lemma 10.6 and the assumption $\sup_{|z|=r}|f(z)-z_0| = O(r^{-1})$, we complete the proof. □

11 Appendix

11.1 Proof of Lemma 3.4 and Lemma 3.5.

Proof of Lemma 3.4. Let $\{f_{\mathsf{p}}\}$ be a $\mathcal{E}_{r,k}^{m,a,\mu}$-Cauchy sequence in $\mathcal{D}_\infty^{a,\mu}$ such that $\lim \|f_{\mathsf{p}}\|_{L^2(\mathsf{S},\mu_{r,k}^m)} = 0$. Then from (3.3) and (3.4), we see that $\{f_{\mathsf{p}}\}$ satisfies

$$\lim_{\mathsf{p},\mathsf{q}\to\infty} \int_{\mathsf{S}} \mathcal{E}_{r,k,\mathsf{s}}^{m,a,\mu}(f_{\mathsf{p}}-f_{\mathsf{q}}, f_{\mathsf{p}}-f_{\mathsf{q}})\mu_{r,k}^m(d\mathsf{s}) = 0 \tag{11.1}$$

$$\lim_{\mathsf{p}\to\infty} \int_{\mathsf{S}} \|f_{\mathsf{p}}\|^2_{L^2(\mathsf{S}_r^m, \mu_{r,k,\mathsf{s}}^m)} \mu_{r,k}^m(d\mathsf{s}) = 0. \tag{11.2}$$

We prove that $\lim_{\mathsf{p}\to\infty}\mathcal{E}^{m,a,\mu}_{r,k}(f_{\mathsf{p}},f_{\mathsf{p}})=0$. For this purpose, it is enough to show that, for any subsequence $\{f_{1,\mathsf{p}}\}$ of $\{f_{\mathsf{p}}\}$, we can choose a subsequence $\{f_{2,\mathsf{p}}\}$ of $\{f_{1,\mathsf{p}}\}$ such that

$$\lim_{\mathsf{p}\to\infty}\mathcal{E}^{m,a,\mu}_{r,k}(f_{2,\mathsf{p}},f_{2,\mathsf{p}})=0. \tag{11.3}$$

Therefore, let $\{f_{1,\mathsf{p}}\}$ be any subsequence of $\{f_{\mathsf{p}}\}$. Then by (11.1) and (11.2), we can choose a subsequence $\{f_{2,\mathsf{p}}\}$ such that $\mu^m_{r,k}(\mathsf{A}_{\mathsf{p}})\le 2^{-k}$ and $\mu^m_{r,k}(\mathsf{B}_{\mathsf{p}})\le 2^{-k}$, where

$$\mathsf{A}_{\mathsf{p}}=\{\mathsf{s}\,;\,\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f_{2,\mathsf{p}}-f_{2,\mathsf{p}+1},f_{2,\mathsf{p}}-f_{2,\mathsf{p}+1})\ge 2^{-2k}\}$$
$$\mathsf{B}_{\mathsf{p}}=\{\mathsf{s}\,;\,\|f_{\mathsf{p}}\|^2_{L^2(\mathsf{S}^m_r,\mu^m_{r,k,\mathsf{s}})}\ge 2^{-2k}\}.$$

Hence, from Borel-Cantelli's lemma, we see that $\mu^m_{r,k}(\limsup\mathsf{A}_{\mathsf{p}})=\mu^m_{r,k}(\limsup\mathsf{B}_{\mathsf{p}})=0$. This means that, for $\mu^m_{r,k}$-a.s. s, the sequence $\{f_{2,\mathsf{p}}\}$ is an $\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}$-Cauchy sequence converging to 0 in $L^2(\mathsf{S}^m_r,\mu^m_{r,k,\mathsf{s}})$ as $\mathsf{p}\to\infty$. Therefore, by assumption, we have

$$\lim_{\mathsf{p}\to\infty}\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f_{2,\mathsf{p}},f_{2,\mathsf{p}})=0\quad\text{ for }\mu^m_{r,k}\text{-a.s. }\mathsf{s}. \tag{11.4}$$

Let $\check{\mu}^m_{r,k,\mathsf{s}}$ be the symmetric measure on S^m_r such that $\check{\mu}^m_{r,k,\mathsf{s}}\circ\iota^{-1}=\mu^m_{r,k,\mathsf{s}}$. For $f_{2,\mathsf{p}}$, there exists a function $f^{r,m}_{2,\mathsf{p}}:S^m_r\times\mathsf{S}\to\mathbb{R}$ such that $f^{r,m}_{2,\mathsf{p}}(\mathbf{x},\mathsf{s})$ is symmetric in $\mathbf{x}=(x_1,\dots,x_m)$ for each $\mathsf{s}\in\mathsf{S}$ and that $f^{r,m}_{2,\mathsf{p}}(\mathbf{x},\mathsf{s})=f_{2,\mathsf{p}}(\mathsf{s})$ for $\mathsf{s}\in\mathsf{S}^m_r$ decomposed as $\mathsf{s}=\iota(\mathbf{x})+\pi_{S^c_r}(\mathsf{s})$. Let $x_l=(x_{l1},\dots,x_{ld})\in\mathbb{R}^d$. Then

$$\int_{\mathsf{S}^m_r}\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}}(f_{2,\mathsf{p}}-f_{2,\mathsf{p}+1},f_{2,\mathsf{p}}-f_{2,\mathsf{p}+1})\mu^m_{r,k}(d\mathsf{s})=$$
$$\int_{S^m_r\times\mathsf{S}}\frac{1}{2}\sum_{l=1}^m\sum_{i,j=1}^d a_{ij}(\mathsf{s},x_l)\frac{\partial(f^{r,m}_{2,\mathsf{p}}-f^{r,m}_{2,\mathsf{p}+1})}{\partial x_{li}}\cdot\frac{\partial(f^{r,m}_{2,\mathsf{p}}-f^{r,m}_{2,\mathsf{p}+1})}{\partial x_{lj}}\check{\mu}^m_{r,k,\mathsf{s}}(d\mathbf{x})\mu^m_{r,k}(d\mathsf{s}).$$

Hence, by (11.1), we see that the vector-valued function $(\nabla_{x_l}f^{r,m}_{2,\mathsf{p}})_{l=1,\dots,m}:S^m_r\times\mathsf{S}\to(\mathbb{R}^d)^m$ is a Cauchy sequence in $L^2(S^m_r\times\mathsf{S}\to(\mathbb{R}^d)^m,\check{\mu}^m_{r,k,\mathsf{s}})$, where we equip $L^2(S^m_r\times\mathsf{S}\to(\mathbb{R}^d)^m,\check{\mu}^m_{r,k,\mathsf{s}})$ with the inner product

$$(\mathbf{f},\mathbf{g})=\int_{S^m_r\times\mathsf{S}}\sum_{l=1}^m\{f_l(\mathbf{x},\mathsf{s})g_l(\mathbf{x},\mathsf{s})a_0(\iota(\mathbf{x})+\pi_{S^c_r}(\mathsf{s}),x_l)\}\check{\mu}^m_{r,k,\mathsf{s}}(d\mathbf{x})\mu^m_{r,k}(d\mathsf{s}).$$

Here $\mathbf{f}=(f_1,\dots,f_m)$, and a_0 is the function in (2.3). Combining this with (11.1) and (11.4), we obtain (11.3), which completes the proof. □

Proof of Lemma 3.5. By (2.3), we deduce the closability of $(\mathcal{E}^{m,a,\mu}_{r,k,\mathsf{s}},\mathcal{D}^{a,\mu}_\infty)$ on $L^2(\mathsf{S}^m_r,\mu^m_{r,k,\mathsf{s}})$ from that of $(\mathcal{E}^{m,a_0I,\mu}_{r,k,\mathsf{s}},\mathcal{D}^{a_0I,\mu}_\infty)$ on $L^2(\mathsf{S}^m_r,\mu^m_{r,k,\mathsf{s}})$. Here I is the $d\times d$ unit matrix.

Let $\check{\mu}^m_{r,k,\mathsf{s}}$ be as in the proof of Lemma 3.4. Then by (2.8), $\check{\mu}^m_{r,k,\mathsf{s}}$ has a density $\check{\sigma}(\mathbf{x})$ w.r.t. $e^{-\mathcal{H}_r(\mathbf{x})}d\mathbf{x}$. Here $d\mathbf{x}$ denotes the Lebesgue measure on S^m_r and we regard $e^{-\mathcal{H}_r}$ as a symmetric function on S^m_r in an obvious manner. In the following we use the same convention for functions on the configuration space S^m_r. We note that according to (2.8), $\check{\sigma}$ is uniformly positive and bounded on S^m_r.

Let $O_p=\{\mathbf{x}\in S^m_r\,;\,p^{-1}<a_0(\mathbf{x})\}\cap\{\mathbf{x}\in S^m_r\,;\,p^{-1}<e^{-\mathcal{H}_r(\mathbf{x})}\}$ $(p\in\mathbb{N})$. Recall that a_0 and $e^{-\mathcal{H}_r}$ are lower semicontinuous (the latter claim follows from the assumption that Φ and Ψ are upper semicontinuous), which implies that O_p is an open set. Moreover, $\{O_p\}$ is nondecreasing in p. Let ε_p be the bilinear form on S^m_r defined by

$$\varepsilon_p(f,g)=\int_{O_p}\mathbb{D}^{a_0I}[f,g]\check{\sigma}e^{-\mathcal{H}_r}d\mathbf{x}=\int_{O_p}\mathbb{D}[f,g]a_0\check{\sigma}e^{-\mathcal{H}_r}d\mathbf{x}.$$

Recall that $a_0\check{\sigma}e^{-\mathcal{H}_r}$ and $\check{\sigma}e^{-\mathcal{H}_r}$ are bounded on S_r^m and greater than or equal to p^{-2} on O_p. Hence, $(\varepsilon_p, C_b^\infty(S_r^m))$ is closable on $L^2(S_r^m, \check{\sigma}e^{-\mathcal{H}_r}d\mathbf{x}) = L^2(S_r^m, \check{\mu}_{r,k,\mathsf{s}}^m)$. Since $\{O_p\}$ is nondecreasing, the sequence of closable bilinear forms $(\varepsilon_p, C_b^\infty(S_r^m))$ is nondecreasing. Hence, the limit bilinear form $(\varepsilon_\infty, C_b^\infty(S_r^m))$ is also closable on $L^2(S_r^m, \check{\mu}_{r,k,\mathsf{s}}^m)$ (see [11, Prop. 3.7, 30 p.]). We used here $\{f \in C_b^\infty(S_r^m); \varepsilon_\infty(f,f) < \infty\} = C_b^\infty(S_r^m)$.

It is easy to see that the closability of $(\varepsilon_\infty, C_b^\infty(S_r^m))$ on $L^2(S_r^m, \check{\mu}_{r,k,\mathsf{s}}^m)$ implies the closability of $(\mathcal{E}_{r,k,\mathsf{s}}^{m,a_0 I,\mu}, \mathcal{D}_\infty^{a_0 I,\mu})$ on $L^2(\mathsf{S}_r^m, \mu_{r,k,\mathsf{s}}^m)$. Hence, we complete the proof. □

11.2 The weak convergence of $\{\mu^N\}$.

In Section 4, we considered the fact that the measures $\{\mu^N\}$ in (A.2) converge weakly to μ. For the sake of completeness we give a proof of this. Let $\tilde{S}_r = \{x \in S\,;\, |x| < r\}$ and $\tilde{S}_r^n = \prod_{m=1}^n \{|x_m| < r\}$ as before.

Lemma 11.1. *Assume* (4.1) *and* (4.2) *in* (A.4). *Then* $\lim_{N\to\infty} \mu^N = \mu$ *weakly.*

Proof. A permutation invariant function $m_r^n : \tilde{S}_r^n \to \mathbb{R}$ is by definition the n-density function of μ if, for any bounded $\sigma[\pi_{\tilde{S}_r}]$-measurable function f,

$$\int_{\tilde{\mathsf{S}}_r^n} \mathsf{f} d\mu = \frac{1}{n!} \int_{\tilde{S}_r^n} f_r^n m_r^n dx_1 \cdots dx_n,$$

where $\tilde{\mathsf{S}}_r^n = \{\mathsf{x} \in \mathsf{S}\,;\, \mathsf{x}(\tilde{S}_r) = n\}$, and $f_r^n : \tilde{S}_r^n \to \mathbb{R}$ is the permutation invariant function such that $f_r^n(x_1, \ldots, x_n) = \mathsf{f}(\mathsf{x})$ for $\mathsf{x} \in \tilde{\mathsf{S}}_r^n$ such that $\pi_{\tilde{S}_r}(\mathsf{x}) = \sum_i \delta_{x_i}$.

Let $m_{N,r}^n(x_1, \ldots, x_n)$ (resp. $m_r^n(x_1, \ldots, x_n)$) be the n-density function of μ^N (resp. μ) on $\tilde{S}_r$. Then by (4.2), we easily see that

$$m_{N,r}^n(x_1, \ldots, x_n) = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{\tilde{S}_r^n} \rho_N^{n+k}(x_1, \ldots, x_{n+k}) dx_{n+1} \cdots dx_{n+k}. \tag{11.5}$$

Combining (4.1) and (4.2) with (11.5) and the same equality as (11.5) for μ and applying the bounded convergence theorem, we obtain for each $r, n \in \mathbb{N}$

$$\sup_N \sup_{\tilde{S}_r^n} |m_{N,r}^n(x_1, \ldots, x_n)| < \infty, \quad \lim_{N\to\infty} m_{N,r}^n(x_1, \ldots, x_n) = m_r^n(x_1, \ldots, x_n) \quad \text{a.e.}.$$

From this, we see that the measures satisfy $\lim_{N\to\infty} \mu^N \circ \pi_{\tilde{S}_r}^{-1} = \mu \circ \pi_{\tilde{S}_r}^{-1}$ weakly in $\pi_{\tilde{S}_r}(\mathsf{S})$ for all r. Hence, it only remains to prove that the sequence $\{\mu^N\}$ is tight in S.

Now we recall a closed subset S_0 in S is compact if and only if there exists an increasing sequence $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}}$ of natural numbers such that $\sup_{\mathsf{s}\in\mathsf{S}_0} \mathsf{s}(\tilde{S}_r) \le a_r$ for all $r \in \mathbb{N}$ [19, Sect. 3.4]. Let $\mathsf{K}(r,a) = \{\mathsf{s};\ \mathsf{s}(\tilde{S}_r) \le a\}$. Set $\mathsf{K}(\mathbf{a}) = \cap_{r\in\mathbb{N}} \mathsf{K}(r, a_r)$ for $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}}$. We then see that the set $\mathsf{K}(\mathbf{a})$ is compact in S because of the equivalence condition given above.

Let $\epsilon > 0$ be fixed. Note that $\pi_{\tilde{S}_r}(\mathsf{S})$ is also a Polish space because $\tilde{S}_r$ is Polish [19, Prop. 3.17]. Since $\{\mu^N \circ \pi_{\tilde{S}_r}^{-1}\}$ is tight as probability measures in $\pi_{\tilde{S}_r}(\mathsf{S})$, there exists a compact set K_r in $\pi_{\tilde{S}_r}(\mathsf{S})$ such that

$$\sup_N \mu^N \circ \pi_{\tilde{S}_r}^{-1}(\mathsf{K}_r^c) \le \epsilon 2^{-r}. \tag{11.6}$$

Moreover there exists an $a_r \in \mathbb{N}$ such that $\mathsf{K}_r \subset \mathsf{K}(r, a_r)$ because K_r is compact. We can and do take $a_r \in \mathbb{N}$ in such a way that $a_r < a_{r+1}$. By (11.6) and $\mathsf{K}_r \subset \mathsf{K}(r, a_r)$, we have

$\sup_N \mu^N(\mathsf{K}(r,a_r)^c) \le \epsilon 2^{-r}$. Hence, for $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}}$, we have

$$\sup_N \mu^N(\mathsf{K}(\mathbf{a})^c) = \sup_N \mu^N(\bigcup_{r\in\mathbb{N}} \mathsf{K}(r,a_r)^c) \le \sup_N \sum_{r\in\mathbb{N}} \mu^N(\mathsf{K}(r,a_r)^c) \le \epsilon.$$

This implies $\{\mu^N\}$ is tight, which completes the proof. □

11.3 Proof of (9.32) and (9.33)

In this subsection, we prove (9.32) and (9.33). Let $J_N(x) = I_N(x) - \frac{1}{2}\mathrm{sgn}(x)$. Note that S_N is an even function and I_N, D_N, and J_N are odd functions. By (9.5) for S_N, D_N, and I_N,

(11.7)
$$\begin{aligned} &\mathsf{K}^N_{\mathrm{dys},1}(x)\mathsf{K}^N_{\mathrm{dys},1}(-x) \\ &= \Theta(\begin{bmatrix} S_N(x) & D_N(x) \\ J_N(x) & S_N(x) \end{bmatrix} \begin{bmatrix} S_N(-x) & D_N(-x) \\ J_N(-x) & S_N(-x) \end{bmatrix}) \\ &= \Theta(\begin{bmatrix} S_N(x)^2 - D_N(x)J_N(x) & 0 \\ 0 & S_N(x)^2 - D_N(x)J_N(x) \end{bmatrix}. \end{aligned}$$

Hence, by (9.1) and (9.30), we have $\mathcal{T}_1^N = S_N^2 - D_N J_N$ .This combined with (9.27)–(9.29) yields (9.32). We consider (9.33) next. By (9.7) for S_N, D_N, and I_N, we see that

$$\begin{aligned} &\mathsf{K}^N_{\mathrm{dys},4}(x)\mathsf{K}^N_{\mathrm{dys},4}(-x) \\ &= \Theta(\begin{bmatrix} S_N(2x) & D_N(2x) \\ I_N(2x) & S_N(2x) \end{bmatrix} \begin{bmatrix} S_N(-2x) & D_N(-2x) \\ I_N(-2x) & S_N(-2x) \end{bmatrix}) \\ &= \Theta(\begin{bmatrix} S_N(2x)^2 - D_N(2x)I_N(2x) & 0 \\ 0 & S_N(2x)^2 - D_N(2x)I_N(2x) \end{bmatrix}. \end{aligned}$$

Hence, by (9.1) and (9.30), we have $\mathcal{T}_4^N(x) = S_N(2x)^2 - D_N(2x)J_N(2x)$. This combined with (9.27)–(9.29) yields (9.33).

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