

Right Inverses of Lévy Processes

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Abstract

We call a right continuous increasing process K_x a partial right inverse (PRI) of a given Lévy process X if $X_{K_x} = x$ at least for all x in some random interval $[0, \zeta)$ of positive length. In this paper we give a necessary and sufficient condition for the existence of a PRI in terms of the Lévy triplet.

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1 Introduction and Results.

In this paper a real-valued Lévy process is studied. The problem of existence of a partial right inverse (PRI) is considered and an explicit integral criterion is provided for testing whether any Lévy process possesses a PRI.

We continue work by Evans [3] and Winkel [5]. Evans has introduced the notion of full right inverse and has defined this process K as the minimal increasing process that satisfies $X(K_x) = x$, for all $x \geq 0$, and Winkel in [5] has extended this definition to $X(K_x) = x$ on some random interval $[0, \zeta)$ of positive length, and has named this process a PRI. In these two papers it is shown that if K exists it is a (possibly killed) subordinator.

A Lévy process $X = (X_t; t \geq 0)$ is a stochastic process which possesses stationary and independent increments, starts from zero and whose paths are a.s. right continuous. Each Lévy process is fully characterised by its Lévy triplet (γ, σ, Π) , where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and the Lévy measure Π has the property

$$\int_{-\infty}^{\infty} (1 \wedge x^2) \Pi(dx) < \infty.$$

Also each Lévy process X can be represented as follows;

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$$X_t = \gamma t + \sigma B_t + X_t^{(1)} + \sum_{0 < s \leq t} (X_s - X_{s-}) \mathbf{1}_{(|X_s - X_{s-}| > 1)}, \quad (1)$$

where B is a standard Brownian motion, $X^{(1)}$ is a pure jump zero mean martingale and all the components in (1) are independent. In the class of Lévy processes we distinguish between Lévy processes with bounded variation and Lévy processes with unbounded variation. The first are those for which $\sigma = 0$ and $\int_{-\infty}^{\infty} (1 \wedge |x|) \Pi(dx) < \infty$. In this case X can be represented as follows

$$X_t = bt + X_t^+ + X_t^-, \quad (2)$$

where b is the drift coefficient and X^+ and X^- are independent driftless subordinators (i.e. increasing Lévy processes). In our setting as well as in many other situations these two classes of processes exhibit quite different behaviour and need separate attention.

Write $R_t = \sup_{s \leq t} X_s - X_t$. It is shown in [1], Chapter 6, that R is a strong Markov process and it possesses a local time at zero, $L(t)$, and correspondingly an inverse local time $L^{-1}(t) = \inf\{s : L(s) > t\}$, such that $(L^{-1}(t); X(L^{-1}(t)))$ is a bivariate subordinator: we denote its Lévy measure by $\mu^{(+)}(dt; dy)$ and we use in particular $\mu^{(+)}(dy) = \mu^{+}((0; \infty); dy)$. Also we use the notation $H^+(t) := X(L^{-1}(t))$, and call H^+ the upwards ladder height process. Similarly we can define $Z_t = X_t - \inf_{s \leq t} X_s$ and using the same arguments we have an associated inverse local time $L_-^{-1}(t)$ and downwards ladder height process $H^-(t) := X(L_-^{-1}(t))$. We denote the Lévy measure of H^- by $\mu^{(-)}(dy)$. Finally with each of the subordinators H^+ and H^- we associate the so-called renewal measure defined as follows:

$$U_+(x) = E \int_0^{\infty} \mathbf{1}_{\{H_t^+ \leq x\}} dt, \quad U_-(x) = E \int_0^{\infty} \mathbf{1}_{\{H_t^- \leq x\}} dt. \quad (3)$$

We refer to Bertoin [1] or Doney [2] for more information on Lévy processes.

Next we discuss briefly the definition of a PRI, i.e. $K = (K_x, x \geq 0)$. We follow an approach developed in Evans [3]. Define, for each $n \geq 1$ and $k \geq 0$, the following stopping times

$$T_0 = 0, \quad T_n^{k+1} = \inf\{t \geq T_n^k : X_t = \frac{k+1}{2^n}\}, \quad (4)$$

and processes

$$K_x^n = T_n^k, \quad \frac{k}{2^n} \leq x < \frac{k+1}{2^n}.$$

Then a pathwise argument shows that

$$K_x = \inf_{y > x} \sup_{n \geq 0} K_y^n. \quad (5)$$

It is possible that, for each $x > 0$, the definition above gives $K_x \stackrel{a.s.}{=} \infty$, and in this case we say that a PRI does not exist. The question of existence of a

PRI has been studied by Evans in [3] and Winkel in [5]. Evans has shown that for any symmetric Lévy process with $\sigma > 0$ a full right inverse exists. Then Winkel [5] showed that the same result holds for any oscillating Lévy process with $\sigma > 0$, and also described all Lévy processes with bounded variation having a PRI. Moreover, in the unbounded variation case he provided a necessary and sufficient condition (NASC) for the existence of a PRI, but this NASC is not satisfactory since it requires knowledge about the second derivative at zero of the so-called q-potentials of the given Lévy process, which are generally unknown. Therefore the main aim of this paper is to supply a NASC for existence of a PRI in terms of the Lévy triplet, i.e. (γ, σ, Π) , in the unbounded variation case. In fact our method, which is probabilistic in nature, also deals with the bounded variation case, and gives the following result.

Theorem 1 *Let X be a Lévy process with a Lévy measure Π such that $\Pi(\mathbb{R}) > 0$. Then*

(i) *If X has unbounded variation it has a partial right inverse (PRI) iff $\sigma > 0$ or $\sigma = 0$, $\Pi(\mathbb{R}^-) = \infty$, and $J < \infty$, where, with $\bar{\Pi}^{(-)}(s) = \int_{-\infty}^{-s} \Pi(dx)$,*

$$J = \int_0^1 \frac{x^2 \Pi(dx)}{\left(\int_0^x \int_y^1 \bar{\Pi}^{(-)}(s) ds dy \right)^2}. \quad (6)$$

(ii) *If X has bounded variation then it has a PRI iff $\Pi(\mathbb{R}^+) < \infty$ and X has a drift coefficient $b > 0$.*

Remark 2 *If $\Pi(\mathbb{R}) = 0$ then $X_t = \gamma t + \sigma B_t$ is a continuous process and $T_x = \inf\{t : X_t = x\}$ will be a PRI on the set $\{T_x < \infty\}$. Note that in this case $\{T_x < \infty\}$ will be the empty set iff $\sigma = 0$ and $\gamma < 0$.*

Remark 3 *A Lévy process X is said to "creep upwards" if $P(X(T_x^+) = x) > 0$ for some (and then all) $x > 0$, where $T_x^+ = \inf\{t > 0 : X_t > x\}$. It is known that this happens iff the ladder height process H^+ has drift $\delta_+ > 0$: see e.g. Theorem 19, p 174 of [1]. Since always $\sigma^2 = 2\delta_+\delta_-$, where δ_- is the drift of H^- , this certainly happens when $\sigma > 0$. If $\sigma = 0$ and $J < \infty$ then clearly the integral*

$$L = \int_0^1 \frac{x^2 \Pi(dx)}{\int_0^x \int_y^1 \bar{\Pi}^{(-)}(s) ds dy} \quad (7)$$

is finite, and it is shown in [4] that this is the NASC for $\delta_+ > 0$ in the unbounded variation case when $\sigma = 0$. (See also section 6.4 of [2] for an alternative proof of this result.) Finally in the bounded variation case $b > 0$ is clearly equivalent to $\delta_+ > 0$. So we conclude that our theorem is consistent with the intuitively obvious claim that "upward creeping" is necessary, but not sufficient, for the existence of a PRI.

The next Corollary illustrates how our Theorem yields specific information in special cases. Here and throughout the paper we use the notation $f \approx g$ to

denote the existence of constants $0 < c < C < \infty$ with $cg(x) \leq f(x) \leq Cg(x)$, for all x sufficiently small.

Corollary 4 *Let X be a Lévy process with $\sigma = 0$ and Lévy measure Π such that $\bar{\Pi}^+(x) = \int_x^\infty \Pi(dy) \approx x^{-\beta}$ and $\bar{\Pi}^-(x) \approx x^{-\alpha}$, where $1 \leq \alpha < 2$ and $0 \leq \beta < 2$. Then X has a PRI iff $\beta < 2\alpha - 2$.*

Remark 5 *This result extends Proposition 2 and Theorem 6 in [5].*

2 Proofs

Recall that we denote by H^+ the ascending ladder height process of a given Lévy process X . We use δ_+ to denote the drift of H^+ , and $\mu^{(+)}(dy)$ to denote its Lévy measure. We also use U_+ and U_- which are defined in (3). We start the proof by disposing of some special cases.

Suppose first that $\Pi(\mathbb{R}) < \infty$. Then $V = \inf\{t > 0 : X_t - X_{t-} \neq 0\} > 0$ a.s. since it is an exponentially distributed random variable with parameter $\Pi(\mathbb{R})$, and the given process coincides up to time V with the process we get by removing all its jumps. The resulting process will be of the form $\sigma B_t + bt$, which possesses a PRI iff $\sigma > 0$ or $\sigma = 0$ and $b > 0$, in accord with Theorem 1. Suppose next that $\Pi(\mathbb{R}) = \infty$ but $\Pi(\mathbb{R}^+) < \infty$. Then removing all the positive jumps gives a spectrally negative Lévy process \tilde{X} . If \tilde{X} has unbounded variation, or has bounded variation and a positive drift b , then it passes continuously over positive levels. Then with $\tilde{T}(x) = \inf\{t > 0 : \tilde{X}_t = x\}$ we obviously have $\tilde{X}_{\tilde{T}(x)} = x$ on $\{\tilde{T}(x) < \infty\}$ and we can choose $K_x = \tilde{T}(x)$. Alternatively \tilde{X} has bounded variation and a drift $b \leq 0$, and clearly no PRI exists for \tilde{X} or X in this case. Noting that in the unbounded variation case we have $\int_0^1 \bar{\Pi}^{(-)}(s) ds = \infty$ so we necessarily have $J < \infty$, we see that these results also accord with Theorem 1. Next suppose that $\Pi(\mathbb{R}) = \infty$ but $\Pi(\mathbb{R}^-) < \infty$. If X has bounded variation, then removing all the negative jumps gives us a spectrally positive process of the form $\tilde{X}_t = X_t^+ + bt$, where X^+ is a driftless subordinator. If $b \geq 0$ then \tilde{X} has monotone paths and the assumption that $\Pi(\mathbb{R}^+) = \infty$ implies the existence of points $x_n \downarrow 0$ with $P(T(x_n) = \infty) = 1$, which verifies Theorem 1 in this case. Finally, if $b < 0$ or if X has unbounded variation then the decreasing ladder height process is a pure drift, possibly killed at an exponential time, and we see that the hypothesis of Proposition 7 below holds.

The rest of our proof uses the following simple consequence of the construction of K due to Evans, [3].

Lemma 6 *Let X be an arbitrary Lévy process and put $T_x = \inf\{t > 0 : X_t = x\}$ and $p_x = P(T_x = \infty) = P(X \text{ does not visit } x)$. Then*

(i) *a PRI exists for X if*

$$\limsup_{x \downarrow 0} \frac{1 - E(e^{-\theta T_x})}{x} < \infty \text{ for some } \theta > 0, \quad (8)$$

(ii) no PRI exists for X if

$$\lim_{x \downarrow 0} x^{-1} p_x = \infty. \quad (9)$$

Proof. First note that the sequence $K^{(n)} := T_n^{2^n}$, $n \geq 1$, where T_n^k are defined in (4), is monotone increasing. If we denote its limit by \tilde{K} , then it is immediate from (5) that $K_1 \leq \tilde{K} \leq K_2$. Since we know that K is a (possibly killed) subordinator, we see that existence of a PRI for X is equivalent to $P(\tilde{K} < \infty) > 0$. But this is equivalent to

$$\lim_{n \rightarrow \infty} E(e^{-\theta K^{(n)}}) = E(e^{-\theta \tilde{K}} : \tilde{K} < \infty) > 0$$

for some (and then all) $\theta > 0$. Since $K^{(n)}$ is the sum of 2^n independent random variables distributed as $T_{2^{-n}}$,

$$\log E(e^{-\theta \tilde{K}} : \tilde{K} < \infty) = \lim_{n \rightarrow \infty} 2^n \log E(e^{-\theta T_{2^{-n}}}),$$

and this is clearly finite for any θ for which (8) holds. Since $1 - E(e^{-\theta T_x}) \geq p_x$ we see that this limit is $-\infty$ for all $\theta > 0$ whenever (9) holds, and the result follows. ■

The crux of our proof is contained in the following result, in which $\overline{\mu}^+(x) = \mu((x, \infty))$ for $x > 0$.

Proposition 7 *Let X be a Lévy process having $\Pi(\mathbb{R}^+) = \infty$ and $U_-(dx) > 0$ for all small enough $x > 0$. Then X has a PRI iff $\delta_+ > 0$ and $I < \infty$, where*

$$I = \int_0^1 \overline{\mu}^+(x) U_-(dx) = \int_0^1 \mu^+(dx) U_-(x). \quad (10)$$

Proof. Since the existence of PRI is a local property we can truncate the Lévy measure so that it is contained in $[-1; 1]$. Indeed the first jump of X larger than 1 in absolute value occurs after an exponential time ζ and K_x is a subordinator and therefore $K_x < \zeta$ pathwise for all x small enough. This shows that the existence of K is independent of the large jumps, so we will assume, WLOG, that $\Pi([1, \infty)) = \Pi((-\infty, -1]) = 0$. Moreover the value of δ_+ is also a local property, so this is also unchanged by any alteration of the Lévy measure on closed intervals which do not contain 0. Note that our assumptions imply that $I > 0$, and that these alterations do not change the finiteness/infiniteness of I . Let us introduce some notation. For $x > 0$ we put $T_x^+ = \inf\{t > 0 : X_t > x\}$ and $T_x^- = \inf\{t > 0 : X_t < -x\}$ for the first passage times above x and below $-x$, and $O^+(x) = X_{T_x^+} - x$, $O^-(x) = x - X_{T_x^-}$ for the overshoot above x and the undershoot below $-x$. Noting that $O^+(x)$ is also the overshoot of H^+ above x , we can use Proposition 2, p76 in [1] to deduce that for $x > 0, y > 0$,

$$\begin{aligned} \overline{\mu}^{(+)}(x+y)U_+(x) &\leq P(O^+(x) > y) = \int_0^x \overline{\mu}^{(+)}(x+y-z)U_+(dz) \\ &\leq \overline{\mu}^{(+)}(y)U_+(x). \end{aligned} \quad (11)$$

To prove the result in one direction, we alter the Lévy measure by adding a mass at $\{1\}$, if necessary, to make X drift to $+\infty$. We then have the estimate

$$\begin{aligned}
p_x &\geq P(O_x^+ > 0, \text{ and } X \text{ stays above } x) \\
&= \int_0^1 P(O^+(x) \in dy) P(T_y^- = \infty) \\
&= c \int_0^1 P(O^+(x) \in dy) U_-(y) \\
&= c \int_0^1 P(O^+(x) > y) U_-(dy),
\end{aligned}$$

where the fact that $P(T_y^- = \infty) = cU_-(y)$ comes from Proposition 17, p172 of [1]. (It is obvious that in fact $c = 1/U_-(\infty)$, since $P(T_y^- = \infty) \rightarrow 1$ as $y \rightarrow \infty$.) Then from (11) it follows that

$$\begin{aligned}
\liminf_{x \downarrow 0} x^{-1} p_x &\geq c \liminf_{x \downarrow 0} x^{-1} U_+(x) \int_0^1 \bar{\mu}^{(+)}(x+y) U_-(dy) \\
&\geq cI \liminf_{x \downarrow 0} x^{-1} U_+(x).
\end{aligned}$$

Finally we recall from Proposition 1, p. 74 in [1] that $U_+(x) \approx x/(\delta_+ + \int_0^x \bar{\mu}^{(+)}(y) dy)$, so that $x^{-1} U_+(x) \approx 1/\delta_+$ as $x \downarrow 0$, and thus (9) holds and no PRI exists, whenever $\delta_+ = 0$, or $\delta_+ > 0$ and $I = \infty$. To argue in the other direction, we assume that $\delta_+ > 0$ and $I < \infty$; then without loss of generality we can take $\delta_+ = 1$. Next we denote by P^θ the law of this process killed at an independent exponential time τ with parameter θ , and note that

$$p_x^\theta := P^\theta(T_x = \infty) = P(T_x > \tau) = 1 - E(e^{-\theta T_x}).$$

Our aim is to show that $\exists \theta > 0$ such that

$$\limsup_{x \downarrow 0} x^{-1} p_x^\theta < \infty, \tag{12}$$

since then the existence of a PRI for X will follow from Lemma 6. We decompose p_x^θ according to the number of upcrossings and downcrossings of level x that occur. To do so we denote by $T^+(x, n)$ the time of n -th crossing above x , $T^-(x, n)$ the time of n -th crossing below x and for $n \geq 1$ put

$$\begin{aligned}
p_x^\theta(n) &= P^\theta\{T_x = \infty, T^+(x, n) < \infty, T^-(x, n) = \infty\}. \\
q_x^\theta(n) &= P^\theta\{T_x = \infty, T^-(x, n) < \infty, T^+(x, n+1) = \infty\}.
\end{aligned}$$

Then since X creeps upwards, it is easy to see that

$$p_x^\theta = P^\theta(T_x^+ = \infty) + \sum_1^\infty p_x^\theta(n) + \sum_1^\infty q_x^\theta(n). \tag{13}$$

We start by noting that

$$P^\theta(T_x^+ = \infty) = c^+(\theta)U_+^\theta(x), \text{ where } c^+(\theta) = \frac{1}{U_+^\theta(\infty)},$$

and $U_+^\theta(x)$ is the renewal function of the ladder height process H^+ under P^θ . Of course, under P^θ , H^+ is killed at some rate $k^+(\theta) > 0$, and has Lévy measure $\mu^+(\theta, dx) \leq \mu^+(dx)$, but as we have mentioned its drift is unchanged, and $= 1$. Using a version of Erickson's bound for killed subordinators, which can be found in [4], we therefore have

$$U_+^\theta(x) \leq \frac{c_0 x}{1 + \int_0^x \bar{\mu}^+(y, \theta) dy + x k^+(\theta)} \leq c_0 x, \quad (14)$$

where c_0 is an absolute constant. Also

$$U_+^\theta(\infty) = \lim_{y \rightarrow \infty} \int_0^\infty e^{-tk^+(\theta)} P(H_t^+ \leq y) dt = \frac{1}{k^+(\theta)},$$

and this gives the bound

$$P^\theta(T_x^+ = \infty) \leq c_0 k^+(\theta) x. \quad (15)$$

Next, using a similar notation, we see that

$$\begin{aligned} p_x^{(\theta)}(1) &= \int_0^1 P^\theta(O_+(x) \in dy) P^\theta(T_y^- = \infty) \\ &= c^-(\theta) \int_0^1 P^\theta(O_+(x) \in dy) U_-^\theta(y) \\ &= c^-(\theta) \int_0^1 P^\theta(O_+(x) > y) U_-^\theta(dy) \\ &\leq c^-(\theta) U_+^\theta(x) \int_0^1 \bar{\mu}^+(\theta, y) U_-^\theta(dy) \\ &: = c^-(\theta) I(\theta) U_+^\theta(x), \end{aligned}$$

where we have used the P^θ version of (11). Using (14) again gives the bound

$$p_x^{(\theta)}(1) \leq x c_0 c^-(\theta) I(\theta). \quad (16)$$

Then writing $O_\pm(n, x)$ for the successive overshoots upwards and downwards over level x , we have

$$p_x^{(\theta)}(n) = \int_0^1 P^\theta(O_-(n-1, x) \in dz) p_z^{(\theta)}(1) \leq c_0 c^-(\theta) I(\theta) E^\theta(O_-(n-1, x)).$$

Also Wald's identity gives $E^\theta(O^-(y)) \leq m_-^\theta U_-^\theta(y)$, where $m_-^\theta = E^\theta(H_1^-)$, and

so we have

$$\begin{aligned}
E^\theta(O_-(n-1, x)|O_-(n-2, x) = y) &= E^\theta(O^-(y)) \\
&\leq m_-(\theta) \int_0^1 P^\theta(O_y^+ \in dz) U_-^\theta(z) \\
&= m_-(\theta) \int_0^1 P^\theta(O_y^+ > z) U_-^\theta(dz) \\
&\leq m_-(\theta) U_+^\theta(y) \int_0^1 U_-^\theta(dz) \bar{\mu}^+(\theta, z) \\
&\leq c_0 m_-(\theta) I(\theta) y,
\end{aligned}$$

where we have used (11) again. Iterating this gives

$$E^\theta(O_-(n-1, x)) \leq \{c_1(\theta)\}^{n-1} x, \quad (17)$$

where $c_1(\theta) = c_0 m_-(\theta) I(\theta)$, and thus

$$p_x^{(\theta)}(n) \leq c_0 c^-(\theta) I(\theta) \{c_1(\theta)\}^{n-1} x, \quad n \geq 1.$$

Moreover, using (15) and (17) we get the bound

$$\begin{aligned}
q_x^{(\theta)}(n) &= \int_0^1 P^\theta(O_-(n, x) \in dz) P^\theta(T_z^+ = \infty) \\
&\leq c_0 k^+(\theta) E^\theta(O_-(n, x)) \leq c_0 k^+(\theta) \{c_1(\theta)\}^{n-1} x.
\end{aligned}$$

So (12) will follow provided θ can be chosen such that

$$c_1(\theta) = c_0 m_-(\theta) I(\theta) < 1. \quad (18)$$

To see this we need to note first that $m_-(\theta) \leq E(H_1^-)$. Also, provided that $k^-(\theta) \rightarrow \infty$, by applying bound (14) to H^- , we get $U_-^\theta(z) \rightarrow 0$ for each $z \in (0, 1]$ as $\theta \rightarrow \infty$, and since $U_-^\theta(z) \leq U_-(z)$ and $I < \infty$, dominated convergence will give

$$I(\theta) = \int_0^1 U_-^\theta(z) \mu^+(\theta, dz) \leq \int_0^1 U_-(z) \mu^+(dz) \rightarrow 0 \text{ as } \theta \rightarrow \infty.$$

To see that $k^-(\theta) \rightarrow \infty$ note that the killing time of H^- under P^θ is the same as that of the ladder time subordinator L_-^{-1} , and this has the distribution of $L_-(\tau)$, which is $\exp(\kappa_-(\theta))$, where κ_- is the Laplace exponent of L_- under P . The assumption that $U_-(dx) > 0$ for all small $x > 0$ implies that L_- is not a compound Poisson process, so by Corollary 3, p 17 of [1], $\kappa_-(\infty) = \infty$, and thus if we choose θ large enough, (18) will hold, and the proof is complete. ■

Proposition 8 (i) Let X be a Lévy process having $\Pi(\mathbb{R}^+) = \infty$ and $\sigma > 0$: then a PRI exists.

(ii) Let X be a Lévy process having $\sigma = 0$, $\Pi(\mathbb{R}^+) = \infty$, and $\Pi(\mathbb{R}^-) < \infty$; then no PRI exists .

Proof. (i) Here $\delta_+ > 0$ and $\delta_- > 0$, so $U_-(x) \sim x/\delta_-$ and since $\int_0^1 x\mu^+(dx)$ is automatically finite we have $I < \infty$.

(ii) By the argument preceding Lemma 6 we can take $\Pi(\mathbb{R}^-) = 0$ and assume that $\delta_- > 0$, so that again I is necessarily finite. But $\sigma = 0$ and $\delta_- > 0$ imply $\delta_+ = 0$, and the result follows. ■

To deal with the remaining situations, we need

Lemma 9 *Let X be an oscillating Lévy process whose Lévy measure is supported by $[-1, 1]$ and satisfies $\Pi([-1, 0]) = \Pi((0, 1]) = \infty$. Suppose additionally $\sigma = 0$ and $\delta^+ > 0$. Then $I = \int_0^1 \bar{\mu}^+(x)U_-(dx) < \infty$ iff*

$$J = \int_0^1 \frac{x^2\Pi(dx)}{\left(\int_0^x \int_y^1 \bar{\Pi}^{(-)}(s)dsdy\right)^2} < \infty. \quad (19)$$

Proof. We use Vigons' équation amicale inversée, see [4], which, since our Lévy measure lives on $[-1; 1]$, takes the form

$$\bar{\mu}^+(x) = \int_0^\infty \bar{\Pi}^+(x+y)U_-(dy) = \int_x^1 U_-(y-x)\Pi(dy).$$

Then we use this in the following computation

$$\begin{aligned} I &= \int_0^1 \bar{\mu}^+(x)U_-(dx) < \infty = \int_0^1 \int_x^1 U_-(y-x)\Pi(dy)U_-(dx) \\ &= \int_0^1 \Pi(dy) \int_0^y U_-(y-x)U_-(dx) = \int_0^1 U_-^{*2}(y)\Pi(dy). \end{aligned}$$

Next we recall that the potential function $U_-(x)$ is increasing in x . This is enough to show that

$$(U_-(y/2))^2 \leq U_-^{*2}(y) = \int_0^y U_-(y-x)U_-(dx) \leq (U_-(y))^2.$$

Moreover since X oscillates, H_- is an unkilled subordinator with zero drift, and we have that $U_-(y) \approx y/A(y)$, where $A(y) = \int_0^y \bar{\mu}^{(-)}(s)ds$ satisfies $A(y)/2 \leq A(y/2) \leq A(y)$. This implies that $U_-(y) \approx U_-(y/2)$ and therefore $U_-^{*2}(y) \approx (U_-(y))^2$. Therefore we conclude that

$$I = \int_0^1 U_-^{*2}(y)\Pi(dy) < \infty \iff \int_0^1 \frac{y^2\Pi(dy)}{A(y)^2} < \infty.$$

Next we need the équation amicale intégrée of Vigon, see [4], which in our case takes the form

$$\bar{\bar{\Pi}}^{(-)}(x) = \int_x^1 \bar{\Pi}^{(-)}(y)dy = \int_0^1 \bar{\mu}^{(+)}(y)\bar{\mu}^{(-)}(x+y)dy + \delta_+\bar{\mu}^{(-)}(x).$$

Our assumptions imply that $\overline{\overline{\Pi}}^{(-)}(0+) > 0$. If $\overline{\overline{\Pi}}^{(-)}(0+) < \infty$, it is obvious that $0 < \overline{\mu}^{(-)}(0+) < \infty$, and if $\overline{\overline{\Pi}}^{(-)}(0+) = \infty$ it is easy to deduce that $\overline{\mu}^{(-)}(0+) = \infty$, and then from dominated convergence that

$$\lim_{x \downarrow 0} \frac{\overline{\overline{\Pi}}^{(-)}(x)}{\overline{\mu}^{(-)}(x)} = \delta_+.$$

Thus, in both cases $A(y) \approx \int_0^y \overline{\overline{\Pi}}^{(-)}(z) dz$, and the result follows. ■

Proof of Theorem 1. We have already covered all cases except those having $\sigma = 0$ and $\Pi(\mathbb{R}^+) = \Pi(\mathbb{R}^-) = \infty$. By the standard argument we can find another process \tilde{X} which oscillates and whose Lévy measure $\tilde{\Pi}$ agrees with Π on $(-1, 1)$ and is supported by $[-1, 1]$, and is such that a PRI exists for X iff a PRI exists for \tilde{X} . Note that $\tilde{\Pi}([-1, 0)) = \tilde{\Pi}((0, 1]) = \infty$, and that, in the obvious notation, $\tilde{J} < \infty$ iff $J < \infty$. Then Proposition 7 and Lemma 9 apply and show that a PRI exists iff $\delta_+ > 0$ and $J < \infty$. If X has bounded variation, then $\overline{\overline{\Pi}}^{(-)}(0+) \in (0, \infty)$, and then $J = \infty$ is automatic. If X has unbounded variation, as previously noted, $J < \infty$ implies $\delta_+ > 0$, and this completes the proof.

Proof of Corollary 4. Since $\overline{\overline{\Pi}}^-(x) \approx x^{-\alpha}$, where $1 < \alpha < 2$, we are in the unbounded variation case, and we need only check the value of the integral (6). Clearly $\int_0^x \int_y^1 \overline{\overline{\Pi}}^{(-)}(s) ds dy \approx x^{2-\alpha}$ so this reduce to checking whether

$$\int_0^1 x^{2\alpha-2} \Pi(dx) = (2\alpha - 2) \int_0^1 x^{2\alpha-3} \overline{\overline{\Pi}}^+(x) dx < \infty,$$

and this holds iff $\beta < 2\alpha - 2$.

Remark 10 A similar calculation for the integral L in (7) shows that in this example X creeps upwards iff $\beta < \alpha$.

3 The excursion measure

Evans [3] and Winkel [5] both observed that we can associate an excursion theory with K .

They introduced $\Lambda_t = \inf\{x : K_x > t\}$, $Z = X - \Lambda$, and showed that Z is a strong Markov process with Λ as a local time at zero. It is clear that excursions away from 0 of Z evolve in the same way as excursions away from 0 of X , viz they have the same semigroup, but their entrance laws will be different. For example, if $X = B$, all excursions of Z are negative, and the characteristic measure n^Z is n^X restricted to negative excursion paths.

Winkel showed that when $\sigma > 0$, n^Z is the restriction of n^X to the set of excursion paths which **start** negative. (To do this he had to demonstrate that all excursion paths either start negative or start positive, i.e. cannot leave 0 in an oscillatory fashion.) So n^Z is absolutely continuous wrt n^X .

However, this depends on both δ_+ and δ_- being positive. When $\sigma = 0$ and $\delta_+ > 0$, we have $\delta_- = 0$, which means that excursions of X have to return to 0 from below. By time-reversal, this means they must start positive, and since excursions of Z start negative, the two measures must be mutually singular whenever $\sigma = 0$. We believe that the problem of describing the excursion measure n^Z in this case is both interesting and difficult.

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