Submitted to the Annals of Probability
arXiv: math.PR/0812.2399

GIBBSIANNESS AND NON-GIBBSIANNESS IN DIVIDE AND COLOUR MODELS

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For parameters $p \in [0,1]$ and $q > 0$ such that the random-cluster measure $\Phi^{\mathbb{Z}^d}_{p,q}$ for $\mathbb{Z}^d$ with parameters p and q is unique, the q-divide and colour (DaC(q)) model on $\mathbb{Z}^d$ is defined as follows. First we draw a bond configuration with distribution $\Phi^{\mathbb{Z}^d}_{p,q}$. Then to each FK cluster (i.e., to every vertex in the FK cluster), independently for different FK clusters, we assign a spin value from the set $\{1, 2, \ldots, s\}$ in such a way that spin i has probability a_i.

In this paper we prove that the resulting measure on spin configurations is a Gibbs measure for small values of p, and is not a Gibbs measure for large p, except in the special case of $q \in \{2, 3, \ldots\}$, $a_1 = a_2 = \ldots = a_s = 1/q$, when the DaC($q$) model coincides with the q-state Potts model.

1. Introduction.

The random-cluster representations of various models have played an important role in the study of physical systems and phase transitions. They provide a different viewpoint of the physical models, and many problems in the Ising and Potts models can indeed be solved by using their random-cluster representations, see, e.g., [10, 15, 11].

For $\beta \geq 0$ and an integer $q \geq 2$, a spin configuration in the q-state Potts model at inverse temperature β can be obtained as follows. Draw a bond configuration according to a random-cluster measure with parameters $p = 1 - e^{-2\beta}$ and q (for definitions, see Section 2), then assign to each vertex a spin from the set $\{1, 2, \ldots, q\}$ in such a way that all spins have equal probability and that vertices that are connected in the bond configuration get the same spin. If the spin is chosen from a set $\{1, 2, \ldots, s\}$ with an integer $1 < s < q$, and the probability of spin i is k_i/q with positive integers $k_1, k_2, \ldots, k_s$ such that $\sum_{i=1}^{s} k_i = q$, then we get the so-called fuzzy Potts model [25, 18]. Recent papers (see [16, 17, 9, 21, 1, 2, 12]) have shown that generalisations of the above constructions with different values of q and s, and more general rules of spin assignment are also of interest. From a mathematical viewpoint, such models are natural examples of a dependent site percolation model with a

AMS 2000 subject classifications: 60K35, 82B20, 82B43

Keywords and phrases: DaC models, Gibbs measures, non-Gibbsianness, quasilocality, random-cluster measures

simple definition but nontrivial behaviour. The study of such models may also lead to a better understanding of models of primary physical importance, as it was the case in [2] where an informative, new perspective of the high temperature Ising model on the triangular lattice was given.

The model treated here is defined as follows. Let $G = (\mathcal{V}, \mathcal{E})$ be a (finite or infinite) locally finite graph. Fix parameters $p \in [0,1], q > 0$ in such a way that there exists exactly one random-cluster measure for G with parameters p and q. We denote this measure by $\Phi^G_{p,q}$. Fix also an integer $s \geq 2$, and $a_1, a_2, \ldots, a_s \in (0,1)$ such that $\sum_{i=1}^s a_i = 1$, and define the **single-spin space** $S = \{1, 2, \ldots, s\}$, and the **state space** $\Omega^G = \Omega^G_C \times \Omega^G_D$ with $\Omega^G_C = S^{\mathcal{V}}$ and $\Omega^G_D = \{0,1\}^{\mathcal{E}}$. Let Y be a random bond configuration taking values in Ω^G_D with distribution $\Phi^G_{p,q}$. Given $Y = \eta$ for some $\eta \in \Omega^G_D$, we construct a random Ω^G_C-valued spin configuration X by assigning spin $i \in S$ with probability a_i to each connected component in η (i.e., the same spin i to each vertex in the component), independently for different components. We write $\mathbb{P}^G_{p,q,(a_1,a_2,\ldots,a_s)}$ for the joint distribution of (X,Y) on Ω^G, and $\mu^G_{p,q,(a_1,a_2,\ldots,a_s)}$ for the marginal of $\mathbb{P}^G_{p,q,(a_1,a_2,\ldots,a_s)}$ on Ω^G_C. This definition is a slight generalisation of the fractional fuzzy Potts model defined in [16], p.1156 (see also [1], Section 1.2). However, we shall call this model the **q-divide and colour (DaC(q)) model** to emphasise that we look at it rather as a generalisation of the model introduced in [17] by Häggström (which is the DaC(1) model in the present terminology) than of the fuzzy Potts model of [25, 18].

Let us now consider the (hypercubic) lattice with vertex set $\mathbb{Z}^d$ and edge set $\mathcal{E}^d$ with edges between vertices at Euclidean distance 1. With an abuse of notation, we shall denote this graph by $\mathbb{Z}^d$, and the sets $\Omega^{\mathbb{Z}^d}_D, \Omega^{\mathbb{Z}^d}_C$ and $\Omega^{\mathbb{Z}^d}$ by Ω_D, Ω_C and Ω, respectively. The present work is focused on the Gibbs properties and k-Markovianness of the measure $\mu^{\mathbb{Z}^d}_{p,q,(a_1,a_2,\ldots,a_s)}$ in $d \geq 2$ dimensions. Since the cases $p = 0$ or 1 are trivial, we henceforth assume that $p \in (0,1)$. We give results for $q \geq 1$ only, since much more is known about random-cluster measures with $q \geq 1$ than with $q < 1$.

We shall prove that, except in the special case of $q = s$ and $a_1 = a_2 = \ldots = a_s$ (when the DaC(q) model coincides with the q-state Potts model on $\mathbb{Z}^d$ at inverse temperature $\beta = -1/2 \log(1-p)$), the DaC(q) model is not k-Markovian for any k. For large values of p, $\mu^{\mathbb{Z}^d}_{p,q,(a_1,a_2,\ldots,a_s)}$ is not even quasilocal and is therefore not a Gibbs measure, again with the exception of the Potts case. This shows that the Gibbsianness of the Potts model at low temperatures is very sensitive with respect to perturbations in the assignment of the spin probabilities, even the smallest change makes the model non-quasilocal. By demonstrating the special role of the q-state Potts Gibbs measure among DaC(q) models, our result supports the view expressed in

[5, 6] that, especially at low temperatures, Gibbsianness of measures is rather the exception than the rule. For a related general result, see [20], where Israel proved that in the set of all translation-invariant measures Gibbsianness is exceptional in a topological sense. However, if p is small enough, then Gibbsianness does hold. The proof of this fact uses the idea that at small values of p (which corresponds to high temperatures in the (fuzzy) Potts model), the DaC(q) model is close in spirit to independent site percolation on $\mathbb{Z}^d$.

These results are in line with those in [18] (see also [25] and [19]) concerning the fuzzy Potts model, and in some cases, essentially the same proofs work in the current, more general situation. Therefore, at places only a sketch of the proof is given, and the reader is referred to [18] for the details. Note, however, that such similarities are not immediate from the definition of the models. More importantly, in the DaC(q) model, a distinction must be made between the case when $a_i \geq 1/q$ for all i, and when there exists j with $a_j < 1/q$. In the former (of which the fuzzy Potts model is a special case), a rather complete picture can be given, whereas in the latter, there is an interval in p where we do not know whether $\mu^{\mathbb{Z}^d}_{p,q,(a_1,a_2,\ldots,a_s)}$ is a Gibbs measure.

Finally, we give a sufficient (but not necessary) condition for the almost sure quasilocality of $\mu^{\mathbb{Z}^d}_{p,q,(a_1,a_2,\ldots,a_s)}$, and as an application, we obtain this weak form of Gibbsianness in the two-dimensional case for a large range of parameters. Some intuition behind our main results will be given after Remark 3.8.

2. Definitions and main results.

2.1. *Random-cluster measures.* In this section, we recall the definition of the Fortuin-Kasteleyn (FK) random-cluster measures, and those properties of these measures that are important for the rest of the paper. For the proofs and much more on random-cluster measures, see, e.g., [14].

DEFINITION 2.1. *For a finite graph $G = (\mathcal{V}, \mathcal{E})$ and parameters $p \in [0,1]$ and $q > 0$, the* ***random-cluster measure*** $\Phi^G_{p,q}$ *is the measure on Ω^G_D which assigns to a bond configuration $\eta \in \Omega^G_D$ probability*

$$\Phi^G_{p,q}(\eta) = \frac{q^{k(\eta)}}{Z^G_{p,q}} \prod_{e\in\mathcal{E}} p^{\eta(e)}(1-p)^{1-\eta(e)}, \tag{1}$$

where $k(\eta)$ is the number of connected components in the graph with vertex set $\mathcal{V}$ and edge set $\{e \in \mathcal{E} : \eta(e) = 1\}$ (we call such components ***FK clusters*** *throughout, edges with state 1* ***open****, and edges with state 0* ***closed****), and $Z^G_{p,q}$ is the appropriate normalising factor.*

This definition is not suitable for infinite graphs. In that case, we shall require that certain conditional probabilities are the same as in the finite case. The relevant definition, given below, will formally contain conditioning on an event with probability 0, which should be understood as conditioning on the appropriate σ-algebra. We shall frequently use this simplification in order to keep the notation as simple as possible.

A graph is called **locally finite** if every vertex has a bounded degree. We shall denote bond configurations throughout by η and ζ. For the restriction of a bond configuration η to an edge set H, we write η_H. For vertices v and w, we denote the edge between v and w by $\langle v, w\rangle$. The following definition is taken from [18], and its equivalence with a more common definition (where arbitrary finite edge sets and not only single edges are considered) is stated, e.g., in Lemma 6.18 of [11].

Definition 2.2. *For an infinite, locally finite graph $G = (\mathcal{V}, \mathcal{E})$ and parameters $p \in [0,1], q > 0$, a measure ϕ on Ω_D^G is called a* ***random-cluster measure for G with parameters p and q*** *if for each edge $e = \langle x, y\rangle \in \mathcal{E}$ and edge configuration $\zeta \in \{0,1\}^{\mathcal{E}\setminus\{e\}}$ outside e, we have that*

$$\phi(\{\eta \in \Omega_D^G : \eta(e) = 1\} \mid \{\eta \in \Omega_D^G : \eta_{\mathcal{E}\setminus\{e\}} = \zeta\}) = \begin{cases} p & \text{if } x \overset{\zeta}{\leftrightarrow} y, \\ \frac{p}{p+(1-p)q} & \text{otherwise,} \end{cases}$$

where $x \overset{\zeta}{\leftrightarrow} y$ denotes that there exists a path of edges between x and y in which every edge has ζ-value 1.

It is not difficult to prove that one gets the same conditional probabilities for random-cluster measures on finite graphs, hence Definition 2.2 is a reasonable extension of Definition 2.1 to infinite graphs.

It is not clear from the definition that such measures exist. However, for $\mathbb{Z}^d$ and for $q \geq 1$, two random-cluster measures can be constructed as follows. For a vertex set $H \subset \mathbb{Z}^d$, let ∂H denote the **vertex boundary** of the set, that is, $\partial H = \{v \in \mathbb{Z}^d \setminus H : \exists w \in H \text{ such that } \langle v, w\rangle \in \mathcal{E}^d\}$. Define, for $n \in \{1, 2, \ldots\}$, the set $\Lambda_n = \{-n, \ldots, n\}^d$, the graph $G_n = (\mathcal{V}_n, \mathcal{E}_n)$ with vertex set $\mathcal{V}_n = \Lambda_n \cup \partial\Lambda_n$ and edge set $\mathcal{E}_n = \{e \in \mathcal{E}^d : \text{both endvertices of } e$ are in $\mathcal{V}_n\}$. For $n \in \{1, 2, \ldots\}$, let W_n be the event that all edges with both endvertices in $\partial\Lambda_n$ are open, and let $\Phi_{p,q}^{G_n,1}$ be the measure $\Phi_{p,q}^{G_n}$ conditioned on W_n. Then both $\Phi_{p,q}^{G_n}$ and $\Phi_{p,q}^{G_n,1}$ converge weakly as $n \to \infty$; we denote the limiting measures by $\Phi_{p,q}^{\mathbb{Z}^d,0}$ and $\Phi_{p,q}^{\mathbb{Z}^d,1}$, respectively. $\Phi_{p,q}^{\mathbb{Z}^d,0}$ is called the **free**, and $\Phi_{p,q}^{\mathbb{Z}^d,1}$ is called the **wired** random-cluster measure for $\mathbb{Z}^d$ with parameters p and q. These measures are indeed random-cluster measures in the sense of Definition 2.2, moreover, they are extremal among such measures in the following sense.

A natural partial order on the set $\Omega_D = \{0,1\}^{\mathcal{E}^d}$ of edge configurations is given by defining $\eta' \geq \eta$ for $\eta, \eta' \in \Omega_D$ if for all $e \in \mathcal{E}^d$, $\eta'(e) \geq \eta(e)$. We call a function $f : \Omega_D \to \mathbb{R}$ **increasing** if $\eta' \geq \eta$ implies that $f(\eta') \geq f(\eta)$. For probability measures ϕ, ϕ' on Ω_D, we say that ϕ' is **stochastically larger** than ϕ if for all bounded increasing measurable functions $f : \Omega_D \to \mathbb{R}$, we have that

$$\int_{\Omega_D} f(\eta)\ d\phi'(\eta) \geq \int_{\Omega_D} f(\eta)\ d\phi(\eta).$$

For later purposes we mention that by Strassen's theorem [28] this is equivalent to the existence of an appropriate coupling of the measures ϕ' and ϕ, that is, the existence of a probability measure Q on $\Omega_D \times \Omega_D$ such that the marginals of Q on the first and second coordinates are ϕ' and ϕ respectively, and $Q(\{(\eta', \eta) \in \Omega_D \times \Omega_D : \eta' \geq \eta\}) = 1$.

It is well-known that $\Phi^{\mathbb{Z}^d,0}_{p,q}$ is the stochastically smallest, and $\Phi^{\mathbb{Z}^d,1}_{p,q}$ is the stochastically largest random-cluster measure for $\mathbb{Z}^d$ with parameters p and q. Therefore, there exists a unique random-cluster measure for $\mathbb{Z}^d$ with parameters p and q if and only if

$$\Phi^{\mathbb{Z}^d,0}_{p,q} = \Phi^{\mathbb{Z}^d,1}_{p,q}. \tag{2}$$

This is the case for any fixed $q \geq 1$, except possibly for at most countably many values of p. It is widely believed that for any $q \geq 1$, there is at most one exceptional p, which can only be the **critical value** $p_c(q,d) = \sup\{p : \Phi^{\mathbb{Z}^d,0}_{p,q}(\{\eta \in \Omega_D : \mathbf{0} \text{ is in an infinite FK cluster in } \eta\}) = 0\}$, where $\mathbf{0}$ denotes the origin in $\mathbb{Z}^d$. It is not difficult to show that the choice of $\Phi^{\mathbb{Z}^d,0}_{p,q}$ in the definition is not crucial. That is, for any random-cluster measure ϕ for $\mathbb{Z}^d$ with parameters p and q, we have that

$$\phi(\{\eta \in \Omega_D : \mathbf{0} \text{ is in an infinite FK cluster in } \eta\}) \begin{cases} = 0 & \text{if } p < p_c(q,d), \\ > 0 & \text{if } p > p_c(q,d). \end{cases}$$

For the rest of the paper, we shall assume without further mentioning that parameters d, p, q for the DaC(q) model on $\mathbb{Z}^d$ are always chosen in such a way that (2) holds, and denote the unique random-cluster measure by $\Phi^{\mathbb{Z}^d}_{p,q}$.

Another important feature of the random-cluster measures $\Phi^{\mathbb{Z}^d,0}_{p,q}$ and $\Phi^{\mathbb{Z}^d,1}_{p,q}$ with $q \geq 1$ is that they satisfy the FKG inequality for increasing events [8] (an event $A \subset \Omega_D$ is called **increasing** if its indicator function is increasing, i.e., if $\eta \in A$ and $\eta' \geq \eta$ implies that $\eta' \in A$). This in particular means that for $d \geq 2, p \in [0,1], q \geq 1$, for any edge set $E \subset \mathcal{E}^d$, configuration

$\zeta \in \{0,1\}^E$ on E, and increasing events $A_1, A_2 \subset \Omega_D$, we have denoting $B = \{\eta \in \Omega_D : \eta_E = \zeta\}$ that

$$\Phi^{\mathbb{Z}^d,0}_{p,q}(A_1 \cap A_2 \mid B) \geq \Phi^{\mathbb{Z}^d,0}_{p,q}(A_1 \mid B)\Phi^{\mathbb{Z}^d,0}_{p,q}(A_2 \mid B). \tag{3}$$

Finally, for our main result, we need to consider the critical value in half-spaces as well. Let $\mathcal{H}^+ = \mathcal{H}^+_d$ denote the subset of $\mathbb{Z}^d$ which consists of those vertices whose first coordinate is strictly positive, $\tilde{E} \subset \mathcal{E}^d$ the set of edges that are incident to at least one vertex in $\mathbb{Z}^d \setminus \mathcal{H}^+$, and denote the vertex $(1,0,0,\ldots,0) \in \mathbb{Z}^d$ by u_1. Consider also the event $A_{\mathcal{H}^+} = \{\eta \in \Omega_D : u_1$ is in an infinite open path in η which is contained in $\mathcal{H}^+\}$. For $q \geq 1$, we define $p_c^{\mathcal{H}}(q,d) = \sup\{p : \Phi^{\mathbb{Z}^d,0}_{p,q}(A_{\mathcal{H}^+} \mid \{\eta \in \Omega_D : \eta_{\tilde{E}} \equiv 0\}) = 0\}$.

Using (3), it is easy to see that $p_c^{\mathcal{H}}(q,d) \geq p_c(q,d)$. Equality of the two critical values for $q = 1$ was proved by Barsky, Grimmett and Newman [3], for $q = 2$ by Bodineau [4], and for very large values of q, it follows from the Pirogov-Sinai theory (see the last paragraph of Section 2.3 in [4]). For general $q \geq 1$, equality has been conjectured ([27],[18],[4],[29]), but no definite answer is known thus far. However, an upper bound $p_c^{\mathcal{H}}(q,d) \leq \frac{p_c(1,d)q}{p_c(1,d)q+1-p_c(1,d)}$ can be given easily, using that for $q \geq 1$, $\Phi^{\mathbb{Z}^d,0}_{p,q}$ conditioned on $\{\eta \in \Omega_D : \eta_{\tilde{E}} \equiv 0\}$ is stochastically larger on $\mathcal{E}^d \setminus \tilde{E}$ than $\Phi^{\mathbb{Z}^d,0}_{\frac{p}{p+(1-p)q},1}$ and that $p_c^{\mathcal{H}}(1,d) = p_c(1,d)$. Note that $p_c(1,d)$ is the critical value for Bernoulli bond percolation on $\mathbb{Z}^d$. It is well-known (see, e.g., [13]) that for all $d \geq 2$, $0 < p_c(1,d) < 1$. This implies that the above upper bound for $p_c^{\mathcal{H}}(q,d)$ is nontrivial.

2.2. *Main results.* Before stating the main results, let us give the relevant definitions. In this section, μ denotes a probability measure on $\Omega_C = S^{\mathbb{Z}^d}$. Spin configurations will be denoted throughout by ξ, σ, and κ, and the restriction of a spin configuration ξ to a vertex set W by ξ_W. For a set $W \subset \mathbb{Z}^d$ and a spin configuration $\sigma \in S^W$ on W, we denote $K_W^\sigma = \{\xi \in \Omega_C : \xi_W = \sigma\}$. We shall use $A \subset\subset B$ to denote that "A **is a finite subset of** B" throughout. We denote the graph-theoretic distance on $\mathbb{Z}^d$ by $dist$, and define the **distance** of a vertex $v \in \mathbb{Z}^d$ and a vertex set $H \subset \mathbb{Z}^d$ by $dist(v,H) = \min\{dist(v,w) : w \in H\}$. For $k \in \{1,2,\ldots\}$, let $\partial_k H$ denote the k**-neighbourhood** of H, that is, $\partial_k H = \{v \in \mathbb{Z}^d : 1 \leq dist(H,v) \leq k\}$. Note that $\partial_1 H = \partial H$.

We usually want to view the DaC(q) model as a dependent spin model on $\mathbb{Z}^d$, in which the only role of the edge configuration is to introduce the dependence. One of the first natural questions one may ask about a spin model is whether the finite energy property of [26] holds. This turns out to be the case, moreover, we can even prove a stronger form of it, called uniform nonnullness. The proofs of all statements in this section will be given in Section 4.

DEFINITION 2.3. *μ is called* ***uniformly nonnull*** *if there exists an $\varepsilon > 0$ such that for all $v \in \mathbb{Z}^d$, $m \in S$, $\sigma \in S^{\mathbb{Z}^d \setminus \{v\}}$, we have that*

$$\mu(K^m_{\{v\}} \mid K^\sigma_{\mathbb{Z}^d \setminus \{v\}}) \geq \varepsilon.$$

PROPOSITION 2.4. *For all $d \in \{1, 2, \ldots\}, q \geq 1, p \in [0, 1)$, and arbitrary values of the other parameters, the measure $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ is uniformly nonnull.*

The concept of k-Markovianness is concerned with the following question: conditioning on a spin configuration outside a set W, do vertices farther than k from W have any influence on the spin configuration in W?

DEFINITION 2.5. *For $k \in \{1, 2, \ldots\}$, μ is called* ***k-Markovian*** *if for all $W \subset\subset \mathbb{Z}^d$, $\kappa \in S^W$ and $\sigma, \sigma' \in S^{\mathbb{Z}^d \setminus W}$ such that $\sigma_{\partial_k W} = \sigma'_{\partial_k W}$, we have that*

$$\mu(K^\kappa_W \mid K^\sigma_{\mathbb{Z}^d \setminus W}) = \mu(K^\kappa_W \mid K^{\sigma'}_{\mathbb{Z}^d \setminus W}).$$

A weaker notion is that of quasilocality, where the above conditional probabilities do not need to be equal for any k, just their difference is required to tend to 0 as $k \to \infty$. Due to the compactness of $S^{\mathbb{Z}^d}$ in the product topology, this amounts to the following.

DEFINITION 2.6. *μ is called* ***quasilocal*** *if for all $W \subset\subset \mathbb{Z}^d$, $\kappa \in S^W$ and $\sigma \in S^{\mathbb{Z}^d \setminus W}$, we have that*

$$\lim_{k \to \infty} \sup_{\substack{\sigma' \in \mathbb{Z}^d \setminus W \\ \sigma'_{\partial_k W} = \sigma_{\partial_k W}}} |\mu(K^\kappa_W \mid K^\sigma_{\mathbb{Z}^d \setminus W}) - \mu(K^\kappa_W \mid K^{\sigma'}_{\mathbb{Z}^d \setminus W})| = 0.$$

If the above equation holds for μ-almost all $\sigma \in S^{\mathbb{Z}^d \setminus W}$, then μ is called ***almost surely quasilocal****.*

Finally, we need to say what we mean by Gibbsianness. Instead of the usual definition with absolutely summable interaction potentials (see, e.g., [10, 6]), we shall use a well-known characterisation (see [6], Theorem 2.12), namely that μ is a **Gibbs measure** if and only if it is quasilocal and uniformly nonnull.

We are now ready to state our main result concerning k-Markovianness and Gibbsianness of the DaC(q) model. The cases $p = 0, 1$ are trivial, therefore we assume that $p \in (0, 1)$. For fixed q, s, and $a_1, \ldots, a_s$, recall that

$S = \{1, 2, \ldots, s\}$, and define $S_{1/q} = \{i \in S : a_i = 1/q\}$. The case $S = S_{1/q}$ is well understood since $S = S_{1/q}$ implies that $s = q$ and $a_1 = a_2 = \ldots = a_s$, in which case the procedure defining the DaC(q) model gives the random-cluster representation of the Potts model. Therefore, for all $p \in (0, 1)$, $\mu^{\mathbb{Z}^d}_{p,q,(1/q,1/q,\ldots,1/q)}$ equals a Gibbs measure for the q-state Potts model on $\mathbb{Z}^d$ (at inverse temperature $\beta = -1/2 \log(1-p)$). It follows immediately from the standard definition of Potts Gibbs measures with a Hamiltonian (see e.g. [11] for the definition) that all such measures are Markovian (i.e., 1-Markovian). For an alternative proof of the Markovianness of $\mu^{\mathbb{Z}^d}_{p,q,(1/q,1/q,\ldots,1/q)}$, see Remark 3.8. If $S \neq S_{1/q}$, let $\ell \in S$ be an (for concreteness, the smallest) index such that $a_\ell = \min\{a_i : i \in S \setminus S_{1/q}\}$.

THEOREM 2.7. *Assume that $d \geq 2, q \geq 1$, and that $S \neq S_{1/q}$. Then we have the following.*

1. *For any values of $p, a_1, \ldots, a_s \in (0, 1)$, the measure $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ is not k-Markovian for any $k \in \{1, 2, \ldots\}$.*
2. *If $a_\ell > 1/q$, then*
   - *(a) for $p < p_c(qa_\ell, d)$, $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ is quasilocal, but*
   - *(b) for $p > p_c^{\mathcal{H}}(qa_\ell, d)$, it is not quasilocal.*
3. *If $a_\ell < 1/q$, then*
   - *(a) if $p < \frac{p_c(1,d)qa_\ell}{p_c(1,d)qa_\ell+1-p_c(1,d)}$, then $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ is quasilocal, whereas*
   - *(b) if $p > p_c(1, d)$, it is not.*

Combining Theorem 2.7 with Theorem 2.4 and the earlier mentioned characterisation of Gibbs measures, we conclude the following.

COROLLARY 2.8. *If $S = S_{1/q}$ and in cases 2(a) and 3(a) of Theorem 2.7, the measure $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ is a Gibbs measure, and in cases 2(b) and 3(b) of Theorem 2.7, it is not a Gibbs measure.*

To demonstrate the fundamental difference between the q-state Potts model and other DaC(q) models, let us consider the case with $s = q - 1$ and $a_1 = a_2 = \ldots = a_s = \frac{1}{q-1}$. Intuitively, for very large values of q, the difference between this scenario and the case when $S = S_{1/q}$ should vanish. Nevertheless, while $\mu^{\mathbb{Z}^d}_{p,q,(1/q,1/q,\ldots,1/q)}$ is a Gibbs measure for any p and q, Corollary 2.8 gives that there exists a constant $c = c(d) \in (0, 1)$ such that for all $q \in \{3, 4, \ldots\}$ and $p > c$, $\mu^{\mathbb{Z}^d}_{p,q,(\frac{1}{q-1},\frac{1}{q-1},\ldots,\frac{1}{q-1})}$ is not a Gibbs measure.

This result might seem to contradict Theorem 2.9 in [24], which implies that any sufficiently fine local coarse graining preserves the Gibbs property of the q-state Potts model. Note, however, that an arbitrarily fine coarse graining is available only when the local state space is continuous, which is not the case here.

The question whether quasilocality is "seriously" violated in cases when $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ is not a Gibbs measure (i.e., whether "bad" configurations are exceptional or they actually occur) is related to that of percolation by the following statement, which is a generalisation of Proposition 3.7 in [17].

PROPOSITION 2.9. *Consider the event* $E_\infty = \{\xi \in \Omega_C : \xi$ *contains an infinite connected component of equal spins*$\}$. *If the parameters* $p \in [0,1], q \geq 1, s \in \{2,3,\ldots\}$, *and* $a_1,\ldots,a_s \in (0,1)$ *of the DaC(q) model are chosen in such a way that*

$$\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}(E_\infty) = 0, \tag{4}$$

then $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ *satisfies almost sure quasilocality.*

It is easy to see that (4) is not a necessary condition for almost sure quasilocality. For instance, one can take $d \geq 2, q \geq 1, p = 0, s = 2$, and an $a_1 < 1$ which is greater than the critical value for Bernoulli *site* percolation on $\mathbb{Z}^d$. Then, although (4) fails, $\mu^{\mathbb{Z}^d}_{0,q,(a_1,a_2)}$ is Markovian (and therefore obviously almost surely quasilocal). Despite this, Proposition 2.9 is not useless. We shall demonstrate this below by giving an application in the two-dimensional case. Häggström's results in Section 3 of [16] imply that for $d = 2, q \geq 2, p < p_c(q,d)$, if $a_i \leq 1/2$ for all $i \in S$, then (4) holds. Using the main result in [21], this can be extended to $d = 2, q \geq 1, p < p_c(q,d)$ with the same proof. Combining this with Proposition 2.9, we obtain almost sure quasilocality when $d = 2$ for these parameters.

COROLLARY 2.10. *If* $q \geq 1, p < p_c(q,2)$, *and* $a_i \leq 1/2$ *for all* $i \in S$, *then* $\mu^{\mathbb{Z}^2}_{p,q,(a_1,\ldots,a_s)}$ *is almost surely quasilocal.*

3. Useful tools.

Here we collect the lemmas needed for the proofs of the results in Section 2.2. The statements of the most important ones, Lemma 3.3 and Corollary 3.7, are proved for finite graphs first, then a limit is taken. We shall have an appropriate limiting procedure for $q \geq 1$ only, and this is the reason why we need to restrict to this case in all our results. Throughout this and the next section, we shall use the following notations. For a set $W \subset \mathbb{Z}^d$ and a spin configuration $\sigma \in S^W$ on W, we denote $C^\sigma_W = \{(\xi,\eta) \in$

$\Omega : \xi_W = \sigma\}$. Analogously, for $E \subset \mathcal{E}^d$ and a bond configuration $\zeta \in \{0,1\}^E$ on E, we denote $D_E^\zeta = \{(\xi,\eta) \in \Omega : \eta_E = \zeta\}$.

For fixed parameters $s \in \{2,3,\ldots\}, p, a_1, a_2, \ldots, a_s$, and $q \geq 1$, the measure $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ can be obtained as a limit as follows. Let $G_n = (\mathcal{V}_n, \mathcal{E}_n)$ be as in Section 2.1. Consider the DaC(q) model on G_n with the given parameters as defined in the introduction. Then the corresponding sequence of measures $\mathbb{P}^{G_n}_{p,q,(a_1,\ldots,a_s)}$ converges to $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ as $n \to \infty$, in the sense that probabilities of cylinder sets converge. Note that $q \geq 1$ is needed to ensure the convergence of $\Phi^{G_n}_{p,q}$ to the (unique) random-cluster measure $\Phi^{\mathbb{Z}^d}_{p,q}$, see Section 2.1.

The next two lemmas, which give the conditional edge distribution in the DaC(q) model given any spin configuration, are of crucial importance for the rest of this paper. The statements (and the proofs) are analogues of Proposition 5.1 and Theorem 6.2 in [18]. For a graph $G = (\mathcal{V}, \mathcal{E})$ (where $\mathcal{V}$ and $\mathcal{E}$ are finite or $\mathcal{V} = \mathbb{Z}^d, \mathcal{E} = \mathcal{E}^d$) and a spin configuration $\sigma \in \Omega_C^G$, we define for all $i \in S$ the vertex sets $\mathcal{V}^{\sigma,i} = \{v \in \mathcal{V} : \sigma(v) = i\}$, edge sets $\mathcal{E}^{\sigma,i} = \{e = \langle x,y \rangle : x,y \in \mathcal{V}^{\sigma,i}\}$ and $\mathcal{E}^{\sigma,\mathrm{diff}} = \mathcal{E} \setminus \cup_{i=1}^s \mathcal{E}^{\sigma,i}$, and graphs $G^{\sigma,i} = (\mathcal{V}^{\sigma,i}, \mathcal{E}^{\sigma,i})$.

Lemma 3.1. *Let $G = (\mathcal{V}, \mathcal{E})$ be a finite graph. Fix parameters $p \in [0,1], q > 0, s \in \{2,3,\ldots\}, a_1, a_2, \ldots, a_s \in (0,1)$ in such a way that $\sum_{i=1}^s a_i = 1$, and an arbitrary spin configuration $\sigma \in S^{\mathcal{V}}$, and define the event $A = \{(\xi,\eta) \in \Omega^G : \xi = \sigma\}$. Then we have that*

(a) for all $e \in \mathcal{E}^{\sigma,\mathrm{diff}}$, $\mathbb{P}^G_{p,q,(a_1,\ldots,a_s)}(\{(\xi,\eta) \in \Omega^G : \eta(e) = 0\} \mid A) = 1$, and

(b) for all $i \in S$, independently for different values of i, on the set $\{0,1\}^{\mathcal{E}^{\sigma,i}}$, the conditional distribution of $\mathbb{P}^G_{p,q,(a_1,\ldots,a_s)}$ given A is the random-cluster measure $\Phi^{G^{\sigma,i}}_{p,qa_i}$.

Proof. Statement (a) is immediate from the definition of the model. Now let $\eta \in \Omega_D^G$ be such that $\eta(e) = 0$ for all $e \in \mathcal{E}^{\sigma,\mathrm{diff}}$. Denote by $k^{\sigma,i}(\eta)$ the number of connected components in η that have spin i in σ, and notice that $k(\eta) = \sum_{i=1}^s k^{\sigma,i}(\eta)$. Using this observation, (1), and a rearrangement of the factors, we obtain that

$$\begin{aligned}\mathbb{P}((\sigma,\eta)) &= \Phi^G_{p,q}(\eta) \prod_{i=1}^s a_i^{k^{\sigma,i}(\eta)} \\ &= \frac{(1-p)^{|\mathcal{E}^{\sigma,\mathrm{diff}}|}}{Z^G_{p,q}} \prod_{i=1}^s \left((qa_i)^{k^{\sigma,i}(\eta)} \prod_{e \in \mathcal{E}^{\sigma,i}} p^{\eta(e)} (1-p)^{1-\eta(e)} \right),\end{aligned}$$

where we wrote $\mathbb{P}$ for $\mathbb{P}^G_{p,q,(a_1,\ldots,a_s)}$, and $|\cdot|$ for cardinality. Therefore,

$$\mathbb{P}^G_{p,q,(a_1,\ldots,a_s)}((\sigma,\eta)\mid A)=\prod_{i=1}^{s}\Phi^{G^{\sigma,i}}_{p,qa_i}(\eta_{\mathcal{E}^{\sigma,i}})$$

since the factor $\frac{(1-p)^{|\mathcal{E}^{\sigma,\mathrm{diff}}|}\prod_{i=1}^{s}Z^{G^{\sigma,i}}_{p,qa_i}}{Z^G_{p,q}\mu^G_{p,q,(a_1,\ldots,a_s)}(\sigma)}$ is constant in η, thus it must be 1 to give a probability measure. This proves statement (b). □

Remark 3.2. Let $\sigma\in S^{\mathcal{V}}$ and $A\subset\Omega^G$ be as in Lemma 3.1. The fact that random-cluster measures factorise on disconnected graphs provides a simple way of drawing a random bond configuration Y with distribution $\mathbb{P}^G_{p,q,(a_1,\ldots,a_s)}$ given A. First, set $Y(e)=0$ for all $e\in\mathcal{E}^{\sigma,\mathrm{diff}}$. Then choose any component $C=(\mathcal{V}_C,\mathcal{E}_C)$ in the graph $(\mathcal{V},\mathcal{E}\setminus\mathcal{E}^{\sigma,\mathrm{diff}})$. Notice that C is a maximal monochromatic component in G (with respect to σ); suppose that for all $v\in\mathcal{V}_C$, $\sigma(v)=i$. Then, independently of everything else, draw $Y_{\mathcal{E}_C}$ according to the random-cluster measure Φ^C_{p,qa_i}. Repeat this procedure with a new component in $(\mathcal{V},\mathcal{E}\setminus\mathcal{E}^{\sigma,\mathrm{diff}})$ until there are no more such components. Lemma 3.1 and the observation at the beginning of this paragraph ensure that we get the correct (conditional) distribution.

By using Lemma 3.1 and the limiting procedure for $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$, one obtains analogous statements for $\mathbb{Z}^d$ in case of $q\geq 1$.

Lemma 3.3. *Fix parameters $d,p,q\geq 1,s,(a_1,a_2,\ldots,a_s)$ of the DaC(q) model on $\mathbb{Z}^d$ and a spin configuration $\sigma\in\Omega_C$. Then the conditional distribution of $\mathbb{P}=\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ given $C^\sigma_{\mathbb{Z}^d}$ assigns value 0 to all edges in $\mathcal{E}^{\sigma,\mathrm{diff}}$, and is a random-cluster measure for $G^{\sigma,i}$ with parameters p and qa_i on $\mathcal{E}^{\sigma,i}$, independently for each i. Moreover, for each edge $e\in\mathcal{E}^{\sigma,i}$ and almost every edge configuration $\zeta\in\{0,1\}^{\mathcal{E}^d\setminus\{e\}}$, we have that*

$$\mathbb{P}(\{(\xi,\eta)\in\Omega:\eta(e)=1\}\mid C^\sigma_{\mathbb{Z}^d}\cap D^\zeta_{\mathcal{E}^d\setminus\{e\}})=\begin{cases} p & \text{if } x\overset{\zeta}{\leftrightarrow}y,\\ \frac{p}{p+(1-p)qa_i} & \text{otherwise.}\end{cases}$$

Proof sketch. Unless the edge configuration $\zeta\in\{0,1\}^{\mathcal{E}^d\setminus\{e\}}$ is special in the sense that it contains at least two infinite FK clusters or there exists an edge $f\in\mathcal{E}^d\setminus\{e\}$ such that changing the state of f in ζ would create at least two infinite FK clusters, we see after a certain stage of the limiting construction described at the beginning of this section whether or not $x\overset{\zeta}{\leftrightarrow}y$ occurs, therefore, an equality corresponding to the "moreover" part of Lemma 3.3 can be verified by Lemma 3.1 for all further stages of the limiting construction. Since the above mentioned special edge configurations have $\Phi^{\mathbb{Z}^d}_{p,q}$-measure 0, we are done. For the details, see the proof of Theorem 6.2 in [18]. □

The next lemma, which is a more general form of Lemma 7.3 in [18], and can be proved in the same way, shows that given edge and spin configurations of a certain type (such as the ones that we shall use in the proof of Theorem 2.7 parts 1, 2(b), and 3(b), see Figure 2 before Lemma 4.1), the "price of changing a spin" depends only on the existence or nonexistence of connections in the edge configuration. Since it looks somewhat specialised, and will not be used until Section 4, the reader might choose to skip it for now.

LEMMA 3.4. *Fix parameters $d \geq 2, q \geq 1, p \in [0,1), s, (a_1, a_2, \ldots, a_s)$ of the DaC(q) model, and let $i, j \in S$ be different spin values. Then there exist positive constants $c_1^{i,j} = c_1^{i,j}(p, q, a_i, a_j)$ and $c_2^{i,j} = c_2^{i,j}(p, q, a_i, a_j)$ such that for any $v \in \mathbb{Z}^d$ with nearest neighbours $u_1, u_2, \ldots, u_{2d}$ and the edges between v and u_i denoted by e_i ($i \in \{1, 2, \ldots, 2d\}$), we have for all $\sigma \in S^{\mathbb{Z}^d \setminus \{v\}}$ and $\zeta \in \{0,1\}^{\mathcal{E}^d \setminus \{e_1, e_2, \ldots, e_{2d}\}}$ satisfying*

1. *$\sigma(u_1) = \sigma(u_2) = i$ and $\sigma(u_3) = \sigma(u_4) = \ldots = \sigma(u_{2d}) = j$, and*
2. *no two of $u_3, u_4, \ldots, u_{2d}$ are connected in ζ,*

that

$$\frac{\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}(C^i_{\{v\}} \mid C^\sigma_{\mathbb{Z}^d \setminus \{v\}} \cap D^\zeta_{\mathcal{E}^d \setminus \{e_1,\ldots,e_{2d}\}})}{\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}(C^j_{\{v\}} \mid C^\sigma_{\mathbb{Z}^d \setminus \{v\}} \cap D^\zeta_{\mathcal{E}^d \setminus \{e_1,\ldots,e_{2d}\}})} = \begin{cases} c_1^{i,j} & \text{if } u_1 \overset{\zeta}{\leftrightarrow} u_2, \\ c_2^{i,j} & \text{otherwise.} \end{cases}$$

The exact values of $c_1^{i,j}$ and $c_2^{i,j}$ are

$$c_1^{i,j} = \frac{p^2 q a_i + 2p(1-p) q a_i + (1-p)^2 (q a_i)^2}{(1-p)^2 (q a_i)^2} \cdot \frac{a_i}{a_j} \cdot \left(\frac{(1-p) q a_j}{p + (1-p) q a_j} \right)^{2d-2}$$

and

$$c_2^{i,j} = \frac{p^2 + 2p(1-p) q a_i + (1-p)^2 (q a_i)^2}{(1-p)^2 (q a_i)^2} \cdot \frac{a_i}{a_j} \cdot \left(\frac{(1-p) q a_j}{p + (1-p) q a_j} \right)^{2d-2},$$

and this shows that

$$\begin{cases} c_1^{i,j} > c_2^{i,j} & \text{if and only if } \; q a_i > 1, \\ c_1^{i,j} = c_2^{i,j} & \text{if and only if } \; q a_i = 1, \\ c_1^{i,j} < c_2^{i,j} & \text{if and only if } \; q a_i < 1. \end{cases}$$

Lemma 3.4 will play a role in proving parts 1, 2(b), and 3(b) in Theorem 2.7. For the proof of parts 2(a) and 3(a), we shall need Lemma 3.10, which is preceded by a few definitions and another lemma. The next definition is motivated by Corollary 3.7.

DEFINITION 3.5. *We call an edge set $E = \{e_1, e_2, \dots, e_k\}$ a **barrier** if removing $e_1, e_2, \dots, e_k$ (but not their endvertices) separates the graph $\mathbb{Z}^d$ into two or more disjoint connected subgraphs. Note that exactly one of the resulting subgraphs is infinite, which we call the **exterior** of E, and denote by $ext(E)$. We denote the vertex set of $ext(E)$ by $\mathcal{V}_{ext(E)}$, and the edge set of $ext(E)$ by $\mathcal{E}_{ext(E)}$. We call the union of the finite subgraphs the **interior** of E, and denote it by $int(E)$, and use $\mathcal{V}_{int(E)}$ and $\mathcal{E}_{int(E)}$ to denote its vertex and edge set, respectively. $E = \{e_1, e_2, \dots, e_k\}$ is called a **closed barrier** in a configuration $(\xi, \eta) \in \Omega$ if E is a barrier and $\eta(e_i) = 0$ holds for all $i \in \{1, 2, \dots, k\}$, and it is called **quasi-closed barrier** if for all edges $e = \langle x, y\rangle \in E$ such that $\eta(e) = 1$, it is true that $\xi(x) = \xi(y) \in S_{1/q}$.*

For a vertex set $W \subset \mathbb{Z}^d$, we define the **edge boundary** ΔW of W by $\Delta W = \{\langle x, y\rangle \in \mathcal{E}^d : x \in W, y \in \mathbb{Z}^d \setminus W\}$. Note that the edge boundary of a union of finite spin clusters is a closed barrier and that all closed barriers are quasi-closed.

According to Lemma 3.3, the states of edges in spin i clusters where $a_i = 1/q$ are chosen independently of everything else, hence they should play no role in issues of dependence. We prove a formal statement concerning this in the following lemma, and we show in Corollary 3.7 a way to make use of this feature of the model. For an event A, we denote the indicator random variable of A by $\mathbb{I}_A$.

LEMMA 3.6. *Let $G = (\mathcal{V}, \mathcal{E})$ be a finite graph, $V_1, V_2 \subset \mathcal{V}$ a partition of $\mathcal{V}$, and for $i \in \{1, 2\}$, define edge sets $E_i = \{e \in \mathcal{E}$: both endvertices of e are in $V_i\}$, and graphs $G_i = (V_i, E_i)$. Define also the edge set $B = \{e \in \mathcal{E} : e$ has one endvertex in V_1 and one in $V_2\}$, a subset $B^0 \subset B$, and for $i \in \{1, 2\}$, W_i as the set of endvertices of edges in $B \setminus B^0$ that are in V_i, see Figure 1 below. Fix parameters $p, q > 0, s, (a_1, a_2, \dots, a_s)$ of the DaC(q) model on G, and a spin configuration $\sigma \in S_{1/q}^{W_1 \cup W_2}$ such that for all $e = \langle x, y\rangle \in B \setminus B^0$ we have that $\sigma(x) = \sigma(y)$. Considering the events $C(B, \sigma) = \{(\xi, \eta) \in \Omega^G : \eta_{B^0} \equiv 0, \xi_{W_1 \cup W_2} = \sigma\}$, $K_1 = \{(\xi, \eta) \in \Omega^{G_1} : \xi_{W_1} = \sigma_{W_1}\}$, $K_2 = \{(\xi, \eta) \in \Omega^{G_2} : \xi_{W_2} = \sigma_{W_2}\}$, and $Z(B^0) = \{\eta \in \{0,1\}^{B^0} : \eta \equiv 0\}$, we have for each*

$(\xi,\eta)\in\Omega^G$ *that*

$$\begin{aligned}\mathbb{P}^G_{p,q,(a_1,a_2,\dots,a_s)}((\xi,\eta)\mid C(B,\sigma)) &= \mathbb{P}^{G_1}_{p,q,(a_1,a_2,\dots,a_s)}((\xi_{V_1},\eta_{E_1})\mid K_1)\\ &\quad\times\mathbb{P}^{G_2}_{p,q,(a_1,a_2,\dots,a_s)}((\xi_{V_2},\eta_{E_2})\mid K_2)\\ &\quad\times\mathbb{I}_{Z(B^0)}(\eta_{B^0})\prod_{e\in B\setminus B^0}p^{\eta(e)}(1-p)^{1-\eta(e)}.\end{aligned}$$

This implies in particular the conditional independence given $C(B,\sigma)$ *of the random configurations on* G_1 *and on* G_2.

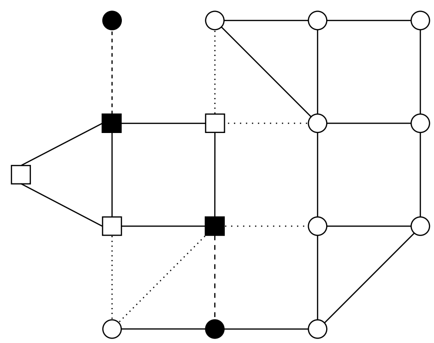

FIG 1. *Illustration of the situation considered in Lemma 3.6. The circles represent the vertices in* V_1*, and the squares are the vertices in* V_2*. The union of the dotted and dashed edges make up* B*, with the dotted ones being in* B_0*. Accordingly, the black circles are the vertices in* W_1*, and the black squares represent the vertices in* W_2*.*

Proof. Let us fix $(\xi,\eta)\in\Omega^G$. Note that

$$\mathbb{I}_{C(B,\sigma)}(\xi,\eta)=\mathbb{I}_{K_1}(\xi_{V_1},\eta_{E_1})\mathbb{I}_{K_2}(\xi_{V_2},\eta_{E_2})\mathbb{I}_{Z(B_0)}(\eta_{B_0}),$$

hence if $(\xi,\eta)\notin C(B,\sigma)$ then we have that both sides of the equation that we want to prove are 0, thus for all such configurations we indeed have equality of the two sides. Therefore, let us assume that $(\xi,\eta)\in C(B,\sigma)$. Define the event $A=\{(\kappa,\zeta)\in\Omega^G:$ there is no edge $e=\langle x,y\rangle\in\mathcal{E}$ with $\zeta(e)=1$ and $\kappa(x)\neq\kappa(y)\}$, and denote the analogously defined subsets of Ω^{G_1} and Ω^{G_2} by A_1 and A_2 respectively. Since $(\xi,\eta)\in C(B,\sigma)$, we have that

$$\mathbb{I}_A(\xi,\eta)=\mathbb{I}_{A_1}(\xi_{V_1},\eta_{E_1})\mathbb{I}_{A_2}(\xi_{V_2},\eta_{E_2}).$$

Therefore, if $(\xi,\eta)\notin A$, we have 0 on both sides of the desired equation in Lemma 3.6 by the definition of the model, so let us assume that $(\xi,\eta)\in A$.

Now denote by n the total number of FK clusters in η, and for all $i\in S$, $j\in\{1,2\}$ the number of FK clusters in η that contain a vertex in V_j with spin i in ξ but no vertex in W_j by n_j^i, and for each $i \in S$ the number of FK clusters in η that contain a vertex in $W_1 \cup W_2$ with spin i in ξ by n_3^i. Throughout this proof, we shall omit the subscripts of the joint measures in the DaC(q) models, e.g., we write $\mathbb{P}^G$ for the measure $\mathbb{P}^G_{p,q,(a_1,a_2,\dots,a_s)}$. Since $(\xi, \eta) \in C(B, \sigma) \cap A$, it immediately follows from the definition of $\mathbb{P}^G$ and the definition (1) of random-cluster measures that

$$\mathbb{P}^G((\xi,\eta)) = \frac{q^n}{Z^G_{p,q}} \Big(\prod_{e \in \mathcal{E}} p^{\eta(e)}(1-p)^{1-\eta(e)}\Big)\Big(\prod_{i=1}^{s} a_i^{n_1^i + n_2^i + n_3^i}\Big). \tag{5}$$

Note that $\mathcal{E} = E_1 \cup E_2 \cup B^0 \cup (B \setminus B^0)$. Since $(\xi, \eta) \in C(B, \sigma) \cap A$, we have that $\prod_{e \in B^0} p^{\eta(e)}(1-p)^{1-\eta(e)} = (1-p)^{|B^0|}$ where $|\cdot|$ denotes cardinality, and $n = \sum_{i=1}^s n_1^i + n_2^i + n_3^i$. Furthermore, it is the case that

$$\prod_{i=1}^{s} (qa_i)^{n_1^i + n_2^i + n_3^i} = \prod_{i=1}^{s} (qa_i)^{n_1^i + n_2^i},$$

since for all $i \notin S_{1/q}$ we have $n_3^i = 0$, whereas for all $i \in S_{1/q}$, we have $qa_i = 1$, so the factor $\prod_{i=1}^s (qa_i)^{n_3^i}$ is indeed 1. Using these observations, we can factorise the expression in (5). Indeed, denoting by c the quantity $(1-p)^{|B^0|}/(Z^G_{p,q}\mathbb{P}^G(C(B,\sigma)))$ which does not depend on (ξ, η), we have that

$$\begin{aligned}
\mathbb{P}^G((\xi,\eta) \mid C(B,\sigma)) &= \frac{\mathbb{P}^G((\xi,\eta))}{\mathbb{P}^G(C(B,\sigma))} \\
&= c\left[\prod_{e \in E_1} p^{\eta(e)}(1-p)^{1-\eta(e)} \prod_{i=1}^{s} (qa_i)^{n_1^i}\right] \\
&\quad \times \left[\prod_{e \in E_2} p^{\eta(e)}(1-p)^{1-\eta(e)} \prod_{i=1}^{s} (qa_i)^{n_2^i}\right] \\
&\quad \times \left[\prod_{e \in B \setminus B_0} p^{\eta(e)}(1-p)^{1-\eta(e)}\right].
\end{aligned}$$

The last part of the proof, namely to show that the expressions between the first and second pair of square brackets are $c_1\mathbb{P}^{G_1}((\xi_{V_1}, \eta_{E_1}) \mid K_1)$ and $c_2\mathbb{P}^{G_2}((\xi_{V_2}, \eta_{E_2}) \mid K_2)$ respectively where c_1 and c_2 are constants (i.e., they do not depend on ξ or η) will be easy. It is sufficient to show the first one, since then the second one follows by relabeling V_1 and V_2. Let n_4 denote the total number of FK clusters in η_{E_1}, and for each $i \in S$, n_5^i the number of FK clusters in η_{E_1} that contain a vertex in W_1 with spin i in ξ_{V_1}. Since

$(\xi,\eta)\in C(B,\sigma)\cap A$, we have that $n_4=\sum_{i=1}^s n_1^i+n_5^i$. Similarly as in the paragraph after (5), we have $n_5^i=0$ for all $i\notin S_{1/q}$ and $qa_i=1$ for all $i\in S_{1/q}$, hence

$$\prod_{i=1}^s (qa_i)^{n_1^i+n_5^i}=\prod_{i=1}^s (qa_i)^{n_1^i}.$$

Denoting $Z_{p,q}^{G_1}\mathbb{P}^{G_1}(K_1)$ by c_1, the above observations imply that

$$\begin{aligned}\mathbb{P}^{G_1}((\xi_{V_1},\eta_{E_1})\mid K_1) &= \frac{\mathbb{P}^{G_1}((\xi_{V_1},\eta_{E_1}))}{\mathbb{P}^{G_1}(K_1)}\\ &= \frac{q^{n_4}}{c_1}\prod_{e\in E_1}p^{\eta(e)}(1-p)^{1-\eta(e)}\prod_{i=1}^s a_i^{n_1^i+n_5^i}\\ &= \frac{1}{c_1}\prod_{e\in E_1}p^{\eta(e)}(1-p)^{1-\eta(e)}\prod_{i=1}^s (qa_i)^{n_1^i}.\end{aligned}$$

Finally, notice that none of c,c_1,c_2 depends on (ξ,η), hence the product cc_1c_2 must be equal to 1 to make $\mathbb{P}^G(\cdot\mid C(B,\sigma))$ a probability measure. This observation completes the proof of Lemma 3.6. □

Lemma 3.6 combined with the limiting procedure for $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ yields the following result, which shows why quasi-closed barriers are useful.

COROLLARY 3.7. *Fix parameters $d,p,q\ge 1,s$, and $(a_1,a_2,\ldots,a_s)$ of the DaC(q) model on $\mathbb{Z}^d$. Let (X,Y) be a random configuration in Ω with distribution $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$, B a barrier, and $C(B)$ the event that B is quasi-closed. Then, given $C(B)$, $(X_{\mathcal{V}_{int(B)}},Y_{\mathcal{E}_{int(B)}})$ and $(X_{\mathcal{V}_{ext(B)}},Y_{\mathcal{E}_{ext(B)}})$ are conditionally independent. In particular, for a set $H\subset\mathbb{Z}^d$ and a spin configuration $\sigma\in S^H$, we have that the conditional distribution of $(X_{\mathcal{V}_{int(B)}},Y_{\mathcal{E}_{int(B)}})$ given $C(B)$ and $\{(\xi,\eta)\in\Omega:\xi_H=\sigma_H\}$ is $\mathbb{P}^{int(B)}_{p,q,(a_1,\ldots,a_s)}$ conditioned on $\{(\xi,\eta)\in\Omega^{int(B)}:\xi_{H\cap\mathcal{V}_{int(B)}}=\sigma_{H\cap\mathcal{V}_{int(B)}}\}$.*

REMARK 3.8. As a first application of Corollary 3.7, we give a proof of the Markovianness of the measure $\mu^{\mathbb{Z}^d}_{p,q,(1/q,1/q,\ldots,1/q)}$ that does not use the connection between the DaC(q) and Potts models. Fix a finite subset W of $\mathbb{Z}^d$. For any spin configuration $\sigma\in S^{\mathbb{Z}^d\setminus W}$ outside W, we have that the edge set $B=\{e\in\mathcal{E}^d: e \text{ has one endvertex in } \partial W \text{ and one in } \partial_2 W\setminus\partial W\}$ is a quasi-closed barrier since for each edge $e=\langle x,y\rangle\in B$, it is either the case that $\sigma(x)\neq\sigma(y)$ and therefore e is closed, or $\sigma(x)=\sigma(y)\in S_{1/q}$ since $S=S_{1/q}$. Therefore, we have by Corollary 3.7 that for any spin configurations $\sigma, \sigma' \in S^{\mathbb{Z}^d \setminus W}$ with $\sigma'_{\partial W} = \sigma_{\partial W}$, the conditional distributions $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(1/q,1/q,\ldots,1/q)}$ given $K^{\sigma}_{\mathbb{Z}^d \setminus W}$ and $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(1/q,1/q,\ldots,1/q)}$ given $K^{\sigma'}_{\mathbb{Z}^d \setminus W}$ are the same in $S^{\mathcal{V}_{int(B)}} \times \{0,1\}^{\mathcal{E}_{int(B)}}$. The statement follows.

Corollary 3.7 enables us to give some intuition behind our main results. Roughly speaking, quasilocality means that conditioning on a spin configuration $\sigma \in S^{\mathbb{Z}^d \setminus H}$ outside a set H, the spin distribution in H does not depend on spins very far away from H. Corollary 3.7 shows that this is the case if H is surrounded by a quasi-closed barrier. In particular, the presence of such a quasi-closed barrier is automatic if $S = S_{1/q}$, as witnessed in Remark 3.8. It is also easy to see that if there is no percolation in σ in any spin (i.e., there exists no infinite connected component of equal spins), then there exists a closed barrier surrounding H, namely the edge boundary of the (finite) union of H and the spin clusters in σ that contain at least one vertex in ∂H. This reasoning will be used in the proof of Proposition 2.9.

However, if there exists an infinite spin i cluster C in σ with $i \in S \setminus S_{1/q}$ that contains a vertex in ∂H, then it cannot be decided whether a quasi-closed barrier surrounding H exists or not by looking at the spin configuration only, but one also needs to check the edge configuration in C. Clearly, if we see an infinite open edge component in C that contains a vertex in ∂H, then there is no quasi-closed barrier that surrounds $H \cup \partial H$. Since by Lemma 3.3 the conditional edge distribution in the spin i cluster C is a random-cluster measure with parameters p and qa_i, the question is whether such measures percolate.

Now recall the definition of $\ell \in S$ right before Theorem 2.7, and consider the case when $a_\ell > 1/q$. Note that the condition $p < p_c(qa_\ell, d)$, which appears in part 2(a) of Theorem 2.7, ensures that there is no infinite edge cluster in any spin j cluster where $j \in S \setminus S_{1/q}$ by the definition of ℓ, using the well-known fact (see, e.g., [14]) that if $q_1 \geq q_2$, then $p_c(q_1, d) \geq p_c(q_2, d)$. We shall show in Section 4 that for all such p there exists a quasi-closed barrier surrounding H given any spin configuration $\sigma \in S^{\mathbb{Z}^d \setminus H}$ with arbitrarily high probability, and hence quasilocality holds.

This argument suggests that the best candidate for a spin configuration in which spins arbitrarily far away from H still have a significant influence on the spin distribution in H (thereby implying non-quasilocality) are those with an infinite spin ℓ cluster, and that quasilocality might fail for all $p > p_c(qa_\ell, d)$. We did not manage to prove this, but got very close by proving non-quasilocality for all $p > p_c^{\mathcal{H}}(qa_\ell, d)$ in Section 4. Indeed, this is equivalent to the full statement under the widely accepted conjecture that for random-cluster measures the critical value and the half-space critical value coincide.

The case of $a_\ell < 1/q$ is more problematic, mainly because then the random-cluster measures with parameters p and qa_ℓ do not satisfy positive association of the edge variables; in fact, it is easy to see that for such random-cluster measures, the conditional probability given in Definition 2.2 is non-increasing in ζ. Since for $a_\ell > 1/q$ positive association plays a role both in proving quasilocality for small p and in proving non-quasilocality for large p, without that we had to resort to comparing the conditional edge configuration given a spin configuration to Bernoulli bond percolation. This, however, yielded worse upper, respectively lower bounds for the quasilocality and the non-quasilocality regime, leaving a gap between the bounds.

The following definition will help us to localise quasi-closed barriers.

Definition 3.9. *Let (X,Y) be an $S^{\mathbb{Z}^d}\times\{0,1\}^{\mathcal{E}^d}$-valued random pair with distribution $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$. Given that $(X,Y)=(\xi,\eta)$ for some $(\xi,\eta)\in\Omega$, let $\hat{Y}\in\{0,1\}^{\mathcal{E}^d}$ be defined by setting, for each $e=\langle x,y\rangle\in\mathcal{E}^d$,*

$$\hat{Y}(e)=\begin{cases} 0 & \text{if } \xi(x)=\xi(y)\in S_{1/q},\\ \eta(e) & \text{otherwise.}\end{cases}$$

We write $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ for the induced joint distribution of $(X,Y,\hat{Y})$ on $\hat{\Omega}=S^{\mathbb{Z}^d}\times\{0,1\}^{\mathcal{E}^d}\times\{0,1\}^{\mathcal{E}^d}$.

The next lemma, which is a generalisation of Lemma 9.5 in [18], compares the conditional distribution of $\hat{Y}$ given a spin configuration and $\hat{Y}$ outside a finite edge set F to a random-cluster measure for $\mathbb{Z}^d$ with parameters p and qa_ℓ in case of $a_\ell > 1/q$, and the conditional distribution of Y given a spin configuration and $\hat{Y}$ outside a finite edge set F to Bernoulli bond percolation with parameter $\frac{p}{p+(1-p)qa_\ell}$ if $a_\ell < 1/q$.

Lemma 3.10. *Suppose that $q\ge 1$ and $S\neq S_{1/q}$. For any spin configuration $\sigma\in S^{\mathbb{Z}^d}$, edge set $F\subset\subset\mathcal{E}^d$, and edge configurations $\zeta,\zeta'\in\{0,1\}^{\mathcal{E}^d\setminus F}$ such that $\zeta'\ge\zeta$, denoting $\hat{A}=\{(\xi,\eta,\hat{\eta})\in\hat{\Omega}:\xi=\sigma,\hat{\eta}_{\mathcal{E}^d\setminus F}=\zeta\}$ and $A'=\{\eta\in\Omega_D:\eta_{\mathcal{E}^d\setminus F}=\zeta'\}$, we have the following.*

1. *If $a_\ell > 1/q$, let ϕ be a random-cluster measure for $\mathbb{Z}^d$ with parameters p and qa_ℓ. Then the conditional distribution of ϕ given A' is stochastically larger than the marginal on $\hat{Y}$ of $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ given $\hat{A}$.*
2. *If $a_\ell < 1/q$, then the conditional distribution of the product measure $\Phi^{\mathbb{Z}^d,1}_{\frac{p}{p+(1-p)qa_\ell},1}$ given A' is stochastically larger than the marginal on Y of $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ given $\hat{A}$.*

Proof. First we prove part 1. By Holley's theorem on stochastic domination (see [11], Theorem 4.8), it is sufficient to prove that for all $a \in \{0,1\}$, $e = \langle x, y\rangle \in F$, $\zeta_g, \zeta_s \in \{0,1\}^{F\setminus\{e\}}$ such that $\zeta_g \geq \zeta_s$, denoting $B_g = \{\eta \in \Omega_D : \eta_{F\setminus\{e\}} = \zeta_g\}$ and $\hat{B}_s = \{(\xi, \eta, \hat{\eta}) \in \hat{\Omega} : \hat{\eta}_{F\setminus\{e\}} = \zeta_s\}$, we have that

$$\phi(\{\eta \in \Omega_D : \eta(e) \geq a\} \mid A' \cap B_g) \tag{6}$$

is greater than or equal to

$$\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(\{(\xi,\eta,\hat{\eta}) \in \hat{\Omega} : \hat{\eta}(e) \geq a\} \mid \hat{A} \cap \hat{B}_s). \tag{7}$$

This is obvious for $a = 0$. For $a = 1$, using the notation $(\eta_1; \eta_2)$ for an edge configuration which agrees with η_1 on $\mathcal{E}^d \setminus F$ and with η_2 on $F \setminus \{e\}$, we have by Definition 2.2 of random-cluster measures that (6) equals

$$\begin{cases} p & \text{if } x \overset{(\zeta';\zeta_g)}{\leftrightarrow} y, \\ \frac{p}{p+(1-p)qa_\ell} & \text{if } x \overset{(\zeta';\zeta_g)}{\not\leftrightarrow} y. \end{cases}$$

For (7), we need to check first what the spins of the endvertices x, y of e are in σ. Indeed, if $\sigma(x) \neq \sigma(y)$ or $\sigma(x) = \sigma(y) \in S_{1/q}$, then (7) $= 0$ by Definition 3.9. Let us assume that $\sigma(x) = \sigma(y) = j \notin S_{1/q}$, and denote the maximal monochromatic component (with respect to σ) in the graph $\mathbb{Z}^d$ which contains x by G_x. By Lemma 3.3, the conditional distribution of Y given σ is a random-cluster measure on G_x with parameters p and qa_j. Moreover, since $j \notin S_{1/q}$, we have that $\hat{Y}$ and Y agree on G_x. Keeping these observations in mind, it follows that (7) equals

$$\begin{cases} 0 & \text{if } \sigma(x) \neq \sigma(y) \text{ or } \sigma(x) = \sigma(y) \in S_{1/q}, \\ p & \text{if } \sigma(x) = \sigma(y) \notin S_{1/q} \text{ and } x \overset{(\zeta;\zeta_s)}{\leftrightarrow} y, \\ \frac{p}{p+(1-p)qa_j} & \text{if } \sigma(x) = \sigma(y) = j \notin S_{1/q} \text{ and } x \overset{(\zeta;\zeta_s)}{\not\leftrightarrow} y. \end{cases}$$

Since due to the assumption $a_\ell \geq 1/q$, we have for all $j \in S$ that

$$p \geq \frac{p}{p+(1-p)qa_j}$$

and by the definition of a_ℓ we have for all $j \in S \setminus S_{1/q}$ that

$$\frac{p}{p+(1-p)qa_\ell} \geq \frac{p}{p+(1-p)qa_j},$$

we obtain the desired result by noting that $x \overset{(\zeta';\zeta_g)}{\not\leftrightarrow} y$ implies $x \overset{(\zeta;\zeta_s)}{\not\leftrightarrow} y$.

Part 2 can also be proved by a direct application of Holley's theorem, noticing that due to the definition of ℓ and the assumption $qa_\ell < 1$ we have that

$$\frac{p}{p+(1-p)qa_\ell} \geq \max\{p, \max_{i\in S} \frac{p}{p+(1-p)qa_i}\}.$$

□

Although in Lemma 3.10 the set F had to be finite so that we could use Holley's theorem in the proof, it will not be difficult to deduce an analogous statement corresponding to $F = \mathcal{E}^d$, see below.

COROLLARY 3.11. *Suppose that $q \geq 1$ and $S \neq S_{1/q}$, and let $\sigma \in S^{\mathbb{Z}^d}$ be an arbitrary spin configuration. Defining $\hat{A} = \{(\xi,\eta,\hat{\eta}) \in \hat{\Omega} : \xi = \sigma\}$, we have the following.*

1. *If $a_\ell > 1/q$, then the wired random-cluster measure $\Phi^{\mathbb{Z}^d,1}_{p,qa_\ell}$ is stochastically larger than the marginal on $\hat{Y}$ of $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ given $\hat{A}$.*
2. *If $a_\ell < 1/q$, then the product measure $\Phi^{\mathbb{Z}^d,1}_{\frac{p}{p+(1-p)qa_\ell},1}$ is stochastically larger than the marginal on Y of $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ given $\hat{A}$.*

Proof. We only give the proof of part 1 since part 2 can be proved analogously. Assume that $a_\ell > 1/q$, and let ϕ be a random-cluster measure for $\mathbb{Z}^d$ with parameters p and qa_ℓ. For $n \in \{1,2,\ldots\}$, let $\mathcal{E}_n$ and W_n be as in Section 2.1, and define ϕ_n as ϕ conditioned on $W_n \cap \{\eta \in \Omega_D : \eta_{\mathcal{E}^d\setminus\mathcal{E}_n} \equiv 1\}$. It follows from Lemma 3.10 that for each n, ϕ_n is stochastically larger than the marginal on $\hat{Y}$ of $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ given $\hat{A}$. On the other hand, ϕ_n coincides on $\mathcal{E}_n$ with $\phi^{G_n,1}_{p,qa_\ell}$ (which is defined in Section 2.1), therefore it converges to $\Phi^{\mathbb{Z}^d,1}_{p,qa_\ell}$ as $n \to \infty$. Since stochastic domination is preserved under weak limits, this observation finishes the proof. □

Note that if $S_{1/q} = \emptyset$, then $\hat{Y}$ can be replaced by Y in part 1 of Lemma 3.10 and Corollary 3.11. Also, since $\hat{Y} \leq Y$ by definition, we could write $\hat{Y}$ instead of Y in part 2.

4. Proofs of the main results.

After all the preparation in Section 3, we are now ready to prove our main results. The proof of Proposition 2.4 is not difficult. In fact, one can use the same idea as the one behind the proof of Lemma 5.6 in [17], namely that any vertex can be isolated (i.e., incident to closed edges only) in the edge configuration (given any spin configuration) with probability bounded away from 0, in which case it can be assigned any spin in S, independently of everything else. A formal proof goes as follows.

Proof of Proposition 2.4. Fix $v \in \mathbb{Z}^d, m \in S$ and $\sigma \in S^{\mathbb{Z}^d \setminus \{v\}}$, and recall the definition for $W \subset \mathbb{Z}^d$ of the event $K_W^\sigma \subset \Omega_C$ at the beginning of Section 2.2 and of the analogous event $C_W^\sigma \subset \Omega$ at the beginning of Section 3. Denote by E_v the event that all $2d$ edges incident to v are closed. We have that

$$\begin{aligned} \mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(K^m_{\{v\}} \mid K^\sigma_{\mathbb{Z}^d\setminus\{v\}}) \quad &\geq \quad \mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(C^m_{\{v\}} \mid C^\sigma_{\mathbb{Z}^d\setminus\{v\}} \cap E_v) \\ &\quad \times \mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(E_v \mid C^\sigma_{\mathbb{Z}^d\setminus\{v\}}). \end{aligned} \tag{8}$$

Obviously (or as a special case of Corollary 3.7), we have that the first term on the right hand side of (8) is a_m, since given E_v, v is assigned a spin independently of everything else.

On the other hand,

$$\begin{aligned} \mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(E_v \mid C^\sigma_{\mathbb{Z}^d\setminus\{v\}}) \quad &= \quad \sum_{b\in S} \mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(E_v \mid C^\sigma_{\mathbb{Z}^d\setminus\{v\}} \cap C^b_{\{v\}}) \\ &\quad \times \mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(C^b_{\{v\}} \mid C^\sigma_{\mathbb{Z}^d\setminus\{v\}}). \end{aligned}$$

Now, whatever value $b \in S$ takes, the full spin configuration is given in the first factor on the right hand side, so we can apply Lemma 3.3. Under any random-cluster measure with parameters p and $\tilde{q} > 0$, the probability of E_v is bounded away from 0: a lower bound for $\tilde{q} \geq 1$ is $(1-p)^{2d}$, while for $\tilde{q} < 1$ it is $(1 - \frac{p}{p+(1-p)\tilde{q}})^{2d}$. Since here the parameter $\tilde{q}$ equals qa_b for some b, we get the lower bound

$$\min\{(1-p)^{2d}, (1 - \frac{p}{p + (1-p)q \min_{i \in S} a_i})^{2d}\}$$

for the first factor, which is uniform in b. As $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(C^b_{\{v\}} \mid C^\sigma_{\mathbb{Z}^d\setminus\{v\}})$ for $b \in S$ sum up to 1, we get the same bound for $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(E_v \mid C^\sigma_{\mathbb{Z}^d\setminus\{v\}})$.

Combining this with (8) and the remark thereafter, we have that

$$\varepsilon = \left(\min_{i\in S} a_i\right) \left(\min\{1-p, 1 - \frac{p}{p + (1-p)q \min_{i\in S} a_i}\}\right)^{2d}$$

is a lower bound for $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(K^m_{\{v\}} \mid K^\sigma_{\mathbb{Z}^d\setminus\{v\}})$. Since ε does not depend on v, m or σ and is positive for any values of $p \in [0,1), q \geq 1, a_1, \dots, a_s \in (0,1)$, we conclude that $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ is uniformly nonnull for such parameters. □

The proof of Theorem 2.7 consists of many parts. For the proof of parts 1, 2(b), and 3(b), we use a counterexample that is very similar to the one given in [17, 18] (see also [7, 22, 23]), defined below. After the definition, we give

Lemma 4.1, after which it will not be difficult to prove parts 1, 2(b), and 3(b). Finally, we prove parts 2(a) and 3(a). From this point on, we assume that $S \neq S_{1/q}$.

Recall first the definitions of Λ_n and ℓ from Sections 2.1 and 2.2, respectively. Fix an arbitrary spin $m \in S$ such that $m \neq \ell$, and define an auxiliary spin configuration $\sigma^* \in S^{\mathbb{Z}^d}$ by setting, for each $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$,

$$\sigma^*(x) = \begin{cases} m & \text{if } x_1 = 0, |x_2| + |x_3| + \ldots + |x_d| = 1 \\ & \text{or } x_1 = -1, |x_2| + |x_3| + \ldots + |x_d| > 1, \text{ and} \\ \ell & \text{otherwise,} \end{cases}$$

and for $k \in \{1, 2, \ldots\}$, spin configurations $\sigma^{k,\ell}, \sigma^{k,m} \in S^{\mathbb{Z}^d \setminus \{\mathbf{0}\}}$ (see Figure 2 below) by

$$\sigma^{k,\ell}(x) = \begin{cases} \ell & \text{for } x \in \mathbb{Z}^d \setminus \Lambda_k, \\ \sigma^*(x) & \text{otherwise,} \end{cases}$$

and

$$\sigma^{k,m}(x) = \begin{cases} m & \text{for } x \in \Lambda_{k+1} \setminus \Lambda_k, \\ \sigma^{k,\ell}(x) & \text{otherwise.} \end{cases}$$

ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	ℓ		ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	ℓ
ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ	ℓ

m	m	m	m	m	m	m	m	m
m	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	m
m	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	m
m	ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	m
m	ℓ	ℓ	ℓ		ℓ	ℓ	ℓ	m
m	ℓ	ℓ	ℓ	m	ℓ	ℓ	ℓ	m
m	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	m
m	ℓ	ℓ	m	ℓ	ℓ	ℓ	ℓ	m
m	m	m	m	m	m	m	m	m

FIG 2. *Restriction of $\sigma^{3,\ell}$ (to the left) and $\sigma^{3,m}$ (to the right) to $\Lambda_4 \setminus \{\mathbf{0}\}$ in two dimensions. For all $x \in \mathbb{Z}^2 \setminus \Lambda_4$, $\sigma^{3,\ell}(x) = \sigma^{3,m}(x) = \ell$.*

Denote the two nearest neighbours of $\mathbf{0}$ in $\mathbb{Z}^d$ with σ^*-spin ℓ by $u_1 = (1, 0, 0, \ldots, 0)$ and $u_2 = (-1, 0, 0, \ldots, 0)$, the other nearest neighbours by $u_3, u_4, \ldots, u_{2d}$, and for $i \in \{1, 2, \ldots, 2d\}$, the edges between $\mathbf{0}$ and u_i by e_i. Most of the work needed for the proof of parts 1, 2(b), and 3(b) of Theorem 2.7 is contained in the following lemma.

LEMMA 4.1. *Fix parameters $d \geq 2, q \geq 1, p, a_1, a_2, \ldots, a_s \in (0, 1)$ of the DaC(q) model on $\mathbb{Z}^d$ in such a way that $S \neq S_{1/q}$. Considering the events*

$A = \{(\xi, \eta) \in \Omega :$ *there exists an open path in* $\eta_{\mathcal{E}^d \setminus \{e_1, e_2, \ldots, e_{2d}\}}$ *between* u_1 *and* $u_2\}$ *and* $O^{\ell,m} = \{(\xi, \eta) \in \Omega : \xi(\mathbf{0}) \in \{\ell, m\}\}$, *we have the following.*

1. *If for a fixed* $k \in \{1, 2, \ldots\}$ *we have that*

$$\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}(A \mid O^{\ell,m} \cap C^{\sigma^{k,\ell}}_{\mathbb{Z}^d \setminus \{\mathbf{0}\}}) > 0,$$

then $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ *is not* k*-Markovian.*

2. *If there exists* $\gamma > 0$ *such that, for all* $k \in \{1, 2, \ldots\}$,

$$\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}\ (A \mid O^{\ell,m} \cap C^{\sigma^{k,\ell}}_{\mathbb{Z}^d \setminus \{\mathbf{0}\}}) > \gamma,$$

then $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ *is not quasilocal.*

Proof. In order to simplify the notation, in this proof we denote $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ by $\mathbb{P}$, $C^{\sigma^{k,\ell}}_{\mathbb{Z}^d \setminus \{\mathbf{0}\}}$ by $L = L^k$, and $C^{\sigma^{k,m}}_{\mathbb{Z}^d \setminus \{\mathbf{0}\}}$ by $M = M^k$. The first step in the proof is to derive, for all $k \in \{1, 2, \ldots\}$, inequality (10). Consider the expression

$$\left|\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid L) - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid M)\right|. \tag{9}$$

Note that we have that

$$\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid L) = \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L)\mathbb{P}(O^{\ell,m} \mid L),$$

and similarly with M. Using this, we obtain by basic algebra (i.e., first subtracting, then adding a dummy term $\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M)\mathbb{P}(O^{\ell,m} \mid L)$ in (9) between the absolute values and finally using that $|a - b| \geq |a| - |b|$) that (9) is greater than or equal to

$$\left|\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L) - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M)\right| \mathbb{P}(O^{\ell,m} \mid L) \\ - \left|\mathbb{P}(O^{\ell,m} \mid L) - \mathbb{P}(O^{\ell,m} \mid M)\right| \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M).$$

Since $\mathbb{P}(O^{\ell,m} \mid L) = \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid L) + \mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid L)$, we have by uniform nonnullness (i.e., Proposition 2.4) that there exists $\delta > 0$ such that, uniformly in k, $\left|\mathbb{P}(O^{\ell,m} \mid L)\right| \geq \delta$. This observation, noting that $\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M) \leq 1$, and that $\mathbb{P}(O^{\ell,m} \mid L) = \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid L) + \mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid L)$ and similarly with M, and applying the triangle inequality yields that (9) is greater than or equal to

$$\left|\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L) - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M)\right| \delta \\ - \left|\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid L) - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid M)\right| - \left|\mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid L) - \mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid M)\right|.$$

After a rearrangement of the terms, this gives that

$$
\begin{aligned}
& 2\left|\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid L) - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid M)\right| + \left|\mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid L) - \mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid M)\right| \\
\geq \; & \delta\left|\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L) - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M)\right|. \qquad (10)
\end{aligned}
$$

From now on, we shall be working on bounding the right hand side of (10) from below. Elementary calculations and an application of Lemma 3.4 with $i = \ell, j = m, v = \mathbf{0}$ show that

$$
\begin{aligned}
\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L \cap A) &= \frac{\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L \cap A)}{\mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L \cap A)} \mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L \cap A) \\
&= c_1^{\ell,m}(1 - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L \cap A)),
\end{aligned}
$$

and therefore

$$
\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L \cap A) = \frac{c_1^{\ell,m}}{c_1^{\ell,m} + 1}. \qquad (11)
$$

By similar considerations, we obtain that

$$
\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L \cap A^c) = \frac{c_2^{\ell,m}}{c_2^{\ell,m} + 1}, \qquad (12)
$$

and that

$$
\begin{aligned}
\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M) &= \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap M \cap A^c) \\
&= \frac{c_2^{\ell,m}}{c_2^{\ell,m} + 1}. \qquad (13)
\end{aligned}
$$

Using (11) and (12), we get that

$$
\begin{aligned}
\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid O^{\ell,m} \cap L) &= \frac{c_1^{\ell,m}}{c_1^{\ell,m} + 1} \mathbb{P}(A \mid O^{\ell,m} \cap L) \\
&\quad + \frac{c_2^{\ell,m}}{c_2^{\ell,m} + 1} \mathbb{P}(A^c \mid O^{\ell,m} \cap L) \\
&= \frac{c_2^{\ell,m}}{c_2^{\ell,m} + 1} + \left(\frac{c_1^{\ell,m}}{c_1^{\ell,m} + 1} - \frac{c_2^{\ell,m}}{c_2^{\ell,m} + 1}\right) \\
&\quad \times \mathbb{P}(A \mid O^{\ell,m} \cap L). \qquad (14)
\end{aligned}
$$

Applying (14) and (13) in (10) yields that, for any k, we have that

$$
\begin{aligned}
& 2\left|\mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid L^k) - \mathbb{P}(C^{\ell}_{\{\mathbf{0}\}} \mid M^k)\right| + \left|\mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid L^k) - \mathbb{P}(C^{m}_{\{\mathbf{0}\}} \mid M^k)\right| \\
\text{(15)} \geq \quad & \delta \left| \frac{c_1^{\ell,m}}{c_1^{\ell,m}+1} - \frac{c_2^{\ell,m}}{c_2^{\ell,m}+1} \right| \mathbb{P}(A \mid O^{\ell,m} \cap L^k).
\end{aligned}
$$

Since $a_\ell \neq 1/q$ by definition, we have that $c_1^{\ell,m} \neq c_2^{\ell,m}$. This implies that the first two factors on the right hand side of (15) are positive constants, neither of which depends on k. Now suppose that $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ is k-Markovian for some k. In that case the left hand side of (15) is 0 since $\sigma^{k,\ell}_{\Lambda_k\setminus\{\mathbf{0}\}} = \sigma^{k,m}_{\Lambda_k\setminus\{\mathbf{0}\}}$, therefore $\mathbb{P}(A \mid O^{\ell,m} \cap L^k) = 0$. This proves part 1 of Lemma 4.1. Similarly, if $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ is quasilocal, then the limit of the left hand side of (15) is 0 as $k \to \infty$, which cannot be the case if $\mathbb{P}(A \mid O^{\ell,m} \cap L^k)$ is bounded away from 0, uniformly in k. This concludes the proof of part 2. □

Proof of Theorem 2.7, parts 1, 2(b), and 3(b). For this proof, recall the notion of an increasing event on Ω_D (see Section 2.1). Let $d \geq 2, q \geq 1, p, a_1, a_2, \dots, a_s \in (0,1)$ be arbitrary parameters of the DaC(q) model on $\mathbb{Z}^d$ in such a way that $S \neq S_{1/q}$, and let $\sigma^{k,\ell}$, A, and $O^{\ell,m}$ be as in Lemma 4.1. For $k \in \{1,2,\dots\}$, define the edge sets $E^{d,k} = \{e \in \mathcal{E}^d : e$ is incident to $\mathbf{0}$ or to some $v \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ with $\sigma^{k,\ell}(v) = m\}$. For a parameter $\tilde{p} \in (0,1)$ and each $k \in \{1,2,\dots\}$, we define an inhomogeneous bond percolation measure $P_{\tilde{p},k}$ on Ω_D which assigns value 0 to all $e \in E^{d,k}$, and independently to each $e \in \mathcal{E}^d \setminus E^{d,k}$ value 1 with probability $\tilde{p}$ and 0 with probability $1-\tilde{p}$. It follows from Lemma 3.3 and Definition 2.2 that the marginal on Y of the conditional distribution $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ given $O^{\ell,m} \cap C^{\sigma^{k,\ell}}_{\mathbb{Z}^d\setminus\{\mathbf{0}\}}$ is stochastically larger than $P_{\tilde{p},k}$ with $\tilde{p} = \frac{p}{p+(1-p)qa_\ell}$ if $qa_\ell \geq 1$, and with $\tilde{p} = p$ if $qa_\ell \leq 1$. Therefore, denoting the projection of A on Ω_D by A_D (note that $A_D \subset \Omega_D$ is increasing), we have for any k that

$$
\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(A \mid O^{\ell,m} \cap C^{\sigma^{k,\ell}}_{\mathbb{Z}^d\setminus\{\mathbf{0}\}}) \geq P_{\tilde{p},k}(A_D). \tag{16}
$$

Since $\min\{\frac{p}{p+(1-p)qa_\ell}, p\} > 0$ for all $p \in (0,1)$, we obviously have for any fixed $k \in \{1,2,\dots\}$ that $P_{\tilde{p},k}(A_D) > 0$. This proves non-k-Markovianness of the measure $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ according to (16) and part 1 of Lemma 4.1.

For the proof of part 2(b), recall the definition of the vertices $u_1, u_2 \in \mathbb{Z}^d$ and edges $e_1, e_2, \dots, e_{2d} \in \mathcal{E}^d$ (right before Lemma 4.1), and that $\mathcal{H}^+$ denotes the set of vertices in $\mathbb{Z}^d$ whose first coordinate is strictly positive. Define $\mathcal{H}^-$ as the set of vertices in $\mathbb{Z}^d$ whose first coordinate is strictly negative. Consider

the following events:
$A_{\mathcal{H}^+} = \{\eta \in \Omega_D : \exists$ an infinite open path in $\eta_{\mathcal{H}^+}$ which contains $u_1\}$,
$A_{\mathcal{H}^-} = \{\eta \in \Omega_D : \exists$ an infinite open path in $\eta_{\mathcal{H}^-}$ which contains $u_2\}$,
$U = \{\eta \in \Omega_D :$ there is at most one infinite open cluster in $\eta_{\mathcal{E}^d \setminus \{e_1, e_2, \ldots, e_{2d}\}}\}$,
and note that $A_{\mathcal{H}^+} \cap A_{\mathcal{H}^-} \cap U \subset A_D$.

Now assume that $a_\ell > 1/q$. This implies that $qa_\ell \geq 1$ and hence the free random-cluster measure $\Phi^{\mathbb{Z}^d,0}_{p,qa_\ell}$ exists and is the stochastically smallest random-cluster measure for $\mathbb{Z}^d$ with parameters p and qa_ℓ (see Section 2.1). Let us denote by $\Phi^{(c)}_k$ the measure $\Phi^{\mathbb{Z}^d,0}_{p,qa_\ell}$ conditioned on the event $\{\eta \in \Omega_D : \eta_{E^{d,k}} \equiv 0\}$. Due to the above mentioned extremality of $\Phi^{\mathbb{Z}^d,0}_{p,qa_\ell}$ with respect to stochastic ordering, Lemma 3.3 implies that the marginal on Y of the measure $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}$ conditioned on $O^{\ell,m} \cap C^{\sigma^{k,\ell}}_{\mathbb{Z}^d \setminus \{\mathbf{0}\}}$ is stochastically larger than $\Phi^{(c)}_k$. Therefore, we have that

$$\begin{aligned} \mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\ldots,a_s)}(A \mid O^{\ell,m} \cap C^{\sigma^{k,\ell}}_{\mathbb{Z}^d \setminus \{\mathbf{0}\}}) &\geq \Phi^{(c)}_k(A_D) \\ &\geq \Phi^{(c)}_k(A_{\mathcal{H}^+} \cap A_{\mathcal{H}^-} \cap U). \end{aligned} \tag{17}$$

Under the measure $\Phi^{\mathbb{Z}^d,0}_{p,qa_\ell}$ the event U has probability one, and the event one conditions on to obtain $\Phi^{(c)}_k$ has positive probability, so it follows that $\Phi^{(c)}_k(U) = 1$. Hence, we have that

$$\begin{aligned} \Phi^{(c)}_k(A_{\mathcal{H}^+} \cap A_{\mathcal{H}^-} \cap U) &= \Phi^{(c)}_k(A_{\mathcal{H}^+} \cap A_{\mathcal{H}^-}) \\ &\geq \Phi^{(c)}_k(A_{\mathcal{H}^+})\Phi^{(c)}_k(A_{\mathcal{H}^-}) \end{aligned} \tag{18}$$

by (3), since $A_{\mathcal{H}^+}$ and $A_{\mathcal{H}^-}$ are increasing events.

Recalling from Section 2.1 that $\tilde{E} \subset \mathcal{E}^d$ is the set of edges that are incident to at least one vertex in $\mathbb{Z}^d \setminus \mathcal{H}^+$, we have, again by the FKG inequality, that

$$\Phi^{(c)}_k(A_{\mathcal{H}^+}) \geq \Phi^{\mathbb{Z}^d,0}_{p,qa_\ell}(A_{\mathcal{H}^+} \mid \{\eta \in \Omega_D : \eta_{\tilde{E}} \equiv 0\}). \tag{19}$$

Similarly, defining the half-space $\mathcal{H}^{\geq -1}$ as the set of vertices in $\mathbb{Z}^d$ whose first coordinate is at least -1, and denoting by $\tilde{E}'$ the set of edges in $\mathcal{E}^d$ that are incident to a vertex in $\mathcal{H}^{\geq -1} \setminus \{u_2\}$, we have by the FKG inequality that

$$\begin{aligned} \Phi^{(c)}_k(A_{\mathcal{H}^-}) &\geq \Phi^{\mathbb{Z}^d,0}_{p,qa_\ell}(A_{\mathcal{H}^-} \mid \{\eta \in \Omega_D : \eta_{\tilde{E}'} \equiv 0\}) \\ &= \frac{p}{p + (1-p)qa_\ell} \Phi^{\mathbb{Z}^d,0}_{p,qa_\ell}(A_{\mathcal{H}^+} \mid \{\eta \in \Omega_D : \eta_{\tilde{E}} \equiv 0\}). \end{aligned} \tag{20}$$

Here we have also used that, conditioning on $\{\eta \in \Omega_D : \eta_{\tilde{E}'} \equiv 0\}$, $A_{\mathcal{H}^-}$ can occur only if the edge between u_2 and $(-2, 0, \ldots, 0)$ is open (which has conditional probability $\frac{p}{p+(1-p)qa_\ell}$ by Definition 2.2), and that the state of edges incident to u_2 are conditionally independent of the event that $(-2,0,\dots,0)$ is in an infinite open edge component in the corresponding half-space. It follows from (19),(20), and the definition of $p_c^{\mathcal{H}}(qa_\ell,d)$ that for all $p > p_c^{\mathcal{H}}(qa_\ell,d)$, both $\Phi_k^{(c)}(A_{\mathcal{H}^+})$ and $\Phi_k^{(c)}(A_{\mathcal{H}^-})$ are bounded away from 0, uniformly in k. Therefore, by (17) and (18), $\mathbb{P}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(A \mid O^{\ell,m} \cap C^{\sigma^{k,\ell}}_{\mathbb{Z}^d\setminus\{\mathbf{0}\}})$ is bounded away from 0 for such values of p, which implies non-quasilocality of $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ by part 2 of Lemma 4.1. This concludes the proof of part 2(b).

In case of $a_\ell < 1/q$, as remarked above, (16) holds with $\tilde{p} = p$. On the other hand, if $p > p_c(1,d)$, then $\tilde{p} > p_c(1,d)$, hence by Lemma 8.2 in [18] (whose proof is based on a computation similar to (17) and (18)), we have

$$\lim_{k\to\infty} P_{\tilde{p},k}(A_D) > 0.$$

By this, (16), and part 2 of Lemma 4.1, it follows that $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ is not quasilocal, proving part 3(b). □

Our proof of Theorem 2.7 part 2(a), i.e., quasilocality of $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ for small p when $a_\ell > 1/q$, will be a straightforward generalisation of the proof of Theorem 4.4 part (i) in [18]. Although slightly more care is required when $a_\ell < 1/q$, a similar argument will work in that case as well. Therefore, we will be able to give a proof below that deals with both cases simultaneously.

Proof of Theorem 2.7, parts 2(a) and 3(a). For the proof, recall the definition of $\hat{Y}$ and $\hat{\Omega}$ (Definition 3.9), and for a set $W \subset \mathbb{Z}^d$ and a spin configuration $\kappa \in S^W$, the definition of K^κ_W (Section 2.2), and define the analogous event $\hat{C}^\kappa_W = \{(\xi,\eta,\hat{\eta}) \in \hat{\Omega} : \xi_W = \kappa\}$. Fix parameters $d \geq 2, q \geq 1, a_1, a_2, \dots, a_s \in (0,1)$ of the DaC(q) model on $\mathbb{Z}^d$, then p in such a way that $p < p_c(qa_\ell,d)$ if $a_\ell > 1/q$, and $p < \frac{p_c(1,d)qa_\ell}{p_c(1,d)qa_\ell+1-p_c(1,d)}$ if $a_\ell < 1/q$. Fix an arbitrary $W \subset\subset \mathbb{Z}^d$, $\kappa \in S^W$, and $\varepsilon > 0$. We shall show the existence of $N = N(\varepsilon, W)$ such that for all $n \geq N$, if $\sigma,\sigma' \in S^{\mathbb{Z}^d\setminus W}$ are spin configurations that agree on $\Lambda_n \setminus W$, then

$$|\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(K^\kappa_W \mid K^\sigma_{\mathbb{Z}^d\setminus W}) - \mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(K^\kappa_W \mid K^{\sigma'}_{\mathbb{Z}^d\setminus W})| \tag{21}$$

is smaller than or equal to ε.

In order to find such an N, we consider a "dominating measure" ϕ^{dom} on Ω_D: we define $\phi^{\text{dom}} = \Phi^{\mathbb{Z}^d,1}_{p,qa_\ell}$ in case of $a_\ell > 1/q$, and $\phi^{\text{dom}} = \Phi^{\mathbb{Z}^d,1}_{\frac{p}{p+(1-p)qa_\ell},1}$ if $a_\ell < 1/q$. By Corollary 3.11, ϕ^{dom} is stochastically larger than the conditional distribution of the modified random edge configuration $\hat{Y}$ given any spin configuration. Note that the parameters are chosen in such a way that $\phi^{\rm dom}$-a.s. there exists no infinite open edge cluster (for the case $a_\ell < 1/q$, notice that $p < \frac{p_c(1,d)qa_\ell}{p_c(1,d)qa_\ell+1-p_c(1,d)}$ ensures that $\frac{p}{p+(1-p)qa_\ell} < p_c(1,d)$). Therefore, it is possible to choose an N so large that

$$\phi^{\rm dom}(\{\partial W \leftrightarrow \partial\Lambda_N\}) \le \varepsilon, \tag{22}$$

where $\{\partial W \leftrightarrow \partial\Lambda_N\} = \{\eta \in \Omega_D :$ there exists a path between ∂W and $\partial\Lambda_N$ along which all edges are open in $\eta\}$. Fix an arbitrary $n \ge N$, and let $\sigma, \sigma' \in S^{\mathbb{Z}^d\setminus W}$ be two arbitrary spin configurations such that $\sigma_{\Lambda_n\setminus W} = \sigma'_{\Lambda_n\setminus W}$. An informal overview of the proof that (21) $\le \varepsilon$ is as follows.

Let $\hat{Y}$ (respectively $\hat{Y}'$) be the modified random edge configuration when the spin configuration σ (respectively σ') is given. We would like to show that $\hat{Y}$ and $\hat{Y}'$ can be coupled in such a way that there exists a barrier B with a high enough (at least $1-\varepsilon$) probability such that (a) $\hat{Y}_B = \hat{Y}'_B \equiv 0$, and (b) B separates ∂W and $\partial\Lambda_n$. By the definition of $\hat{Y}$, a barrier B satisfying (a) is a quasi-closed barrier in the case when the spin configuration σ is given. Therefore, if B also satisfies (b), then by Corollary 3.7, the spin configuration in W does not depend on $\sigma_{\mathbb{Z}^d\setminus\Lambda_n} \subset \sigma_{ext(B)}$. Clearly, the same argument holds with σ'. Since we have that $\sigma'_{\Lambda_n\setminus W} = \sigma_{\Lambda_n\setminus W}$, we see that finding a barrier B that satisfies (a) and (b) ensures that the conditional spin distribution in W is the same given either of σ or σ'. Therefore, finding such a barrier with probability at least $1-\varepsilon$ yields that (21) $\le \varepsilon$.

In order to find such a barrier, we will couple $\hat{Y}$ and $\hat{Y}'$ together with an auxiliary random edge configuration $Y^{\rm dom}$ with distribution $\phi^{\rm dom}$. We shall show below how one can repeatedly use Lemma 3.10 to simultaneously construct $\hat{Y}, \hat{Y}'$, and $Y^{\rm dom}$ with the correct distributions in such a way that $Y^{\rm dom} \ge \hat{Y}$ and $Y^{\rm dom} \ge \hat{Y}'$ holds at all stages of the construction. By the choice of N in (22), we will find with probability $1-\varepsilon$ a barrier B satisfying (b) with $Y^{\rm dom}_B \equiv 0$. The point is that since $Y^{\rm dom} \ge \hat{Y}$ and $Y^{\rm dom} \ge \hat{Y}'$, this implies that (a) also holds for B, so we are done. It is important to add that our construction will find an appropriate barrier B when such a barrier exists by assigning $\hat{Y}$-, $\hat{Y}'$-, and $Y^{\rm dom}$-values to edges in $B \cup \mathcal{E}_{ext(B)}$ only. Therefore, although $\hat{Y}$ and $\hat{Y}'$ may take different values on such edges, the conditional spin distributions given σ and the explored part of $\hat{Y}$, respectively given σ' and the explored part of $\hat{Y}'$ are indeed the same in $W \subset \mathcal{V}_{int(B)}$.

The formal implementation of this idea goes through essentially the same coupling as used in [18], but we give it for completeness. We shall define below a probability measure $\mathbb{Q}$ on $\hat{\Omega} \times \hat{\Omega} \times \Omega_D$ that is a coupling of

(i) an $\hat{\Omega}$-valued random triple $(X, Y, \hat{Y})$ with distribution $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ conditioned on $\hat{C}^{\sigma}_{\mathbb{Z}^d \setminus W}$,

(ii) an $\hat{\Omega}$-valued random triple $(X', Y', \hat{Y}')$ with distribution $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ conditioned on $\hat{C}^{\sigma'}_{\mathbb{Z}^d \setminus W}$, and

(iii) an Ω_D-valued random edge configuration Y^{dom} with distribution ϕ^{dom}.

Then it follows from the coupling inequality (Proposition 4.2 in [11]) that (21) $\leq \mathbb{Q}(X_W \neq X'_W)$, hence showing that $\mathbb{Q}(X_W = X'_W) \geq 1 - \varepsilon$ would complete the proof. We define $\mathbb{Q}$ in three stages, as follows.

I. Recall that $\mathcal{E}_n$ is the set of edges with both endvertices in $\Lambda_n \cup \partial\Lambda_n$. It follows from Corollary 3.11 and Strassen's theorem (see Section 2.1) that the set $Q = \{\mu : \mu \text{ is a coupling of (i),(ii), and (iii) satisfying that } \mu(\hat{Y}_{\mathcal{E}^d \setminus \mathcal{E}_n} \leq Y^{\mathrm{dom}}_{\mathcal{E}^d \setminus \mathcal{E}_n} \text{ and } \hat{Y}'_{\mathcal{E}^d \setminus \mathcal{E}_n} \leq Y^{\mathrm{dom}}_{\mathcal{E}^d \setminus \mathcal{E}_n}) = 1\}$ of probability measures on $\hat{\Omega} \times \hat{\Omega} \times \Omega_D$ is nonempty. We shall choose $\mathbb{Q}$ from this set, and we will specify in stages II and III which element of Q we pick.

II. Fix an arbitrary deterministic ordering of $\mathcal{E}_n$, and let $(U_e : e \in \mathcal{E}_n)$ be a collection of independent random variables with uniform distribution on the interval $[0,1]$. The following algorithm will determine $\hat{Y}, \hat{Y}'$ and Y^{dom} on a subset of $\mathcal{E}_n$ given that they are known in $\mathcal{E}^d \setminus \mathcal{E}_n$ by drawing $\hat{Y}$-, $\hat{Y}'$- and Y^{dom}-values for one edge at a time, as follows.

1. Let $e \in \mathcal{E}_n$ be the first edge in the previously fixed deterministic ordering which has not been selected in any previous step of the algorithm and is incident to some vertex in $\partial\Lambda_n$ or some previously selected edge f with $Y^{\mathrm{dom}}(f) = 1$.
2. Let us denote by $\mathbb{P}^{(c)}$ the probability measure $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ conditioned on $\hat{C}^{\sigma}_{\mathbb{Z}^d \setminus W}$ and what we have seen so far of $\hat{Y}$, by $\mathbb{P}^{(c)\prime}$ the measure $\hat{\mathbb{P}}^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ conditioned on $\hat{C}^{\sigma'}_{\mathbb{Z}^d \setminus W}$and what we have seen so far of $\hat{Y}'$, and by $\phi^{(c)}$ the measure ϕ^{dom} conditioned on what we have seen so far of Y^{dom}. We define

$$\hat{Y}(e) = \begin{cases} 1 & \text{if } U_e < \mathbb{P}^{(c)}(\hat{Y}(e) = 1), \\ 0 & \text{otherwise,} \end{cases}$$

analogously,

$$\hat{Y}'(e) = \begin{cases} 1 & \text{if } U_e < \mathbb{P}^{(c)\prime}(\hat{Y}'(e) = 1), \\ 0 & \text{otherwise,} \end{cases}$$

and finally,

$$Y^{\mathrm{dom}}(e) = \begin{cases} 1 & \text{if } U_e < \phi^{(c)}(Y^{\mathrm{dom}}(e) = 1), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if we had $Y^{\mathrm{dom}} \geq \hat{Y}$ and $Y^{\mathrm{dom}} \geq \hat{Y}'$ before step 2 of the algorithm (which is μ-a.s. the case for any $\mu \in Q$ before the beginning of this algorithm), then Lemma 3.10 implies that these inequalities are preserved by step 2.

3. If determining $Y^{\mathrm{dom}}(e)$ in step 2 created either an open path in Y^{dom} between $\partial\Lambda_n$ and ∂W or a barrier B such that $W \cup \partial W \subset \mathcal{V}_{int(B)} \subset \Lambda_n$ and $Y^{\mathrm{dom}}_B \equiv 0$, then stop the algorithm, otherwise go back to step 1.

Note that this algorithm terminates at latest once all edges in $ext(\Delta W)$ have been selected, and that it does not select any edge in ΔW or in $int(\Delta W)$.

III. If the algorithm in stage II ended by finding an open path in Y^{dom} between $\partial\Lambda_n$ and ∂W, then draw the rest of $(X, Y, \hat{Y})$, $(X', Y', \hat{Y}')$, and Y^{dom} arbitrarily with the correct conditional distributions, given what we have seen so far of them. This will possibly give that $X_W \neq X'_W$, but that is not a problem since by inequality (22) this case occurs with probability at most ε, and otherwise we will always be able to ensure that $X_W = X'_W$.

Indeed, let as assume that the above algorithm found a barrier B such that $W \cup \partial W \subset \mathcal{V}_{int(B)} \subset \Lambda_n$ and $Y^{\mathrm{dom}}_B \equiv 0$. Since the inequalities $Y^{\mathrm{dom}} \geq \hat{Y}$ and $Y^{\mathrm{dom}} \geq \hat{Y}'$ were retained throughout the whole algorithm (as remarked in step 2), it follows from $Y^{\mathrm{dom}}_B \equiv 0$ that B is closed in $\hat{Y}$ and $\hat{Y}'$ as well. Since B is a barrier which closed in $\hat{Y}$, it is a quasi-closed barrier in (X, Y), therefore Corollary 3.7 implies that the conditional distribution of (X, Y) on $int(B)$ given $X_{\mathbb{Z}^d \setminus W} = \sigma$ and what we have seen of $\hat{Y}$ is $\mathbb{P}^{int(B)}_{p,q,(a_1,\ldots,a_s)}$ conditioned on $\{(\xi, \eta) \in \Omega^{int(B)} : \xi_{\mathcal{V}_{int(B)} \setminus W} = \sigma_{\mathcal{V}_{int(B)} \setminus W}\}$. By similar considerations, the conditional distribution of (X', Y') given $X'_{\mathbb{Z}^d \setminus W} = \sigma'$ and what we have seen of $\hat{Y}'$ is $\mathbb{P}^{int(B)}_{p,q,(a_1,\ldots,a_s)}$ conditioned on $\{(\xi, \eta) \in \Omega^{int(B)} : \xi_{\mathcal{V}_{int(B)} \setminus W} = \sigma'_{\mathcal{V}_{int(B)} \setminus W}\}$. Since $\mathcal{V}_{int(B)} \subset \Lambda_n$ and $\sigma_{\Lambda_n \setminus W} = \sigma'_{\Lambda_n \setminus W}$, we can take $(X_{\mathcal{V}_{int(B)}}, Y_{\mathcal{E}_{int(B)}}, \hat{Y}_{\mathcal{E}_{int(B)}}) = (X'_{\mathcal{V}_{int(B)}}, Y'_{\mathcal{E}_{int(B)}}, \hat{Y}'_{\mathcal{E}_{int(B)}})$ in our coupling. This already implies that $X_W = X'_W$ since $W \subset \mathcal{V}_{int(B)}$, so the coupling can be finished by drawing the rest of $(X, Y, \hat{Y})$, $(X', Y', \hat{Y}')$, and Y^{dom} arbitrarily with the correct conditional distributions.

These considerations yield that with a coupling $\mathbb{Q}$ of (i),(ii), and (iii) as specified in stages I, II, and III, we have that $\mathbb{Q}(X_W = X'_W) \geq 1 - \varepsilon$, which concludes the proof as noted above. □

The proof of Proposition 2.9 is an easier application of the concept that the existence of a (quasi-)closed barrier "blocks the information from outside". Since the proof is virtually the same as the proof of Proposition 3.7 in [17], i.e., the analogous statement for the DaC(1) model, we will just sketch it for the reader's convenience.

Proof sketch of Proposition 2.9. Fix $W \subset \mathbb{Z}^d$ and $\sigma \in S^{\mathbb{Z}^d \setminus W}$ such that none of the spins in σ percolates. By the assumption (4), this is true for almost every spin configuration. Let $W' \subset \mathbb{Z}^d \setminus W$ be the union of all spin components in $\sigma_{\mathbb{Z}^d \setminus W}$ that intersect the vertex boundary ∂W. Since there is no infinite spin component in σ, we have that W' is a finite set, hence the edge boundary $B = \Delta(W \cup W')$ is a closed barrier. Therefore, it follows from Lemma 3.7 that the conditional distribution of $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ given $K^{\sigma}_{\mathbb{Z}^d \setminus W}$ is $\mu^{int(B)}_{p,q,(a_1,\dots,a_s)}$ conditioned on $\{\xi \in \Omega^{int(B)}_C : \xi_{W'} = \sigma_{W'}\}$.

Now recall the definition of $\partial_n W$, the n-neighbourhood of W, in Section 2.2. If k is so large that $W' \subset \partial_{k-1} W$, and $\sigma' \in S^{\mathbb{Z}^d \setminus W}$ is such that $\sigma'_{\partial_k W} = \sigma_{\partial_k W}$, then it is clear by the same argument that the conditional distribution of $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$ given $K^{\sigma'}_{\mathbb{Z}^d \setminus W}$ is $\mu^{int(B)}_{p,q,(a_1,\dots,a_s)}$ conditioned on $\{\xi \in \Omega^{int(B)}_C : \xi_{W'} = \sigma'_{W'}\}$. Since $\sigma_{W'} = \sigma'_{W'}$, the above conditions are the same, therefore for any $\kappa \in S^W$, we have that

$$|\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(K^{\kappa}_W \mid K^{\sigma}_{\mathbb{Z}^d \setminus W}) - \mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}(K^{\kappa}_W \mid K^{\sigma'}_{\mathbb{Z}^d \setminus W})| = 0.$$

This proves almost sure quasilocality of $\mu^{\mathbb{Z}^d}_{p,q,(a_1,\dots,a_s)}$. □

Acknowledgements I am grateful to Federico Camia and Ronald Meester for a careful reading of an earlier version of this paper and numerous suggestions, and to Aernout van Enter for drawing my attention to [4], and other valuable comments. I would also like to thank the two anonymous referees for their remarks and corrections, which have definitely improved on this paper.

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