Multiple Stratonovich integral and Hu–Meyer formula for Lévy processes [1]

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Abstract. In the framework of vector measures and the combinatorial approach to stochastic multiple integral introduced by Rota and Wallstrom [*Stochastic integrals: A combinatorial approach*, The Annals of Probability, **25** (1997) 1257–1283], we present an Itô multiple integral and a Stratonovich multiple integral with respect to a Lévy process with finite moments up to a convenient order. In such a framework, the Stratonovich multiple integral is an integral with respect to a product random measure whereas the Itô multiple integral corresponds to integrate with respect to a random measure that gives zero mass to the diagonal sets. A general Hu–Meyer formula that gives the relationship between both integrals is proved. As particular cases, the classical Hu–Meyer formulas for the Brownian motion and for the Poisson process are deduced. Furthermore, a pathwise interpretation for the multiple integrals with respect to a subordinator is given.

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1 Introduction

Let $W = \{W_t,\ t \geq 0\}$ be a standard Brownian motion. Itô [10] defined the multiple stochastic integral of a function $f \in L^2(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n), (dt)^{\otimes n})$,

$$I_n(f) = \int \cdots \int_{\mathbb{R}_+^n} f(t_1, \ldots, dt_n) dW_{t_1} \cdots dW_{t_n},$$

taking care to ensure that the diagonal sets, like $\{(s_1, \ldots, s_n) \in \mathbb{R}_+^n,\ s_1 = s_2\}$, do not contribute at all. For this reason the integral has very good properties and is easy to work with. However, on a function of the form

$$(g_1 \otimes \cdots \otimes g_n)(t_1, \ldots, t_n) := g(t_1) \cdots g(t_n),$$

we have that, in general,

$$I_n(g_1 \otimes \cdots \otimes g_n) \neq I_1(g_1) \cdots I_1(g_n).$$

That means, the Itô multiple integral does not behave like the integral with respect to a product measure.

Many years later, Hu and Meyer [8] introduced (although they believed that this integral was already known [8, page 75]) a multiple integral, $I_n^S(f)$, which followed the ordinary rules of multiple integration. They called it the multiple Stratonovich integral. Furthermore, Hu and Meyer stated the relationship between the Itô and Stratonovich integrals, the celebrated Hu–Meyer formula, adding

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the contribution of the diagonals to the Itô integral: for a function $f(t_1,\dots,t_n)$ symmetric with good properties,

$$I_n^S(f)=\sum_{j=0}^{[n/2]}\frac{n!}{(n-2j)!j!2^j}\,I_{n-2j}\Big(\int_{\mathbb{R}_+^j} f(\cdot,t_1,t_1,t_2,t_2,\dots,t_j,t_j)\,dt_1\cdots dt_j\Big).$$

This formula is simple because the quadratic variation of the Brownian motion is t, and the integral over coincidences of order three or superior are zero. Following their ideas, Solé and Utzet [28] proved a Hu–Meyer formula for the Poisson process. Again, in that case, the formula is relatively simple because the variations of any order of the process can always be written in terms of the Poisson process and t.

From another point of view, Engel [7], working with a general process with independent increments, related the (Itô) multiple stochastic integral with the theory of vector valued measures, and Masani [16], using also vector valued measures and starting from the Wiener's original ideas, developed both the Itô and Stratonovich integrals (with respect to the Brownian motion) and proved many profound results. The vector measures approach is no simple matter and Engel's work covers 82 pages, and Masani's covers 160. An important and clarifying contribution was made by Rota and Wallstrom [24] who used combinatorial techniques to show the features of the multiple stochastic integration. They did not really work with integrals, but with products of vector measures. However, the path towards a general theory of multiple stochastic integration had been laid. See also Pérez–Abreu [22] for an interesting generalization to Hilbert space valued random measures. Further, Vershik and Tsilevich [30], in a more algebraic context, constructed a Fock factorization for a Lévy process, and some important subspaces can be described through Rota and Wallstrom concepts. We should also mention the very complete survey by Peccati and Taqqu [21] in which a unified study of multiple integrals, moments, cumulants and diagram formulas, as well as applications to some new Central Limit Theorems, is presented.

It is worth remarking that Rota and Walstrom's [24] combinatorial approach to multiple integration has been extended to the context of free probability in a very interesting and fertile field of research, started by Anshelevich, see [1, 2, 3, 4, 5] and the references therein. In fact, Rota and Walstrom's ideas fit very well with the combinatorics of free probability (see Nica and Speicher [19]) and noncommutative Lévy processes. Our renewed interest in Rota and Walstrom's paper [24] was motivated by Anshelevich's work.

In the present paper we use the powerful Rota and Wallstrom's [24] combinatorial machinery to study the Stratonovich integral (the integral with respect to the product random measure) with respect to a Lévy processes with finite moments up to a convenient order. The key point is to understand how the product of stochastic measures works on the diagonal sets, and that leads to the diagonal measures defined by Rota and Wallstrom [24]. For a Lévy process those measures are related to the powers of the jumps of the process, and hence to a family of martingales introduced by Nualart and Schoutens [20], called Teugels martingales, which offer excellent properties. Specifically, these martingales have deterministic predictable quadratic variation and this makes it possible to easily construct an Itô multiple stochastic integral with respect to different integrators, which can be interpreted as an integral with respect to a random measure that gives zero mass to the diagonal sets. With all these ingredients we prove a general Hu–Meyer formula. The paper uses arduous combinatorics because of our need to work with stochastic multiple integrals with respect to the different powers of the jumps of the process, and such integrals can be conveniently handled through the lattice of the partitions of a finite set.

As in the Brownian case (see, for example, [12, 27, 9, 16]), there are alternative methods to construct a multiple Stratonovich integral based on approximation procedures, and it is possible to relax the conditions on the integrator process by assuming more regularity on the integrand function. Such regularity is usually expressed in terms of the existence of *traces* of the function in a convenient sense. The advantage of using Lévy processes with finite moments lies in the fact that simple $L^2(\Omega)$ estimates for the multiple stochastic integral of simple functions can be obtained, and then the multiple Stratonovich integral can be defined in an L^2 space with respect to a measure that controls the behaviour of the functions on the diagonal sets. In this way, the problem of providing a manageable definition of the traces is avoided.

We would like to comment that an impressive body of work on multiple stochastic integrals with respect to Lévy processes has been done by Kallenberg, Kwapien, Krakowiak, Rosinski, Szulga, Woyczinski and many others (see [13, 14, 15, 23] and the references therein). However, their approach is very different from ours, and assumes different settings to those used in this work. For this reason, we have only used a few results of their results.

The paper is organized as follows. In Section 2 we review some combinatorics concepts and the basics of the stochastic measures as vector valued measures. In Section 3 we introduce the random measures induced by a Lévy process, and we identify the diagonal measures in such a case. In Section 4 we study the relationship between the product and Itô measures of a set, and we obtain a Hu–Meyer formula for measures. In Section 5 we define the multiple Itô stochastic integral and the multiple Stratonovich integral and also prove the general Hu–Meyer formula for integrals. In Section 6, as particular cases, we deduce the classical Hu–Meyer formulas for the Brownian motion and for the Poisson process. We also study the case where the Lévy process is a subordinator, and also prove that both the multiple Itô stochastic integral and the multiple Stratonovich integral can be computed in a pathwise sense. Finally, in order to make the paper lighter, some of the combinatorial results are included as an appendix.

2 Preliminaries

2.1 Partitions of a finite set

We need some notations of the combinatorics of the partitions of a finite set; for details we refer to Stanley [29, Chapter 3] or Rota and Wallstrom [24].

Let F be a finite set. A partition of F is a family $\pi = \{B_1, \dots, B_m\}$ of non-void subsets of F, pairwise disjoint, such that $F = \bigcup_{i=1}^m B_i$. The elements $B_1, \dots, B_m$ are called the *blocks* of the partition.

Denote by $\Pi(F)$ the set of all partitions of F, and write Π_n for $\Pi(\{1, \dots, n\})$. Given $\sigma, \pi \in \Pi(F)$, we write $\sigma \leq \pi$ if each block of σ is contained in some block of π; we then say that σ is a refinement of π. This relationship defines a partial order that is called the *reversed refinement order*, and it makes $\Pi(F)$ a lattice. We write $\widehat{0} = \big\{\{x\}, \, x \in F\big\}$, which is the minimal element, and $\widehat{1} = \{F\}$ the maximal one.

We say that a partition $\pi \in \Pi(F)$ is of type $(1^{r_1} 2^{r_2} \cdots n^{r_n})$ if π has exactly r_1 blocks with 1 element, exactly r_2 blocks with 2 elements, and so on. In the same way, for $\sigma \leq \pi$, $\#\sigma = m$ and $\#\pi = k$, we say that the segment $[\sigma, \pi]$ is of type $(1^{r_1} 2^{r_2} \cdots m^{r_m})$ if there are exactly r_1 blocks of π in σ; there are exactly r_2 blocks of π that each one gives rise to 2 blocks of σ, etc. Necessarily,

$$\sum_{j=1}^{m} r_j = k \quad \text{and} \quad \sum_{j=1}^{m} j r_j = m.$$

In that situation, the Möbius function of $[\sigma, \pi]$ is

$$\mu(\sigma, \pi) = (-1)^{m-k} (2!)^{r_3} \cdots ((m-1)!)^{r_m}.$$

We use the Möbius inversion formula, that in the context of the lattice of the partitions of a finite set, says that for two functions $f, g : \Pi(F) \longrightarrow \mathbb{R}$,

$$g(\sigma) = \sum_{\pi \geq \sigma} f(\pi), \quad \forall \sigma \in \Pi(F),$$

if and only if

$$f(\sigma) = \sum_{\pi \geq \sigma} \mu(\sigma, \pi) g(\pi), \quad \forall \sigma \in \Pi(F). \tag{1}$$

(see [29, Proposition 3.7.2]).

2.2 Diagonal sets induced by a partition

As we commented in the Introduction, we will introduce two random measures on a n-dimensional space, and the diagonal sets will play an essential role. Diagonal sets can be conveniently described through the partitions of the set $\{1, \dots, n\}$. We use the notations introduced by Rota and Wallstorm [24].

Let S be an arbitrary set and consider $C \subset S^n$. Given $\pi \in \Pi_n$, we write $i \sim_\pi j$ if i and j belong to the same block of π. Put

$$C_{\geq \pi} = \{(s_1, \dots, s_n) \in C : \; s_i = s_j \; \text{ if } i \sim_\pi j\},$$

and

$$C_\pi = \{(s_1, \dots, s_n) \in C : \; s_i = s_j \; \text{ if and only if } i \sim_\pi j\}.$$

The sets C_π are called *diagonal sets*. Note that $C_\pi = C \cap S^n_\pi$ and $C_{\geq \pi} = C \cap S^n_{\geq \pi}$.

For example, for $n = 4$ and $\pi = \big\{\{1\}, \{2\}, \{3, 4\}\big\}$, we have

$$C_{\geq \pi} = \{(s_1, s_2, s_3, s_4) \in C : \; s_3 = s_4\}$$

and

$$C_\pi = \{(s_1, s_2, s_3, s_4) \in C : \; s_3 = s_4, \; s_1 \neq s_2, \; s_1 \neq s_3, \; s_2 \neq s_3\}.$$

The sets corresponding to the minimal and maximal partitions are specially important:

$$C_{\widehat{0}} = \{(s_1, \dots, s_n) \in C : \; s_i \neq s_j, \; \forall i \neq j\} \quad \text{and} \quad C_{\widehat{1}} = \{(s_1, \dots, s_n) \in C : \; s_1 = \dots = s_n\}.$$

If $\sigma \neq \pi$, then

$$C_\sigma \cap C_\pi = \emptyset \quad \text{and} \quad \big(C_\pi\big)_\sigma = \emptyset. \tag{2}$$

The above notation $C_{\geq \pi}$ is coherent with the reversed refinement order:

$$C_{\geq \pi} = \bigcup_{\sigma \geq \pi} C_\sigma \quad \text{(disjoint union)}. \tag{3}$$

In particular, $C = C_{\geq \widehat{0}} = \bigcup_{\sigma \in \Pi_n} C_\sigma$.

2.3 Random measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. In this paper, a random measure Φ on a measurable space $(S, \mathcal{S})$ is an $L^2(\Omega)$-valued σ-additive vector measure, that means, a map $\Phi : \mathcal{S} \to L^2(\Omega)$ such that for every sequence $\{A_n,\ n \geq 1\} \subset \mathcal{S}$, such that $A_n \cap A_m = \emptyset,\ n \neq m$,

$$\Phi(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Phi(A_n), \quad \text{convergence in} \quad L^2(\Omega).$$

The σ–additive vector measures defined on a σ–field inherit some basic properties of the ordinary measures, but not all. So, for a sake of easy reference, we write here a uniqueness property translated to our setting. The proof is the same as the one for ordinary measures.

Proposition 2.1. *Let Φ and Ψ be two random measures on $(S, \mathcal{S})$, and consider a family of sets $\mathcal{C} \subset \mathcal{S}$ closed under finite intersection and such that $\sigma(\mathcal{C}) = \mathcal{S}$. Then*

$$\Phi = \Psi, \ \text{ on } \mathcal{C} \quad \Longrightarrow \quad \Phi = \Psi, \ \text{ on } \mathcal{S}.$$

2.4 Product and Itô stochastic measures

Assume that the measurable space $(S, \mathcal{S})$ satisfies that for every set $C \in \mathcal{S}^{\otimes n}$ and every $\pi \in \Pi_n$, we have $C_\pi \in \mathcal{S}^{\otimes n}$. As Rota and Wallstrom [24] point out, this condition is satisfied if S is a Polish space and $\mathcal{S}$ its Borel σ-algebra. We extend the definition of *good random measure* introduced by Rota and Wallstorm [24] to a family of measures; specifically, we say that the random measures $\Phi_1, \ldots, \Phi_k$ over a measurable space $(S, \mathcal{S})$ are ***jointly good random measures*** if the finite additive product vector *measure* $\Phi_1 \otimes \cdots \otimes \Phi_k$ defined on the product sets by

$$\big(\Phi_1 \otimes \cdots \otimes \Phi_k\big)(A_1 \times \cdots \times A_k) = \prod_{j=1}^{k} \Phi_i(A_i), \quad A_1, \ldots, A_k \in \mathcal{S},$$

can be extended to a (unique) σ-additive random measure on $(S^n, \mathcal{S}^{\otimes n})$. This extension, obvious for ordinary measures, is in general not transferred to arbitrary vector measures, see Engel [7], Masani [16], and Kwapien and Woyczynski [15].

Given a good random measure Φ (in the sense that the n-fold product $\Phi \otimes \cdots \otimes \Phi = \Phi^{\otimes n}$ satisfies the above condition), the starting point of Rota and Wallstrom [24, Definition 1] is to consider new random measures given by the restriction over the diagonal sets; specifically, for $\pi \in \Pi_n$ they define

$$\Phi_\pi^{\otimes n}(C) := \Phi^{\otimes n}(C_{\geq \pi}) \quad \text{and} \quad St_\pi^{[n]}(C) := \Phi^{\otimes n}(C_\pi), \quad \text{for} \quad C \in \mathcal{S}^{\otimes n}.$$

The following definitions are the extension of these concepts to a family of random measures.

Definition 2.2. *Let $\Phi_{r_1}, \ldots, \Phi_{r_n}$ be jointly good random measures on $(S, \mathcal{S})$. For a partition $\pi \in \Pi_n$, define*

$$\big(\Phi_{r_1} \otimes \cdots \otimes \Phi_{r_n}\big)_\pi(C) = \big(\Phi_{r_1} \otimes \cdots \otimes \Phi_{r_n}\big)(C_{\geq \pi}), \quad C \in \mathcal{S}^{\otimes n}, \tag{4}$$

and

$$St_\pi^{(r_1, \ldots, r_n)}(C) = \big(\Phi_{r_1} \otimes \cdots \otimes \Phi_{r_n}\big)(C_\pi), \quad C \in \mathcal{S}^{\otimes n}. \tag{5}$$

In agreement with the notations in Rota and Wallstrom [24], when $\Phi_{r_1} = \cdots = \Phi_{r_n} = \Phi$, we simply write $\Phi_\pi^{\otimes n}$ for $\big(\Phi \otimes \cdots \otimes \Phi\big)_\pi$ and $St_\pi^{[n]}$ for the corresponding measure given in (5). Since

$C_{\geq \widehat{0}} = C$, then $\Phi_{\widehat{0}}^{\otimes n} = \Phi^{\otimes n}$, that is the product measure. The measure $St_{\widehat{0}}^{(r_1,\dots,r_n)}$ is called ***the Itô multiple stochastic measure*** relative to $\Phi_{r_1}, \dots, \Phi_{r_n}$.

As the ordinary multiple Itô integral, the Itô multiple stochastic measure gives zero mass to every diagonal set different from $C_{\widehat{0}}$:

Proposition 2.3. *Let $\pi \in \Pi_n$ such that $\pi > \widehat{0}$. For every $C \in \mathcal{S}^{\otimes n}$, we have*

$$St_{\widehat{0}}^{(r_1,\dots,r_n)}(C_\pi) = 0, \text{ a.s.}$$

Proof. From (2) we have $(C_\pi)_{\widehat{0}} = \emptyset$. □

The basic result of Rota and Wallstrom [24, Proposition 1], is transferred to this situation:

Proposition 2.4.

$$\big(\Phi_{r_1} \otimes \cdots \otimes \Phi_{r_n}\big)_\pi = \sum_{\sigma \geq \pi} St_\sigma^{(r_1,\dots,r_n)}, \tag{6}$$

and

$$St_\pi^{(r_1,\dots,r_n)} = \sum_{\sigma \geq \pi} \mu(\pi,\sigma)\big(\Phi_{r_1} \otimes \cdots \otimes \Phi_{r_n}\big)_\sigma, \tag{7}$$

where $\mu(\pi,\sigma)$ is the Möbius function defined in Subsection 2.1.

Proof. The equality (6) is deduced from (3) and the definitions (4) and (5). The equality (7) follows from (6) and the Möebius inversion formula (1). □

3 Random measures induced by a Lévy process

Let $X = \{X_t,\, t \in [0,T]\}$ be a Lévy process, that is, X has stationary and independent increments, is continuous in probability, is cadlag and $X_0 = 0$. In all the paper we assume that X has moments of all orders; however, if the interest is restricted to multiple integral up to order $n \geq 2$, then it is enough to assume that the process has moments up to order $2n$.

Denote the Lévy measure of X by ν, and by σ^2 the variance of its Gaussian part. The existence of moments of X_t of all orders implies that $\int_{\{|x|>1\}} |x|\, \nu(dx) < \infty$ and $\int_{\mathbb{R}} |x|^n\, \nu(dx) < \infty,\ \forall n \geq 2$. Write

$$K_1 = E[X_1], \quad K_2 = \sigma^2 + \int_{\mathbb{R}} x^2\, \nu(dx) \quad \text{and} \quad K_n = \int_{\mathbb{R}} x^n\, \nu(dx) < \infty,\ n \geq 3. \tag{8}$$

From now on, take $S = [0,T]$ and $\mathcal{S} = \mathcal{B}([0,T])$. The basic random measure ϕ that we consider is the measure induced by the process X itself, defined on the intervals by

$$\phi(]s,t]) = X_t - X_s,\ 0 \leq s \leq t \leq T, \tag{9}$$

and extended to $\mathcal{B}([0,T])$. The measure ϕ is an independently scattered random measure, that is, if $A_1, \dots, A_n \in \mathcal{B}([0,T])$ are pairwise disjoint, then $\phi(A_1), \dots, \phi(A_n)$ are independent.

The random measures induced by the powers of the jumps of the process, $\Delta X_t = X_t - X_{t-}$, are also used. Consider the *variations* of the process X (see Meyer [17, page 319]):

$$\begin{aligned} X_t^{(1)} &= X_t, \\ X_t^{(2)} &= [X,X]_t = \sum_{0<s\leq t} \big(\Delta X_s\big)^2 + \sigma^2 t \\ X_t^{(n)} &= \sum_{0<s\leq t} \big(\Delta X_s\big)^n, n \geq 3. \end{aligned} \tag{10}$$

The processes $X^{(1)}, \dots, X^{(n)}$, ... are Lévy processes such that

$$\mathbb{E}[X_t^{(n)}] = K_n\, t,\ \forall n \geq 1.$$

So, the centered processes,

$$Y_t^{(n)} = X_t^{(n)} - K_n\, t, \quad n \geq 1,$$

are square integrable martingales, called *Teugels martingales* (see Nualart and Schoutens [20]), with predictable quadratic covariation

$$\langle Y^{(n)}, Y^{(m)} \rangle_t = K_{n+m}\, t, \quad n,\, m \geq 1.$$

Notations 3.1. We denote by ϕ_n the random measure induced by $X^{(n)}$, and for $n = 1$, $\phi_1 = \phi$ (we indistinctly use both ϕ_1 and ϕ). Every ϕ_n is a independently scattered random measure. For $A,\, B \in \mathcal{B}([0,T])$,

$$\mathbb{E}[\phi_n(A)\phi_m(B)] = K_{n+m} \int_{A\cap B} dt + K_n K_m \int_A dt \int_B dt.$$

We stress the following property, which is the basis of all the paper, and is a consequence of Theorem 10.1.1 by Kwapien and Woyczynski [15].

Theorem 3.2. *For every $r_1, \dots, r_n \geq 1$, the random measures $\phi_{r_1}, \dots, \phi_{r_n}$ are jointly good random measures on $\big([0,T]^n, \mathcal{B}\big([0,T]^n\big)\big)$.*

3.1 The diagonal measures

Rota and Wallstrom [24] define the diagonal measure of order n of ϕ as the random measure on $[0,T]$ given by

$$\mathbf{\Delta}_n(A) = \phi^{\otimes n}(A^n_{\hat{1}}),\ A \in \mathcal{B}([0,T]). \tag{11}$$

To identify the diagonal measures is a necessary step to study the stochastic multiple integral. In the case of a random measure generated by a Lévy process we show that the diagonal measures are the measures generated by the variations of the process.

Proposition 3.3. *For every $A \in \mathcal{B}([0,T])$ and $n \geq 1$,*

$$\mathbf{\Delta}_n(A) = \phi_n(A), \tag{12}$$

where ϕ_n is the random measure induced by $X^{(n)}$.

Proof. Since both $\mathbf{\Delta}_n$ and ϕ_n are random measures, by Proposition 2.1 it is enough to check the equality for $A = (0,t]$. Consider an increasing sequence of equidistributed partitions of $[0,t]$ with the mesh going to 0, for example, take $t_k^{(m)} = tk/2^m$ and let

$$\mathcal{P}_m = \{t_k^{(m)},\ k = 0\dots, 2^m\}.$$

To short the notation, write t_k instead of $t_k^{(m)}$. Consider the sets

$$A_m = \big(0, t_1\big]^n \cup \big(t_1, t_2\big]^n \cup \dots \cup \big(t_{2^m-1}, t\big]^n.$$

Random measures are sequentially continuous and $A_m \searrow (0,t]^n_{\hat{1}}$, when $m\to\infty$, so we have that

$$\boldsymbol{\Delta}_n((0,t]) = \lim_m \sum_{k=0}^{2^m-1} \Big(\phi\big((t_k,t_{k+1}]\big)\Big)^n = \lim_m \sum_{k=0}^{2^m-1} \big(X_{t_{k+1}} - X_{t_k}\big)^n,$$

in $L^2(\Omega)$. For $n=2$,

$$\lim_m \sum_{k=0}^{2^m-1} \big(X_{t_{k+1}} - X_{t_k}\big)^2 = [X,X]_t = \phi_2\big((0,t]\big), \text{in probability,}$$

so the proposition is true in this case. For $n>2$, by Itô formula,

$$\begin{aligned}
&\sum_{k=0}^{2^m-1} \big(X_{t_{k+1}} - X_{t_k}\big)^n \\
&\quad = n\sum_{k=0}^{2^m-1} \int_{t_k}^{t_{k+1}} \big(X_{s-} - X_{t_k}\big)^{n-1}\, dX_s \\
&\quad + \frac{1}{2}\, n(n-1) \sum_{k=0}^{2^m-1} \int_{t_k}^{t_{k+1}} \big(X_s - X_{t_k}\big)^{n-2}\, ds \\
&\quad + \sum_{k=0}^{2^m-1} \sum_{t_k<s\le t_{k+1}} \Big[\big(X_s - X_{t_k}\big)^n - \big(X_{s-} - X_{t_k}\big)^n - n\big(X_{s-} - X_{t_k}\big)^{n-1}\big(X_s - X_{s-}\big)\Big]
\end{aligned}$$

$$= n\int_0^t \Big(\sum_{k=0}^{2^m-1} \big(X_{s-} - X_{t_k}\big)^{n-1} \mathbf{1}_{(t_k,t_{k+1}]}(s)\Big)\, dX_s \tag{a}$$

$$+ \binom{n}{2} \int_0^t \Big(\sum_{k=0}^{2^m-1} \big(X_{s-} - X_{t_k}\big)^{n-2} \mathbf{1}_{(t_k,t_{k+1}]}(s)\Big)\, d[X,X]_s \tag{b}$$

$$+ \sum_{j=3}^{n} \sum_{k=0}^{2^m-1} \sum_{t_k<s\le t_{k+1}} \binom{n}{j} \big(X_{s-} - X_{t_k}\big)^{n-j} \big(\Delta X_s\big)^j. \tag{c}$$

For $j=3,\dots,n-1$, the corresponding term in (c) is

$$\binom{n}{j} \int_0^t \Big(\sum_{k=0}^{2^m-1} \big(X_{s-} - X_{t_k}\big)^{n-j} \mathbf{1}_{(t_k,t_{k+1}]}(s)\Big)\, dX_s^{(j)}, \tag{d}$$

Hence, (a), (b) and (d) have the same structure

$$\int_0^t H_s^{(m)}\, dZ_s,$$

where $H_s^{(m)} = \sum_{k=0}^{2^m-1} \big(X_{s-} - X_{t_k}\big)^r \mathbf{1}_{(t_k,t_{k+1}]}(s)$ is a predictable process and Z is a semimartingale. Since X_{s-} is left continuous,

$$\lim_m H_s^{(m)} = 0, \text{ a.s.}$$

Moreover,

$$\big|H_s^{(m)}\big| \le C \sup_{0\le u\le s} |X_u|^r,$$

and the process $\left\{ \sup_{0\leq u\leq s} |X_u|^r,\ s\in[0,t]\right\}$ is cadlag and adapted, and as a consequence, it is prelocally bounded (see pages 336 and 340 in Dellacherie and Meyer [6]). By the dominated convergence theorem for stochastic integrals, Dellacherie and Meyer [6, Theoreme 14, page 338],

$$\lim_m \int_0^t H_s^{(m)}\, dZ_s = 0, \text{ in probability.}$$

Finally, for $j=n$, the term in (c) is $\sum_{0<s\leq t}\big(\Delta X_s\big)^n = X_t^{(n)}$, and the proposition is proved. □

Diagonal measures associated to a random measure of the form $\phi_{r_1}\otimes\cdots\otimes\phi_{r_n}$ are needed. This is an extension of the previous proposition, and it is a key result for the sequel.

Theorem 3.4. *Let* $r_1,\dots,r_n\geq 1$*,* $n\geq 2$*, and* $A\in\mathcal{B}([0,T])$*. Then*

$$\big(\phi_{r_1}\otimes\cdots\otimes\phi_{r_n}\big)(A_{\widehat{1}}^n) = \mathbf{\Delta}_{r_1+\cdots+r_n}(A) = \phi_{r_1+\cdots+r_n}(A).$$

Proof.

As in the proof of the last proposition and with the same notations, it suffices to prove that for all $t>0$,

$$\lim_m \sum_{k=0}^{2^m-1}\big(X_{t_{k+1}}^{(r_1)} - X_{t_k}^{(r_1)}\big)\cdots\big(X_{t_{k+1}}^{(r_n)} - X_{t_k}^{(r_n)}\big) = \phi_{r_1+\cdots+r_n}\big((0,t]\big), \text{ in probability.}$$

This convergence follows from Proposition 3.3 by polarization. □

4 The Hu–Meyer formula: measures

The Hu–Meyer formula gives the relationship between the product measure $\phi^{\otimes n}$ and the Itô stochastic measures $St_{\widehat{0}}^{\mathbf{r}}$. In this section we obtain this formula for measures and in the next one we extend it to the corresponding integrals.

The idea of Hu–Meyer formula is the following. Given $C\in\mathcal{B}([0,T]^n)$, we can decompose

$$C = \bigcup_{\sigma\in\Pi_n} C_\sigma.$$

So

$$\phi^{\otimes n}(C) = \sum_{\sigma\in\Pi_n}\phi^{\otimes n}(C_\sigma).$$

Next step is to express each $\phi^{\otimes n}(C_\sigma)$ as a multiple Itô stochastic measure. For example, take $n=3$, $\sigma=\big\{\{1\},\{2,3\}\big\}$ and $C=A^3$. Then,

$$A_\sigma^3 = \{(s,t,t),\ \ s,t\in A,\ s\neq t\},$$

and we will prove that

$$\phi^{\otimes 3}(A_\sigma^3) = St_{\widehat{0}}^{(1,2)}(A^2).$$

That is, both the product measure and the product set on the last two variables collapse to produce a diagonal measure, and since $s\neq t$, we get an Itô measure. To handle in general this property, we need some notations.

Given a partition $\sigma \in \Pi_n$ with blocks $B_1, \dots, B_m$, we can order the blocks in agreement with the minimum element of each block. When necessary, we assume that the blocks have been ordered with that procedure and we simply say that $B_1, \dots, B_m$ are ordered. In that situation, we write

$$\overline{\sigma} = (\#B_1, \dots, \#B_m), \tag{13}$$

We start considering a set $C = A^n$, with $A \in \mathcal{B}([0,T])$, and later we extend the Hu–Meyer formula to an arbitrary set $C \in \mathcal{B}([0,T]^n)$.

Theorem 4.1. *Let $A \in \mathcal{B}([0,T])$. Then*

$$\phi^{\otimes n}(A^n) = \sum_{\sigma \in \Pi_n} St_{\widehat{0}}^{\overline{\sigma}}(A^{\#\sigma}). \tag{14}$$

To prove this theorem we need two lemmas. The first one is an invariance type property of product measures under permutations. We remember some standard notations:

Notations 4.2. We denote by $\mathfrak{G}_n$ the set of permutations of $1, \dots, n$. Consider $p \in \mathfrak{G}_n$.

1. For a partition $\sigma \in \Pi_n$ with blocks $B_1, \dots, B_m$, we write $p(\sigma)$ for the partition with blocks $W_j = p(B_j) = \{p(i),\ i \in B_j\}$. Note that in general the blocks $W_1, \dots, W_m$ are not ordered, even when $B_1, \dots, B_m$ are.

2. For a vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write

$$p(\boldsymbol{x}) = \big(x_{p(1)}, \dots, x_{p(n)}\big).$$

Given $C \subset \mathbb{R}^n$, we put

$$p(C) = \{p(\boldsymbol{x}),\ \text{for } \boldsymbol{x} \in C\}.$$

Lemma 4.3. *Let $p \in \mathfrak{G}_n$ and $r_1, \dots, r_n \geq 1$. Then for every $C \in \mathcal{B}([0,T]^n)$,*

$$\big(\phi_{r_{p(1)}} \otimes \cdots \otimes \phi_{r_{p(n)}}\big)(p(C)) = \big(\phi_{r_1} \otimes \cdots \otimes \phi_{r_n}\big)(C), \tag{15}$$

and

$$St_{\widehat{0}}^{p(\boldsymbol{r})}(p(C)) = St_{\widehat{0}}^{\boldsymbol{r}}(C). \tag{16}$$

Proof.
Define the vector measure

$$\Psi(C) = \big(\phi_{r_{p(1)}} \otimes \cdots \otimes \phi_{r_{p(n)}}\big)(p(C)).$$

For $C = A_1 \times \cdots \times A_n$, we have that

$$p(A_1 \times \cdots \times A_n) = A_{p(1)} \times \cdots \times A_{p(n)},$$

and it is clear that

$$\Psi(C) = \big(\phi_{r_1} \otimes \cdots \otimes \phi_{r_n}\big)(C).$$

Then, equality (15) follows from Proposition 2.1.

To prove (16), first note that, by definition, the Itô stochastic measure satisfies

$$St_{\widehat{0}}^{\boldsymbol{r}}(C) = St_{\widehat{0}}^{\boldsymbol{r}}(C_{\widehat{0}}).$$

Moreover $\big(p(C)\big)_{\widehat{0}} = p\big(C_{\widehat{0}}\big)$. So it suffices to prove (16) for a set $C = C_{\widehat{0}}$. Denote by $\mathcal{B}^n_{\widehat{0}}$ the σ-algebra trace of $\mathcal{B}([0,T]^n)$ with $[0,T]^n_{\widehat{0}}$, which is composed by all sets $C_{\widehat{0}}$, with $C \in \mathcal{B}([0,T]^n)$. This σ-algebra is generated (on $[0,T]^n_{\widehat{0}}$) by the family of rectangles $A_1 \times \cdots \times A_n$, with $A_1, \dots, A_n$ pairwise disjoint. By Proposition 2.1, we only need to check (16) for this type of rectangles, and the property reduces to (15) $\square$

The next lemma is an important step in proving Theorem 4.1. To have an insight on its meaning, consider the following example: Let $n = 4$ and $\sigma = \big\{\{1\},\{2\},\{3,4\}\big\}$. With a slight abuse of notation, we can write,

$$A^4_{\geq\sigma} = \{(s,t,u,u) : s,t,u \in A\} = A^2 \times A^2_{\widehat{1}}.$$

By Theorem 3.4,

$$\big(\phi_{r_1} \otimes \phi_{r_2} \otimes \phi_{r_3} \otimes \phi_{r_4}\big)\big(A^4_{\geq\sigma}\big) = \phi_{r_1}(A)\phi_{r_2}(A)\big(\phi_{r_3} \otimes \phi_{r_4}\big)\big(A^2_{\widehat{1}}\big) = \phi_{r_1}(A)\phi_{r_2}(A)\phi_{r_3+r_4}(A).$$

However, if you consider $\tau = \big\{\{1,\,3\},\,\{2\},\,\{4\}\big\}$, even though τ and σ have the same number of blocks with 1 element and the same number of blocks with 2 elements (they have the same type), the computation of $\big(\phi_{r_1} \otimes \phi_{r_2} \otimes \phi_{r_3} \otimes \phi_{r_4}\big)\big(A^4_{\geq\tau}\big)$ is not so straightforward. The lemma gives such computation. Its proof demands some combinatorial results and it is transferred to Appendix A.3.

Lemma 4.4. *Let $r_1, \dots, r_n \geq 1$, $\sigma \in \Pi_n$ with blocks $B_1, \dots, B_m$ (ordered), and $A \in \mathcal{B}([0,T])$. Then*

$$\big(\phi_{r_1} \otimes \cdots \otimes \phi_{r_n}\big)\big(A^n_{\geq\sigma}\big) = \prod_{j=1}^m \phi_{\sum_{i\in B_j} r_i}(A).$$

Proof of Theorem 4.1. By Proposition 2.4,

$$\phi^{\otimes n}(A^n) = \sum_{\sigma\in\Pi_n} St^{[n]}_\sigma(A^n).$$

So it suffices to prove that

$$St^{[n]}_\sigma(A^n) = St^{\overline{\sigma}}_{\widehat{0}}(A^{\#\sigma}).$$

By the second statement in Proposition 2.4 we have

$$St^{[n]}_\sigma(A^n) = \sum_{\pi\in[\sigma,\widehat{1}]} \mu(\sigma,\pi)\phi^{\otimes n}_\pi(A^n) = \sum_{\pi\in[\sigma,\widehat{1}]} \mu(\sigma,\pi)\phi^{\otimes n}\big(A^n_{\geq\pi}\big). \tag{17}$$

By Lemma 4.4,

$$\phi^{\otimes n}\big(A^n_{\geq\pi}\big) = \prod_{V\in\pi} \phi_{\#V}(A). \tag{18}$$

Let $B_1, \dots, B_m$ be the blocks of $\sigma \in \Pi_n$ (ordered) and write

$$\overline{\sigma} = (\#B_1, \dots, \#B_m) = (s_1 \dots, s_m).$$

The partition $\pi \in [\sigma, \widehat{1}]$, with blocks $V_1, \dots, V_k$, induces a unique partition of $\pi^* \in \Pi_m$, with blocks $W_1, \dots, W_k$ such that

$$V_i = \bigcup_{j\in W_i} B_j,$$

see Proposition A.1 in the Appendix. Hence, for $i = 1, \dots, k$,

$$\phi_{\#V_i}(A) = \phi_{\sum_{j\in W_i}\#B_j}(A) = \phi_{\sum_{j\in W_i} s_j}(A).$$

Thus, from (18) and Lemma 4.4,

$$\phi^{\otimes n}\big(A^n_{\geq\pi}\big) = \prod_{W_i\in\pi*} \phi_{\sum_{j\in W_i} s_j}(A) = \big(\phi_{s_1}\otimes\cdots\otimes\phi_{s_m}\big)(A^m_{\geq\pi^*}). \tag{19}$$

By (17) and (19) using again the bijection between $[\sigma, \widehat{1}]$ and Π_m stated in Proposition A.1 in the Appendix, and Proposition 2.4, we obtain

$$\begin{aligned} St^{[n]}_\sigma(A^n) &= \sum_{\pi\in[\sigma,\widehat{1}]} \mu(\sigma,\pi)\big(\phi_{s_1}\otimes\cdots\otimes\phi_{s_m}\big)(A^m_{\geq\pi^*}) \\ &= \sum_{\rho\in\Pi_m} \mu(\widehat{0},\rho)\big(\phi_{s_1}\otimes\cdots\otimes\phi_{s_m}\big)(A^m_{\geq\rho}) = St^{\overline{\sigma}}_{\widehat{0}}(A^{\#\sigma}). \quad \square \end{aligned}$$

In order to extend the Hu–Meyer formula for a general set in $\mathcal{B}([0,T]^n)$, we use a set function to express for an arbitrary set the contraction from A^n to $A^{\#\sigma}$. That is, given a partition $\sigma \in \Pi_n$, with blocks $B_1,\ldots,B_m$ ordered, we want to contract a set $C \in \mathcal{B}([0,T]^n)$ into a set of $\mathcal{B}([0,T]^{\#\sigma})$ according to the structure of the σ-diagonal sets. With this purpose, define the function

$$\begin{aligned} q_\sigma : [0,T]^{\#\sigma} &\longrightarrow [0,T]^n \\ (x_1,\ldots,x_m) &\rightarrow (y_1,\ldots,y_n) \end{aligned} \tag{20}$$

where $y_i = x_j$, if $i \in B_j$. For example, if $n = 4$ and $\sigma = \big\{\{1\},\{2,4\},\{3\}\big\}$,

$$q_\sigma(x_1,x_2,x_3) = (x_1,x_2,x_3,x_2).$$

Note that

$$q_\sigma^{-1}(A^n) = A^{\#\sigma}.$$

See Appendix A.4 for more details.

Theorem 4.5. *Let $C \in \mathcal{B}([0,T]^n)$. Then*

$$\phi^{\otimes n}(C) = \sum_{\sigma\in\Pi_n} St^{\overline{\sigma}}_{\widehat{0}}(q_\sigma^{-1}(C)). \tag{21}$$

Proof of Theorem 4.5. We separate the proof in two steps. In the first one, we show that it is enough to prove the theorem for a rectangle of the form

$$C = A_1^{r_1}\times\cdots\times A_\ell^{r_\ell},$$

where $A_1,\ldots,A_\ell$ are pairwise disjoint. In the second step we check formula (21) for those rectangles.

First step. By Proposition 2.1, it suffices to prove the theorem for a rectangle $A_1\times\cdots\times A_n$. Since every rectangle can be written as a disjoint union of rectangles such that every two components are either equal or disjoint, we consider one of this rectangles, $C = A_1\times\cdots\times A_n$, where for every i, j, $A_i = A_j$ or $A_i\cap A_j = \emptyset$. Now we show that the formula (21) applied to C is invariant by permutations: specifically, we see that for any permutation $p \in \mathfrak{S}_n$,

$$\phi^{\otimes n}(p(C)) = \phi^{\otimes n}(C) \quad \text{and} \quad \sum_{\sigma\in\Pi_n} St^{\overline{\sigma}}_{\widehat{0}}(q_\sigma^{-1}(p(C))) = \sum_{\sigma\in\Pi_n} St^{\overline{\sigma}}_{\widehat{0}}(q_\sigma^{-1}(C)).$$

The first equality is deduced from (15). For the second one, applying Proposition A.4 (i), we have

$$St_{\widehat{0}}^{\overline{\sigma}}(q_\sigma^{-1}(p(C))) = St_{\widehat{0}}^{\overline{\sigma}}(p_1^{-1}(q_{p(\sigma)}^{-1}(C))),$$

where $p_1 \in \mathfrak{G}_{\#\sigma}$ is the permutation that gives the correct order of the blocks of $p(\sigma)$ (see the lines before Proposition A.4). By Lemma 4.3

$$St_{\widehat{0}}^{\overline{\sigma}}(p_1^{-1}(q_{p(\sigma)}^{-1}(C))) = St_{\widehat{0}}^{p_1(\overline{\sigma})}(q_{p(\sigma)}^{-1}(C)) = St_{\widehat{0}}^{\overline{p(\sigma)}}(q_{p(\sigma)}^{-1}(C)),$$

where the last equality is due that $p_1(\overline{\sigma}) = \overline{p(\sigma)}$ by the definition of p_1 (see (37)). Finally,

$$\sum_{\sigma\in\Pi_n} St_{\widehat{0}}^{\overline{\sigma}}(q_\sigma^{-1}(C)) = \sum_{\sigma\in\Pi_n} St_{\widehat{0}}^{\overline{p(\sigma)}}(q_{p(\sigma)}^{-1}(C)),$$

because we are adding over all the set $\Pi_n = \{p(\sigma),\ \sigma \in \Pi_n\}$.

Second step. Consider

$$C = A_1^{r_1} \times \cdots \times A_\ell^{r_\ell},$$

with $A_1 \ldots, A_\ell$ pairwise disjoint and $\sum_{i=1}^{\ell} r_i = n$. By Theorem 4.1,

$$\begin{aligned}\phi^{\otimes n}(A_1^{r_1} \times \cdots \times A_\ell^{r_\ell}) &= \prod_{i=1}^{\ell} \phi^{\otimes r_i}(A_i^{r_i}) = \prod_{i=1}^{\ell} \sum_{\sigma_i\in\Pi_{r_i}} St_{\widehat{0}}^{\overline{\sigma}_i}(A_i^{\#\sigma_i}) \\ &= \sum_{\sigma_1\in\Pi_{r_1},\ldots,\sigma_\ell\in\Pi_{r_\ell}} St_{\widehat{0}}^{\overline{\sigma}_1,\cdots,\overline{\sigma}_\ell}\big(A_1^{\#\sigma_1} \times \cdots \times A_\ell^{\#\sigma_\ell}\big).\end{aligned}$$

where the last equality is due to the fact that

$$\big(A_1^{\#\sigma_1} \times \cdots \times A_\ell^{\#\sigma_\ell}\big)_{\widehat{0}} = \big(A_1^{\#\sigma_1}\big)_{\widehat{0}} \times \cdots \times \big(A_\ell^{\#\sigma_\ell}\big)_{\widehat{0}},$$

and the definition of the Itô measure $St_{\widehat{0}}^{\overline{\sigma}_1,\cdots,\overline{\sigma}_\ell}$.

Let $\tau \in \Pi_n$ be the partition with blocks

$$\begin{aligned} F_1 &= \{1, \ldots, r_1\}, \\ F_2 &= \{r_1 + 1, \ldots, r_1 + r_2\}, \\ &\vdots \\ F_\ell &= \{r_1 + \cdots + r_{\ell-1} + 1, \ldots, n\}. \end{aligned}$$

There is a bijection between the elements $\sigma \in \Pi_n$, with $\sigma \leq \tau$, and $(\sigma_1, \ldots, \sigma_\ell) \in \Pi_{r_1} \times \cdots \times \Pi_{r_\ell}$ such that

$$\overline{\sigma} = (\overline{\sigma}_1, \ldots, \overline{\sigma}_\ell) \quad \text{and} \quad q_\sigma^{-1}(A_1^{r_1} \times \cdots \times A_\ell^{r_\ell}) = A_1^{\#\sigma_1} \times \cdots \times A_\ell^{\#\sigma_\ell},$$

where we use equality (36) in the Appendix. Then,

$$\phi^{\otimes n}(A_1^{r_1} \times \cdots \times A_\ell^{r_\ell}) = \sum_{\sigma\in\Pi_n,\ \sigma\leq\tau} St_{\widehat{0}}^{\overline{\sigma}}(q_\sigma^{-1}(A_1^{r_1} \times \cdots \times A_\ell^{r_\ell})) = \sum_{\sigma\in\Pi_n} St_{\widehat{0}}^{\overline{\sigma}}(q_\sigma^{-1}(A_1^{r_1} \times \cdots \times A_\ell^{r_\ell})),$$

where the last equality is due to the fact that if $\sigma \not\leq \tau$, then $q_\sigma^{-1}(A_1^{r_1} \times \cdots \times A_\ell^{r_\ell}) = \emptyset$ (see (36)). $\square$

5 Multiple Itô and Stratonovich integral, and the corresponding Hu–Meyer formula

We extend Theorem 4.5 to integrals with respect to the random measures involved. We first define an Itô type multiple integral and an integral with respect to the product measure.

5.1 Multiple Itô Stochastic integral

We generalize the multiple Itô integral with respect to the Brownian motion (Itô [10], see also [11]) to a multiple integral with respect to the Lévy processes $X^{(r_1)}, \dots, X^{(r_n)}$. As we will prove, that integral can be interpreted as the integral with respect to the Itô stochastic measure. The ideas used to construct this integral are mainly the Itô's ones, however, the fact that these processes (in general) are not centered obstructs the classical isometry property, being substituted by an inequality.

Write $L_n^2 = L^2([0,T]^n, \mathcal{B}([0,T]^n), (dt)^{\otimes n})$. Denote by $\mathcal{E}_n^{\mathrm{Ito}}$ the set of the so-called Itô-elementary functions, having the form

$$f(t_1, \dots, t_n) = \sum_{i_1, \dots, i_n = 1}^{m} a_{i_1, \dots, i_n} \mathbf{1}_{A_{i_1} \times \cdots \times A_{i_n}}(t_1, \dots, t_n),$$

where $A_1, \dots, A_m \in \mathcal{B}([0,T])$ are pairwise disjoint, and $a_{i_1,\dots,i_n}$ is zero if two indices are equal. It is well known (see Itô [10]) that $\mathcal{E}_n^{\mathrm{Ito}}$ is dense in L_n^2. Consider $f \in \mathcal{E}_n^{\mathrm{Ito}}$ and define the multiple Itô integral of f with respect to $X^{(r_1)}, \dots, X^{(r_n)}$ by

$$I_n^{(r_1,\dots,r_n)}(f) = \sum_{i_1, \dots, i_n = 1}^{m} a_{i_1,\dots,i_n} \phi_{r_1}(A_{i_1}) \cdots \phi_{r_n}(A_{i_n}).$$

Lemma 5.1. *Let $f \in \mathcal{E}_n^{\mathrm{Ito}}$ and $\boldsymbol{r} = (r_1, \dots, r_n)$. Then*

$$\mathbb{E}\big[(I_n^{\boldsymbol{r}}(f))^2\big] \le \alpha_{\boldsymbol{r}} \int_{[0,T]^n} f^2(t_1, \dots, t_n)\, dt_1 \cdots dt_n,$$

where $\alpha_{\boldsymbol{r}}$ is a constant that depends on $r_1, \dots, r_n$ but not on f.

Proof.

The proof follows exactly the same steps as that of Theorem 4.1 in Engel [7]. The key point is that the measures ϕ_{r_i} can be written as

$$\phi_{r_i}(A) = \overline{\phi_{r_i}}(A) + K_{r_i} \int_A dt,$$

where $\overline{\phi_{r_i}}$ is the centered and independently scattered random measure corresponding to $Y^{(r_i)}$. □.

The extension of the multiple Itô stochastic integral to L_n^2, stated below, is proved as in the Brownian case, see Itô [10].

Theorem 5.2. *The map*

$$\begin{aligned} I_n^{\boldsymbol{r}} : \mathcal{E}_n^{\mathrm{Ito}} &\longrightarrow L^2(\Omega) \\ f &\longrightarrow I_n^{\boldsymbol{r}}(f) \end{aligned}$$

can be extended to a unique linear continuous map from L_n^2 to $L^2(\Omega)$. In particular, $I_n^{\boldsymbol{r}}(f)$ satisfies the inequality

$$\mathbb{E}\big[(I_n^{\boldsymbol{r}}(f))^2\big] \le \alpha_{\boldsymbol{r}} \int_{[0,T]^n} f^2(t_1, \dots, t_n)\, dt_1 \cdots dt_n. \tag{22}$$

As in the Brownian case, it is useful to express the multiple integral in terms of iterated integrals of the form

$$\int_0^T \Big(\int_0^{t_{i_1}-} \cdots \Big(\int_0^{t_{i_{n-1}}-} f(t_{i_1}, \dots, t_{i_n})\, dX_{t_{i_n}}^{(r_{i_n})} \Big) \cdots dX_{t_{i_2}}^{(r_{i_2})} \Big) dX_{t_{i_1}}^{(r_{i_1})},$$

where $i_1, \dots, i_n$ is a permutation of $1, \dots, n$. This integral is properly defined for $f \in L_n^2$. This can be checked using the decomposition of $X^{(r_i)}$ as a special semimartingale $X_t^{(r_i)} = K_{r_i}\, t + Y_t^{(r_i)}$, where, as we said in Section 3, $Y^{(r_i)}$ is a square integrable martingale with predictable quadratic variation $\langle Y^{(r_i)}, Y^{(r_i)} \rangle_t = K_{2r_i} t$. The previous iterated integral then reduces to a linear combination of iterated integrals of type

$$\int_0^T \Big(\int_0^{t_{i_1}-} \cdots \Big(\int_0^{t_{i_{n-1}}-} f(t_1, \dots, t_n)\, dZ_{t_{i_n}}^{(n)} \Big) \cdots dZ_{t_{i_2}}^{(2)} \Big) dZ_{t_{i_1}}^{(1)},$$

being $Z_t^{(j)}$ either t or $Y_t^{(r_j)}$. Hence, at each iteration, the integrability condition

$$\mathbb{E}\Big[\int_0^t g^2\, d\langle Z^{(i)}, Z^{(i)} \rangle \Big] < \infty$$

of a predictable process g with respect to $Z^{(i)}$ can be easily verified.

Next proposition gives the precise expression of the multiple integral as a sum of iterated integrals. Since we are integrating with respect to different processes, we need to separate the space $[0, T]^n$ into simplexs.

Proposition 5.3. *Let $f \in L_n^2$. Then*

$$I_n^{(r_1, \dots, r_n)}(f) = \sum_{p \in \mathfrak{G}_n} \int \cdots \int_{p(\Sigma_n)} f(t_1, \dots, t_n)\, dX_{t_1}^{(r_1)} \cdots dX_{t_n}^{(r_n)},$$

where $\Sigma_n = \{0 < t_1 < \cdots < t_n < T\}$, and the integrals on the right hand side are interpreted as iterated integrals.

Proof

By linearity and density arguments, it suffices to consider a function

$$f = \mathbf{1}_{A_1 \times \cdots \times A_n},$$

where $A_i = (s_i, t_i]$ are pairwise disjoint, and a computation gives the result. □

When $r_1 = \cdots = r_n = 1$, we write $I_n(f)$ instead of $I_n^{(1, \dots, 1)}(f)$; in that case, the multiple Itô integral enjoys nicer properties.

Proposition 5.4.

1. *Let $f \in L_n^2$. Then*

$$I_n(f) = I_n\big(\widetilde{f}\,\big),$$

where $\widetilde{f}$ is the symmetrization of f:

$$\widetilde{f} = \frac{1}{n!} \sum_{p \in \mathfrak{G}_n} f \circ p. \tag{23}$$

2. *Assume* $\mathbb{E}[X_t]=0$. *For* $f,g\in L_n^2$,

$$\mathbb{E}[I_n(f)I_m(g)]=\delta_{n,m}K_2^n\, n!\int_{[0,T]^n}\widetilde{f}\,\widetilde{g}\,d\boldsymbol{t},$$

where $\delta_{n,m}=1,$ *if* $n=m$, *and 0 otherwise.*

3. *Let* $f\in L_n^2$ *be a symmetric function. Then*

$$I_n(f)=n!\int_0^T\Big(\int_0^{t_1-}\cdots\Big(\int_0^{t_{n-1}-}f(t_1,\dots,t_n)\,dX_{t_n}\Big)\cdots dX_{t_2}\Big)dX_{t_1}.$$

We now state the relationship between the Itô stochastic measure $St_{\widehat{0}}^{\boldsymbol{r}}$ and the Itô multiple integral $I_n^{\boldsymbol{r}}$.

Proposition 5.5. *Let* $C\in\mathcal{B}([0,T]^n)$ *and* $\boldsymbol{r}=(r_1,\dots,r_n)$. *Then*

$$St_{\widehat{0}}^{\boldsymbol{r}}(C)=I_n^{\boldsymbol{r}}(\mathbf{1}_C). \tag{24}$$

Proof.

Thanks to (22) the map $C\mapsto I_n^{\boldsymbol{r}}(\mathbf{1}_C)$ defines a vector measure on $\mathcal{B}([0,T]^n)$. On the left hand of (24), the Itô measure satisfies

$$St_{\widehat{0}}^{\boldsymbol{r}}(C)=St_{\widehat{0}}^{\boldsymbol{r}}(C_{\widehat{0}}). \tag{25}$$

Now, look at right hand side of (24). For $\pi\in\Pi_n$, we have that $C_\pi=C\cap[0,T]_\pi^n$. For all $\pi>\widehat{0}$,

$$\mathbb{E}\big[\big(I_n^{\boldsymbol{r}}(\mathbf{1}_{C_\pi})\big)^2\big]\le\alpha_n\int_{[0,T]^n}\mathbf{1}_{C_\pi}dt_1\cdots dt_n\le\alpha_n\int_{[0,T]^n}\mathbf{1}_{[0,T]_\pi^n}\,dt_1\cdots dt_n=0.$$

Hence,

$$I_n^{\boldsymbol{r}}(\mathbf{1}_C)=I_n^{\boldsymbol{r}}(\mathbf{1}_{C_{\widehat{0}}}). \tag{26}$$

From (25) and (26), it suffices to prove (24) for a set $C=C_{\widehat{0}}$. As in the proof of the second part of Lemma 4.3, this can be reduced to check that equality for a rectangle $(s_1,t_1]\times\cdots\times(s_n,t_n]$, with the intervals pairwise disjoint. This follows from the fact that both sides of (24) are equal to $\phi_{r_1}((s_1,t_1])\cdots\phi_{r_n}((s_n,t_n])$. $\square$

The property $I_n(f)=I_n\big(\widetilde{f}\big)$ is lost when the integrators are different. However, from Proposition 5.5 and (16) we can deduce the following useful property:

Proposition 5.6. *Let* $f\in L_n^2$, *and* $\boldsymbol{r}=(r_1,\dots,r_n)$, *where* $r_1,\dots,r_n\ge1$. *Consider* $p\in\mathfrak{S}_n$. *Then*

$$I_n^{\boldsymbol{r}}(f)=I_n^{p(\boldsymbol{r})}\big(f\circ p^{-1}\big).$$

5.2 Multiple Stratonovich Integral and Hu–Meyer formula

Given $f:[0,T]^n\to\mathbb{R}$ (the hypothesis will be added later), the integral with respect to the product measure $\phi^{\otimes n}$ is called the multiple Stratonovich integral, and denoted by $I_n^S(f)$. Its basic property is that the integral of a product function factorizes:

$$I_n^S(g_1\otimes\cdots\otimes g_n)=I_1^S(g_1)\cdots I_1^S(g_n),$$

where

$$I_1^S(g) = I_1(g) = \int_0^T g(t)\, dX_t.$$

In order to construct this integral, we consider ordinary simple functions of the measurable space $([0,T]^n, \mathcal{B}([0,T]^n))$. Specifically, denote by $\mathcal{E}_n^{\text{Strato}}$ the set of functions with the form

$$f = \sum_{i=1}^k a_i \mathbf{1}_{C_i},$$

where $C_i \in \mathcal{B}([0,T]^n),\ i = 1, \dots, k$. For such f, define the multiple Stratonovich integral by

$$I_n^S(f) = \sum_{i=1}^k a_i \phi^{\otimes n}(C_i).$$

The integral of an simple function does not depend on its representation, and it is linear. Moreover,

Proposition 5.7. *Let $f \in \mathcal{E}_n^{\text{Strato}}$. Then we have the Hu–Meyer formula*

$$I_n^S(f) = \sum_{\sigma \in \Pi_n} I_{\#\sigma}^{\overline{\sigma}}(f \circ q_\sigma), \tag{27}$$

where the function $q_\sigma : [0,T]^{\#\sigma} \to [0,T]^n$ is introduced in (20), $\overline{\sigma} = (\#B_1, \dots, \#B_m)$ is the vector whose components are the sizes of the ordered blocks of σ, and $I_m^{(s_1,\dots,s_m)}$ is the multiple Itô integral of order m with respect to the measures $\phi_{s_1}, \dots, \phi_{s_m}$.

Proof. By linearity, it suffices to consider $f = \mathbf{1}_C$, where $C \in \mathcal{B}([0,T]^n)$. A generic term on the right hand side of (27) is

$$I_{\#\sigma}^{\overline{\sigma}}(\mathbf{1}_C \circ q_\sigma),$$

and

$$\mathbf{1}_C \circ q_\sigma = \mathbf{1}_{q_\sigma^{-1}(C)}.$$

Hence, by Proposition 5.5,

$$I_{\#\sigma}^{\overline{\sigma}}(\mathbf{1}_C \circ q_\sigma) = St_0^{\overline{\sigma}}(q_\sigma^{-1}(C)),$$

and (27) follows from Theorem 4.5. □

Let $\sigma \in \Pi_n$, with $\#\sigma = m$, and denote by λ_σ the image measure of the Lebesgue measure $(dt)^{\otimes m}$ by the function $q_\sigma : [0,T]^m \to [0,T]^n$. The image measure theorem implies that for $f : [0,T]^n \to \mathbb{R}$ measurable, positive or λ_σ–integrable,

$$\int_{[0,T]^n} f(t_1, \dots, t_n)\, d\lambda_\sigma(t_1, \dots, t_n) = \int_{[0,T]^m} f(q_\sigma(t_1, \dots, t_m))\, dt_1 \cdots dt_m. \tag{28}$$

Define on $\mathcal{B}([0,T]^n)$ the measure

$$\Lambda_n = \sum_{\sigma \in \Pi_n} \lambda_\sigma,$$

and write $L^2(\Lambda_n)$ for $L^2([0,T]^n, \mathcal{B}([0,T]^n), \Lambda_n)$.

In order to extend the multiple Stratonovich integral we need the following inequality of norms:

Lemma 5.8. *Let $f \in \mathcal{E}_n^{\mathrm{Strato}}$. Then*

$$\mathbb{E}\Big[\big(I_n^S(f)\big)^2\Big] \le C \int_{[0,T]^n} f^2 \, d\Lambda_n, \tag{29}$$

where C is a constant.

Proof. By (27), (22), and (28),

$$\mathbb{E}\Big[\big(I_n^S(f)\big)^2\big)\Big] \le C \sum_{\sigma\in\Pi_n} \mathbb{E}\Big[\big(I_{\#\sigma}^{\overline{\sigma}}(f\circ q_\sigma)\big)^2\Big] \le C \sum_{\sigma\in\Pi_n} \int_{[0,T]^{\#\sigma}} (f\circ q_\sigma)^2 dt_1 \cdots dt_{\#\sigma}$$
$$= C \sum_{\sigma\in\Pi_n} \int_{[0,T]^n} f^2 d\lambda_\sigma = C \int_{[0,T]^n} f^2 d\Lambda_n. \quad \square$$

The main result of the paper is the following theorem:

Theorem 5.9. *The map $I_n^S : \mathcal{E}_n^{\mathrm{Strato}} \to L^2(\Omega)$ can be extended to a unique linear continuous map from $L^2(\Lambda_n)$ to $L^2(\Omega)$, and we have the Hu–Meyer formula*

$$I_n^S(f) = \sum_{\sigma\in\Pi_n} I_{\#\sigma}^{\overline{\sigma}}(f\circ q_\sigma). \tag{30}$$

Proof.
The extension of I_n^S to a continuous map on $L^2(\Lambda_n)$ is proved using a density argument and the inequality (29). To prove the Hu–Meyer formula, let $f \in L^2(\Lambda_n)$ and $\{f_k,\, k \ge 1\} \subset \mathcal{E}_n^{\mathrm{Strato}}$ such that $\lim_k f_k = f$ in $L^2(\Lambda_n)$. For every $\sigma \in \Pi_n$, we have $\lim_k f_k\circ q_\sigma = f\circ q_\sigma$ in $L^2_{\#\sigma}$; hence, from Theorem 5.2 the Itô integrals on the right hand side of (30) converge, and the formula follows from Proposition 5.7. $\square$

Remarks 5.10.

(1) Let $g_1, \dots, g_n \in L^{2n}([0,T], dt)$. Then $g_1 \otimes \cdots \otimes g_n \in L_n^2(\Lambda_n)$ and

$$I_n^S(g_1 \otimes \cdots \otimes g_n) = I_1^S(g_1) \cdots I_1^S(g_n).$$

This result is easily checked for simple functions $g_1, \dots, g_n$ and extended to the general case by a density argument.

(2) In order to prove the Hu–Meyer formula for I_n^S it is enough to assume that the process X has moments up to order $2n$.

(3) For $\sigma \in \Pi_n,\ \sigma > \widehat{0}$, the measure λ_σ is singular with respect to the Lebesgue measure on $[0,T]^n$. For example, for $n = 2$ and $\sigma = \widehat{1}$, let $D = \{(t,t),\, t \in [0,T]\}$ be the diagonal of $[0,T]^2$. Then $\lambda_{\widehat{1}}$ is concentrated in D, that has zero Lebesgue measure, but $\lambda_{\widehat{1}}$ is non zero:

$$\lambda_{\widehat{1}}(D) = \int_{[0,T]^2} \mathbf{1}_D(s,t)\, d\lambda_{\widehat{1}}(s,t) = \int_{[0,T]} \mathbf{1}_D(t,t)\, dt = T.$$

(4) As in the Brownian case (see [12, 27, 9, 16] and the references therein), there are other procedures to construct the multiple Stratonovich integral. The main difficulty in every approach is that the usual condition $f \in L_n^2$ in Itô's theory is not sufficient to guarantee the multiple Stratonovich integrability of f. The reason is that one needs to control the behaviour of f on the diagonal sets $[0,T]^n_\sigma$, that have zero Lebesgue measure when $\sigma > \widehat{0}$. We solve this difficulty using the norm induced by the measure Λ_n, which seems to be the appropriate to deal with the diagonal sets, avoiding in this way the difficulty of a manageable definition of the *traces*.

When the function $f \in L^2(\Lambda_n)$ is symmetric, the Hu–Meyer formula can be considerably simplified. We show that we can assume that symmetry on f without loss of generality.

Proposition 5.11. *Let $f \in L^2(\Lambda_n)$. Then $I_n^S(f) = I_n^S(\widetilde{f})$, where $\widetilde{f}$ is the symmetrization of f (see (23)).*

Proof.
The proof is straightforward for $f = \mathbf{1}_C$, $C \in \mathcal{B}([0,T]^n)$, using Lemma 4.3. By linearity is extended to $\mathcal{E}_n^{\text{Strato}}$ and by density to $L^2(\Lambda_n)$. $\square$

Next we show the Hu–Meyer formula for a symmetric function f. In general (for f symmetric), the function $f \circ q_\sigma$ is non symmetric, but as we will see in the proof of the next theorem, its multiple Itô integral depends only on the block structure of σ (the type of σ). For example, with $n = 3$, $f(t_1, t_2, t_3) = t_1 t_2 t_3$ and $\sigma = \big\{\{1\},\, \{2,3\}\big\}$, we have that

$$f(q_\sigma(t_1,t_2)) = t_1\, t_2^2,$$

that is non symmetric. Its integral is

$$I_{\#\sigma}^{\overline{\sigma}}(f \circ q_\sigma) = I_2^{(1,2)}(f \circ q_\sigma) = I_2^{(1,2)}(t_1\, t_2^2).$$

Take $\pi = \big\{\{1,3\},\, \{2\}\big\}$. Then $f(q_\pi(t_1,t_2)) = t_1^2\, t_2$ and

$$I_{\#\pi}^{\overline{\pi}}(f \circ q_\pi) = I_2^{(2,1)}(t_1^2\, t_2) = I_2^{(1,2)}(t_1\, t_2^2),$$

where the last equality is due to Proposition 5.6.

We use the following notations: Given nonnegative integers $r_1, \ldots, r_k$ such that $\sum_{i=1}^k i r_i = n$, we write

$$[r_1, r_2, \ldots, r_k] = (\underbrace{1, \ldots, 1}_{r_1}, \underbrace{2 \ldots, 2}_{r_2}, \ldots).$$

Note that this corresponds to $\overline{\sigma}$ when

$$\sigma = \big\{\{1\}, \ldots, \{r_1\},\, \{r_1+1, r_1+2\}, \ldots, \{r_1+2r_2-1, r_1+2r_2\}, \ldots\big\}.$$

We also write $q_{r_1,\ldots,r_k}$ for q_σ, with σ the above partition.

Theorem 5.12. *Let $f \in L_n^2(\Lambda_n)$ be a symmetric function. Then*

$$I_n^S(f) = \sum \frac{n!}{r_1!(2!)^{r_2} r_2! \cdots (k!)^{r_k} r_k!} I_{r_1+\cdots+r_k}^{[r_1,\ldots,r_k]}\big(f \circ q_{r_1,\ldots,r_k}\big), \tag{31}$$

where the sum is extended over all nonnegative integers $r_1, \ldots, r_k$ such that $\sum_{i=1}^k i r_i = n$, for $k = 1, \ldots, n$.

Proof. Let $f \in L^2(\Lambda_n)$ symmetric. For every $\sigma \in \Pi_n$ and $p \in \mathfrak{G}_n$,

$$I_{\#p(\sigma)}^{\overline{p(\sigma)}}(f \circ q_{p(\sigma)}) =_{(a)} I_{\#p(\sigma)}^{\overline{p(\sigma)}}(f \circ p^{-1} \circ q_\sigma \circ p_1^{-1}) =_{(b)} I_{\#p(\sigma)}^{\overline{p(\sigma)}}(f \circ q_\sigma \circ p_1^{-1}) =_{(c)} I_{\#\sigma}^{\overline{\sigma}}(f \circ q_\sigma),$$

where (a) is due to Proposition A.4 (ii), the equality (b) follows from the symmetry of f, and (c) from Proposition 5.6 and the fact that p_1 gives the correct order of $p(\sigma)$ (see (37)). This implies that all the partitions that have the same number of blocks of 1 element, the same number with two elements, etc.

(that is, they have the same type) give the same Itô multiple integral in the Hu–Meyer formula. To obtain (31) it suffices to count the number of partitions of $\{1,\ldots,n\}$ with r_1 blocks with 1 element, r_2 blocks with 2 elements,..., r_k blocks with k elements, which is

$$\frac{n!}{r_1!(2!)^{r_2}r_2!\cdots(k!)^{r_k}r_k!}. \quad \square$$

Final remark. One may expect that by decomposing the Lévy process into a sum of two independent processes, one with the small jumps and the other with the large ones, the assumption of the existence of moments could be avoided. However, this decomposition introduces dramatic changes to the context of the work, and such an extension is beyond the scope and purposes of the present paper.

6 Special cases

6.1 Brownian motion

When $X=W$ is a standard Brownian motion,

$$\phi_2([0,t])=t, \quad \text{and} \quad \phi_n=0,\ n\geq 3.$$

It follows that in the Hu–Meyer formula only the partitions with all blocks of cardinality 1 or 2 give a contribution, and all the Itô integrals are a mixture of multiple stochastic Brownian integrals and Lebesgue integrals. We can organize the sum according the number of blocks of two elements. For a partition having j blocks of 2 elements, and $f\in L^2_n(\Lambda_n)$ symmetric, the multiple Itô integral is

$$\begin{aligned} I_{n-j}^{[n-2j,j]}(f) &= \int_{[0,T]^{n-j}} f(s_1,\ldots,s_{n-2j},t_1,t_1,t_2,t_2,\ldots,t_j,t_j)\,dW_{s_1}\cdots dW_{s_{n-2j}}dt_1\cdots dt_j \\ &= I_{n-2j}\Big(\int_{[0,T]^j} f(\cdot,t_1,t_1,t_2,t_2,\ldots,t_j,t_j)\,dt_1\cdots dt_j\Big), \end{aligned}$$

where the last equality is due to a Fubini type theorem. Therefore,

$$I_n^S(f)=\sum_{j=0}^{[n/2]}\frac{n!}{(n-2j)!j!2^j}\,I_{n-2j}\Big(\int_{[0,T]^j} f(\cdot,t_1,t_1,t_2,t_2,\ldots,t_j,t_j)\,dt_1\cdots dt_j\Big) \tag{32}$$

which is the classical Hu–Meyer formula, see [8].

On the other hand, in the measure Λ_n only participate the measures λ_σ corresponding to the partitions above mentioned. Consider the measure $\ell_2=\lambda_{\widehat{1}}$ on $[0,T]^2$, that is, for a positive or ℓ_2 integrable function h,

$$\int_{[0,T]^2} h(s,t)\,d\ell_2(s,t)=\int_{[0,T]} h(t,t)dt.$$

Given the partition $\sigma\in\Pi_n$,

$$\sigma=\big\{\{1\},\ldots,\{n-2j\},\{n-2j+1,n-2j+2\},\ldots,\{n-1,n\}\big\}$$

we have

$$\lambda_\sigma=(dt)^{\otimes(n-2j)}\otimes\ell_2^{\otimes j}.$$

6.2 Poisson process

Let N_t be a standard Poisson process with intensity 1, and consider the process $X_t = N_t - t$. For every $n \geq 2$,

$$X_t^{(n)} = N_t = X_t + t.$$

and hence, a multiple Itô integral can be reduced to a linear combination of multiple integrals where all the integrators are dX or dt. For $f \in L_n^2(\Lambda_n)$ symmetric, each integral $I_{r_1+\cdots+r_k}^{[r_1,\ldots,r_k]}(f \circ q_{r_1,\ldots,r_k})$ in (31) can be expressed in terms of the number of Lebesgue integrals that appear:

$$\begin{aligned} &I_{r_1+\cdots+r_k}^{[r_1,\ldots,r_k]}(f \circ q_{r_1,\ldots,r_k}) \\ &\qquad = \sum_{j=0}^{r_2+\cdots+r_k} I_{r_1+\cdots+r_k-j}\Big(\int_{[0,T]^j} \big(\sum_{\substack{l_1\ldots,l_j=r_1+1 \\ \text{different}}}^{r_1+\cdots+r_k} (f \circ q_{r_1,\ldots,r_k})(t_1,\ldots,t_{r_1+\cdots+r_k})\big) dt_{l_1}\cdots dt_{l_j}\Big), \end{aligned}$$

and the Hu–Meyer formula of Solé and Utzet [28] can be deduced from this expression.

6.3 Gamma process and subordinators

A subordinator is a Lévy process with increasing paths. An important example of a subordinator with moments of all orders is the Gamma process, denoted by $\{G_t,\, t \geq 0\}$, which is the Lévy process corresponding to an exponential law of parameter 1. Its Lévy measure is

$$\nu(dx) = \frac{e^{-x}}{x}\,\mathbf{1}_{\{x>0\}}(x)\,dx.$$

The law of G_t is Gamma with mean t and scale parameter equal to one. A Gamma process can be represented as the sum os its jumps, that are all positive,

$$G_t = \sum_{0<s\leq t} \Delta G_s.$$

The Lévy measure of $G^{(n)}$ is (see Schoutens [26])

$$\nu_n(dx) = \frac{e^{-x^{1/n}}}{nx}\mathbf{1}_{\{x>0\}}(x)\,dx, \quad n \geq 1,$$

and the Teugels martingales are

$$Y_t^{(n)} = \sum_{\{0<s\leq t\}} \big(\Delta G_s\big)^n - (n-1)!\,t, \quad n \geq 1.$$

In this case, unlike the Brownian motion and the Poisson process, the Hu-Meyer formula does not simplify, due to the fact that the diagonal measures cannot be expressed in a simple way in terms of, say, the process and a deterministic measure. However, for Gamma process, and in general, for a subordinator without drift (see below for the definition) with moments of all orders, both the multiple Itô and Stratonovich integrals can be computed pathwise integrating with respect to an ordinary measure. This is a multivariate extension of the property that states that the stochastic integral and the pathwise Lebesgue-Stieljes integral with respect to a semimartingale of bounded variation are equal; such property was proved for the integral with respect to a Lévy process of bounded variation by Millar [18] under weak conditions, and part of our proof follows his scheme.

Let $X = \{X_t,\, t \geq 0\}$ be a subordinator. The Lévy-Itô representation of X takes the form

$$X_t = \gamma_0 t + \sum_{0<s\leq t} \Delta X_s,$$

with $\gamma_0 \geq 0$, see Sato [25, Theorems 21.5 and 19.3]. The number γ_0 is called the *drift* of the subordinator, and we will assume that $\gamma_0 = 0$. Consider the sequence of stopping times $\{T_k,\, k \geq 1\}$ with disjoint graphs that exhausts the jumps of X: $\Delta X_{T_k} \neq 0$, $\forall k \geq 1$, and X only has jumps on these times; see, *e.g.*, Dellacherie and Meyer [6, Theoreme B, pag. XIII] for a construction of this sequence. Denote by J_n the set of n-tuples $(T_{i_1}, \dots T_{i_n})$, with $T_{i_j} \leq T$, and all entries different. For $r_1, \dots, r_n \geq 1$, define a measure on $[0,T]^n$ by

$$m_{r_1,\dots,r_n} = \sum_{(T_{i_1},\dots,T_{i_n})\in J_n} \big(\Delta X_{T_{i_1}}\big)^{r_1} \cdots \big(\Delta X_{T_{i_n}}\big)^{r_n} \delta_{(T_{i_1},\dots,T_{i_n})},$$

where δ_a is a Dirac measure at point a. We have the following property:

Proposition 6.1. *Let $X = \{X_t,\, t \geq 0\}$ be a subordinator without drift and with moments of all orders. With the preceding notations, for every $f \in L_n^2$,*

$$I_n^{(r_1,\dots,r_n)}(f) = \int_{[0,T]^n} f\, dm_{r_1,\dots,r_n}, \text{ a.s.} \tag{33}$$

Proof.

First, note two facts:

(a) $m_{r_1,\dots,r_n}$ is a finite measure:

$$\begin{aligned} m_{r_1,\dots,r_n}([0,T]^n) &= \sum_{(T_{i_1},\dots T_{i_n})\in J_n} \big(\Delta X_{T_{i_1}}\big)^{r_1} \cdots \big(\Delta X_{T_{i_n}}\big)^{r_n} \\ &\leq \sum_{T_{i_1}\leq T,\dots,T_{i_n}\leq T} \big(\Delta X_{T_{i_1}}\big)^{r_1} \cdots \big(\Delta X_{T_{i_n}}\big)^{r_n} = X_T^{(r_1)} \cdots X_T^{(r_n)} < \infty. \end{aligned}$$

(b) If the intervals $(s_1,t_1], \dots, (s_n,t_n]$ are pairwise disjoint, then (33) is true for $f = \mathbf{1}_{(s_1,t_1]\times\cdots\times(s_n,t_n]}$. The proof is straightforward.

We separate the proof of the proposition in two steps.

Step 1. Formula (33) is true for every map $f : [0,T]^n \to \mathbb{R}$ $\mathcal{B}_0$-measurable and bounded, where $\mathcal{B}_0$ is the σ-field on $[0,T]^n$ generated by the rectangles $(s_1,t_1]\times\cdots\times(s_n,t_n]$, with $(s_1,t_1], \dots, (s_n,t_n]$ pairwise disjoint. To prove this claim we use a convenient monotone class theorem. Denote by $\mathcal{H}$ the family of functions that satisfy (33); it is a vector space such that

(i) $1 \in \mathcal{H}$.

(ii) If $f_m \in \mathcal{H}$, $0 \leq f_m \leq K$ for some constant K, and $f_m \nearrow f$, then $f \in \mathcal{H}$.

To see (i), consider the dyadic partition of $[0,T]$ with mesh 2^{-k}, write

$$B_j = \big((j-1)T2^{-k}, jT2^{-k}\big], \quad j = 1, \dots, 2^k,$$

and define

$$f_k = \sum_{\substack{j_1,\dots,j_n \\ \text{different}}} \mathbf{1}_{B_{j_1}\times\cdots\times B_{j_n}}.$$

By the remark (b) at the beginning of the proof,

$$I_n^{(r_1,\dots,r_n)}(f_k) = \int_{[0,T]^n} f_k \, dm_{r_1,\dots,r_n}.$$

Moreover, $f_k \nearrow 1$ out off the diagonal sets $[0,T]^n_\sigma$, with $\sigma \neq \widehat{0}$, and then $f_k \nearrow 1$ a.e. with respect to the Lebesgue measure, and in L^2_n. Therefore,

$$\lim_k I_n^{(r_1,\dots,r_n)}(f_k) = I_n^{(r_1,\dots,r_n)}(1).$$

On the other hand, for every ω, the measure $m_{r_1,\dots,r_n}$ does not charge on any of the above mentioned diagonal sets. Thus, the convergence $f_k \nearrow 1$ is also $m_{r_1,\dots,r_n}$–a.e. By the monotone convergence Theorem,

$$\lim_k \int_{[0,T]^n} f_k \, dm_{r_1,\dots,r_n} = \int_{[0,T]^n} 1 \, dm_{r_1,\dots,r_n}.$$

and (i) follows.

The point (ii) is deduced directly from the monotone convergence Theorem and taking into account that under the conditions in (ii) we have $f_m \to f$ in L^2_n.

Again by remark (b) above, the indicator of a set $(s_1,t_1] \times \cdots \times (s_n,t_n]$, with $(s_1,t_1],\dots,(s_n,t_n]$ pairwise disjoint, is in $\mathcal{H}$, and this family of sets is closed by intersection. By the monotone class Theorem, it follows that all bounded $\mathcal{B}_0$-measurable functions are in $\mathcal{H}$.

Step 2. Extension of (33) to all $f \in L^2_n$. First, note that $\mathcal{B}_0$ is the σ-field generated by the Borelian sets $B \in \mathcal{B}([0,T]^n)$ such that $B \subset [0,T]^n_{\widehat{0}}$. Then, given $B \in \mathcal{B}([0,T]^n)$, the indicator $\mathbf{1}_{B \cap [0,T]^n_{\widehat{0}}}$ is $\mathcal{B}_0$ measurable. Let $f \in L^2_n$, and assume $f \geq 0$. There is a sequence of simple (and then bounded) functions such that $0 \leq f_m \nearrow f$. Define $f^0_m = f_m \, \mathbf{1}_{[0,T]^n_{\widehat{0}}}$, which is $\mathcal{B}_0$ measurable, and $f^0_m \nearrow f$ a.e. with respect to the Lebesgue measure. The convergence is also in L^2_n, and then $\lim_m I_n(f^0_m) = I(f)$. On the other hand, $f^0_m \nearrow f$, $m_{r_1,\dots,r_n}$–a.e. so

$$\lim_m \int_{[0,T]^n} f^0_m \, dm_{r_1,\dots,r_n} = \int_{[0,T]^n} f \, dm_{r_1,\dots,r_n}.$$

By Step 1, we get the result. For a general $f \in L^2_n$, decompose $f = f^+ - f^-$. $\square$

Finally, for a subordinator without drift and with moments of all orders, the multiple Stratonovich measure can be identified with the n-fold product measure of $\phi = \sum_k \Delta X_{T_k} \delta_{T_k}$. So for $f \in \mathcal{E}^{\text{Srato}}_n$, by definition,

$$I^S_n(f) = \int_{[0,T]^n} f \, d\phi^{\otimes n}.$$

Using similar arguments as in the previous proposition, but easier, it is proved that

$$I^S_n(f) = \int_{[0,T]^n} f \, d\phi^{\otimes n}, \quad \forall f \in L^2(\Lambda_n).$$

Then, the Hu-Meyer formula can be transferred to a pathwise context.

Appendix

A.1 The isomorphism $[\sigma, \widehat{1}] \simeq \Pi_{\#\sigma}$

Fix a partition $\sigma \in \Pi_n$, with blocks $B_1, \dots, B_m$. Let $\pi \geq \sigma$, with blocks $V_1, \dots, V_k$; each block V_i is the union of some of the blocks $B_1, \dots, B_m$. Hence, we can consider the partition $\pi^* \in \Pi_m$ that gives the relationship between the V_i's and the B_j's, that is, π^* has blocks $W_1, \dots, W_k$ defined by

$$V_i = \bigcup_{j \in W_i} B_j, \ i = 1, \dots, k.$$

Proposition A.1. *Let $\sigma \in \Pi_n$ with $\#\sigma = m$. With the above notations, the map*

$$\begin{aligned} [\sigma, \widehat{1}] &\longrightarrow \Pi_m \\ \pi &\mapsto \pi^* \end{aligned}$$

is a bijection and, for $\pi, \tau \in [\sigma, \widehat{1}]$,

$$\pi \leq \tau \quad \Longleftrightarrow \quad \pi^* \leq \tau^*.$$

Moreover,

$$\mu^{(n)}(\sigma, \pi) = \mu^{(m)}(\widehat{0}, \pi^*),$$

where $\mu^{(r)}$ is the Môbius function on Π_r.

The proof is straightforward.

A.2 Permutations and partitions

Let $p : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ be a permutation. This application induces a bijection on Π_n, and a bijection on $\mathbb{R}^n$. Specifically,

1. For a subset $B \subset \{1, \dots, n\}$ we denote by $p(B)$ the image of B by p:

$$p(B) = \{p(j), \text{for } j \in B\}.$$

Given a partition $\sigma \in \Pi_n$, with blocks $B_1, \dots, B_m$, let $p(\sigma)$ be the partition with blocks $W_1, \dots, W_m$ defined by $W_j = p(B_j)$. Note that in general the blocks $W_1, \dots, W_m$ are not ordered. The application

$$\begin{aligned} p : \Pi_n &\longrightarrow \Pi_n \\ \sigma &\mapsto p(\sigma) \end{aligned}$$

is a bijection and for $\sigma, \tau \in \Pi_n$,

$$\sigma \leq \tau \quad \Longleftrightarrow \quad p(\sigma) \leq p(\tau).$$

This last property is clear, because if $V \in \tau$, and $V = B_{r_1} \cup \dots \cup B_{r_k}$, then

$$p(V) = p(B_{r_1}) \cup \dots \cup p(B_{r_k}).$$

Further, this application is compatible with the relationship introduced in Subsection 2.2:

$$i \sim_\sigma j \quad \Longleftrightarrow \quad p(i) \sim_{p(\sigma)} p(j). \tag{34}$$

2. For a vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write

$$p(\boldsymbol{x}) = \big(x_{p(1)}, \dots, x_{p(n)}\big),$$

and the application $\boldsymbol{x} \mapsto p(\boldsymbol{x})$ determines a bijection on $\mathbb{R}^n$, that we also denote by p. For a set $C \subset \mathbb{R}^n$, we write

$$p(C) = \{p(\boldsymbol{x}),\ \text{for}\ \boldsymbol{x} \in C\}.$$

In particular, for $A_1, \dots, A_n \subset \mathbb{R}$,

$$p(A_1 \times \cdots \times A_n) = A_{p(1)} \times \cdots \times A_{p(n)}.$$

Notice that if we look for the position of a particular set, say A_1, in $p(A_1 \times \cdots \times A_n)$, we find it at place $p^{-1}(1)$:

$$\begin{array}{c} A_{i_1} \times \cdots \times \boxed{A_1} \times \cdots \times A_{i_n} \\ \uparrow \\ p^{-1}(1) \end{array}$$

This last observation gives some light to the next property:

Proposition A.2. *Consider* $p \in \mathfrak{S}_n$, $C \subset \mathbb{R}^n$ *and* $\sigma \in \Pi_n$.

(i) $p(C_\sigma) = \big(p(C)\big)_{p^{-1}(\sigma)}$.

(ii) $p(C_{\geq\sigma}) = \big(p(C)\big)_{\geq p^{-1}(\sigma)}$. *In particular,* $p(A^n_{\geq\sigma}) = A^n_{\geq p^{-1}(\sigma)}$.

Proof.

(i) Let $\boldsymbol{x} = (x_1, \dots, x_n) \in p(C_\sigma) \subset p(C)$. Write $p^{-1}(\boldsymbol{x}) = (x_{p^{-1}(1)}, \dots, x_{p^{-1}(n)}) = (y_1, \dots, y_n) = \boldsymbol{y} \in C_\sigma$. Therefore,

$$\boldsymbol{y} \in C, \quad \text{and} \quad y_i = y_j \quad \Longleftrightarrow \quad i \sim_\sigma j.$$

The condition on the right is equivalent to $p^{-1}(i) \sim_{p^{-1}(\sigma)} p^{-1}(j)$ (see (34)). So, returning to the $\boldsymbol{x}$'s,

$$\boldsymbol{x} \in p(C), \quad \text{and} \quad x_{p^{-1}(i)} = x_{p^{-1}(j)} \quad \Longleftrightarrow \quad p^{-1}(i) \sim_{p^{-1}(\sigma)} p^{-1}(j).$$

Call $p^{-1}(i) = r$ and $p^{-1}(j) = s$. We have

$$\boldsymbol{x} \in p(C), \quad \text{and} \quad x_r = x_s \quad \Longleftrightarrow \quad r \sim_{p^{-1}(\sigma)} s.$$

Hence, $\boldsymbol{x} \in \big(p(C)\big)_{p^{-1}(\sigma)}$.

The reciprocal inclusion is analogous.

(ii) Applying (i),

$$p(C_{\geq\sigma}) = p\big(\bigcup_{\pi\geq\sigma} C_\pi\big) = \bigcup_{\pi\geq\sigma} p(C_\pi) = \bigcup_{\pi\geq\sigma} \big(p(C)\big)_{p^{-1}(\pi)} = \bigcup_{\tau\geq p^{-1}(\sigma)} \big(p(C)\big)_\tau = \big(p(C)\big)_{\geq p^{-1}(\sigma)}. \qquad \square$$

Consider a partition $\sigma \in \Pi_n$ with blocks $B_1, \dots, B_m$ (ordered). If the elements of each block are consecutive numbers, then,

$$A^n_{\geq\sigma} = \bigtimes_{j=1}^m A_{\widehat{1}}^{\# B_j}. \tag{35}$$

When σ does not fulfill the previous condition, the expression (35) in not valid. However, since we are interested in computing $\big(\phi_{r_1} \otimes \cdots \otimes \phi_{r_n}\big)(A^n_{\geq\sigma})$, thanks to Lemma 4.3, we fortunately can permute both the set and the product measure to make things work. The next proposition is essential for this purpose.

Proposition A.3. *Let $A \subset \mathbb{R}$ and $\sigma \in \Pi_n$ be a partition, with blocks $B_1, \dots, B_m$ (ordered). There is a permutation $p \in \mathfrak{S}_n$ such that*

$$p\big(A^n_{\geq\sigma}\big) = \bigtimes_{j=1}^{m} A_{\hat{1}}^{\#B_j}.$$

Proof. Write $s_j = \#B_j,\ j = 1, \dots, n$, and let $p' \in \mathfrak{S}_n$ such that

$$\begin{aligned} p'(B_1) &= \{1, \dots, s_1\} \\ p'(B_2) &= \{s_1 + 1, \dots, s_1 + s_2\} \\ &\vdots \end{aligned}$$

Take $p = (p')^{-1}$ and apply Proposition A.2 (ii). $\square$

A.3 Proof of Lemma 4.4

We prove the following lemma:

Lemma 4.4. *Let $r_1, \dots, r_n \geq 1$, $\sigma \in \Pi_n$ with blocks $B_1, \dots, B_m$ (ordered), and $A \in \mathcal{B}[0, T]$. Then*

$$\big(\phi_{r_1} \otimes \cdots \otimes \phi_{r_n}\big)(A^n_{\geq\sigma}) = \prod_{j=1}^{m} \phi_{\sum_{i\in B_j} r_i}(A).$$

Proof. Let $B_1, \dots, B_m$ be the blocks of σ ordered. If σ is such that $A^n_{\geq\sigma} = \bigtimes_{j=1}^{m} A_{\hat{1}}^{\#B_j}$, by Theorem 3.4,

$$\big(\phi_{r_1} \otimes \cdots \otimes \phi_{r_n}\big)(A^n_{\geq\sigma}) = \prod_{j=1}^{m} \Big(\otimes_{i\in B_j} \phi_{r_i}\big(A_{\hat{1}}^{\#B_j}\big)\Big) = \prod_{j=1}^{m} \phi_{\sum_{i\in B_j} r_i}(A).$$

For the general case, let p be the permutation given by Proposition A.3 and write $V_j = p^{-1}(B_j), j = 1, \dots, m$. By Proposition A.3 (first), and $\#B_j = \#V_j$ (second), we have

$$p\big(A^n_{\geq\sigma}\big) = \bigtimes_{j=1}^{m} A_{\hat{1}}^{\#B_j} = \bigtimes_{j=1}^{m} A_{\hat{1}}^{\#V_j}.$$

By Lemma 4.3 and the first part of the proof,

$$\begin{aligned} \big(\phi_{r_1} \otimes \cdots \otimes \phi_{r_n}\big)(A^n_{\geq\sigma}) &= \big(\phi_{r_{p(1)}} \otimes \cdots \otimes \phi_{r_{p(n)}}\big)(p(A^n_{\geq\sigma})) \\ &= \big(\phi_{u_1} \otimes \cdots \otimes \phi_{u_n}\big)\Big(\bigtimes_{j=1}^{m} A_{\hat{1}}^{\#V_j}\Big) = \prod_{j=1}^{m} \phi_{\sum_{i\in V_j} u_i}(A), \end{aligned}$$

where $u_i = r_{p(i)}$. For every $j = 1, \dots, m$,

$$\sum_{i\in V_j} u_i = \sum_{i\in V_j} r_{p(i)} = \sum_{i\in p^{-1}(B_j)} r_{p(i)} = \sum_{i\in B_j} r_i. \quad \square$$

A.4 The function q_σ

Given a partition $\sigma \in \Pi_n$, with blocks $B_1, \dots, B_m$ (ordered), the function q_σ (see (20)) is defined by

$$\begin{aligned} q_\sigma : [0,T]^m &\longrightarrow [0,T]^n \\ (x_1, \dots, x_m) &\to \ (y_1, \dots, y_n) \end{aligned}$$

where $y_i = x_j$, if $i \in B_j$. This function is a bijection between $[0,T]^m$ and $[0,T]^n_{\geq \sigma}$, and it is Borel measurable because

$$q_\sigma^{-1}(A_1 \times \cdots \times A_n) = \Big(\bigcap_{i \in B_1} A_i \Big) \times \cdots \times \Big(\bigcap_{i \in B_m} A_i \Big). \tag{36}$$

Given a partition $\sigma \in \Pi_n$, with blocks (ordered) $B_1, \dots, B_m$ and a permutation $p \in \mathfrak{S}_n$, as we commented, the blocks of $p(\sigma)$ in general are not ordered. It is convenient to consider the permutation $p_1 \in \mathfrak{S}_m$ that gives the correct order of the blocks of $p(\sigma)$, that means, $p_1(1) = i$ if $p(B_i)$ is the first block of $p(\sigma)$, $p_1(2) = j$ if $p(B_j)$ is the second block, and so on; in other words,

$$p(B_{p_1(1)}), \dots, p(B_{p_1(m)})$$

are the blocks of $p(\sigma)$ ordered. Remember that we defined (see (13)) the m–dimensional vector $\overline{\sigma} = (\#B_1, \dots, \#B_m)$. Then

$$p_1(\overline{\sigma}) = \overline{p(\sigma)}. \tag{37}$$

Proposition A.4. *Consider $\sigma \in \Pi_n$, with $\#\sigma = m$, $p \in \mathfrak{S}_n$, and let $p_1 \in \mathfrak{S}_m$ be the permutation that gives the correct order of the blocks of $p(\sigma)$.*

(i) For $A_1, \dots, A_n \in \mathcal{B}([0,T])$,

$$q_\sigma^{-1}\big(p(A_1 \times \cdots \times A_n)\big) = p_1^{-1}\big(q_{p(\sigma)}^{-1}(A_1 \times \cdots \times A_n)\big).$$

(ii) $p^{-1} \circ q_\sigma = q_{p(\sigma)} \circ p_1$.

Proof.

(i) Let $B_1, \dots, B_m$ the blocks of σ (ordered). We have

$$\begin{aligned} q_\sigma^{-1}\big(p(A_1 \times \cdots \times A_n)\big) &= q_\sigma^{-1}(A_{p(1)} \times \cdots \times A_{p(n)}) = \Big(\bigcap_{i \in B_1} A_{p(i)} \Big) \times \cdots \times \Big(\bigcap_{i \in B_m} A_{p(i)} \Big) \\ &= \Big(\bigcap_{i \in p(B_1)} A_i \Big) \times \cdots \times \Big(\bigcap_{i \in p(B_m)} A_i \Big) = G_1 \times \cdots \times G_m, \end{aligned}$$

where

$$G_j = \bigcap_{i \in p(B_j)} A_i, \ j = 1, \dots, m.$$

Since p_1 gives the correct order of $p(B_1), \dots, p(B_m)$,

$$\begin{aligned} q_{p(\sigma)}^{-1}(A_1 \times \cdots \times A_n) &= \Big(\bigcap_{i \in p(B_{p_1(1)})} A_i \Big) \times \cdots \times \Big(\bigcap_{i \in p(B_{p_1(m)})} A_i \Big) \\ &= G_{p_1(1)} \times \cdots \times G_{p_1(m)} = p_1(G_1 \times \cdots \times G_m), \end{aligned}$$

and then

$$p_1^{-1}\big(q_{p(\sigma)}^{-1}(A_1 \times \cdots \times A_n)\big) = G_1 \times \cdots \times G_m.$$

(ii) Consider $\boldsymbol{y} = (y_1, \ldots, y_n) \in [0,T]^n$. Since $\{\boldsymbol{y}\} = \{y_1\} \times \cdots \times \{y_n\}$,

$$(p^{-1} \circ q_\sigma)^{-1}(\{y\}) = q_\sigma^{-1}(\{p(\boldsymbol{y})\}) =_{(*)} p_1^{-1}(q_{p(\sigma)}^{-1}(\{\boldsymbol{y}\})) = (q_{p(\sigma)} \circ p_1)^{-1}(\{y\}),$$

where the equality (*) is due to part (i). $\square$

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