MARTIN BOUNDARY OF A KILLED RANDOM WALK ON A QUADRANT

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Abstract. A complete representation of the Martin boundary of killed random walks on the quadrant $\mathbb{N}^* \times \mathbb{N}^*$ is obtained. It is proved that the corresponding full Martin compactification of the quadrant $\mathbb{N}^* \times \mathbb{N}^*$ is homeomorphic to the closure of the set $\{w = z/(1+|z|) \ : \ z \in \mathbb{N}^* \times \mathbb{N}^*\}$ in $\mathbb{R}^2$. The method is based on a ratio limit theorem for local processes and large deviation techniques.

1. Introduction

The concept of Martin boundary was first introduced for Brownian motion by Martin [15] and next extended for countable discrete time Markov chains by Doob [8] and Hunt [10]. For a Markov chain $(Z(t))$ on a countable set E with the Green function $G(z,z')$, the Martin compactification E_M is the smallest compactification of the set E for which the Martin kernels $K(z,\cdot) = G(z,\cdot)/G(z_0,\cdot)$ extend continuously. See the book of Woess [20] (Chapter IV) or Rogers and Williams [18] (Section III.28) for example. The Martin boundary for homogeneous random walks in $\mathbb{Z}^d$ was obtained by Ney and Spitzer [16].

We identify the Martin boundary of a killed random walk $(Z_+(t))$ on the positive quadrant $\mathbb{N}^* \times \mathbb{N}^*$. Such a random walk has a sub-stochastic transition matrix $(p(z,z') = \mu(z'-z), \ z,z' \in \mathbb{N}^* \times \mathbb{N}^*)$ with some probability measure μ on $\mathbb{Z}^2$, it is identical to a homogeneous random walk $(S(t))$ on the 2-dimensional lattice $\mathbb{Z}^2$ before it first exits from the quadrant $\mathbb{N}^* \times \mathbb{N}^*$ and is killed at the time

$$\tau \doteq \inf\{n \geq 0 \ : S(n) \notin \mathbb{N}^* \times \mathbb{N}^*\}.$$

The random walk $(Z_+(t))$ is therefore not homogeneous : transition probabilities on the boundary of the quadrant $\mathbb{N}^* \times \mathbb{N}^*$ are not the same as in the interior. For non-homogeneous Markov processes, the problem of Martin boundary identification is usually non-trivial and there are few examples where it was resolved :

Cartier [3] described the Martin boundary of random walks on non-homogeneous trees, Doney [7] identified the Martin boundary of a homogeneous random walk $(Z(n))$ on $\mathbb{Z}$ killed on the negative half-line $\{z : z < 0\}$. Alili and Doney [1] identified the Martin boundary for space-time random walk $S(n) = (Z(n), n)$ for a homogeneous random walk $Z(n)$ on $\mathbb{Z}$ killed on the negative half-line $\{z : z < 0\}$. All these results were obtained by using a special linear structure of the processes. The Martin boundary of Brownian motion on a half-space was obtained in the book of Doob [8] by using an explicit form of the Green function.

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In Kurkova and Malyshev [14] the full Martin compactification is obtained by using methods of complex analysis for nearest neighbors random walks on a half-plane $\mathbb{Z}\times\mathbb{N}$ and in the quadrant $\mathbb{Z}^2_+ = \mathbb{N}\times\mathbb{N}$. In a recent paper of Raschel [17], the Martin boundary is obtained for nearest neighbor random walks in $\mathbb{N}\times\mathbb{N}$ with an absorption condition on the boundary also by using methods of complex analysis. Because of the use of the specific algebraic setting of elliptic curves, these methods seem to be difficult to apply when the jump sizes are more general.

The results of Kurkova and Malyshev [14] exhibit a formal similarity between the limiting behavior of the Martin kernel and the optimal large deviation trajectories obtained by Ignatyuk, Malyshev and Scherbakov [13]. A natural idea is then to study the Martin compactification by using large deviation methods. The large deviation approach was first proposed in the papers of Ignatiouk-Robert [12, 11] in order to identify the Martin boundary for partially homogeneous random walks on a half-space $\mathbb{Z}^{d-1}\times\mathbb{N}$. The minimal harmonic functions were determined there by using the methods of Choquet-Deny theory (see Woess [20]) and then the limiting behavior of the Martin kernel was obtained by using an explicit representation of the harmonic functions combined with the large deviation estimates of the Green function and the ratio limit theorem of Markov-additive processes. Unfortunately, the methods of Choquet-Deny theory and the ratio limit theorem are valid only for Markov-additive processes, i.e. when transition probabilities are invariant with respect to the translations on some directions. In the setting of the present paper, for a random walk in the quadrant $\mathbb{N}^*\times\mathbb{N}^*$, such an invariance property cannot hold. Our paper is the first step towards a more ambitious program : to identify the Martin boundary for general partially homogeneous random walks in $\mathbb{N}^n$.

The main idea of our method is the following : to study the asymptotic behavior of the Martin kernels $K(z, z_n)$ for a sequence of points z_n which tends to infinity with $\lim_n z_n/|z_n| = q$, one should consider a twisted random walk conditioned to go to infinity in the direction q. For a non-zero vector $q \in \mathbb{R}^2_+$, such a twisted homogeneous random walk will visit at least one of the boundaries $(-\mathbb{N})\times\mathbb{Z}$ or $\mathbb{Z}\times(-\mathbb{N})$ only a finite number of times. If the corresponding boundary $\{0\}\times\mathbb{N}$ [resp. $\mathbb{N}\times\{0\}$] is removed, the resulting process is then identical to the homogeneous random walk $(S(t))$ before the first time when it hits the set $\mathbb{Z}\times(-\mathbb{N})$ [resp. $(-\mathbb{N})\times\mathbb{Z}$]. The limiting behavior of the Martin kernel of this process corresponding to the direction q is already known in such a setting. The limiting behavior of the Martin kernel of the original process $(Z_+(t))$ should be essentially the same but with a correction given by a potential function. When both coordinates of q are positive, this idea is transformed into a rigorous proof with the aid of large deviation estimates and a generalization of a ratio limit theorem of the paper [12]. When one of the coordinates of q is zero, i.e. when the process is conditioned to go to infinity along one of the boundaries, our proof is much more complicated. In this case we combine large deviation techniques and the ratio limit theorem with delicate estimates obtained from the Harnack inequalities.

We assume that the probability measure μ on $\mathbb{Z}^2$ satisfies the following conditions :

(H1) *The homogeneous random walk $S(t) = (S_1(t), S_2(t))$ on $\mathbb{Z}^2$ having transition probabilities $p_S(z, z') = \mu(z'-z)$ is irreducible and*

$$m \doteq \sum_{z\in\mathbb{Z}^d} z\,\mu(z) \;\neq\; 0,$$

(H2) *The killed random walk* $(Z_+(t))$ *is irreducible on* $\mathbb{N}^* \times \mathbb{N}^*$.
(H3) *The jump generating function*

$$\varphi(a) \doteq \sum_{z\in\mathbb{Z}^2} \mu(z)\ \exp(a\cdot z) \tag{1.1}$$

(H4) $(S_1(t))$ and $(S_2(t))$ are aperiodic random walks on $\mathbb{Z}$. *is finite everywhere on* $\mathbb{R}^2$.

Under the above assumptions, the set

$$D \doteq \{a \in \mathbb{R}^2 :\ \varphi(a) \le 1\}$$

is compact and strictly convex, the gradient $\nabla\varphi(a)$ exists everywhere on $\mathbb{R}^2$ and does not vanish on the boundary $\partial D = \{a \in \mathbb{R}^2 :\ \varphi(a) = 1\}$, the mapping

$$a \to q(a) \doteq \nabla\varphi(a)/|\nabla\varphi(a)| \tag{1.2}$$

determines a homeomorphism from ∂D to the unit 2-dimensional sphere $\mathcal{S}^2 = \{q \in \mathbb{R}^2 :\ |q| = 1\}$ (see [9]). We denote by $q \to a(q)$ the inverse mapping of (1.2) and we let $a(q) = a(q/|q|)$ for a non-zero $q \in \mathbb{R}^2$. According to this notation, $a(q)$ is the only point in ∂D where the vector q is normal to the convex set D. Throughout this paper we denote by $\mathbb{N}$ the set of all non-negative integers and we let $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. The set of all non-negative real numbers is denoted by $\mathbb{R}_+ = [0,+\infty[$ and $\mathbb{R}_+^* =]0,+\infty[$ denotes the set of all strictly positive real numbers. It is convenient moreover to introduce the following notations : $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$,

$$\tau \doteq \inf\{n \ge 0\ : S(n) \notin \mathbb{N}^*\times\mathbb{N}^*\}$$

is the first time when the random walk $(S(t))$ exits from the quadrant $\mathbb{N}^*\times\mathbb{N}^*$,

$$\mathcal{S}^2_+ \doteq \{q \in \mathbb{R}^2_+\ :\ |q| = 1\} \quad \text{and} \quad \Gamma_+ \doteq \{a \in \partial D : q(a) \in \mathcal{S}^2_+\}.$$

For $a \in \Gamma_+$ and $z = (x_1, x_2) \in \mathbb{N}^*\times\mathbb{N}^*$ we set

$$h_a(z) \doteq \begin{cases} x_1 \exp(a\cdot z) - \mathbb{E}_z\big(S_1(\tau)\exp(a\cdot S(\tau)),\ \tau<\infty\big) & \text{if } q(a) = (0,1),\\ x_2 \exp(a\cdot z) - \mathbb{E}_z\big(S_2(\tau)\exp(a\cdot S(\tau)),\ \tau<\infty\big) & \text{if } q(a) = (1,0),\\ \exp(a\cdot z) - \mathbb{E}_z\big(\exp(a\cdot S(\tau)),\ \tau<\infty\big) & \text{otherwise.}\end{cases} \tag{1.3}$$

$G_+(z,z')$ denotes the Green function of the process $(Z(t))$:

$$G_+(z,z') = \sum_{n=0}^{\infty} \mathbb{P}_z(Z_+(n) = z').$$

The main result of our paper is the following theorem.

Theorem 1. *Under the hypotheses (H1)-(H4), for any* $q \in \mathcal{S}^2_+$ *and any sequence of points* $z_n \in \mathbb{N}^*\times\mathbb{N}^*$ *with* $\lim_n |z_n| = \infty$ *and* $\lim_n z_n/|z_n| = q$,

$$\lim_{n\to\infty} G_+(z,z_n)/G_+(z_0,z_n) = h_{a(q)}(z)/h_{a(q)}(z_0) \tag{1.4}$$

for all $z \in \mathbb{N}^*\times\mathbb{N}^*$.

Remark that the conditions (H1) and (H2) are essential for our approach, our method does not work when at least one of them is not satisfied. The hypotheses (H3) and (H4) are required by the paper [12], we use its results to get (1.4) for $q \in \{(1,0),(0,1)\}$. When the coordinates q_1 and q_2 of the vector $q = \lim_n z_n/|z_n|$ are non-zero, the assumption (H4) is not needed and the hypotheses (H3) can be replaces by a less restrictive condition of Ney and Spitzer [16] where the jump generating function (1.1) is assumed to be finite only in a neighborhood of the set D.

Recall that a sequence z_n is said to converge to a point on the Martin boundary $\partial_M(\mathbb{N}^*\times\mathbb{N}^*)$ of $\mathbb{N}^*\times\mathbb{N}^*$ determined by the Markov process $(Z_+(t))$ if and only if the sequence of functions $z\to G_+(z,z_n)/G_+(z_0,z_n)$ converges point-wise on $\mathbb{N}^*\times\mathbb{N}^*$. According to this definition, Theorem 1 implies the following statement.

Corollary 1.1. *Under the hypotheses (H1)-(H4), the following assertions hold :*
1) A sequence of points $z_n\in\mathbb{N}^*\times\mathbb{N}^*$ with $\lim_n |z_n| = +\infty$ converge to a point of the Martin boundary for the Markov process $Z_+(t)$ if and only if $z_n/|z_n|\to q$ for some point $q\in\mathcal{S}^2_+$.
2) The full Martin compactification of the quadrant $\mathbb{N}^*\times\mathbb{N}^*$ is homeomorphic to the closure of the set $\{w = z/(1+|z|)\ :\ z\in\mathbb{N}^*\times\mathbb{N}^*\}$ in $\mathbb{R}^2$.

Our paper is organized as follows : in Section 2, the main idea of the proof of our result is sketched. Section 3 is devoted to the preliminary results. In Section 4 we prove that the functions h_a with $a\in\Gamma_+$ defined by (1.3) are finite, harmonic for the Markov process $(Z_+(t))$ and strictly positive. Section 5 is devoted to the large deviation results. It is shown that the family of scaled processes $Z^\varepsilon_+(t) = \varepsilon Z_+([t/\varepsilon])$ satisfies sample path large deviation principle. The logarithmic estimates of the Green function are obtained from the corresponding large deviation bounds. In Section 6 the large deviation estimates are used to decompose the Green function $G_+(z,z_n)$ into a main part corresponding to an optimal large deviation way to go from z to z_n and the negligible part. In Section 7, we generalize the ratio limit theorem of Ignatiouk-Robert [12]. The decomposition into a main and a negligible parts of the Green function $G_+(z,z_n)$ and the ratio limit theorem are next combined in Section 8 in order to complete the proof of Theorem 1.

2. Local processes and renewal equations: a sketch of proofs

The main steps of our method can be summarized as follows :

1) For a sequence $(z_n)\in\mathbb{N}^*\times\mathbb{N}^*$ with $\lim_n z_n/|z_n| = q$ and $\lim_n |z_n| = +\infty$, the Green function $G_+(z,z_n)$ of the Markov process $(Z_+(t))$ is represented in terms of a local random walk which is Markov-additive and has the same transition probabilities as the original random walk $(Z_+(t))$ in a neighborhood of the point $q|z_n|$.

2) Next, large deviation estimates are used to decompose $G_+(z,z_n)$ into a main part corresponding to an optimal large deviation way to go from z to z_n and the negligible part. Such a decomposition allows us to get the limit of the Martin kernel

$$\lim_n G_+(z,z_n)/G_+(z_0,z_n)$$

from the limiting behavior and the uniform bounds of the Martin kernel of the corresponding local process.

When the coordinates of the vector $q=(q_1,q_2)$ are non-zero, the local Markov-additive process is simply a homogeneous random walk $(S(t))$ on $\mathbb{Z}^2$ having transition probabilities $p(z,z') = \mu(z'-z)$. This is the simplest case in our proof. The following renewal equation represents the Green function $G_+(z,z')$ of the Markov process $(Z_+(t))$ in terms of the Green function $G(z,z')$ of the random walk $(S(t))$:

$$G_+(z,z') \ = \ G(z,z') \ - \ \mathbb{E}_z\Big(G(S(\tau),z'),\ \tau<\infty\Big) \tag{2.1}$$

Ney and Spitzer [16] proved that for any $q \in \mathcal{S}^2_+$ and any sequence of points $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ with $\lim_n |z_n| = \infty$ and $\lim_n z_n/|z_n| = q$,

$$\lim_{n\to\infty} G(z, z_n)/G(0, z_n) = \exp(a(q) \cdot z) \tag{2.2}$$

for all $z \in \mathbb{Z}^2$ (see also Section 7 in [12] for an alternative simple proof of this result). Using the renewal equation (2.1) one can get therefore the equality

$$\lim_{n\to\infty} \frac{G_+(z, z_n)}{G(0, z_n)} = \exp(a(q) \cdot z) - \mathbb{E}_z\Big(\exp(a(q) \cdot S(\tau)),\ \tau < \infty\Big) \doteq h_{a(q)}(z), \tag{2.3}$$

if one can prove the exchange of limits

$$\lim_{n\to\infty} \mathbb{E}_z\left(\frac{G(S(\tau), z_n)}{G(0, z_n)},\ \tau < \infty\right) = \mathbb{E}_z\left(\lim_{n\to\infty} \frac{G(S(\tau), z_n)}{G(0, z_n)},\ \tau < \infty\right) \tag{2.4}$$

Relation (1.4) will follow finally from the relation (2.3) because the function h_a is strictly positive on $\mathbb{N}^* \times \mathbb{N}^*$ (see Proposition 4.1 below). Equality (2.4) is therefore a key relation for our problem.

While the above idea seems quite simple, the proof of (2.4) is non-trivial because the convergence (2.2) is not uniform and the classical convergence theorems are here difficult to use. With our approach, for a sequence of points $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ with $\lim_n |z_n| = \infty$ and $\lim_n z_n/|z_n| = q$ we first decompose the right hand side of (2.1) into a main part

$$\Xi^q_\delta(z, z_n) \doteq G(z, z_n)\ - \mathbb{E}_z\Big(G(S(\tau), z_n),\ \tau < \infty,\ |S(\tau)| < \delta|z_n|\Big)$$

and the corresponding negligible part by using the large deviation estimates of the Green function $G(z, z')$ and $G_+(z, z')$. Next, we get the estimates

$$\sup_n\ 1_{\{|z|<\delta|z_n|\}} G(z, z_n)/G(z_0, z_n) \le C(z) \tag{2.5}$$

such that $\mathbb{E}_z(C(S(\tau)),\ \tau < \infty) < \infty$ and finally, using the point-wise convergence (2.2) and dominated convergence theorem we obtain (1.4). The estimates (2.5) are obtained in Section 7 with a suitable exponential function $C(z)$ by using the ratio limit theorem applied to the random walk $(S(t))$.

The case when one of the coordinates of the vector q is equal to zero, i.e. when the sequence (z_n) tends to infinity along one of the boundaries of the domain, is much more delicate to handle. First of all, we cannot use here the renewal equation (2.1) because the function $\exp(a(q)\cdot z) - \mathbb{E}_z\big(\exp(a(q)\cdot S(\tau)),\ \tau < \infty\big)$ is in this case identical to zero. If $q = (1, 0)$, one should consider a Markov-additive process having the same statistical behavior as the process $(Z_+(t))$ near the boundary $\mathbb{N} \times \{0\}$ and far from the boundary $\{0\} \times \mathbb{N}$. This is a random walk $(Z^1_+(t))$ on $\mathbb{Z} \times \mathbb{N}^*$ having a sub-stochastic transition matrix $(p_1(z, z') = \mu(z' - z),\ z, z' \in \mathbb{Z} \times \mathbb{N}^*)$. It is identical to the random walk $(S(t))$ before the time $\tau_2 \doteq \inf\{t \ge 0 :\ S_2(t) \le 0\}$ and killed at the time τ_2. Our Markov process $(Z_+(t))$ is therefore identical to $(Z^1_+(t))$ before the time $\tau_1 \doteq \inf\{t \ge 0 :\ S_1(t) \le 0\}$. Since clearly $\tau = \min\{\tau_1, \tau_2\}$, the Green function $G_+(z, z')$ of the Markov process $(Z_+(t))$ is related to the Green function $G^1_+(z, z')$ of the process $(Z^1_+(t))$ as follows :

$$G_+(z, z')\ =\ G^1_+(z, z')\ - \mathbb{E}_z\Big(G^1_+(S(\tau), z'),\ \tau = \tau_1 < \tau_2\Big). \tag{2.6}$$

Theorem 1 of [12] proves that for any sequence of points $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ with $\lim_n |z_n| = \infty$ and $\lim_n z_n/|z_n| = q = (1,0)$,

$$\lim_{n\to\infty} G^1_+(z,z_n)/G^1_+(z_0,z_n) = h^1_{a(q),+}(z)/h^1_{a(q),+}(z_0), \quad \forall z \in \mathbb{Z}\times\mathbb{N}^* \tag{2.7}$$

with a strictly positive function $h^1_{a(q),+}$ on $\mathbb{Z}\times\mathbb{N}^*$ defined by

$$h^1_{a(q),+}(z) \;=\; x_2 \exp(a(q)\cdot z) - \mathbb{E}_z\Big(S_2(\tau_2)\exp(a(q)\cdot S(\tau_2)),\ \tau_2 < \infty\Big).$$

Similarly to the previous case we decompose the right hand side of the renewal equation (2.6) into a main part

$$G^1_+(z,z_n) \ - \mathbb{E}_z\Big(G^1_+(S(\tau),z_n),\ \tau = \tau_1 < \tau_2,\ |S(\tau)| < \delta |z_n|\Big)$$

and the corresponding negligible part by using the large deviation estimates of the Green functions $G_+(z,z')$ and $G^1_+(z,z')$ and we show there are $\delta > 0$ and a function $C^1_+(z)$ with

$$\mathbb{E}_z(C^1_+(S(\tau)),\ \tau = \tau_1 < \tau_2) \ < \ \infty$$

such that

$$\sup_n \ 1_{\{|z|<\delta|z_n|\}} G^1_+(z,z_n)/G^1_+(z_0,z_n) \ \le \ C^1_+(z). \tag{2.8}$$

The proof of these estimates is the most delicate part of our work.

3. Preliminary results

For a given $a \in D \doteq \{a \in \mathbb{R}^2 : \varphi(a) \le 1\}$, let us consider a new twisted homogeneous random walk $(S^a(t))$ on $\mathbb{Z}^2$ having transition probabilities

$$p_a(z,z') \ = \ \mu(z'-z)\exp(a\cdot(z'-z)). \tag{3.1}$$

According to the definition of the set D, the transition matrix of such a random walk is sub-stochastic. Recall that

$$\tau \ = \ \tau_1 \wedge \tau_2$$

where $\tau_1 \doteq \inf\{n \ge 0 :\ S(n) \notin \mathbb{N}^*\times\mathbb{Z}\}$ and $\tau_2 \doteq \inf\{n\ge 0 :\ S(n) \notin \mathbb{Z}\times\mathbb{N}^*\}$.

Proposition 3.1. *For every $a \in D$, the quantity $\mathbb{E}_z\big(\exp(a\cdot(S(\tau)-z)),\ \tau<\infty\big)$ is equal to the probability that the twisted random walk $(S^a(t))$ starting at the point z ever exits from the positive quadrant $\mathbb{N}^*\times\mathbb{N}^*$.*

Proof. Indeed, let τ^a denote the first time when the twisted random walk $(S^a(t))$ exits from the quadrant $\mathbb{N}^*\times\mathbb{N}^*$. Then for any $t\in\mathbb{N}$,

$$\mathbb{P}_z(S^a(t) = z',\ \tau = t) \ = \ \exp(a\cdot(z'-z))\mathbb{P}_z(S(t)=z',\ \tau=t) \quad \forall z,z' \in \mathbb{Z}^2$$

and consequently, $\mathbb{P}_z(\tau^a < \infty) \ = \ \mathbb{E}_z\big(\exp(a\cdot(S(\tau)-z)),\ \tau<\infty\big)$. □

The set $\Gamma_+ = \{a \in \partial D :\ q(a) \in \mathcal{S}^2_+\}$ endowed with a topology induced by the usual topology of $\mathbb{R}^2$ is homeomorphic to a segment with the end points in $a(1,0)$ and $a(0,1)$. The points $a(1,0)$ and $a(0,1)$ are said to be critical.

Proposition 3.2. *Every non-critical point of Γ_+ has a neighborhood where the functions $a \to \mathbb{E}_z(\exp(a\cdot S(\tau)),\ \tau<\infty)$ are finite for all $z\in\mathbb{N}^*\times\mathbb{N}^*$.*

Proof. By Proposition 3.1, the function $a \to \mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau < \infty)$ is finite on $D \dot{=} \{a \in \mathbb{R}^2 : \ \varphi(a) \leq 1\}$. Furthermore, let us consider the critical points $a(1,0) = (a'_1, a'_2)$ and $a(0,1) = (a''_1, a''_2)$. Recall that under the hypotheses (H1) and (H3) the set D is compact and strictly convex, and according to the definition of the mapping $q \to a(q)$,

$$\nabla\varphi(a'_1, a'_2) = |\nabla\varphi(a'_1, a'_2)|(1,0) \quad \text{and} \quad \nabla\varphi(a''_1, a''_2) = |\nabla\varphi(a''_1, a''_2)|(0,1).$$

Every non-critical point of Γ_+ has therefore a neighborhood where for any point $a = (a_1, a_2) \notin D$ there exist two points $\hat{a} = (\hat{a}_1, \hat{a}_2)$ and $\tilde{a} = (\tilde{a}_1, \tilde{a}_2)$ on the boundary of the set D with $\hat{a}_1 = a_1$, $\hat{a}_2 < a_2$ and $\tilde{a}_1 < a_1$, $\tilde{a}_2 = a_2$ (see figure 1).

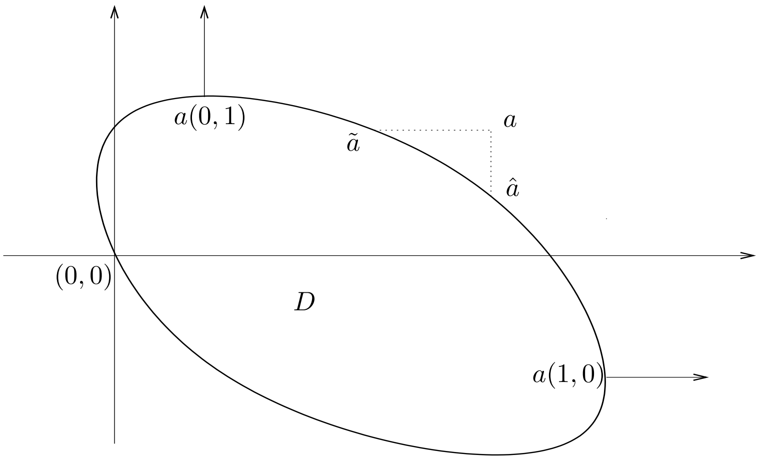

FIGURE 1

Since $S_1(\tau) \leq 0$ on the event $\{\tau = \tau_1 < +\infty\}$, and $S_2(\tau) \leq 0$ on the event $\{\tau = \tau_2 < +\infty\}$, from this it follows that

$$\begin{aligned}
&\mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau < +\infty) \\
&\qquad \leq \ \mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau = \tau_1 < +\infty) + \mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau = \tau_2 < +\infty) \\
&\qquad \leq \ \mathbb{E}_z(\exp(\tilde{a} \cdot S(\tau)),\ \tau = \tau_1 < +\infty) + \mathbb{E}_z(\exp(\hat{a} \cdot S(\tau)),\ \tau = \tau_2 < +\infty) \\
&\qquad \leq \ \mathbb{E}_z(\exp(\tilde{a} \cdot S(\tau)),\ \tau < +\infty) + \mathbb{E}_z(\exp(\hat{a} \cdot S(\tau)),\ \tau < +\infty) \ < \ +\infty.
\end{aligned}$$

□

Proposition 3.3. *The critical point $a(1,0) = (a'_1, a'_2)$ has a neighborhood where the functions $a \to \mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau = \tau_1 < \tau_2)$ are finite for all $z \in \mathbb{N}^* \times \mathbb{N}^*$. Moreover, for any $\delta > 0$ small enough there is a point $\hat{a} = (\hat{a}_1, \hat{a}_2) \in \partial D$ with $\hat{a}_1 < a'_1$ and $\hat{a}_2 = a'_2 + \delta$ such that*

$$\mathbb{E}_z(\exp(a(1,0) \cdot S(\tau) + \delta S_2(\tau)),\ \tau = \tau_1 < \tau_2) \ \leq \ \exp(\hat{a} \cdot z) \tag{3.2}$$

for all $z \in \mathbb{N}^* \times \mathbb{N}^*$.

Proof. The proof of this proposition use essentially the same arguments as the proof of Proposition 3.2. For $a \in D$,

$$\mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau = \tau_1 < \tau_2) \ \leq \ \exp(a \cdot z), \quad \forall z \in \mathbb{N}^* \times \mathbb{N}^*$$

because the quantity $\mathbb{E}_z(\exp(a\cdot(S(\tau)-z)),\ \tau=\tau_1<\tau_2)$ is equal to the probability that the twisted sub-stochastic homogeneous random walk $(S^a(t))$ starting at z hits the set $(-\mathbb{N})\times\mathbb{Z}$ before hitting the set $\mathbb{Z}\times(-\mathbb{N})$. This proves that the functions $a\to\mathbb{E}_z(\exp(a\cdot S(\tau)),\ \tau=\tau_1<\tau_2)$ are finite on D for all $z\in\mathbb{N}^*\times\mathbb{N}^*$. Moreover, let us consider the points $a(0,1)=(a_1'',a_2'')$ and $a(0,-1)=(a_1''',a_2''')$ on the boundary ∂D of D. Then the set $\Omega\doteq\{a=(a_1,a_2)\in\mathbb{R}^2:\ a_1>\max\{a_1'',a_1'''\},\ a_2'''<a_2<a_2''\}$ is an open neighborhood of the point $a(1,0)$ and for any $a=(a_1,a_2)\in\Omega\setminus D$ there is a point $\hat a=(\hat a_1,\hat a_2)$ on the boundary of the set D with $\hat a_2=a_2$ and $\hat a_1<a_1$ (see Figure 2).

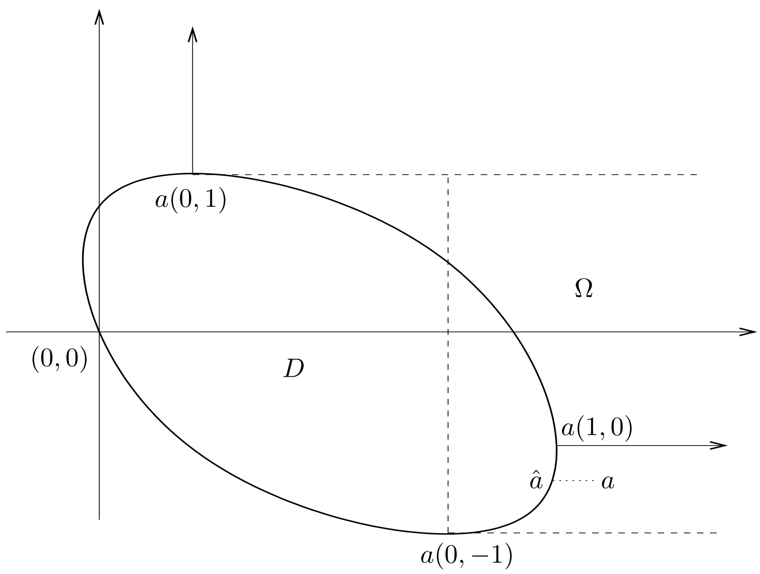

FIGURE 2

Since on the event $\{\tau=\tau_1<\tau_2\}$, $S_1(\tau)\le 0$ we conclude that for any $z\in\mathbb{N}^*\times\mathbb{N}^*$,

$$\mathbb{E}_z(\exp(a\cdot S(\tau)),\ \tau=\tau_1<\tau_2)\ \le\ \mathbb{E}_z(\exp(\hat a\cdot S(\tau)),\ \tau=\tau_1<\tau_2)\ \le\ \exp(\hat a\cdot z).$$

The functions $a\to\mathbb{E}_z(\exp(a\cdot S(\tau)),\ \tau=\tau_1<\tau_2)$ are therefore finite on Ω for all $z\in\mathbb{N}^*\times\mathbb{N}^*$. Finally, for $\delta>0$ small enough, $a=a(1,0)+(0,\delta)\in\Omega$ and hence, the last inequality proves also (3.2). □

A straightforward consequence of Proposition 3.3 is the following statement.

Corollary 3.1. *For $a=a(1,0)$, the function*

$$z\to\mathbb{E}_z(|S_2(\tau)|\exp(a\cdot S(\tau)),\ \tau=\tau_1<\tau_2) \tag{3.3}$$

is finite on $\mathbb{N}^*\times\mathbb{N}^*$.

Proof. Indeed, on the event $\tau=\tau_1<\tau_2$, for any $\delta>0$, one has

$$0\ <\ S_2(\tau)\ \le\ \frac{1}{\delta}\exp(\delta S_2(\tau))$$

and consequently, for $a = a(1,0)$,

$$\begin{aligned}\mathbb{E}_z(|S_2(\tau)|\exp(a\cdot S(\tau)), \tau=\tau_1<\tau_2) &= \mathbb{E}_z(S_2(\tau)\exp(a\cdot S(\tau)),\ \tau=\tau_1<\tau_2)\\ &\le \frac{1}{\delta}\mathbb{E}_z(\exp(a\cdot S(\tau)+\delta S_2(\tau)),\ \tau=\tau_1<\tau_2).\end{aligned}$$

Since by Proposition 3.3, the right hand side of the last relation is finite for all $z\in\mathbb{N}^*\times\mathbb{N}^*$ and $\delta>0$ small enough , we conclude that the function (3.3) is finite on $\mathbb{N}^*\times\mathbb{N}^*$. □

To show that the functions (1.3) are well defined we will need moreover the following statement.

Lemma 3.1. *For a random walk $(\xi(t))$ on $\mathbb{Z}$ having zero mean and transition probabilities $P(x,x') = P(0,x'-x)$ such that for some $\delta>0$,*

$$\sum_x e^{-\delta x}P(0,x) \ < \ \infty \quad \text{and} \quad \sum_x |x|P(0,x) \ < \ \infty,$$

the function $f(x) \ = \ \mathbb{E}_x(|\xi(T_0)|)$ with $T_0 = \inf\{t\ge 0 : \xi(t)\le 0\}$ is finite everywhere on $\mathbb{N}^*$.

This elementary lemma has been proved in the proof of Lemma 5.3 in Ignatiouk [12]. A more general related result can also be found in Chow [4]. Corollary 3.1 combined with Lemma 3.1 implies the following proposition.

Proposition 3.4. *The function $z\to\mathbb{E}_z(|S_2(\tau)|\exp(a(1,0)\cdot S(\tau)),\ \tau<\infty)$ is finite on $\mathbb{N}^*\times\mathbb{N}^*$.*

Proof. To prove this proposition let us first notice that

$$\begin{aligned}\mathbb{E}_z(|S_2(\tau)|\exp(a\cdot S(\tau)),\tau<\infty) \ &= \ \mathbb{E}_z(|S_2(\tau)|\exp(a\cdot S(\tau)),\tau=\tau_1<\tau_2)\\ &\quad + \mathbb{E}_z(|S_2(\tau)|\exp(a\cdot S(\tau)),\ \tau=\tau_2<\infty)\end{aligned}$$

where for $a=a(1,0)$, by Corollary 3.1,

$$\mathbb{E}_z(|S_2(\tau)|\exp(a(1,0)\cdot S(\tau)),\tau=\tau_1<\tau_2) \ < \ \infty, \quad \forall z\in\mathbb{N}^*\times\mathbb{N}^*.$$

To prove that the function $z\to\mathbb{E}_z(|S_2(\tau)|\exp(a(1,0)\cdot S(\tau)),\ \tau<\infty)$ is finite on $\mathbb{N}^*\times\mathbb{N}^*$ it is therefore sufficient to show that

$$\mathbb{E}_z(|S_2(\tau)|\exp(a(1,0)\cdot S(\tau)),\ \tau=\tau_2<\infty) \ < \ \infty, \quad \forall z\in\mathbb{N}^*\times\mathbb{N}^*. \tag{3.4}$$

Next, we consider a twisted random walk $(S^a(t))$ on $\mathbb{Z}^2$ with transition probabilities $p_a(z,z') \ = \ \mu(z'-z)\exp(a\cdot(z'-z))$ for $a=a(1,0)$. The second coordinate $(S^a_2(t))$ of $(S^a(t))$ is a random walk on $\mathbb{Z}$ having a mean

$$\mathbb{E}_0(S_2^a(1)) \ = \ \frac{\partial}{\partial a_2}\varphi(a_1,a_2)\Big|_{(a_1,a_2)=a(1,0)} \ = \ 0$$

and satisfying the conditions of Lemma 3.1. This lemma applied with $\xi(t)=S^a_2(t)$ and $T_0=\tau^a_2 \doteq \inf\{n\ge 0:\ S^a_2(n)\le 0\}$ proves that the function $\mathbb{E}_x(|S^a_2(\tau^a_2)|)$ is finite on $\mathbb{N}^*$. Since for any $z=(x_1,x_2)\in\mathbb{N}^*\times\mathbb{N}^*$,

$$\mathbb{E}_z(|S_2(\tau_2)|\exp(a(1,0)\cdot S(\tau_2)),\ \tau_2<\infty) \ = \ \mathbb{E}_{x_2}(|S^a_2(\tau^a_2)|)$$

we conclude that (3.4) holds. Proposition 3.4 is therefore proved. □

4. Harmonic functions.

The main result of this section is the following proposition.

Proposition 4.1. *For every $a \in \Gamma_+$, the functions h_a defined by* (1.3) *is finite, strictly positive on $\mathbb{N}^* \times \mathbb{N}^*$ and harmonic for the Markov process $(Z_+(t))$.*

Before proving this proposition we consider the following lemmas.

Lemma 4.1. *For $a \in \Gamma_+$, the function $z \to 1 - \mathbb{E}_z\left(\exp\bigl(a \cdot (S(\tau) - z)\bigr),\, \tau < \infty\right)$ is strictly positive on $\mathbb{N}^* \times \mathbb{N}^*$ when $q(a) \not\in \{(1,0),(0,1)\}$ and is identically zero when $q(a) \in \{(1,0),(0,1)\}$.*

Proof. Indeed, for any $a \in \Gamma_+$, the twisted random walk $S^a(t) = (S^a_1(t), S^2_2(t))$ has a stochastic transition matrix $(p_a(z,z') = \exp(a \cdot (z'-z))\mu(z'-z),\ z,z' \in \mathbb{Z}^2)$, a non-zero mean

$$m(a) \;=\; \sum_{z\in\mathbb{Z}^2} z\exp(a\cdot z)\mu(z) \;=\; \nabla\varphi(a) \;=\; |\nabla\varphi(a)|q(a)$$

and a finite variance. If $q(a) = (0,1)$, the first coordinate $S^a_1(t)$ of $S^a(t)$ is therefore a recurrent random walk on $\mathbb{Z}$, the first time when $S^a_1(t)$ becomes negative or zero is almost surely finite for any starting point $S^a(0) = z \in \mathbb{N}^* \times \mathbb{N}^*$ and consequently, the twisted random walk $(S^a(t))$ almost surely exits from the quadrant $\mathbb{N}^* \times \mathbb{N}^*$. By Proposition 3.1 from this it follows that

$$1 - \mathbb{E}_z\left(\exp\bigl(a\cdot(S(\tau)-z)\bigr),\, \tau<\infty\right) \;=\; \mathbb{P}_z(\tau^a = \infty) \;=\; 0, \quad \forall z \in \mathbb{N}^*\times\mathbb{N}^*.$$

The same arguments but with a recurrent random walk $(S^a_2(t))$ prove this equality when $q(a) = (1,0)$.

Suppose now that $q(a) \not\in \{(1,0),(0,1)\}$. Then by the strong law of large numbers, $S^a(t)/t \to m(a)$ almost surely as $t \to \infty$ for any initial state $S^a(0) = z$. From this it follows that for any $S^a(0) = z$ and $\varepsilon > 0$ there is an almost surely finite positive random variable $N_{z,\varepsilon}$ such that $|S^a(t) - m(a)t| < \varepsilon t$ for all $t \geq N_{z,\varepsilon}$. Since $q(a) \not\in \{(1,0),(0,1)\}$, the both coordinates of the mean vector $m(a)$ are positive and non-zero and consequently, there exist $N > 0$ and $\hat\varepsilon > 0$ for which the set

$$\bigl\{z \in \mathbb{Z}^2 :\ |z - m(a)t| < \hat\varepsilon t \text{ for some } t \geq N\bigr\}$$

is included to the quadrant $\mathbb{N}^* \times \mathbb{N}^*$. For the initial state $S^a(0) = 0$, from this it follows that almost surely $S^a(t) \in \mathbb{N}^* \times \mathbb{N}^*$ for all $t \geq \hat N \doteq \max\{N_{0,\hat\varepsilon}, N\}$. The minimums

$$\min_{t\in\mathbb{N}} S^a_1(t) \quad\text{and}\quad \min_{t\in\mathbb{N}} S^a_2(t)$$

are therefore almost surely finite and consequently, for some $\hat z = (\hat x, \hat y) \in \mathbb{N}^* \times \mathbb{N}^*$,

$$\mathbb{P}_{\hat z}(\tau^a = +\infty) \;=\; \mathbb{P}_0\left(\min_{t\in\mathbb{N}} S^a_1(t) > -\hat x \;\text{ and }\; \min_{t\in\mathbb{N}} S^a_2(t) > -\hat y\right) \;>\; 0.$$

The last inequality combined with Proposition 3.1 shows that

$$1 - \mathbb{E}_{\hat z}\bigl(\exp(a\cdot(S(\tau)-\hat z)),\ \tau<\infty\bigr) \;=\; \mathbb{P}_{\hat z}(\tau^a = +\infty) \;>\; 0$$

for some $\hat z = (\hat x, \hat y) \in \mathbb{N}^* \times \mathbb{N}^*$. To complete our proof it is now sufficient to notice that under the hypotheses (H2), for any $z \in \mathbb{N}^* \times \mathbb{N}^*$, the probability that the random walk $(S^a(t))$ starting at z hits the point $\hat{z}$ before the first exit from the quadrant $\mathbb{N}^* \times \mathbb{N}^*$ is non-zero and consequently, for some $t = t(z, \hat{z}) \in \mathbb{N}$,

$$\begin{aligned} 1 - \mathbb{E}_z\big(\exp(a \cdot (S(\tau) - z)),\ \tau < \infty\big) &= \ \mathbb{P}_z(\tau^a = +\infty) \\ &\geq \ \mathbb{P}_z(S^a(t) = \hat{z},\ \tau^a > t)\mathbb{P}_{\hat{z}}(\tau^a = +\infty)\ >\ 0. \end{aligned}$$

Lemma 4.1 is therefore proved. □

Lemma 4.2. *The function*

$$z = (x_1, x_2)\ \to\ x_2 \exp(a(1,0) \cdot z) - \mathbb{E}_z\big(S_2(\tau) \exp(a(1,0) \cdot S(\tau)),\ \tau < \infty\big) \tag{4.1}$$

is well defined and non-negative on $\mathbb{N}^* \times \mathbb{N}^*$.

Proof. Indeed, Proposition 3.4 proves that the function (4.1) is well defined. To prove that this function is non-negative on $\mathbb{N}^* \times \mathbb{N}^*$ let us notice that by dominated convergence theorem from Proposition 3.4 it follows that

$$\mathbb{E}_z\big(S_2(\tau) \exp(a \cdot S(\tau)),\ \tau < \infty\big)\ =\ \lim_{n\to\infty} \mathbb{E}_z\big(S_2(\tau) \exp(a \cdot S(\tau)),\ \tau \leq n\big). \tag{4.2}$$

Moreover, the function $z = (x_1, x_2) \to x_2 \exp(a(1,0) \cdot z)$ is harmonic for the random walk $S(t)$ because according to the definition of the point $a(1,0)$, for any $z = (x_1, x_2)$,

$$\mathbb{E}_z\big(S_2(1) \exp(a(1,0) \cdot S(1))\big) - x_2 \exp(a(1,0) \cdot z)\ =\ \frac{\partial \varphi(a_1, a_2)}{\partial a_2}\Big|_{(a_1,a_2)=a(1,0)}\ =\ 0.$$

Hence, for $a = a(1,0)$, the sequence $S_2(n) \exp(a \cdot S(n))$ is a martingale relative to the natural filtration of $(S(n))$ and by the stopping-time theorem, for any $z = (x_1, x_2) \in \mathbb{N}^* \times \mathbb{N}^*$,

$$\begin{aligned} &\mathbb{E}_z\big(S_2(\tau) \exp(a \cdot S(\tau)),\ \tau \leq n\big) \\ &\quad = \ \mathbb{E}_z\big(S_2(\tau \wedge n) \exp(a \cdot S(\tau \wedge n))\big) - \mathbb{E}_z\big(S_2(n) \exp(a \cdot S(n)),\ \tau > n\big) \\ &\quad = \ x_2 \exp(a \cdot z) - \mathbb{E}_z\big(S_2(n) \exp(a \cdot S(n)),\ \tau > n\big)\ \leq\ x_2 \exp(a \cdot z) \end{aligned}$$

where the last relation holds because on the event $\{\tau > n\}$ one has $S_2(n) > 0$. The last inequality combined with (4.2) proves that the function (4.1) is non-negative on $\mathbb{N}^* \times \mathbb{N}^*$. □

Proof of Proposition 4.1. Suppose first that $a \notin \{a(1,0), a(0,1)\}$. Then by Lemma 4.1, the function $h_a(z)\ =\ \exp(a \cdot z) - \mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau < \infty)$ is finite and strictly positive on $\mathbb{N}^* \times \mathbb{N}^*$. For the homogeneous random walk $(S(t))$ on $\mathbb{Z}^2$, the exponential function $z \to \exp(a \cdot z)$ is harmonic and the function

$$f(z)\ =\ \mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau < \infty)$$

satisfies the equality $\mathbb{E}_z(f(S(1))\ =\ f(z)$ for all $z \in \mathbb{N}^* \times \mathbb{N}^*$. The function $h_a(z) = \exp(a \cdot z) - f(z)$ satisfies therefore the equality

$$\mathbb{E}_z(h_a(S(1)))\ =\ h_a(z)$$

for all $z \in \mathbb{N}^* \times \mathbb{N}^*$. Moreover, for $z \in \mathbb{Z} \times \mathbb{Z} \setminus (\mathbb{N}^* \times \mathbb{N}^*)$, $\mathbb{P}_z$-almost surely, $\tau\ =\ 0$ and $S(\tau) = z$ from which it follows that

$$h_a(z)\ =\ \exp(a \cdot z) - \mathbb{E}_z(\exp(a \cdot S(\tau)),\ \tau < \infty)\ =\ 0, \quad \forall z \in \mathbb{Z} \times \mathbb{Z} \setminus (\mathbb{N}^* \times \mathbb{N}^*).$$

Since $Z_+(t)$ is killed at to the first time τ when $S(t)$ exits from $\mathbb{N}^* \times \mathbb{N}^*$ and is identical to $S(t)$ for $t \le \tau$, we conclude that the function h_a is harmonic for the random walk $(Z_+(t))$. For $a \notin \{a(1,0), a(0,1)\}$, Proposition 4.1 is therefore proved.

Consider now the case when $a = a(1,0) = (a'_1, a'_2)$. Then by Lemma 4.2, the function $h_a(z) = x_2 \exp(a\cdot z) - \mathbb{E}_z\big(S_2(\tau)\exp(a\cdot S(\tau)),\ \tau < \infty\big)$ is well defined and non-negative on $\mathbb{N}^* \times \mathbb{N}^*$. To prove that this function is harmonic for the Markov process $(Z_+(t))$ it is sufficient to notice that

$$\mathbb{E}_z(h_a(Z_+(1))) \;=\; \mathbb{E}_z(h_a(S(1)),\ \tau > 1) \;=\; \mathbb{E}_z(h_a(S(1))) \;=\; h_a(z), \quad \forall z \in \mathbb{N}^* \times \mathbb{N}^*$$

because $h_a(z) = 0$ for all $z \in \mathbb{Z}\times\mathbb{Z} \setminus (\mathbb{N}^* \times \mathbb{N}^*)$ and $Z_+(1) = S(1)$ whenever $\tau > 1$. To prove that the function h_a is strictly positive we first notice that

$$\begin{aligned} h_a(z)\exp(-a\cdot z) \;=\; & x_2 - \mathbb{E}_z\big(S_2(\tau)\exp(a\cdot(S(\tau)-z)),\ \tau = \tau_2 < \infty\big) \\ & - \mathbb{E}_z\big(S_2(\tau)\exp(a\cdot(S(\tau)-z)),\ \tau = \tau_1 < \tau_2 \le \infty\big) \end{aligned}$$

where

$$x_2 - \mathbb{E}_z\big(S_2(\tau)\exp(a\cdot(S(\tau)_z)),\ \tau = \tau_2 < \infty\big) \;\ge\; x_2 \;>\; 0$$

because on the event $\{\tau = \tau_2\}$ one has $S_2(\tau) = S_2(\tau_2) \le 0$. Moreover, by Proposition 3.3, for $a = a(1,0) = (a'_1, a'_2)$ and any $\delta > 0$ small enough there is a point $\hat a = (\hat a_1, \hat a_2) \in \partial D$ with $\hat a_1 < a'_1$ and $\hat a_2 = a'_2 + \delta$ such that

$$\begin{aligned} &\mathbb{E}_z\big(S_2(\tau)\exp(a\cdot(S(\tau)-z)),\ \tau = \tau_1 < \tau_2 \le \infty\big) \\ &\qquad \le\; \frac{1}{\delta}\mathbb{E}_z\big(\exp(a\cdot(S(\tau)-z) + \delta S_2(\tau)),\ \tau = \tau_1 < \tau_2 \le \infty\big) \\ &\qquad \le\; \frac{1}{\delta}\exp((\hat a - a)\cdot z) \;=\; \frac{1}{\delta}\exp(-(a'_1 - \hat a_1)x_1 + \delta x_2) \end{aligned}$$

Since the right hand side of the last inequality tends to zero as $x_1 \to \infty$, this proves that $h_a(z) > 0$ for $z = (x_1, x_2) \in \mathbb{N}^* \times \mathbb{N}^*$ with $x_2 = 1$ and $x_1 > 0$ large enough. Since by the Harnack inequality,

$$h_a(z') \;\ge\; h_a(z)\mathbb{P}_{z'}(Z_+(n) = z \text{ for some } n \in \mathbb{N}\,) \;\ge\; 0, \quad \forall z, z' \in \mathbb{N}^* \times \mathbb{N}^*,$$

using (H2), we conclude that $h_a(z') > 0$ for all $z' \in \mathbb{N}^* \times \mathbb{N}^*$. Proposition 4.1 is therefore proved.

5. Large deviation results

In this section we obtain large deviation results for the family of scaled process and we deduce from them the logarithmic asymptotics of the Green function. To get the large deviation results for scaled processes $(\varepsilon Z_+([t/\varepsilon])$ we need to show that the original non-scaled process $(Z_+(t))$ satisfies the following communication condition.

5.1. Communication condition.

Definition 5.1. *A discrete time Markov chain $(\mathcal{Z}(t))$ on a countable state space $E \subset \mathbb{Z}^d$ is said to satisfy the communication condition on $E_0 \subset E$ if there exist $\theta > 0$ and $C > 0$ such that for any $z \ne z'$, $z, z' \in E_0$ there is a sequence of points $z_0, z_1, \ldots, z_n \in E_0$ with $z_0 = z$, $z_n = z'$ and $n \le C|z'-z|$ such that*

$$|z_i - z_{i-1}| \;\le\; C \quad \text{and} \quad \mathbb{P}_{z_{i-1}}(\mathcal{Z}(1) = z_i) \;\ge\; \theta, \qquad \forall\, i = 1, \ldots, n.$$

Proposition 5.1. *Under the hypotheses (H2), the random walk $(Z_+(t))$ satisfies the communication condition on the hole space $\mathbb{N}^* \times \mathbb{N}^*$.*

Proof. Indeed, under the hypotheses (H2), for $\hat{x} = (1,1)$ and any unit vector $e \in \{(1,0),(0,1)\}$ there is $n_e \in \mathbb{N}$ such that $\mathbb{P}_{\hat{x}}(Z_+(n_e) = \hat{x}+e) > 0$. Hence, there are $u_1^e, \dots, u_{n_e}^e \in \mathbb{Z}^2$ with $u_1^e + \cdots + u_{n_e}^e = e$ such that $\mu(u_k^e) > 0$ and $\hat{x} + u_1^e + \cdots + u_k^e \in \mathbb{N}^* \times \mathbb{N}^*$ for all $k \in \{1,\dots,n_e\}$. Similarly, for any unit vector $e \in \{(-1,0),(0,-1)\}$ there is $n_e \in \mathbb{N}$ such that $\mathbb{P}_{\hat{x}-e}(Z_+(n_e) = \hat{x}) > 0$ and consequently, there are $u_1^e, \dots, u_{n_e}^e \in \mathbb{Z}^2$ with $u_1^e + \cdots + u_{n_e}^e = e$ such that $\mu(u_k^e) > 0$ and $\hat{x} - e + u_1^e + \cdots + u_k^e \in \mathbb{N}^* \times \mathbb{N}^*$ for all $k \in \{1,\dots,n_e\}$. This proves that for any $z, z' \in \mathbb{N}^* \times \mathbb{N}^*$ with $|z'-z| = 1$ there are $n_e \in \mathbb{N}^*$ and $u_1^e, \dots, u_{n_e}^e \in \mathbb{Z}^2$ with $z + u_1^e + \cdots + u_{n_e}^e = z'$ such that $\mu(u_k^e) > 0$ and $z + u_1^e + \cdots + u_k^e \in \mathbb{N}^* \times \mathbb{N}^*$ for all $k \in \{1,\dots,n_e\}$ and consequently, the communication condition is satisfied with

$$\theta \;=\; \min_e \min_{i=1,\dots,n_e} \mu(u_i^e) > 0 \quad \text{and} \quad C \;=\; \max_e \left\{ n_e,\ \max_{i=1,\dots,n_e} |u_i^e| \right\}.$$

□

5.2. **Large deviation properties of scaled processes.** Before formulating our large deviations results we recall the definition of the sample path large deviation principle.

Let $D([0,T],\mathbb{R}^2)$ denote the set of all right continuous functions with left limits from $[0,T]$ to $\mathbb{R}^2$ endowed with Skorohod metric (see Billingsley [2]).

Definition 5.2. *1) A mapping $I_{[0,T]} : D([0,T],\mathbb{R}^2) \to [0,+\infty]$ is a good rate function on $D([0,T],\mathbb{R}^2)$ if for any $c \geq 0$ and any compact set $V \subset \mathbb{R}^2$, the set*

$$\{\phi \in D([0,T],\mathbb{R}^2) : \ \phi(0) \in V \ \text{ and } \ I_{[0,T]}(\phi) \leq c\}$$

is compact in $D([0,T],\mathbb{R}^2)$. According to this definition, a good rate function is lower semi-continuous.

2) Let $(Z(t))$ be a Markov process on $E \subset \mathbb{Z}^2$ and let $Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon])$ for $\varepsilon > 0$. When $\varepsilon \to 0$, the family of scaled processes $(Z^\varepsilon(t) = \varepsilon Z([t/\varepsilon]),\, t \in [0,T])$, is said to satisfy a sample path large deviation principle with a rate function $I_{[0,T]}$ on $D([0,T],\mathbb{R}^2)$ if for any $T > 0$ and $z \in \mathbb{R}^2$

$$\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \inf_{z' \in \varepsilon E : |z'-z|<\delta} \varepsilon \log \mathbb{P}_{[z'/\varepsilon]}\left(Z^\varepsilon(\cdot) \in \mathcal{O}\right) \geq - \inf_{\phi \in \mathcal{O} : \phi(0) = z} I_{[0,T]}(\phi), \tag{5.1}$$

for every open set $\mathcal{O} \subset D([0,T],\mathbb{R}^2)$, and

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \sup_{z' \in \varepsilon E : |z'-z|<\delta} \varepsilon \log \mathbb{P}_{[z'/\varepsilon]}\left(Z^\varepsilon(\cdot) \in F\right) \leq - \inf_{\phi \in F : \phi(0) = z} I_{[0,T]}(\phi) \tag{5.2}$$

for every closed set $F \subset D([0,T],\mathbb{R}^2)$.

$\mathbb{P}_{[z/\varepsilon]}$ denotes here the distribution of the Markov process $(Z(t))$ corresponding to the initial state $Z(0) = [z/\varepsilon]$ where $[z/\varepsilon]$ is the nearest lattice point to z/ε in E. For $t \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $[t/\varepsilon]$ the integer part of t/ε.

By Mogulskii's theorem (see [5]), under the hypotheses (H1)-(H3), the family of scaled random walks $S^\varepsilon(t) = \varepsilon S([t/\varepsilon])$ satisfies the sample path large deviation principle with a good rate function

$$I_{[0,T]}(\phi) \;=\; \begin{cases} \int_0^T (\log\varphi)^*(\dot{\phi}(t))\,dt, & \text{if } \phi \text{ is absolutely continuous,} \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3}$$

The convex conjugate $(\log\varphi)^*$ of the function $\log\varphi$ is defined by

$$(\log\varphi)^*(v) \doteq \sup_{a\in\mathbb{R}^2}\Big(a\cdot v - \log\varphi(a)\Big).$$

Under the hypotheses (H4), $(\log\varphi)^*(v) = a\cdot v - \log\varphi(a)$ whenever $v = \nabla\varphi(a)$ because the function $(\log\varphi)$ is convex and differentiable everywhere in $\mathbb{R}^2$ (see Lemma 2.2.31 of the book of Dembo and Zeitouni [5]).

Consider now the local processes $(Z^1_+(t))$ and $(Z^2_+(t))$. Recall that $(Z^1_+(t))$ is the random walk on $\mathbb{Z}\times\mathbb{N}^*$ with transition probabilities $p_1(z,z') = \mu(z'-z)$ which is killed at hitting the half-plane $\mathbb{Z}\times(-\mathbb{N})$. Similarly, $(Z^2_+(t))$ is the random walk on $\mathbb{N}^*\times\mathbb{Z}$ with transition probabilities $p_2(z,z') = \mu(z'-z)$ which is killed at hitting the half-plane $(-\mathbb{N})\times\mathbb{Z}$. The sample path large deviation principle for the scaled processes $\varepsilon(Z^1_+([t/\varepsilon]))$ and $\varepsilon(Z^2_+([t/\varepsilon]))$ is proved by Proposition 4.1 of Ignatiouk-Robert [12] :

Proposition 5.2. *Under the hypotheses (H1)-(H3), the family of scaled processes* $Z^{\varepsilon,1}_+(t) = \varepsilon Z^1_+([t/\varepsilon])$ *and* $Z^{\varepsilon,2}_+(t) = \varepsilon Z^2_+([t/\varepsilon])$ *satisfies the sample path large deviation principle with the good rate functions*

$$(5.4)\qquad I^{1,+}_{[0,T]}(\phi) = \begin{cases} \int_0^T (\log\varphi)^*(\dot\phi(t))\,dt, & \text{if } \phi \text{ is absolutely continuous and} \\ & \phi(t)\in\mathbb{R}\times\mathbb{R}_+ \text{ for all } t\in[0,T], \\ +\infty & \text{otherwise.} \end{cases}$$

and

$$(5.5)\qquad I^{2,+}_{[0,T]}(\phi) = \begin{cases} \int_0^T (\log\varphi)^*(\dot\phi(t))\,dt, & \text{if } \phi \text{ is absolutely continuous and} \\ & \phi(t)\in\mathbb{R}_+\times\mathbb{R} \text{ for all } t\in[0,T], \\ +\infty & \text{otherwise.} \end{cases}$$

respectively.

For the random walk $(Z_+(t))$ killed at the first exit from the quadrant $\mathbb{N}^*\times\mathbb{N}^*$, wit the same arguments as in the proof of Proposition 4.1 of Ignatiouk-Robert [12], one gets the following statement.

Proposition 5.3. *Under the hypotheses (H1)-(H3), the families of scaled processes* $Z^\varepsilon_+(t) = \varepsilon Z_+([t/\varepsilon])$ *satisfies the sample path large deviation principle with the good rate function*

$$(5.6)\qquad I^{+}_{[0,T]}(\phi) = \begin{cases} \int_0^T (\log\varphi)^*(\dot\phi(t))\,dt, & \text{if } \phi \text{ is absolutely continuous and} \\ & \phi(t)\in\mathbb{R}_+\times\mathbb{R}_+ \text{ for all } t\in[0,T], \\ +\infty & \text{otherwise.} \end{cases}$$

5.3. **Large deviation estimates of the Green function.** The large deviation properties of scaled processes imply the large deviation estimates of the Green function. Recall that $G(z,z')$ denotes the Green function of the homogeneous random walk $(S(t))$, $G^i_+(z,z')$ denotes the Green function of the Markov process $(Z^i_+(t))$, for $i = 1,2$, and the Green function of the random walk $(Z_+(t))$ is denoted by $G_+(z,z')$.

Proposition 5.4. *For any $q \in \mathbb{R}^2_+$, $z \in \mathbb{N}^* \times \mathbb{N}^*$ and any sequences $\varepsilon_n > 0$ and $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ with $\lim_n \varepsilon_n = 0$ and $\lim_n \varepsilon_n z_n = q$, the following relations hold*

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \inf_{z\in\mathbb{N}^*\times\mathbb{N}^*:\varepsilon_n|z|<\delta} \varepsilon_n \log G_+(z, z_n) \;\geq\; -a(q)\cdot q, \tag{5.7}$$

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \inf_{z\in\mathbb{Z}^2:\; \varepsilon_n|z|<\delta} \varepsilon_n \log G(z, z_n) \;\geq\; -a(q)\cdot q, \tag{5.8}$$

and for every $i \in \{1, 2\}$,

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \inf_{z\in\mathbb{Z}\times\mathbb{N}^*:\varepsilon_n|z|<\delta} \varepsilon_n \log G^i_+(z, z_n) \qquad \geq\; -a(q)\cdot q. \tag{5.9}$$

The proof of this proposition uses the communication condition of Proposition 5.1 and the lower large deviation bound (5.1) for the families of scaled processes $\varepsilon(Z_+([t/\varepsilon]))$, $\varepsilon(S([t/\varepsilon]))$, $\varepsilon(Z^1_+([t/\varepsilon]))$ and $\varepsilon(Z^2_+([t/\varepsilon]))$ respectively. It is quite similar to the proof of Proposition 4.2 of Ignatiouk-Robert [12].

6. Principal part of the renewal equations

For $\delta > 0$ and a sequence of points $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ with $\lim_n |z_n| = +\infty$ and $\lim_n z_n/|z_n| = q = (q_1, q_2)$ we define the sequence of functions $\Xi^q_\delta(z, z_n)$ by letting

$$\Xi^q_\delta(z, z_n) \;=\; G(z, z_n) \;-\mathbb{E}_z\Big(G(S(\tau), z_n),\; \tau < \infty,\; |S(\tau)| < \delta|z_n|\Big) \tag{6.1}$$

if the coordinates q_1 and q_2 of the vector q are non-zero. For $q = (1, 0)$ we put

$$\Xi^q_\delta(z, z_n) \;=\; G^1_+(z, z_n) \;-\mathbb{E}_z\Big(G^1_+(S(\tau), z_n),\; \tau = \tau_1 < \tau_2,\; |S(\tau)| < \delta|z_n|\Big) \tag{6.2}$$

and for $q = (0, 1)$ we let

$$\Xi^q_\delta(z, z_n) \;=\; G^2_+(z, z_n) \;-\mathbb{E}_z\Big(G^2_+(S(\tau), z_n),\; \tau = \tau_2 < \tau_1,\; |S(\tau)| < \delta|z_n|\Big). \tag{6.3}$$

Recall that $G(z, z')$ denotes the Green function of the homogeneous random walk $(S(t))$ on $\mathbb{Z}^2$ having transition probabilities $p_S(z, z') = \mu(z' - z)$. The Green function of the random walk $(Z^1_+(t))$ on $\mathbb{Z}\times\mathbb{N}^*$ having a sub-stochastic transition matrix $(p_1(z, z') \;=\; \mu(z'-z),\; z, z' \in \mathbb{Z}\times\mathbb{N}^*)$ is denoted by $G^1_+(z, z')$. Similarly, $G^2_+(z, z')$ denotes the Green function of the random walk $(Z^2_+(t))$ on $\mathbb{N}^* \times \mathbb{N}^*$ with a sub-stochastic transition matrix $(p_2(z, z') \;=\; \mu(z'-z),\; z, z' \in \mathbb{N}^*\times\mathbb{Z})$. The main result of this section proves that for any $z \in \mathbb{N}^* \times \mathbb{N}^*$ and $\delta > 0$, the quantity $\Xi^q_\delta(z, z_n)$ represents the principal part of right hand side of the renewal equations (2.1) and (2.6) for $z' = z_n$ when $\lim_n |z_n| = +\infty$ and $\lim_n z_n/|z_n| = q$. This is a subject of the following proposition.

Proposition 6.1. *Under the hypotheses (H1)-(H3), for any $z \in \mathbb{N}^* \times \mathbb{N}^*$, $\delta > 0$ and any sequence $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ with $\lim_n |z_n| = +\infty$ and $\lim_n z_n/|z_n| = q$,*

$$\lim_{n\to\infty} G_+(z, z_n)/\Xi^q_\delta(z, z_n) \;=\; 1. \tag{6.4}$$

To prove this proposition we need to investigate the function

$$\lambda_\varepsilon(q, w) \;=\; a(w)\cdot w + a(q-w)\cdot(q-w) - \varepsilon|w|.$$

Lemma 6.1. *Under the hypotheses (H1) and (H3), for any $q \in \mathcal{S}^2_+$ and $\delta > 0$, there is a small $\varepsilon > 0$ such that*

$$\inf_{w\in\mathbb{R}^2:\inf_{\theta>0}|w-\theta q|\geq\delta} \lambda_\varepsilon(q, w) \;>\; a(q)\cdot q. \tag{6.5}$$

Proof. Under the hypotheses (H1) and (H3), the set $D \doteq \{a \in \mathbb{R}^2 : \varphi(a) \leq 1\}$ is compact and strictly convex (see [9]), and according to the definition (1.2) of the mapping $q \to a(q)$, the point $a(q)$ is the only point on the boundary of the set D where the vector q is normal to D. For any non-zero vector $q \in \mathbb{R}^2$, the point $a(q)$ is therefore the only point in D where the linear function $a \to a \cdot q$ achieves its maximum over $a \in D$. Hence, for any $w \in \mathbb{R}^2$,

$$a(w) \cdot w + a(q-w) \cdot (q-w) \;\geq\; a(q) \cdot w + a(q) \cdot (q-w) \;=\; a(q) \cdot q$$

where the inequality holds with the equality if and only if $a(w) = a(q) = a(q-w)$. Since the mapping $w \to a(w)$ from the unit sphere $\mathcal{S}^2$ to $\partial D = \{a \in \mathbb{R}^2 : \; \varphi(a) = 1\}$ is one to one, this proves that

$$a(w) \cdot w + a(q-w) \cdot (q-w) \;>\; a(q) \cdot q \quad \text{if} \quad w \notin \{\theta q : \theta \geq 0\}. \tag{6.6}$$

Moreover, the set $D = \{a \in \mathbb{R}^2 : \; \varphi(a) \leq 1\}$ being compact, the function $w \to a(w) \cdot w + a(q-w) \cdot (q-w)$ is convex, finite and therefore continuous on $\mathbb{R}^2$. Hence, for any $R > 0$ and $\delta > 0$,

$$\varepsilon(R,\delta) \doteq \inf_{\substack{w \in \mathbb{R}^2 : \; |w| \leq R \\ \inf_{\theta > 0} |w - \theta q| \geq \delta}} \Big(a(w) \cdot w + a(q-w) \cdot (q-w) \Big) - a(q) \cdot q \;>\; 0$$

and consequently, for $0 < \varepsilon < \varepsilon(R,\delta)/R$,

$$\inf_{w \in \mathbb{R}^2 : \; |w| \leq R, \, \inf_{\theta > 0} |w - \theta q| \geq \delta} \lambda_\varepsilon(q, w) \;>\; a(q) \cdot q.$$

To get (6.5) it is now sufficient to show that for any $\varepsilon > 0$ small enough, there is $R > 0$ such that

$$\inf_{w \in \mathbb{R}^2 : \; |w| \geq R, \, \inf_{\theta > 0} |w - \theta q| \geq \delta} \lambda_\varepsilon(q, w) \;>\; a(q) \cdot q. \tag{6.7}$$

Here, we use the following estimates : for any $w \in \mathbb{R}^2$ and $q \in \mathcal{S}^2_+$,

$$\begin{aligned} a(w) \cdot w + a(q-w) \cdot (q-w) - a(q) \cdot q &= \sup_{a \in D} a \cdot w + \sup_{a \in D} a \cdot (q-w) - a(q) \cdot q \\ &\geq a(w) \cdot w + a(-w) \cdot (q-w) - a(q) \cdot q \\ &\geq a(w) \cdot w + a(-w) \cdot (-w) - 2 \max_{a \in D} |a|. \end{aligned} \tag{6.8}$$

The function $\lambda(w) \doteq a(w) \cdot w + a(-w) \cdot (-w)$ is continuous and positively homogeneous :

$$\lambda(w) \;=\; |w| \lambda(w/|w|). \tag{6.9}$$

Moreover, the same arguments as in the proof of the inequality (6.6) show that

$$\lambda(w) \;>\; a(w) \cdot w + a(w) \cdot (-w) \;=\; 0 \quad \text{whenever} \quad a(w) \neq a(-w),$$

and consequently $\lambda(w) > 0$ for all $w \neq 0$. Hence, letting

$$\varepsilon_0 \doteq \frac{1}{2} \min_{w \in \mathbb{R}^2 : \; |w| = 1} \lambda(w) > 0 \quad \text{and} \quad c \doteq 2 \max_{a \in D} |a|$$

and using (6.9) at the right hand side of (6.8) we get

$$\lambda_\varepsilon(q, w) - a(q) \cdot q \;\geq\; 2\varepsilon_0 |w| - c - \varepsilon |w| \;\geq\; \varepsilon_0 |w| - \varepsilon |w| \;>\; 0$$

for all $0 < \varepsilon < \varepsilon_0$ and $w \in \mathbb{R}^2$ with $|w| > c/\varepsilon_0$. The inequality (6.7) holds therefore for $R = c/\varepsilon_0$ and $0 < \varepsilon < \varepsilon_0$, and the inequality (6.5) is satisfied for $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon(c/\varepsilon_0, \delta)\}$. □

Proof of Proposition 6.1. Let a sequence of points $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ be such that $\lim_n |z_n| = +\infty$ and $\lim_n z_n/|z_n| = q$. Then by Proposition 5.4,

$$\liminf_{n\to\infty} \frac{1}{|z_n|} \log G_+(z, z_n) \;\geq\; -a(q)\cdot q, \quad \forall z \in \mathbb{N}^* \times \mathbb{N}^*,$$

and hence, to get (6.4) it is sufficient to show that

$$\limsup_{n\to\infty} \frac{1}{|z_n|} \log\left(\Xi^q_\delta(z, z_n) - G_+(z, z_n)\right) \;<\; -a(q)\cdot q, \quad \forall z \in \mathbb{N}^* \times \mathbb{N}^*. \tag{6.10}$$

Moreover, since the quantities $\Xi^q_\delta(z, z_n) - G_+(z, z_n)$ are decreasing with respect to $\delta > 0$, it is sufficient to prove this relation for small $\delta > 0$. For this the following estimates are used : for any $\delta > 0$, $z \in \mathbb{N}^* \times \mathbb{N}^*$ and $n \in \mathbb{N}$,

$$\begin{aligned}\Xi^q_\delta(z, z_n) - G_+(z, z_n) \;&=\; \mathbb{E}_z\Big(G(S(\tau), z_n),\; \tau < \infty,\; |S(\tau)| \geq \delta|z_n|\Big)\\ &\leq \sum_{w \in \mathbb{Z}^2 \setminus (\mathbb{N}^* \times \mathbb{N}^*):\; |w| \geq \delta |z_n|} G(z, w) G(w, z_n)\end{aligned}$$

when the coordinates of the vector $q \in \mathcal{S}^2_+$ are non-zero. Similarly,

$$\Xi^q_\delta(z, z_n) - G_+(z, z_n) \;\leq \sum_{w \in (-\mathbb{N}) \times \mathbb{N}^*:\; |w| \geq \delta |z_n|} G(z, w) G(w, z_n)$$

for $q = (1, 0)$ and

$$\Xi^q_\delta(z, z_n) - G_+(z, z_n) \;\leq \sum_{w \in \mathbb{N}^* \times (-\mathbb{N}):\; |w| \geq \delta |z_n|} G(z, w) G(w, z_n)$$

for $q = (0, 1)$. These estimates show that for any $q = (q_1, q_1) \in \mathcal{S}^2_+$, $\delta > 0$, $z \in \mathbb{N}^* \times \mathbb{N}^*$ and $n \in \mathbb{N}$,

$$\Xi^q_\delta(z, z_n) - G_+(z, z_n) \;\leq \sum_{w \in \mathbb{Z}^2:\; \inf_{\theta > 0} |w - \theta q| \geq \kappa\delta |z_n|} G(z, w) G(w, z_n) \tag{6.11}$$

with

$$\kappa \;=\; \begin{cases} 1 & \text{if } q = (q_1, q_2) \in \{(1, 0), (0, 1)\},\\ \min\{q_1, q_2\} & \text{otherwise.}\end{cases}$$

Remark furthermore that for all $a, a' \in \partial D$ and $z, w, z_n \in \mathbb{Z}^2$

$$G(z, w) G(w, z_n) = \; \exp(-a\cdot(w - z) - a'\cdot(z_n - w)) G^a(z, w) G^{a'}(w, z_n)$$

where $G^a(z, z')$ denotes the Green function of the twisted random walk $(S^a(t))$ on $\mathbb{Z}^2$ with transition probabilities (3.1). Since clearly $G^a(z, w) \leq G^a(w, w) = G(0, 0)$ and $G^{a'}(w, z_n) \leq G^{a'}(z_n, z_n) \leq G(0, 0)$ from this it follows that

$$\begin{aligned} G(z, w) G(w, z_n) &\leq \; \exp(-a\cdot(w - z) - a'\cdot(z_n - w))\,(G(0, 0))^2\\ &\leq \exp\big(-a\cdot w - a'\cdot(|z_n| q - w) + a\cdot z - a'\cdot(z_n - |z_n| q)\big)(G(0, 0))^2\\ &\leq \exp\big(-a\cdot w - a'\cdot(|z_n| q - w) + c(|z| + |z_n - |z_n| q|)\big)(G(0, 0))^2\end{aligned}$$

with $c \doteq \max_{a\in D} |a|$. Letting moreover $a = a(w/|z_n|) = a(w)$ and $a' = a(q - w/|z_n|)$ and using the last inequality at the right hand side of (6.11) we obtain

$$\Big(\Xi_\delta^q(z, z_n) - G_+(z, z_n)\Big) \exp\left(-c|z| - c|z_n - |z_n|q|\right) / (G(0,0))^2$$
$$\le \sum_{w\in\mathbb{Z}^2:\ \inf_{\theta>0} |w-\theta q| \ge \kappa\delta|z_n|} \exp\Big(-a(w)\cdot w - a\big(q - w/|z_n|\big)\cdot(q|z_n| - w)\Big)$$
$$\le \sum_{w\in\mathbb{Z}^2:\ \inf_{\theta>0} |w-\theta q| \ge \kappa\delta|z_n|} \exp\Big(-|z_n|\lambda_\varepsilon\big(q, w/|z_n|\big) - \varepsilon|w|\Big)$$

for any $\varepsilon > 0$. Since $\lim_n z_n/|z_n| = q$ and the series $\sum_{w\in\mathbb{Z}^2} \exp(-\varepsilon|w|)$ converge for every $\varepsilon > 0$, from this inequality it follows that the left hand side of (6.10) does not exceed

$$\limsup_{n\to\infty} \frac{1}{|z_n|} \log \sum_{w\in\mathbb{Z}^2:\inf_{\theta>0} |w-\theta q|\ge\kappa\delta|z_n|} \exp\Big(-|z_n|\lambda_\varepsilon\big(q, w/|z_n|\big) - \varepsilon|w|\Big)$$
$$\le -\inf_{w\in\mathbb{R}^2:\inf_{\theta>0} |w-\theta q|\ge\kappa\delta} \lambda_\varepsilon(q, w)$$

When combined with Lemma 6.1 the last inequality proves (6.10).

7. Uniform ratio limit theorem for Markov-additive processes

In this section we improve the ratio limit theorem of the paper [12]. This result is next applied to get the desirable estimates (2.5) and (2.8) for the local processes $(S(t))$ and $(Z^1_+(t))$.

7.1. **Uniform ratio limit theorem for general Markov-additive processes.**

Recall that a Markov chain $\mathcal{Z}(t) = (A(t), M(t))$ on a countable set $\mathbb{Z}^d \times E$ with transition probabilities $p\big((x,y),(x',y')\big)$ is called *Markov-additive* if

$$p\big((x,y),(x',y')\big) = p\big((0,y),(x'-x,y')\big) \quad \text{for all } x, x' \in \mathbb{Z}^d,\ y, y' \in E.$$

The first component $A(t)$ of $\mathcal{Z}(t) = (A(t), M(t))$ is said to be an *additive* part of the process $\mathcal{Z}(t)$, and the second component $M(t)$ is its *Markovian part.* The assumptions we need on the Markov-additive process $(\mathcal{Z}(t) = (A(t), M(t)))$ are the following :

(A1) *The Markov chain $(\mathcal{Z}(t))$ is irreducible on $\mathbb{Z}^d \times E$.*

(A2) *$E \subset \mathbb{R}^l$ for some $l \in \mathbb{N}$ and the function*

$$\hat\varphi(a) = \sup_{z\in\mathbb{Z}^d\times E} \mathbb{E}_z\big(\exp(a\cdot(\mathcal{Z}(1) - z))\big) \tag{7.1}$$

is finite in a neighborhood of zero in $\mathbb{R}^{d+l}$.

Remark that the Markov-additive process $(\mathcal{Z}(t))$ is not assumed to be stochastic : its transition matrix can be strictly sub-stochastic in some points $z = (x,y) \in \mathbb{Z}^d \times E$.

The following property of Markov-additive processes is essential in our analysis. $\mathcal{G}(z, z')$ denotes here the Green function of the Markov process $(\mathcal{Z}(t))$.

Proposition 7.1. *Let the Markov-additive processes $\mathcal{Z}(t) = (A(t), M(t))$ be transient and satisfy the conditions (A1) and (A2). Suppose moreover that for given $w, w' \in \mathbb{Z}^d \times \{0\}$ the inequality*

$$\inf_{z\in\mathbb{Z}^d\times E} \min\ \{\mathbb{P}_z(\mathcal{Z}(n) = z + w),\ \mathbb{P}_z(\mathcal{Z}(n) = z + w')\} > 0 \tag{7.2}$$

holds with some $n > 0$. *Then for any* $0 < \sigma < 1$ *and* $r > 0$ *there are* $C > 0$ *and* $\theta > 0$ *such that*

$$\mathcal{G}(z,z') \;\le\; \frac{1+\sigma+C/|z'|}{1-\sigma}\,\mathcal{G}(z+w-w',z') + C\exp(-\theta|z'|+r|z|) \tag{7.3}$$

for all $z, z' \in \mathbb{Z}^d \times E$.

Proof. In a particular case, for $n = 1$, this statement was proved in the core of the proof of Proposition 3.2 of the paper [12] by using the method of Bernoully part decomposition due to Foley and McDonald [6]. When $n > 1$, for the Green function

$$\tilde{\mathcal{G}}(z,z') \;=\; \sum_{t=0}^{\infty}\mathbb{P}_z(\mathcal{Z}(nt) \;=\; z'), \quad z,z'\in\mathbb{Z}^d\times E,$$

of the included Markov chain $\tilde{\mathcal{Z}}(t) = \mathcal{Z}(nt)$, this result proves that for any $r > 0$ and $0 < \sigma < 1$ there are $\tilde{C} > 0$ and $\tilde{\theta} > 0$ such that

$$\tilde{\mathcal{G}}(z,z') \;\le\; \frac{1+\sigma+\tilde{C}/|z'|}{1-\sigma}\,\tilde{\mathcal{G}}(z+w-w',z')+\tilde{C}\exp(-\tilde{\theta}|z'|+r|z|), \quad \forall z,z'\in\mathbb{Z}^d\times E.$$

Since clearly, $\tilde{\mathcal{G}}(z+w-w',z') = \tilde{\mathcal{G}}(z,z'+w'-w)$ for all $w,w' \in \mathbb{Z}^d\times\{0\}$ and

$$\mathcal{G}(z,z') \;=\; \sum_{t=0}^{n-1}\sum_{z''\in\mathbb{Z}^d\times E}\mathbb{P}_z(\mathcal{Z}(t)=z'')\tilde{\mathcal{G}}(z,z'), \quad \forall z,z'\in\mathbb{Z}^d\times E$$

from this it follows that

$$\begin{aligned}
\mathcal{G}(z,z') \;&\le\; \frac{1+\sigma+\tilde{C}/|z'|}{1-\sigma}\sum_{t=0}^{n-1}\sum_{z''\in\mathbb{Z}^d\times E}\mathbb{P}_z(\mathcal{Z}(t)=z'')\tilde{\mathcal{G}}(z'',z'+w'-w)\\
&\qquad + \tilde{C}\sum_{t=0}^{n-1}\sum_{z''\in\mathbb{Z}^d\times E}\mathbb{P}_z(\mathcal{Z}(t)=z'')\exp(-\tilde{\theta}|z'|+r|z''|)\\
&\le\; \frac{1+\sigma+\tilde{C}/|z'|}{1-\sigma}\,\mathcal{G}(z,z'+w'-w)\\
&\qquad + \tilde{C}\exp(-\tilde{\theta}|z'|)\sum_{t=0}^{n-1}\sum_{z''\in\mathbb{Z}^d\times E}\mathbb{P}_z(\mathcal{Z}(t)=z'')\exp(r|z''|)
\end{aligned}$$

where

$$\begin{aligned}
\sum_{z''\in\mathbb{Z}^d\times E}\mathbb{P}_z(\mathcal{Z}(t)=z'')\exp(r|z''|) \;&\le\; 4\max_{a\in\mathbb{R}^2:|a|=r}\sum_{z''\in\mathbb{Z}^d\times E}\mathbb{P}_z(\mathcal{Z}(t)=z'')\exp(a\cdot z'')\\
&\le\; 4\max_{a\in\mathbb{R}^2:\;|a|\le r}\hat{\varphi}(a)^t, \quad \forall t\in\mathbb{N}.
\end{aligned}$$

When $n > 1$, the inequality (7.3) holds therefore for $r > 0$ small enough with $\theta = \tilde{\theta}$ and

$$C \;=\; 4\tilde{C}\sum_{t=0}^{n-1}\max_{a\in\mathbb{R}^2:\;|a|\le r}\hat{\varphi}(a)^t \;<\; \infty.$$

To complete the proof of this proposition it is now sufficient to notice that the right hand side of (7.3) is increasing with respect to $r > 0$. Hence, if the inequality (7.3) holds with some $C > 0$ and $\theta > 0$ for a small $r > 0$, then it is also satisfied for large $r > 0$ with the same constants C and θ. □

The following statement is an immediate consequence of Proposition 7.1. From now on, for the sake of simplicity of expressions, we will use the following notations

$$\underline{\mathrm{Lim}}_{\,\delta,n,z} = \lim_{\delta\to 0}\liminf_{n\to\infty}\inf_{z\in\mathbb{Z}^d\times E:\ |z|<\delta|z_n|}$$

(7.4)

$$\overline{\mathrm{Lim}}_{\,\delta,n,z} = \lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{z\in\mathbb{Z}^d\times E:\ |z|<\delta|z_n|}$$

Proposition 7.2. *Let a sequence $z_n \in \mathbb{Z}^d \times E$ be such that $\lim_n |z_n| = \infty$ and*

$$\underline{\mathrm{Lim}}_{\,\delta,n,z} \frac{1}{|z_n|}\log\mathcal{G}\big(z, z_n\big) \;\geq\; 0. \tag{7.5}$$

Then under the hypotheses of Proposition 7.1,

$$\underline{\mathrm{Lim}}_{\,\delta,n,z}\frac{\mathcal{G}(z+w',z_n)}{\mathcal{G}(z+w,z_n)} \;=\; \overline{\mathrm{Lim}}_{\,\delta,n,z}\frac{\mathcal{G}(z+w',z_n)}{\mathcal{G}(z+w,z_n)} \;=\; 1. \tag{7.6}$$

Proof. Indeed, by Proposition 7.1, for any $r > 0$ and $0 < \sigma < 1$ there are $C > 0$ and $\theta > 0$ such that

$$\mathcal{G}(z+w',z_n) \;\leq\; \frac{1+\sigma+C/|z_n|}{1-\sigma}\,\mathcal{G}(z+w,z_n) + C\exp(-\theta|z_n| + r|z|)$$

for all $z, z' \in \mathbb{Z}^d \times E$ and consequently,

$$\overline{\mathrm{Lim}}_{\,\delta,n,z}\frac{\mathcal{G}(z+w',z_n)}{\mathcal{G}(z+w,z_n)} \;\leq\; \frac{1+\sigma}{1-\sigma} + C\,\overline{\mathrm{Lim}}_{\,\delta,n,z}\frac{\exp(-\theta|z_n| + r\delta|z_n|)}{\mathcal{G}(z+w,z_n)} \tag{7.7}$$

Moreover, (7.5) shows that the sequence $\exp(-\theta|z_n|+r\delta|z_n|)$ tends to zero as $n\to\infty$ faster than the sequence $1/\mathcal{G}(z+w,z_n)$. From this it follows that the second term of the right hand side of (7.7) is equal to zero and hence, letting $\sigma\to 0$ we conclude that

$$\overline{\mathrm{Lim}}_{\,\delta,n,z}\ \mathcal{G}(z+w',z_n)/\mathcal{G}(z+w,z_n) \;\leq\; 1.$$

The prove the inequality

$$\underline{\mathrm{Lim}}_{\,\delta,n,z}\ \mathcal{G}(z+w',z_n)/\mathcal{G}(z+w,z_n) \;\geq\; 1$$

it is now sufficient to exchange the roles of w and w'. The equalities (7.6) are therefore verified.

□

Suppose now that the Markov process $(\mathcal{Z}(t))$ satisfies the communication condition 5.1 on $\mathbb{Z}^d \times E$. Then there is a bounded function $n_0 : E \to \mathbb{N}^*$ such that for any $z = (x,y) \in \mathbb{Z}^d \times E$,

$$\mathbb{P}_{(x,y)}\big(\mathcal{Z}(n_0(y)) = (x,y)\big) \geq \theta^{n_0(y)} > 0$$

and hence, there is $k \in \mathbb{N}^*$ (for instance, $k = n!$ with $n = \max_y n_0(y)$) such that

$$\mathbb{P}_z\big(\mathcal{Z}(k) = z\big) \geq \theta^k, \qquad \forall z \in \mathbb{Z}^d \times E.$$

We denote by $\hat{k}$ the greatest common divisor of the set of all integers $k > 0$ for which

$$\inf_{z\in Z^d\times E}\mathbb{P}_z\big(\mathcal{Z}(k) = z\big) \;>\; 0. \tag{7.8}$$

The following statement is a refined version of the ratio limit theorem obtained in [12].

Proposition 7.3. *Let a Markov-additive process $\mathcal{Z}(t) = (A(t), M(t))$ be transient and satisfy the communication condition 5.1 and the condition (A2). Suppose moreover that a sequence of points $z_n \in \mathbb{Z}^d \times E$ satisfies the inequality* (7.5) *with* $\lim_n |z_n| = \infty$. *Then*

$$\underline{\mathrm{Lim}}_{\,\delta,n,z}\ \mathcal{G}(z + \hat{k}w, z_n)/\mathcal{G}(z, z_n) \ = \ \overline{\mathrm{Lim}}_{\,\delta,n,z}\ \mathcal{G}(z + \hat{k}w, z_n)/\mathcal{G}(z, z_n) \ = \ 1$$

for all $w \in \mathbb{Z}^d \times \{0\}$.

Proof. Indeed, let $\mathcal{K}$ be the set of all integers for which the inequality (7.8) holds. Because of the communication condition 5.1, for any $w \in \mathbb{Z} \times \{0\}$ there are $\varepsilon > 0$ and a bounded function $n : E \to \mathbb{N}^*$ such that

$$\inf_{z \in \mathbb{Z}^d \times E} \mathbb{P}_z(\mathcal{Z}(n(y)) = z + w) \ \geq \ \varepsilon.$$

Using Markov property, we get therefore

$$\inf_{z \in \mathbb{Z}^d \times E} \mathbb{P}_z(\mathcal{Z}(kn(y)) = z + kw) \ \geq \ \varepsilon^k$$

for any $k \in \mathbb{N}^*$ and consequently,

$$\inf_{z \in \mathbb{Z}^d \times E} \ \min\Big\{\mathbb{P}_z(\mathcal{Z}(kn(y)) = z + kw),\ \mathbb{P}_z(\mathcal{Z}(kn(y)) = z)\Big\} \ > \ 0, \quad \forall k \in \mathcal{K}.$$

By Proposition 7.2, from this it follows that

$$\underline{\mathrm{Lim}}_{\,\delta,n,z}\ \mathcal{G}(z + kw, z_n)/\mathcal{G}(z, z_n) \ = \ \overline{\mathrm{Lim}}_{\,\delta,n,z}\ \mathcal{G}(z + kw, z_n)/\mathcal{G}(z, z_n) \ = \ 1 \tag{7.9}$$

for all $w \in \mathbb{Z}^d \times \{0\}$ and $k \in \mathcal{K}$. Consider now the subgroup $\langle\mathcal{K}\rangle$ of $\mathbb{Z}$ generated by $\mathcal{K}$. Since (7.9) is satisfied for all $w \in \mathbb{Z}^d \times \{0\}$ one can replace w in the left hand side of (7.9) by $-w$ and hence, (7.9) holds also for any $k \in -\mathcal{K}$. Moreover, if (7.9) is satisfied for some $k = k_1$ and $k = k_2$ then the same relation is clearly satisfied for $k = k_1 + k_2$. This proves that (7.9) holds for any $k \in \langle\mathcal{K}\rangle$ and in particular for $k = \hat{k}$ because $\hat{k} \in \langle\mathcal{K}\rangle$ (see Lemma A.1 of Seneta [19]). □

7.2. **Applications to local processes.** According to the above definition, our homogeneous random walk $(S(t))$ on $\mathbb{Z}^2$ is Markov-additive : its additive part is the process $S(t)$ itself and the Markovian part is empty. The quantity $\hat{k}$ is here the period of the random walk $(S(t))$. Proposition 7.3 applied for the process $(S(t))$ with $d = 2$ and $E = \emptyset$ and combined with the estimates (5.8) yields the following statement.

Proposition 7.4. *For any sequence of points $z_n \in \mathbb{Z}^2$ with $\lim_n |z_n| = +\infty$ and $\lim_n z_n/|z_n| = q \in \mathcal{S}^2$,*

$$\begin{aligned} &\underline{\mathrm{Lim}}_{\,\delta,n,z}\ \exp(-a(q)\cdot \hat{k}w) G(z + \hat{k}w, z_n)/G(z, z_n) \ = \\ &\qquad \overline{\mathrm{Lim}}_{\,\delta,n,z}\ \exp(-a(q)\cdot \hat{k}w) G(z + \hat{k}w, z_n)/G(z, z_n) \ = \ 1 \end{aligned} \tag{7.10}$$

for all $w \in \mathbb{Z}^2$.

Proof. Indeed, for any $a \in \partial D$, the twisted homogeneous random walk $(S^a(t))$ defined by (3.1) satisfies the communication condition 5.1 and the condition (A2). The condition (A2) is satisfied because of the assumption (H3), and the communication condition 5.1 is satisfied because the random walk $(S^a(t))$ is irreducible (see the proof of Lemma 4.1 in [12] for more details). Moreover, the Green function $G^a(z,z')$ of the twisted random walk $(S^a(t))$ satisfies the equality

$$G^a(z,z') \;=\; G(z,z')\exp(a\cdot(z'-z)), \quad \forall z,z'\in\mathbb{Z}^2. \tag{7.11}$$

Hence, for any sequence of points $z_n\in\mathbb{Z}^2$ with $\lim_n|z_n|=+\infty$ and $\lim_n z_n/|z_n| = q\in\mathcal{S}^2$, using (5.8) we get

$$\underline{\mathrm{Lim}}_{\,\delta,n,z}\,\frac{1}{|z_n|}\log G^{a(q)}(z,z_n) \;\geq\; 0.$$

and consequently, by Proposition 7.3,

$$\underline{\mathrm{Lim}}_{\,\delta,n,z}G^{a(q)}(z+\hat{k}w,z_n)/G^{a(q)}(z,z_n) = \overline{\mathrm{Lim}}_{\,\delta,n,z}G^{a(q)}(z+\hat{k}w,z_n)/G^{a(q)}(z,z_n) = 1$$

The last relations combined with (7.11) prove (7.10).

□

We need the following consequence of this proposition.

Corollary 7.1. *Let a sequence of points $z_n\in\mathbb{Z}^2$ be such that $\lim_n|z_n|=+\infty$ and $\lim_n z_n/|z_n|=q\in\mathcal{S}^2$. Then for any $\sigma>0$ there are $C'>0$, $C''>0$, $\delta>0$ and $N>0$ such that*

$$C'\exp(a(q)\cdot z-\sigma|z|) \;\leq\; G(z,z_n)/G(0,z_n) \;\leq\; C''\exp(a(q)\cdot z+\sigma|z|) \tag{7.12}$$

for all $n\geq N$ and $z\in\mathbb{Z}^2$ with $|z|<\delta|z_n|$.

Proof. Indeed, the equalities (7.10) show that for any $\sigma>0$ there are $\delta>0$ and $N>0$ such that

$$\exp(a(q)\cdot\hat{k}e-\hat{k}\sigma/2) \;\leq\; G(u+\hat{k}e,z_n)/G(u,z_n) \;\leq\; \exp(a(q)\cdot\hat{k}e+\hat{k}\sigma/2) \tag{7.13}$$

for any unit vector $e\in\mathbb{Z}^2$ and all $n\geq N$, $u\in\mathbb{Z}^2$ with $|z|<\delta|z_n|$. Remark that for any $z\in\mathbb{Z}^2$ there are unit vectors $e_1\in\{(-1,0),(1,0)\}$ and $e_2\in\{(0,-1),(0,1)\}$, non-negative integers $n_1,n_2\in\mathbb{N}$ and real numbers $r_1,r_2\in[0,1[$ such that

$$z=\hat{k}(n_1+r_1)e_1+\hat{k}(n_2+r_2)e_2 \quad\text{ and }\quad \hat{k}(n_1+r_1)+\hat{k}(n_2+r_2)\leq 2|z|$$

If $|z|<\delta|z_n|$ then letting $u_0=\hat{k}r_1e_1+\hat{k}r_2e_2$ and

$$u_k=\begin{cases}u_0+k\hat{k}e_1 & \text{for } 1\leq k\leq n_1\\ u_0+n_1\hat{k}e_1+(k-n_1)\hat{k}e_2 & \text{for } n_1<k\leq n_1+n_2\end{cases}$$

we get $|u_k|\leq(n_1+r_1)\hat{k}+(n_2+r_2)\hat{k}\leq 2|z|<2\delta|z_n|$ for all $k=0,\dots,n_1+n_2-1$. The inequalities (7.13) applied with $u=u_k$ for each $k=0,\dots,n_1+n_2-1$ prove therefore that

$$\begin{aligned}
1_{\{|z|<\delta|z_n|\}}\,\frac{G(z,z_n)}{G(0,z_n)} \;&\leq\; \frac{G(u_0,z_n)}{G(0,z_n)}\prod_{k=0}^{n_1+n_2-1}1_{\{|u_k|<2\delta|z_n|\}}\frac{G(u_{k+1},z_n)}{G(u_k,z_n)}\\
&\leq\; \frac{G(u_0,z_n)}{G(0,z_n)}\prod_{k=0}^{n_1+n_2-1}\exp(a(q)\cdot(u_k-u_{k-1})+\hat{k}\sigma/2)\\
&\leq\; \frac{G(u_0,z_n)}{G(0,z_n)}\exp(a(q)\cdot(z-z_0)+\hat{k}\sigma(n_1+n_2)/2)\\
&\leq\; \frac{G(u_0,z_n)}{G(0,z_n)}\exp(a(q)\cdot z+2\hat{k}|a(q)|+\sigma|z|) \qquad (7.14)
\end{aligned}$$

and similarly,

(7.15)
$$1_{\{|z|<\delta|z_n|\}} \frac{G(z,z_n)}{G(0,z_n)} \;\geq\; 1_{\{|z|<\delta|z_n|\}} \frac{G(u_0,z_n)}{G(0,z_n)} \;\exp(a(q)\cdot z - 2\hat{k}|a(q)| - \sigma|z|)$$

for all $n \geq N$. Remark finally that for any $u \in \mathbb{Z}^2$,

$$\mathbb{P}_u(S(t) = 0 \;\text{ for some } t > 0) \;\leq\; \frac{G(u,z_n)}{G(0,z_n)} \;\leq\; \frac{1}{\mathbb{P}_0(S(t) = u \;\text{ for some } t > 0)}$$

where $\mathbb{P}_u(S(t) = 0 \;\text{ for some } t > 0) > 0$ and $\mathbb{P}_0(S(t) = u \;\text{ for some } t > 0) > 0$ because by assumption (H1), our random walk $(S(t))$ is irreducible. Using this relations together with (7.14) and (7.15) we conclude that (7.12) holds with

$$C' \;=\; \inf_{u\in\mathbb{Z}^2 : |u|\leq 2\hat{k}} \mathbb{P}_u(S(t) = 0 \;\text{ for some } t > 0) \;\exp(-2|a(q)|\hat{k})$$

and

$$C'' \;=\; \sup_{u\in\mathbb{Z}^2 : |u|\leq 2\hat{k}} \frac{1}{\mathbb{P}_0(S(t) = u \;\text{ for some } t > 0)} \;\exp(2|a(q)|\hat{k})$$

□

Consider now the random walk $(Z^1_+(t))$ on $\mathbb{Z}\times\mathbb{N}^*$. Recall that $(Z^1_+(t))$ is identical to $(S(t))$ for $t < \tau_2 \doteq \inf\{n \geq 0 : \; S(n) \not\in \mathbb{Z}\times\mathbb{N}^*\}$ and killed at the time τ_2. Such a process $(Z^1_+(t))$ is Markov additive, its additive and Markovian parts are respectively the first and the second coordinates of $(Z^1_+(t))$. To apply Proposition 7.3 in this case we need to identify the greatest common divisor of the set of all integers $k > 0$ for which

(7.16)
$$\inf_{z\in Z\times\mathbb{N}^*} \mathbb{P}_z\big(Z^1_+(k) = z\big) \;>\; 0.$$

This is a subject of the following lemma.

Lemma 7.1. *The greatest common divisor of the set of all integers $k > 0$ for which (7.16) holds is equal to the period $\hat{k}$ of the random walk $(S(t))$.*

Proof. Indeed, if $\mathbb{P}_0(S(k) = 0) > 0$ for some $k \in \mathbb{N}^*$ then there is a sequence of points $u_0, u_1, \ldots, u_k \in \mathbb{Z}^2$ with $u_0 = u_k = 0$ such that

$$\mathbb{P}_{u_{i-1}}(S(1) = u_i) > 0 \quad \text{ for all } i = 1,\ldots,k.$$

Moreover, without any restriction of generality one can assume that for some $l \in \{1,\ldots,k\}$, the second coordinate of the vectors $e_1 = u_1 - u_0, \ldots, e_l = u_l - u_{l-1}$ is positive and the second coordinate of the vectors $e_{l+1} = u_{l+1} - u_l, \ldots, e_k = u_k - u_{k-1}$ is negative or zero. Then $(0,1) + u_i \in \mathbb{Z}\times\mathbb{N}^*$ for all $i = 0,\ldots,k$ and

$$\begin{aligned}\inf_{z\in\mathbb{Z}\times\mathbb{N}^*} \mathbb{P}_z(Z^1_+(k) = z) \geq\; & \mathbb{P}_{(0,1)}(Z^1_+(k) = (0,1)) \\ \geq\; & \mathbb{P}_{(0,1)}\Big(Z^1_+(t) = (0,1) + u_t, \; \forall t = 1,\ldots k\Big) \\ =\; & \mathbb{P}_0\big(S(t) = u_t, \; \forall t = 1,\ldots k\big) \;>\; 0.\end{aligned}$$

Since according to the definition of the process $(Z^1_+(t))$,

$$\inf_{z\in Z\times\mathbb{N}^*} \mathbb{P}_z\big(Z^1_+(k) = z\big) \;\leq\; \mathbb{P}_z\big(S(k) = z\big) \;=\; \mathbb{P}_0(S(k) = 0), \quad \forall k \in \mathbb{N}^*,$$

we conclude that (7.16) holds if and only if $\mathbb{P}_0(S(k)=0)>0$ and consequently, the greatest common divisor of the set of all integers $k>0$ for which (7.16) holds is equal to the period $\hat{k}$ of the random walk $(S(t))$. Lemma 7.1 is therefore proved. □

From Proposition 7.3 applied with $d=1$ and $E=\mathbb{N}^*$, using the estimates (5.9) and Lemma 7.1 we get the following statement :

Proposition 7.5. *For any sequence of points* $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ *with* $\lim_n |z_n| = +\infty$ *and* $\lim_n z_n/|z_n| = q = (1,0)$,

$$\underline{\operatorname{Lim}}_{\,\delta,n,z}\ \exp(-a(q)\cdot\hat{k}w)G^1_+(z+\hat{k}w,z_n)/G^1_+(z,z_n) \ = \ \overline{\operatorname{Lim}}_{\,\delta,n,z}\ \exp(-a(q)\cdot\hat{k}w)G^1_+(z+\hat{k}w,z_n)/G^1_+(z,z_n) \ = \ 1 \tag{7.17}$$

for all $w\in\mathbb{Z}\times\{0\}$.

Proof. The proof of this proposition is quite similar to the proof of Proposition 7.4. Proposition 7.3 is applied here for the twisted random walk $(Z^{a,1}_+(t))$ on $\mathbb{Z}\times\mathbb{N}^*$ with $a=a(q)$, which is identical to $(S^a(t))$ for

$$t \ < \ \tau_2^a \ \doteq \ \inf\{n\geq 0 :\ S^a(n)\not\in \mathbb{Z}\times\mathbb{N}^*\}$$

and killed at the time τ^a_2. Lemma 4.1 of [12] proves that such a random walk satisfies the communication condition 5.1. The condition (A2) is satisfied here because by assumption (H3), for any $a'\in\mathbb{R}^2$,

$$\sup_{z\in\mathbb{Z}\times\mathbb{N}^*} \ \mathbb{E}_z\big(\exp(a'\cdot(Z^{a,1}_+(1)-z))\big) \ \leq \ \mathbb{E}_z\big(\exp(a'\cdot(S^a(1)-z))\big) \ = \ \varphi(a'+a) \ < \ +\infty.$$

The greatest common divisor of the set of all integers $k>0$ for which

$$\inf_{z\in Z\times\mathbb{N}^*} \mathbb{P}_z\big(Z^{a,1}_+(k)=z\big) \ > \ 0,$$

is clearly the same as for the original process $(Z^1_+(t))$. By Lemma 7.1, this is the period $\hat{k}$ of the random walk $(S(t))$. Finally, the Green function $G^{a,1}_+(z,z')$ of the twisted random walk $(Z^{a,1}_+(t))$ is related to the Green function $G^1_+(z,z')$ of the original random walk $(Z^1_+(t))$ as follows :

$$G^{a,1}_+(z,z') \ = \ G^1_+(z,z')\exp(a\cdot(z'-z)), \quad \forall z,z'\in\mathbb{Z}^2. \tag{7.18}$$

Using this relation together with (5.9), we conclude that for any sequence of points $z_n\in\mathbb{N}^*\times\mathbb{N}^*$ with $\lim_n|z_n|=+\infty$ and $\lim_n z_n/|z_n| = q = (1,0)$,

$$\underline{\operatorname{Lim}}_{\,\delta,n,z}\ \frac{1}{|z_n|}\log G^{a(q),1}_+(z,z_n) \ \geq \ 0$$

and consequently, by Proposition 7.3, for any $w\in\mathbb{Z}\times\{0\}$,

$$\underline{\operatorname{Lim}}_{\,\delta,n,z}\ G^{a(q),1}_+(z+\hat{k}w,z_n)/G^{a(q),1}_+(z,z_n) \ = \ \overline{\operatorname{Lim}}_{\,\delta,n,z}\ G^{a(q),1}_+(z+\hat{k}w,z_n)/G^{a(q),1}_+(z,z_n) \ = \ 1.$$

The last relation combined with (7.18) proves (7.17).

□

From Proposition 7.5, using the same arguments as in the proof of Corollary 7.1 we get the following statement.

Corollary 7.2. *Let a sequence of points* $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ *be such that* $\lim_n |z_n| = +\infty$ *and* $\lim_n z_n/|z_n| = q = (1,0)$*. Then for any* $\sigma > 0$ *there are* $C' > 0$*,* $C'' > 0$*,* $\delta > 0$ *and* $N > 0$ *such that*

$$C' \exp(a(q) \cdot w - \sigma|w|) \;\le\; G^1_+(z+w, z_n)/G^1_+(z, z_n) \;\le\; C'' \exp(a(q)\cdot w + \sigma|w|)$$

for all $n \ge N$*,* $z \in \mathbb{Z} \times \mathbb{N}^*$ *and* $w \in \mathbb{Z} \times \{0\}$ *with* $\max\{|z|, |w|\} < \delta|z_n|$*.*

For the proof of Theorem 1 we need moreover the following stronger statement.

Proposition 7.6. *Let a sequence* $z_n \in \mathbb{N}^* \times \mathbb{N}^*$ *be such that* $\lim_n |z_n| = +\infty$ *and* $\lim_n z_n/|z_n| = q = (1,0)$*. Then for any* $\sigma > 0$ *there are* $C > 0$*,* $\delta > 0$ *and* $N > 0$ *such that*

$$G^1_+(z, z_n)/G^1_+(z_0, z_n) \;\le\; C \exp(a(q)\cdot z + \sigma|z|)$$

for all $n \ge N$ *and* $z \in \mathbb{Z} \times \mathbb{N}^*$ *with* $|z| < \delta|z_n|$*.*

The proof of this proposition uses Corollary 7.2 and the following results.

Lemma 7.2. *Let* $(\xi(t))$ *be an irreducible homogeneous random walk on* $\mathbb{Z}$ *with a zero mean and a finite variance. Denote* $T_0 \doteq \inf\{t \ge 0 : \xi(t) \le 0\}$ *and let* $T \doteq \inf\{t \ge 0 : \xi(t) = \xi(0) + 1\}$*. Then* $\lim_{n\to\infty} \mathbb{P}_n(T < T_0) = 1$*.*

Proof. Indeed, under the hypotheses of this lemma, $T \doteq \inf\{t \ge 0 : \xi(t) = \xi(0)+1\}$ is an almost surely finite stopping time relative to the natural filtration of $(\xi(t))$ and $\mathbb{P}_{n+1}(T < T_0) = \mathbb{P}_1(\xi(t) > -n \text{ for all } 0 \le t \le T)$ for any $n \in \mathbb{N}$. Hence, by monotone convergence theorem

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}_n(T < T_0) &= \lim_{n\to\infty} \mathbb{P}_1\left(\inf_{0\le t\le T} \xi(t) > -n\right) = \mathbb{P}_1\left(\inf_{0\le t\le T}\xi(t) > -\infty\right) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_1(T = n, \inf_{0\le t\le n} \xi(t) > -\infty) = \sum_{n=0}^{\infty} \mathbb{P}_1(T=n) = 1.\end{aligned}$$

□

For the random walk $(S(t))$, this lemma implies the following statement.

Lemma 7.3. *Let* $\hat\tau = \inf\{t \ge 0 : S_2(t) = S_2(0) + 1\}$*. Then for* $a = a(1,0)$*,* $\mathbb{E}_{(0,k)}\left(\exp(a \cdot (S(\hat\tau) - (0,k))), \hat\tau < \tau_2\right) \to 1$ *as* $k \to \infty$*.*

Proof. Indeed, consider the twisted random walk $(S^a(t))$ on $\mathbb{Z}^2$ having transition probabilities $p_a(z,z') = \exp(a\cdot(z'-z))$ with $a = a(1,0)$. Then the same arguments as in the proof of Proposition 3.1 show that

$$\mathbb{E}_{(0,k)}\left(\exp(a\cdot(S(\hat\tau) - (0,k))),\ \hat\tau < \tau_2\right) \;=\; \mathbb{P}_{(0,k)}\left(T^a < T^a_0\right)$$

with $T^a_0 = \inf\{n \ge 0 : S^a_2(t) \le 0\}$ and $T^a = \inf\{n \ge 0 : S^a_2(t) = S^a_2(0) + 1\}$. Moreover, for $a = a(1,0)$, the second coordinate $S^a_2(t)$ of $S^a(t)$ is a homogeneous random walk on $\mathbb{Z}$ with zero mean

$$\mathbb{E}_0(S^a_2(1)) \;=\; \mathbb{E}_0\big(S_2(1)\exp(a\cdot S(1))\big) \;=\; \frac{\partial}{\partial a_2}\varphi(a_1,a_2)\bigg|_{(a_1,a_2)=a(1,0)} \;=\; 0$$

and a finite variance because according to the assumption (H3), the jump generating function

$$\alpha \to \mathbb{E}_0\big(\exp(\alpha S^a_2(1))\big) \;=\; \varphi\big(a + (0,\alpha)\big)$$

of $S_2^a(t)$ is finite everywhere in $\mathbb{R}$. Lemma 7.2 applied with $\xi(t) = S_2^a(t)$, $T = T^a$ and $T_0 = T_0^a$ proves therefore that

$$\lim_{k\to\infty} \mathbb{E}_{(0,k)} \left(\exp(a\cdot(S(\hat\tau) - (0,k))),\ \hat\tau < \tau_2\right) \;=\; \lim_{k\to\infty} \mathbb{P}_{(0,k)}\left(T < T_0\right) \;=\; 1$$

□

Lemma 7.4. *Under the hypotheses of Lemma 7.3, for any $\varepsilon > 0$ there are $N_\varepsilon > 0$, $k_\varepsilon > 0$ and $\sigma_\varepsilon > 0$ such that for all $N \geq N_\varepsilon$, $k \geq k_\varepsilon$ and $0 < \sigma \leq \sigma_\varepsilon$,*

$$\mathbb{E}_{(0,k)}\Big(\exp\big(a(1,0)\cdot S(\hat\tau) - \sigma|S_1(\hat\tau)|\big),\ \hat\tau < \tau_2,\ |S_1(\hat\tau)| < N\Big) \geq \exp(-\varepsilon + a(1,0)\cdot(0,k))$$

Proof. Indeed, for any $x \in \mathbb{Z}$, the sequence $k \to \mathbb{P}_{(0,k)}\big(S(\hat\tau) = (x, k+1),\ \hat\tau < \tau_2\big)$ is increasing because for any $k \in \mathbb{N}^*$,

$$\begin{aligned} &\mathbb{P}_{(0,k+1)}\big(S(\hat\tau) = (x, k+2),\ \hat\tau < \tau_2\big) \\ &\quad\geq\; \mathbb{P}_{(0,k+1)}\left(S(\hat\tau) = (x, k+2) \text{ and } S_2(t) > 1 \text{ for all } t \leq \hat\tau\right) \\ &\quad=\; \mathbb{P}_{(0,k)}\left(S(\hat\tau) = (x, k+1) \text{ and } S_2(t) > 0 \text{ for all } t \leq \hat\tau\right) \\ &\quad=\; \mathbb{P}_{(0,k)}\big(S(\hat\tau) = (x, k+1),\ \hat\tau < \tau_2\big). \end{aligned}$$

By monotone convergence theorem from this it follows that

$$\begin{aligned} &\mathbb{E}_{(0,k)}\Big(\exp\big(a(1,0)\cdot(S(\hat\tau) - (0,k)) - \sigma|S_1(\hat\tau)|\big),\ \hat\tau < \tau_2,\ |S_1(\hat\tau)| < N\Big) \\ &\quad= \sum_{x\in\mathbb{Z}:\ |x|<N} \exp\big(a(1,0)\cdot(x,1) - \sigma|x|\big)\mathbb{P}_{(0,k)}\left(S(\hat\tau) = (x, k+1),\ \hat\tau < \tau_2\right) \\ &\quad\to \sum_{x\in\mathbb{Z}} \exp\big(a(1,0)\cdot(x,1)\big) \lim_{k\to\infty} \mathbb{P}_{(0,k)}\left(S(\hat\tau) = (x, k+1),\ \hat\tau < \tau_2\right) \end{aligned}$$

as $k \to \infty$, $\sigma \to 0$ and $\mathbb{N} \to \infty$. Moreover, using again monotone convergence theorem we get

$$\begin{aligned} &\sum_{x\in\mathbb{Z}} \exp\big(a(1,0)\cdot(x,1)\big) \lim_{k\to\infty} \mathbb{P}_{(0,k)}\left(S(\hat\tau) = (x, k+1),\ \hat\tau < \tau_2\right) \\ &\qquad= \lim_{k\to\infty} \sum_{x\in\mathbb{Z}} \exp\big(a(1,0)\cdot(x,1)\big)\mathbb{P}_{(0,k)}\left(S(\hat\tau) = (x, k+1),\ \hat\tau < \tau_2\right) \\ &\qquad= \lim_{k\to\infty} \mathbb{E}_{(0,k)}\Big(\exp\big(a(1,0)\cdot(S(\hat\tau) - (0,k))\big),\ \hat\tau < \tau_2\Big). \end{aligned}$$

Since by Lemma 7.3, the right hand side of the last relation is equal to 1 we conclude that

$$\mathbb{E}_{(0,k)}\Big(\exp\big(a(1,0)\cdot(S(\hat\tau) - (0,k)) - \sigma|S_1(\hat\tau)|\big),\ \hat\tau < \tau_2,\ |S_1(\hat\tau)| < N\Big) \;\to\; 1$$

as $k \to \infty$, $\sigma \to 0$ and $\mathbb{N} \to \infty$, and consequently, for any $\varepsilon > 0$ there are $N_\varepsilon > 0$, $k_\varepsilon > 0$ and $\sigma_\varepsilon > 0$ such that for all $N \geq N_\varepsilon$, $k \geq k_\varepsilon$ and $0 < \sigma \leq \sigma_\varepsilon$,

$$\mathbb{E}_{(0,k)}\Big(\exp\big(a(1,0)\cdot(S(\hat\tau) - (0,k)) - \sigma|S_1(\hat\tau)|\big),\ \hat\tau < \tau_2,\ |S_1(\hat\tau)| < N\Big) \;\geq\; \exp(-\varepsilon).$$

Lemma 7.4 is therefore proved. □

Consider now an increasing sequence of stopping times $\hat\tau_k$ defined as follows : $\hat\tau_0 \doteq 0$, $\hat\tau_1 \doteq \hat\tau$ and $\hat\tau_k \doteq \inf\{t \geq \hat\tau_{k-1} :\ S_2(t) = S_2(0) + k\}$ for $k \geq 2$. Then from Lemma 7.4 using strong Markov property we obtain the following statement.

Lemma 7.5. *Let $a = a(1,0)$. Then for any $\varepsilon > 0$ there are $C_\varepsilon > 0$, $N_\varepsilon > 0$ and $\sigma_\varepsilon > 0$ such that for all $N \geq N_\varepsilon$, $0 < \sigma \leq \sigma_\varepsilon$ and $k \geq 1$,*

$$\mathbb{E}_{(0,1)}\Big(\exp\big(a\cdot(S(\hat\tau_k)-(0,1))-\sigma|S_1(\hat\tau_k)|\big),\ \hat\tau_k < \tau_2,\ |S_1(\hat\tau_k)| < N(k-1)\Big) \geq C_\varepsilon \exp(-k\varepsilon) \tag{7.19}$$

Proof. Indeed, by strong Markov property, the left hand side of the above inequality is greater than

$$\begin{aligned}
&\mathbb{E}_{(0,1)}\left(\prod_{l=1}^{k-1}\exp\big(a\cdot(S(\hat\tau_{l+1}-S(\hat\tau_l))-\sigma|S_1(\hat\tau_{l+1}-S_1(\hat\tau_l)|\big),\ \hat\tau_k<\tau_2,\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad |S_1(\hat\tau_{l+1}-S_1(\hat\tau_l)| < N,\ \forall 1\leq l\leq k-1\Big)\\
&\geq \prod_{l=1}^{k-1}\mathbb{E}_{(0,l)}\Big(\exp\big(a\cdot(S(\hat\tau)-(0,l))-\sigma|S_1(\hat\tau)|\big),\ \hat\tau<\tau_2,\ |S_1(\hat\tau)|<N\Big)
\end{aligned}$$

and hence, for any $\varepsilon > 0$ with the same quantities $N_\varepsilon > 0$, $\sigma_\varepsilon > 0$ and $k_\varepsilon > 0$ as in Lemma 7.4, the inequality (7.19) holds for all $N \geq N_\varepsilon$, $0 < \sigma \leq \sigma_\varepsilon$ and $k \geq 1$ with

$$C_\varepsilon = \exp(\sigma_\varepsilon k_\varepsilon)\prod_{l=1}^{k_\varepsilon-1}\mathbb{E}_{(0,l)}\Big(\exp\big(a\cdot(S(\hat\tau)-(0,l))-\sigma|S_1(\hat\tau)|\big),\ \hat\tau<\tau_2,\ |S_1(\hat\tau)|<N\Big).$$

□

Proof of Proposition 7.6. Let a sequence of points $z_n \in \mathbb{N}^*\times\mathbb{N}^*$ be such that $\lim_n |z_n| = +\infty$ and $\lim_n z_n/|z_n| = q = (1,0)$. To simplify the notations, we denote throughout the proof of Proposition 7.6

$$a(1,0) = a.$$

Then by Corollary 7.2, for any $\sigma > 0$ there are $C' > 0$, $C'' > 0$, $\delta_\sigma > 0$ and $n_\sigma > 0$ such that

$$C'\exp\big(a\cdot(x,0)-\sigma|x|\big) \leq \frac{G^1_+((x,k),z_n)}{G^1_+((0,k),z_n)} \leq C''\exp\big(a\cdot(x,0)+\sigma|x|\big) \tag{7.20}$$

for all those $n \geq n_\sigma$, $x \in \mathbb{Z}$ and $k \in \mathbb{N}^*$ for which

$$\max\{|x|,k\} < \delta_\sigma|z_n|.$$

Furthermore, recall that the process $(Z^1_+(t))$ is identical to the homogeneous random walk $(S(t))$ on $\mathbb{Z}^2$ before the first time when the second coordinate $S_2(t)$ of $S(t)$ becomes zero or negative and is killed at the time $\tau_2 \doteq \inf\{t \geq 0 : S_2(t) \leq 0\}$. Hence, for any $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$

$$\begin{aligned}
\frac{G^1_+((0,1),z_n)}{G^1_+((0,k),z_n)} &\geq \sum_{x\in\mathbb{Z}}\mathbb{P}_{(0,1)}\left(S(\hat\tau)=(x,k),\ \hat\tau<\tau_2\right)\frac{G^1_+((x,k),z_n)}{G^1_+((0,k),z_n)}\\
&\geq \sum_{x\in\mathbb{Z}:\ |x|<N(k-1)}\mathbb{P}_{(0,1)}\left(S(\hat\tau)=(x,k),\ \hat\tau_k<\tau_2\right)\frac{G^1_+((x,k),z_n)}{G^1_+((0,k),z_n)}
\end{aligned}$$

and consequently, for all $n \geq n_\sigma$, $N > 0$ and $k \geq 1$ satisfying the inequalities $0 < N(k-1) < \delta_\sigma|z_n|$ and $1 < k < \delta_\sigma|z_n|$, using the first inequality of (7.20) we get

$$\frac{G^1_+((0,1),z_n)}{G^1_+((0,k),z_n)} \geq \sum_{x\in\mathbb{Z}:\ |x|<N(k-1)} C'\,\mathbb{P}_{(0,1)}\left(S(\hat\tau)=(x,k),\hat\tau<\tau_2\right)\exp\bigl(a\cdot(x,0)-\sigma|x|\bigr)$$

Moreover, the right hand side of the above inequality is equal to

$$C'\exp(-a\cdot(0,k))\mathbb{E}_{(0,1)}\Bigl(\exp\bigl(a\cdot S(\hat\tau)-\sigma|S_1(\hat\tau)|\bigr),\ \hat\tau<\tau_2,\ |S_1(\hat\tau)|<N(k-1)\Bigr)$$

and hence, using Lemma 7.5 we conclude that for any $\varepsilon>0$, there are $C_\varepsilon>0$, $N_\varepsilon>0$ and $\sigma_\varepsilon>0$ such that

$$G^1_+((0,1),z_n)/G^1_+((0,k),z_n) \ \geq\ C'C_\varepsilon\exp(-a\cdot(0,k-1)-k\varepsilon)$$

whenever

$$0<\sigma<\sigma_\varepsilon,\ n\geq n_\sigma,\ 1\leq k<\delta_\sigma|z_n| \ \text{ and } \ \delta_\sigma|z_n|>(1-k)N_\varepsilon.$$

Since $|z_n|\to+\infty$, this proves that for any $\varepsilon>0$ there are $\hat C_\varepsilon>0$, $\hat\delta_\varepsilon>0$ and $\hat n_\varepsilon>0$ such that

$$1_{\{k<\hat\delta_\varepsilon|z_n|\}}G^1_+((0,k),z_n)/G^1_+((0,1),z_n) \ \leq\ \hat C_\varepsilon\exp(a\cdot(0,k)+\varepsilon k) \tag{7.21}$$

for all $n\geq\hat n_\varepsilon$ and $k\in\mathbb{N}^*$.

To complete the proof of our proposition we combine now the estimates (7.21) with (7.20). From now on $\varepsilon>0$ and $\sigma>0$ are arbitrary and independent from each other. For $n\geq\max\{n_\sigma,\hat n_\varepsilon\}$ and $z=(x,k)\in\mathbb{Z}\times\mathbb{N}^*$ satisfying the inequalities $|x|\leq\delta_\sigma|z_n|$ and $k\leq\hat\delta_\varepsilon|z_n|$, the second inequality of (7.20) together with (7.21) imply that

$$\frac{G^1_+((x,k),z_n)}{G^1_+((0,1),z_n)} \leq \frac{G^1_+((x,k),z_n)}{G^1_+((0,k),z_n)}\times\frac{G^1_+((0,k),z_n)}{G^1_+((0,1),z_n)} \leq \hat C_\varepsilon C''\exp(a\cdot(x,k)+\sigma|x|+\varepsilon k)$$

and consequently,

$$\begin{aligned}\frac{G^1_+((x,k),z_n)}{G^1_+(z_0,z_n)} \ &\leq\ \frac{G^1_+(z_0,z_n)}{G^1_+((0,1),z_n)}\times\hat C_\varepsilon C''\exp(a\cdot(x,k)+\sigma|x|+\varepsilon k)\\ &\leq\ \frac{\hat C_\varepsilon C''}{\mathbb{P}_{z_0}(Z^1_+(t)=(0,1)\text{ for some }t>0)}\exp(a\cdot(x,k)+\sigma|x|+\varepsilon k).\end{aligned}$$

When $\varepsilon=\delta$, the last inequality proves Proposition 7.6 with $\delta=\min\{\hat\delta_\varepsilon,\delta_\sigma\}>0$, $N=\max\{n_\sigma,\hat n_\varepsilon\}>0$ and $C=C''\hat C_\varepsilon/\mathbb{P}_{z_0}(Z^1_+(t)=(0,1)\text{ for some }t>0)$.

8. Proof of Theorem 1

Let a sequence of point $z_n\in\mathbb{N}^*\times\mathbb{N}^*$ be such that $\lim_n|z_n|=+\infty$ and $\lim_n z_n/|z_n|=q\in\mathcal{S}^2_+$. Recall that by Proposition 6.1, for any $z\in\mathbb{N}^*\times\mathbb{N}^*$ and $\delta>0$,

$$\lim_{n\to\infty}G_+(z,z_n)/\Xi^q_\delta(z,z_n)\ =\ 1.$$

To prove (1.4) it is therefore sufficient to show that for some $\delta>0$,

$$\lim_{n\to\infty}\Xi^q_\delta(z,z_n)/\Xi^q_\delta(z_0,z_n)\ =\ h_{a(q)}(z)/h_{a(q)}(z_0),\quad\forall z\in\mathbb{N}^*\times\mathbb{N}^*. \tag{8.1}$$

Consider first the case when the coordinates q_1 and q_2 of the vector q are non-zero. In this case the quantities $\Xi^q_\delta(z,z_n)$ are defined by (6.1) and to get (8.1) it is sufficient to show that for some $\delta > 0$,

$$\lim_{n\to\infty}\frac{\Xi^q_\delta(z,z_n)}{G(0,z_n)} \doteq \lim_{n\to\infty}\frac{G(z,z_n)}{G(0,z_n)} - \lim_{n\to\infty}\mathbb{E}_z\left(\frac{G(S(\tau),z_n)}{G(0,z_n)},\ \tau<\infty,\ |S(\tau)|<\delta|z_n|\right)$$
$$= h_{a(q)}(z), \qquad \forall z\in\mathbb{N}^*\times\mathbb{N}^*. \tag{8.2}$$

The proof of this relation uses dominated convergence theorem, Proposition 3.2, Corollary 7.1 and the results of Ney and Spitzer [16]. Ney and Spitzer [16] proved that

$$\lim_{n\to\infty} G(z,z_n)/G(0,z_n) = \exp(a(q)\cdot z), \quad \forall z\in\mathbb{Z}^2,$$

by Corollary 7.1, for any $\sigma>0$, there are $C>0$ and $\delta>0$ for such that

$$1_{\{|z|<\delta|z_n|\}}G(z,z_n)/G(z_0,z_n) \le C\exp(a(q)\cdot z + \sigma|z|)$$

for all $n\in\mathbb{N}$ and $z\in\mathbb{Z}^2\setminus(\mathbb{N}^*\times\mathbb{N}^*)$, and by Proposition 3.2,

$$\mathbb{E}_z(\exp(a(q)\cdot S(\tau)+\sigma|S(\tau)|),\ \tau<\infty)\ <\ \infty \tag{8.3}$$

if $\sigma>0$ is small enough. Hence, by dominated convergence theorem,

$$\lim_{n\to\infty}\mathbb{E}_z\left(\frac{G(S(\tau),z_n)}{G(0,z_n)},\ \tau<\infty,\ |S(\tau)|<\delta|z_n|\right) = \mathbb{E}_z(\exp(a(q)\cdot S(\tau)),\ \tau<\infty)$$

and consequently, (8.2) holds. When the coordinates of $\lim_n z_n/|z_n| = q$ are non-zero, the equality (8.1) is therefore proved.

Suppose now that $\lim_n z_n/|z_n| = q = (1,0)$. For such a vector q, the quantities $\Xi^q_\delta(z,z_n)$ are defined by (6.2) and to get (8.1) it is sufficient to show that for some $\delta>0$ and $C_0>0$,

$$\lim_{n\to\infty}\frac{G^1_+(z,z_n)}{G^1_+(z_0,z_n)} - \lim_{n\to\infty}\mathbb{E}_z\left(\frac{G^1_+(S(\tau),z_n)}{G^1_+(z_0,z_n)},\ \tau=\tau_1<\tau_2,\ |S(\tau)|<\delta|z_n|\right)$$
$$= C_0 h_{a(q)}(z), \qquad \forall z\in\mathbb{N}^*\times\mathbb{N}^*. \tag{8.4}$$

The proof of this equality uses the same arguments as above but with the help of Proposition 3.3, Proposition 7.6 and the results of the paper [12]. Theorem 1 of [12] proves the point-wise convergence

$$\lim_{n\to\infty} G^1_+(z,z_n)/G^1_+(z_0,z_n) = h^1_{a(q),+}(z)/h^1_{a(q),+}(z_0)$$

with a strictly positive function $h^1_{a(q),+}$ on $\mathbb{Z}\times\mathbb{N}^*$ defined by

$$h^1_{a(q),+}(z) \ =\ x_2\exp(a(q)\cdot z) - \mathbb{E}_z\Big(S_2(\tau_2)\exp(a(q)\cdot S(\tau_2)),\ \tau_2<\infty\Big). \tag{8.5}$$

By Proposition 3.3, (8.3) holds if $\sigma>0$ is small enough and by Proposition 7.6, for any $\sigma>0$ there are $C>0$ and $\delta>0$ such that

$$1_{\{|z|<\delta|z_n|\}}G^1_+(z,z_n)/G^1_+(z_0,z_n)\ \le\ C\exp(a(q)\cdot z+\sigma|z|)$$

for all $n\in\mathbb{N}$ and $z\in\mathbb{Z}\times\mathbb{N}^*$. By dominated convergence theorem from this it follows that the left hand side of (8.4) is equal to

$$\frac{1}{h^1_{a(q),+}(z_0)}\Big(h^1_{a(q),+}(z)\ -\ \mathbb{E}_z\Big(h^1_{a(q),+}(S(\tau)),\ \tau=\tau_1<\tau_2\Big)\Big).$$

Finally, for any $z = (x_1, x_2) \in \mathbb{N}^* \times \mathbb{N}^*$, from (8.5) it follows that

$$
\begin{aligned}
h^1_{a(q),+}(z) \; - \; & \mathbb{E}_z\big(h^1_{a(q),+}(S(\tau)),\ \tau = \tau_1 < \tau_2\big) \\
&= x_2 \exp(a(q)\cdot z) - \mathbb{E}_z\Big(S_2(\tau_2)\exp(a(q)\cdot S(\tau_2)),\ \tau_2 < \infty\Big) \\
&\quad - \mathbb{E}_z\big(S_2(\tau)\exp(a(q)\cdot S(\tau)),\ \tau = \tau_1 < \tau_2\big) \\
&\quad + \mathbb{E}_z\Big(\mathbb{E}_{S(\tau)}\Big(S_2(\tau_2)\exp(a(q)\cdot S(\tau_2)),\ \tau_2 < \infty\Big),\ \tau = \tau_1 < \tau_2\Big).
\end{aligned}
$$

By strong Markov property, the last term of the right hand side of this relation is equal to

$$
\begin{aligned}
&\mathbb{E}_z\left(S_2(\tau_2)\exp(a(q)\cdot S(\tau_2)),\ \tau_1 < \tau_2 < \infty\right) \; = \\
&\mathbb{E}_z\left(S_2(\tau_2)\exp(a(q)\cdot S(\tau_2)),\ \tau_2 < \infty\right) - \mathbb{E}_z\left(S_2(\tau_2)\exp(a(q)\cdot S(\tau_2)),\ \tau = \tau_2 \le \tau_1\right)
\end{aligned}
$$

from which if follows that

$$
\begin{aligned}
h^1_{a(q),+}(z) - & \mathbb{E}_z\big(h^1_{a(q),+}(S(\tau)),\ \tau = \tau_1 < \tau_2\big) \\
&= x_2\exp(a(q)\cdot z) - \mathbb{E}_z\Big(S_2(\tau)\exp(a(q)\cdot S(\tau)),\ \tau < \infty\Big) \\
&= h_{a(q)}(z)
\end{aligned}
$$

and consequently, the left hand side of (8.4) is equal to $h_{a(q)}(z)/h^1_{a(q),+}(z_0)$. The equality (8.4) holds therefore with $C_0 \; = \; 1/h^1_{a(q),+}(z_0) > 0$ and hence, for $q = (1,0)$, the equality (8.1) is also proved.

The proof of (8.1) for $q = (0,1)$ uses exactly the same arguments as above, it is sufficient to exchange the roles of the first and the second coordinates.

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