Submitted to the Annals of Probability

THE GROWTH OF THE INFINITE LONG-RANGE PERCOLATION CLUSTER

BY PIETER TRAPMAN[1,2]

1) Department of Mathematics, Vrije Universiteit Amsterdam
2) Julius Center for Health Sciences and Primary Care, University Medical Center Utrecht

We consider long-range percolation on $\mathbb{Z}^d$, where the probability that two vertices at distance r are connected by an edge is given by $p(r) = 1-\exp[-\lambda(r)] \in (0,1)$ and the presence or absence of different edges are independent. Here $\lambda(r)$ is a strictly positive, non-increasing regularly varying function. We investigate the asymptotic growth of the size of the k-ball around the origin, $|\mathcal{B}_k|$, i.e. the number of vertices that are within graph-distance k of the origin, for $k \to \infty$ for different $\lambda(r)$. We show that conditioned on the origin being in the (unique) infinite cluster, non-empty classes of non-increasing regularly varying $\lambda(r)$ exist for which respectively

- $|\mathcal{B}_k|^{1/k} \to \infty$ almost surely,
- there exist $1 < a_1 < a_2 < \infty$ such that $\lim_{k\to\infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} < a_2) = 1$,
- $|\mathcal{B}_k|^{1/k} \to 1$ almost surely.

This result can be applied to spatial SIR epidemics. In particular, regimes are identified for which the basic reproduction number, R_0, which is an important quantity for epidemics in unstructured populations, has a useful counterpart in spatial epidemics.

1. Introduction and results.

1.1. *Nearest neighbour and long-range percolation.* Ordinary or Bernoulli nearest-neighbour bond percolation models can be used to construct undirected random graphs in which space is explicitly incorporated. Consider an undirected ground graph $G_{\text{ground}} = (V,E)$, in which V is the set of vertices and E the set of edges between vertices. The random graph $G = G(G_{\text{ground}}, p)$ is obtained by removing the edges in E with probability $1-p$, independently of each other. In percolation theory, properties of the remaining graph are studied. Much effort has been put in understanding the dependence of G on p for $G_{\text{ground}} = \mathbb{L}^d := (\mathbb{Z}^d, E_{nn})$, where $\mathbb{Z}^d$ is the d-dimensional cubic lattice and E_{nn} is the set of edges between nearest

AMS 2000 subject classifications: Primary 60K35, 92D30; Secondary 82B28
Keywords and phrases: long-range percolation, epidemics, chemical distance

imsart-aop ver. 2009/08/13 file: Trapmanannalsrevision.tex date: November 20, 2009

neighbours, i.e., vertices at Euclidean distance 1. See [17] for an extensive account on percolation on this graph.

Long-range percolation is an extension of this model: consider a countable vertex set $V \subset \mathbb{R}^d$. Vertices at distance r (according to some norm) share an edge with probability $p(r) = 1 - e^{-\lambda(r)}$, which only depends on r, and the presence or absence of an edge is independent on the presence or absence of other edges. We refer to $\lambda(r)$ as the *connection function.* Questions similar to the questions in ordinary nearest-neighbour percolation can be asked for properties of the random graph $G = G(V, \lambda(r))$ obtained by long-range percolation. Note that ordinary percolation on $\mathbb{L}^d$ is a special case of long-range percolation with $V = \mathbb{Z}^d$ and $p(r) = p\mathbb{1}(r = 1)$, where $\mathbb{1}$ is the indicator function and the Euclidean distance has been used.

In this paper we consider long-range percolation on $V = \mathbb{Z}^d$ and we investigate properties of the k-ball $\mathcal{B}_k$, the set of vertices within graph (or chemical) distance k of the origin (a definition of the graph distance is provided below). In particular, we analyse the asymptotic behaviour of the size of this k-ball, $|\mathcal{B}_k|$, for $k \to \infty$. We show that there exist non-empty regimes of non-increasing, positive regularly varying connection functions for which respectively

- $|\mathcal{B}_k|^{1/k} \to \infty$ almost surely,
- there exist $1 < a_1 < a_2 < \infty$ such that $\lim_{k\to\infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} < a_2) > 0$,
- $|\mathcal{B}_k|^{1/k} \to 1$ almost surely.

1.2. *The model and notation.* In this paper we will frequently use the following notation: $\mathbb{N}$ is the set of natural numbers, including 0, while $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$ is the set of strictly positive integers. Similarly, $\mathbb{R}_+ = (0, \infty)$ consists of the strictly positive real numbers. The ceiling of a real number x is defined by $\lceil x \rceil := \min\{y \in \mathbb{Z}; x \leq y\}$ and its floor by $\lfloor x \rfloor := \max\{y \in \mathbb{Z}; y \leq x\}$. For $x, y \in \mathbb{R}$, we define $\sum_{i=x}^{y} f(i) := \sum_{i=\lfloor x \rfloor}^{\lceil y \rceil} f(i)$. The cardinality of a set S is denoted by $|S|$.

The probability space for long-range percolation graphs on a countable vertex set $V \subset \mathbb{R}^d$, with *connection function* $\lambda(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$, used in this paper, is denoted by $(\mathcal{G}_V, \mathcal{F}, \mathbb{P})$. Here $\mathcal{G}_V$ is the set of all simple undirected graphs with vertex set V, $\mathcal{F}$ is an appropriate σ-algebra and $\mathbb{P}$ is the product measure defined by $\mathbb{P}(< x, y > \in E) = p(x, y) := 1 - e^{-\lambda(x,y)}$ for $x, y \in V$, where $< x, y > \in E$ denotes the event that the vertices $x \in V$ and $y \in V$ share an edge. We say that long-range percolation system is *homogeneous*, if the connection function only depends on the distance between its arguments, i.e. $\lambda(x,y)=\lambda(\|x-y\|)$. In this paper $\|x\|$ denotes the L^∞ norm of x, i.e. for $x=(x_1,x_2,\dots,x_d)\in\mathbb{R}^d$, $\|x\|:=\max_{1\le i\le d} x_i$ and we only consider homogeneous long-range percolation models. Using the L^∞ norm, is just for mathematical convenience and using the L^1 or Euclidean norm will not cause substantial changes in this paper.

We assume that $\lambda(r)$ is non-increasing and regularly varying, that is, $\lambda(r)$ may be written as $r^{-\beta}L(r)$, for some $\beta\in[0,\infty)$ and $L(r)$ is slowly varying, that is, for every $c>0$, $\lim_{r\to\infty} L(cr)/L(r)=1$.

The random long-range percolation graph is denoted by $G_V=G_V(\lambda(r))$. With some abuse of notation we define $G_K:=G_{\mathbb{Z}^d\cap[\lfloor -K/2\rfloor,\lfloor K/2\rfloor)^d}$, for $K\in\mathbb{N}_+$ and $G:=G_{\mathbb{Z}^d}$.

A path of length n consists of an ordered set of edges $(<v_{i-1},v_i>)_{1\le i\le n}$. Furthermore, if the vertices $\{v_i\}_{0\le i\le n}\in V$ are all different, this path is said to be *self-avoiding*. Vertices x and y are in the same *cluster* if there exists a path from x to y. The graph distance or chemical distance, $D_V(x,y)=D_{G_V}(x,y)$, between x and y is the (random) minimum length of a path from x to y in G_V. If x and y are not in the same cluster, then $D_V(x,y)=\infty$. Furthermore, we set $D_V(x,x)=0$. We use $D(x,y)$ for $D_{\mathbb{Z}^d}(x,y)$ and for $K\in\mathbb{N}_+$, we define $D_K(x,y):=D_{\mathbb{Z}^d\cap[\lfloor -K/2\rfloor,\lfloor K/2\rfloor)^d}(x,y)$.

If the probability that the origin is contained in an infinite cluster (a cluster containing infinitely many vertices) of G is positive, then the long-range percolation system is said to be *percolating*. If a homogeneous long-range percolation system is percolating, Kolmogorov's zero-one law (see e.g. [16, p.290]) gives that G almost surely (a.s) contains at least 1 infinite cluster, while Theorem 0 of [15] (see also [5, Thm. 1.3]), gives that under mild conditions, the infinite cluster is a.s. unique. These mild conditions are satisfied for homogeneous long-range percolation models on $\mathbb{Z}^d$, for which $\lambda(r)$ is non-increasing. This unique infinite cluster is denoted by $\mathcal{C}_\infty$. Throughout, we will only consider percolating systems.

For $x\in\mathbb{Z}^d$, the set $\mathcal{B}_k(x)$ is defined by $\mathcal{B}_k(x):=\{y\in\mathbb{Z}^d; D(x,y)\le k\}$ and $\mathcal{B}_k:=\mathcal{B}_k(0)$. We define (as in [23]):

$$\underline{R_*} := \liminf_{k\to\infty} (\mathbb{E}(|\mathcal{B}_k|))^{1/k}, \tag{1}$$

$$\overline{R_*} := \limsup_{k\to\infty} (\mathbb{E}(|\mathcal{B}_k|))^{1/k}. \tag{2}$$

If $\underline{R_*}=\overline{R_*}$, then $R_*:=\underline{R_*}=\overline{R_*}$.

1.3. *The main results.*

Theorem 1.1 *Consider a percolating homogeneous long-range percolation model as defined in Section 1.2 with vertex set $\mathbb{Z}^d$ and non-increasing connection function $\lambda(r) = r^{-\beta}L(r)$, where $L(r)$ is slowly varying and $\beta \in \mathbb{R}_+$.*
(a) *If either $\beta < d$, or $\beta = d$ and $\int_1^\infty L(r)r^{-1}dr = \infty$, then $\mathbb{P}(\mathcal{B}_1 = \infty) = 1$. In particular, $|\mathcal{B}_k|^{1/k} = \infty$ a.s., for $k \in \mathbb{N}_+$. So, $R_* = \infty$.*
(b) *If $\beta = d$, there exists a $K > 1$ such that $L(r)$ is non-increasing on $[K, \infty)$ and the following conditions are satisfied*

$$\int_1^\infty \frac{L(r)}{r} dr < \infty, \tag{3}$$

$$-\int_K^\infty \frac{\log[L(r)]}{r(\log[r])^2} dx < \infty, \tag{4}$$

then there exist constants $1 < a_1 \leq a_2 < \infty$, such that

$$\lim_{k\to\infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} < a_2 | 0 \in \mathcal{C}_\infty) = 1.$$

Furthermore, $1 < \underline{R_*} \leq \overline{R_*} < \infty$.
(c) *If $\beta > d$, then for $k \to \infty$, $|\mathcal{B}_k|^{1/k} \to 1$ a.s. Furthermore, $R_* = 1$.*

Part (a) of this theorem is almost trivial and is only stated for reasons of completeness. A function which satisfies all of the conditions in part (b) is $\lambda(r) = r^{-d}(\log[1+r])^{-\gamma}$ for $\gamma > 1$. Part (b) is the main result and perhaps the most surprising result of the paper. Part (c) is not surprising if one knows the results of [8]. However, some work has to be done. We prove part (c) by using that $\mathbb{P}(D(0,x) \leq n)$ decreases faster than $\|x\|^{-\beta'}$ if $\|x\| \to \infty$. This is a result of the following stronger theorem, which proof also provides a simplification of the proof of the main result in [8]:

Theorem 1.2 *Consider a percolating homogeneous long-range percolation model as defined in Section 1.2 with vertex set $\mathbb{Z}^d$ and non-increasing connection function $\lambda(r) = r^{-\beta}L(r)$, where $L(r)$ is slowly varying and $\beta > d$. Let the constants α, β' and β'' be such that $d < \beta' < \beta'' < \min(2d, \beta)$ and for all $r \geq 1$, it holds that $\lambda(r) \leq \alpha r^{-\beta''}$. There exists a positive constant $c = c(\alpha, \beta'', \beta')$ such that for $\gamma := \frac{\log(2d/\beta')}{\log 2} < 1$, $K(n) := \exp[cn^\gamma] + 1$, all $n \in \mathbb{N}$ and all $x \in \{x \in \mathbb{Z}^d; \|x\| > K(n)\}$, it holds that*

$$\mathbb{P}(D(0,x) \leq n) \leq [K(n)]^{\beta'} \|x\|^{-\beta'}. \tag{5}$$

1.4. *Motivation from epidemiology.* We consider an SIR (Susceptible $\rightarrow$ Infectious $\rightarrow$ Recovered) epidemic with a fixed infectious period (which without loss of generality will be taken to be of length 1) in a homogeneous randomly mixing population of size n. In this model pairs of individuals contact each other according to independent Poisson processes with rate λ/n. If an infectious individual contacts a susceptible one, the latter becomes infectious as well. An infectious individual stays infectious for one time unit and then recovers and stays immune forever. Usually it is assumed that initially there is 1 infectious individual, with a remaining infectious period of 1 time unit, and all other individuals are initially susceptible.

The basic reproduction number, R_0 of an SIR epidemic process in a large homogeneous randomly mixing population of size n is defined as the expected number of individuals infected by a single infectious individual in a further susceptible population [13]. To proceed, we define X_0^n as the set of initially infected individuals in a population of size n. These individuals are said to enter X_0^n at time 0. For $k \in \mathbb{N}$, an individual not in $\cup_{j=0}^k X_j^n$ enters X_{k+1}^n at the first instance it is contacted by an individual which itself entered X_k^n at most 1 time unit ago. We define $\mathcal{B}_k^n := \cup_{j=0}^k X_j^n$. Note that the actual chain of infections that has caused the infectiousness of an individual in X_k^n might be longer than length k, because it is possible that the time needed to traverse this longer infection chain is less than the time needed to traverse the chain of k contacts that caused the individual to be in X_k^n.

It has been known for a long ting (see e.g. [2]), that in randomly mixing populations, SIR epidemics can be coupled to branching processes, in the sense that we can simultaneously define a Galton-Watson process $\{Z_k\}_{k\in\mathbb{N}}$ (for a definition see [18]), and an epidemic processes $\{|X_k^n|\}_{k\in\mathbb{N}}$, for all $n \in \mathbb{N}$ on one probability space, such that for every $k \in \mathbb{N}$ and $n \to \infty$, $\mathbb{P}(|X_k^n| \to Z_k) = 1$. In this approximation R_0, corresponds to the offspring mean $m := \lim_{n\to\infty} \mathbb{E}(Z_1|Z_0 = 1)$ of the Galton-Watson process. From the theory of branching processes we know that under mild conditions, $m > 1$, implies that $m^{-k}\sum_{i=0}^k Z_i$ converges a.s. to an a.s. finite random variable, which is strictly positive with non-zero probability. By the relationship between R_0 and the offspring mean m, we deduce that if $R_0 > 1$ in large populations the expectation $\mathbb{E}(|\mathcal{B}_k^n|)$ will initially grow exponentially in k (with base R_0), and $|\mathcal{B}_k^n|$ will also grow exponentially (with base R_0) with positive probability [18]. In particular, it holds that

$$\lim_{k\to\infty}\lim_{n\to\infty}(\mathbb{E}(|\mathcal{B}_k^n|))^{1/k} = \max(R_0, 1). \tag{6}$$

In this paper we investigate whether it is possible to define a quantity with similar properties as R_0 for spatial epidemics.

Assume that the individuals in the population are located at $\mathbb{Z}^d$ and that the epidemic starts with one infectious individual at the origin and all other individuals are initially susceptible. A pair of individuals at L^∞distance r will make contacts, according to independent Poisson processes with rate $\lambda(r)$. The poisson processes governing the contacts are independent. The probability that an infectious individual makes at least 1 contact to a given individual at distance r during its infectious period is given by $p(r) = 1 - e^{-\lambda(r)}$. For this spatial epidemic let X_k be defined as X_k^n is defined above. It is easy to see that the law of $\cup_{j=0}^{k} X_j$ is the same as the law of $\mathcal{B}_k$ in the long-range percolation model with connection function $\lambda(r)$ (see [10] for an exposition on this relationship for nearest-neighbour bond percolation).

It is possible to define R_0 for spatial epidemics by the usual definition $R_0 = \mathbb{E}(|X_1|||X_0| = 1)$. However, this definition is of no practical use, because there is no reason to assume that $\mathbb{E}(|X_1|||X_0| = 1) = 1$ is a threshold above which a large epidemic is possible and below which it is impossible. Indeed, if $p(x) = p$ for x at Euclidean distance 1 of the origin and 0 otherwise, then it is known that on $\mathbb{Z}^2$, $p = 1/2$ is a threshold [17, 19], which corresponds to $\mathbb{E}(|X_1|||X_0| = 1) = 2$. For more results on the growth of the nearest neighbour bond percolation cluster see [1].

The definitions (1) and (2) and, if it exist, the corresponding R_*, might be useful and provide information about the spread of the spatial epidemic. These definitions are inspired by (6). Theorem 1.1 gives that regimes of $\lambda(r)$ exist in which $R_* = \infty$, $R_* = 1$ and $1 < \underline{R_*} \leq \overline{R_*} < \infty$.

Note that only if $1 < \underline{R_*} \leq \overline{R_*} < \infty$, the quantities $\underline{R_*}$ and $\overline{R_*}$, seem to be informative, because $R_* = 1$ does not even contain information on whether an epidemic survives with positive probability or not. While, for $R_* = \infty$ the number of infected individuals will be immense within a few generations, and R_* does not really tell anything about the asymptotic behaviour of the spread.

A real-life application of long-range percolation for the spread of epidemics can be found in [12], where the spread of plague among great gerbils in Kazakhstan is modelled using techniques from (long-range) percolation theory. This present paper may be seen as the mathematical rigorous counterpart of the paper by Davis et al.

2. Remarks and discussion.

- Without costs in the proof we could replace Theorem 1.1 by the following more general, but less elegant theorem:

Theorem 2.1 *Consider a percolating homogeneous long-range percolation model as defined in Section 1.2 with vertex set $\mathbb{Z}^d$ and non-increasing connection function $\lambda(r)$.*
(a) *If $\sum_{x \in \mathbb{Z}^d} 1 - e^{-\lambda(\|x\|)} = \infty$, then $\mathbb{P}(\mathcal{B}_1 = \infty) = 1$. Therefore, $|\mathcal{B}_k|^{1/k} = \infty$ a.s., for $k \in \mathbb{N}_+$ and $R_* = \infty$.*
(b) *If $\lambda(r) > r^{-d} L'(r)$ is non-increasing, $\sum_{x \in \mathbb{Z}^d} 1 - e^{-\lambda(\|x\|)} < \infty$, $L'(r)$ is positive, non-increasing and slowly varying and satisfies*

$$- \int_K^\infty \frac{\log[L'(r)]}{r(\log[r])^2} dx < \infty,$$

then there exist constants $a_1 > 1$ and $a_2 < \infty$, such that

$$\lim_{k \to \infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} < a_2 | 0 \in \mathcal{C}_\infty) = 1.$$

Furthermore, $1 < \underline{R_*} \le \overline{R_*} < \infty$.
(c) *If $\liminf_{x \to \infty} -\log[\lambda(r)]/\log[r] > d$, then for $k \to \infty$, $|\mathcal{B}_k|^{1/k} \to 1$ a.s. Furthermore, $R_* = 1$.*

Contrary to Theorem 2.1(b) part (b) of this theorem includes a class of connection functions that are constant on $[n, n+1)$ for every $n \in \mathbb{Z}$ and some other piecewise constant connection functions (for which there exists no K such that $L(r) = r^d \lambda(r)$ is non-increasing on $[K, \infty)$).

- Condition (4) is annoying, because this assumption makes that this paper does not deal with all possible non-increasing regularly varying connection functions. An example of a function, which does not satisfy (4), but satisfies the other assumptions in Theorem 1.1 (c), is

$$\lambda(r) = r^{-d} L(r) = r^{-d} \exp\left(-\frac{\log[r]}{\log[1 + \log[r]]}\right) \qquad \text{for } r > 1.$$

However, as stated above, for $\gamma > 1$, functions of the form

$$\lambda(r) = r^{-d} (\log[1 + r])^{-\gamma},$$

satisfy all of the conditions of Theorem 1.1 (b). So, the class of functions treated in the second statement of the theorem is not empty. We do not know whether Theorem 1.1 (b) still holds without condition (4).

- Theorem 1.1(b) gives rise to some other questions, such as

  1. Does R_* exist for long-range percolation models with connection functions in the regime of Theorem 1.1(b)?

2. Does the long-range percolation graph obtained in Theorem 1.1(b) have a non-amenable subgraph $G' = (V', E')$? i.e., does the following hold?

$$\inf_{W \subset V'; 0 < |W| < \infty} \frac{|\delta W|}{|W|} > 0, \tag{7}$$

where δW is the set of edges in G', with 1 end-vertex in W and 1 end vertex in $V' \setminus W$.

- The assumption $p(r) < 1$ (i.e. $\lambda(r) < \infty$) is only used for ease of exposition. All results of this paper are equally valid if we relax this assumption and replace Condition (3) by $\int_{R_1+1}^{\infty} \lambda(r) r^{d-1} dr = \infty$, where $R_1 := \inf\{r \in \mathbb{R}; p(r) < 1\}$. So, we may allow for $p(1) = 1$, in order to guarantee that it is possible to have an infinite component for any dimension d and any β for long-range percolation on $\mathbb{Z}^d$. Indeed, if $d = 1$ and $\beta > 2$, an infinite component only exists if $p(1) = 1$ [21].
  It is tempting to add the assumption $p(1) = 1$ in Theorem 1.1 (b). With that extra condition, the proofs in this paper will become easier. However, without this extra assumption, the results of Theorem 1.1 can be generalised to the random connection model [20], i.e. long-range percolation, where the vertex set is generated by a homogeneous Poisson point process on $\mathbb{R}^d$. This is important in biological applications, where exact lattice structures will not appear and models in which the individuals/vertices are located according to a Poisson point process might be more realistic (see e.g. [12]).
- Up to now, in literature, most effort has been put in investigating the scaling behaviour of the maximum diameter of the clusters of a homogeneous long-range percolation graph defined on the block, $V_K = \mathbb{Z}^d \cap [\lfloor -K/2 \rfloor, \lfloor K/2 \rfloor)^d$, i.e. in obtaining

$$D_K := \max_{x,y \in V_K; D(x,y) < \infty} D_K(x,y).$$

  See e.g. [3, 6, 8, 11]. Some of the results have been proven, under the extra assumption that $p(1) = 1$.
  Benjamini, Kesten, Peres and Schramm [4] proved that for $\lambda(r) = r^{-\beta} L(r)$, where $\beta < d$ and $L(r)$ is slowly varying, $\lim_{K \to \infty} D_K = \lceil d/(d-\beta) \rceil$, a.s. (see also [3]). Coppersmith, Gamarnik and Sviridenko [11] showed that for $\lambda(r) = \alpha r^{-d}$ and $K \to \infty$, the quantity $D_K \log[\log[K]] / \log[K]$ is a.s. bounded away from 0 and ∞.
  We define $\mathcal{C}_K$ as the (random) largest cluster of the long-range percolation graph G_K (recall that $G_K := G_{\mathbb{Z}^d \cap [\lfloor -K/2 \rfloor, \lfloor K/2 \rfloor)^d}$). In case

of a tie, $\mathcal{C}_K$ is chosen uniformly at random from the largest clusters. Note that if $D_K = k$ and there exist a ρ such that $|\mathcal{C}_K| > \rho K^d$ with probability tending to 1 if $K \to \infty$, then $|\mathcal{B}_k| > \rho K^d$ with positive probability for $k \to \infty$. So, there is an obvious relation between the diameter of a long-range percolation cluster on V_K and the rate at which $\mathcal{B}_k$ grows. However, this relation and the results stated above do not help us directly for obtaining Theorem 1.1(b) and (c), because the regime of part (b) is not even considered in the papers cited above, and the proof of the statement that $D_K \log[\log[K]]/\log[K]$ is bounded away from 0 an ∞ for $\lambda(r) = \alpha r^{-d}$ in [11], critically depends on the fact that $\sum_{i\in\mathbb{N}_+} \lambda(i)i^{d-1} = \infty$. Although results on the diameter of a long-range percolation cluster on V_K may provide a lower bound for the number of vertices that are within graph distance k of the origin, they do not provide an upper bound. So, these results are of no direct help to prove the final statement of the theorem.

- Biskup proved the following theorem (Here given in our notation):

  Theorem 2.2 (Biskup, [8]) *Consider a percolating homogeneous long-range percolation model as defined in Section 1.2, with vertex set* $\mathbb{Z}^d$ *and non-increasing connection function* $\lambda(r) = r^{-\beta}L(r)$ *with* $\beta \in (d, 2d)$ *and* $L(r)$ *is positive and slowly varying. Then for* $\Delta = \frac{\log[2]}{\log[2d/\beta]}$ *and every* $\epsilon > 0$*,*

  $$\lim_{\|x\|\to\infty} \mathbb{P}\Big(\Delta - \epsilon \le \frac{\log[D(0,x)]}{\log[\log[\|x\|]]} \le \Delta + \epsilon \big| 0, x \in \mathcal{C}_\infty\Big) = 1. \tag{8}$$

  Note that $\Delta > 1$. This theorem implies that for every $\epsilon > 0$ and every sequence of vertices $\{x_k; x_k \in \mathbb{Z}^d\}$,

  $$\lim_{k\to\infty} \mathbb{1}(\|x_k\| > \exp[k^{(\Delta-\epsilon)^{-1}}])\mathbb{P}(D(0, x_k) \le k) = 0,$$

  but it does not give results on the rate at which this probability decreases to 0. This rate is needed to prove whether $|\mathcal{B}_k|^{1/k} \to 1$ or not. Theorem 1.2 contains

  $$\lim_{\|x\|\to\infty} \mathbb{P}\Big(\frac{\log[D(0,x])}{\log[\log[\|x\|]]} \ge \Delta - \epsilon\Big) = 1, \tag{9}$$

  from [8] as Corollary 3.2. The proofs of Theorem 1.2 and Corollary 3.2 are shorter and arguably more straightforward than the proof of the lower bound in Theorem 2.2 as provided in [8] (cf. [9]).
- For $\beta > 2d$, Berger [6] proves that

  $$\liminf_{\|x\|\to\infty} \Big(\frac{D(0,x)}{\|x\|}\Big) > 0, \tag{10}$$

almost surely. This implies that with probability 1, the growth of $|\mathcal{B}_k|$ is at most of order k^d.

- In a recent manuscript Biskup [9] proved that if $p(1) = 1$ and β and Δ are as in Theorem 2.2, then for every $\epsilon > 0$

$$\lim_{L\to\infty} \mathbb{P}\Big((\log[L])^{\Delta-\epsilon} \leq D_L \leq (\log[L])^{\Delta+\epsilon}\Big) = 1.$$

Furthermore, he proves that for $p(1) = 1$ and $\Lambda(r) := \mathbb{Z}^d \cap [-r, r]^d$, it holds that for every $\epsilon > 0$,

$$\lim_{k\to\infty} \mathbb{P}\Big(\Lambda(\exp[k^{\Delta^{-1}-\epsilon}]) \subset \mathcal{B}_k \subset \Lambda(\exp[k^{\Delta^{-1}+\epsilon}])\Big) = 1.$$

We note that an alternative proof of the statement

$$\lim_{k\to\infty} \mathbb{P}\Big(\mathcal{B}_k \subset \Lambda(\exp[k^{\Delta^{-1}+\epsilon}])\Big) = 1,$$

might be obtained by a slight change in the proof of Theorem 1.1(c): If we replace the definition "$A_k(\epsilon)$ is the event that $\mathcal{B}_k$ contains a vertex at distance more than $(1+\epsilon)^k$ from the origin" by "$A'_k(\epsilon)$ is the event that $\mathcal{B}_k$ contains a vertex at distance more than $\exp[ck^{\gamma+\epsilon}]$ from the origin", the proof essentially does not change.

3. Proofs of Theorems 1.1 and 1.2.

3.1. *The $R_* = \infty$ regime: proof of Theorem 1.1(a).* We consider long-range percolation with non-increasing connection function $\lambda(r) = r^{-\beta}L(r)$, with $L(r)$ strictly positive and slowly varying and $\beta < d$ or $\beta = d$ and $\int_1^\infty L(r)/r dr = \infty$. We prove that in the cases under consideration in Theorem 1.1(a), $\sum_{x\in\mathbb{Z}^d\setminus\{0\}} p(0,x) = \infty$, and therefore by the second Borel-Cantelli lemma (see e.g. [16, p288]) we obtain immediately that $\mathbb{P}(|\mathcal{B}_1| = \infty) = 1$ a.s.

By [7, Thm. 1.3.6] we know that for all $c > 0$ $\lim_{r\to\infty} r^c L(r) = \infty$, and therefore for both cases under consideration and for all $R > 0$, we obtain

$$\int_R^\infty \lambda(r) r^{d-1} dr = \int_R^\infty (r^{d-\beta}L(r))/r dr = \infty.$$

Furthermore, note that for $x < 1$, it holds that $1 - e^{-x} \geq x - x^2/2 \geq x/2$. and that for large enough r, $\lambda(r) < 1$ for both cases under consideration.

So, constants $R > 0$ and $c' > 0$ exist, such that

$$(11) \quad \sum_{x\in\mathbb{Z}^d\setminus 0} p(0,x) \geq \sum_{x\in\mathbb{Z}^d;\|x\|>R} p(0,x) \geq \frac{1}{2} \sum_{x\in\mathbb{Z}^d;\|x\|>R} \lambda(\|x\|) \geq \geq c' \int_{R+1}^{\infty} r^{d-1}\lambda(r)dr = \infty,$$

which proves that $|\mathcal{B}_1| = \infty$ a.s. in the regimes of Theorem 1.1(a). □

3.2. *The $R_* = 1$ regime: proofs of Theorem 1.2 and Theorem 1.1(c).* In this subsection we prove that if $\lambda(r) = r^{-\beta}L(r)$, with $\beta > d$ and $L(r)$ positive and slowly varying, then $|\mathcal{B}_k|^{1/k} \to 1$ a.s. and $\mathbb{E}(|\mathcal{B}_k|^{1/k}) \to 1$, for $k \to \infty$. To do this we first show that Theorem 1.2 implies Theorem 1.1(c):

Proof of Theorem 1.1(c): Note that $|\mathcal{B}_k| \geq 1$ and therefore, $|\mathcal{B}_k|^{1/k} \geq 1$ for all k. So, $\liminf_{k\to\infty}(\mathbb{E}(|\mathcal{B}_k|))^{1/k} \geq 1$. Furthermore it is immediate from Theorem 1.2 that there exist a constant C such that

$$(12) \quad \mathbb{E}(|\mathcal{B}_k|) = \sum_{x\in V} \mathbb{P}(D(0,x) \leq k) \leq \leq (2K(k)+1)^d + \sum_{x\in V;\|x\|\geq K(k)} K(k)^{\beta'}\|x\|^{-\beta'} \leq CK(k)^d.$$

Because $\gamma < 1$, $\lim_{k\to\infty} K(k)^{1/k} = 1$. This implies that $\limsup_{k\to\infty}(\mathbb{E}(|\mathcal{B}_k|))^{1/k} = 1$. So $R_* = 1$.

From Theorem 1.2 we obtain that if $d < \beta' < \beta'' < \min(\beta, 2d)$, then for all $\epsilon > 0$ there exists a constant $N_1 = N_1(\epsilon)$ such that for all $k > N_1$, $K(k) < (1+\epsilon)^k$. Let $A_k = A_k(\epsilon)$ be the event that $\mathcal{B}_k$ contains a vertex at distance more than $(1+\epsilon)^k$ from the origin. For $k > N_1$, it holds that

$$(13) \quad \mathbb{P}(A_k) \leq \sum_{x\in\mathbb{Z}^d;\|x\|>(1+\epsilon)^k} [K(k)]^{\beta'}\|x\|^{-\beta'} \leq c_1 \sum_{n=(1+\epsilon)^k}^{\infty} [K(k)]^{\beta'} n^{d-1-\beta'} \leq \leq c_2[K(k)]^{\beta'}(1+\epsilon)^{(d-\beta')k},$$

where c_1 and c_2 are positive constants. Note that for $d < \beta' < 2d$ there exist constants $N_2 > N_1$ and $c_3 > 0$ such that for all $k > N_2$,

$$c_2 \exp[c\beta' k^{\gamma}](1+\epsilon)^{(d-\beta')k} < (1+\epsilon)^{-c_3 k},$$

by $\gamma := \frac{\log(2d/\beta')}{\log(2)} < 1$. This implies that for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(A(n)) < N_2 + \sum_{n=N_2}^{\infty} (1+\epsilon)^{-c_3 n} < \infty,$$

and so, $\sum_{k=1}^{\infty} \mathbb{P}(|\mathcal{B}_k|^{1/k} - 1 > \epsilon) < \infty$, which in turn implies by the Borel-Cantelli lemma (see e.g. [16, p.277]) that Theorem 1.1(c) holds. □

Before providing the proof of Theorem 1.2, we state a useful lemma:

Lemma 3.1 *Consider the long-range percolation model defined in Section 1.2, with vertex set $\mathbb{Z}^d$ and connection function $\lambda(r)$, which satisfies $\lambda(r) < \alpha r^{-\beta''}$ for all $r \geq 1$ and a constant $\alpha > 0$, then*

$$\mathbb{P}(D(0,x) \leq k) \leq \sum_{i=1}^{k} \mathbb{E}(|\mathcal{B}_{i-1}|)\mathbb{E}(|\mathcal{B}_{k-i}|)\alpha(\|x\|/k)^{-\beta''}.$$

Proof of Lemma 3.1: If a self-avoiding path between 0 and x of length at most k exists, this path will contain at least 1 edge shared by vertices at distance $\lceil \|x\|/k \rceil$ or more of each other. Let $N(k,x)$ be the number of edges shared by vertices at distance at least $\lceil \|x\|/k \rceil$ of each other, that are contained in at least 1 self-avoiding path between vertices 0 and x of length at most k. For $1 \leq j \leq k$, let $N(k,x;j)$ be the number of edges shared by vertices at distance at least $\lceil \|x\|/k \rceil$ of each other, that are contained as the j-th edge in at least 1 self-avoiding path from 0 to x of length at most k.

By Markov's inequality we obtain

$$\mathbb{P}(D(0,x) \leq k) \leq \mathbb{P}(N(k,x) \geq 1) \leq \mathbb{E}(N(k,x)) \leq \sum_{j=1}^{k} \mathbb{E}(N(k,x;j)).$$

We further note that if we define $D(0,0) = 0$, then by observing that for $x > 0$, $p(x) < \lambda(x)$, it holds for $k \geq 1$ that:

$$\begin{aligned}
&\mathbb{E}(N(k,x;j)) \\
\leq\ & \sum_{\substack{x_1,x_2 \in \mathbb{Z}^d \\ \|x_1 - x_2\| \geq \lceil \|x\|/k \rceil}} \mathbb{P}(D(0,x_1) = j-1)p(x_1,x_2)\mathbb{P}(D(x_2,x) \leq k-j) \\
\leq\ & \sum_{x_1,x_2 \in \mathbb{Z}^d} \mathbb{P}(D(0,x_1) = j-1)p(\lceil \|x\|/k \rceil)\mathbb{P}(D(x_2,x) \leq k-j) \\
=\ & \sum_{x_1 \in \mathbb{Z}^d} \mathbb{P}(D(0,x_1) = j-1)p(\lceil \|x\|/k \rceil) \sum_{x_2 \in \mathbb{Z}^d} \mathbb{P}(D(x_2,x) \leq k-j) \\
\leq\ & \mathbb{E}(|\mathcal{B}_{j-1}|)\mathbb{E}(|\mathcal{B}_{k-j}|)\alpha(\|x\|/k)^{-\beta''}.
\end{aligned}$$

□

Proof of Theorem 1.2: We prove this theorem by induction and using Lemma 3.1. Let the constant c be such that,

$$c > \max\Big(\frac{\log[\alpha]}{\beta'}, \frac{\beta''+1}{(\beta''-\beta')\gamma} + \frac{3d\log[5] + 2\log[1 + d/(\beta'-d)] + \log[\alpha]}{\beta''-\beta'}\Big),$$

The induction hypothesis is that for all $j \leq k$ and $x \in \{x \in \mathbb{Z}^d; x > K(j)\}$,

$$\mathbb{P}(D(0,x) \leq j) \leq (K(j))^{\beta'} \|x\|^{-\beta'}.$$

Note that the assumption holds for $k = 0$ and $k = 1$, because $K(0) = 2$, $K(1) = e^c + 1$ and $c > (\beta')^{-1} \log[\alpha]$. A straightforward computation yields that the induction hypothesis implies that for all $j \leq k$,

$$\begin{aligned}
\mathbb{E}(|\mathcal{B}_j|) &\leq (2K(j)+3)^d + \sum_{x \in \mathbb{Z}^d; \|x\| > K(j)+1} (K(j))^{\beta'} \|x\|^{-\beta'} \\
&\leq (2K(j)+3)^d + \sum_{i=K(j)+2}^{\infty} 2d(2i+1)^{d-1} (K(j))^{\beta'} i^{-\beta'} \\
&\leq (2K(j)+3)^d + 2d(K(j))^{\beta'} 3^{d-1} \int_{K(j)}^{\infty} x^{d-\beta'-1} dx \\
&= (2K(j)+3)^d + 2d(K(j))^{\beta'} 3^{d-1} (\beta'-d)^{-1} K(j)^{d-\beta'} \\
&\leq K(j)^d (5^d + 2d3^{d-1}(\beta'-d)^{-1}) \\
&\leq 5^d (1 + d(\beta'-d)^{-1}) K(j)^d,
\end{aligned}$$

We now observe that by Lemma 3.1,

$$\begin{aligned}
&\mathbb{E}(N(k+1,x;j)) \\
\leq\ & \mathbb{E}(|\mathcal{B}_{j-1}|)\mathbb{E}(|\mathcal{B}_{k-j}|)\alpha(\|x\|/(k+1))^{-\beta''} \\
\leq\ & 5^{2d}\Big(1 + \frac{d}{\beta'-d}\Big)^2 (K(j-1)K(k+1-j))^d \alpha \Big(\frac{k+1}{\|x\|}\Big)^{\beta''} \\
\leq\ & 5^{2d}\Big(1 + \frac{d}{\beta'-d}\Big)^2 (e^{c(j-1)^\gamma} + 1)^d (e^{c(k+1-j)^\gamma} + 1)^d \alpha \Big(\frac{k+1}{\|x\|}\Big)^{\beta''} \\
\leq\ & 5^{3d}\Big(1 + \frac{d}{\beta'-d}\Big)^2 e^{2^{1-\gamma} dck^\gamma} \alpha \Big(\frac{k+1}{\|x\|}\Big)^{\beta''} \\
=\ & 5^{3d}\Big(1 + \frac{d}{\beta'-d}\Big)^2 (K(k))^{\beta'} \alpha \Big(\frac{k+1}{\|x\|}\Big)^{\beta''},
\end{aligned}$$

where we have used that for $\gamma \leq 1$ and $0 \leq y \leq x$, $y^\gamma + (x-y)^\gamma \leq 2(x/2)^\gamma$, where the right-hand side is equal to $x^\gamma \beta'/d$, by the definition of γ.

Using this for $\|x\| > K(k+1)$, we obtain

$$
\begin{aligned}
&\mathbb{P}(D(0,x)) \le k+1) \\
\le\ & \sum_{j=1}^{k+1} \mathbb{E}(N(k+1,x;j)) \\
\le\ & (k+1)5^{3d}\Big(1+\frac{d}{\beta'-d}\Big)^2 K(k)^{\beta'}\alpha\Big(\frac{k+1}{\|x\|}\Big)^{\beta''} \\
=\ & (k+1)^{\beta''+1}5^{3d}\Big(1+\frac{d}{\beta'-d}\Big)^2 K(k)^{\beta'}\alpha\|x\|^{-(\beta''-\beta')}\|x\|^{-\beta'} \\
\le\ & (k+1)^{\beta''+1}5^{3d}\Big(1+\frac{d}{\beta'-d}\Big)^2 \alpha(K(k+1))^{-(\beta''-\beta')}(K(k+1))^{\beta'}\|x\|^{-\beta'}.
\end{aligned}
$$

Because

$$
c > \frac{\beta''+1}{(\beta''-\beta')\gamma} + \frac{3d\log[5]+2\log[1+d/(\beta'-d)]+\log[\alpha]}{\beta''-\beta'},
$$

and $\mathbb{P}(D(0,x) \le k+1) \le 1$, we deduce after some straightforward computations,

$$
(k+1)^{\beta''+1}5^{3d}(1+d(\beta'-d)^{-1})^2\alpha\exp[-(\beta''-\beta')c(k+1)^{\gamma}] \le 1.
$$

for $k \ge 0$, and that if the induction hypothesis holds, then it also holds that if $\|x\| > K(k+1)$

$$
\mathbb{P}(D(0,x) \le k+1) \le (K(k+1))^{\beta'}\|x\|^{-\beta'}.
$$

Which proves the theorem. □

Theorem 1.2 contains a part of Theorem 2.2 as a corollary:

Corollary 3.2 *Consider homogeneous long-range percolation on $\mathbb{Z}^d$ as in Section 1.2, with non-increasing connection function $\lambda(r) = r^{-\beta}L(r)$ with $d < \beta < 2d$ and $L(r)$ positive and slowly varying. For $\Delta = \frac{\log[2]}{\log[2d/\beta]}$ and every $\epsilon > 0$,*

$$
\lim_{\|x\|\to\infty} \mathbb{P}\Big(\Delta - \epsilon < \frac{\log[D(0,x])}{\log[\log[\|x\|]]}\Big) = 1. \tag{14}
$$

Proof of Corollary 3.2: Observe that (14) can be rewritten as

$$
\lim_{\|x\|\to\infty} \mathbb{P}\Big(D(0,x) \le (\log[\|x\|])^{\Delta-\epsilon}\Big) = 0.
$$

We choose $\beta' < \beta$ such that

$$(\Delta - \epsilon)\gamma = \Big(\frac{\log[2]}{\log[2d/\beta]} - \epsilon\Big)\Big(\frac{\log[2]}{\log[2d/\beta']}\Big)^{-1} < 1,$$

which can be done for every $\epsilon > 0$. By filling in $k = (\log[\|x\|])^{\Delta-\epsilon}$, into

$$\mathbb{P}(D(0,x) \leq k) \leq [K(k)]^{\beta'}\|x\|^{-\beta'} \qquad \text{for } x \in \{x \in \mathbb{Z}^d; \|x\| > K(k)\}.$$

and $K(k) = 1 + \exp[ck^\gamma]$, we obtain that

$$\mathbb{P}(D(0,x) \leq (\log[\|x\|])^{\Delta-\epsilon}) \leq (1 + \exp[c(\log[\|x\|])^{(\Delta-\epsilon)\gamma}])^{\beta'}\|x\|^{-\beta'},$$

for $x \in \{x \in \mathbb{Z}^d; \|x\| > 1 + \exp[c(\log[\|x\|])^{(\Delta-\epsilon)\gamma}]\}$. If $\|x\|$ is large enough, then

$$\|x\| > 1 + \exp[c(\log[\|x\|])^{(\Delta-\epsilon)\gamma}] \Leftrightarrow \|x\| > 1 + \|x\|^{c(\log[\|x\|])^{(\Delta-\epsilon)\gamma-1}}$$

holds. Therefore, for $\|x\| \to \infty$,

$$\begin{aligned}\mathbb{P}(D(0,x) \leq (\log[\|x\|])^{\Delta-\epsilon}) &\leq (1 + \exp[c\log[\|x\|]^{(\Delta-\epsilon)\gamma}])^{\beta'}\|x\|^{-\beta'} \\ &= (\|x\|^{-1} + \|x\|^{c(\log[\|x\|])^{(\Delta-\epsilon)\gamma-1}-1})^{\beta'} \\ &\to 0,\end{aligned}$$

which proves the corollary. □

3.3. *The non-trivial R_* regime: proof of Theorem 1.1(b).* In this subsection we consider percolating homogeneous long-range percolation with non-increasing connection function $\lambda(r) = r^{-d}L(r)$, where $L(r)$ is non-negative, slowly varying and satisfies

$$\int_1^\infty r^{-1}L(r)dr < \infty \qquad \text{and} \qquad -\int_R^\infty \frac{\log[L(r)]}{r(\log[r])^2}dr < \infty$$

for some constant $R > 0$. We investigate the growth behaviour of $|\mathcal{B}_k|$ for $k \to \infty$. In particular, we show that $\lambda(r)$ exists, satisfying these conditions, such that $\lim_{k\to\infty} |\mathcal{B}_k|^{1/k} > 1$ with positive probability.

In the first subsection we provide a straightforward and almost trivial proof for the upper bounds of the growth of $|\mathcal{B}_k|$ given in Theorem 1.1(b). After that we provide some useful lemmas that will be used in the proof of the lower bound of the growth of $|\mathcal{B}_k|$. Then we give an outline of the proof and in the final subsection the full proof of $\lim_{k\to\infty} \mathbb{P}(|\mathcal{B}_k|^{1/k} > a_1 | 0 \in \mathcal{C}_\infty) = 1$ for some $a_1 > 1$ is given. In this proof renormalisation arguments are used.

3.3.1. *The upper bound: proof of $\overline{R_*} < \infty$.*

Lemma 3.3 *Consider the percolating homogeneous long-range percolation model as defined in Section 1.2 with vertex set $\mathbb{Z}^d$ and non-increasing connection function $\lambda(r) = r^{-d}L(r)$ where $L(r)$ is positive, slowly varying and satisfies condition (3). There exists a positive and finite constant a_2, such that $\overline{R_*} < a_2$ and*

$$\lim_{k\to\infty} \mathbb{P}(|\mathcal{B}_k|^{1/k} < a_2) = 1.$$

Proof of Lemma 3.3: Assign independent Poisson processes to every pair of vertices in $\mathbb{Z}^d$, denoting the contacts between the pair of vertices. The density of the Poisson process of vertices at distance $r > 0$ is $\lambda(r)$. We observe that the probability that at least 1 contact is made between two vertices at distance r at the interval $(0,1)$ is $p(r)$. If the pairs of individuals that make at least 1 contact in the interval $(0,1)$ are joined by an edge, the long-range percolation graph under consideration is re-obtained.

We obtain after some basic computations that

$$\sum_{x\in\mathbb{Z}^d\setminus\{y\}} \lambda(\|x-y\|) < \infty,$$

for all $y \in \mathbb{Z}^d$. It is straightforward to couple the k-ball, $|\mathcal{B}_k|$ to the number of individuals in the first k generations of a supercritical branching random walk with a Poisson distributed offspring size distribution and $\overline{R_*} < a_2$ for some $a_2 > 1$ follows immediately. The branching random walk is the process in which initially one individual (or particle) lives at the origin. This individual stays there forever, although it can only give birth to new individuals during the first time unit of its life. This individual gives births to individuals at vertex x, according to a Poisson process with rate $\lambda(\|x\|)$.

The set $\mathcal{B}_k$ is created by killing upon birth, all individuals that are born on a vertex that is already occupied by another individual. From the theory of branching processes [18] we know that there exist a random variable, W which is almost surely finite and a constant a_2', such that for all $k \in \mathbb{N}_+$, the number of individuals in the first k generations of such a branching random walk is a.s. bounded above by $W(a_2')^k$.

In the coupled process $|\mathcal{B}_k|$ is bounded above by the number of individuals in the first k generations of the branching random walk, which proves that $\lim_{k\to\infty} \mathbb{P}(|\mathcal{B}_k|^{1/k} < a_2) = 1$ for $a_2 > a_2'$. □

3.3.2. *the lower bound, preliminary lemmas and definitions.* In order to prove that for the given connection function, $\lim_{k\to\infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} | 0 \in \mathcal{C}_\infty) = 1$ holds, we need the following lemmas.

Lemma 3.4 *For positive slowly varying $L(x)$, which is non-increasing and strictly less than 1 on $[K,\infty)$ for some $K > 1$, condition (4) is equivalent to the following condition:*
For every $\delta > 0$, a $K_1 := K_1(\delta) > K$ exists such that for $r > K_1$,

$$\sum_{k=1}^{\infty} \frac{\log[L(r^{2^k})]}{2^k(\log[r])} > -\delta$$

and therefore,

$$\prod_{k=1}^{\infty} [L(r^{2^k})]^{2^{-k}} > r^{-\delta}.$$

Lemma 3.5 *For positive slowly varying $L(x)$, which is non-increasing on $[K,\infty)$ for some $K > 1$, and satisfies condition (3), there exists $K_2 \geq K$ such that $L(r) < 1$ for all $r \geq K_2$.*

Lemma 3.6 *Consider a percolating homogeneous long-range percolation graph as in Section 1.2 with non-increasing connection function $\lambda(r) = r^{-d}L(r)$, where $L(r)$ is positive and slowly varying. Let $\mathcal{C}'_K$ be the largest cluster of G restricted to vertices in $V_K := \mathbb{Z}^d \cap [\lceil -K/2 \rceil, \lceil K/2 \rceil)^d$ and edges shared by vertices both in V_K. For every $\epsilon > 0$, there exist numbers $\rho > 0$ and $K_3 = K_3(\epsilon) < \infty$ such that for every $r \geq K_3$*

$$\mathbb{P}(|\mathcal{C}'_r| < \rho r^d) \leq \epsilon, \tag{15}$$

$$\mathbb{P}(|\mathcal{C}'_r| < \rho r^d, 0 \in \mathcal{C}'_r | 0 \in \mathcal{C}_\infty) \leq \epsilon. \tag{16}$$

Proof of Lemma 3.4: Since $L(x)$ is non increasing and strictly less than 1 for $x \geq K$, it is enough to show that $-\int_K^\infty \frac{\log[L(x)]}{x(\log[x])^2} dx < \infty$ is equivalent with $-\sum_{k=1}^\infty \frac{\log[L(K^{2^k})]}{2^k(\log[K])} < \infty$.

From [7] we know that for slowly varying, eventually decreasing $L(x)$, there exist a function $\delta(x)$, converging to a finite number and a non-negative function $\epsilon(x)$ converging to 0 for $x \to \infty$, such that

$$L(x) = \exp[\delta(x) - \int_K^x \frac{\epsilon(t)}{t} dt].$$

Filling this in gives

$$\begin{aligned} \int_K^\infty \frac{\log[L(x)]}{x(\log[x])^2} dx &= \int_K^\infty \frac{\delta(x) - \int_K^x \frac{\epsilon(t)}{t} dt}{x(\log[x])^2} dx = \\ &= \int_K^\infty \frac{\delta(x)}{x(\log[x])^2} dx - \int_K^\infty \int_t^\infty \frac{\epsilon(t)}{t} \frac{1}{x(\log[x])^2} dx dt = \\ &= \int_K^\infty \frac{\delta(x)}{x(\log[x])^2} dx - \int_K^\infty \frac{\epsilon(t)}{t \log[t]} dt. \end{aligned} \tag{17}$$

Since $\delta(x)$ converges to a finite number and $0 < L(x) < 1$ for $x \geq K$, $\delta(x)$ is bounded away from infinity and the first term is finite. So, we obtain that $-\int_K^\infty \frac{\log[L(x)]}{x(\log[x])^2} dx < \infty$ is equivalent to $\int_K^\infty \frac{\epsilon(t)}{t\log[t]} dt < \infty$.

Note that

$$\sum_{k=1}^{\infty} \frac{\log[L(K^{2^k})]}{2^k(\log[K])} = \sum_{k=1}^{\infty} \frac{\delta(K^{2^k})}{2^k(\log[K])} - \sum_{k=1}^{\infty} \frac{\int_K^{K^{2^k}} \frac{\epsilon(t)}{t} dt}{2^k(\log[K])},$$

and

$$\sum_{k=1}^{\infty} \frac{\int_K^{K^{2^k}} \frac{\epsilon(t)}{t} dt}{2^k(\log[K])} = \sum_{k=1}^{\infty} \sum_{l=1}^{k} \frac{\int_{K^{2^{l-1}}}^{K^{2^l}} \frac{\epsilon(t)}{t} dt}{2^k(\log[K])} = \tag{18}$$

$$= \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \frac{\int_{K^{2^{l-1}}}^{K^{2^l}} \frac{\epsilon(t)}{t} dt}{2^k(\log[K])} = \sum_{l=1}^{\infty} \frac{\int_{K^{2^{l-1}}}^{K^{2^l}} \frac{\epsilon(t)}{t} dt}{2^{l-1}(\log[K])}.$$

The final term of (18) is bounded below by

$$\sum_{l=1}^{\infty} \int_{K^{2^{l-1}}}^{K^{2^l}} \frac{\epsilon(t)}{t\log[t]} dt = \int_K^\infty \frac{\epsilon(t)}{t\log[t]} dt.$$

Similarly we deduce that this term is bounded above by $2\int_K^\infty \frac{\epsilon(t)}{t\log[t]} dt$. Furthermore, $\sum_{k=1}^\infty \frac{\delta(K^{2^k})}{2^k(\log[K])}$ is finite since $\delta(x)$ converges and is bounded away from infinity. Therefore,

$$-\sum_{k=1}^{\infty} \frac{\log[L(K^{2^k})]}{2^k(\log[K])} < \infty \Leftrightarrow \int_K^\infty \frac{\epsilon(t)}{t\log[t]} dt < \infty$$

and the lemma follows. □

Proof of Lemma 3.5: Because $L(x)$ is not increasing for $x > K$ we know that $\lim_{x\to\infty} L(x)$ exists. If $\lim_{x\to\infty} L(x) > 0$ then $\int_K^\infty L(x)x^{-1}dx = \infty$ and violates condition (3). Together with the assumption that $L(x) > 0$ for all $x > 0$, this leads to $\lim_{x\to\infty} L(x) = 0$ and therefore there exists $K_2 \in \mathbb{R}_+$ such that $L(x) < 1$ for all $x > K_2$. □

Proof of Lemma 3.6: This Lemma immediately follows from Theorem 3.2 and Corollary 3.3 in [8] together with $\mathbb{P}(0 \in \mathcal{C}_\infty) > 0$. □

In the remaining proof of Theorem 1.1(b) we use a construction for which the following definitions are needed.

We define a hierarchy of blocks of vertices in $\mathbb{Z}^d$ as follows. For every $i \in \mathbb{N}$ and every $\bar{n} \in \mathbb{Z}^d$, we define $\Lambda_i(\bar{n}) := \mathbb{Z}^d \cap [-(l_i-1)/2, (l_i-1)/2] + \bar{n} l_i$, where $l_i := (l_0)^{2^i}$, l_0 is odd and

$$l_0 > \max[(\frac{100}{\rho^2})^{1/(d-2/5)}, \frac{100}{16d}, K_1(1/5), K_2, K_3(1/25)].$$

The constants $\rho, K_1(1/5), K_2$ and $K_3(1/25)$ are as in the preceding lemmas. We say that $\Lambda_i(\bar{n})$ is a level i block, and note that for every $i \in \mathbb{N}$, the level i blocks form a partition of $\mathbb{Z}^d$. Every level i block is entirely contained in a level $i+1$ block and every level $i+1$ block contains $(l_0)^{2^i d}$ level i blocks. We use $\bar{n}_i(x)$ to denote the index of the level-i block containing vertex x, that is, we define for $x \in V$, $\bar{n}_i(x) \in \mathbb{Z}^d$ such that $x \in \Lambda_i(\bar{n}_i(x))$.

Let G be the long-range percolation-graph under consideration, and $G_i(\bar{n})$ is defined as G restricted to $\Lambda_i(\bar{n})$, that is, $G_i(\bar{n})$ is the graph consisting of vertex set $\Lambda_i(\bar{n})$ and those edges of G for which both end-vertices are in $\Lambda_i(\bar{n})$. Let $\mathcal{D}_i(x)$ be the set of vertices in $\Lambda_i(\bar{n}_i(x))$, that are within graph distance

$$h_i := (l_0)^d 2^i + 2^i - 1 = ((l_0)^d + 1)2^i - 1$$

of x in the graph $G_i(\bar{n}_i(x))$.

A vertex $x \in \mathbb{Z}^d$ is *good up to level* 0 if

$$|\mathcal{D}_0(x)| \geq m_0 := \rho (l_0)^d.$$

For $x \in \mathbb{Z}^d$ and $S \subset \mathbb{Z}^d$, let $x \leftrightarrow S$ denote the event that there is a vertex $y \in S$ such that $< x, y > \in E$. Furthermore, let

$$\bar{\mathcal{D}}_{i+1}(x) := \{y \in \mathbb{Z}^d \setminus (\Lambda_i(0) \cup \Lambda_i(\bar{n}_i(x))) | y \leftrightarrow \mathcal{D}_i(x),\ y \text{ good up to level } i\},$$

be the set of vertices not in $(\Lambda_i(0) \cup \Lambda_i(\bar{n}_i(x)))$ that share an edge with vertices in $\mathcal{D}_i(x)$ and that are good up to level i. A vertex $x \in \mathbb{Z}^d$ is *good up to level* $i+1$ if x is good up to level i, and if

$$\begin{aligned} (19) \quad & A_{i+1}(x) := \\ & := |\{\bar{n} \in \mathbb{Z}^d | \Lambda_i(\bar{n}) \subset \Lambda_{i+1}(\bar{n}_{i+1}(x)), \exists y \in \Lambda_i(\bar{n}),\ \text{s.t. } y \in \bar{\mathcal{D}}_{i+1}(x)\}| \geq \\ & \geq m_{i+1} := c_0 L(l_{i+1}) M_i. \end{aligned}$$

Here $c_0 := 2\rho/25$, $M_i := \prod_{j=0}^{i} m_j$ and for $i \in \mathbb{N}_+$ the constants m_i are defined recursively. In words this means that the number of level i blocks in $\Lambda_{i+1}(\bar{n}_{i+1}(x) \setminus (\Lambda_i(0) \cup \Lambda_i(\bar{n}_i(x)))$, that contains at least 1 vertex that shares an edge with a vertex in $D_i(x)$ is at least m_{i+1}. Some algebra gives that

$m_{i+1} = L(l_{i+1})(c_0 m_0 \prod_{j=1}^i [L(l_j)]^{2^{-j}})^{2^i} = L(l_{i+1})(c_0\rho(l_0)^d \prod_{j=1}^i [L(l_j)]^{2^{-j}})^{2^i}$ for $i \in \mathbb{N}$. Since $l_0 > K_1(1/5)$ and $l_0 > K_2$, Lemmas 3.4 and 3.5 give

$$\prod_{j=1}^{i} [L(l_j)]^{2^{-j}} \geq \prod_{j=1}^{\infty} [L(l_j)]^{2^{-j}} \geq (l_0)^{-1/5},$$

while Lemma 3.5 gives

$$\prod_{j=1}^{i} [L(l_j)]^{2^{-j}} \leq 1.$$

Combining these observations gives that for $i \in \mathbb{N}_+$,

$$L((l_0)^{2^i})(c_0\rho(l_0)^{d-1/5})^{2^{i-1}} \leq m_i \leq L((l_0)^{2^i})(c_0\rho(l_0)^d)^{2^{i-1}}.$$

Since $L(x)$ is slowly varying this implies that there exist constants $1 < c_0' < c_0'' < \infty$ and $0 < \hat{c}_0' < \hat{c}_0'' < \infty$ such that $\hat{c}_0'(c_0')^{2^i} < m_i < \hat{c}_0''(c_0'')^{2^i}$ holds for all $i \in \mathbb{N}_+$.

A vertex is *ultimately good*, if it is good up to every level $i \in \mathbb{N}$.

3.3.3. *Outline of proof of lower bound in Theorem 1.1(b).* As may be guessed from the definitions above, the proof will follow a renormalisation scheme. The following steps are made.

- We observe that if x is good up to level $i+1$, then $|\mathcal{D}_{i+1}(x)| \geq m_{i+1}|\mathcal{D}_i(x)|$, and x is good up to all levels $0 \leq j \leq i$. Therefore,

  $$\begin{aligned} (20) \quad |\mathcal{D}_i(x)| &\geq M_i = m_{i+1}(c_0 L(l_{i+1})^{-1} = \\ &= \frac{1}{c_0}\Big(c_0 \prod_{j=1}^{i}[L(l_j)]^{2^{-j}} m_0\Big)^{2^i} = \frac{1}{c_0}\Big(c_0\rho \prod_{j=1}^{i}[L(l_j)]^{2^{-j}}(l_0)^d\Big)^{2^i} \geq \\ &\geq \frac{1}{c_0}\Big(c_0\rho \prod_{j=1}^{\infty}[L(l_j)]^{2^{-j}}(l_0)^d\Big)^{2^i} \geq \frac{1}{c_0}\Big(c_0\rho(l_0)^{d-1/5}\Big)^{2^i}. \end{aligned}$$

  Here we have used that $l_0 > K_2$. Note that by $l_0 > (100/\rho^2)^{1/(d-2/5)}$,

  $$(21) \qquad c_0\rho(l_0)^{d-1/5} > (2\rho^2/25)(100/\rho^2)^{(d-1/5)/(d-2/5)} > 8.$$

- Recall that $\mathcal{B}_k(x)$ is the set of vertices in $\mathbb{Z}^d$ within graph distance k (in G) of x. Note that $\mathcal{D}_i(x) \subset \mathcal{B}_{h_i}(x)$. We show that if x is ultimately good, then $|\mathcal{B}_{h_i}(x)|^{1/h_{i+1}} > a_1'$ for some $a_1' > 1$, which in turns implies that if x is ultimately good then $|\mathcal{B}_k(x)|^{1/k} > a_1$ for all $k \geq 1$ and some $a_1 > 1$.

- We show that l_0 is large enough to guarantee that the probability that x is ultimately good is positive and $\rho c_0(l_0)^{d-1/5} > 1$.
- We use a zero-one law to prove that the number of ultimately good vertices is infinite.
- Finally, we show that $|\mathcal{B}_j|^{1/j} := |\mathcal{B}_j(0)|^{1/j} > a_1$ if $0 \in \mathcal{C}_\infty$.

3.3.4. *Proof of* $\underline{R_*} > 1$. Now we are ready to state a lemma, which will lead to the proof of Theorem 1.1(b).

Lemma 3.7 *Consider a percolating homogeneous long-range percolation model as defined in Section 1.2 with vertex set* $\mathbb{Z}^d$ *and connection function* $\lambda(r) = r^{-d}L(r)$*, where* $L(r)$ *is positive, slowly varying, decreasing on* $[K, \infty)$ *for some* $K > 0$ *and satisfies (3) and (4). If* ρ, c_0 *and* l_0 *are as above, then the number of ultimately good vertices in* $\mathbb{Z}^d$ *is a.s. infinite.*

Proof of Theorem 1.1 (b):
Note that $\rho c_0(l_0)^{d-1/5} > 1$ by (21) and if x is ultimately good, then (20) implies that for $2^i(1+(l_0)^d) \leq k < 2^{i+1}(1+(l_0)^d)$,

$$|\mathcal{B}_k(x)| \geq \frac{1}{c_0}\left(\rho c_0(l_0)^{d-1/5}\right)^{2^i} \geq c_0^{-1}(\rho c_0(l_0)^{d-1/5})^{(1+(l_0)^d)^{-1}k/2},$$

where we have used that $l_0 > \max(K_1(1/5), K_2)$.

Lemma 3.7 implies that there is at least one ultimately good vertex in $\mathbb{Z}^d$. By the construction of $\mathcal{D}_i(x)$, it is clear that an ultimately good vertex is in an infinite cluster of G. By the uniqueness of the infinite cluster of G, we know that conditioned on $\{0 \in \mathcal{C}_\infty\}$, the random variable $Y := \min\{D(0,x); x \in \mathbb{Z}^d, x \text{ is ultimately good}\}$ is a.s. finite. Therefore,

$$\begin{aligned}(\mathcal{B}_{k+Y})^{1/(k+Y)} &\geq \left((c_0)^{-1}(\rho c_0(l_0)^{d-1/5}))^{(1+(l_0)^d)^{-1}k/2}\right)^{1/(k+Y)} \\ &\geq (c_0)^{-1/(k+Y)}(\rho c_0(l_0)^{d-1/5}))^{(1+(l_0)^d)^{-1}k/(2(k+Y))},\end{aligned}$$

which converges to $a_1' := (\rho c_0(l_0)^{d-1/5})^{(2+2(l_0)^d)^{-1}} > 1$ and therefore there exists a constant $a_1 > 1$ such that

$$\lim_{k\to\infty} \mathbb{P}(a_1 < |\mathcal{B}_k|^{1/k} | 0 \in \mathcal{C}_\infty) = 1,$$

which proves the theorem. □

For the proof of Lemma 3.7 we need a bound for

$$\mathbb{P}(x \text{ is good up to level } i+1 | x \text{ is good up to level } 0).$$

We obtain this bound by using the following lemma.

Lemma 3.8 *Consider a percolating homogeneous long-range percolation model as defined in Section 1.2 with vertex set $\mathbb{Z}^d$ and connection function $\lambda(r) = r^{-d}L(r)$, where $L(r)$ is positive, slowly varying, non-increasing on $[K,\infty)$ for some $K > 0$ and satisfies (3) and (4). If ρ, c_0 and l_0 are as above, then for $i \in \mathbb{N}$,*

$$\mathbb{P}(x \textit{ is good up to level } i+1 | x \textit{ is good up to level } i) \geq 1 - 4^{-2^i}.$$

Proof of Lemma 3.8: Assume that the statement holds for $j < i$, then

$$\begin{aligned}(22)\quad & \mathbb{P}(x \text{ is good up to level } i | x \text{ is good up to level } 0) \geq \\ & \geq 1 - \sum_{j=0}^{i-1} \mathbb{P}(x \text{ is not good up to level } j+1 | x \text{ is good up to level } j) \geq \\ & \geq 1 - \sum_{j=0}^{i-1} 4^{-2^j} \geq 1 - \sum_{j=0}^{i-1} 4^{-(j+1)} = 1 - \frac{1-4^{-i}}{3} \geq 2/3.\end{aligned}$$

Furthermore, note that if the random variable X is binomially distributed with parameters n and p, then by Chebychev's inequality

$$\mathbb{P}\Big(X < \frac{\mathbb{E}(X)}{2}\Big) \leq \frac{4Var(X)}{(\mathbb{E}(X))^2} = \frac{4(1-p)}{np} \leq \frac{4}{np}. \tag{23}$$

Observe that if x and y are not in the same level-i block, then the events $\{y \text{ is good up to level i}\}$ and $\{y \leftrightarrow \mathcal{D}_i(x)\}$ are independent, because different edges are involved. We already know by (20) that if x is good up to level i, then

$$|\mathcal{D}_i(x)| \geq \frac{1}{c_0}\Big(c_0 \rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}} (l_0)^d\Big)^{2^i}. \tag{24}$$

Furthermore, all vertices in $\Lambda_{i+1}(\bar{n}_{i+1}(x)) \setminus (\Lambda_i(0) \cup \Lambda_i(\bar{n}_i(x)))$ have probability exceeding $1 - \exp[-|\mathcal{D}_i(x)|\lambda(l_{i+1})]$, to share an edge with a vertex in $\mathcal{D}_i(x)$. Therefore, the probability that a given level-i block, $\Lambda_i(\bar{n}') \subset \Lambda_{i+1}(\bar{n}_{i+1}(x)) \setminus (\Lambda_i(0) \cup \Lambda_i(\bar{n}_i(x)))$, contains a vertex (say y) that is good up to level i and shares an edge with a vertex in $\mathcal{D}_i(x)$ is bounded below by:

$$\begin{aligned}& \mathbb{P}\Big(y \text{ is good up to level } 0 | y \text{ is chosen u.a.r. from } \Lambda_i(\bar{n}_i(y))\Big) \\ \times\ & \mathbb{P}\Big(y \text{ is good up to level } i | y \text{ is good up to level } 0\Big) \\ \times\ & \mathbb{P}^*\Big(\Lambda_i(\bar{n}_i(y)) \leftrightarrow \mathcal{D}_i(x) \big| |\mathcal{D}_i(x)| = \frac{1}{c_0}\Big(c_0 \rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}} (l_0)^d\Big)^{2^i}\Big).\end{aligned}$$

Here $\mathbb{P}^*$ is the product measure for which a pair of vertices $x, y \in \mathbb{Z}^d$ share an edge with probability $1 - e^{-\lambda(l_{R(x,y)})}$, where

$$R(x,y) = \inf\{i \in \mathbb{N}; y \in \Lambda_i(\bar{n}_i(x))\}.$$

Note that

$$\begin{aligned} & \mathbb{P}^*\Big(\Lambda_i(\bar{n}_i(y)) \leftrightarrow \mathcal{D}_i(x) \big| |\mathcal{D}_i(x)| = \frac{1}{c_0}\Big(c_0\rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}} (l_0)^d\Big)^{2^i}\Big) \\ \leq\ & \mathbb{P}\Big(\Lambda_i(\bar{n}_i(y)) \leftrightarrow \mathcal{D}_i(x) \big| |\mathcal{D}_i(x)| = \frac{1}{c_0}\Big(c_0\rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}} (l_0)^d\Big)^{2^i}\Big). \end{aligned}$$

By Lemma 3.6 and $l_0 > K_3(1/25)$ we see,

$$\mathbb{P}\Big(y \text{ is good up to level } 0 | y \text{ is chosen u.a.r. from } \Lambda_i(\bar{n}_i(y))\Big) \geq \frac{24}{25}\rho. \tag{25}$$

Furthermore,

$$\begin{aligned} & \mathbb{P}^*\Big(\Lambda_i(\bar{n}_i(y)) \leftrightarrow \mathcal{D}_i(x) \big| |\mathcal{D}_i(x)| = \frac{1}{c_0}\Big(c_0\rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}} (l_0)^d\Big)^{2^i}\Big) \\ =\ & 1 - \exp[-\frac{1}{c_0}\Big(c_0\rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}} (l_0)^d\Big)^{2^i} (l_0)^{d2^i} \lambda(l_{i+1})] \\ =\ & 1 - \exp[-\frac{1}{c_0}\Big(c_0\rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}}\Big)^{2^i} L(l_{i+1})] \\ \geq\ & \frac{L(l_{i+1})}{2c_0}\Big(c_0\rho \prod_{j=1}^{i} [L(l_j)]^{2^{-j}}\Big)^{2^i}, \end{aligned}$$

where we have used that $1 - e^{-x} \geq x/2$ for $0 < x \leq 1$ and that $\rho, c_0 < 1$ and $l_0 > K_2$.

Observe that $A_{i+1}(x)$ is dominated by a random variable which is binomially distributed with parameters n_i and p_i, where

$$n_i = (l_{i+1}/l_i)^d - 1 = (l_0)^{d2^i} - 1 \geq (l_0)^{d2^i}/2$$

and

$$p_i > \frac{24}{25}\rho\frac{2}{3}(1 - \exp[-(l_0)^{d2^i} |\mathcal{D}_i(x)| \lambda(l_{i+1})]),$$

by (22) and (25).

If x is good up to level i, then by (24), it holds that

$$\begin{aligned} p_i &> (\frac{16\rho}{25}(1-\exp[-(l_0)^{d2^i}\frac{1}{c_0}\Big(c_0\rho\prod_{j=1}^{i}[L(l_j)]^{2^{-j}}(l_0)^d\Big)^{2^i}\lambda(l_{i+1})]) \\ &\geq \frac{8\rho L(l_{i+1})}{25c_0}\Big(c_0\rho\prod_{j=1}^{i}[L(l_j)]^{2^{-j}}\Big)^{2^i} \\ &= \frac{8\rho}{25c_0}\Big(c_0\rho[L(l_{i+1})]^{2^{-(i+1)}}\prod_{j=1}^{i+1}[L(l_j)]^{2^{-j}}\Big)^{2^i} \\ &\geq \frac{8\rho}{25c_0}\Big(c_0\rho[\prod_{j=1}^{\infty}[L(l_j)]^{2^{-j}}]^2\Big)^{2^i}. \end{aligned}$$

The second line above together with $c_0 = 2\rho/25$ imply that

$$m_{i+1} = L(l_{i+1})(c_0\rho(l_0)^d\prod_{j=1}^{i}[L(l_j)]^{2^{-j}})^{2^i} = \frac{25c_0}{4\rho}\frac{(l_0)^{d2^i}}{(l_0)^{d2^i}-1}\frac{n_ip_i}{2} < \frac{n_ip_i}{2}.$$

By

$$\frac{n_ip_i}{2} \geq \frac{1}{4}(l_0)^{d2^i}\frac{8\rho}{25c_0}\Big(c_0\rho[\prod_{j=1}^{\infty}[L(l_j)]^{2^{-j}}]^2\Big)^{2^i} = \frac{2\rho}{25c_0}\Big(c_0\rho[\prod_{j=1}^{\infty}[L(l_j)]^{2^{-j}}]^2(l_0)^d\Big)^{2^i}, \tag{26}$$

together with $l_0 > (100/\rho^2)^{1/(d-2/5)}$, $c_0 := 2\rho/25$ and (23) we obtain,

$$\begin{aligned} &\mathbb{P}(A_{i+1}(x) \geq m_{i+1}|x \text{ is good up to level } i) \\ \geq\ &\mathbb{P}(A_{i+1}(x) \geq \frac{n_ip_i}{2}|x \text{ is good up to level } i) \\ \geq\ &1-\frac{4}{n_ip_i} \\ \geq\ &1-\frac{25c_0}{\rho}\Big(c_0\rho[\prod_{j=1}^{\infty}[L(l_j)]^{2^{-j}}]^2(l_0)^d\Big)^{-2^i} \\ \geq\ &1-\Big(\rho^2/25[\prod_{j=1}^{\infty}[L(l_j)]^{2^{-j}}]^2(l_0)^d\Big)^{-2^i} \\ \geq\ &1-\frac{1}{4^{2^i}}. \end{aligned}$$

This completes the proof of Lemma 3.8. □

Proof of Lemma 3.7: The first step in the proof is the observation that the event

$$\mathcal{E} := \{\text{the number of ultimately good vertices in } \mathbb{Z}^d \text{ is infinite}\}$$

is independent of any finite set of edges. Indeed, for every finite set of edges E_0, there is an $i \in \mathbb{N}$ such that all edges in E_0 are shared by vertices in $\Lambda_i(0)$. However, whether a vertex x with $r(0,x) > i$ is ultimately good does not depend on edges with at least one end-vertex in $\Lambda_i(0)$. So, $\mathcal{E}$ does not depend on E_0.

By a Kolmogorov-like zero-one law (see e.g. [16, p. 289]), we know that the probability that there will be infinitely many ultimately good vertices is either 0 or 1. We will prove that with positive probability every annulus of the form $\Lambda_{i+1}(0) \setminus \Lambda_i(0)$ with $i \in \mathbb{N}$, contains at least one ultimately good vertex. Which will prove the lemma.

Note that by Lemma 3.8,

$$\mathbb{P}(x \text{ is ultimately good}|x \text{ is good up to level } 0) \geq 1 - \sum_{i=0}^{\infty} \frac{1}{4^{2^i}} \geq 2/3$$

and thus by Lemma 3.6,

$$\begin{aligned} & \mathbb{P}(x \text{ is ultimately good}|x \in \mathcal{C}_\infty) \\ \geq\ & \mathbb{P}(x \text{ is ultimately good}|x \text{ is good up to level } 0) \\ & \times \mathbb{P}(x \text{ is good up to level } 0|x \in \mathcal{C}_\infty) \\ \geq\ & \frac{2}{3}\frac{24}{25}. \end{aligned}$$

The probability that in the annulus $\Lambda_{i+1}(0) \setminus \Lambda_i(0)$ contains no vertex that is good up to level i, is given by

$$(\mathbb{P}(\Lambda_i(0) \text{ contains no vertex that is good up to level } i))^{(l_{i+1}/l_i)-1},$$

where we have used that the events that vertices in different level i blocks are good up to level i are independent. Note that by Lemma 3.6 and $L_0 > L_3(1/25)$

$$\begin{aligned} & \mathbb{P}(\Lambda_i(0) \text{ contains no vertex that is good up to level } i) \\ \leq\ & 1 - \mathbb{P}(x \text{ is ultimately good}|x \text{ is good up to level } 0) \\ & \times \mathbb{P}(\Lambda_0(0) \text{ contains at least one vertex that is good up to level } 0) \\ \leq\ & 1 - 48/75 \\ =\ & 9/25. \end{aligned}$$

So the probability that in the annulus $\Lambda_{i+1}(0) \setminus \Lambda_i(0)$ contains no vertex that is good up to level i, is less than or equal to $(9/25)^{(d(l_0)^{2^i}-1)}$, which in turn is less than $e^{-(16d/50)(l_0)^{2^i}}$ by $1-x \leq e^{-x}$. Furthermore by Lemma 3.8 it holds that,

$$\mathbb{P}(x \text{ is ultimately good}|x \text{ is good up to level } i) \geq 1 - \sum_{j=i}^{\infty} 4^{-2^j} \geq \geq 1 - \sum_{j=i+1}^{\infty} 4^{-j} \geq 1 - (1/3)4^{-i}. \tag{27}$$

For every $i \in \mathbb{N}$ the event that the annulus $\Lambda_{i+1}(0) \setminus \Lambda_i(0)$ contains at least 1 ultimately good vertex, is increasing (for a definition of increasing events see [17, p.32]), by the FKG inequality [14, 17] we obtain:

$$\begin{aligned}
& \mathbb{P}(\bigcap_{i\in\mathbb{N}} \{(\Lambda_{i+1}(0) \setminus \Lambda_i(0)) \text{ contains at least one ultimately good vertex}\}) \\
\geq\ & \prod_{i\in\mathbb{N}} \mathbb{P}(\Lambda_{i+1}(0) \setminus \Lambda_i(0) \text{ contains at least one ultimately good vertex}) \\
\geq\ & \prod_{i\in\mathbb{N}} (1 - e^{-(16d/50)(l_0)^{2^i}})(1-(1/3)4^{-i}) \\
\geq\ & \prod_{i\in\mathbb{N}} (1 - e^{-(16d/50)l_0 2^i})(1-(1/3)4^{-i}) \\
\geq\ & 1 - \sum_{i\in\mathbb{N}} e^{-(16d/50)l_0 2^i} - \sum_{i\in\mathbb{N}} (1/3)4^{-i} \\
\geq\ & 1 - \sum_{i\in\mathbb{N}_+} e^{-(16d/50)l_0 i} - 4/9 \\
\geq\ & 1 - \frac{e^{-(16d/50)l_0}}{1-e^{-(16d/50)l_0}} - 4/9 \\
\geq\ & 1/18,
\end{aligned}$$

where we have used that $l_0 > 100/(16d)$ and $\frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x-1} < \frac{1}{x}$ for $x > 0$ in the final inequality. □

Acknowledgments. The author wishes to thank Artem Sapozhnikov and Ronald Meester for helpful discussions.

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Pieter Trapman:
Vrije Universiteit Amsterdam
Department of Mathematics
De Boelelaan 1081a
1081 HV Amsterdam
The Netherlands
E-mail: ptrapman@few.vu.nl