THE ASYMPTOTIC SHAPE THEOREM FOR GENERALIZED FIRST PASSAGE PERCOLATION

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Abstract. We generalize the asymptotic shape theorem in first passage percolation on $\mathbb{Z}^d$ to cover the case of general semimetrics. We prove a structure theorem for equivariant semimetrics on topological groups and an extended version of the maximal inequality for $\mathbb{Z}^d$-cocycles by D. Boivin and Y. Derriennic in the vector-valued case. This inequality will imply a very general form of Kingman's subadditive ergodic theorem. For certain classes of generalized first passage percolation we prove further structure theorems and provide rates of convergence in the asymptotic shape theorem. We also establish a general form of the multiplicative ergodic theorem by A. Karlsson and F. Ledrappier for cocycles with values in separable Banach spaces with the Radon-Nikodým property.

1. Introduction

First passage percolation was introduced by J.T. Hammersley and D.J.A. Welsh in the paper [16]. A detailed description of the model is given subsection 2.3. The theory can be roughly described as the study of the generic large scale geometry of semimetric spaces, where the semimetric is allowed to vary measurably. The classical case deals with the space $\mathbb{Z}^d$ and semimetrics induced from random weights on the edges of the standard Cayley graph of $\mathbb{Z}^d$. However, the setup easily extends to general groups.

In this paper, we introduce the notion of a random semimetric. Let G be a locally compact group, and suppose G acts on a probability space (X, μ) where μ is invariant under the action of G. We say that the action is *ergodic* if the invariant sets are either null or conull, and *quasi-invariant* if it preserves the measure class of μ. Suppose (Y, ν) is a σ-finite measure space. A *random semimetric* on Y, modelled on the G-space X, is a map $\rho : X \times Y \times Y \to [0, \infty)$ such that ρ_x is a semimetric for almost every x in X and

$$\rho_{g.x}(y, y') = \rho_x(g.y, g.y')$$

for all y, y' in Y, g in G and $x \in X$, and for all y, y' in Y, the map

$$x \mapsto \rho_x(y, y')$$

is measurable. In general, these objects are very complicated, and form the basis of subadditive ergodic theory. However, it turns out that all random semimetrics can be realized as norms of additive cocycles with values in large Banach spaces. The definition of a Gelfand cocycle is given in subsection 2.4 and is rather technical, but turns out to be useful in view of the following theorem.

Theorem 1.1 (Structure Theorem). *Let G be a locally compact, second countable group. Suppose (X, μ) is a probability measure space with a G–invariant ergodic measure μ. Suppose (Y, ν) is a G–space with a quasi–invariant σ–finite measure ν. If ρ is a random*

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G–equivariant semimetric on Y, modelled on the G–space (X,μ), there exists a Gelfand $L^1(Y,\nu)$–cocycle, with respect to the left-regular representation of G on $L^\infty(Y,\nu)$ on the G–space (X,μ) such that

$$\rho_x(y,y') = ||s_x(y,y')||_{L^\infty(Y,\nu)}.$$

We will refer to a random semimetric ρ on a space Y as *generalized first passage percolation* on Y. In view of Theorem 1.1, the study of generalized first passage percolation is equivalent to the study of Gelfand cocycles with values in $L^\infty(Y,\nu)$. However, any Gelfand cocycle with values in the dual of a Banach space B defines a random semimetric. In subsections 3.3 and 3.5, we will restrict the class of Banach spaces under consideration, and this will allow us to establish certain structure theorems which are not known for classical first passage percolation. For instance, we determine the horofunctions of random semimetric spaces, when the cocycles take values in separable Hilbert spaces, and we prove an analogue of H. Kesten's celebrated inequality for classical first passage percolation in this context.

However, the main result of this paper is the following extension of D. Boivin's asymptotic shape theorem to general random semimetrics.

Theorem 1.2 (Asymptotic Shape Theorem)**.** *Suppose ρ is a random $\mathbb{Z}^d$–semimetric modelled on an ergodic $\mathbb{Z}^d$–space (X,μ), where μ is a probability measure. Suppose $\rho(0,n)$ is in $L^{d,1}(X,\mu)$ for every $n \in \mathbb{Z}^d$. Then there exists a seminorm L on $\mathbb{R}^d$ such that*

$$\lim_{|n|\to\infty} \frac{\rho_x(0,n) - L(n)}{|n|} = 0$$

almost everywhere on (X,μ).

This result was only known for a certain class of *inner* random semimetrics on $\mathbb{Z}^d$ [7]. It can be proved [11] that the integrability condition on the cocycle to belong to the Lorentz space $L^{d,1}(X)$ is sharp. The unit ball of the semimetric L roughly describes the generic asymptotic shape of large balls in $\mathbb{Z}^d$ with the random semimetric ρ. In the general ergodic situation essentially all convex shapes can be attained as asymptotic shapes. This is a result by O. Häggström and R. Meester [17].

2. Generalized First Passage Percolation

2.1. **Bochner-Lorentz Spaces.** In the following sections we will make use of certain classes of function spaces introduced by G. Lorentz in [23]. It is straightforward to extend the definition to cover the case of vector-valued functions, and we will do so. Before we give the definition of the necessary function spaces, we recall some basic notions and useful facts about measurability of vector-valued functions. Let B be a Banach space, and $(X,\mathfrak{F},\mu)$ a measure space. A *simple* function $f : X \to B$ is a function on the form

$$f = \sum_{k=1}^{n} c_k \chi_{A_k},$$

where A_k are elements of $\mathfrak{F}$ and c_k are elements in B. A function $f : X \to B$ is *Bochner measurable* (or strongly measurable) if there is a sequence of simple functions $f_n : X \to B$ such that $||f_n - f||_B \to 0$. A function $f : X \to B$ is *weakly measurable* if

$$x \mapsto \langle \lambda, f(x) \rangle$$

is measurable for every λ in B^*, where B^* is the dual of B. A function $f : X \to B^*$ is *weak*-measurable* if

$$x \mapsto \langle \lambda, f(x) \rangle$$

is measurable for every λ in B, canonically identified with an element of B^{**}.

We now turn to the definition of the function spaces. Let $1 \leq p, q \leq \infty$, and suppose f is a complex-valued measurable function on X. We define

$$f^*(t) = \inf\{s > 0 \mid d_f(s) \leq t\},$$

where d_f is the distribution function of f, i.e.

$$d_f(\alpha) = \mu(\{x \in X \mid |f(x)| > \alpha\}), \quad \alpha \geq 0.$$

We define the norm

$$||f||_{p,q} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty. \end{cases}$$

We denote the set of all f with $||f||_{p,q} < \infty$ by $L^{p,q}(X)$. With the above norm it is a Banach space, usually referred to as the *Lorentz space* with indices p and q. For instance, we see that $L^{p,p}(X) = L^p(X)$.

The extension to vector-valued functions is straightforward: We say that a weak*-measurable function $f : X \to B$ is in $L^{p;q}_{w^*}(X, B^*)$ if there is a non-negative function g on X with finite $L^{p,q}(X)$-norm such that $||f(x)||_B \leq g(x)$ almost everywhere. Note that $||f||_B$ is not necessarily measurable on X. If f is in the space $L^{p;q}_{w^*}(X, B)$ we define the norm $||f||_{L^{p,q}(X,B^*)}$ as the infinum of the $L^{p;q}_{w^*}(X)$–norms of all non-negative functions g such that $||f||_B \leq g$ almost everywhere on X. It can be proved that this defines a Banach space structure (See Chapter 1 in [9]). If $f : X \to B$ is Bochner-measurable, and B^* is separable, we say that f is in $L^{p,q}(X, B)$ if the *measurable* function

$$x \mapsto ||f(x)||_B$$

is in $L^{p,q}(X)$. We will refer to $L^{p,q}(X, B)$ as the *Bochner-Lorentz space* with indices p and q.

2.2. **Random Semimetric Spaces.** We will recall some basic notions from the ergodic theory of subadditive cocycles. Classically, a *subadditive cocycle* over a measurable $\mathbb{Z}$-action T on a probability measure $(X, \mathfrak{F}, \mu)$ is a measurable map $a : \mathbb{Z} \times X \to \mathbb{R}$ such that

$$a(n + m, x) \leq a(n, x) + a(m, T_n x), \quad \forall\, n, m \in \mathbb{Z}.$$

A celebrated theorem by J.F.C. Kingman [21] asserts that, if $a(n, \cdot)$ is integrable with respect to μ for all n in $\mathbb{Z}$, then there is a T-invariant real-valued measurable function A on X such that

$$\lim_{n\to+\infty} \frac{a(n, x) - nA(x)}{n} = 0,$$

almost everywhere on (X, μ). If the action T is assumed to be ergodic, A is necessarily constant. Furthermore, in this case

$$A = \inf_{n>0} \frac{1}{n} \int_X a(n, x)\, d\mu(x).$$

In this paper we will be concerned with a generalization of this theorem to measurable $\mathbb{Z}^d$-actions. We will need the following definition.

Definition 2.1 (Random Semimetric)**.** Let G be a locally compact and second countable group. Suppose $(X, \mathfrak{F})$ is a measurable space on which G acts measurably and with an invariant probability measure μ. Let (Y, ν) be a σ-finite measure space, where ν is a quasi-invariant measure under the action of G. A *random semimetric on Y, modelled on the G-space X*, is a map $\rho : X \times Y \times Y \to [0, \infty)$ such that the following conditons hold:

(i) (Symmetry) For all $x \in X$ and y, y' in Y,

$$\rho_x(y, y') = \rho_x(y', y) \quad \text{and} \quad \rho_x(y, y) = 0.$$

(ii) (Triangle inequality) For all $x \in X$ and y, y', y'' in Y,

$$\rho_x(y, y') \leq \rho_x(y, y'') + \rho_x(y'', y').$$

(iii) (Equivariance) For all $x \in X$ and $g \in G$ and y, y' in Y,

$$\rho_{gx}(y, y') = \rho_x(gy, gy').$$

Remark. Let (Z, d) be a metric space and suppose that $c : G \times X \to \mathrm{Isom}(Z, d)$ is a measurable map which satisfy the equations,

$$c(gg', x) = c(g, x)c(g', gx), \quad \forall\, g, g' \in G \quad \text{and} \quad x \in X.$$

It is easy to see that

$$\rho_x(g, g') = d(c(g, x).z_0, c(g', x).z_0),$$

defines a random semimetric on G, modelled on the G-space X, for any choice of base point z_0 in Z. Indeed, by the cocycle property of c, we have

$$\begin{aligned} \rho_{gx}(g', g'') &= d(c(g', gx).z_0, c(g'', gx) \\ &= d(c(g, x)c(g', gx).z_0, c(g, x)c(g'', gx).z_0) \\ &= d(c(gg', x).z_0, c(gg'', x)z_0) \\ &= \rho_x(gg', gg''), \end{aligned}$$

for all $x \in X$ and g, g', g'' in G.

2.3. **Classical First Passage Percolation.** First passage percolation was first defined by J.M. Hammersley and D.J.A. Welsh in [16] and has served as one of the main inspirations to the early developments of subadditive ergodic theory. Let $(X, \mathfrak{F}, \mu)$ be a probability space on which the group $\mathbb{Z}^d$ acts ergodically and preserving the measure μ. We denote the action by T. Let $f_1, \ldots, f_d$ be non-negative measurable functions on X, and define, for an edge $\bar{e} = (n, n + e_k)$ in the standard Cayley graph of $\mathbb{Z}^d$, the weight

$$t_x(\bar{e}) = f_k(T_n x), \quad x \in X,\ k \in \{1, \ldots, d\},$$

where e_k denotes the k-th standard basis vector in $\mathbb{Z}^d$. We define the weight $t_x(\gamma)$ of a path γ by summing the individual weights on the edges of the path. For two points m, n in $\mathbb{Z}^d$ we define

$$\rho_x(m, n) = \inf \{t_x(\gamma) \mid \gamma \text{ is a path from } m \text{ to } n.\}$$

It is clear from the construction that this defines a measurable map from X into the convex cone of semimetrics on $\mathbb{Z}^d$, equipped with the Borel structure coming from topology of pointwise convergence. Note that the relation $t_{T_k x}(\gamma) = t_x(\gamma + k)$ for $k \in \mathbb{Z}^d$ implies that

$$\rho_{Tx}(m, n) = \rho_x(m + k, n + k),$$

and thus ρ is a random semimetric on $\mathbb{Z}^d$, modelled on the $\mathbb{Z}^d$-space (X, μ). By construction, the semimetric ρ is inner. The random semimetric space $(\mathbb{Z}^d, \rho)$ modelled on the $\mathbb{Z}^d$-space X is known as the *classical first passage percolation* model.

Note that in the case when $d = 1$, we essentially recover the absolute value of the *Birkhoff sum*

$$\sum_{k=0}^{n-1} f_1(T_k x),$$

and thus the almost sure asymptotic behaviour of the random semimetric can be analysed using Birkhoff's ergodic theorem. When $d \geq 2$, the situation is more involved and new techniques are needed. The main part of this paper is concerned with a generalization to general semimetrics on $\mathbb{Z}^d$ of the following theorem by D. Boivin. [7]

Theorem 2.1 (Boivin). *Suppose that $f_1, \ldots, f_d$ are in $L^{d,1}(X)$. Then there is a seminorm L on $\mathbb{R}^d$ such that*

$$\lim_{n\to\infty} \frac{\rho_x(0,n) - L(n)}{|n|} = 0,$$

almost everywhere on (X, μ).

Remark. This theorem had earlier been established for independent and identically distributed edge–weights by J.T. Cox and R. Durrett [10] ($d = 2$) and by H. Kesten [20] ($d \geq 2$) under weaker integrability conditions. However, it can be shown [8], that $L^{d,1}(X)$ is a sharp condition in the general ergodic case.

The definition of classical first passage percolation described above extends naturally to a more general situation. Let G be a finitely generated group, and suppose S is a finite subset of G such that S and S^{-1} are disjoint, and $S \cup S^{-1}$ generates G as a group. Suppose $\{f_s\}_{s\in S}$ is a set of non-negative measurable functions on a probability measure space (X, μ) with a measure-preserving *right* action by G. For every g in G and edge (g, gs) in the Cayley graph of $(G, S \cup S^{-1})$, we define the random weight $t_x(g, gs) = f_s(xg)$. In analogy with the scheme above, we define the distance ρ between to points g and g' in G as the infinum of the weights over all paths between g and g'. By construction, ρ is a semimetric and

$$\rho_x(hg, hg') = \rho_{xh}(g, g')$$

for all g, g', h in G and x in X. It is not clear that D. Boivin's *proof* of Theorem 2.1 immediately extends to the case when $G = \mathbb{Z}^d$ and S is *not* the standard generating set. Note however that Theorem 1.2 covers this case.

2.4. **Cohomology of Borel Groupoids.** In this subsection we will define various important types of cocycles. A more conceptual explaination can be given in the language of groupoids; however we will refrain from making very general statements, and restrict our attention to the first order cohomology of a groupoid.

Definition 2.2 (Borel cocycle)**.** Let (Z, d) be a metric space, and G a topological group. Suppose X is a G-space. A map $c : G \times X \to \mathrm{Isom}(Z, d)$ such that

$$(g, x) \mapsto c(g, x).z,$$

is measurable for all z in Z, with respect to the Borel σ-algebra on Z, and

$$c(gg', x) = c(g, x)c(g', gx) \quad \forall\, g, g' \in G,\ x \in X,$$

is called a *Borel cocycle* over the G-space X.

Definition 2.3 (Gelfand cocycle)**.** Let B be a Banach space, and G a locally compact and second countable group. Let (X, μ) be a probability measure space, where μ is a G–invariant measure, and (Y, ν) a σ-finite measure space, where ν is a quasi–invariant measure under the action of G. Let $c : G \times X \to \mathrm{Isom}(B^*)$ be a Borel cocycle. A map $s : X \times Y \times Y \to B^*$ is called a *Gelfand B–cocycle with respect to the Borel cocycle* c, if the following conditions hold:

(i) (Additivity) For all $x \in X$,

$$s_x(y, y'') + s_x(y'', y') = s_x(y, y'), \quad \forall\, y, y', y'' \in Y.$$

(ii) (Equivariance) For all $x \in X$,

$$c(g, x).s_{gx}(y, y') = s_x(gy, gy'), \quad \forall\, y, y' \in Y \text{ and } g \in G.$$

(iii) (Measurability) The maps

$$x \mapsto s_x(y, y') \quad \text{and} \quad x \mapsto ||s_x(y, y')||_{B^*}$$

are weak*–measurable for every $y, y' \in Y$.

We say that s is in the Lorentz space $L^{p,q}_{w*}(X, B)$ if the map $x \mapsto ||s_x(y, y')||_{p,q}$ is in $L^{p,q}(X)$ for all y, y' in Y.

Remark. If the cocycle is trivial, i.e. if there is an isometric representation π of G on B^* such that $c(g, x) = \pi(g)$ for all $x \in X$ and $g \in G$, we will refer to s as a *Gelfand B–cocycle* with respect to the representation π.

We also define two related types of cocycles, where stronger versions of measurability are assumed.

Definition 2.4 (Pettis cocycle)**.** A map $s : X \times \mathbb{Z}^d \times \mathbb{Z}^d \to B$ is called a *Pettis B–cocycle with respect to the Borel cocycle c* if it is a Gelfand cocycle with respect to the cocycle c and the maps

$$x \mapsto s_x(y, y'),$$

are weakly measurable for all y, y' in Y.

Remark. Note that we in the definition of a Pettis cocycle do not insist that s takes values in the *dual* of a Banach space B. Thus, the formulation of the definition is slightly misleading, but we hope that this will not cause any confusion for the reader.

In subsection 2.8, we will need the following cocycles which are measurable in a strong sense.

Definition 2.5 (Bochner cocycle)**.** A map $s : X \times \mathbb{Z}^d \times \mathbb{Z}^d \to B$ is called a *Bochner B–cocycle with respect to the Borel cocycle c* if it is a Gelfand cocycle with respect to c and the maps

$$x \mapsto s_x(y, y'),$$

are Bochner measurable for all y, y' in Y.

Remark. If B is a separable Banach space, it follows from Pettis' measurability theorem (See e.g. chapter 1 in [12]) that every Pettis cocycle is a Bochner cocycle. The converse is obvious.

One connection between Gelfand B-cocycles and random semimetrics on Y is suggested by the following proposition.

Proposition 2.1. *Let G be a locally compact group. Suppose $s : X \times Y \times Y \to B^*$ is a Gelfand B–cocycle with respect to a Borel cocycle c. Then*

$$\rho_x(y, y') = ||s_x(y, y')||_{B^*}, \quad y, y' \in Y,$$

is a random semimetric on Y, modelled on on the G-space X.

Proof. The measurability is clear from the definition of s. From the additivity property of s it follows that

$$s_x(y,y') = -s_x(y',y) \quad \text{for all } x,y,y'.$$

Thus, ρ_x is symmetric and $\rho_x(y,y)=0$. For the triangle inequality, we observe that

$$\begin{aligned} ||s_x(y,y')||_{B^*} &= ||s_x(y,y'')+s_x(y'',y')||_{B^*} \\ &\leq ||s_x(y,y'')||_{B^*}+||s_x(y'',y')||_{B^*} \\ &= \rho_x(y,y'')+\rho_x(y'',y'), \end{aligned}$$

for all y,y',y'' in Y. Finally, to prove equivariance, we note that, since c takes values in the isometry group of B^*,

$$\begin{aligned} \rho_{gx}(y,y') &= ||s_{gx}(y,y')||_{B^*} = ||c(g,x).s_{gx}(y,y')||_{B^*} \\ &= ||s_x(gy,gy')||_{B^*} = \rho_x(gy,gy'), \end{aligned}$$

for all $g\in G$ and y,y' in Y. In the second to last equality, the equivariance property of s was used. □

Remark. We will see in Theorem 2.2 that these examples of random semimetrics are the only examples. This observation will be one of the crucial steps in the proof of Theorem 2.4.

2.5. **Representation of Subadditive Cocycles.** In this subsection we will prove the following important structure theorem.

Theorem 2.2. *Let G be a locally compact, second countable group. Suppose (X,μ) is a probability measure space with a G–invariant ergodic measure μ. Suppose (Y,ν) is a G–space with a quasi–invariant σ–finite measure. If ρ is a random G–equivariant semimetric on Y, modelled on the G–space (X,μ), there exists a Gelfand $L^1(Y,\nu)$–cocycle, with respect to the left-regular representation λ of G on $L^\infty(Y,\nu)$ on the G–space (X,μ) such that*

$$\rho_x(y,y') = ||s_x(y,y')||_{L^\infty(Y,\nu)}.$$

Proof. The proof is based on the following trivial observation:

$$\rho_x(y,y') = \sup_{y''\in Y} |\rho_x(y,y'')-\rho_x(y'',y')|,$$

which is a direct consequence of the triangle inequality. We define

$$s_x(y,y') = \rho_x(y,\cdot)-\rho_x(\cdot,y') \in L^\infty(Y,\nu).$$

Note that

$$s_x(y,y'')+s_x(y'',y') = \rho_x(y,\cdot)-\rho_x(\cdot,y'')+\rho_x(\cdot,y'')-\rho_x(\cdot,y') = s_x(y,y')$$

and

$$\lambda(g).s_{gx}(y,y') = \rho_{gx}(y,g^{-1}\cdot)-\rho_{gx}(g^{-1}\cdot,y') = s_x(gy,gy').$$

To prove measurabilty, we first note that the map $x\mapsto ||s_x(y,y')||_{L^\infty(Y,\nu)}$ is measurable by definition. Thus, we only need to prove that the map $s_x(y,y')$ is weak*–measurable. Pick $\eta\in L^1(Y,\nu)$, then

$$\langle \eta, s_x(y,y')\rangle = \int_Y (\rho_x(y,y'')-\rho_x(y'',y'))\eta(y'')\,d\nu(y'')$$

is measurable by Fubini's theorem, since, by definition, the map

$$(x,y'')\mapsto (\rho_x(y,y'')-\rho_x(y'',y'))\eta(y'')$$

is clearly measurable on the probability measure space $(X\times Y,\mu\times\nu)$ for almost every choice of $y,y'\in Y$ with respect to $\nu\times\nu$. □

Remark. In the paper [21], J.F.C. Kingman asked the natural question whether every subadditive cocycle a on $\mathbb{Z}$-space X has a representation of the form

$$a(n,x) = \sup_{i\in I} \sum_{k=0}^{n-1} f_i(T^k x), \quad n \in \mathbb{N},$$

where $\{f_i\}_{i\in I}$ is a set of real-valued measurable function on X and I is some countable index set. This question was later answered in the negative by J.M. Hammersley in [15]. Theorem 2.2 gives a positive answer to an extended version Kingman's question, where the functions f_i are allowed to be Banach space valued, and the supremum is replaced by the corresponding Banach norm. However, it is certainly an inconvenience that the proof requires the Banach space B^* to be non-separable. Therefore, it seems appropriate to ask the following question:

Question. *Can every random G-equivariant semimetric ρ on a G-space Y, quasi-invariant under the action of G and modelled on a G-space (X,μ), be represented as the norm of a Gelfand B-cocycle, where B^* is a* separable *Banach space?*

2.6. **Asymptotic Shape Theorems.** We will now outline the main steps in the proof of theorem 1.2. We first make some preliminary observations and remarks.

Proposition 2.2. *Suppose ρ is a random G–equivariant semimetric on Y, modelled on the G–space (X,μ). Then the function*

$$r(y,y') = \int_X \rho_x(y,y')\, d\mu(x)$$

is a G–invariant semimetric on Y.

Proof. The axioms for a semimetric are easily verified. The rest of the proof consists of the following simple calculation:

$$r(gy,gy') = \int_X \rho_x(gy,gy')\, d\mu(x) = \int_X \rho_{gx}(y,y')\, d\mu(x) = r(y,y').$$

□

The study of the almost sure asymptotic geometry of random semimetric spaces will be referred to as *generalized first passage percolation.* We begin by describing some general features of this theory. Suppose Y is a locally compact space, and r is dominated by a G–invariant metric η, such that

$$\liminf_{y\to\infty} \eta(y,o) = +\infty,$$

for every choice of a base point $o \in Y$. We say that the random G–equivariant semimetric on Y satisfies an *asymptotic shape theorem* (with respect to the G–invariant metric η) if there exists a measurable function $L : Y \to [0,\infty)$ such that

$$\limsup_{y\to\infty} \left| \frac{\rho_x(o,y) - L(y)}{\eta(o,y)} \right| = 0.$$

This paper is concerned with a general asymptotic shape theorem for actions of the group $\mathbb{Z}^d$ on probability spaces. We will specialize the above situation to the case when $G = \mathbb{Z}^d$ and $Y = \mathbb{Z}^d$. In this case, L can be realized as a norm on $\mathbb{R}^d$, and η will be taken to be the standard word-metric on $\mathbb{Z}^d$.

The following important lemma is proved in [7].

Lemma 2.1 (Boivin's Lemma)**.** *Suppose ρ is a random $\mathbb{Z}^d$–equivariant semimetric on $\mathbb{Z}^d$, modelled on an ergodic $\mathbb{Z}^d$–space (X,μ), where μ is a probability measure. If there is a positive constant C such that*

$$\mu(\{x \in X \mid \sup_{n\neq 0} \frac{\rho_x(0,n)}{|n|} \geq \lambda\}) \leq \frac{C}{\lambda^d}, \quad \forall\, \lambda > 1,$$

then there exists a seminorm L on $\mathbb{R}^d$ such that

$$\lim_{|n|\to\infty} \frac{\rho_x(0,n) - L(n)}{|n|} = 0,$$

almost everywhere on (X,μ).

Thus, in order to prove an asymptotic shape theorem, we will need the a maximal inequality. Let s be a Gelfand B-cocycle and define the function,

$$Ms(x) = \sup_{n\neq 0} \frac{||s_x(0,n)||_{B^*}}{|n|}, \quad x \in X.$$

We prove the following maximal inequality.

Theorem 2.3 (Maximal Inequality)**.** *Let B be a Banach space. Suppose s is a Gelfand B–cocycle on an ergodic $\mathbb{Z}^d$–space (X,μ), where μ is a probability measure. Suppose $s(0,n)$ is in $L^{d,1}_{w^*}(X,\mu,B^*)$ for every $n \in \mathbb{Z}^d$. There exists a positive constant C such that*

$$\mu(\{x \in X \mid Ms(x) \geq \lambda\}) \leq \frac{C}{\lambda^d}||s||_{L^{d,1}_{w^*}(X,B)}$$

for all $\lambda \geq 1$.

The proof of this theorem will be presented in subsection 2.7. An immediate corollary of this result is

Theorem 2.4 (Asymptotic Shape Theorem)**.** *Suppose ρ is a random $\mathbb{Z}^d$–semimetric modelled on an ergodic $\mathbb{Z}^d$–space (X,μ), where μ is a probability measure. Suppose $\rho(0,n)$ is in $L^{d,1}(X,\mu)$ for every $n \in \mathbb{Z}^d$. Then there exists a seminorm L on $\mathbb{R}^d$ such that*

$$\lim_{|n|\to\infty} \frac{\rho_x(0,n) - L(n)}{|n|} = 0$$

almost everywhere on (X,μ).

Remark. This theorem was proved by D. Boivin in [7] in the case of certain *inner* random semimetrics on $\mathbb{Z}^d$. The proof is slightly different and does not seem to extend to the general situation. Note that when $d = 1$, Boivin's theorem is essentially equivalent to Birkhoff's ergodic theorem, while our theorem is strictly stronger.

2.7. **Maximal Inequalities.** The goal of this section is to establish Theorem 2.3. The proof closely follows the arguments outlined by Y. Derriennic and D. Boivin in [8]. We begin to recall the basic combinatorial lemma used by Boivin and Derriennic in their proof. A detailed proof can be found in [8].

Lemma 2.2. *For every $n \in \mathbb{Z}^d$, $n \neq 0$, let*

$$P_n = \{m \in \mathbb{Z}^d \mid |n-m| \leq |n|/2\}.$$

Let H be a coordinate–hyperplane of $\mathbb{Z}^d$ such that $H \cap P_x = \emptyset$. Let $(H_j)_{j=1}^{d-1}$ be an increasing sequence of coordinate–subspaces of $\mathbb{Z}^d$ such that $\dim H_j = j$ and $H_{d-1} = H$. There exists a set $\mathcal{E}_n$ of elementary paths γ in $\mathbb{Z}^d$, joining 0 to n, such that *(i) the cardinality of* $\mathcal{E}_n$ *is* $|n|^{d-1}$,

(ii) each γ *is entirely included in the set*

$$\{m \in \mathbb{Z}^d \mid |m| \leq 2|n|\},$$

(iii) for every $m \in P_n$ *and* $m \neq n$,

$$|\{\gamma \in \mathcal{E}_n \mid m \in \gamma\}| \leq C \left(\frac{|n|}{|n-m|} \right)^{d-1},$$

(iv) for every $m \notin P_n$, *with* $|m| \leq 2|n|$,

$$|\{\gamma \in \mathcal{E}_n \mid m \in \gamma\}| \leq |n|^{d-j(m)},$$

for $j(m) = \sup\{j = 1, \ldots, d-1 \mid m \in H_j\}$,

(v) for every $m \notin H \cup P_n$ *with* $|m| \leq 2|n|$

$$|\{\gamma \in \mathcal{E}_x \mid m \in \gamma\}| \leq 1.$$

For an elementary path $\gamma = \{n_1, \ldots, n_r\}$ between 0 and n in $\mathbb{Z}^d$, we define

$$A_\gamma f(x) = \sum_{k=0}^{r-1} f(T_{n_k} x),$$

where $f : X \to \mathbb{R}$ is a measurable function. The following lemma was proved in [8]:

Lemma 2.3. *Suppose* f *is a nonnegative and measurable function on* X. *Then, for all* $n \neq 0$,

$$\begin{aligned} \frac{1}{|\mathcal{E}_n|} \Big| \sum_{\gamma \in \mathcal{E}_n} A_\gamma f(x) \Big| \quad &\leq \quad C \Big[\sum_{H \in \mathcal{H}} \frac{1}{|n|^{\dim H}} \sum_{\substack{m \in H \\ |m| \leq 2|n|}} f(T_m x) + \\ &+ \quad \frac{1}{|n|} \left(f(T_n x) + \sum_{0 < |n-m| \leq |n|/2} \frac{f(T_m x)}{|m-n|^{d-1}} \right) \Big], \end{aligned}$$

where $\mathcal{H}$ *denotes the collection of all coordinate subspaces of* $\mathbb{Z}^d$.

This lemma readily implies the following estimate:

Proposition 2.3. *For every non-zero* $n \in \mathbb{Z}^d$, *we have*

$$\begin{aligned} \frac{||s_x(0,n)||_{B^*}}{|n|} \quad &\leq \quad C \Big[\sum_{H \in \mathcal{H}} \frac{1}{|n|^{\dim H}} \sum_{\substack{m \in H \\ |m| \leq 2|n|}} f(T_m x) + \\ &+ \quad \frac{1}{|n|} \left(f(T_n x) + \sum_{0 < |n-m| \leq |n|/2} \frac{f(T_m x)}{|m-n|^{d-1}} \right) \Big], \end{aligned}$$

where $\mathcal{H}$ *denotes the collection of all coordinate subspaces of* $\mathbb{Z}^d$, *and*

$$f(x) = \sup_{k=1,\ldots,d} \max\left(||s_x(0,e_k)||_{B^*}, ||s_x(0,-e_k)||_{B^*}\right).$$

Proof. For every elementary path $\gamma_n = \{n_1, \ldots, n_r\}$ from 0 to n, we write

$$s_x(0,n) = \sum_{k=0}^{r-1} s_x(n_k, n_{k+1}) = \sum_{k=0}^{r-1} \lambda(n_k).s_{T_{n_k} x}(0, n_{k+1} - n_k),$$

where λ is the left-regular representation of $L^\infty(\mathbb{Z}^d)$. Thus, since $|n_{k+1} - n_k| = 1$ for all k, we have

$$||s_x(0,n)||_{B^*} \leq \sum_{k=0}^{r-1} f(T_{n_k}x) = A_\gamma f(x).$$

We now take the average over the set $\mathcal{E}_n$. By Lemma 2.2 and 2.3, we have

$$\begin{aligned}\frac{||s_x(0,n)||_{B^*}}{|n|} &\leq C\Big[\sum_{H\in\mathcal{H}} \frac{1}{|n|^{\dim H}} \sum_{\substack{m\in H\\ |m|\leq 2|n|}} f(T_m x) + \\ &+ \frac{1}{|n|}\left(f(T_n x) + \sum_{0<|n-m|\leq |n|/2} \frac{f(T_m x)}{|m-n|^{d-1}}\right)\Big]\end{aligned}$$

□

The following lemma was proved in [8] for general actions of $\mathbb{Z}^d$ on probability spaces.

Lemma 2.4. *Suppose f is a nonnegative and measurable function on X. Then there is a constant $C > 0$ such that*

$$\mu\Big(x \in X \mid \sup_{n\in\mathbb{Z}^d} \frac{1}{|n|}\Big(f(T_n x) + \sum_{0<|n-m|\leq|n|/2} \frac{f(T_m x)}{|n-m|^{d-1}} > \lambda\Big)\Big) \leq C\left(\frac{1}{\lambda}||f||_{d,1}\right)^d.$$

for all $\lambda \geq 1$.

Define the maximal function

$$Ms(x) = \sup_{n\neq 0} \frac{||s_x(0,n)||_{B^*}}{|n|},$$

for a Gelfand B–cocycle s. The maximal inequality for the first terms in the estimate in Lemma 2.3 are taken care of by Wiener's maximal inequality [26]. Proposition 2.3 and Lemma 2.4 now imply the following theorem:

Theorem 2.5 (Maximal Inequality)**.** *Let B be a Banach space. Suppose s is a Gelfand B–cocycle on an ergodic $\mathbb{Z}^d$–space (X,μ), where μ is a probability measure. Suppose $s(0,n)$ is in $L^{d,1}_{w^*}(X,\mu,B^*)$ for every $n \in \mathbb{Z}^d$. There exists a positive constant C such that*

$$\mu(\{x \in X \mid Ms(x) \geq \lambda\}) \leq \Big(\frac{C}{\lambda}||s||_{L^{d,1}_{w^*}(X,B)}\Big)^d$$

for all $\lambda \geq 1$.

2.8. **Ergodic Theorems for Bochner Cocycles.** In this section we will be concerned with a slight generalization of the ergodic theorem of D. Boivin and Y. Derriennic in [8] to vector-valued cocycles. We will see an application of this theorem to horofunctions in random media in subsection 3.3. The main ingredient of the proof is a result by R.S. Phillips, ensuring that the Bochner-Lorentz space $L^{d,q}(X,B)$ is a reflexive Banach space if $1 < q < \infty$. This will allow for standard splitting theorems to be used (see chapter 2 of U. Krengel's book [22] for more references).

Theorem 2.6. *Let B be a reflexive Banach space. Suppose s is Bochner B–cocycle on an ergodic $\mathbb{Z}^d$–space (X,μ), where μ is a probability measure. Let $q > 1$ and suppose that the cocycle $s(0,n)$ is in $L^{d,q}_s(X,B^*)$ for every $n \in \mathbb{Z}^d$. Then there is a linear and continuous map $L : \mathbb{R}^d \to B^*$ such that*

$$\lim_{|n|\to\infty} \frac{s_x(0,n) - L(n)}{|n|} = 0,$$

almost everywhere on (X, μ).

Before we turn to the proof of Theorem 2.6, we recall some basic splittings theorems for $\mathbb{Z}^d$-actions on Bochner-Lorentz spaces. The following theorem is due to R.S. Phillips [25].

Theorem 2.7. *Let* $(X, \mathcal{F}, \mu)$ *be a measure space. Suppose* B *is a reflexive Banach space and suppose* $1 \leq p < \infty$, *and* $1 < q < \infty$. *Then* $L^{p,q}(X, B)$ *is reflexive.*

By the well-developed splitting theory of semi-groups of isometries on reflexive Banach spaces (see e.g. chapter 2 in [22]), this implies that every Bochner cocycle s in $L^{d,1}(X, B^*)$ can be written as a limit in $L^{d,1}(X, B)$ of

$$s = \lim_{j \to \infty} r + c^j,$$

where r is an invariant cocycle and c^j is a sequence of coboundaries, i.e. cocycles on the form

$$c_x^j(0, n) = g_j(x) - g_j(T_n x), \quad g_j \in L^{d,q}(X, B^*),\ q > 1.$$

and extended by equivariance.

Proof of Theorem 2.6. Note that the theorem is trivial for invariant cocycles and coboundaries. By Banach's principle and Theorem 2.3, the set of all cocycles for which the theorem holds is closed in $L^{d,q}(X, B)$. Since the span of invariant cocycles and coboundaries is dense in $L^{d,q}(X, B^*)$, we are done. □

3. Applications

3.1. **Random Schödinger Operators.** Let $(X, \mathcal{F}, \mu, T)$ be an ergodic $\mathbb{Z}$-space and suppose that S be a measurable map from $\mathbb{Z} \times X \to GL_d(\mathbb{R})$ which satisfy the equations $S(0, \cdot) = I$ and

$$S(n + m, x) = S(n, T^m x) S(m, x) \quad \forall\, m, n \in \mathbb{Z}$$

almost everywhere on X. The asymptotic behavior of the random semimetric

$$\rho_x(m, n) = \max\left(\log^+(||S_n(x)\, S_m(x)^{-1}||, \log^+(||S_m(x) S_n(x)^{-1}||\right), \quad \forall\, n, m \in \mathbb{Z}.$$

has been of subject to a detailed study in the theory of random Schrödinger operators over the years. The first convergence result, prior to Kingman's paper, is due to Furstenberg and Kesten [14], where the almost sure limit

$$A = \lim_{n \to \infty} \frac{\rho_x(0, n)}{|n|}$$

is established. Before we start discussing multiparameter analogues, we first describe the connection to random Schrödinger operators. Let $(X, \mathcal{F}, \mu, T)$ be an ergodic probability measure preserving system, and suppose that V is a real-valued measurable function on X. We consider, for a fixed x in X, the following discrete analogue of the Schrödinger equation,

$$v_{n+1} + v_{n-1} + V(T^n x) v_n = \lambda v_n, \quad \forall\, n \in \mathbb{Z},$$

with $v_o = a$ and $v_1 = b$, where λ is assumed to be real. If we introduce the vectors $u_n = (v_n, v_{n+1})^t$, the equation can be written in the following equivalent form,

$$u_{n+1} = \begin{pmatrix} 0 & 1 \\ -1 & \lambda - V(T^n x) \end{pmatrix} u_n, \quad u_0 = \begin{pmatrix} a \\ b \end{pmatrix},$$

and thus,

$$u_n = S(n, x) u_o, \quad \forall\, n \in \mathbb{Z},$$

with S is generated by

$$S(1,x) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda - V(x) \end{pmatrix}.$$

Hence the generic (in terms of the measure space $(X, \mathcal{F}, \mu)$) asymptotic behavior of the solutions of random Schrödinger operators on $\mathbb{Z}$ is governed by the random semimetric ρ_x defined above. By a remarkable tour de force, Furstenberg and Kesten established the almost sure limit

$$A = \lim_{n \to \infty} \frac{\rho_x(0,n)}{|n|}.$$

This result predates Kingman's subadditive ergodic theorem, and the methods of Furstenberg and Kesten were indeed quite different from the ones Kingman later used.

This example leads to a natural generalization for $\mathbb{Z}^d$-actions. Let $(X, \mathcal{F}, \mu, T)$ be an ergodic $\mathbb{Z}^d$-space, and let $S : \mathbb{Z}^d \times X \to GL_k(\mathbb{R})$ be a measurable map which satisfy $S(0, \cdot) = I$ and

$$S(n+m, x) = S(n, T^m x) S(m, x), \quad \forall\, m, n \in \mathbb{Z}^d,$$

almost everywhere on X. Define the random semimetric,

$$\rho_x(m,n) = \max\left(\log^+(||S_n(x)\, S_m(x)^{-1}\,||, \log^+(||S_m(x) S_n(x)^{-1}||)\right), \quad \forall\, n, m \in \mathbb{Z}^d.$$

Note that the sequence $u_n = S(T^n x)u$ is the solution to the random difference equation,

$$\sum_{|e|=1} u_{n+e} = \Big(\sum_{|e|=1} S(e, T^n x) \Big) u_n, \quad \forall\, n \in \mathbb{Z}^d,$$

with $u_o = u$, where $| \cdot |$ denotes the ℓ^∞-metric on $\mathbb{Z}^d$. Note that existence of a map S with the above properties is not obvious, and indeed we do not expect any new examples for $d \leq 2$. However, any embeddding of $\mathbb{Z}^d$ into $GL_k(\mathbb{R})$ for sufficiently large k, will give rise to maps S with the above properties, and hence the class of new difference equations which can be solved by this method is non-trivial. In this class, the following theorem can be deduced from Theorem 1.2.

Theorem 3.1. *Suppose that for some $\varepsilon > 0$,*

$$\int_X \left(\log^+ ||S_n(x)|| \right)^{d+\varepsilon} d\mu(x) < +\infty,$$

for all $|n| = 1$. Then there is a seminorm L on $\mathbb{R}^d$ with the property that

$$\lim_{n \to \infty} \frac{\rho_x(0,n) - L(n)}{|n|} = 0,$$

almost everywhere on X.

The class of difference equations which can be solved by the above scheme can probably be considerably enlarged if Theorem 1.2 is extended to more general linear groups, which motivates the study of extensions of Boivin and Derriennic's result to more general groups.

3.2. **A Multiplicative Ergodic Theorem.** In this subsection we will establish a multiplicative ergodic theorem for general Pettis $\mathbb{Z}$–cocycles on ergodic probability measure space (X, μ) with values in separable Banach spaces B with the Radon-Nikodým property. The formulation is close to the celebrated Karlsson-Ledrappier ergodic theorem [18]. However, their paper is concerned with a special kind of a Pettis cocycle with values in the Banach space of continuous functions on an infinite compact metrizable space, which unfortunately does not possess the Radon-Nikodým property [12]. It is not unlikely that our ergodic theorem holds in greater generality (e.g. non-separable or weakly compact generated Banach spaces). For the present proof and methods, the Radon-Nikodým assumption seems to be sharp.

We begin by recalling the definition and some basic facts about Banach spaces with the Radon-Nikodým property. Recall that a vector measure ν is μ-continuous if

$$\lim_{\mu(E)\to 0} \nu(E) = 0.$$

Definition 3.1 (Radon-Nikodým Property)**.** A Banach space B has the *Radon-Nikodým property with respect to the measure space* $(X, \mathcal{F}, \mu)$ if for each μ–continuous vector measure $\nu : \mathcal{F} \to B$ of bounded variation there exists $g \in L^1(X, B)$ such that

$$\nu(E) = \int_E g \, d\mu, \quad \forall\, E \in \mathcal{F},$$

in the sense of Bochner integrals. A Banach space B has the *Radon-Nikodým property* if B has the Radon-Nikodým property with respect to any finite measure space.

Classical examples of Banach spaces with the Radon-Nikodým property are reflexive Banach space and Banach spaces with separable dual spaces. Examples of Banach space without the Radon-Nikodým property are $L^1([0,1])$ and $C(H)$, where H is an infinite compact Hausdorff space. The notion of a Radon-Nikodým space is now fairly well-understood, and a very readable account of results and techniques can be found in [12].

We will need the following theorem by S. Bochner and A.E. Taylor [6]:

Theorem 3.2. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space, $1 \leq p < \infty$, and B a Banach space. Then $L^p(X,B)^* = L^q(X, B^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$, if and only if B^* has the Radon-Nikodým property with respect to μ.*

In particular, this implies that, if B has the Radon-Nikodym property, then

$$||f||_{L^1(X,B)} = \sup_{||\lambda||_{\infty,B^*} \leq 1} \int_X \langle \lambda(x), f(x) \rangle \, d\mu(x),$$

for every Bochner measurable function $f : X \to B$, and where $|| \cdot ||_{\infty,B^*}$ denotes the $L^\infty(X, B^*)$–norm. More generally, we will write $|| \cdot ||_{q,\mathcal{C}}$ if we restrict the elements in $L^q(X, B^*)$ to take values in $\mathcal{C}$, for $q > 1$.

Suppose s is a Pettis B-cocycle on a probability measure space (X, μ) with respect to a $\mathbb{Z}$-action T and Borel cocycle c, where the Banach space B is supposed to be separable and have the Radon-Nikodym property. Note that this implies that s is also Bochner integrable. We also assume that the function $x \mapsto ||s_x(0,n)||_B$ is integrable for all $n \in \mathbb{Z}$. Suppose that there is a weak*-compact subset $\mathcal{C}$ of B_1^*, which is invariant under the dual action of the cocycle c, such that

$$||s_x(m,n)||_{1,B} = \sup_{||\lambda||_{\infty,\mathcal{C}} \leq 1} \langle \lambda, s_x(m,n) \rangle, \quad \forall\, m,n \in \mathbb{Z}.$$

It is a wellknown fact that (See e.g. chapter V.5.1. in [13]) $\mathcal{C}$ is metrizable, and thus separable. By subadditivity, the following non-negative limit exists:

$$A := \lim_{n\to\infty} \frac{1}{n} ||s(0,n)||_{L^1(X,B)} = \inf_{n>0} \sup_{||\lambda||_{\infty,\mathcal{C}} \leq 1} \frac{1}{n} \int_X \langle \lambda(x), s_x(0,n) \rangle \, d\mu(x).$$

We define the skew-product $\mathbb{Z}$-action $\hat{T}$ on the measurable space $X \times \mathcal{C}$ with the product σ-algebra by

$$\hat{T}_n(x,y) = (T_n x, c(n,x)^*.\lambda), \quad x \in X,\ \lambda \in \mathcal{C}.$$

Note that, if $n \geq 0$ and $\lambda \in \mathcal{C}$, then

$$\langle \lambda, s_x(0,n) \rangle = \sum_{k=0}^{n-1} \langle \lambda, c(k,x).s_x(0,1) \rangle = \sum_{k=0}^{n-1} F(\hat{T}_k(x,\lambda)),$$

where $F(x,\lambda) = \langle \lambda, s_x(0,1) \rangle$, and thus

$$\begin{aligned} A &= \inf_{n>0} \sup_{||\lambda||_{\infty,\mathcal{C}} \leq 1} \frac{1}{n} \int_X \sum_{k=0}^{n-1} F(\hat{T}_k(x,\xi))\, d\delta_{\lambda(x)}(\xi) d\mu(x) \\ &= \inf_{n>0} \sup_{\hat{\mu} \in M_\mu^1(X \times \mathcal{C})} \frac{1}{n} \int_{X \times \mathcal{C}} \sum_{k=0}^{n-1} F(\hat{T}_k(x,\xi))\, d\hat{\mu}(x,\xi), \end{aligned}$$

where $M_\mu(X \times \mathcal{C})$ denotes the space of probability measures on $X \times \mathcal{C}$ which projects onto μ under the canonical map from $X \times \mathcal{C}$ to X. By standard disintegration theory (See e.g. [3]), this space can be given a compact metrizable topology coming from the duality of $L^1(X, C(\mathcal{C}))$. Following the outline of the proof in [18], we take a sequence of elements $\hat{\mu}_n$ in $M^1(X \times \mathcal{C})$ such that

$$\frac{1}{n} \int_X \sum_{k=0}^{n-1} F(T_k(x,\xi))\, d\hat{\mu}_n(x,\xi) \geq A, \quad \forall\, n \geq 1.$$

This is possible due to the compactness of $M^1(X \times \mathcal{C})$. Define

$$\hat{\nu}_n = \frac{1}{n} \sum_{k=0}^{n-1} \hat{T}_*^k \hat{\mu}_n, \quad n \geq 1.$$

By sequential compactness, there is a convergent subsequent and a $\hat{T}$-invariant limit probability measure $\hat{\nu}_0$ in $M_\mu^1(X \times \mathcal{C})$ such that

$$\int_{X \times \mathcal{C}} F(x,\xi)\, d\hat{\nu}_0(x,\xi) \geq A,$$

and thus the set of $\hat{T}$–invariant probability measures on $X \times \mathcal{C}$ which projects onto μ and satisfy the above inequality, is a compact and convex subset of $M_\mu^1(X \times \mathcal{C})$. By Krein-Milman's theorem, there must be an extremal point ν in this set, and by a standard argument (See e.g. [3]), this point is an ergodic measure for $\hat{T}$. By Birkhoff's theorem, and the obvious inequality,

$$|\langle \xi, s_x(0,n) \rangle| \leq ||s_x(0,n)||_B,$$

for all ξ in the unit ball of B^*, we can conclude that

$$A = \lim_{n \to \infty} \frac{1}{n} ||s_x(0,n)||_B = \lim_{n \to \infty} \frac{1}{n} \langle \xi, s_x(0,n) \rangle$$

for a co–null subset of $X \times \mathcal{C}$ with respect to the measure ν. If we assume that $(X, \mathfrak{F}, \mu)$ is standard Borel space, then we can use the Von Neumann selection theorem (in complete analogy with [18]) and establish the existence of a *measurable* map $\xi : X \to \mathcal{C}$ such that

$$\lim_{n \to \infty} \frac{\langle \xi(x), s_x(0,n) \rangle}{n} = A,$$

for all x in a co-null subset of X. We have established the following theorem

Theorem 3.3. *Suppose $(X, \mathfrak{F}, \mu)$ is a standard measure space with an ergodic $\mathbb{Z}$-action. Suppose B is a separable Banach space with the Radon-Nikodým property, and s an integrable Pettis cocycle with respect to a Borel cocycle c. Suppose there is a weak*-compact subset of B_1^* such that*

$$||s(0,n)||_{1,B} = \sup_{||\lambda||_{\infty,B^*} \leq 1} \int_X \langle \lambda(x), s_x(0,n) \rangle \, d\mu(x), \quad \forall \, n \geq 1.$$

Then there is a measurable map $\xi : X \to \mathcal{C}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \langle \xi(x), s_x(0,n) \rangle = \lim_{n\to\infty} \frac{1}{n} \int_X ||s_x(0,n)||_B \, d\mu(x),$$

almost everywhere on (X, μ).

Remark. The main reason for including the proof above is an application to Kingman decompositions of subadditive cocycles which will be described below. Note that the restrictions on the Banach space B and the measurability of s are fairly severe and exclude many interesting applications. For instance; note that the case of Pettis cocycles for $B = C(H)$, where H is a compact metrizable space, would generalize the celebrated multiplicative ergodic theorem by V.I. Oseledec [24]. In this situation, Theorem 3.3 was established for a certain class of cocycles by A. Karlsson and F. Ledrappier in [18]. One important feature with these cocycles is an obvious choice of a sequence of weakly *measurable* maps $\eta_n : X \to B^*$ such that

$$||s_x(0,n)||_B = \langle \eta_n(x), s_x(0,n) \rangle$$

for $n \in \mathbb{Z}$. This is no longer true for general cocycles in Banach spaces. The Radon-Nikodým assumption on B is a convenient way to circumvent this problem.

An extension of Theorem 3.3 to conservative and ergodic actions of $\mathbb{Z}$ on σ-finite measure spaces can be proved using the same techniques as in [5], where Karlsson-Ledrappier's ergodic theorem is extended to the σ-finite situation.

We now turn to the proof of an alternative Kingman decomposition for random semi-metrics induced by Pettis cocycles on reflexive and separable Banach spaces. Let η denote the disintegration of ν with respect to the canonical projection onto measure μ. Let $g : X \to \mathrm{Isom}(B)$ be the generator of the Borel cocycle c, i.e.

$$c(n,x) = g(x) \cdots g(T^{n-1}x),$$

for $n \geq 0$. For all $f \in L^1(X, B)$, we have

$$\begin{aligned} \langle \nu, \hat{T} f \rangle &= \int_X \langle \eta(x), g(x) f(Tx) \rangle \, d\mu(x) \\ &= \int_X \langle (g(x))^* \eta(x), f(Tx) \rangle \, d\mu(x) \\ &= \int_X \langle (g(T^{-1}x)^*) \eta(T^{-1}x), f(x) \rangle \, d\mu(x) \\ &= \int_X \langle \eta(x), f(x) \rangle \, d\mu(x) = \langle \nu, f \rangle. \end{aligned}$$

Thus, if B is a reflexive Banach space, we conclude that

$$\eta(Tx) = (g(x)^*)^{-1} \eta(x).$$

or equivalently,

$$\eta(T^k x) = c(k,x)^* \eta(x), \quad \forall \, k \geq 1.$$

Thus, we can rewrite the Birkhoff sum above as

$$\langle \eta(x), s_x(0,n) \rangle = \sum_{k=0}^{n-1} \langle \eta(x), c(k,x).f(T^k x) \rangle = \sum_{k=0}^{n-1} \varphi(T^k x).$$

where $\varphi(x) = \langle \eta(x), f(x) \rangle$ satisfies

$$\int_X \varphi(x)\, d\mu(x) = A.$$

Furthermore, we obviously have

$$||s_x(0,n)||_B \geq \sum_{k=0}^{n-1} \varphi(T^k x), \quad n \geq 1.$$

We have proved the following weak version of Kingman's decomposition of subadditive cocycles:

Theorem 3.4 (Kingman Decomposition)**.** *Suppose s is an integrable Pettis cocycle with values in a separable and reflexive Banach space, defined on a standard probability measure space with an ergodic $\mathbb{Z}$-action. Then the random semimetric defined by*

$$\rho_x(m,n) = ||s_x(m,n)||_B, \quad n, m \in \mathbb{Z},$$

decomposes as

$$\rho_x(0,n) = \sum_{k=0}^{n-1} \varphi(T^k x) + r_n(x),$$

where φ is integrable on (X,μ) such that $\int_X \varphi(x)\, d\mu(x)$ equals the drift of ρ and r_n is a non-negative subadditive cocycle with drift 0.

Remark. J.F.C. Kingman [21] established a more general decomposition theorem for integrable subadditive cocycles. Note however that Theorem 3.4 provides more information about the decomposition. The restrictions on the measurability of s and the Banach space B in the theorem above seem to be necessary for the methods described. However, it seems natural to ask for a canonical class $\mathcal{M}$ of Gelfand cocycles on ergodic G-spaces and with values in Banach spaces with separable pre-duals, such that for any s in $\mathcal{M}$ with values in B, there is an G-equivariant and weakly*–measurable map $\eta : X \to B^*$ such that

$$||s_x(e,g)||_B = p(g)\langle \eta_x, s_x(e,g) \rangle + r_x(e,g),$$

where $p : G \to \mathbb{R}$ is a weight function and r_x is negligable with respect to s in a certain sense. In the case when $G = \mathbb{Z}^d$ and the seminorm L in Theorem 1.2 is non-degenerate, this would have interesting implications for generalized first passage percolation. Indeed, this would imply a multiparameter version of Oseledec's theorem with possible applications to infinite geodesics in random metric spaces.

3.3. **Horofunctions in Random Media.** Suppose $\mathcal{H}$ is a separable Hilbert space and suppose $s : X \times \mathbb{Z}^d \times \mathbb{Z}^d \to \mathcal{H}$ is a Bochner-cocycle in $L^{d,1}(X,\mathcal{H})$. Recall that

$$\rho_x(m,n) = ||s_x(m,n)||_{\mathcal{H}}, \quad n, m \in \mathbb{Z}^d,$$

defines random semimetric on $\mathbb{Z}^d$. Suppose m is in $\mathbb{Z}^d$ and define the horofunction at the point m, with respect to the random semimetric ρ, by

$$h_m(n) = \rho(m,n) - \rho(m,0), \quad n \in \mathbb{Z}^d.$$

We want to study the behaviour of h_m as m leaves finite subsets of $\mathbb{Z}^d$. We will see that the limit exists along the sequence m^j if and only if there is an element η in the unit ball of $\ell^1(\mathbb{Z}^d)$ such that

$$\lim_{j\to\infty} \frac{m_k^j}{|m|} = \eta_k, \quad k = 1, \ldots, d.$$

It will follow from the proof that the limit point is unique, i.e. independent of the particular sequence which converges to η. We will denote the limit point by h_η, and refer to it as the horofunction located at η. Before we give the proof, we establish the following simple lemma.

Lemma 3.1. *Suppose m^j is a sequence in $\mathbb{Z}^d$ such that there is an element η in the unit ball of $\ell^1(\mathbb{R}^d)$, such that $m_k^j/|m| \to \eta_k$ for $k = 1, \ldots, d$, where $|\cdot|$ denotes the ℓ^1–metric. Suppose s is Bochner-cocycle in $L^1(X, \mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, and (X, μ) is an ergodic $\mathbb{Z}^d$-space. Then,*

$$\lim_{m\to\eta} \frac{||s_x(0,m)||_{\mathcal{H}}}{|m|} = ||\sum_{k=1}^{d} \eta_k L_k||_{\mathcal{H}},$$

almost everywhere on X with respect to μ. Here $L_k = L(e_k)$, $k = 1, \ldots, d$ and L is the continuous linear map in Theorem 2.6. Conversely, the limit

$$\lim_{m\to\eta} \frac{||s_x(0,m)||_{\mathcal{H}}}{|m|}$$

exists almost everywhere on X if and only if $m_k/|m|$ converges to η.

Proof. By Theorem 2.6,

$$\lim_{|m|\to\infty} \frac{||s_x(0,m) - L(m)||_{\mathcal{H}}}{|m|} = 0,$$

almost everywhere on X. Thus,

$$\lim_{m\to\eta} \frac{||s_x(0,m)||_{\mathcal{H}}}{|m|} = \lim_{m\to\eta} \frac{||s_x(0,m) - L(m) + L(m)||_{\mathcal{H}}}{|m|} = ||\eta_k L_k||_{\mathcal{H}},$$

since $L(m) = \sum_{k=1}^{d} m_k L_k$ for all $m \in \mathbb{Z}^d$. □

In general, if (Y, d) is a semimetric space, we define the horofunction at a point y in Y by

$$h_y(y') = d(y, y') - d(y, 0), \quad y' \in Y.$$

If d is a metric, the map $y \mapsto h_y$ is injective. Furthermore, if (Y, d) is a proper metric space, i.e. closed and bounded sets are compact, then the closure of the set $\{h_y\}_{y\in Y}$ in $C(Y)$ is compact by the Arzela-Ascoli theorem. In our case, the semimetric ρ is in general not a metric, nor is the topology it induces proper. However, the notion of a horofunction is still well-defined. We will study the asymptotic behaviour of the horofunctions with respect to the random semimetric ρ defined above in terms of $\mathcal{H}$-valued cocycles. It turns out that a nice description is possible in this situation.

Theorem 3.5. *Suppose (X, μ) is an ergodic $\mathbb{Z}^d$-space, and $s : X \times \mathbb{Z}^d \times \mathbb{Z}^d \to \mathcal{H}$ is Bochner-cocycle in $L^{d,1}(X, \mathcal{H})$, where $\mathcal{H}$ is a real separable Hilbert space. Let*

$$\rho(m, n) = ||s(m, n)||_{\mathcal{H}}, \quad n, m \in \mathbb{Z}^d,$$

denote the associated random semimetric on $\mathbb{Z}^d$*. If* η *is an element in* $\ell^1(\mathbb{R}^d)$*, such that* $\xi = \sum_{k=1}^d \eta_k L_k$ *is a non-trivial element in* $\mathcal{H}$*, where* L *is the continuous linear map in Theorem 2.6, then*

$$h_\eta(n) = \frac{2\langle s(0,n), \xi\rangle}{||\xi||_{\mathcal{H}}}, \quad n \in \mathbb{Z}^d,$$

almost everywhere on X *with respect to* μ*.*

Proof. The proof is a straightforward modification of the standard method to compute horofunctions on a Hilbert space. Suppose $s_x(0,n)$ and $s_x(0,m)$ are both non-trivial elements of $\mathcal{H}$, then

$$\begin{aligned} ||s_x(n,m)||_{\mathcal{H}} - ||s_x(m,0)||_{\mathcal{H}} &= \frac{||s_x(n,0) + s_x(0,m)||^2_{\mathcal{H}} - ||s_x(0,m)||^2_{\mathcal{H}}}{||s_x(n,0)||_{\mathcal{H}} + ||s_x(m,0)||_{\mathcal{H}}} \\ &= \frac{||s_x(n,0)||^2_{\mathcal{H}} + 2\langle s_x(n,0), s_x(0,m)\rangle_{\mathcal{H}}}{|m|} \cdot \\ &\cdot \frac{|m|}{||s_x(n,0)||_{\mathcal{H}} + ||s_x(m,0)||_{\mathcal{H}}}. \end{aligned}$$

By Lemma 3.1,

$$\lim_{m\to\eta} ||s_x(n,m)||_{\mathcal{H}} - ||s_x(m,0)||_{\mathcal{H}} = 2\langle s_x(0,n), \hat{\xi}\rangle_{\mathcal{H}},$$

almost everywhere on X, where $\hat{\xi} = \xi/||\xi||_{\mathcal{H}}$. □

Remark. It is still an open problem to compute the horofunctions at infinity for the classical first passage percolation metrics. This would give a more refined knowledge of the asymptotic geometry of these semimetric spaces. It is expected that these horofunctions can be arbitrarily wild; indeed, by a celebrated result of R. Meester and O. Häggström [17], essentially any convex shape in $\mathbb{R}^d$ can be attained as an asymptotic shape of a classical first passage percolation, generated by ergodic $\mathbb{Z}^d$–actions.

3.4. **Reproducing Kernel Hilbert Spaces.** In this subsection we will describe natural examples of Bochner cocycles with values in separable Hilbert spaces. Let $(\mathcal{H}, K, o)$ be a pointed reproducing Hilbert space. This means that there $\mathcal{H}$ is a Hilbert space of measurable functions on a measurable space $(Y, \mathcal{G})$ with a fixed base point o in Y and $K : Y \times Y \to \mathbb{C}$ is a positive definite reproducing kernel, i.e. for all finitely supported sequences (c_i, y_i) in $\mathbb{C} \times Y$ we have the inequality,

$$\sum_{i,j} c_i\, \overline{c_j}\, K(y_i, y_j) \geq 0,$$

and for all y in Y we have

$$\langle K(y,\cdot), f\rangle_{\mathcal{H}} = f(y), \quad \forall\, f \in \mathcal{H},$$

where $\langle \cdot,\cdot\rangle_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$.

Suppose that a locally compact group G acts measurably on $(Y, \mathcal{G})$. In many cases, the action of G on Y can be lifted to an isometric action of G on $\mathcal{H}$ so that the measurable metric

$$d(y, y') = ||K(y,\cdot) - K(y',\cdot)||^2_{\mathcal{H}}, \quad y, y' \in Y,$$

is invariant under the action. Let $(X, \mathcal{F}, \mu, T)$ be a $\mathbb{Z}$-action and suppose that π is a unitary representation of G on $\mathcal{H}$. Given a measurable map $g : X \to G$, we define the isometry on $L^2(X, \mathcal{H})$ by

$$\hat{T} f(x) = \pi(g(x)).f(Tx), \quad f \in L^2(X, \mathcal{H}),$$

almost everywhere on X, and we let

$$f^*(x) = K(g(x)o, \cdot) - K(o, \cdot), \quad x \in X.$$

Let s be the Bocher cocycle generated by f^* and the action $\hat{T}$. We will describe a situation where π can be chosen so that

$$d(Z_n(x)o, o) = ||s_x(0, n)||^2_{\mathcal{H}}, \quad \forall\, n \in \mathbb{Z},$$

where Z_n is the Borel cocycle generated by g and T. We believe that this is a fairly general phenomena.

Let $\mathbb{D}$ be the Poincaré disk, i.e. the unit disk in $\mathbb{C}$ with the distance function β given by

$$\beta(o, z) = \log \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D},$$

where o is the origin, and extended to all pairs (z, z') in $\mathbb{D} \times \mathbb{D}$ by isometry. The isometry group G of $\mathbb{D}$ is isomorphic to the Möbius group $PSL_2(\mathbb{R})$. The large scale behavior of β is can be equivalently described by the metric (see [2] for a more details),

$$d(z, z') = ||K(z, \cdot) - K(z', \cdot)||^2_{\mathcal{H}},$$

where $(\mathcal{H}, K)$ is the normalized Dirichlet reproducing kernel Hilbert space [2] on $\mathbb{D}$, i.e. the reproducing Hilbert space of holomorphic functions ϕ on $\mathbb{D}$ with $\phi(o) = 0$ and subject to the integrability condition,

$$||\phi||_{\mathcal{H}} = \Big(\int_{\mathbb{D}} |\phi'(z)|^2 \, dA(z) \Big)^{1/2} < \infty,$$

where A is the euclidean area measure on $\mathbb{D}$ and

$$K(z, z') = -\log(1 - z\overline{z'}), \quad (z, z') \in \mathbb{D}.$$

The precise relation between the metrics β and d is discussed, in a slightly different language, in the paper [2]. In this example, the representation π can be chosen to be

$$\pi(g).\phi(z) = \phi(g^{-1}z) - \phi(o), \quad z \in \mathbb{D}.$$

For a discussion about the relevance of the metric β and the Borel cocycle Z to random Schrödinger equations can be found in [18].

3.5. **Rates of Convergence.** In this subsection we will prove quantitative statements about the convergence to a limit shape under certain conditions. Our results will not apply to classical first passage percolation, where deep results have been established in a series of paper (see e.g. [4], [19] and [27]). We will restrict the study to Bochner cocycles with values in Hilbert spaces. This allows for certain spectral measure computations to be performed, and the methods will not generalize beyond uniformly convex Banach spaces. In particular, L^∞-spaces, which would be the relevant spaces for classical first passage percolation, are certainly out of reach.

Let s denote a Bochner cocycle on a $\mathbb{Z}^d$-space X with values in a Hilbert space $\mathcal{H}$. By the additivity and equivariance properties of s, we note that

$$s_x(0, ne_1) = \sum_{k=0}^{n-1} s_x(ke_1, (k+1)e_1) = \sum_{k=0}^{n-1} \lambda(k).s_{T_{ke_1}x}(0, e_1), \quad \forall\, n \in \mathbb{Z}^d,$$

where λ is an isometric representation of $\mathbb{Z}^d$ on $\mathcal{H}$. For notational convenience, we define $f(x) = s_x(0, e_1)$. By standard Hilbert space calculations, we have

$$\begin{aligned} \frac{1}{n^2}\int_X ||s_x(0,ne_1)||^2_{\mathcal{H}}\,d\mu(x) &= \frac{1}{n^2}\sum_{j,k=0}^{n-1}\int_X \langle \lambda(j).f(T_{ke_1}x), \lambda(k).f(T_{ke_1}x)\rangle_{\mathcal{H}}\,d\mu(x) \\ &= \frac{1}{n^2}\sum_{j,k=0}^{n-1}\langle f(x), \lambda(k-j).f(T_{(k-j)e_1}x)\rangle_{\mathcal{H}}\,d\mu(x) \\ &= \sum_{k=-n}^{n}\frac{(n-|k|)}{n^2}\int_X \langle f(x), \lambda(k).f(T_{ke_1}x)\rangle_{\mathcal{H}}\,d\mu(x). \end{aligned}$$

We introduce the unitary operator U on $L^2(X,\mathcal{H})$, defined by $U^k f(x) = \lambda(k).f(T^k x)$. We note that the calculations above establish the following proposition.

Proposition 3.1. *Let U be the unitary operator defined above. Then,*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=-n}^{n}(n-|k|)U^k f = Pf,$$

where P is the projection onto the space of U-invariant vectors in $L^2(X,\mathcal{H})$.

Remark. It should be remarked that the proposition is true for any unitary operator on $L^2(X,\mathcal{H})$. This is an immediate consequence of Von Neumann's mean ergodic theorem. We included the calculation above for later references.

A slight reformulation of the above proposition is contained in the following lemma.

Lemma 3.2. *Suppose s is a Bochner cocycle on a $\mathbb{Z}^d$-space X with values in a Hilbert space $\mathcal{H}$. Then,*

$$\lim_{n\to\infty}\frac{1}{n}||s(0,ne_1)||_{L^2(X,\mathcal{H})} = ||Ps(0,e_1)||_{L^2(X,\mathcal{H})},$$

where P is the projection onto the space U–invariant vectors in $L^2(X,\mathcal{H})$.

Suppose $||f||_{L^2(X,\mathcal{H})} = 1$ and let ν_f denote the probability measure on $\mathbb{T}$ such that $\hat{\nu}_f(n) = \langle U^n f, f\rangle$ for all $n \in \mathbb{Z}$. We are interested in asymptotic behavior of the sequence

$$R_n = ||\sum_{k=0}^{n-1}U^k f||^2_{L^2(X,\mathcal{H})} - n^2||Pf(x)||^2_{L^2(X,\mathcal{H})}.$$

By Lemma 3.2, we may assume that $Pf = 0$ in $L^2(X,\mathcal{H})$. Several papers have been written on the analogous situation in the case of classical first passage percolation. See e.g. the papers [1], [4] and [27]. In our situation, we prove the following analogue of Kesten's inequality in [19].

Theorem 3.6. *Suppose ν_f is absolutely continuous with respect to the Haar measure m on $\mathbb{T}$ and suppose $\frac{d\nu_f}{dm}$ is continuous at 0. Then there is a constant C such that*

$$\Big|\,||\sum_{k=0}^{n-1}U^k f(x)||^2_{L^2(X,\mathcal{H})} - n^2||Pf(x)||^2_{L^2(X,\mathcal{H})}\Big| \le Cn, \quad \forall\, n \in \mathbb{N}.$$

Proof. By Lemma 3.2, we can without loss of generality, assume that $Pf = 0$ as an element of $L^2(X, \mathcal{H})$. Thus, by the calculation above, we have

$$\frac{||\sum_{k=0}^{n-1} U^k f||_{L^2(X,H)}}{\sqrt{n}} = \left(\int_{\mathbb{T}} \sum_{|k| \leq n} \left(1 - \frac{|k|}{n}\right) e^{2\pi i k \theta} \, d\nu_f(\theta) \right)^{1/2} = \left(\int_{\mathbb{T}} F_n(\theta) \, d\nu_f(\theta) \right)^{1/2}$$

where F_n denotes the Fejér kernel. Thus, if $\frac{\nu_f}{dm}$ is continuous at 0, then the limit stays bounded for large n, and we are done. □

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