Infinite Rate Mutually Catalytic Branching

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Abstract

Consider the mutually catalytic branching process with finite branching rate γ. We show that as $\gamma \to \infty$, this process converges in finite dimensional distributions (in time) to a certain discontinuous process. We give descriptions of this process in terms of its semigroup, the infinitesimal generator and as the solution of a martingale problem. We also give a strong construction in terms of a planar Brownian motion from which we infer a path property of the process.

This is the first paper of a trilogy where we also construct an interacting version of this process and study its long-time behaviour.

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1 Introduction and Main Results

1.1 Motivation

In [5], Dawson and Perkins introduced a population dynamic model of two populations that live on a countable site space S. The individuals migrate between sites and at any given site, perform a critical branching process with a branching rate depending linearly on the respective other type.

More precisely, Dawson and Perkins considered the system of coupled stochastic differential equations (taking non-negative values)

$$dY_{i,t}(k) = (\mathcal{A}Y_{i,t})(k)dt \; + \; \sqrt{\gamma Y_{1,t}(k)Y_{2,t}(k)}\, dW_{i,t}(k), \quad i = 1, 2,\, k \in S. \tag{1.1}$$

Here $\mathcal{A}(k,l) = a(k,l) - \mathbb{1}_{\{k\}}(l)$ is the q-matrix of a Markov chain on S with symmetric jump kernel a, $(W_i(k),\, k \in S,\, i = 1,2)$ is an independent family of Brownian motions and $\gamma \geq 0$ is a parameter.

Dawson and Perkins showed that there exists a unique weak solution of this SDE taking values in a suitable subspace of $([0,\infty)^2)^S$ with some growth condition. Furthermore, this process is a strong Markov process. While existence of a weak solution is rather standard due to the procedure proposed by Shiga and Shimizu [16],

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weak uniqueness was shown using a certain self-duality of the process established in [13]. We will describe the duality in detail below in (2.4).

A main result of Dawson and Perkins is a dichotomy in the longtime behaviour of the solutions depending on whether $\mathcal{A}$ is recurrent or transient (assuming some mild regularity condition on $\mathcal{A}$). For recurrent $\mathcal{A}$ (fulfilling the regularity assumption), the types segregate, while for transient $\mathcal{A}$ there is coexistence of types. More precisely, let

$$M_{i,t} = \sum_{k \in S} Y_{i,t}(k)$$

denote the total mass processes ($i = 1, 2$) and assume $M_{1,0}$, $M_{2,0} < \infty$. Then M_1 and M_2 are continuous orthogonal nonnegative L^2-martingales. Let $M_{i,\infty} = \lim_{t\to\infty} M_{i,t}$ denote the almost sure limit. Dawson and Perkins show that $\mathbf{E}[M_{1,\infty}M_{2,\infty}] = 0$ if $\mathcal{A}$ is recurrent and $\mathbf{E}[M_{1,\infty}M_{2,\infty}] = M_{1,0}M_{2,0}$ if $\mathcal{A}$ is transient. Furthermore, in the recurrent case, the joint distribution of $(M_{1,\infty}, M_{2,\infty})$ equals $Q_{(M_{1,0},M_{2,0})}$ where for $x \in [0,\infty)^2$, Q_x is the harmonic measure of planar Brownian motion in $[0,\infty)^2$. That is, if $B = (B_1, B_2)$ is a Brownian motion in $\mathbb{R}^2$ started in x and $\tau = \inf\{t > 0 : B_t \notin (0,\infty)^2\}$, then Q_x is the probability measure on

$$E := [0,\infty)^2 \setminus (0,\infty)^2$$

given by

$$Q_x = \mathbf{P}_x[B_\tau \in \cdot\,]. \tag{1.2}$$

The explicit form of the densities of Q_x can be found in (2.5).

Via the self-duality of the mutually catalytic branching process, its total mass behaviour for finite initial conditions provides information on the local behaviour if the initial condition is infinite and sufficiently homogeneous. For $x \in [0,\infty)^2$, let $\underline{x}$ denote the state in $([0,\infty)^2)^S$ with $\underline{x}_i(k) = x_i$ for all $k \in S$, $i = 1, 2$. Assume $Y_0 = \underline{x}$. Then

$$\lim_{t\to\infty} \mathbf{P}_{\underline{x}}[Y_{1,t}(0)Y_{2,t}(0) > 0] > 0$$

if $\mathcal{A}$ is transient, that is, types can coexist locally. On the other hand, for recurrent $\mathcal{A}$, the distribution of Y_t converges weakly to $\int \delta_{\underline{y}} Q_x(dy)$, that is to a spatially homogeneous point $\underline{y}$, where y is sampled according to the distribution Q_x. Hence in the recurrent case, the two types segregate locally and form clusters. The assumption that the initial point is constant can be weakened to an ergodic random initial condition (see [3]).

The starting point for this work was the wish to get a quantitative description of the cluster growth in the recurrent case. We only briefly give the heuristics. Dawson and Perkins also constructed a version of their process in continuous space $\mathbb{R}$ instead of S as the solution of a stochastic partial differential equation

$$\frac{dY_{i,t}(r)}{dt} = \Delta Y_{i,t}(r) + \sqrt{\gamma Y_{1,t}(r)Y_{2,t}(r)}\,\dot{W}_i(t,r), \qquad r \in \mathbb{R},\ i = 1, 2, \tag{1.3}$$

where $\dot{W}_1$ and $\dot{W}_2$ are independent space time white noises and Δ is the Laplace operator. As Δ on $\mathbb{R}$ is recurrent, here also types segregate. Now due to Brownian scaling, if we denote by Y^γ the solution of (1.3) with that given value of γ, we obtain

$$\mathbf{P}_{\underline{x}}\big[(Y_T^\gamma(r\sqrt{T}))_{r\in\mathbb{R}} \in \cdot\,\big] = \mathbf{P}_{\underline{x}}\big[(Y_1^{\gamma T}(r))_{r\in\mathbb{R}} \in \cdot\,\big]. \tag{1.4}$$

Equation (1.4) shows that clusters of $Y_{1,T}$ grow like $\sqrt{T}$ and that a better understanding of the precise cluster formation can be obtained by letting $\gamma \to \infty$ for fixed time. Hence we aim at constructing a model X that in some sense is the limit of Y^γ as $\gamma \to \infty$.

In this paper we construct X in the simple case where S is a singleton and where the migration between colonies is replaced by an interaction with a time-invariant mean field. This is a first step towards the investigation of the infinitely many sites model. We give characterisations of the process X via an infinitesimal generator, as the solution of a well-posed martingale problem, and as the limit of Y^γ as $\gamma \to \infty$. Finally, we give a strong construction of the process via a time-changed planar Brownian motion. This will also serve to derive path properties.

In two forthcoming papers we construct the infinite rate process on a countable site space S via a stochastic differential equation with jump type noise and give a characterisation via a martingale problem [9]. Furthermore, we will investigate the longtime behaviour and give conditions for segregation and for coexistence of types [10]. An alternative construction via a Trotter product approach is carried out in [14] and [11].

1.2 Results

We now describe the one-colony process which is the subject of investigation of this paper. Assume that S is a singleton and that immigration and emigration come from and go to some colony that is thought infinitely big and whose effective population size (for immigration) is $\theta \in [0,\infty)^2$. Furthermore, let $c \geq 0$ be the rate of migration. Hence we consider the solution $Y = Y^{\gamma,c,\theta}$ of the stochastic differential equation

$$dY_{i,t} = c\big(\theta_i - Y_{i,t}\big)\,dt \;+\; \sqrt{\gamma Y_{1,t}Y_{2,t}}\,dW_{i,t}, \quad i = 1,2. \tag{1.5}$$

This model can be thought of as a version of the model defined in (1.1) where the migration between colonies is replaced by an interaction with a time invariant mean field θ or with an infinitely large reservoir whose types have proportions θ_1 and θ_2. (In fact, in [2] it was shown (Proposition 1.1) that $Y^{\gamma,c,\theta}$ arises as the McKean-Vlasov limit of solutions of (1.1) with symmetric interaction on a complete graph S.) More formally, the interaction term $\mathcal{A}Y$ is replaced by a drift $c\big(\theta_i - Y_{i,t}\big)$. It is this simplification of the interaction that allows for a tractable exposition in this article. Note that, as $t \to \infty$, the process without drift ($c = 0$) converges almost surely to some random $x \in E$. Hence in the case $c = 0$, if we let $\gamma \to \infty$, then the limiting process would be trivial: if it starts at $x \in E$ then it stays at x forever. See Section 2 for a more detailed description of the process Y solving (1.5) (finite γ process).

On a heuristic level, as the stochastic term in (1.5) defines an isotropic two-dimensional diffusion, that is, a time-transformed planar Brownian motion, if we let $\gamma \to \infty$, we should end up with a process where the stochastic part is a planar Brownian motion at infinite speed, stopped when it reaches the boundary of the upper right quadrant. That is, the limiting process X should be a Markov process with values in E. When x is the current state and the drift moves it to $x + c(\theta - x)dt$, this point should instantaneously be replaced by a random point chosen according to $Q_{x+c(\theta-x)dt}$. We will in fact be able to describe this infinitesimal dynamics both in terms of a martingale problem and in terms of a generator of Markov transition kernels. However, first we define X by an explicit transition semigroup and show that it is the limit of $Y^{\gamma,c,\theta}$ as $\gamma \to \infty$. Let

$$C_l(E) := \Big\{ f : E \to \mathbb{C} \text{ is continuous and } \lim_{u\to\infty} f(u,0) = \lim_{v\to\infty} f(0,v) \text{ is finite} \Big\}, \tag{1.6}$$

equipped with the supremum norm $\|f\|_\infty = \sup_{x\in E} |f(x)|$.

Definition 1.1. *Let $c \geq 0$ and $\theta \in [0,\infty)^2$. For $t \geq 0$ and $x \in E$, define the stochastic kernel p_t by*

$$p_t(x,\,\cdot\,) := p_t^{c,\theta}(x,\,\cdot\,) := Q_{e^{-ct}x + (1-e^{-ct})\theta}.$$

Define the contraction semigroup $\mathcal{S} = (\mathcal{S}_t)_{t\geq 0}$ on $C_l(E)$ by

$$\mathcal{S}_t f(x) = \int_E f(y)\, p_t(x,dy).$$

The Markov process $X = X^{c,\theta}$ with state space E, càdlàg paths, and with transition kernels $(p_t)_{t\geq 0}$ is called the infinite rate mutually catalytic branching process (IMUB) with parameters (c,θ).

In order that this definition makes sense, we will show in Proposition 3.2 that $(\mathcal{S}_t)_{t\geq 0}$ is in fact a Markov semigroup.

Proposition 1.2. *$X^{c,\theta}$ is a Feller process and has the strong Markov property. It is ergodic and the unique invariant measure is Q_θ.*

Proof. The map $x \mapsto Q_x$ is continuous, hence also $x \mapsto p_t(x, \cdot)$ is continuous, i.e., $X^{c,\theta}$ is a Feller process. Since $Q_x = \delta_x$ for $x \in E$, the semigroup $\mathcal{S}$ is strongly continuous. Hence by the general theory of Markov processes, there exists a càdlàg version of X that is strong Markov (see, e.g., [15, Chapter III.7 and 8]).

Ergodicity and the explicit form of the invariant measure are trivial. □

Theorem 1.3 ($X^{c,\theta}$ as infinite rate process). *Assume that* $Y_0^{\gamma,c,\theta} = X_0^{c,\theta} = x \in E$ *for all* $\gamma \geq 0$. *As* $\gamma \to \infty$, *the finite dimensional distributions of* $Y^{\gamma,c,\theta}$ *converge to those of* $X^{c,\theta}$.

Note that in Theorem 1.3, trivially we do not have convergence in the Skorohod path space, since continuous processes do not converge to discontinuous processes in that topology.

In addition to the convergence of the finite dimensional distributions, we also have convergence of the p-th moments for $p \in [1, 2)$ (but not for $p = 2$, of course, since for $x \in (0, \infty)^2$, the measure Q_x does not possess finite second moments as can be derived easily from its density formula (2.5)). Hence on a suitable probability space, we have L^p convergence of $Y^{\gamma,c,\theta}$ to $X^{c,\theta}$.

Theorem 1.4 (L^p convergence). *Assume* $Y_0^{\gamma,c,\theta} = X_0^{c,\theta} = x \in E$ *for all* $\gamma \geq 0$ *and let* $p \in [1, 2)$ *and* $t \geq 0$.

(i) *For every* $\gamma \geq 0$ *and* $i = 1, 2$, *we have*

$$\mathbf{E}_x\left[(Y_{i,t}^{\gamma,c,\theta})^p\right] \leq \mathbf{E}_x\left[(X_{i,t}^{c,\theta})^p\right] < \infty.$$

(ii) *On a suitable probability space, for* $i = 1, 2$, *we have*

$$Y_{i,t}^{\gamma,c,\theta} \overset{\gamma\to\infty}{\longrightarrow} X_{i,t}^{c,\theta} \qquad \text{in } L^p.$$

It can be seen from the proofs of Theorems 1.3 and 1.4 that the statements of these theorems also hold for $Y_0^{\gamma,c,\theta} = x \in [0, \infty)^2$ and $t > 0$ if we replace $X_0^{c,\theta}$ by a random points chosen according to Q_x.

Remark 1.5 (Trotter product approach). While in the one-colony case considered in the paper it is easy to write down explicitly the semigroup for the infinite rate mutually catalytic branching process $X^{c,\theta}$, it is less obvious how to construct an interacting version of the process on a countable site space. One possibility is the Trotter product approach that is used in [14] and [11]. Here we briefly sketch it for $X^{c,\theta}$.

In the classical setting the Trotter product approach works as follows. In order to construct a solution $Y^{\gamma,c,\theta}$ of (1.5), in time intervals of length ε one could alternate between a solution of the pure drift equation ($\gamma = 0$) and the pure stochastic noise equation ($c = 0$). As $\varepsilon \downarrow 0$ this process converges to a solution of (1.5).

If we let $\gamma \to \infty$, the noise term results in an instantaneous jump to a point in E chosen according to Q_y where y is the value of Y at the end of the preceding "drift interval". More formally, let $(\xi(k, x), k \in \mathbb{N}, x \in [0, \infty)^2)$ be an independent family of E valued random variables with distribution $\mathcal{L}[\xi(k, x)] = Q_x$. For $t \in [k\varepsilon, (k + 1)\varepsilon)$ let X_t^ε be the solution of the differential equation

$$dX_t^\varepsilon = c(\theta - X_t)dt,$$

that is,

$$X_t^\varepsilon = e^{-c(t-k\varepsilon)} X_{k\varepsilon}^\varepsilon + \left(1 - e^{-c(t-k\varepsilon)}\right)\theta.$$

Let

$$X_{(k+1)\varepsilon-}^\varepsilon := \lim_{t\uparrow(k+1)\varepsilon} X_t^\varepsilon = e^{-c\varepsilon} X_{k\varepsilon}^\varepsilon + (1 - e^{-c\varepsilon})\theta$$

and define

$$X_{(k+1)\varepsilon}^\varepsilon = \xi\left(k + 1, X_{(k+1)\varepsilon-}^\varepsilon\right).$$

One can prove that X^ε converges in distribution in the Skorohod topology on the space of càdlàg paths to $X^{c,\theta}$ (see [14] and [11]). ◇

While in Definition 1.1 we gave an explicit formula for the transition kernels of X, it is interesting to characterise the process X via its infinitesimal dynamics also. In Section 5 we investigate the generator $\bar{\mathcal{G}}$ of the semigroup $\mathcal{S}$. For a certain class $C_l^2(E) \subset C_l(E)$ of smooth functions f (see Definition 5.1), we give an explicit formula for $\bar{\mathcal{G}}f$ as an integro-differential operator. Using the classical Hille-Yoshida theorem, we show that the restricted operator $\mathcal{G} = \bar{\mathcal{G}}\big|_{C_l^2(E)}$ uniquely defines $(\mathcal{S}_t)_{t\geq 0}$ (Theorem 5.3). Furthermore, we show that $\mathcal{G}$ restricted to an even smaller space V of functions that appear in the duality for X still uniquely defines the process X via a martingale problem (Theorem 5.4). To define $\mathcal{G}$ it is crucial to study (for suitable functions f) the limit

$$\lim_{t\downarrow 0} t^{-1}(\mathcal{S}_t f(x) - f(x)) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\bigg(\int f\, dQ_{x+\varepsilon c(\theta - x)} - f(x)\bigg)$$

which will also clarify the jump structure of the process X. The description of the exact form of the operator $\mathcal{G}$ and of the precise statements of the theorems are a bit technical and are deferred to Section 5.

While for Proposition 1.2 we used general construction principles of Markov processes, here we provide an explicit strong construction of the process X in terms of a given planar Brownian motion B. This construction also allows the investigation of certain path properties.

Assume $B_0 = 0$. For $z \in \mathbb{R}^2$, we write

$$[z, \infty) = [z_1, \infty) \times [z_2, \infty)$$

for the rectangular cone north-east of z. For $x \in [0,\infty)^2$ let

$$\tau_x := \inf\big\{t > 0 : \, B_t \notin [-x, \infty)\big\} \tag{1.7}$$

and

$$D_x := B_{\tau_x} + x \, \in E. \tag{1.8}$$

For $x, y \in \mathbb{R}^2$, we write $y \leq x$ if $x \in [y, \infty)$, that is, if $y_1 \leq x_1$ and $y_2 \leq x_2$. For $x \in [0,\infty)^2$, define the σ-algebra

$$\mathcal{F}_x^D = \sigma(D_y : \, y \leq x). \tag{1.9}$$

We will show in Lemma 3.1 that D is a Markov process with respect to $(\mathcal{F}_x^D)_{x\in[0,\infty)^2}$.

Let $\bar{\theta} : [0,\infty) \to [0,\infty)^2$ and $\bar{c} : [0,\infty) \to [0,\infty)$ be measurable and locally integrable. For $0 \leq s \leq t$ define

$$C(s,t) = \exp\bigg(-\int_s^t \bar{c}(r)\, dr\bigg) \quad \text{ and } \quad \Xi(s,t) = \int_s^t \frac{\bar{\theta}(r)}{C(0,r)}\, dr. \tag{1.10}$$

Theorem 1.6. *Let $x \in E$ and define the process $X^{\bar{c},\bar{\theta}}$ by*

$$X_t^{\bar{c},\bar{\theta}} = C(0,t) D_{x+\Xi(0,t)}, \qquad t \geq 0.$$

Then $X^{\bar{c},\bar{\theta}}$ is a time-inhomogeneous Markov process on E with càdlàg paths and with transition probabilities

$$p_{s,t}(z, \boldsymbol{\cdot}) = Q_{C(s,t)z + C(0,t)\Xi(s,t)} \quad \text{for } \; 0 \leq s < t, \, z \in E. \tag{1.11}$$

In particular, for $\bar{\theta} \equiv \theta \in [0,\infty)^2$ and $\bar{c} \equiv c > 0$,

$$X_t^{c,\theta} = e^{-ct} D_{x+(e^{ct}-1)\theta} \tag{1.12}$$

is an infinite rate mutually catalytic branching process with parameter (c, θ).

It is tempting to use this strong construction of $X^{\bar{c},\bar{\theta}}$ in order to define an interacting version of the infinite rate mutually catalytic branching process on a countable site space S, where $c\theta_k(t)$ at site $k \in S$ reflects the migration from neighbouring sites to k. However, in this paper, we do not pursue this topic. Rather we use the strong construction in order to derive a path property of $X^{c,\theta}$ via a result of Le Gall and Meyre [12] on the cone points of planar Brownian motion.

Recall that a measurable set $A \subset E$ is called *polar* for $X^{c,\theta}$ if for all $x \in E$, we have

$$\mathbf{P}_x[X_t^{c,\theta} \in A \;\text{ for some }\; t > 0] = 0.$$

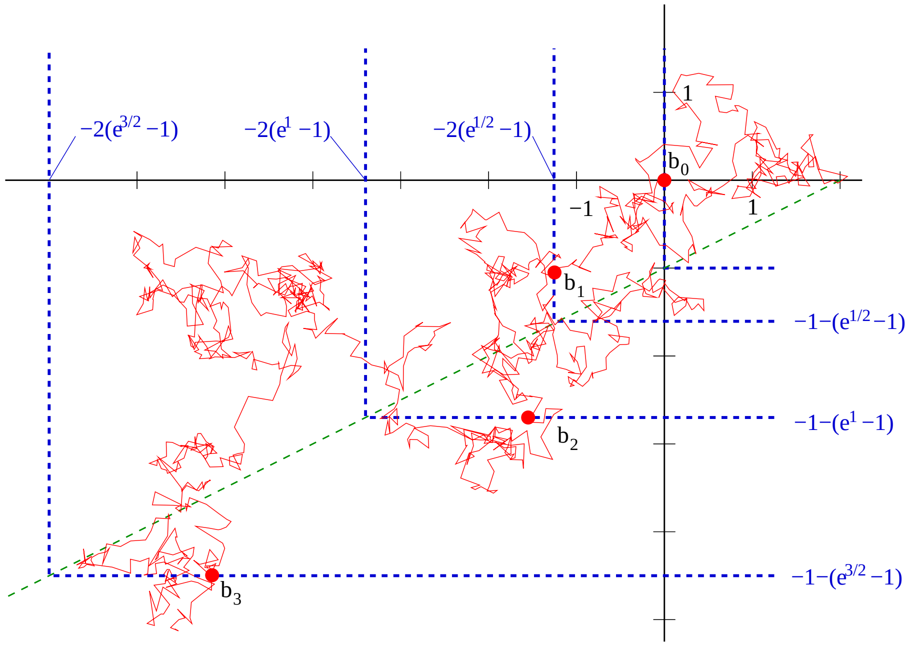

Figure 1: Strong construction of $X^{1/2,(2,1)}$ with $X_0 = x = (0,1)$ via a planar Brownian motion. Here $X_t^{1/2,(2,1)} = e^{-t/2}((0,1) + b_t + (2,1)(e^{t/2} - 1))$ for $t = 0,\ 1,\ 2,\ 3$.

Theorem 1.7. *The point $0 \in E$ is polar for $X^{c,\theta}$.*

1.3 Organisation of the paper

In Section 2, we give a detailed description of the duality for the process with finite branching rate. In Section 3, we establish a similar duality for the infinite rate process and use it in order to show the convergence in Theorems 1.3 and 1.4. In Section 4, we justify the strong construction of Theorem 1.6 and also prove Theorem 1.7. Finally, in Section 5, we describe the infinite rate process in terms of its infinitesimal dynamics and state and prove the theorem on the construction via the Hille-Yoshida theory (Theorem 5.3) and via a martingale problem (Theorem 5.4).

2 Duality of the finite γ process

A major tool for the investigation of mutually catalytic branching processes is a self-duality for the process. As it turns out to be crucial also for the limiting case of infinite branching rate ($\gamma = \infty$), we describe this duality here in more detail. For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$, we introduce the *lozenge product*

$$x \diamond y \ := \ -(x_1 + x_2)(y_1 + y_2) \ + \ i(x_1 - x_2)(y_1 - y_2) \tag{2.1}$$

(with $i = \sqrt{-1}$) and define

$$F(x,\, y) = \exp(x \diamond y). \tag{2.2}$$

Note that $x \diamond y = y \diamond x$. Furthermore, define the "scalar product"

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 \quad \text{ for } \ x, y \in [0, \infty)^2. \tag{2.3}$$

For $x=(x(k))_{k\in S}$ and $y=(y(k))_{k\in S}$, we write

$$H(x,\,y)=\exp\left(\sum_{k\in S}x(k)\diamond y(k)\right).$$

If Y is the process defined in (1.1) started in state y and $\tilde Y$ is the process started in some suitable $\tilde y$ (such that all sums are finite), then the duality reads (see [13, Equation (2.5)])

$$\mathbf{E}_y\big[H\big(Y_t,\tilde y\big)\big]=\mathbf{E}_{\tilde y}\big[H\big(y,\tilde Y_t\big)\big]. \tag{2.4}$$

In fact, this duality also holds for asymmetric $\mathcal{A}$ if $\tilde Y$ is a solution of (1.1) with $\mathcal{A}$ replaced by its transpose $\mathcal{A}^*$. As this mixed Laplace and Fourier transform H is measure determining ([13, Lemma 2.5]), the duality yields uniqueness of the solutions of (1.1). Furthermore, it provides a tool for translating local properties of the solutions into global properties and vice versa. If $x=(u,v)\in(0,\infty)^2$, then the harmonic measure Q_x (recall (1.2)) has a one-dimensional Lebesgue density on

$$E:=\big([0,\infty)\times\{0\}\big)\cup\big(\{0\}\times[0,\infty)\big)$$

that can be computed explicitly

$$Q_{(u,v)}\big(d(\bar u,\bar v)\big)=\begin{cases}\dfrac{4}{\pi}\,\dfrac{uv\,\bar u}{4u^2v^2+\big(\bar u^2+v^2-u^2\big)^2}\,d\bar u, & \text{if } \bar v=0,\\[2ex] \dfrac{4}{\pi}\,\dfrac{uv\,\bar v}{4u^2v^2+\big(\bar v^2+u^2-v^2\big)^2}\,d\bar v, & \text{if } \bar u=0.\end{cases} \tag{2.5}$$

Furthermore, trivially we have

$$Q_x=\delta_x\quad\text{if }\ x\in E. \tag{2.6}$$

We now turn to the situation of only one colony. We consider the solution $Z=(Z_1,Z_2)$ of

$$dZ_{i,t}=\sqrt{\gamma Z_{1,t}Z_{2,t}}\,dW_{i,t},\qquad i=1,2,\quad Z_0=z\in[0,\infty)^2. \tag{2.7}$$

By Theorem 1 of [4], there is the unique strong solution to the above equation.

Clearly, Z_1 and Z_2 are orthogonal L^2-martingales and hence they converge almost surely to some random variable $Z_\infty=(Z_{1,\infty},Z_{2,\infty})$. As Z is an isotropic diffusion on $[0,\infty)^2$, it is a time-transformed Brownian motion. Thus Z_∞ has the same distribution as a planar Brownian motion B started in z and stopped (at time τ) upon leaving $(0,\infty)^2$, that is (see (2.5)),

$$\mathcal{L}_z[Z_\infty]=\mathcal{L}_z[B_\tau]=Q_z.$$

(We denote by $\mathcal{L}_x[X_t]=\mathbf{P}_x[X_t\in\,\cdot\,]=\mathbf{P}[X_t\in\,\cdot\,|\,X_0=x]$ the distribution of the process X at time t when started at x.) It is easy to see that in fact

$$\tau^Z:=\inf\{t>0:Z_t\in E\}<\infty\quad\text{almost surely}$$

and that

$$Z_t=Z_{\tau^Z}\qquad\text{for all }\ t>\tau^Z.$$

Clearly, increasing γ amounts to speeding up the process. Hence in the limit we would have a process that instantaneously jumps from z to a random point (picked according to Q_z) and then stays there. In order to obtain a more interesting limiting process, and with a view towards interacting colonies, we introduce a drift term and consider the equation (which was analysed in some more detail in [2, Proposition 1.1 and 1.2])

$$dY_{i,t}=c\big(\theta_i-Y_{i,t}\big)\,dt\;+\;\sqrt{\gamma Y_{1,t}Y_{2,t}}\,dW_{i,t},\quad i=1,2. \tag{2.8}$$

Here $c \geq 0$ and $\theta \in [0,\infty)^2$ are parameters of the process. It is standard to show that (2.8) has a weak solution. Weak uniqueness can be obtained via duality. We first outline the general picture for the duality that comes from the interacting colonies case and then give an explicit computation for our special situation.

Let us consider a two colonies model with site space $S = \{1, 2\}$ where Y is the size of the population at site 1 and the size of the population at site 2 is constant and equals θ. This amounts to a migration matrix

$$\mathcal{A} = \begin{pmatrix} -c & c \\ 0 & 0 \end{pmatrix} \tag{2.9}$$

and to branching rates $\gamma(1) = \gamma$ (at site 1) and $\gamma(2) = 0$ (at site 2). Note that the approach of Dawson and Perkins does not require that the branching rate be constant nor that the migration matrix be symmetric nor a q-matrix. (At least if S is finite, otherwise certain regularity conditions have to be imposed.) Dawson and Perkins use a duality with respect to a process $\tilde{Y}$ with migration matrix $\mathcal{A}^*$ (the transpose of $\mathcal{A}$) and with the same branching rates as Y to show weak uniqueness of Y.

Let us now construct the dual process explicitly. We will later use this approach in order to construct a dual for the $\gamma = \infty$ limiting process. Let $\tilde{y} = (\tilde{y}(1), \tilde{y}(2)) \in ([0,\infty)^2)^2$ and let Z be the unique strong (by Theorem 1 of [4]) $[0,\infty)^2$-valued solution of

$$dZ_{i,t} = \sqrt{\gamma Z_{1,t} Z_{2,t}}\, dW_{i,t}, \qquad i = 1, 2, \quad Z_0 = \tilde{y}(1). \tag{2.10}$$

Define a process $\tilde{Y}$ on $([0,\infty)^2)^2$ by

$$\tilde{Y}_t(1) = e^{-ct} Z_t \quad \text{ and } \quad \tilde{Y}_t(2) = \tilde{y}(2) + \int_0^t c\, e^{-cr} Z_r\, dr. \tag{2.11}$$

Note that this $\tilde{Y}$ is a solution of (1.1) with $S = \{1, 2\}$ and with site dependent branching rate $\gamma(1) = \gamma$, $\gamma(2) = 0$ and with $\mathcal{A}$ from (2.9) replaced by $\mathcal{A}^*$. In particular, $\tilde{Y}$ is a time-homogeneous Markov process. We get the time-homogeneous Markov property also by an explicit computation:

$$\begin{aligned} \tilde{Y}_{t+s} &= \Big(e^{-c(t+s)} Z_{t+s},\, \tilde{y}(2) + \int_0^{t+s} c\, e^{-cr} Z_r\, dr \Big) \\ &= \Big(e^{-cs}\, (e^{-ct} Z_{t+s}),\, \tilde{y}(2) + \int_0^t c\, e^{-cr} Z_r\, dr \,+\, \int_0^s c\, e^{-cr} (e^{-ct} Z_{t+r})\, dr \Big) \\ &= \Big(e^{-cs}\, Z'_s,\, \tilde{y}'(2) + \int_0^s c\, e^{-cr} Z'_r\, dr \Big), \end{aligned}$$

where $Z'_r = e^{-ct} Z_{t+r}$ and $\tilde{y}'(2) = \tilde{Y}_t(2) = \tilde{y}(2) + \int_0^t c\, e^{-cr} Z_r\, dr$. Clearly, Z' has the distribution of a solution of (2.7) with $\tilde{y}'(1) := Z'_0 = \tilde{Y}_t(1)$.

For $x, x', y, y' \in [0,\infty)^2$, recall that

$$H\big((x, x'), (y, y')\big) = F(x, y)\, F(x', y'). \tag{2.12}$$

Proposition 2.1 (Duality)**.** *Let Y and $\tilde{Y}$ be defined by (2.8) and (2.11), respectively. Then for all $y \in [0,\infty)^2$, $\tilde{y} \in ([0,\infty)^2)^2$ and $t \geq 0$ we have*

$$\mathbf{E}_y\big[H\big((Y_t, \theta), \tilde{y}\big)\big] = \mathbf{E}_{\tilde{y}}\big[H\big((y, \theta), \tilde{Y}_t\big)\big]. \tag{2.13}$$

In particular, if Z is a solution of (2.10) with $Z_0 = z \in [0,\infty)^2$, then

$$\mathbf{E}_y[F(Y_t, z)] = \mathbf{E}_z\Big[F(y, e^{-ct} Z_t)\, F\Big(\theta, \int_0^t c\, e^{-cr} Z_r\, dr\Big)\Big]. \tag{2.14}$$

A similar duality was derived in [2, Lemma 4.2]. Before we prove the proposition, we have to collect some properties of the derivatives of F. We omit the proof of the following lemma.

Lemma 2.2 (Derivatives of the duality function)**.** *Denote the partial derivatives of F and the Laplace operator by*

$$\nabla_1 F(x,y) := \frac{d}{dx} F(x,y), \qquad \nabla_2 F(x,y) := \frac{d}{dy} F(x,y)$$

and

$$\Delta_1 F(x,y) := \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] F(x,y), \qquad \Delta_2 F(x,y) := \left[\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right] F(x,y).$$

Then for all $x, y, z \in [0,\infty)^2$, we have (recall (2.1) and (2.3))

$$\begin{aligned} \big\langle z,\, \nabla_1 F(x,y)\big\rangle &= (z \diamond y)\, F(x,y), \\ \big\langle z,\, \nabla_2 F(x,y)\big\rangle &= (z \diamond x)\, F(x,y), \\ \Delta_1 F(x,y) &= 8 y_1 y_2 F(x,y), \\ \Delta_2 F(x,y) &= 8 x_1 x_2 F(x,y). \end{aligned}$$

Proof of Proposition 2.1. We use Itô's formula and Lemma 2.2 to compute the derivatives of both sides in (2.13) at $t = 0$:

$$\begin{aligned} \frac{d}{dt}\mathbf{E}_y\big[H\big((Y_t,\theta),\tilde{y}\big)\big]\big|_{t=0} &= \big\langle c(\theta - y),\, \nabla_1 F(y,\tilde{y}(1))\big\rangle\, F(\theta,\tilde{y}(2)) \\ &\quad + \frac{1}{2}\gamma\, y_1 y_2\, \Delta_1 F(y,\tilde{y}(1))\, F(\theta,\tilde{y}(2)) \\ &= H\big((y,\theta),\tilde{y}\big)\, \big[c(\theta - y)\diamond \tilde{y}(1) \,+\, 4\gamma y_1 y_2 \tilde{y}_1(1)\tilde{y}_2(1)\big] \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \frac{d}{dt}\mathbf{E}_{\tilde{y}}\big[H\big((y,\theta),\tilde{Y}_t\big)\big]\big|_{t=0} &= \Big(\big\langle -c\,\tilde{y}(1),\, \nabla_2 F(y,\tilde{y}(1))\big\rangle + \frac{\gamma}{2}\tilde{y}_1(1)\tilde{y}_2(1)\Delta_2 F(y,\tilde{y}(1))\Big)\, F(\theta,\tilde{y}(2)) \\ &\quad + F(y,\tilde{y}(1))\, \big\langle c\tilde{y}(1),\, \nabla_2 F(\theta,\tilde{y}(2))\big\rangle \\ &= H\big((y,\theta),\tilde{y}\big)\, \big[c(\theta - y)\diamond \tilde{y}(1) \,+\, 4\gamma y_1 y_2 \tilde{y}_1(1)\tilde{y}_2(1)\big]. \end{aligned} \tag{2.16}$$

Since both derivatives coincide, (2.13) holds (see Corollary 4.4.13 of [6] with $\alpha = \beta = 0$). Equation (2.14) is a direct consequence of (2.13). □

Corollary 2.3. *Recall Z from (2.10).*
(i) *Taking $c = 0$, Proposition 2.1 implies that Z is self-dual:*

$$\mathbf{E}_x\big[F\big(Z_t, y\big)\big] = \mathbf{E}_y\big[F\big(x, Z_t\big)\big] \quad \textit{for all } x,y \in [0,\infty)^2,\ t \geq 0.$$

(ii) *Letting $t \to \infty$ in (i) and recalling $\mathcal{L}_x\big[Z_t\big] \overset{t\to\infty}{\longrightarrow} Q_x$, we get by dominated convergence the duality relation for the harmonic measure*

$$\int_E F(z,y)\, Q_x(dz) = \int_E F(x,z)\, Q_y(dz) \quad \textit{for all } x,y \in [0,\infty)^2.$$

(iii) *In particular (since $Q_x = \delta_x$ for $x \in E$ and due to the symmetry of F), we have*

$$\int_E F(x,z)\, Q_y(dz) = F(x,y) = F(y,x) = \int_E F(z,x)\, Q_y(dz) \quad \textit{for all } x \in E,\ y \in [0,\infty)^2.$$

Corollary 2.4. *(i) The family of functions $\mathcal{F}_0 = \{[0,\infty)^2 \to \mathbb{C} : x \mapsto F(x,y),\ y \in [0,\infty)^2\}$ is measure determining for $[0,\infty)^2$.*

(ii) The vector space

$$V := \left\{ \sum_{m=1}^{n} \lambda_m F(\cdot, z_m) : n \in \mathbb{N},\ \lambda_1, \dots, \lambda_n \in \mathbb{C},\ z_1, \dots, z_n \in E \right\} \tag{2.17}$$

spanned by $\mathcal{F} := \{E \to \mathbb{C} : x \mapsto F(x,z),\ z \in E\}$ *is dense in* $C_l(E)$*. In particular,* $\mathcal{F}$ *is measure determining for* E*.*

Proof. Let $\mathcal{D}_0$ be the algebra generated by $\mathcal{F}_0$. Clearly, $\mathcal{F}_0$ separates points of $[0,\infty)^2$, contains $1 = F(\cdot, 0)$ and is closed under multiplication and under complex conjugation since $\overline{F(x,(y_1,y_2))} = F(x,(y_2,y_1))$. Hence by the Stone-Weierstraß theorem, $\mathcal{D}_0$ is dense in the space $C_l([0,\infty)^2)$ of functions $[0,\infty)^2 \to \mathbb{C}$ that are continuous and have a limit at infinity. As $\mathcal{F}_0$ is closed under multiplication, $\mathcal{D}_0$ is the vector space spanned by $\mathcal{F}_0$, and thus $\mathcal{F}_0$ is measure determining on $[0,\infty)^2$.

Let $\mathcal{F}_E = \{f\big|_E : f \in \mathcal{F}_0\} \supset \mathcal{F}$ and let $\mathcal{D}_E = \{f\big|_E : f \in \mathcal{D}_0\}$ denote the algebra generated by $\mathcal{F}_E$. By the above argument, $\mathcal{D}_E \subset C_l(E)$ is dense. Now by Corollary 2.3(iii), an element $F(\cdot, y) \in \mathcal{F}_E$ can be written as the integral $F(x,y) = \int F(x,z) Q_y(dz)$ where the integrand functions are in $\mathcal{F}$. The integral can be approximated (uniformly in x) by finite sums, that is, by elements of V. Hence V is dense in $\mathcal{D}_E$ and thus also in $C_l(E)$. $\square$

Apparently, Y is ergodic and has a unique invariant distribution with a Lebesgue density on $(0,\infty)^2$. Unlike for the analogous one-dimensional equation

$$dU_t = c(b - U_t)dt + \sqrt{\gamma U_t}\, dW_t,$$

where the invariant distribution is known to be the Gamma distribution $\Gamma_{2c/\gamma, 2cb/\gamma}$, here the explicit form of the density is unknown. It is known (see, e.g., [7, Example IV.8.2, page 237]) that U hits 0 if and only if $2cb/\gamma < 1$. Hence we may expect that $Y = Y^{\gamma,c,\theta}$ hits E only at $((2c\theta_2/\gamma, \infty) \times \{0\}) \cup (\{0\} \times (2c\theta_1/\gamma, \infty))$. Compare this with the fact that $0 \in E$ is not hit by the infinite γ process $X^{c,\theta}$ (see Theorem 1.7).

3 Convergence as $\gamma \to \infty$, Proofs of Theorems 1.3 and 1.4

3.1 Construction of the process

Recall the definition of p_t, $\mathcal{S}$ and $X^{c,\theta}$ in Definition 1.1. In order that the definition makes sense, we still have to show in Proposition 3.2 below that p_t is indeed a Markov kernel and that the Chapman-Kolmogorov equation holds. We prepare for Proposition 3.2 with a lemma.

Recall the definition of C, Ξ, D and $\mathcal{F}^D$ in (1.8), (1.9) and (1.10).

Lemma 3.1. *(i)* D *has the Markov property, that is, for* $x, y \in [0,\infty)^2$ *and* $A \subset E$ *measurable, we have*

$$\mathbf{P}[D_{x+y} \in A \,|\, \mathcal{F}^D_x] = Q_{y+D_x}(A).$$

(ii) For $f : E \to \mathbb{C}$ *bounded and measurable and* $r \geq 0$*, we have*

$$\int_E f(rz)\, Q_x(dz) = \int_E f dQ_{rx}.$$

(iii) Furthermore,

$$\int_E Q_x(dz)\, Q_{rz+y} \;=\; Q_{rx+y}.$$

Proof. **(i)** Let $\mathcal{F}^B$ denote the filtration generated by the Brownian motion B and let $\mathcal{F}^B_{\tau_x}$ denote the σ-algebra of the τ_x past of B (recall (1.7)). Note that $\mathcal{F}^B_{\tau_x} \supset \mathcal{F}^D_x$.

For $x' \in [0,\infty)^2$, denote by $\mathbf{P}_{-x'}$ the law of B when started in $B_0 = -x'$. Hence by spatial homogeneity, for $x' \le x$, we have

$$\mathbf{P}_{-x'}\big[B_{\tau_{x+y}} + (x+y) \in A\big] = Q_{y+(x-x')}(A).$$

Choosing $x' = -B_{\tau_x}$, we infer

$$\mathbf{P}_{B_{\tau_x}}\big[B_{\tau_{x+y}} + (x+y) \in A\big] = Q_{y+D_x}(A).$$

Now we apply the strong Markov property of B to obtain

$$\begin{aligned}\mathbf{P}[D_{x+y} \in A \,|\, \mathcal{F}^D_x] &= \mathbf{E}\big[\mathbf{P}_0[B_{\tau_{x+y}} + (x+y) \in A \,|\, \mathcal{F}^B_{\tau_x}]\,\big|\,\mathcal{F}^D_x\big] \\ &= \mathbf{E}\big[\mathbf{P}_{B_{\tau_x}}[B_{\tau_{x+y}} + (x+y) \in A]\,\big|\,\mathcal{F}^D_x\big] \\ &= \mathbf{E}\big[Q_{y+D_x}(A)\,\big|\,\mathcal{F}^D_x\big] = Q_{y+D_x}(A).\end{aligned}$$

(ii) This follows from spatial homogeneity of B.

(iii) Recall that D_{rx} has distribution Q_{rx}. Hence by (ii) and (i), we get

$$\int_E Q_x(dz)\, Q_{rz+y}(A) \;=\; \int_E Q_{rx}(dz)\, Q_{z+y}(A) = \mathbf{E}[Q_{y+D_{rx}}(A)] = \mathbf{P}[D_{rx+y} \in A] = Q_{rx+y}(A). \qquad \square$$

Proposition 3.2. $(\mathcal{S}_t)_{t\ge 0}$ *defined in Definition 1.1 is a Markov semigroup.*

Proof. Recall that $x \mapsto Q_x$ is a continuous map. Hence for open sets $A \subset E$, the map $x \mapsto Q_x(A)$ is lower semicontinuous by the Portemanteau theorem (see, e.g., [8, Theorem 13.16]) and is hence measurable. Hence $x \mapsto Q_x(A)$ is measurable for all Borel sets $A \subset E$. It remains to check the Chapman-Kolmogorov equation for (p_t). By Lemma 3.1(iii) we infer

$$\begin{aligned}\int_E p_t(x,dy)\, p_s(y,\,\boldsymbol{\cdot}\,) &= \int_E Q_{e^{-ct}x+(1-e^{-ct})\theta}(dy)\, Q_{e^{-cs}y+(1-e^{-cs})\theta} \\ &= Q_{e^{-c(t+s)}x+e^{-cs}(1-e^{-ct})\theta+(1-e^{-cs})\theta} \\ &= Q_{e^{-c(t+s)}x+(1-e^{-c(t+s)})\theta} \\ &= p_{t+s}(x,\,\boldsymbol{\cdot}\,). \qquad \square\end{aligned}$$

3.2 Duality and proof of the fdd convergence, Theorem 1.3

In this section we prove the convergence of the finite dimensional distributions of $Y^{\gamma,c,\theta}$ to those of $X = X^{c,\theta}$ by means of a duality relation. For $Y^{\gamma,c,\theta}$, we have established the duality already in Proposition 2.1. Now we come to the duality for X. Recall the definition of $\tilde{Y}$ from (2.11). We will need as initial values only $\tilde{y} \in E \times [0,\infty)^2$. Note that in this case, the process Z is constant in time and the process $\tilde{Y}$ is given by the deterministic equation

$$\tilde{Y}_t = \big(e^{-ct}\tilde{y}(1),\, (1-e^{-ct})\tilde{y}(1) + \tilde{y}(2)\big). \tag{3.1}$$

Hence $\tilde{Y}$ can be understood as a deterministic Markov process with state space $E \times [0,\infty)^2$. Recall H from (2.12) and F from (2.2).

Proposition 3.3. *X and $\tilde{Y}$ are dual in the sense that for all initial conditions $X_0 = x \in E$ and $\tilde{Y}_0 = \tilde{y} \in E \times [0,\infty)^2$ and for all $t \geq 0$, we have*

$$\mathbf{E}_x\big[H\big((X_t,\theta),\tilde{y}\big)\big] = \mathbf{E}_{\tilde{y}}\big[H\big((x,\theta),\tilde{Y}_t\big)\big]. \tag{3.2}$$

In particular, we get

$$\mathbf{E}_x[F(X_t,z)] = F\big(x,e^{-ct}z\big)\,F\big(\theta,\big(1-e^{-ct}\big)z\big) \quad \text{for } \ x\in[0,\infty)^2, z\in E, \tag{3.3}$$

and the distribution of X_t is determined by (3.3).

Proof. As $\tilde{Y}$ is deterministic, (3.2) and (3.3) are equivalent and hence we show (3.3) only. Since $z \in E$, by Corollary 2.3(iii), the left hand side of (3.3) equals

$$\begin{aligned}\int_E F(y,z)\,Q_{e^{-ct}x+(1-e^{-ct})\theta}(dy) &= F\big(e^{-ct}x+(1-e^{-ct})\theta,z\big)\\ &= F\big(x,e^{-ct}z\big)\,F\big(\theta,(1-e^{-ct})z\big).\end{aligned}$$

By Corollary 2.4, equation (3.3) determines the distribution of X_t. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. As both $X^{c,\theta}$ and $Y^{\gamma,c,\theta}$ are Markov processes, it is easy to see that for convergence of finite dimensional distributions, it is enough to show that for any $t \geq 0$, $x \in E$ and $(x_\gamma)_{\gamma\geq 0}$ in $[0,\infty)^2$ such that $\lim_{\gamma\to\infty} x_\gamma \to x$, we have

$$\mathcal{L}_{x_\gamma}\big[Y_t^{\gamma,c,\theta}\big] \overset{\gamma\to\infty}{\longrightarrow} \mathcal{L}_x\big[X_t^{c,\theta}\big] \quad \text{weakly.} \tag{3.4}$$

As shown in the proof of Corollary 2.4(i), $\mathcal{D}_0$ is dense in $C_l([0,\infty)^2)$. Hence it is enough to consider $F(\,\cdot\,,z)$, $z \in [0,\infty)^2$, as test functions. Denote by Z^γ the process defined in (2.10) started in $Z_0^\gamma = z$. For $\gamma = 1$ we drop the superscript, that is, $Z := Z^1$. Denote by Z_∞ the almost sure limit of Z_t as $t \to \infty$ and recall that its distribution is Q_z. Note that due to Brownian scaling $(Z_t^\gamma)_{t\geq 0} \overset{\mathcal{D}}{=} (Z_{\gamma t})_{t\geq 0}$. Hence by Proposition 2.1, we have

$$\begin{aligned}\mathbf{E}_{x_\gamma}\big[F\big(Y_t^{\gamma,c,\theta},z\big)\big] &= \mathbf{E}_z\Big[F\big(x_\gamma,\,e^{-ct}Z_{\gamma t}\big)F\Big(\theta,\,\int_0^t c\,e^{-cr}Z_{\gamma r}\,dr\Big)\Big]\\ &\overset{\gamma\to\infty}{\longrightarrow} \mathbf{E}_z\big[F\big(x,\,e^{-ct}Z_\infty\big)F\big(\theta,\,(1-e^{-ct})Z_\infty\big)\big]\\ &= \int_E F\big(x,e^{-ct}y\big)F\big(\theta,\,(1-e^{-ct})y\big)\,Q_z(dy)\\ &= \int_E \mathbf{E}_x[F(X_t,y)]\,Q_z(dy)\\ &= \mathbf{E}_x\Big[\int_E F(X_t,y)\,Q_z(dy)\Big]\\ &= \mathbf{E}_x\big[F(X_t,z)\big],\end{aligned}$$

where the fourth line follows by (3.3) and the last equality follows by Corollary 2.3(iii). □

Remark 3.4. We could define $X^{c,\theta}$ in Definition 1.1 also for initial values $x \in [0,\infty)^2$ (instead of E only). This means that $X^{c,\theta}$ starts life with a jump from x to a random point on E chosen according to Q_x and then continues with the usual dynamics. Clearly, this process does not have a càdlàg version (due to the jump at time 0) and its transition semigroup is not strongly continuous at 0. Nevertheless, the proof of Theorem 1.3 shows that that theorem also holds for this process and hence for $Y_0^{\gamma,c,\theta} = X_0^{c,\theta} = x \in [0,\infty)^2$. ◇

3.3 Proof of the L^p-convergence, Theorem 1.4

We prepare for the proof of Theorem 1.4 with two lemmas.

Lemma 3.5. *Let $B = (B_1, B_2)$ be a planar Brownian motion started in $(B_{1,0}, B_{2,0}) = (u, v) \in [0,\infty)^2$ and let*

$$\tau = \inf\{t > 0 : B_t \notin (0,\infty)^2\}.$$

Then for any $p \in [1, 2)$, we have

$$\mathbf{E}\big[\tau^{p/2}\big] \le \frac{2}{2-p}\left(\frac{2}{\pi}\right)^{p/2}(uv)^{p/2} \;<\; \infty.$$

More generally, one could show for the exit time of a cone with angle 2α (here: $\alpha = \pi/4$) that $\mathbf{E}[\tau^{p/2}] < \infty$ if and only if $p\alpha < \pi/2$ (see [1, equation (3.8)]). We give the short proof here in order to be self contained.

Proof. By the reflection principle and independence of B_1 and B_2, we get

$$\mathbf{P}[\tau > t] = 4\mathcal{N}_{0,t}(0, u)\mathcal{N}_{0,t}(0, v),$$

where $\mathcal{N}_{0,t}(a, b) = (2\pi t)^{-1/2}\int_a^b e^{-r^2/2t}dr$ is the centred normal distribution with variance t. Hence

$$\mathbf{E}\big[\tau^{p/2}\big] = \int_0^\infty \mathbf{P}\big[\tau > t^{2/p}\big]\,dt \le \int_0^\infty 1 \wedge \Big(\frac{2}{\pi}\,uv\,t^{-2/p}\Big)\,dt = \frac{2}{2-p}\left(\frac{2}{\pi}\right)^{p/2}(uv)^{p/2}. \qquad \square$$

Lemma 3.6. *For every $(u, v) \in [0,\infty)^2$, every $p \in [1, 2)$ and every $i = 1, 2$, we have*

$$\int_E x_i^p\, Q_{(u,v)}(dx) \le \big|u^2 - v^2\big|^{p/2} + \frac{2^{p/2}(uv)^{p/2}}{\cos(p\pi/4)} < \infty.$$

Proof. This can be verified by an explicit computation using the density formula of $Q_{(u,v)}$ in (2.5). $\square$

Note that finiteness of the expression on the left hand side in Lemma 3.6 (which is what we need in the proof of Theorem 1.4) could be inferred also without computations by the Burkholder-Davis-Gundy inequality and Lemma 3.5.

Proof of Theorem 1.4. **(i)** By Lemma 3.6, we have

$$\mathbf{E}\big[(X^{c,\theta}_{i,t})^p\big] = \int_E y_i^p\, Q_{e^{-ct}x+(1-e^{-ct})\theta}(dy) \;<\infty.$$

Fix $t > 0$ and define

$$M^t_{i,s} := e^{-ct}x_i + (1 - e^{-ct})\theta_i + \int_0^s e^{c(r-t)}\sqrt{\gamma Y^{\gamma,c,\theta}_{1,r} Y^{\gamma,c,\theta}_{2,r}}\, dW_{i,r}.$$

Let $\langle M^t_1\rangle = \langle M^t_2\rangle$ denote the square variation process of both M^t_1 and M^t_2. Note that $M^t_{i,t} = Y^{\gamma,c,\theta}_{i,t} \ge 0$ and that M^t_i is a martingale and thus

$$M^t_{i,s} = \mathbf{E}[M^t_{i,t}\,|\,M^t_{i,s}] \ge 0 \quad \text{for all } s \in [0, t]. \tag{3.5}$$

Now $(M^t_s)_{s\ge0}$ is an isotropic diffusion in $\mathbb{R}^2$ and is hence a time-transformed planar Brownian motion. That is, there exists a planar Brownian motion B (with respect to some right-continuous complete filtration $\mathcal{F}$) started in $B_0 = e^{-ct}x + (1-e^{-ct})\theta$ such that each $\langle M^t_{1,\,\cdot}\rangle_s$ is an $\mathcal{F}$ stopping time and such that $B_{\langle M^t_{1,\,\cdot}\rangle_s} = M^t_s$ for all $s \geq 0$.

Define the $\mathcal{F}$ stopping times

$$\tau := \inf\big\{s > 0 :\ B_s \notin (0,\infty)^2\big\} \quad \text{and} \quad \tau_0 := \inf\big\{s > 0 :\ B_s \notin [0,\infty)^2\big\}.$$

Clearly, we have $\tau = \tau_0$ almost surely, and hence by (3.5)

$$\langle M^t_{1,\,\cdot}\rangle_t \leq \tau_0 = \tau \quad \text{a.s.}$$

Using the Burkholder-Davis-Gundy inequality for the martingale $(B_{i,s})_{s\geq 0}$ yields (see Lemma 3.5)

$$\mathbf{E}\Big[\sup_{s\leq\tau} B^p_{i,s}\Big] \leq 2^{p-1}\big(B^p_{i,0} + (4p)^p\,\mathbf{E}[\tau^{p/2}]\big) < \infty.$$

Hence $(|B_{i,\tau\wedge s}|^p)_{s\geq 0}$ is uniformly integrable and we can apply the optional sampling theorem to obtain

$$\mathbf{E}\big[(Y^{\gamma,c,\theta}_{i,t})^p\big] = \mathbf{E}\big[(B_{i,\langle M^t_{1,\,\cdot}\rangle_t})^p\big] \leq \mathbf{E}\big[(B_{i,\tau})^p\big] = \mathbf{E}\big[(X^{c,\theta}_{i,t})^p\big].$$

(ii) By Theorem 1.3 and the Skorohod embedding theorem, we may construct all processes on one probability space such that $Y^{\gamma,c,\theta}_t \to X^{c,\theta}_t$ almost surely as $\gamma\to\infty$. By Part (i), the pth moments of $Y^{\gamma,c,\theta}_{i,t}$, $\gamma \geq 0$, are uniformly integrable and hence we have the desired L^p-convergence. □

4 The strong construction, Proofs of Theorems 1.6 and 1.7

Recall the definition of C, Ξ and D in (1.8) and (1.10).

Lemma 4.1. *The map $x \mapsto D_x$ is càdlàg.*

Proof. This follows from continuity of B and the definition of τ_x. □

[Proof of Theorem 1.6.] From Lemma 3.1 and 4.1 we infer that $X^{\bar c,\bar\theta}$ has the Markov property and has càdlàg paths. It remains to show (1.11).

By Lemma 3.1, for $x, z \in E$, $A \subset E$ measurable and $0 \leq s < t$, we have (with $\mathbf{P}_x$ denoting the probability law of $X^{\bar c,\bar\theta}_t$ as defined in Theorem 1.6)

$$\begin{aligned} p_{s,t}(z,A) &= \mathbf{P}_x\big[X^{\bar c,\bar\theta}_t \in A \,\big|\, X^{\bar c,\bar\theta}_s = z\big] \\ &= \mathbf{P}\big[C(0,t)\,D_{x+\Xi(0,t)} \in A \,\big|\, D_{x+\Xi(0,s)} = C(0,s)^{-1}z\big] \\ &= Q_{C(0,s)^{-1}z+\Xi(s,t)}\big(C(0,t)^{-1}A\big) \\ &= Q_{C(s,t)z+C(0,t)\Xi(s,t)}\big(A\big). \end{aligned}$$

□

Proof of Theorem 1.7. If $c\theta = 0$, then $X^{c,\theta}$ is the deterministic process $X^{c,\theta}_t = e^{-ct}X^{c,\theta}_0$ and hence 0 is polar.

Now assume that $c\theta \neq 0$. Le Gall and Meyre [12] show that almost surely, for all $z \in (0,\infty)^2$, the planar Brownian motion B does not leave the cone $[-z,\infty)$ first at $-z$. More formally, consider the event

$$A := \big\{B_{\tau_z} \neq -z \ \text{ for all } \ z \in (0,\infty)^2\big\}.$$

Then Theorem 1 of [12] implies that $\mathbf{P}[A] = 1$ (in fact, they show that *no* rectangular cone is first left at its vertex, not only north-east cones $[z, \infty)$). Now, by (1.12), we have

$$\begin{aligned}\big\{X_t^{c,\theta} \neq 0 \ \text{ for all } \ t > 0\big\} &= \big\{D_{x+r\theta} \neq 0 \ \text{ for all } \ r > 0\big\} \\ &= \big\{B_{\tau_{x+r\theta}} \neq x + r\theta \ \text{ for all } \ r > 0\big\} \supset A.\end{aligned}$$

This shows the claim of Theorem 1.7. □

5 The infinitesimal dynamics of $X^{c,\theta}$

In this section we give a description and construction of the infinite rate mutually catalytic branching process X in terms of its infinitesimal characteristics. To this end, we will define a linear operator $\mathcal{G}^{c,\theta}$ that

(i) defines the contraction semigroup of X in the sense of the Hille-Yoshida theory (Theorem 5.3)

(ii) defines a well-posed martingale problem whose unique solution is X (Theorem 5.4).

5.1 Results

Recall from Definition 1.1 that the linear operator $\mathcal{S}_t$ on $C_l(E)$ is defined by

$$\mathcal{S}_t f(x) := \int_E f(y)\, p_t(x, dy) = \int_E f(y)\, Q_{e^{-ct}x+(1-e^{-ct})\theta}(x, dy).$$

In order to define the generator of $\mathcal{S} = (\mathcal{S}_t)_{t\geq 0}$, we will need to study (for suitable functions f) the limit

$$\lim_{t\downarrow 0} t^{-1}(\mathcal{S}_t f(x) - f(x)) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\bigg(\int f\, dQ_{x+\varepsilon c(\theta - x)} - f(x)\bigg). \tag{5.1}$$

In the sequel we will use the short hand notation

$$\partial_1 f(u, v) := \frac{\partial}{\partial u} f(u, v) \quad \text{ and } \quad \partial_2 f(u, v) := \frac{\partial}{\partial v} f(u, v).$$

In order to define what we mean by a suitable function, we introduce the subspace $C_l^2(E) \subset C_l(E)$.

Definition 5.1. *Let $C_l^2(E) \subset C_l(E)$ be the subspace of such functions $f \in C_l(E)$*

(i) whose partial derivatives $\partial_1 f$ and $\partial_2 f$ exist on $(0, \infty) \times \{0\}$ and $\{0\} \times (0, \infty)$, respectively, are continuous, can continuously be extended to $\{0\} \times [0, \infty)$, and fulfill

$$\lim_{u\to\infty} u\partial_1 f(u, 0) = \lim_{v\to\infty} v\partial_2 f(0, v) = 0, \tag{5.2}$$

(ii) and whose partial second derivatives $\partial_1^2 f$ and $\partial_2^2 f$ exist on $(0, \infty) \times \{0\}$ and $\{0\} \times (0, \infty)$, respectively, and are such that

$$\|f\|_{2,\infty} := \sup_{r\in(0,\infty)} r\big(|\partial_1^2 f(r, 0)| + |\partial_2^2 f(0, r)|\big) < \infty. \tag{5.3}$$

Note that for $f \in C_l^2(E)$, we have

$$\|f\|_{1,\infty} := \sup_{r\in[0,\infty)} \big(|\partial_1 f(r, 0)| + |\partial_2 f(0, r)|\big) < \infty. \tag{5.4}$$

In order to get an explicit formula for the limit in (5.1), we define the vague limits (for $u, v > 0$)

$$\nu_{(0,v)} := \text{v-}\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} Q_{(\varepsilon,v)} \quad \text{and} \quad \nu_{(u,0)} := \text{v-}\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} Q_{(u,\varepsilon)}.$$

$\nu_{(u,0)}$ can be thought of as the "Lévy measure" of the next jump when the actual position is $(u, 0)$ and the drift is in direction of $(0, 1)$. In order to formalise this, for the drift in direction $(0, 1)$, we define the linear operator $\mathcal{G}_2$ on $C_l^2(E)$ by $\mathcal{G}_2 f(x) = \partial_2 f(x)$ if $x_1 = 0$ and

$$\mathcal{G}_2 f(x) = \int_E [f(y) - f(x) - (y_1 - x_1)\partial_1 f(x)]\, \nu_x(dy) \quad \text{if } x_1 > 0.$$

For the drift in direction $(1, 0)$, we define $\mathcal{G}_1$ similarly. Note that ν_x is not a finite measure and that the integral of $y_1 - u$ with respect to $\nu_{(u,0)}$ is well defined only as a Cauchy principal value and as such equals zero. Hence this term in the integral is needed in order that the integral is well-defined in the usual sense. We will show in Lemma 5.5 below that $\mathcal{G}_1 f$ and $\mathcal{G}_2 f$ are in fact well defined and are in $C_l(E)$.

Due to spatial homogeneity of planar Brownian motion, we have a scaling relation that helps getting rid of the many different ν_x in the definition of $\mathcal{G}_1$ and $\mathcal{G}_2$:

$$\int_E f(x)\, \nu_{(u,0)}(dx) \;=\; \frac{1}{u} \int_E f(ux)\, \nu_{(1,0)}(dx).$$

Furthermore, letting $f^\dagger((x_1, x_2)) := f((x_2, x_1))$, by symmetry, we have

$$\int_E f(x)\, \nu_{(0,v)}(dx) \;=\; \int_E f^\dagger(x)\, \nu_{(v,0)}(dx) \;=\; \frac{1}{v} \int_E f^\dagger(vx)\, \nu_{(1,0)}(dx).$$

Hence, we can express $\mathcal{G}_1$ and $\mathcal{G}_2$ in terms of

$$\nu := \nu_{(1,0)}. \tag{5.5}$$

Using the explicit form of the density of $Q_{(1,\varepsilon)}$ in (2.5) and letting $\varepsilon \to 0$, we get that the σ-finite measure ν on E has a one-dimensional Lebesgue density given by

$$\nu\big(d(u, v)\big) = \begin{cases} \dfrac{4}{\pi} \dfrac{u}{(1-u)^2\,(1+u)^2}\, du, & \text{if } v = 0, \\[2ex] \dfrac{4}{\pi} \dfrac{v}{\big(1+v^2\big)^2}\, dv, & \text{if } u = 0. \end{cases} \tag{5.6}$$

$\mathcal{G}_1$ and $\mathcal{G}_2$ can now be written as

$$\mathcal{G}_2 f\,(x) = \begin{cases} \partial_2 f(x), & \text{if } x_1 = 0, \\[1ex] \dfrac{1}{x_1} \displaystyle\int_E \big[f(x_1 y) - f(x) - x_1(y_1 - 1)\,\partial_1 f(x)\big]\, \nu(dy), & \text{if } x_1 > 0, \end{cases} \tag{5.7}$$

and

$$\mathcal{G}_1 f = (\mathcal{G}_2 f^\dagger)^\dagger. \tag{5.8}$$

Finally, we define the operator $\mathcal{G}^{c,\theta}$ on $C_l^2(E)$ with domain $\mathcal{D}(\mathcal{G}^{c,\theta}) = C_l^2(E)$ that determines the infinitesimal characteristics of the process $X = X^{c,\theta}$:

$$\mathcal{G}^{c,\theta} f(x) = \sum_{i=1}^{2} c(\theta_i - x_i)\mathcal{G}_i f(x). \tag{5.9}$$

Lemma 5.2. *The operator $\mathcal{G}^{c,\theta}$ is well-defined. That is, for $f \in C_l^2(E)$, the expressions in (5.9) and (5.7) are well-defined and we have $\mathcal{G}^{c,\theta} f \in C_l(E)$.*

This lemma will be proved in Section 5.2.

Theorem 5.3 ($X^{c,\theta}$ via its generator). *(i) For every $f \in C_l^2(E)$, we have pointwise for all $x \in E$,*

$$\mathcal{G}^{c,\theta} f(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \bigg(\int_E f \, dQ_{x+\varepsilon\, c(\theta - x)} \; - f(x) \bigg) = \lim_{t \downarrow 0} \frac{\mathcal{S}_t f(x) - f(x)}{t}. \tag{5.10}$$

(ii) The operator $\mathcal{G}^{c,\theta}$ on $C_l(E)$ is closable and its closure generates the contraction semigroup $\mathcal{S}$ of the process $X^{c,\theta}$.

The theorem will be proved in Section 5.2 using the classical Hille-Yoshida theory.

A different and more modern approach to constructing Markov processes from their infinitesimal dynamics is the martingale problem technique due to Stroock and Varadhan.

Recall from (2.17) that $V \subset C_l^2(E)$ is the vector space spanned by $\{F(\,\cdot\,, z),\ z \in E\}$. Define the linear operator $\mathcal{G}^{c,\theta}$ on V by (5.9) and (5.7). By Theorem 5.3(i), we obtain for $z \in E$ (using Corollary 2.3(iii) in the second line and Lemma 2.2 in the last line)

$$\begin{aligned}
\mathcal{G}^{c,\theta} F(\,\cdot\,, z)(x) &= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \bigg(\int_E F(y,z) \, dQ_{x+\varepsilon\, c(\theta - x)}(dy) \; - F(x,z) \bigg) \\
&= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \Big(F\big(x + \varepsilon\, c(\theta - x), z\big) - F\big(x, z\big) \Big) \\
&= \big\langle c\,(\theta - x),\, \nabla_1 F(x,z) \big\rangle \\
&= F\big(x, y\big)\, [c(\theta - x) \diamond z].
\end{aligned} \tag{5.11}$$

Hence (5.11) is enough to define $\mathcal{G}^{c,\theta}$ on V and we do not really need the measure ν from (5.7) here.

A solution of the $(\mathcal{G}^{c,\theta}, V)$ martingale problem is an E valued measurable stochastic process X such that

$$M_t := F(X_t, z) - \int_0^t (c(\theta - X_s) \diamond z) F(X_s, z)\, ds$$

is a ($\mathbb{C}$ valued) martingale. A martingale problem is said to be well-posed, if for every probability measure μ on E, there exists a solution X with $\mathcal{L}[X_0] = \mu$ (existence) and any two solutions have the same finite dimensional distributions (uniqueness). In this case, X is a Markov process (see [6, Theorem 4.4.2(a)]).

Theorem 5.4 (Martingale problem characterisation of $X^{c,\theta}$). *The martingale problem $(\mathcal{G}^{c,\theta}, V)$ is well-posed and its unique solution is $X^{c,\theta}$.*

This theorem will be proved in Section 5.3.

5.2 The Hille Yoshida-Approach, Proof of Theorem 5.3

Lemma 5.2 and part (i) of Theorem 5.3 are direct consequences of the following two lemmas.

Let $e_1 = (1,0)$ and $e_2 = (0,1)$.

Lemma 5.5. *For $f \in C_l^2(E)$, $x \in E$ and $i = 1, 2$, the expression $\mathcal{G}_i f(x)$ from (5.7) and (5.8) is well-defined, and we have*

$$\lim_{\varepsilon \downarrow 0} \; \varepsilon^{-1} \bigg(\int_E f \, dQ_{x+\varepsilon e_i} \; - f(x) \bigg) = \mathcal{G}_i f(x). \tag{5.12}$$

Lemma 5.6. *For $f \in C_l^2(E)$, we have $\mathcal{G}^{c,\theta} f \in C_l(E)$.*

Proof of Lemma 5.5. For $x=(0,0)$, since $Q_{\varepsilon e_i}=\delta_{\varepsilon e_i}$, this is the very definition of $\mathcal{G}_i$. For $u\neq(0,0)$, by linear scaling and symmetry, it is enough to consider the case $x=(1,0)$. If $i=1$, then the left hand side of (5.12) equals

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\big(f(1+\varepsilon,0)-f(1,0)\big)=\partial_1 f(1,0)=(\mathcal{G}_1 f)(1,0).$$

Now consider $i=2$. It is a simple exercise to compute that for every $\varepsilon>0$,

$$\frac{4}{\pi}\int_0^\infty \frac{r(r-1)}{4\varepsilon^2+(r^2+\varepsilon^2-1)^2}\,dr \;=\; \frac{2}{\pi}\,\varepsilon^{-1}\arctan(\varepsilon) \;=\; \frac{4}{\pi}\int_0^\infty \frac{s}{4\varepsilon^2+(s^2-\varepsilon^2+1)^2}\,ds.$$

Hence if we let $g(y):=(y_1-1)\partial_1 f(1,0)$, then for every $\varepsilon>0$,

$$\int_E \big(g(y)-g(1,0)\big)\,Q_{(1,\varepsilon)}\big(dy\big) \;=\; 0.$$

Hence we can replace f by $f-g$. Now $f-g$ is twice differentiable, has at most linear growth and $\partial_1(f-g)(1,0)=0$. Hence

$$\sup_{u\geq 0,\,u\neq 1}\frac{|(f-g)(u,0)-f(1,0)|}{(u-1)^2}<\infty.$$

This allows to use dominated convergence in the following computation to obtain

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\left(\int_E f\,dQ_{(1,0)+\varepsilon e_2}\;-f(1,0)\right)\\
&\quad=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int\big[f(x)-f(1,0)-(x_1-1)\partial_1 f(1,0)\big]Q_{(1,\varepsilon)}(dx)\\
&\quad=\lim_{\varepsilon\downarrow 0}\left(\frac{4}{\pi}\int_0^\infty \frac{u[(f-g)(u,0)-f(1,0)]}{4\varepsilon^2+\big(u^2+\varepsilon^2-1\big)^2}\,du\;+\;\frac{4}{\pi}\int_0^\infty \frac{v[(f-g)(0,v)-f(1,0)]}{4\varepsilon^2+\big(v^2-\varepsilon^2+1\big)^2}\,dv\right)\\
&\quad=\lim_{\varepsilon\downarrow 0}\left(\frac{4}{\pi}\int_0^\infty \frac{u[(f-g)(u,0)-f(1,0)]}{4\varepsilon^2+\varepsilon^4+2\varepsilon^2(u+1)(u-1)+(u+1)^2(u-1)^2}\,du\right.\\
&\qquad\qquad\left.+\;\frac{4}{\pi}\int_0^\infty \frac{v[(f-g)(0,v)-f(1,0)]}{4\varepsilon^2+\big(v^2-\varepsilon^2+1\big)^2}\,dv\right)\\
&\quad=\frac{4}{\pi}\int_0^\infty \frac{u[(f-g)(u,0)-f(1,0)]}{(u^2-1)^2}\,du\;+\;\frac{4}{\pi}\int_0^\infty \frac{v[(f-g)(0,v)-f(1,0)]}{(v^2+1)^2}\,dv\\
&\quad=\int_E\big[f(y)-f(1,0)-(y_1-1)\partial_1 f(1,0)\big]\,\nu(dy)\\
&\quad=\mathcal{G}_2 f(1,0).
\end{aligned}$$

□

Proof of Lemma 5.6. We have to show that for any $f\in C_l^2(E)$, $\mathcal{G}^{c,\theta}f(x)$ is continuous in $x\in E$ and has a limit at ∞. By (5.9), it is enough to derive these properties for $G_i(x):=(\theta_i-x_i)\mathcal{G}_i f(x)$, $i=1,2$. We will give the proof only for the case of $i=2$ since the case $i=1$ is analogous.

For $x_1=0$, we have

$$G_2(x)=G_2(0,x_2)=(\theta_2-x_2)\partial_2 f(0,x_2). \tag{5.13}$$

This expression is clearly continuous in $x_2\in[0,\infty)$ and by (5.2), we have

$$\lim_{x_2\to\infty}G_2(x)=0. \tag{5.14}$$

Now consider the case $x_1 > 0$. Hence by (5.7),

$$G_2(x) = \int g(x,y)\,\nu(dy),$$

where

$$g(x,y) := \frac{\theta_2}{x_1}\big[f(x_1 y) - f(x) - x_1(y_1 - 1)\,\partial_1 f(x)\big].$$

Since $f \in C_l^2(E)$, for all $y \in E$, we have

(i) $x \mapsto g(x,y)$ is continuous on $(0,\infty)\times\{0\}$,

(ii) $\lim_{x_1\to\infty} g(x,y) = 0$, and

(iii) $\lim_{x_1\downarrow 0} g(x,y) = \theta_2 \partial_2 f(0,0) y_2$.

In order to find an integrable dominating function for g, define $h : E \to [0,\infty)$ by (recall (5.3) and (5.4))

$$h(y) := \begin{cases} \theta_2\,\|f\|_{2,\infty}\,(y_1-1)^2, & \text{if } y_1 \in \left(\frac{1}{2},\frac{3}{2}\right), \\ 2\theta_2\,\|f\|_{1,\infty}\,(y_1+y_2+1), & \text{otherwise.} \end{cases}$$

Note that the density of $\nu(dy)$ decays like $1/(y_1+y_2)^3$ as $y \to \infty$. Furthermore,

$$(y_1-1)^2\frac{\nu(dy)}{dy_1} = \frac{4}{\pi}\frac{y_1}{(1+y_1)^2}$$

is bounded on $(1/2,3/2)\times\{0\}$. Hence we have $\int h\,d\nu < \infty$.
For all $y \in E$, $x_1 > 0$, we have

$$|g(x,y)| \le \frac{\theta_2}{x_1}\big(|f(x_1 y) - f(0,0)| + |f(x) - f(0,0)| + x_1(y_1+1)|\partial_1 f(x)|\big) \le 2\theta_2(y_1+y_2+1)\,\|f\|_{1,\infty}.$$

Furthermore, recalling (5.3), for $y_1 \in (1/2,3/2)$, we get by Taylor's formula

$$\begin{aligned} |g(x,y)| &= \frac{\theta_2}{x_1}\big|f((y_1-1)x + x) - f(x) - x_1(y_1-1)\partial_1 f(x)\big| \\ &\le \frac{\theta_2}{2}(y_1-1)^2 \sup_{u \ge x_1/2} x_1\,|\partial_1^2 f(u,0)| \\ &\le \theta_2\,\|f\|_{2,\infty}\,(y_1-1)^2. \end{aligned}$$

Hence, in fact $|g(x,y)| \le h(y)$ for all $y \in E, x \in (0,\infty)\times\{0\}$ and the dominated convergence theorem yields that G_2 shares the properties (i)-(ii) of $g(x,\,\cdot\,)$ and that

$$\lim_{x_1\downarrow 0} G_2(x) \;=\; \theta_2\,\partial_2 f(0,0)\int y_2\,\nu(dy) \;=\; \theta_2\,\partial_2 f(0,0) \;=\; G_2(0,0).$$

Combining this with (5.13) and (5.14), we have $G_2 \in C_l(E)$. □

In order to show part (ii) of Theorem 5.3, we will apply the Hille-Yoshida theory for generators of contraction semigroups. Recall from Corollary 2.4 that V is dense in $C_l(E)$. Also, by Lemma 2.2 one can easily check that

$$V \subset C_l^2(E).$$

For each $z \in E$, define the map $u_y : [0,\infty) \to C_l(E)$ by $u_y(t) := \mathcal{S}_t F(\,\cdot\,,y)$.
By [6, Proposition 1.3.4], the operator $\mathcal{G}^{c,\theta}$ on $C_l(E)$ with domain $\mathcal{D}(\mathcal{G}^{c,\theta}) = C_l^2(E)$ is closable and its closure generates (uniquely) the semigroup $(\mathcal{S}_t)_{t\ge 0}$ on $C_l(E)$ if the following conditions are fulfilled:

(a) $\mathcal{G}^{c,\theta}$ is dissipative,

(b) $u_y(t) \in \mathcal{D}(\mathcal{G}^{c,\theta})$ for all $t > 0$,

(c) the map $(0,\infty) \to C_l(E)$, $t \mapsto \mathcal{G}^{c,\theta} u_y(t)$ is continuous,

(d) and for all $t > 0$,

$$u_y(t) - u_y(0) = \int_0^t \mathcal{G}^{c,\theta} u_y(s)\, ds. \tag{5.15}$$

Hence in order to prove part (ii) of Theorem 5.3, it remains to check (a)–(d).

(a) Let $f \in C_l^2(E)$ and assume that f assumes its maximum at $x \in E \cup \{\infty\}$. Since $\mathcal{S}_t f(x) \le f(x)$ for all $t \ge 0$, equation (5.10) implies $\mathcal{G}^{c,\theta} f(x) \le 0$. Hence $\mathcal{G}^{c,\theta}$ fulfills the positive maximum principle and is thus dissipative (see, e.g., [6, Lemma 4.2.1]).

(b) By Proposition 3.3, for any $y \in E$, $x \in E$, and $t > 0$, we have

$$u_y(t)(x) = \mathcal{S}_t F(\,\cdot\,, y)(x) = F\big(x, e^{-ct} y\big)\, F\big(\theta, \big(1 - e^{-ct}\big) y\big). \tag{5.16}$$

As $F\big(\,\cdot\,, e^{-ct} y\big)$ is in $C_l^2(E)$, so is $\mathcal{S}_t F(\,\cdot\,, y)$.

(c) By (5.10), we have

$$\mathcal{G}^{c,\theta} u_y(t)(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \big(\mathcal{S}_{t+\varepsilon} - \mathcal{S}_t\big) F(\,\cdot\,, y)(x) = \frac{d}{dt}\big(u_y(t)(x)\big).$$

Using (5.16) and Lemma 2.2, for every $x \in E$, we get

$$\begin{aligned}
&\mathcal{G}^{c,\theta} \mathcal{S}_t F(\,\cdot\,, y)(x) \\
&\quad = \langle -c\, e^{-ct} y, \nabla_2 F(c, e^{-ct} y) \rangle F(\theta, (1 - e^{-ct}) y) + F(x, e^{-ct} y) \langle c\, e^{-ct} y, \nabla_2 F(\theta, (1 - e^{-ct}) y) \rangle \\
&\quad = \big[c\, e^{-ct}\, (\theta - x) \diamond y\big]\, F\big(x, e^{-ct} y\big) F\big(\theta, \big(1 - e^{-ct}\big) y\big).
\end{aligned}$$

Hence $t \mapsto \mathcal{G}^{c,\theta} u_y(t)$ is clearly continuous (in $C_l(E)$).

(d) As $t \mapsto \mathcal{G}^{c,\theta} u_y(t)$ is continuous, it is integrable, and

$$\left(\int_0^t \mathcal{G}^{c,\theta} u_y(s)\, ds \right)(x) = \int_0^t \mathcal{G}^{c,\theta} u_y(s)(x)\, ds = u_y(t)(x) - u_y(0)(x)$$

implies (5.15).

5.3 The Martingale Problem, Proof of Theorem 5.4

Before we prove this theorem, we derive a duality relation for processes satisfying the martingale problem $(\mathcal{G}^{c,\theta}, V)$. Recall the definition of $\tilde{Y}$ from (2.11).

Lemma 5.7. *Let μ be a probability measure on E. Let X be any solution of the martingale problem $(\mathcal{G}^{c,\theta}, V)$ with $\mathcal{L}[X_0] = \mu$. Then X and $\tilde{Y}$ are dual in the sense that for any $\tilde{y} \in E \times [0,\infty)^2$, we have*

$$\mathbf{E}_\mu\big[H\big((X_t, \theta), \tilde{y}\big)\big] = \int_E \mathbf{E}_{\tilde{y}}\big[H\big((x,\theta), \tilde{Y}_t\big)\big] \mu(dx) \quad \textit{for all } \; t \ge 0.$$

Proof. As X is a solution of the martingale problem, we have that

$$\begin{aligned} H\big((X_t,\theta),\tilde y\big) &- \int_0^t H\big((X_s,\theta),\tilde y\big)[c(\theta-X_s)\diamond\tilde y(1)]\,ds \\ &= F(\theta,\tilde y(2))\left(F(X_t,\tilde y(1)) - \int_0^t F(X_s,\tilde y(1))[c(\theta-X_s)\diamond\tilde y(1)]\,ds\right) \end{aligned}$$

is a martingale. On the other hand, by (2.16) (since $\tilde y(1)\in E$ one term vanishes),

$$\begin{aligned} \frac{d}{dt}\mathbf{E}_{\tilde y}\big[H\big((x,\theta),\tilde Y_t\big)\big]\big|_{t=0} &= \big\langle -c\,\tilde y(1),\,\nabla_2 F(x,\tilde y(1))\big\rangle\,F(\theta,\tilde y(2)) + \big\langle c\,\tilde y(1),\,\nabla_2 F(\theta,\tilde y(2))\big\rangle\,F(x,\tilde y(1)) \\ &= H\big((\theta,x),\tilde y\big)\,[c(\theta-x)\diamond\tilde y(1)]. \end{aligned} \tag{5.17}$$

Since $\tilde Y$ is deterministic, we get that

$$H\big((x,\theta),\tilde Y_t\big) - \int_0^t H\big((\theta,x),\tilde Y_s\big)c(\theta-x)\diamond\tilde Y_s(1)\,ds = H\big((x,\theta),\tilde y\big)$$

is the trivial martingale. By [6, Corollary 4.4.13], this implies

$$\mathbf{E}_\mu\big[H\big((X_t,\theta),\tilde y\big)\big] = \int \mathbf{E}_y\big[H\big((x,\theta),\tilde Y_t\big)\big]\,\mu(dx),$$

and we are done. □

Proof of Theorem 5.4. By Theorem 5.3(ii) and (5.11) and since $V\subset C_l(E)$, by definition of $X^{c,\theta}$, the process $X^{c,\theta}$ is in fact a solution of the martingale problem $(\mathcal{G}^{c,\theta},V)$.

Now assume that X and X' are two solutions with $\mathcal{L}[X_0]=\mathcal{L}[X_0']=\mu$. By Lemma 5.7, we get

$$\mathbf{E}_\mu\big[F\big(X_t,y\big)\big] = \mathbf{E}_\mu\big[F\big(X_t',y\big)\big] \quad \text{for all } t\ge 0 \text{ and } y\in E.$$

By Corollary 2.4, $\{F(\,\cdot\,,y),\ y\in E\}$ is measure determining on E. Hence $\mathcal{L}_\mu[X_t]=\mathcal{L}_\mu[X_t']$ for all $t\ge 0$. By [6, Theorem 4.4.2], this implies that the finite dimensional distributions of X and X' coincide. □

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