

GAUSSIAN MULTIPLICATIVE CHAOS REVISITED

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ABSTRACT. In this article, we extend the theory of multiplicative chaos for positive definite functions in \mathbb{R}^d of the form $f(x) = \lambda^2 \ln^+ \frac{R}{|x|} + g(x)$ where g is a continuous and bounded function. The construction is simpler and more general than the one defined by Kahane in 1985. As main application, we give a rigorous mathematical meaning to the Kolmogorov-Obukhov model of energy dissipation in a turbulent flow.

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1. INTRODUCTION

The theory of multiplicative chaos was first defined rigorously by Kahane in 1985 in the article [13]. More specifically, Kahane built a theory relying on the notion of σ -positive type kernel: a generalized function $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is of σ -positive type if there exists a sequence $K_k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ of continuous positive and positive definite kernels such that:

$$K(x, y) = \sum_{k \geq 1} K_k(x, y). \quad (1.1)$$

If K is a σ -positive type kernel with decomposition (1.1), one can consider a sequence of gaussian processes $(X_n)_{n \geq 1}$ of covariance given by $\sum_{k=1}^n K_k$. It is proven in [13] that the sequence of random measures m_n given by:

$$m_n(A) = \int_A e^{X_n(x) - \frac{1}{2}E[X_n(x)^2]} dx, \quad A \in \mathcal{B}(\mathbb{R}^d) \quad (1.2)$$

converges almost surely in the space of Radon measures (equipped with the topology of weak convergence) towards a random measure m and that the limit measure m obtained does not depend on the sequence $(K_k)_{k \geq 1}$ used in the decomposition

(1.1) of K . Thus, the theory enables to give a unique and mathematically rigorous definition to a random measure m in \mathbb{R}^d defined formally by:

$$m(A) = \int_A e^{X(x) - \frac{1}{2}E[X(x)^2]} dx, \quad A \in \mathcal{B}(\mathbb{R}^d) \quad (1.3)$$

where $(X(x))_{x \in \mathbb{R}^d}$ is a "gaussian field" whose covariance K is a σ -positive type kernel. As it will appear later the σ -positive type condition is not easy to check in practice. From where the need to avoid this hypothesis.

The main application of the theory is to give a meaning to the "limit-lognormal" model introduced by Mandelbrot in [17]. In the sequel, we define $\ln^+ x$ for $x > 0$ by the following formula:

$$\ln^+ x = \max(\ln(x), 0).$$

The "limit-lognormal" model corresponds to the choice of a homogeneous K given by:

$$K(x, y) = \lambda^2 \ln^+(R/|x - y|) + O(1) \quad (1.4)$$

where λ^2, R are positive parameters and $O(1)$ is a bounded quantity as $|x - y| \rightarrow 0$. This model has many applications that we now review in the following subchapters.

1.1. Multiplicative chaos in dimension 1: a model for the volatility of a financial asset. If $(X(t))_{t \geq 0}$ is the logarithm of the price of a financial asset, the volatility m of the asset on the interval $[0, t]$ is by definition equal to the quadratic variation of X :

$$m[0, t] = \lim_{n \rightarrow \infty} \sum_{k=1}^n (X(tk/n) - X(t(k-1)/n))^2$$

The volatility m can be viewed as a random measure on \mathbb{R} . The choice for m of multiplicative chaos associated to the kernel $K(s, t) = \lambda^2 \ln^+ \frac{T}{|t-s|}$ satisfies many empirical properties measured on financial markets: lognormality of the volatility, long range correlations (see [6] for a study of the SP500 index and components and [7] for a general review). Note that K is indeed of σ -positive type (see example 2.3 below) so m is well defined. In the context of finance, λ^2 is called the intermittency parameter in analogy with turbulence and T is the correlation length. Volatility modeling and forecasting is an important field of finance since it is related to option pricing and risk forecasting: we refer to [9] for the problem of forecasting volatility with this choice of m .

Given the volatility m , the most natural way to construct a model for the (log) price X is to set:

$$X(t) = B_{m[0,t]} \quad (1.5)$$

where $(B_t)_{t \geq 0}$ is a brownian motion independent of m . Formula (1.5) defines the Multifractal Random Walk (MRW) first introduced in [1] (see [2] for a recent review of the financial applications of the MRW model).

1.2. Multiplicative chaos in dimension 3: a model for the energy dissipation in a turbulent fluid. We refer to [10] for an introduction to the statistical theory of 3 dimensional turbulence. Consider a stationary flow at high Reynolds number; it is believed that at small scales the velocity field of the flow is homogeneous and isotropic in space . By small scales we mean scales much smaller than the integral scale R characteristic of the time stationary force driving the flow. In the work [15] and [19], Kolmogorov and Obukhov propose to model the mean energy dissipation per unit mass in a ball $B(x, l)$ of center x and radius $l \ll R$ by a random variable ϵ_l such that $\ln(\epsilon_l)$ is normal with variance σ_l^2 given by:

$$\sigma_l^2 = \lambda^2 \ln\left(\frac{R}{l}\right) + A$$

where A is a constant and λ^2 is the intermittency parameter. As noted by Mandelbrot ([17]), the only way to define such a model is to construct a random measure ϵ by a limit procedure. Then, one can define ϵ_l by the formula:

$$\epsilon_l = \frac{3 \langle \epsilon \rangle}{4\pi l^3} \epsilon(B(x, l))$$

where $\langle \epsilon \rangle$ is the average mean energy dissipation per unit mass. Formally, one is looking for a random measure ϵ such that:

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \epsilon(A) = \int_A e^{X(x) - \frac{1}{2}E[X(x)^2]} dx \quad (1.6)$$

where $(X(x))_{x \in \mathbb{R}^d}$ is a "gaussian field" whose covariance K is given by $K(x, y) = \lambda^2 \ln^+ \frac{R}{|x-y|}$. The kernel $\lambda^2 \ln^+ \frac{R}{|x-y|}$ is positive definite considered as a tempered distribution (see (2.1) for a definition of positive definite distributions and lemma 3.2 for a proof). Therefore, one can give a rigorous meaning to (1.6) by using theorem-definition 2.1 below.

However, it is not clear if $\lambda^2 \ln^+ \frac{R}{|x-y|}$ is of σ -positive type in \mathbb{R}^3 and therefore, in [13], Kahane considers the σ -positive type kernel $K(x, y) = \int_{1/R}^{\infty} \frac{e^{-u|x-y|}}{u} du$ as an approximation of $\lambda^2 \ln^+ \frac{R}{|x-y|}$: indeed, one can show that $\int_{1/R}^{\infty} \frac{e^{-u|x-y|}}{u} du = \ln^+ \frac{R}{|x-y|} + g(|x-y|)$ where g is a bounded continuous function. Nevertheless, it is important to work with $\lambda^2 \ln^+ \frac{R}{|x-y|}$ since this choice leads to measures which exhibit generalized scale invariance properties (see proposition 3.3).

1.3. Organization of the paper. In section 2, we remind the definition of positive definite tempered distributions and we state theorem-definition 2.1 where we define multiplicative chaos m associated to kernels of the type $\ln^+ \frac{R}{|x|} + O(1)$. In section 3, we review the main properties of the measure m : existence of moments and density with respect to Lebesgue measure, multifractality and generalized scale invariance. In section 4 and 5, we give respectively the proofs of section 2 and 3.

2. DEFINITION OF MULTIPLICATIVE CHAOS

2.1. Positive definite tempered distributions. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of smooth rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions (see [21]). A distribution f in $\mathcal{S}'(\mathbb{R}^d)$ is positive definite if:

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \varphi(x) \overline{\varphi(y)} dx dy \geq 0. \quad (2.1)$$

On $\mathcal{S}'(\mathbb{R}^d)$, one can define the Fourier transform \hat{f} of a tempered distribution by the formula:

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \hat{f}(\xi) \varphi(\xi) d\xi = \int_{\mathbb{R}^d} f(x) \hat{\varphi}(x) dx \quad (2.2)$$

where $\hat{\varphi}(x) = \int_{\mathbb{R}^d} e^{-2i\pi x \cdot \xi} \varphi(\xi) d\xi$ is the Fourier transform of φ . An extension of Bochner's theorem (Schwartz, [21]) states that a tempered distribution f is positive definite if and only if it's Fourier transform is a tempered positive measure.

By definition, a function f in $\mathcal{S}'(\mathbb{R}^d)$ is of σ -positive type if the associated kernel $K(x, y) = f(x-y)$ is of σ -positive type. As mentioned in the introduction, Kahane's theory of multiplicative chaos is defined for σ -positive type functions f . The main problem stems from the fact that definition (1.1) is not practical. Indeed, is there a simple characterization (like the computation of a Fourier transform) of functions whose associated kernel can be decomposed along (1.1)? If such a characterization exists, how does one find the kernels K_n explicitly?

Finally, we recall the following simple implication: If f belongs to $\mathcal{S}'(\mathbb{R}^d)$ and is of σ -positive type, f is positive and positive definite. However, the converse statement is not clear.

2.2. A generalized theory of multiplicative chaos. In this subsection, we construct a theory of multiplicative chaos for positive definite functions of type $\lambda^2 \ln^+ \frac{R}{|x|} + O(1)$ without the assumption of σ -positivity for the underlying function. The theory is therefore much easier to use.

We consider in \mathbb{R}^d a positive definite function f such that

$$f(x) = \lambda^2 \ln^+ \frac{R}{|x|} + g(x) \quad (2.3)$$

where $\lambda^2 \neq 2d$ and $g(x)$ is a bounded continuous function. Let $\theta : \mathbb{R}^d \rightarrow \mathbb{R}$ be some continuous function with the following properties:

- (1) θ is positive definite
- (2) $|\theta(x)| \leq \frac{1}{1+|x|^{d+\gamma}}$ for some $\gamma > 0$.
- (3) $\int_{\mathbb{R}^d} \theta(x) dx = 1$

Here is the main theorem of the article:

Theorem 2.1. (*Definition of multiplicative chaos*)

For all $\epsilon > 0$, we consider the centered gaussian field $(X_\epsilon(x))_{x \in \mathbb{R}^d}$ defined by the convolution:

$$E[X_\epsilon(x) X_\epsilon(y)] = (\theta^\epsilon * f)(y-x),$$

where $\theta^\epsilon = \frac{1}{\epsilon} \theta(\frac{\cdot}{\epsilon})$. Then the associated random measure $m_\epsilon(dx) = e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]} dx$ converges in law in the space of Radon measures (equipped with the topology of weak convergence) as ϵ goes to 0 towards a random measure m independent of the choice of the regularizing function θ with the properties (1), (2), (3). We call the measure m the multiplicative chaos associated to the function f .

We review below two possible choices of the underlying function f . The first example is a d -dimensional generalization of the cone construction considered in [3]. The second example is $\lambda^2 \ln^+ \frac{R}{|x|}$ for $d = 1, 2, 3$ (the case $d = 2, 3$ seems to have never been considered in the litterature). Both examples are in fact of σ -positive type (except perhaps the crucial example of $\lambda^2 \ln^+ \frac{R}{|x|}$ in dimension $d = 3$) and it is easy to show that in these cases theorem-definition 2.1 and Kahane's theory lead to the same limit measure m .

Example 2.2. One can construct a positive definite function f with decomposition (2.3) by generalizing to dimension d the cone construction of [3]. This was performed in [5]. For all x in \mathbb{R}^d , we define the cone $C(x)$ in $\mathbb{R}^d \times \mathbb{R}_+$:

$$C(x) = \{(y, t) \in \mathbb{R}^d \times \mathbb{R}_+; |y - x| \leq \frac{t \wedge R}{2}\}.$$

The function f is given by:

$$f(x) = \lambda^2 \int_{C(0) \cap C(x)} \frac{dy dt}{t^{d+1}} \quad (2.4)$$

One can show that f has decomposition (2.3) (see [5]). The function f is of σ -positive type in the sense of Kahane since one can write $f = \sum_{n \geq 1} f_n$ with f_n given by:

$$f_n(x) = \lambda^2 \int_{C(0) \cap C(x); \frac{1}{n} \leq t < \frac{1}{n-1}} \frac{dy dt}{t^{d+1}}.$$

In dimension $d = 1$, we get the simple formula $f(x) = \lambda^2 \ln^+ \frac{R}{|x|}$.

Example 2.3. In dimension $d = 1, 2$, the function $f(x) = \ln^+ \frac{R}{|x|}$ is of σ -positive type in the sense of Kahane and in particular positive definite. Indeed, one has by straightforward calculations:

$$\ln^+ \frac{R}{|x|} = \int_0^\infty (t - |x|)_+ \nu_R(dt)$$

where $\nu_R(dt) = 1_{[0, R]}(t) \frac{dt}{t^2} + \frac{\delta_R}{R}$. For all $\mu > 0$, we have:

$$\ln^+ \frac{R}{|x|} = \frac{1}{\mu} \ln^+ \frac{R^\mu}{|x|^\mu} = \frac{1}{\mu} \int_0^\infty (t - |x|^\mu)_+ \nu_{R^\mu}(dt).$$

We are therefore led to considering the $\mu > 0$ such that $(1 - |x|^\mu)_+$ is positive definite (the so called Kuttner-Golubov problem: see [11] for an introduction).

For $d = 1$, it is straightforward to show that $(1 - |x|)_+$ is of positive type. One can thus write $f = \sum_{n \geq 1} f_n$ with f_n given by:

$$f_n(x) = \int_{\frac{R}{n}}^{\frac{R}{n-1}} (t - |x|)_+ \nu_R(dt).$$

For $d = 2$, the function $(1 - |x|^{1/2})$ is positive definite (Pasenchenko, [20]). One can thus write $f = \sum_{n \geq 1} f_n$ with f_n given by:

$$f_n(x) = \int_{\frac{R^{1/2}}{n}}^{\frac{R^{1/2}}{n-1}} (t - |x|^{1/2})_+ \nu_{R^{1/2}}(dt).$$

In dimension $d = 3$, the function $\ln^+ \frac{R}{|x|}$ is positive definite (see lemma 3.2 below) but it is an open question whether it is of σ -positive type.

3. MAIN PROPERTIES OF MULTIPLICATIVE CHAOS

In the sequel, we will consider the structure functions ζ_p defined for all p in \mathbb{R} by:

$$\zeta_p = (d + \frac{\lambda^2}{2})p - \frac{\lambda^2 p^2}{2}. \quad (3.1)$$

3.1. Multiplicative chaos is equal to 0 for $\lambda^2 > 2d$. The following proposition shows that multiplicative chaos is non trivial only for sufficiently small values of λ^2 .

Proposition 3.1. *If $\lambda^2 > 2d$, the limit measure is equal to 0.*

3.2. Generalized scale invariance. In this subsection and the following, in view of proposition 3.1, we will suppose that $\lambda^2 < 2d$:

Let m be a homogeneous random measure on \mathbb{R}^d ; we remind that this means that for all x , the measures m and $m(x + \cdot)$ are equal in law. We note $B(0, R)$ the ball of center 0 and radius R in \mathbb{R}^d . We say m has the generalized scale invariance property with integral scale $R > 0$ if for all c in $]0, 1]$ the following equality in law holds:

$$(m(cA))_{A \subset B(0, R)} \stackrel{(Law)}{=} e^{\Omega_c} (m(A))_{A \subset B(0, R)} \quad (3.2)$$

where Ω_c is a random variable independent from m . Let ν_t denote the law of $\Omega_{e^{-t}}$. If m is different from 0, then it is straightforward to prove that the laws $(\nu_t)_{t \geq 0}$ satisfy the following convolution property: $\nu_{t+t'} = \nu_t * \nu_{t'}$. Therefore, one can find a Levy process $(L_t)_{t \geq 0}$ such that, for each t , ν_t is the law of L_t . In the context of gaussian multiplicative chaos, the process $(L_t)_{t \geq 0}$ will be Brownian motion with drift.

In order to get scale invariance with integral scale R , one can choose $f = \ln^+ \frac{R}{|\cdot|}$. This is possible if and only if $\ln^+ \frac{R}{|\cdot|}$ is positive definite. This motivates the following lemma:

Lemma 3.2. *Let $d \geq 1$ be the dimension of the space and $R > 0$ the integral scale. We consider the function $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ defined by:*

$$f(x) = \ln^+ \frac{R}{|x|}.$$

The function f is positive definite if and only if $d \leq 3$.

The above choice of f leads to measures that have the generalized scale invariance property.

Proposition 3.3. *Let d be less or equal to 3 and m be the gaussian multiplicative chaos with kernel $\lambda^2 \ln^+ \frac{R}{|x|}$. Then m is scale invariant; for all c in $]0, 1]$, the following equality holds:*

$$(m(cA))_{A \subset B(0,R)} \stackrel{(Law)}{=} e^{\Omega_c} (m(A))_{A \subset B(0,R)}, \quad (3.3)$$

where Ω_c is a gaussian random variable independent of m with mean $-(d + \frac{\lambda^2}{2}) \ln(1/c)$ and variance $\lambda^2 \ln(1/c)$

The proof of the proposition is straightforward.

Remark 3.4. *It remains an open question to construct homogeneous measures in dimension greater or equal to 4 which are scale invariant.*

3.3. Existence of moments and multifractality. We remind that we suppose that $\lambda^2 < 2d$: this ensures the existence of $\epsilon > 0$ such that $\zeta_{1+\epsilon} > d$. Therefore, there exists a unique $p_* > 1$ such that $\zeta_{p_*} = d$. The following two propositions establish the existence of positive and negative moments for the limit measure.

Proposition 3.5. *(Positive Moments)*

Let p belong to $]0, p_[$ and m be the gaussian multiplicative chaos associated to the function f given by (2.3). For all bounded A in $\mathcal{B}(\mathbb{R}^d)$,*

$$E[m(A)^p] < \infty \quad (3.4)$$

Let θ be some function satisfying the conditions (1), (2), (3) of section 2.2. With the notations of theorem 2.1, we consider the random measure m_ϵ associated to θ . We have the following convergence for all bounded A in $\mathcal{B}(\mathbb{R}^d)$:

$$E[m_\epsilon(A)^p] \xrightarrow{\epsilon \rightarrow 0} E[m(A)^p]. \quad (3.5)$$

Proposition 3.6. *(Negative Moments)*

Let p belong to $] -\infty, 0]$ and m be the gaussian multiplicative chaos associated to the function f given by (2.3). For all $c > 0$,

$$E[m(B(0, c))^p] < \infty \quad (3.6)$$

Let θ be some function satisfying the conditions (1), (2), (3) of section 2.2. With the notations of theorem 2.1, we consider the random measure m_ϵ associated to θ . We have the following convergence for all $c > 0$:

$$E[m_\epsilon(B(0, c))^p] \xrightarrow{\epsilon \rightarrow 0} E[m(B(0, c))^p]. \quad (3.7)$$

The following proposition states the existence of the structure functions.

Proposition 3.7. *Let p belong to $] -\infty, p_*[$. Let m be the gaussian multiplicative chaos associated to the function f given by (2.3). There exists some $C_p > 0$ (independent of g and R in decomposition (2.3): $C_p = C_p(\lambda^2)$) such that we have the*

following multifractal behaviour:

$$E[m([0, c]^d)^p] \underset{c \rightarrow 0}{\sim} e^{\frac{p(p-1)g(0)}{2}} C_p \left(\frac{c}{R}\right)^{\zeta_p}. \quad (3.8)$$

In the next proposition, we will suppose that $d \leq 3$ and that $f(x) = \lambda^2 \ln^+ \frac{R}{|x|}$. In this case, we can prove the existence of a C^∞ density.

Proposition 3.8. *Let d be less or equal to 3 and m be the gaussian multiplicative chaos with kernel $\lambda^2 \ln^+ \frac{R}{|x|}$. For all $c < R$, the variable $m(B(0, c))$ has a C^∞ density with respect to the Lebesgue measure.*

4. PROOF OF THEOREM 2.1

4.1. A few intermediate lemmas. In order to prove the theorem, we start by giving some lemmas we will need in the proof.

Lemma 4.1. *Let θ be some function on \mathbb{R}^d such that there exists $\gamma, C > 0$ with $|\theta(x)| \leq \frac{C}{1+|x|^{d+\gamma}}$. Then we have the following convergence:*

$$\sup_{|z| > A} \left| \int_{\mathbb{R}^d} |\theta(v)| \ln \left| \frac{z}{z-v} \right| dv \right| \xrightarrow{A \rightarrow \infty} 0. \quad (4.1)$$

Proof. We have:

$$\int_{\mathbb{R}^d} |\theta(v)| \ln \left| \frac{z}{z-v} \right| dv = \int_{|v| \leq \sqrt{|z|}} |\theta(v)| \ln \left| \frac{z}{z-v} \right| dv + \int_{|v| > \sqrt{|z|}} |\theta(v)| \ln \left| \frac{z}{z-v} \right| dv.$$

In the sequel of the proof, we suppose that $|z| > 1$.

Consider the first term: we have $1 - \frac{|v|}{|z|} \leq \frac{|z-v|}{|z|} \leq 1 + \frac{|v|}{|z|}$ so that for $|v| \leq \sqrt{|z|}$:

$$1 - \frac{1}{\sqrt{|z|}} \leq \frac{|z-v|}{|z|} \leq 1 + \frac{1}{\sqrt{|z|}},$$

thus we get $\left| \ln \frac{|z-v|}{|z|} \right| \leq \ln \left(\frac{1}{1 - \frac{1}{\sqrt{|z|}}} \right) \leq \frac{1}{\sqrt{|z|-1}}$. We conclude:

$$\int_{|v| \leq \sqrt{|z|}} |\theta(v)| \ln \left| \frac{z}{z-v} \right| dv \leq \frac{1}{\sqrt{|z|-1}} \int_{\mathbb{R}^d} |\theta(v)| dv$$

Consider the second term: we have:

$$\int_{|v| > \sqrt{|z|}} |\theta(v)| \ln \left| \frac{z}{z-v} \right| dv \leq \ln |z| \int_{|v| > \sqrt{|z|}} |\theta(v)| dv + \int_{|v| > \sqrt{|z|}} |\theta(v)| |\ln |z-v|| dv$$

The first term above is obvious; we decompose the second:

$$\int_{|v| > \sqrt{|z|}} |\theta(v)| |\ln |z-v|| dv = \int_{\sqrt{|z|} < |v| < |z|+1} |\theta(v)| |\ln |z-v|| dv + \int_{|v| \geq |z|+1} |\theta(v)| |\ln |z-v|| dv$$

For $|v| \geq |z| + 1$, we have $1 \leq |z-v| \leq |z||v|$ and thus

$$0 \leq \ln |z-v| \leq \ln |z| + \ln |v|$$

which enables to handle the corresponding integral. Let us now estimate the remaining term $I = \int_{\sqrt{|z|} < |v| < |z|+1} |\theta(v)| |\ln |z - v|| dv$. Applying Holder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ gives:

$$I \leq \left(\int_{\sqrt{|z|} < |v| < |z|+1} |\theta(v)|^p dv \right)^{1/p} \left(\int_{\sqrt{|z|} < |v| < |z|+1} |\ln |z - v||^q dv \right)^{1/q},$$

from which we get straightforwardly if p is close to 1:

$$I \leq \frac{C \ln |z|}{|z|^{d/2+\gamma/2-d/2p-d/q}} \xrightarrow{|z| \rightarrow \infty} 0.$$

□

We will also use the following lemma:

Lemma 4.2. *Let λ be a positive number such that $\lambda^2 \neq 2$ and $(X_i)_{1 \leq i \leq n}$ an i.i.d. sequence of centered gaussian variables with variance $\lambda^2 \ln(n)$. For all positive p such that $p < \max(\frac{2}{\lambda^2}, 1)$, there exists $0 < x < 1$ such that:*

$$E \left[\sup_{1 \leq i \leq n} e^{pX_i - p\frac{\lambda^2}{2} \ln(n)} \right] = O(n^{xp}) \quad (4.2)$$

Proof. By Fubini we get:

$$\begin{aligned} & E \left[\sup_{1 \leq i \leq n} e^{pX_i - p\frac{\lambda^2}{2} \ln(n)} \right] \\ &= \int_0^\infty P \left(\sup_{1 \leq i \leq n} e^{pX_i - p\frac{\lambda^2}{2} \ln(n)} > v \right) dv \\ &= \int_0^\infty P \left(\sup_{1 \leq i \leq n} X_i > \frac{\ln(v)}{p} + \frac{\lambda^2}{2} \ln(n) \right) dv \\ &= \int_{-\infty}^\infty p e^{pu} P \left(\sup_{1 \leq i \leq n} X_i > u + \frac{\lambda^2}{2} \ln(n) \right) du \\ &\leq 1 + \int_0^\infty p e^{pu} P \left(\sup_{1 \leq i \leq n} X_i > u + \frac{\lambda^2}{2} \ln(n) \right) du, \end{aligned} \quad (4.3)$$

where we performed the change of variable $u = \frac{\ln(v)}{p}$ in the above identities. If we define $\bar{F}(u) = P(X_1 > u)$ then we have:

$$P \left(\sup_{1 \leq i \leq n} X_i > u + \frac{\lambda^2}{2} \ln(n) \right) = 1 - e^{n \ln(1 - \bar{F}(u + \frac{\lambda^2}{2} \ln(n)))}.$$

Let x be some positive number such that $0 < x < 1$. Using (4.3), we get:

$$\begin{aligned} E \left[\sup_{1 \leq i \leq n} e^{pX_i - p\frac{\lambda^2}{2} \ln(n)} \right] &\leq n^{xp} + p \int_{x \ln(n)}^\infty e^{pu} (1 - e^{n \ln(1 - \bar{F}(u + \frac{\lambda^2}{2} \ln(n)))}) du \\ &\leq n^{xp} + p n^{xp} \int_0^\infty e^{p\tilde{u}} (1 - e^{n \ln(1 - \bar{F}(\tilde{u} + (\frac{\lambda^2}{2} + x) \ln(n)))}) d\tilde{u} \end{aligned} \quad (4.4)$$

We have:

$$\begin{aligned} \bar{F}(\tilde{u} + (\frac{\lambda^2}{2} + x) \ln(n)) &= \frac{1}{\sqrt{2\pi\lambda\sqrt{\ln(n)}}} \int_{\tilde{u} + (\frac{\lambda^2}{2} + x) \ln(n)}^{\infty} e^{-\frac{v^2}{2\lambda^2 \ln(n)}} dv \\ &= \frac{n^{-\frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{\sqrt{2\pi\lambda\sqrt{\ln(n)}}} \int_{\tilde{u}}^{\infty} e^{-(\frac{1}{2} + \frac{x}{\lambda^2})\tilde{v} - \frac{\tilde{v}^2}{2\lambda^2 \ln(n)}} d\tilde{v}, \end{aligned}$$

where we performed the change of variable $\tilde{v} = v - (\frac{\lambda^2}{2} + x) \ln(n)$. Thus, we get:

$$\begin{aligned} &n^{xp} \int_0^{\infty} e^{p\tilde{u}} (1 - e^{n \ln(1 - \bar{F}(\tilde{u} + (\frac{\lambda^2}{2} + x) \ln(n)))}) d\tilde{u} \\ &\leq n^{xp+1} \int_0^{\infty} e^{p\tilde{u}} \bar{F}(\tilde{u} + (\frac{\lambda^2}{2} + x) \ln(n)) d\tilde{u} \\ &\leq \frac{n^{xp+1 - \frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{\sqrt{2\pi\lambda\sqrt{\ln(n)}}} \int_0^{\infty} e^{p\tilde{u}} \left(\int_{\tilde{u}}^{\infty} e^{-(\frac{1}{2} + \frac{x}{\lambda^2})\tilde{v} - \frac{\tilde{v}^2}{2\lambda^2 \ln(n)}} d\tilde{v} \right) d\tilde{u} \\ &\leq \frac{n^{xp+1 - \frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{p\sqrt{2\pi\lambda\sqrt{\ln(n)}}} \int_0^{\infty} e^{p\tilde{u} - (\frac{1}{2} + \frac{x}{\lambda^2})\tilde{u} - \frac{\tilde{u}^2}{2\lambda^2 \ln(n)}} d\tilde{u} \\ &\leq \frac{n^{xp+1 - \frac{(\lambda^2/2+x)^2}{2\lambda^2}}}{p\sqrt{2\pi\lambda\sqrt{\ln(n)}}} \int_{-\infty}^{\infty} e^{p\tilde{u} - (\frac{1}{2} + \frac{x}{\lambda^2})\tilde{u} - \frac{\tilde{u}^2}{2\lambda^2 \ln(n)}} d\tilde{u} \\ &= \frac{n^{xp + \alpha(x, \lambda^2, p)}}{p}, \end{aligned} \tag{4.5}$$

with $\alpha(x, \lambda^2, p) = 1 - \frac{(\lambda^2/2+x)^2}{2\lambda^2} + (p - \frac{1}{2} - \frac{x}{\lambda^2})^2 \frac{\lambda^2}{2}$. We have by combining (4.4) and (4.5):

$$E\left[\sup_{1 \leq i \leq n} e^{pX_i - p\frac{\lambda^2}{2} \ln(n)} \right] \leq n^{xp} + n^{xp + \alpha(x, \lambda^2, p)},$$

We focus on the case $p \in]\frac{1}{2} + \frac{1}{\lambda^2}, \max(\frac{2}{\lambda^2}, 1)[$. This implies inequality (4.2) for $p \leq \frac{1}{2} + \frac{1}{\lambda^2}$; indeed, if inequality (4.2) holds for some p than it holds for all $p' < p$ by applying Jensen's inequality to the concave function $u \rightarrow u^{p'/p}$.

First case: $\lambda^2 < 2$.

Note that $\alpha(1, \lambda^2, \frac{2}{\lambda^2}) = 0$ so if $p < \frac{2}{\lambda^2}$ then there exists $0 < x < 1$ such that $\alpha(x, \lambda^2, p) < 0$.

Second case: $\lambda^2 > 2$.

Note that $\alpha(1, \lambda^2, 1) = 0$ so if $p < 1$ then there exists $0 < x < 1$ such that $\alpha(x, \lambda^2, p) < 0$.

□

4.2. Proof of theorem 2.1. For sake of simplicity, we give the proof in the case $d = 1$, $R = 1$ and the function $f(x) = \lambda^2 \ln^+ \frac{1}{|x|}$. This is no restriction; indeed, the proof in the general case is an immediate adaptation of the following proof.

Uniqueness. Let $\alpha \in]0, 1/2[$. We consider θ and $\tilde{\theta}$ two continuous functions satisfying properties (1), (2) and (3). We note:

$$m(dt) = e^{X(t) - \frac{1}{2}E[X(t)^2]} dt = \lim_{\epsilon \rightarrow 0} e^{X_\epsilon(t) - \frac{1}{2}E[X_\epsilon(t)^2]} dt,$$

where $(X_\epsilon(t))_{t \in \mathbb{R}}$ is a gaussian process of covariance $q_\epsilon(|t-s|)$ with:

$$q_\epsilon(x) = (\theta^\epsilon * f)(x) = \lambda^2 \int_{\mathbb{R}} \theta(v) \ln^+ \left(\frac{1}{|x - \epsilon v|} \right) dv$$

We define similarly the measure \tilde{m} , \tilde{X}_ϵ and \tilde{q}_ϵ associated to the function $\tilde{\theta}$. Note that we suppose that the random measures $m_\epsilon(dt) = e^{X_\epsilon(t) - \frac{1}{2}E[X_\epsilon(t)^2]} dt$ and $\tilde{m}_\epsilon(dt) = e^{\tilde{X}_\epsilon(t) - \frac{1}{2}E[\tilde{X}_\epsilon(t)^2]} dt$ converge in law in the space of Radon measures: this is no restriction since, using Fubini and $E[e^{X_\epsilon(t) - \frac{1}{2}E[X_\epsilon(t)^2]}] = 1$, we get the equality $E[m_\epsilon(A)] = E[\tilde{m}_\epsilon(A)] = |A|$ for all bounded A in $\mathcal{B}(\mathbb{R})$ which implies the measures are tight (see lemma 4.5 in [14]).

We will show that:

$$E[m[0, 1]^\alpha] = E[\tilde{m}[0, 1]^\alpha]$$

for α in the interval $]0, 1/2[$. If we define $Z_\epsilon(t)(u) = \sqrt{t}\tilde{X}_\epsilon(u) + \sqrt{1-t}X_\epsilon(u)$ with $X_\epsilon(u)$ and $\tilde{X}_\epsilon(u)$ independent, we get by using the continuous version of lemma 6.1:

$$E[\tilde{m}_\epsilon[0, 1]^\alpha] - E[m_\epsilon[0, 1]^\alpha] = \frac{\alpha(\alpha-1)}{2} \int_0^1 \varphi_\epsilon(t) dt, \quad (4.6)$$

with $\varphi_\epsilon(t)$ defined by:

$$\varphi_\epsilon(t) = \int_{[0,1]^2} (\tilde{q}_\epsilon(|t_2 - t_1|) - q_\epsilon(|t_2 - t_1|) E[\mathcal{X}_\epsilon(t, t_1, t_2)]) dt_1 dt_2,$$

where $\mathcal{X}_\epsilon(t, t_1, t_2)$ is given by:

$$\mathcal{X}_\epsilon(t, t_1, t_2) = \frac{e^{Z_\epsilon(t)(t_1) + Z_\epsilon(t)(t_2) - \frac{1}{2}E[Z_\epsilon(t)(t_1)^2] - \frac{1}{2}E[Z_\epsilon(t)(t_2)^2]}}{(\int_0^1 e^{Z_\epsilon(t)(u) - \frac{1}{2}E[Z_\epsilon(t)(u)^2]} du)^{2-\alpha}}.$$

We state and prove the following short lemma we will need in the sequel.

Lemma 4.3. *For $A > 0$, we denote $C_A^\epsilon = \sup_{|x| \geq A\epsilon} |q_\epsilon(x) - \tilde{q}_\epsilon(x)|$. We have:*

$$\lim_{A \rightarrow \infty} (\overline{\lim}_{\epsilon \rightarrow 0} C_A^\epsilon) = 0.$$

Proof. Let $|x| \geq A\epsilon$. If $|x| \geq 1/2$ then $q_\epsilon(x)$ and $\tilde{q}_\epsilon(x)$ converge uniformly towards $\lambda^2 \ln^+ \frac{1}{|x|}$ thus $q_\epsilon(x) - \tilde{q}_\epsilon(x)$ converges uniformly to 0 (this a consequence of the fact that $\lambda^2 \ln^+ \frac{1}{|x|}$ is continuous and of compact support for $|x| \geq 1/2$). If $|x| < 1/2$, we write:

$$q_\epsilon(x) = \lambda^2 \left(\ln \frac{1}{\epsilon} + Q(x/\epsilon) + R_\epsilon(x) \right),$$

where $Q(x) = \int_{\mathbb{R}} \ln \frac{1}{|x-z|} \theta(z) dz$ and $R_\epsilon(x)$ converges uniformly to 0 (for $|x| < 1/2$) as $\epsilon \rightarrow 0$ (similarly, we can write $\tilde{q}_\epsilon(x) = \lambda^2 \left(\ln \frac{1}{\epsilon} + \tilde{Q}(x/\epsilon) + \tilde{R}_\epsilon(x) \right)$). This follows from straightforward calculations. Applying lemma 4.1, we get that $Q(x) = \ln \frac{1}{|x|} + \Sigma(x)$

with $\Sigma(x) \rightarrow 0$ for $|x| \rightarrow \infty$. Thus $Q(x) - \tilde{Q}(x)$ is a continuous function such that for $|x| \geq A\epsilon$ and $|x| \leq 1/2$ we have:

$$|q_\epsilon(x) - \tilde{q}_\epsilon(x)| \leq \lambda^2 \sup_{|y| \geq A} |Q(y) - \tilde{Q}(y)| + \lambda^2 \sup_{|x| \leq 1/2} |R_\epsilon(x) - \tilde{R}_\epsilon(x)|$$

The result follows. □

One can decompose expression (4.6) in the following way:

$$E[\tilde{m}_\epsilon[0, 1]^\alpha] - E[m_\epsilon[0, 1]^\alpha] = \frac{\alpha(\alpha - 1)}{2} \int_0^1 \varphi_\epsilon^A(t) dt + \frac{\alpha(\alpha - 1)}{2} \int_0^1 \bar{\varphi}_\epsilon^A(t) dt \quad (4.7)$$

where:

$$\varphi_\epsilon^A(t) = \int_{[0,1]^2, |t_2-t_1| \leq A\epsilon} (\tilde{q}_\epsilon(|t_2 - t_1|) - q_\epsilon(|t_2 - t_1|)) E[\mathcal{X}_\epsilon(t, t_1, t_2)] dt_1 dt_2$$

and

$$\bar{\varphi}_\epsilon^A(t) = \int_{[0,1]^2, |t_2-t_1| > A\epsilon} (\tilde{q}_\epsilon(|t_2 - t_1|) - q_\epsilon(|t_2 - t_1|)) E[\mathcal{X}_\epsilon(t, t_1, t_2)] dt_1 dt_2.$$

With the notations of lemma 4.3, we have:

$$\begin{aligned} |\bar{\varphi}_\epsilon^A(t)| &\leq C_A^\epsilon \int_{[0,1]^2, |t_2-t_1| > A\epsilon} E[\mathcal{X}_\epsilon(t, t_1, t_2)] dt_1 dt_2 \\ &\leq C_A^\epsilon \int_{[0,1]^2} E[\mathcal{X}_\epsilon(t, t_1, t_2)] dt_1 dt_2 \\ &= C_A^\epsilon E\left[\left(\int_0^1 e^{Z_\epsilon(t)(u) - \frac{1}{2}E[Z_\epsilon(t)(u)^2]} du\right)^\alpha\right] \\ &\leq C_A^\epsilon. \end{aligned}$$

Thus, taking the limit as ϵ goes to 0 in (4.7) gives:

$$\overline{\lim}_{\epsilon \rightarrow 0} |E[\tilde{m}_\epsilon[0, 1]^\alpha] - E[m_\epsilon[0, 1]^\alpha]| \leq \frac{\alpha(1 - \alpha)}{2} \overline{\lim}_{\epsilon \rightarrow 0} C_A^\epsilon + \frac{\alpha(1 - \alpha)}{2} \overline{\lim}_{\epsilon \rightarrow 0} \int_0^1 |\varphi_\epsilon^A(t)| dt$$

We will show that $\lim_{\epsilon \rightarrow 0} \varphi_\epsilon^A(0) = 0$ (the general case $\varphi_\epsilon^A(t)$ is similar). There exists a constant $\tilde{C}_A > 0$ independent of ϵ such that:

$$\sup_{|x| \leq A\epsilon} |\tilde{q}_\epsilon(x) - q_\epsilon(x)| \leq \tilde{C}_A.$$

Therefore, we have:

$$\begin{aligned} |\varphi_\epsilon^A(0)| &\leq \tilde{C}_A \int_0^1 \int_{t_1-A\epsilon}^{t_1+A\epsilon} E[\mathcal{X}_\epsilon(0, t_1, t_2)] dt_2 dt_1 \\ &= \tilde{C}_A E\left[\frac{\int_0^1 \int_{t_1-A\epsilon}^{t_1+A\epsilon} e^{X_\epsilon(t_1)+X_\epsilon(t_2) - \frac{1}{2}E[X_\epsilon(t_1)^2] - \frac{1}{2}E[X_\epsilon(t_2)^2]} dt_1 dt_2}{\left(\int_0^1 e^{X_\epsilon(u) - \frac{1}{2}E[X_\epsilon(u)^2]} du\right)^{2-\alpha}}\right] \end{aligned} \quad (4.8)$$

Now we have:

$$\begin{aligned}
& \int_0^1 \int_{t_1 - A\epsilon}^{t_1 + A\epsilon} e^{X_\epsilon(t_1) + X_\epsilon(t_2) - \frac{1}{2}E[X_\epsilon(t_1)^2] - \frac{1}{2}E[X_\epsilon(t_2)^2]} dt_1 dt_2 \\
& \leq \left(\sup_{t_1} \int_{t_1 - A\epsilon}^{t_1 + A\epsilon} e^{X_\epsilon(t_2) - \frac{1}{2}E[X_\epsilon(t_2)^2]} dt_2 \right) \int_0^1 e^{X_\epsilon(t_1) - \frac{1}{2}E[X_\epsilon(t_1)^2]} dt_1 \\
& \leq 2 \left(\sup_{0 \leq i < \frac{1}{2A\epsilon}} \int_{2iA\epsilon}^{2(i+1)A\epsilon} e^{X_\epsilon(t_2) - \frac{1}{2}E[X_\epsilon(t_2)^2]} dt_2 \right) \int_0^1 e^{X_\epsilon(t_1) - \frac{1}{2}E[X_\epsilon(t_1)^2]} dt_1
\end{aligned}$$

In view of (4.8), this implies:

$$\begin{aligned}
|\varphi_\epsilon^A(0)| & \leq 2\tilde{C}_A E \left[\left(\sup_{0 \leq i < \frac{1}{2A\epsilon}} \int_{2iA\epsilon}^{2(i+1)A\epsilon} e^{X_\epsilon(t_2) - \frac{1}{2}E[X_\epsilon(t_2)^2]} dt_2 \right) \left(\int_0^1 e^{X_\epsilon(t_1) - \frac{1}{2}E[X_\epsilon(t_1)^2]} dt_1 \right)^{\alpha-1} \right] \\
& \leq 2\tilde{C}_A E \left[\left(\sup_{0 \leq i < \frac{1}{2A\epsilon}} \int_{2iA\epsilon}^{2(i+1)A\epsilon} e^{X_\epsilon(t_2) - \frac{1}{2}E[X_\epsilon(t_2)^2]} dt_2 \right)^\alpha \right],
\end{aligned}$$

where we used the inequality $\frac{\sup_i a_i}{(\sum_i a_i)^{1-\alpha}} \leq (\sup_i a_i)^\alpha$. For sake of simplicity, we now replace $2A$ by A .

The idea to study the above supremum is to make the approximation $X_\epsilon(t) \approx X_\epsilon(Ai\epsilon)$ for t in $[Ai\epsilon, A(i+1)\epsilon]$. If we define \mathcal{C}_ϵ by:

$$\mathcal{C}_\epsilon = \sup_{\substack{0 \leq i < \frac{1}{A\epsilon} \\ Ai\epsilon \leq u \leq A(i+1)\epsilon}} (X_\epsilon(u) - X_\epsilon(Ai\epsilon)), \quad (4.9)$$

then we have:

$$\begin{aligned}
& E \left[\left(\sup_{0 \leq i < \frac{1}{A\epsilon}} \int_{Ai\epsilon}^{A(i+1)\epsilon} e^{X_\epsilon(t) - \frac{1}{2}E[X_\epsilon(t)^2]} dt \right)^\alpha \right] \\
& \leq E \left[\left(\sup_{0 \leq i < \frac{1}{A\epsilon}} \int_{Ai\epsilon}^{A(i+1)\epsilon} e^{X_\epsilon(Ai\epsilon) - \frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]} dt \right)^\alpha e^{\alpha\mathcal{C}_\epsilon} \right] \\
& = E \left[\left(\epsilon A \sup_{0 \leq i < \frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon) - \frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]} \right)^\alpha e^{\alpha\mathcal{C}_\epsilon} \right] \\
& \leq (\epsilon A)^\alpha E \left[\left(\sup_{0 \leq i < \frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon) - \frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]} \right)^{2\alpha} \right]^{1/2} E \left[e^{2\alpha\mathcal{C}_\epsilon} \right]^{1/2}. \quad (4.10)
\end{aligned}$$

It is straightforward to see that there exists some $c \geq 0$ (independent of ϵ) such that for all s, t in $[0, 1]$:

$$E[X_\epsilon(s)X_\epsilon(t)] \geq -c$$

We introduce a centered gaussian random variable Z independent of X_ϵ and such that $E[Z^2] = c$. Let $(R_i^\epsilon)_{1 \leq i < \frac{1}{A\epsilon}}$ be a sequence of i.i.d gaussian random variables

such that $E[(R_i^\epsilon)^2] = E[X_\epsilon(Ai\epsilon)^2] + c$. By applying corollary 6.3, we get:

$$\begin{aligned} E\left[\left(\sup_{0 \leq i < \frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon) - \frac{1}{2}E[X_\epsilon(Ai\epsilon)^2]} \right)^{2\alpha}\right] &= \frac{1}{e^{2\alpha^2 c - \alpha c}} E\left[\left(\sup_{0 \leq i < \frac{1}{A\epsilon}} e^{X_\epsilon(Ai\epsilon) + Z - \frac{1}{2}E[X_\epsilon(Ai\epsilon)^2] - \frac{c}{2}} \right)^{2\alpha}\right] \\ &\leq \frac{1}{e^{2\alpha^2 c - \alpha c}} E\left[\left(\sup_{0 \leq i < \frac{1}{A\epsilon}} e^{R_i^\epsilon - \frac{1}{2}E[(R_i^\epsilon)^2]} \right)^{2\alpha}\right] \end{aligned}$$

We have $E[(R_i^\epsilon)^2] = \lambda^2 \ln \frac{1}{\epsilon} + C(\epsilon)$ with $C(\epsilon)$ converging to some constant as ϵ goes to 0. Since $2\alpha < 1$, by applying lemma 4.2, there exists $0 < x < 1$ such that:

$$E\left[\left(\sup_{0 \leq i < \frac{1}{A\epsilon}} e^{R_i^\epsilon - \frac{1}{2}E[(R_i^\epsilon)^2]} \right)^{2\alpha}\right] \leq C\left(\frac{1}{\epsilon}\right)^{2\alpha x}$$

and therefore we have:

$$|\varphi_\epsilon^A(0)| \leq C\epsilon^\gamma E\left[e^{2\alpha C_\epsilon}\right]^{1/2}$$

with $\gamma = \alpha(1 - x) > 0$.

One can write $C_\epsilon = \sup_{\substack{0 \leq i < \frac{1}{A\epsilon} \\ 0 \leq v \leq 1}} W_\epsilon^i(v)$ where $W_\epsilon^i(v) = X_\epsilon(Ai\epsilon + A\epsilon v) - X_\epsilon(Ai\epsilon)$. We

have:

$$E[W_\epsilon^i(v)W_\epsilon^i(v')] = g_\epsilon(v - v')$$

where g_ϵ is a continuous function bounded by some constant M independently of ϵ . Let Y be a centered Gaussian random variable independent of W_ϵ^i such that: $E[Y^2] = M$. Thus, we can write:

$$E\left[e^{2\alpha C_\epsilon}\right] = \frac{E\left[e^{2\alpha \sup_{i,v} (W_\epsilon^i(v) + Y)}\right]}{e^{2\alpha^2 M}}.$$

Now let us consider a family $(\overline{W}_\epsilon^i)_{1 \leq i < \frac{1}{A\epsilon}}$ of centered i.i.d. Gaussian processes of law $(W_\epsilon^0(v) + Y)_{0 \leq v \leq 1}$. Applying corollary 6.3 of the appendix, we get:

$$E\left[e^{2\alpha C_\epsilon}\right] \leq \frac{E\left[e^{2\alpha \sup_{i,v} \overline{W}_\epsilon^i(v)}\right]}{e^{2\alpha^2 M}}$$

We now estimate $E\left[e^{2\alpha \sup_{i,v} \overline{W}_\epsilon^i(v)}\right]$. Let us denote $\mathcal{X}_i = \sup_{0 \leq v \leq 1} \overline{W}_\epsilon^i$. Applying Corollary 3.2 of [16] to the continuous Gaussian process $(W_\epsilon^0(v) + Y)_{0 \leq v \leq 1}$, we get that the random variable has a Gaussian tail:

$$P(\mathcal{X}_i > z) \leq C e^{-\frac{z^2}{2\sigma^2}}, \quad \forall z > 0$$

for some C and σ . The above tail inequality gives the existence of some constant $C > 0$ such that:

$$E\left[e^{2\alpha \sup_{0 \leq i < \frac{1}{A\epsilon}} \mathcal{X}_i}\right] \leq C e^{C\sqrt{\ln(\frac{1}{\epsilon})}}.$$

Therefore we have $E\left[e^{2\alpha C_\epsilon}\right] \leq C e^{C\sqrt{\ln(\frac{1}{\epsilon})}}$ and then:

$$|\varphi_\epsilon^A(0)| \leq C \epsilon^\gamma e^{C\sqrt{\ln(\frac{1}{\epsilon})}}.$$

It follows that $\overline{\lim}_{\epsilon \rightarrow 0} |\varphi_\epsilon^A(0)| = 0$ so that for $\alpha < 1/2$:

$$\overline{\lim}_{\epsilon \rightarrow 0} |E[\tilde{m}_\epsilon[0, 1]^\alpha] - E[m_\epsilon[0, 1]^\alpha]| \leq \frac{\alpha(1-\alpha)}{2} \overline{\lim}_{\epsilon \rightarrow 0} C_A^\epsilon.$$

Since $\overline{\lim}_{\epsilon \rightarrow 0} C_A^\epsilon \rightarrow 0$ as A goes to infinity (lemma 4.3), we conclude that:

$$\overline{\lim}_{\epsilon \rightarrow 0} |E[\tilde{m}_\epsilon[0, 1]^\alpha] - E[m_\epsilon[0, 1]^\alpha]| = 0.$$

It is straightforward to check that the above proof can be generalized to show that for all positive $\lambda_1, \dots, \lambda_n$ and intervals I_1, \dots, I_n we have:

$$E\left[\left(\sum_{k=1}^n \lambda_k m(I_k)\right)^\alpha\right] = E\left[\left(\sum_{k=1}^n \lambda_k \tilde{m}(I_k)\right)^\alpha\right]$$

This implies that the random measures m and \tilde{m} are equal (see [8]).

Existence. Let $f(x)$ be a real positive definite function on \mathbb{R}^d (note that this implies that f is symmetric). Let us recall that a centered Gaussian field of correlation $f(x-y)$ can be constructed by the following formula:

$$X(x) = \int_{\mathbb{R}^d} \zeta(x, \xi) \sqrt{\hat{f}(\xi)} W(d\xi),$$

where $\zeta(x, \xi) = \cos(2\pi x \cdot \xi) - \sin(2\pi x \cdot \xi)$ and $W(d\xi)$ is the standard white noise on \mathbb{R}^d (to see this, one can check using the inverse Fourier formula that the above X has the desired correlations). This can also be written:

$$X(x) = \int_{]0, \infty[\times \mathbb{R}^d} \zeta(x, \xi) \sqrt{\hat{f}(\xi)} g(t, \xi) W(dt, d\xi), \quad (4.11)$$

where $W(dt, d\xi)$ is the white noise on $]0, \infty[\times \mathbb{R}^d$ and $\int_0^\infty g(t, \xi)^2 dt = 1$ for all ξ . The interest of the expression (4.11) appears in what follows. Let the function θ be radially symmetric and $\hat{\theta}$ be a decreasing function of $|\xi|$ (for instance take $\theta(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{d/2}}$). Let us consider $g(t, \xi) = \sqrt{-\hat{\theta}'(t|\xi|)|\xi|}$ so that $\int_\epsilon^\infty g(t, \xi)^2 dt = \hat{\theta}(\epsilon|\xi|)$ for $|\xi| \neq 0$. Then if we consider the fields X_ϵ defined by:

$$X_\epsilon(x) = \int_{] \epsilon, \infty[\times \mathbb{R}^d} \zeta(x, \xi) \sqrt{\hat{f}(\xi)} g(t, \xi) W(dt, d\xi) \quad (4.12)$$

we will find:

$$\begin{aligned} E[X_\epsilon(x) X_\epsilon(y)] &= \int_{\mathbb{R}^d} \cos(2\pi(x-y) \cdot \xi) \hat{f}(\xi) \hat{\theta}(\epsilon|\xi|) d\xi \\ &= (f * \theta^\epsilon)(x-y). \end{aligned}$$

The interest of (4.12) is to make the approximation process appear as a martingale. Indeed, if we define the filtration $\mathcal{F}_\epsilon = \sigma\{W(A, B), A \subset]\epsilon, \infty[, B \in \mathcal{B}(\mathbb{R}^d) \text{ and bounded}\}$,

we have that for all $A \in \mathcal{B}(\mathbb{R}^d)$, $(m_\epsilon(A))_{\epsilon>0}$ is a positive \mathcal{F}_ϵ -martingale of expectation $|A|$ so it converges almost surely to a random variable $m(A)$ such that:

$$E[m(A)] \leq |A|. \quad (4.13)$$

This defines a collection $(m(A))_{A \in \mathcal{B}(\mathbb{R}^d)}$ of random variables such that:

- (1) for all disjoint and bounded sets A_1, A_2 in $\mathcal{B}(\mathbb{R}^d)$,

$$m(A_1 \cup A_2) = m(A_1) + m(A_2) \quad a.s.$$

- (2) For any bounded sequence $(A_n)_{n \geq 1}$ decreasing to \emptyset :

$$m(A_n) \xrightarrow[n \rightarrow \infty]{} 0 \quad a.s.$$

By theorem 6.1. VI. in [8], one can consider a version of the collection $(m(A))_{A \in \mathcal{B}(\mathbb{R}^d)}$ such that m is a random measure. It is straightforward that m_ϵ converges almost surely towards m in the space of Radon measures (equipped with the weak topology).

5. PROOFS OF SECTION 3

5.1. Proof of proposition 3.1.

Proof. Since $\zeta_1 = d$, note that $\lambda^2 > 2d$ is equivalent to the existence of $\alpha < 1$ such that $\zeta_\alpha > d$. Let α be fixed and such that $\zeta_\alpha > d$. We will show that $m[[0, 1]^d] = 0$. We partition the cube $[0, 1]^d$ into $\frac{1}{\epsilon^d}$ subcubes $(I_j)_{1 \leq j \leq \frac{1}{\epsilon^d}}$ of size ϵ . One has by subadditivity and homogeneity:

$$\begin{aligned} & E\left[\left(\int_{[0,1]^d} e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]} dx\right)^\alpha\right] \\ &= E\left[\left(\sum_{1 \leq j \leq \frac{1}{\epsilon^d}} \int_{I_j} e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]} dx\right)^\alpha\right] \\ &\leq E\left[\sum_{1 \leq j \leq \frac{1}{\epsilon^d}} \left(\int_{I_j} e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]} dx\right)^\alpha\right] \\ &= \frac{1}{\epsilon^d} E\left[\left(\int_{[0,\epsilon]^d} e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]} dx\right)^\alpha\right] \end{aligned}$$

Let Y_ϵ be a centered Gaussian random variable of variance $\lambda^2 \ln(\frac{1}{\epsilon}) + \lambda^2 c$ where c is such that:

$$\theta^\epsilon * \ln^+ \frac{1}{|x|} \geq \ln \frac{1}{\epsilon} + c$$

for $|x| \leq \epsilon$ and ϵ small enough. By definition of c , we have

$$\forall x, x' \in [0, \epsilon]^d, \quad E[X_\epsilon(x)X_\epsilon(x')] \geq E[Y_\epsilon^2].$$

Using corollary (6.2) in the continuous version, this implies:

$$\begin{aligned}
& E\left[\left(\int_{[0,1]^d} e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]} dx\right)^\alpha\right] \\
& \leq \frac{1}{\epsilon^d} E\left[\left(\int_{[0,\epsilon]^d} e^{Y_\epsilon - \frac{1}{2}E[Y_\epsilon^2]} dx\right)^\alpha\right] \\
& = \frac{\epsilon^{d\alpha}}{\epsilon^d} E\left[(e^{Y_\epsilon - \frac{1}{2}E[Y_\epsilon^2]})^\alpha\right] \\
& = e^{\frac{\alpha^2 - \alpha}{2}c} \epsilon^{\zeta_\alpha - d}
\end{aligned}$$

Taking the limit as ϵ goes to 0 gives $m[[0, 1]^d] = 0$. \square

5.2. Proof of lemma 3.2.

Proof. One has the following general formula for the Fourier transform of radial functions:

$$\hat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{d-2}{2}}} \int_0^\infty \rho^{\frac{d}{2}} J_{\frac{d-2}{2}}(2\pi|\xi|\rho) f(\rho) d\rho, \quad (5.1)$$

where J_ν is the Bessel function of order ν (see [21] for instance).

First case: $d \leq 3$.

It is enough to consider the case $d = 3$. Indeed, consider some function φ in $\mathcal{S}(\mathbb{R}^2)$. We introduce the family of functions $\psi_\epsilon(x_1, x_2, x_3) = \varphi(x_1, x_2)\theta_\epsilon(x_3)$ where θ_ϵ is a smooth function that converges to the Dirac mass δ_0 as ϵ goes to 0. If we take the limit as ϵ goes to 0 in inequality (2.1) applied to ψ_ϵ , we get:

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x-y, 0) \varphi(x) \overline{\varphi(y)} dx dy \geq 0.$$

This shows that $(x_1, x_2) \rightarrow f(x_1, x_2, 0)$ is positive definite. Similarly, one can show that $x \rightarrow f(x, 0, 0)$ is positive definite.

Using the explicit formula $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$, we conclude by integrating by parts:

$$\begin{aligned}
\hat{f}(\xi) &= \frac{2}{|\xi|} \int_0^T \rho \sin(2\pi|\xi|\rho) \ln\left(\frac{T}{\rho}\right) d\rho \\
&= \frac{1}{\pi|\xi|^2} \int_0^T \cos(2\pi|\xi|\rho) \left(\ln\left(\frac{T}{\rho}\right) - 1\right) d\rho \\
&= \frac{1}{2\pi^2|\xi|^3} \left(\int_0^T \frac{\sin(2\pi|\xi|\rho)}{\rho} d\rho - \sin(2\pi|\xi|T) \right) \\
&= \frac{1}{2\pi^2|\xi|^3} (\text{sinc}(2\pi|\xi|T) - \sin(2\pi|\xi|T)),
\end{aligned}$$

where sinc is the sinus cardinal function:

$$\text{sinc}(x) = \int_0^x \frac{\sin(\rho)}{\rho} d\rho.$$

We introduce for $x \geq 0$ the function $l(x) = \text{sinc}(x) - \sin(x)$. Since $\hat{f}(\xi) = \frac{l(2\pi|\xi|T)}{2\pi^2|\xi|^3}$, the non negativity of \hat{f} is equivalent to the non negativity of l . We have $l'(x) = \frac{\sin(x) - x \cos(x)}{x}$. Thus, there exists α in $] \pi, 2\pi[$ such that l is increasing on $]0, \alpha[$ and decreasing on $] \alpha, 2\pi[$. Since $l(0) = 0$ and $l(2\pi) = \int_0^{2\pi} \frac{\sin(\rho)}{\rho} d\rho \geq 0$, we conclude that for all x in $[0, 2\pi]$, $l(x) \geq 0$. A classical computation (Dirichlet integral) gives $\int_0^\infty \frac{\sin(\rho)}{\rho} d\rho = \frac{\pi}{2}$. Thus, we have by an integration by parts:

$$\begin{aligned} \int_0^{2\pi} \frac{\sin(\rho)}{\rho} d\rho &= \frac{\pi}{2} - \int_{2\pi}^\infty \frac{\sin(\rho)}{\rho} d\rho \\ &= \frac{\pi}{2} - \int_{2\pi}^\infty \frac{1 - \cos(\rho)}{\rho^2} d\rho \\ &\geq \frac{\pi}{2} - \frac{1}{2\pi} \\ &\geq 1 \end{aligned}$$

Therefore, if $x \geq 2\pi$, we have:

$$\begin{aligned} l(x) &= \int_0^x \frac{\sin(\rho)}{\rho} d\rho - \sin(x) \\ &\geq \int_0^{2\pi} \frac{\sin(\rho)}{\rho} d\rho - \sin(x) \\ &\geq 0. \end{aligned}$$

Second case: $d \geq 4$. Combining (5.1) with the identity $\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$, we get:

$$\begin{aligned} \hat{f}(\xi) &= \frac{2\pi}{|\xi|^{\frac{d-2}{2}}} \int_0^T \rho^{\frac{d}{2}} J_{\frac{d-2}{2}}(2\pi|\xi|\rho) \ln\left(\frac{T}{\rho}\right) d\rho \\ &= \frac{1}{(2\pi)^{d/2} |\xi|^d} \int_0^{2\pi|\xi|T} x^{\frac{d}{2}} J_{\frac{d-2}{2}}(x) \ln\left(\frac{2\pi|\xi|T}{x}\right) dx \\ &= \frac{1}{(2\pi)^{d/2} |\xi|^d} \int_0^{2\pi|\xi|T} x^{\frac{d}{2}-1} J_{\frac{d}{2}}(x) dx \end{aligned} \quad (5.2)$$

One has the following asymptotic expansion as x goes to ∞ ([12]):

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(1+2\nu)\pi}{4}\right) - \frac{(4\nu^2-1)\sqrt{2}}{8\sqrt{\pi}x^{3/2}} \sin\left(x - \frac{(1+2\nu)\pi}{4}\right) + O\left(\frac{1}{x^{5/2}}\right). \quad (5.3)$$

Combining (5.2) with (5.3), we therefore get the following expansion as $|\xi|$ goes to infinity:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2} |\xi|^d} \left(\sqrt{\frac{2}{\pi}} (2\pi|\xi|T)^{\frac{d-3}{2}} \sin\left(2\pi|\xi|T - \frac{(1+2\nu)\pi}{4}\right) + o(|\xi|^{\frac{d-3}{2}}) \right).$$

Thus $\overline{\lim}_{|\xi| \rightarrow \infty} |\xi|^d \hat{f}(\xi) = -\underline{\lim}_{|\xi| \rightarrow \infty} |\xi|^d \hat{f}(\xi) = +\infty$. In particular, $\hat{f}(\xi)$ takes negative values for some ξ . \square

5.3. Proofs of section 3.3.

Proof of proposition 3.5 and proposition 3.6. We suppose that p belongs to $]1, p_*[$ or $] - \infty, 0[$. Let θ be some function satisfying the conditions (1), (2), (3) of section 2.2 and m_ϵ be the random measure associated to $\theta^\epsilon * f$. Following the notations of example 2.2 for $C(x)$, we consider \tilde{m}_ϵ the random measure associated to \tilde{f}_ϵ where \tilde{f}_ϵ is the function:

$$\tilde{f}_\epsilon(x) = \lambda^2 \int_{C(0) \cap C(x); \epsilon < t < \infty} \frac{dy dt}{t^{d+1}}.$$

One can show that there exists $c, C > 0$ such that for all x we have (see appendix B in [5]):

$$\tilde{f}_\epsilon(x) - c \leq (\theta^\epsilon * f)(x) \leq \tilde{f}_\epsilon(x) + C$$

By using corollary 6.2 of the appendix in the continuous version (with $F(x) = x^p$), we conclude that there exists $c, C > 0$ such that for all ϵ and all bounded A in $\mathcal{B}(\mathbb{R}^d)$:

$$cE[\tilde{m}_\epsilon(A)^p] \leq E[m_\epsilon(A)^p] \leq CE[\tilde{m}_\epsilon(A)^p].$$

First case: p belongs to $]1, p_*[$.

Proposition 3.5 is therefore established if we can show that:

$$\sup_{\epsilon > 0} E[\tilde{m}_\epsilon(A)^p] < \infty.$$

To prove the above inequality for all bounded A , it is enough to suppose that $A = [0, 1]^d$. This is proved in dimension 1 in [3] (theorem 3). One can adapt the dyadic decomposition performed in the proof of theorem 3 in [3] to handle the d dimensional case.

Second case: p belongs to $] - \infty, 0[$.

Proposition 3.5 is therefore established if we can show that for all $c > 0$:

$$\sup_{\epsilon > 0} E[\tilde{m}_\epsilon(B(0, c))^p] < \infty.$$

The above bound can be proved by adapting the proof of proposition 4 in [18] (this is done to prove theorem 3 in [4] where a log Poisson model is considered).

Proof of proposition 3.7. For the sake of simplicity, we consider the case $R = 1$ and we will consider the case $p \in [1, p_*[$. We consider θ a continuous and positive function with compact support $B(0, A)$ satisfying properties (1), (2) and (3) of section 2.2. We note:

$$m_\epsilon(dx) = e^{X_\epsilon(x) - \frac{1}{2}E[X_\epsilon(x)^2]} dx,$$

where $(X_\epsilon(x))_{x \in \mathbb{R}^d}$ is a gaussian field of covariance $q_\epsilon(x - y)$ with:

$$q_\epsilon(x) = (\theta^\epsilon * f)(x) = \int_{\mathbb{R}^d} \theta(z) (\lambda^2 \ln^+ \frac{1}{|x - \epsilon z|} + g(x - \epsilon z)) dz.$$

Let c, c' be two positive numbers in $]0, \frac{1}{2}[$ such that $c < c'$. If ϵ is sufficiently small and u, v belong to $[0, 1]^d$, we get:

$$\begin{aligned} q_{c\epsilon}(c(v-u)) &= \int_{\mathbb{R}^d} \theta(z) \left(\lambda^2 \ln \frac{1}{|c(v-u) - c\epsilon z|} + g(c(v-u) - c\epsilon z) \right) dz \\ &= \lambda^2 \ln\left(\frac{c'}{c}\right) + \int_{\mathbb{R}^d} \theta(z) \left(\lambda^2 \ln \frac{1}{|c'(v-u) - c'\epsilon z|} + g(c(v-u) - c\epsilon z) \right) dz \\ &\leq \lambda^2 \ln\left(\frac{c'}{c}\right) + q_{c'\epsilon}(c'(v-u)) + C_{c,c',\epsilon}, \end{aligned}$$

where

$$C_{c,c',\epsilon} = \sup_{\substack{|z| \leq A \\ |v-u| \leq 1}} |g(c(v-u) - c\epsilon z) - g(c'(v-u) - c'\epsilon z)|.$$

Let $Y_{c,c',\epsilon}$ be some centered gaussian variable with variance $C_{c,c',\epsilon} + \lambda^2 \ln(\frac{c'}{c})$. By using corollary 6.2 of the appendix in the continuous version, we conclude that:

$$\begin{aligned} E[m_{c\epsilon}([0, c]^d)^p] &= E\left[\left(\int_{[0,c]^d} e^{X_{c\epsilon}(x) - \frac{1}{2}E[X_{c\epsilon}(x)^2]} dx\right)^p\right] \\ &= c^{dp} E\left[\left(\int_{[0,1]^d} e^{X_{c\epsilon}(cu) - \frac{1}{2}E[X_{c\epsilon}(cu)^2]} du\right)^p\right] \\ &\leq c^{dp} E\left[\left(\int_{[0,1]^d} e^{X_{c'\epsilon}(c'u) + Y_{c,c',\epsilon} - \frac{1}{2}E[(X_{c'\epsilon}(c'u) + Y_{c,c',\epsilon})^2]} du\right)^p\right] \\ &= c^{dp} \left(\frac{c'}{c}\right)^{\frac{p(p-1)\lambda^2}{2}} e^{\frac{p(p-1)C_{c,c',\epsilon}}{2}} E\left[\left(\int_{[0,1]^d} e^{X_{c'\epsilon}(c'u) - \frac{1}{2}E[X_{c'\epsilon}(c'u)^2]} du\right)^p\right] \\ &= \left(\frac{c}{c'}\right)^{dp - \frac{p(p-1)\lambda^2}{2}} e^{\frac{p(p-1)C_{c,c',\epsilon}}{2}} E\left[\left(\int_{[0,c']^d} e^{X_{c'\epsilon}(x) - \frac{1}{2}E[X_{c'\epsilon}(x)^2]} dx\right)^p\right] \\ &= \left(\frac{c}{c'}\right)^{\zeta_p} e^{\frac{p(p-1)C_{c,c',\epsilon}}{2}} E[m_{c'\epsilon}([0, c']^d)^p] \end{aligned}$$

Taking the limit $\epsilon \rightarrow 0$ in the above inequality leads to:

$$\frac{E[m([0, c]^d)^p]}{c^{\zeta_p}} \leq e^{\frac{p(p-1)C_{c,c'}}{2}} \frac{E[m([0, c']^d)^p]}{c'^{\zeta_p}}, \quad (5.4)$$

where $C_{c,c'} = \sup_{|v-u| \leq 1} |g(c(v-u)) - g(c'(v-u))|$. Similarly, we have:

$$\frac{E[m([0, c']^d)^p]}{c'^{\zeta_p}} \leq e^{\frac{p(p-1)C_{c,c'}}{2}} \frac{E[m([0, c]^d)^p]}{c^{\zeta_p}}. \quad (5.5)$$

Since $C_{c,c'}$ goes to 0 as $c, c' \rightarrow 0$, we conclude by inequality (5.4) and (5.5) that $\left(\frac{E[m([0, c]^d)^p]}{c^{\zeta_p}}\right)_{c>0}$ is a Cauchy sequence as $c \rightarrow 0$ bounded from below and above by positive constants. Therefore, there exists some $c_p > 0$ such that:

$$E[m([0, c]^d)^p] \underset{c \rightarrow 0}{\sim} c_p c^{\zeta_p}.$$

The same method can be applied to show that $\frac{c_p}{e^{\frac{p(p-1)g(0)}{2}}}$ is independent of g . This concludes the proof by setting $C_p = \frac{c_p}{e^{\frac{p(p-1)g(0)}{2}}}$.

Proof of proposition 3.8. We use the scaling relation (3.3) to compute the characteristic function of $m(B(0, c))$ for all ξ in \mathbb{R} :

$$\begin{aligned} E[e^{i\xi m(B(0, c))}] &= E[e^{i\xi e^{\Omega c} m(B(0, R))}] \\ &= E[\mathcal{F}(\xi m(B(0, R)))], \end{aligned}$$

where \mathcal{F} is the characteristic function of $e^{\Omega c}$. It is easy to show that for all $n \in \mathbb{N}$ there exists $C > 0$ such that:

$$|\mathcal{F}(\xi)| \leq \frac{C}{|\xi|^n}.$$

From this, we conclude by proposition 3.6 that:

$$E[e^{i\xi m(B(0, c))}] \leq \frac{C}{|\xi|^n} E\left[\frac{1}{m(B(0, R))^n}\right] \leq \frac{C'}{|\xi|^n}.$$

This implies the existence of a C^∞ density.

6. APPENDIX

We give the following classical lemma first derived in [13].

Lemma 6.1. *Let $(X_i)_{1 \leq i \leq n}$ and $(Y_i)_{1 \leq i \leq n}$ be two independent centered Gaussian vectors and $(p_i)_{1 \leq i \leq n}$ a sequence of positive numbers. If $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is some smooth function with polynomial growth at infinity, we define:*

$$\varphi(t) = E\left[\phi\left(\sum_{i=1}^n p_i e^{Z_i(t) - \frac{1}{2}E[Z_i(t)^2]}\right)\right],$$

with $Z_i(t) = \sqrt{t}X_i + \sqrt{1-t}Y_i$. Then, we have the following formula for the derivative:

$$\varphi'(t) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n p_i p_j (E[X_i X_j] - E[Y_i Y_j]) E\left[e^{Z_i(t) + Z_j(t) - \frac{1}{2}E[Z_i(t)^2] - \frac{1}{2}E[Z_j(t)^2]} \phi''(W_{n,t})\right], \quad (6.1)$$

where:

$$W_{n,t} = \sum_{k=1}^n p_k e^{Z_k(t) - \frac{1}{2}E[Z_k(t)^2]}$$

As a consequence of the above formula, we can derive a similar formula in the continuous case. Let I be a bounded subinterval of \mathbb{R}^d and $(X(u))_{u \in I}$, $(Y(u))_{u \in I}$ be two independent centered continuous gaussian processes. If we define:

$$\varphi(t) = E\left[\phi\left(\int_I e^{Z(t)(u) - \frac{1}{2}E[Z(t)(u)^2]} du\right)\right],$$

with $Z(t)(u) = \sqrt{t}X(u) + \sqrt{1-t}Y(u)$, then we have the following formula for the derivative:

$$\begin{aligned} \varphi'(t) &= \frac{1}{2} \int_I \int_I (E[X(t_1)X(t_2)] - E[Y(t_1)Y(t_2)]) \\ &\quad \times E[e^{Z(t)(t_1)+Z(t)(t_2)-\frac{1}{2}E[Z(t)(t_1)^2]-\frac{1}{2}E[Z(t)(t_2)^2]} \phi''(W_t)] dt_1 dt_2, \end{aligned}$$

where:

$$W_t = \int_I e^{Z(t)(u)-\frac{1}{2}E[Z(t)(u)^2]} du$$

As a consequence of the above lemma, one can derive the following classical comparison principle:

Corollary 6.2. *Let $(p_i)_{1 \leq i \leq n}$ be a sequence of positive numbers. Consider $(X_i)_{1 \leq i \leq n}$ and $(Y_i)_{1 \leq i \leq n}$ two centered Gaussian vectors such that:*

$$\forall i, j \quad E[X_i X_j] \leq E[Y_i Y_j].$$

Then, for all convex function $F : \mathbb{R} \rightarrow \mathbb{R}$, we have:

$$E[F(\sum_{i=1}^n p_i e^{X_i - \frac{1}{2}E[X_i^2]})] \leq E[F(\sum_{i=1}^n p_i e^{Y_i - \frac{1}{2}E[Y_i^2]})]. \quad (6.2)$$

Similarly, we get a comparison in the continuous case. Let I be a bounded subinterval of \mathbb{R}^d and $(X(u))_{u \in I}$, $(Y(u))_{u \in I}$ be two independent centered continuous gaussian processes such that:

$$\forall u, u' \quad E[X(u)X(u')] \leq E[Y(u)Y(u')].$$

Then, for all convex function $F : \mathbb{R} \rightarrow \mathbb{R}$, we have:

$$E[F(\int_I e^{X(u)-\frac{1}{2}E[X(u)^2]} du)] \leq E[F(\int_I e^{Y(u)-\frac{1}{2}E[Y(u)^2]} du)].$$

We will also use another corollary:

Corollary 6.3. *Let $(X_i)_{1 \leq i \leq n}$ and $(Y_i)_{1 \leq i \leq n}$ be two centered Gaussian vectors such that:*

- $\forall i, \quad E[X_i^2] = E[Y_i^2].$
- $\forall i \neq j, \quad E[X_i X_j] \leq E[Y_i Y_j].$

Then, for all increasing function $F : \mathbb{R} \rightarrow \mathbb{R}_+$, we have:

$$E[F(\sup_{1 \leq i \leq n} Y_i)] \leq E[F(\sup_{1 \leq i \leq n} X_i)]. \quad (6.3)$$

Proof. It is enough to show inequality (6.3) for $F = 1_{]x, +\infty[}$ for some $x \in \mathbb{R}$. Let β be some positive parameter. Integrating equality (6.1) applied to the convex function $\phi : u \rightarrow e^{-e^{-\beta x} u}$ and the sequences (βX_i) , (βY_i) , $p_i = e^{\frac{\beta^2}{2}E[X_i^2]}$, we get:

$$E[e^{-\sum_{i=1}^n e^{\beta(X_i-x)}}] \leq E[e^{-\sum_{i=1}^n e^{\beta(Y_i-x)}}]$$

By letting $\beta \rightarrow \infty$, we conclude:

$$P(\sup_{1 \leq i \leq n} X_i < x) \leq P(\sup_{1 \leq i \leq n} Y_i < x).$$

□

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