

# A DIRICHLET PROCESS CHARACTERIZATION OF A CLASS OF REFLECTED DIFFUSIONS

WEINING KANG AND KAVITA RAMANAN

ABSTRACT. For a general class of stochastic differential equations with reflection that admit a Markov weak solution and satisfy a certain  $L^p$  continuity condition,  $p > 1$ , it is shown that the associated reflected diffusion can be decomposed as the sum of a local martingale and a continuous, adapted process of zero  $p$ -variation. In particular, when  $p = 2$ , this implies that the associated reflected diffusion is a Dirichlet processes in the sense of Föllmer. As motivation for such a characterization, it is also shown that reflected diffusions belonging to a specific family within this class are not semimartingales, but are Dirichlet processes. This family of diffusions arise naturally as approximations of certain stochastic networks that use the so-called generalized processor sharing scheduling policy.

---

## CONTENTS

---

1. Introduction	2
1.1. Background and Motivation	2
1.2. Notation	4
2. The Class of Reflected Diffusions	4
2.1. Stochastic Differential Equations with Reflection	5
2.2. Main Assumptions	6
2.3. A Motivating Example and Ramifications of the Assumptions	7
3. Statement of Main Results	10
4. Reflected Diffusions Associated with Class $\mathcal{A}$ SDERs	12
4.1. The Zero Drift Case	12
4.2. The General Drift Case	26
4.3. The Semimartingale Property for $Z$	28
5. Dirichlet Process Characterization	31
5.1. Localization	31
5.2. Proof of Theorem 3.5	32
Appendix A. Elementary Properties of the ESP	37
Appendix B. Auxiliary Results	37
References	38

---

*Date:* June 11, 2008.

The second author was partially supported by NSF grants DMS-0406191, DMS-0405343, CMMI-0728064.

**AMS Subject Classification.** Primary: 60G17, 60J55; Secondary: 60J65

**Key words.** reflected Brownian motion, reflected diffusions, Skorokhod problem, Skorokhod map, extended Skorokhod problem, semimartingale property, Dirichlet process, zero energy, generalized processor sharing, diffusion approximations.

## 1. INTRODUCTION

**1.1. Background and Motivation.** This work identifies fairly general sufficient conditions under which a reflected diffusion can be decomposed as the sum of a local martingale and a continuous adapted process of zero  $p$ -variation, for some  $p$  greater than or equal to one. As motivation for such a characterization, it is also shown that a large class of multidimensional, obliquely reflected diffusions fail to be semimartingales, but are nevertheless Dirichlet processes in the sense of Föllmer [15]. Dirichlet processes are processes that can be expressed (uniquely) as the sum of a local martingale and a continuous process that has zero quadratic variation and, as such, are an extension of the class of continuous semimartingales. As is well-known, semimartingales form an important class of processes for stochastic integration, they are stable under  $\mathcal{C}^2$  transformations and admit an Itô change-of-variable formula. However, there are many natural operations that lead out of the class of semimartingales and motivate the consideration of Dirichlet processes. For example,  $\mathcal{C}^1$  functionals of Brownian motion, certain functionals of stationary symmetric Markov processes associated with Dirichlet forms [18], and Lipschitz functionals of a broad class of semimartingale reflected diffusions in bounded domains [26, 27], are all Dirichlet processes that are in general not semimartingales. Moreover, Dirichlet processes exhibit many nice properties analogous to semimartingales. They admit a natural, Doob-Meyer-type decomposition [7], they are stable under  $\mathcal{C}^1$  transformations (see Proposition 11 of [28] and also [2]) and there are extensions of stochastic calculus and Itô's formula that apply to Dirichlet processes (see [14], [16] and Chapter 4 of [28]) or, more generally, to processes that admit a decomposition as the sum of a local martingale and a continuous, adapted process of bounded  $p$ -variation, for  $p \in (1, 2)$  [2].

The theory of reflected diffusions is most well-developed for semimartingale or symmetric reflected diffusions. In particular, the Skorokhod problem approach to the study of reflected diffusions [29, 10, 22] is automatically limited to semimartingales, while the Dirichlet form approach is best suited to analyze symmetric diffusions (see, e.g., [5], [18]). However, using the submartingale formulation of Stroock and Varadhan [30] or the extended Skorokhod problem [22], it is possible to construct reflected diffusions that are neither semimartingales nor symmetric processes [3, 4, 23, 24, 31]. This leads naturally to the question of determining when these reflected diffusions

are semimartingales and, if they are not semimartingales, whether they belong to some other tractable class of processes such as Dirichlet processes. There has been quite a lot of work that shows that, under a certain condition on the domain and reflection directions (namely, the completely- $\mathcal{S}$  condition and generalizations of it), the associated reflected diffusions are semimartingales [22, 32]. In contrast, it has been a longstanding open problem (see Section 4(iii) of [32]) to develop a theory for multidimensional reflected diffusions for which this condition fails to hold (some results in two dimensions can be found in [3, 4, 31]). As shown in [23, 24], such reflected diffusions arise naturally as approximations of a so-called generalized processor sharing model used in telecommunication networks. Thus the development of such a theory is also of interest from the perspective of applications.

The first main result of this work (Theorem 3.1) shows that multidimensional reflected diffusions belonging to a slight generalization of the family of reflected diffusions obtained as approximations in [23, 24] fail to be semimartingales. In two dimensions and for the case of reflected Brownian motion, this result follows from Theorem 5 of [31] (also see [3] for an alternative proof of this result). However, the analysis in [31] uses constructions in polar coordinates that seem not easily generalizable to higher dimensions. We follow a different approach, which is independent of dimension and allows us to establish the result for uniformly elliptic reflected diffusions, with possibly state-dependent diffusion coefficients, rather than just reflected Brownian motion.

The next main result (Theorem 3.5) shows that a broad class of reflected diffusions admit a decomposition as the sum of a local martingale and a process of zero  $p$ -variation, for some  $p > 1$ . This class consists of Markov, weak solutions to stochastic differential equations with reflection that have locally bounded drift and dispersion coefficients and satisfy a certain  $L^p$  continuity requirement (see Assumption 2). This continuity requirement is satisfied, for example, when the associated extended Skorokhod map is Hölder continuous with exponent greater than or equal to  $2/p$ , but also holds under much weaker conditions that do not even require that the (extended) Skorokhod map be well-defined (see Remark 2.4). For the case when  $p = 2$ , which holds, for example, when the corresponding extended Skorokhod map is Lipschitz continuous, this implies that the associated reflected diffusion is a Dirichlet process. Using this result, it is shown in Corollary 3.6 that the non-semimartingale reflected diffusions considered in Theorem 3.1 are Dirichlet processes.

The paper is organized as follows. Some common notation used throughout the paper is first summarized in Section 1.2. The class of stochastic differential equations with reflection under consideration is then defined in Section 2, while Section 3 contains a rigorous statement of the main results. The proof of Theorem 3.1 is presented in Section 4, while the proof of Theorem 3.5 is given in Section 5. Some elementary results required in the proofs are relegated to the Appendix.

**1.2. Notation.** As usual,  $\mathbb{R}_+$  or  $[0, \infty)$  denote the space of all non-negative reals, and  $\mathbb{N}$  denotes the space of all positive integers. Given two real numbers  $a$  and  $b$ ,  $a \wedge b$  and  $a \vee b$  denote the minimum and maximum, respectively, of  $a$  and  $b$ . For each positive integer  $J \geq 1$ ,  $\mathbb{R}^J$  denotes  $J$ -dimensional Euclidean space and the nonnegative orthant in this space is denoted by  $\mathbb{R}_+^J = \{x \in \mathbb{R}^J : x_i \geq 0 \text{ for } i = 1, \dots, J\}$ . The Euclidean norm of  $x \in \mathbb{R}^J$  is denoted by  $|x|$  and the inner product of  $x, y \in \mathbb{R}^J$  is denoted by  $\langle x, y \rangle$ . The vectors  $(e_1, e_2, \dots, e_J)$  represent the usual orthonormal basis for  $\mathbb{R}^J$ , with  $e_i$  being the  $i$ th coordinate vector. Given a vector  $u \in \mathbb{R}^J$ ,  $u^T$  denotes its transpose, with analogous notation for matrices. For  $x, y \in \mathbb{R}^J$  and a closed set  $A \subset \mathbb{R}^J$ ,  $d(x, y)$  denotes the Euclidean distance between  $x$  and  $y$ , and  $d(x, A) = \inf_{y \in A} d(x, y)$  denotes the distance between  $x$  and the set  $A$ . For each  $r \geq 0$ ,  $N_r(A) = \{x \in \mathbb{R}^J : d(x, A) \leq r\}$ . The unit sphere in  $\mathbb{R}^J$  is represented by  $S_1(0)$ . Given a set  $A \subset \mathbb{R}^J$ ,  $A^\circ$  denotes its interior,  $\bar{A}$  its closure and  $\partial A$  its boundary.

The space of continuous functions on  $[0, \infty)$  that take values in  $\mathbb{R}^J$  is denoted by  $\mathcal{C}[0, \infty)$ , and, given a set  $G \subset \mathbb{R}^J$ ,  $\mathcal{C}_G[0, \infty)$  denotes the subset of functions  $f$  in  $\mathcal{C}[0, \infty)$  such that  $f(0) \in G$ . The spaces  $\mathcal{C}[0, \infty)$  and  $\mathcal{C}_G[0, \infty)$  are assumed to be equipped with the topology of uniform convergence on compact sets. Given  $f \in \mathcal{C}[0, \infty)$  and  $T \in [0, \infty)$ ,  $\mathcal{V}ar_{[0, T]}f$  denotes the  $\mathbb{R}_+ \cup \{\infty\}$ -valued number that equals the variation of  $f$  on  $[0, T]$ . Also, given a real-valued function  $f$  on  $[0, \infty)$ , its oscillation is defined by

$$\text{Osc}(f; [s, t]) = \sup_{s \leq u_1 \leq u_2 \leq t} |f(u_2) - f(u_1)|; \quad 0 \leq s \leq t < \infty.$$

For each  $A \in \mathbb{R}^J$ ,  $\mathbb{I}_A(\cdot)$  denotes the indicator function of the set  $A$ , which takes the value 1 on  $A$  and 0 on the complement of  $A$ .

Given two random variables  $U^{(i)}$  defined on a probability space  $(\Omega^{(i)}, \mathcal{F}^{(i)}, \mathbb{P}^{(i)})$  and taking values in a common Polish space  $S$ ,  $i = 1, 2$ , the notation  $U^{(1)} \stackrel{(d)}{=} U^{(2)}$  will be used to imply that the random variables are equal in distribution. Given a sequence of  $S$ -valued random variables  $\{U^{(n)}, n \in \mathbb{N}\}$  and  $U$ , with  $U^{(n)}$  defined on  $(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbb{P}^{(n)})$  and  $U$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $U^{(n)} \Rightarrow U$  is used to denote weak convergence of the sequence  $U^{(n)}$  to  $U$ . Also, if the sequence of random variables are all defined on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , the notation  $U^{(n)} \xrightarrow{(\mathbb{P})} 0$  is used to denote convergence in probability.

## 2. THE CLASS OF REFLECTED DIFFUSIONS

In Section 2.1, stochastic differential equations with reflection under are introduced, and the basic assumptions on the class under study is introduced in Section 2.2. A motivating example and some useful ramifications of the assumptions are then presented in Section 2.3.

**2.1. Stochastic Differential Equations with Reflection.** The so-called extended Skorokhod problem (ESP), introduced in [22], is a convenient tool for the pathwise construction of reflected diffusions. The data associated with an ESP is the closure  $G$  of an open, connected domain in  $\mathbb{R}^J$  and a set-valued mapping  $d(\cdot)$  defined on  $G$  such that  $d(x) = \{0\}$  for  $x \in G^\circ$ ,  $d(x)$  is a non-empty, closed and convex cone in  $\mathbb{R}^J$  with vertex at the origin for every  $x \in \partial G$  and the graph of  $d(\cdot)$  is closed. Roughly speaking, given a continuous path  $\psi$ , the ESP associated with  $(G, d(\cdot))$  produces a constrained version  $\phi$  of  $\psi$  that is restricted to live within the domain  $G$  by adding to it a “constraining term”  $\eta$  whose increments over any interval lie in the closure of the convex hull of the union of the allowable directions  $d(x)$  at the points  $x$  visited by  $\phi$  during this interval. We now state the rigorous definition of the ESP. (In [22], the ESP was formulated more generally for càdlàg paths, but the formulation below will suffice for our purposes since we consider only continuous processes.)

**Definition 2.1. (Extended Skorokhod Problem)** *Suppose  $(G, d(\cdot))$  and  $\psi \in \mathcal{C}_G[0, \infty)$  are given. Then  $(\phi, \eta) \in \mathcal{C}_G[0, \infty) \times \mathcal{C}[0, \infty)$  are said to solve the ESP for  $\psi$  if  $\phi(0) = \psi(0)$ , and if for all  $t \in [0, \infty)$ , the following properties hold:*

- (1)  $\phi(t) = \psi(t) + \eta(t)$ ;
- (2)  $\phi(t) \in G$ ;
- (3) For every  $s \in [0, t)$

$$(2.1) \quad \eta(t) - \eta(s) \in \overline{\text{co}} \left[ \bigcup_{u \in (s, t]} d(\phi(u)) \right],$$

where  $\overline{\text{co}}[A]$  represents the closure of the convex hull generated by the set  $A$ .

If  $(\phi, \eta)$  is the unique solution to the ESP for  $\psi$ , then we write  $\phi = \bar{\Gamma}(\psi)$ , and refer to  $\bar{\Gamma}$  as the extended Skorokhod map (ESM).

If a unique solution to the ESP exists for all  $\psi \in \mathcal{C}_G[0, \infty)$ , then the associated ESM  $\bar{\Gamma}$  is said to be well-defined on  $\mathcal{C}_G[0, \infty)$ . In this case, it is easily verified (see Lemma A.1) that if  $\phi = \bar{\Gamma}(\psi)$ , then for any  $s \in [0, \infty)$ ,  $\phi^s = \bar{\Gamma}(\psi^s)$ , where

$$(2.2) \quad \psi^s(t) \doteq \phi(s) + \psi(s+t) - \psi(s) \quad \phi^s \doteq \phi(s+t).$$

Moreover, a well-defined ESM is said to be Lipschitz continuous on  $\mathcal{C}_G[0, \infty)$  if for every  $T < \infty$  there exists  $\bar{K}_T < \infty$  such that, for  $i = 1, 2$ , given  $\psi^{(i)} \in \mathcal{C}_G[0, \infty)$  with corresponding solution  $(\phi^{(i)}, \eta^{(i)})$ , we have

$$(2.3) \quad \sup_{s \in [0, T]} |\phi^{(1)}(s) - \phi^{(2)}(s)| \leq \bar{K}_T \sup_{s \in [0, T]} |\psi^{(1)}(s) - \psi^{(2)}(s)|.$$

The ESP is a generalization of the so-called Skorokhod Problem (SP) introduced in [29]. Unlike the SP, the ESP does not require that the constraining term  $\eta$  have finite variation on bounded intervals (compare Definitions 1.1 and 1.2 of [22]). The ESP can be used to define solutions to

stochastic differential equations with reflection (SDERs) associated with a given pair  $(G, d(\cdot))$  and drift and dispersion coefficients  $b : \mathbb{R}^J \mapsto \mathbb{R}^J$  and  $\sigma : \mathbb{R}^J \mapsto \mathbb{R}^J \times \mathbb{R}^N$ .

**Definition 2.2.** *Given  $(G, d(\cdot))$ ,  $b(\cdot)$ ,  $\sigma(\cdot)$ , the triple  $(Z_t, B_t)$ ,  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $\{\mathcal{F}_t\}$  is said to be a weak solution to the associated SDER if and only if*

- (1)  $\{\mathcal{F}_t\}$  is a filtration on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  that satisfies the usual conditions.
- (2)  $\{B_t, \mathcal{F}_t\}$  is an  $N$ -dimensional Brownian motion.
- (3)  $\mathbb{P}\left(\int_0^t |b(Z(s))| ds + \int_0^t |\sigma(Z(s))|^2 ds < \infty\right) = 1 \quad \forall t \in [0, \infty)$ .
- (4)  $\{Z_t, \mathcal{F}_t\}$  is a  $J$ -dimensional, adapted process such that  $\mathbb{P}$  a.s.,  $(Z, Y)$  solves the ESP for  $X$ , where  $Y \doteq Z - X$  and

$$(2.4) \quad X(t) = Z(0) + \int_0^t b(Z(s)) ds + \int_0^t \sigma(Z(s)) dB(s) \quad \forall t \in [0, \infty).$$

- (5) The set  $\{t : Z(t) \in \partial G\}$  has  $\mathbb{P}$  a.s. zero Lebesgue measure. In particular,

$$(2.5) \quad \int_0^\infty \mathbb{I}_{\partial G}(Z(s)) ds = 0.$$

This is similar to the usual definition for weak solutions for SDEs (see, for example, Definitions 3.1 and 3.2 of [20]), except that property 4 is modified to define reflection and property 5 captures the notion of “instantaneous” reflection (see, for example, pages 87–88 of [17]). A strong solution can also be defined in an analogous fashion.

**Definition 2.3.** *Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and an  $N$ -dimensional Brownian motion  $B$  on  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $Z$  is said to be a strong solution to the SDER associated with  $(G, d(\cdot))$ ,  $b(\cdot)$ ,  $\sigma(\cdot)$  and initial condition  $\xi$  if  $\mathbb{P}(Z(0) = \xi) = 1$  and properties 3–5 of Definition 2.2 hold with  $\{\mathcal{F}_t\}$  equal to the completed and augmented filtration generated by the Brownian motion  $B$ .*

For a precise construction of the filtration  $\{\mathcal{F}_t\}$  referred to in Definition 2.3, see (2.3) of [20]). In what follows, given the constraining process  $Y$  in property 4 of Definition 2.2, the quantity  $L$  will denote the associated total variation measure: for  $0 \leq s \leq t < \infty$ , let

$$(2.6) \quad L(s, t) \doteq \text{Var}_{(s, t]} Y \quad \text{and} \quad L(t) \doteq L(0, t].$$

Observe that the process  $L$  in the second definition in (2.6) is  $\{\mathcal{F}_t\}$ -adapted and takes values in the extended non-negative reals,  $\overline{\mathbb{R}}_+$ .

**2.2. Main Assumptions.** We now introduce certain basic assumptions on  $(G, d(\cdot))$ ,  $b(\cdot)$  and  $\sigma(\cdot)$  that will be used in the paper. In Section 2.3 we provide a concrete motivating example of a family of SDERs that arise in applications, which satisfy all the stated assumptions.

The first assumption asserts that Markov, weak solutions of the associated SDER exist.

**Assumption 1.** *There exists a weak solution to the SDER associated with  $(G, d(\cdot))$ ,  $b(\cdot)$ ,  $\sigma(\cdot)$  that has the Markov property.*

Next, we impose a kind of  $L^p$ -continuity condition on the ESM.

**Assumption 2.** *There exist  $p > 1, q \geq 2$  and  $K_T < \infty, T \in (0, \infty)$ , such that the weak solution  $Z$  to the SDER satisfies, for every  $0 \leq s \leq t \leq T$ ,*

$$(2.7) \quad \mathbb{E}[|Y(t) - Y(s)|^p \mid \mathcal{F}_s] \leq K_T \mathbb{E} \left[ \sup_{u \in [s, t]} |X(u) - X(s)|^q \mid \mathcal{F}_s \right],$$

where  $X$  is the process defined by (2.4) and  $Y \doteq Z - X$ .

**Remark 2.4.** Assumption 2 holds under rather mild conditions on the ESP – for example, when the following oscillation inequality is satisfied for any solution  $(\phi, \eta)$  to the ESP for a given  $\psi$ : for every  $0 \leq s \leq t < \infty$ , there exists  $C_{s,t} < \infty$  such that

$$\text{Osc}(\phi, [s, t]) \leq C_{s,t} \text{Osc}(\psi, [s, t]).$$

In this case, since  $(Z, Y)$  solve the ESP for  $X$ , we have for  $0 \leq s \leq t \leq T$ ,

$$|Y(t) - Y(s)| \leq \text{Osc}(Y, [s, t]) \leq C_{s,t} \text{Osc}(X, [s, t]) \leq 2C_T \sup_{u \in [s, t]} |X(u) - X(s)|,$$

where  $C_T = \max_{0 \leq s \leq t \leq T} C_{s,t} < \infty$ , and so Assumption 2 holds with  $p = q = 2$  and  $K_T = 4C_T^2$ . The oscillation inequality can be shown to hold in many situations of interest (see, for example, Lemma 2.1 of [32]). If the ESM associated with  $(G, d(\cdot))$  is well-defined and Lipschitz continuous on  $\mathcal{C}_G[0, \infty)$ , then the oscillation inequality is also automatically satisfied, and so Assumption 2 again holds with  $p = q = 2$ . More generally, if the ESM is well-defined and Hölder continuous on  $\mathcal{C}_G[0, \infty)$  with some exponent  $\alpha \in (0, \infty)$ , then Assumption 2 holds for any  $p > 1 \vee (\alpha/2)$  and  $q = \alpha p$ .

**Assumption 3.** *The coefficients  $b$  and  $\sigma$  are locally bounded, i.e., they are bounded on every compact subset of  $G$ .*

### 2.3. A Motivating Example and Ramifications of the Assumptions.

We now describe a family of multi-dimensional ESPs  $(G, d(\cdot))$  that arise in applications. Fix  $J \in \mathbb{N}$ ,  $J \geq 2$ . The  $J$ -dimensional ESPs in this family have domain  $\mathbb{R}_+^J$  and a constraint vector field  $d(\cdot)$  that is parametrized by a “weight” vector  $\alpha = (\alpha_1, \dots, \alpha_J)$  with  $\alpha_i > 0$  for  $i = 1, \dots, J$  and  $\sum_{i=1}^J \alpha_i = 1$ . Associated with each such  $\alpha$  is the ESP  $(\mathbb{R}_+^J, d(\cdot))$ , where for  $x \in \partial G$ ,

$$d(x) \doteq \left\{ \sum_{i: x_i=0} \beta_i d_i : \beta_i \geq 0 \right\}$$

with

$$(d_i)_j \doteq \begin{cases} -\frac{\alpha_j}{1 - \alpha_i} & \text{for } j \neq i, \\ 1 & \text{for } j = i, \end{cases}$$

for  $i, j = 1, \dots, J$ . As mentioned in the introduction in Section 1.1, reflected diffusions associated with this family were shown in [23, 24] to arise as heavy traffic approximations of the so-called generalized processor sharing (GPS) model in communication networks (see also [9] and [11]). Indeed, the characterization of this class of reflected diffusions serves as one of the motivations for this work.

Next, we define a broad class of SDERs that slightly generalizes the family of GPS ESPs.

**Definition 2.5.** *We will say  $(G, d(\cdot)), b(\cdot), \sigma(\cdot)$  are Class  $\mathcal{A}$  SDERs if they satisfy the following conditions.*

- (1) *The ESP associated with the ESP  $(G, d(\cdot))$  is well-defined and Lipschitz continuous on  $\mathcal{C}_G[0, \infty)$ .*
- (2)  *$G$  is a closed convex cone with vertex at the origin,  $\mathcal{V} = \{0\}$  and there exists  $\vec{v} \in G$  such that*

$$\langle \vec{v}, d \rangle = 0 \quad \text{for all } d \in d(x) \cap S_1(0), x \in \partial G \setminus \{0\}.$$

- (3) *There exists a constant  $\tilde{K} < \infty$  such that for all  $x, y \in G$ ,*

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq \tilde{K}|x - y|$$

and

$$|\sigma(x)| \leq \tilde{K} \quad |b(x)| \leq \tilde{K}(1 + |x|).$$

- (4) *The covariance function  $a : G \rightarrow \mathbb{R}^J \times \mathbb{R}^J$  defined by  $a(\cdot) = \sigma^T(\cdot)\sigma(\cdot)$  is uniformly elliptic, i.e., there exists  $\lambda > 0$  such that*

$$(2.8) \quad u^T a(x) u \geq \lambda |u|^2 \quad \text{for all } u \in \mathbb{R}^J, x \in G.$$

The conditions in property 3 can be relaxed to a local Lipschitz and linear growth condition on both  $b$  and  $\sigma$ , and the main result can still be proved by localization using the current arguments. However, to keep the notation simple, we impose the additional restrictions above.

**Remark 2.6.** ESPs in the GPS family defined above were shown to satisfy properties 1 and 2 (with  $\vec{v} = e_1 + \dots + e_J$ ) of Definition 2.5 in Theorem 3.6 and Lemma 3.4 of [22].

In Theorem 2.7 we summarize some consequences of Assumptions 1–3, showing in particular that Class  $\mathcal{A}$  SDERs satisfy these assumptions. The proof essentially follows from Theorem 4.3 of [22] and Proposition 4.1 of [19]. To state the theorem, we will need to introduce the following set:

$$(2.9) \quad \mathcal{V} \doteq \{x \in \partial G : \text{there exists } d \in S_1(0) \text{ such that } \{d, -d\} \subset d(x)\}.$$

**Theorem 2.7.** *Suppose  $(G, d(\cdot)), b(\cdot)$  and  $\sigma(\cdot)$  satisfy Assumptions 1 and 2, and let  $(Z_t, B_t), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$  be a weak solution to the associated SDER. Then  $Z$  is an  $\mathcal{F}_t$ -semimartingale on  $[0, T_{\mathcal{V}})$ , where*

$$(2.10) \quad T_{\mathcal{V}} \doteq \inf\{t \geq 0 : Z(t) \in \mathcal{V}\},$$

and  $\mathbb{P}$  a.s.,  $Z$  admits the decomposition

$$(2.11) \quad Z(\cdot) = Z(0) + M(\cdot) + A(\cdot),$$

where for  $t \in [0, T_V)$ ,

$$(2.12) \quad M(t) \doteq \int_0^t \sigma(Z(s)) \cdot dB(s), \quad A(t) \doteq \int_0^t b(Z(s)) ds + Y(t),$$

and  $Y$  has finite variation on  $[0, t]$  and satisfies

$$(2.13) \quad Y(t) = \int_0^t \gamma(s) dL(s),$$

where  $L$  is given by (2.6) and  $\gamma(s) \in d(Z(s))$ ,  $dL$  a.e.  $s \in [0, t]$ . Moreover, if  $(G, d(\cdot))$ ,  $b(\cdot)$  and  $\sigma(\cdot)$  satisfy properties 1 and 3 of Definition 2.5, then they also satisfy Assumptions 1, 3 and Assumption 2 with  $p = q = 2$ . In this case,  $Z$  is in fact the pathwise unique strong solution to the SDER, satisfies the strong Markov property and has  $\mathbb{E}[|Z(t)|^2] < \infty$  for every  $t \in (0, \infty)$  if  $\mathbb{E}[|Z(0)|^2] < \infty$ .

*Proof.* By property 4 of Definition 2.2,  $(Z, Z - X)$   $\mathbb{P}$  a.s. satisfy the ESP for  $X$ , where  $X$  is defined by (2.4) and is hence a semimartingale. Theorem 2.9 of [22] then shows that  $Y = Z - X$  has a.s. finite variation on any closed subinterval of  $[0, T_V)$ . When combined with the definition (2.4) of  $X$  and the fact that  $\int_0^\cdot b(Z(s)) ds$  is a process of bounded variation, this establishes the first assertion that  $Z$  is an  $\mathcal{F}_t$ -semimartingale on  $[0, T_V)$  with decomposition as in (2.11)–(2.13).

Next, suppose  $(G, d(\cdot))$ ,  $b(\cdot)$ ,  $\sigma(\cdot)$  satisfy properties 1 and 3 of Definition 2.5. Then property 3 of Definition 2.5 implies Assumption 3, and, by Remark 2.4, property 1 ensures that Assumption 2 holds with  $p = q$ . Moreover, Theorem 4.3 of [22] and Proposition 4.1 of [19] show that, in fact, the associated SDER admits a pathwise unique strong solution  $Z$ , which has the strong Markov property and, so in particular, Assumption 1 is also satisfied. Thus Assumptions 1–3 hold. The last assertion of the theorem can be established using standard techniques, by a modification of the proof in Theorem 4.3 of [22], in the same manner as this result is proved for strong solutions to SDEs, and so we omit the details of the proof.  $\square$

We conclude this section by stating a consequence of property 2 of Definition 2.5 that will be useful in the sequel. Let  $\Gamma_1$  denote the (extended) Skorokhod map associated with the 1-dimensional (extended) Skorokhod problem with  $G = \mathbb{R}_+$  and  $d(0) = \mathbb{R}_+$ ,  $d(x) = 0$  if  $x > 0$ . It is well-known (see, for example, [29] or Lemma 3.6.14 of [20]) that  $\Gamma_1$  is well-defined on  $\mathcal{C}_{\mathbb{R}_+}[0, \infty)$ , and in fact has the explicit form

$$(2.14) \quad \Gamma_1(\psi)(t) = \psi(t) + \sup_{s \in [0, t]} [-\psi(s)] \vee 0.$$

**Lemma 2.8.** *Suppose that  $(G, d(\cdot))$  satisfies property 2 of Definition 2.5. If  $(\phi, \eta)$  solves the associated ESP for  $\psi \in \mathcal{C}_G[0, \infty)$ , then  $\langle \phi, \bar{\mathbf{v}} \rangle = \Gamma_1(\langle \psi, \bar{\mathbf{v}} \rangle)$ .*

The proof of this lemma is exactly analogous to the proof of Corollary 3.5 of [22], and is thus omitted.

### 3. STATEMENT OF MAIN RESULTS

Theorem 2.7 shows that if  $\mathcal{V} = \emptyset$  then  $Z$  is a semimartingale. In fact, it was shown in Theorem 1.3 of [22] that when  $\mathcal{V} = \emptyset$ , the ESM coincides with the SM. Our main focus is to understand the behavior of reflected diffusions  $Z$  associated with ESPs  $(G, d(\cdot))$  for which  $\mathcal{V} \neq \emptyset$ , with the GPS family being a representative example. In [22] it was shown that, for the GPS family of ESPs,  $Z$  is a semimartingale until the first time it hits the origin. However, our first result, Theorem 3.1, shows that  $Z$  is not a semimartingale on  $[0, \infty)$ .

**Theorem 3.1.** *Suppose  $(G, d(\cdot))$ ,  $b(\cdot)$  and  $\sigma(\cdot)$  describe a Class  $\mathcal{A}$  SDER. Then the unique pathwise solution  $Z$  to the associated SDER is not a semimartingale.*

The proof of Theorem 3.1 is given in Section 4.3, building on preliminary results obtained in Sections 4.1 and 4.2. As mentioned in the introduction given in Section 1.1, for the special case when  $G$  is a convex wedge in  $\mathbb{R}^2$  and the directions of constraint on the two faces are constant and point at each other,  $b \equiv 0$  and  $\sigma$  is the identity matrix (i.e.,  $Z$  is a reflected Brownian motion), this result follows from Theorem 5 of [31] (with the parameters  $\alpha = 1$  and the wedge angle  $\pi$  less than  $180^\circ$  therein). The fact that, when  $J = 2$ , the reflected Brownian motion  $Z$  defined here is the same as the reflected Brownian motion defined via the submartingale formulation in [31] follows from Theorem 1.4(2) of [22]. This two-dimensional result can also be viewed as a special case of Proposition 4.13 of [3]. However, the proofs in [3] and [31] do not seem to extend easily to higher dimensions. In this paper, we take a different approach that is applicable in arbitrary dimensions and to more general diffusions, in particular providing a different proof of the two-dimensional result mentioned above.

As is well-known, when a process is a semimartingale,  $\mathcal{C}^2$  functionals of the process can be characterized using Itô's formula. Theorem 3.1 can thus be viewed as a somewhat negative result since it suggests that Class  $\mathcal{A}$  reflected diffusions and, in particular, reflected diffusions associated with the GPS family that arise in applications, may not possess desirable properties. However, we show in Corollary 3.6 that these diffusions are indeed tractable by establishing that they belong to the class of Dirichlet processes (in the sense of Föllmer). This follows as a special case of a more general result, Theorem 3.5.

In order to state the theorem, we first recall the definitions of zero  $p$ -variation processes and Dirichlet processes (see, e.g., [15], Theorem 2).

**Definition 3.2.** For  $p > 0$ , a continuous process  $A$  is of zero  $p$ -variation if and only if for any  $T > 0$ ,

$$(3.15) \quad \sum_{t_i \in \pi^n} |A(t_i) - A(t_{i-1})|^p \xrightarrow{(\mathbb{P})} 0,$$

for any sequence  $\{\pi^n\}$  of partitions of  $[0, T]$  with  $\Delta(\pi^n) \doteq \max_{t_i \in \pi^n} (t_{i+1} - t_i) \rightarrow 0$  as  $n \rightarrow \infty$ . If the process  $A$  satisfies (3.15) with  $p = 2$ , then  $A$  is said to be of zero energy.

**Definition 3.3.** The stochastic process  $Z$  is said to be a Dirichlet process if the following decomposition holds:

$$(3.16) \quad Z = M + A,$$

where  $M$  is an  $\mathcal{F}_t$ -adapted local martingale and  $A$  is a continuous,  $\mathcal{F}_t$ -adapted, zero energy process with  $A(0) = 0$ .

Note that this is weaker than the original definition of a Dirichlet process given by Föllmer [15], which required that  $M$  and  $A$  in the decomposition (3.16) be square integrable and  $A$  satisfy  $\mathbb{E} [\sum_{t_i \in \pi^n} |A_{t_i} - A_{t_{i-1}}|^2] \rightarrow 0$  as  $\Delta\pi^n \rightarrow 0$ , rather than (3.15) with  $p = 2$ . However, our definition coincides with Definition 2.4 of [7] (see also Definition 12 of [28]).

**Remark 3.4.** The decomposition of a Dirichlet process  $Z$ , into a local martingale and a zero energy process starting at 0, is unique. For any  $p > 1$  and partition  $\pi^n$  of  $[0, T]$ ,

$$\sum_{t_i \in \pi^n} |A(t_{i+1}) - A(t_i)|^p \leq \max_{t_i \in \pi^n} |A(t_{i+1}) - A(t_i)|^{p-1} \mathcal{V}ar_{[0, T]}(A).$$

Therefore, it follows that if  $A$  is continuous and of finite variation, then it is also of zero  $p$ -variation, for all  $p > 1$ . In particular, this shows that the class of Dirichlet processes generalizes the class of continuous semimartingales.

**Theorem 3.5.** Suppose  $(G, d(\cdot))$ ,  $b(\cdot)$  and  $\sigma(\cdot)$  satisfy Assumptions 1 and 3, let  $Z$  be an associated weak solution that satisfies Assumption 2 for some  $p > 1$ , and let  $Y = Z - X$ , where  $X$  is defined by (2.4). Then  $Y$  has zero  $p$ -variation.

As an immediate consequence of Theorem 3.5, Definition 3.3 and Theorem 2.7, we have the following result.

**Corollary 3.6.** Suppose  $(G, d(\cdot))$ ,  $b(\cdot)$  and  $\sigma(\cdot)$  satisfy Assumptions 1 and 3, and also Assumption 2 with  $p = 2$ . Then the associated reflected diffusion is a Dirichlet process. In particular, reflected diffusions associated with Class A SDERs are Dirichlet processes.

4. REFLECTED DIFFUSIONS ASSOCIATED WITH CLASS  $\mathcal{A}$  SDERS

Throughout this section we will assume that  $(G, d(\cdot))$ ,  $b(\cdot)$  and  $\sigma(\cdot)$  describe a Class  $\mathcal{A}$  SDER. Let  $B$  be an  $N$ -dimensional Brownian motion on a given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , let  $\{\mathcal{F}_t\}$  be the right-continuous augmentation of the filtration generated by  $B$  (see Definition (2.3) given in [20]). Also, let  $Z$  be the pathwise unique strong solution to the associated SDER (which exists by Theorem 2.7), let  $X$  be defined by (2.4), let  $Y \doteq Z - X$  and let  $L$  be the total variation process of  $Y$  as defined in (2.6). We use  $\mathbb{E}$  to denote expectation with respect to  $\mathbb{P}$  and, for  $z \in G$ ,  $\mathbb{P}_z$  (respectively,  $\mathbb{E}_z$ ) to denote the probability (respectively, expectation) conditioned on  $Z(0) = z$ .

This section is devoted to the proof of our first main result, Theorem 3.1. The key step is to show that the constraining process  $Y$  in the extended Skorokhod decomposition for  $Z$  has  $\mathbb{P}_0$  a.s. infinite variation. More precisely, for a given  $\varepsilon \geq 0$ , consider the hyperplane

$$(4.17) \quad H_\varepsilon \doteq \left\{ x \in \mathbb{R}^d : \langle \vec{\nu}, x \rangle = \varepsilon \right\} \cap G,$$

where  $\vec{\nu}$  is the vector in property 3 of Definition 2.5 and let

$$(4.18) \quad \tau^\varepsilon \doteq \inf\{t \geq 0 : Z(t) \in H_\varepsilon\} \quad \varepsilon \geq 0.$$

Then we establish the following result.

**Theorem 4.1.** *There exists  $T < \infty$  such that  $\mathbb{P}_0(L(T) = \infty) > 0$ .*

A somewhat subtle point to note is that Theorem 4.1 does not immediately establish the fact that  $Z$  is not a semimartingale because we do not know *a priori* that the decomposition  $Z = M + A$  given in (2.11) and (2.12) must be the Doob decomposition of  $Z$  if were a semimartingale. However, in Section 4.3 (see Proposition 4.2) we establish that this is indeed the case, thus obtaining Theorem 3.1 from Theorem 4.1. First, in Section 4.1, we establish Theorem 4.1 for the case when  $b \equiv 0$ . The proof for the general case is obtained from this result via a Girsanov transformation in Section 4.2.

**4.1. The Zero Drift Case.** Throughout this section we assume  $b \equiv 0$  and we prove the following result, which implies Theorem 4.1 since  $\mathbb{P}_0(\tau^1 < \infty) = 1$  in this case (see Lemma 4.10).

**Proposition 4.1.** *If  $b \equiv 0$ , then we have*

$$(4.19) \quad \mathbb{E}_0 \left[ e^{-L(\tau^1)} \right] = 0,$$

and hence,

$$(4.20) \quad L(\tau^1) = \infty \quad \mathbb{P}_0 \text{ a.s.}$$

The proof of Proposition 4.1 is given in Section 4.1.3. First, in Section 4.1.1 we establish a simple upper bound on  $\mathbb{E}_0[e^{-L(\tau^1)}]$ . We then establish a weak convergence result in Section 4.1.2, which is subsequently used to obtain certain estimates in Lemmas 4.8 and 4.9.

4.1.1. *An Upper Bound.* To begin with, we use the strong Markov property of  $Z$  to obtain an upper bound on  $\mathbb{E}_0[e^{-L(\tau^1)}]$ . For  $\varepsilon > 0$ , recursively define two sequences of times  $\{\tau_n^\varepsilon\}_{n \in \mathbb{N}}$  and  $\{\alpha_n^\varepsilon\}_{n \in \mathbb{N}}$  as follows:  $\alpha_0^\varepsilon \doteq 0$  and for  $n \in \mathbb{N}$ ,

$$(4.21) \quad \begin{aligned} \tau_n^\varepsilon &\doteq \inf \{t \geq \alpha_{n-1}^\varepsilon : Z(t) \in H_\varepsilon\} \\ \alpha_n^\varepsilon &\doteq \inf \{t \geq \tau_n^\varepsilon : Z(t) \in H_0\}, \end{aligned}$$

where  $H_0 = \{0\}$  because  $G$  is a closed convex cone with vertex at 0. Also, recall the definition of  $\tau^0$  given in (4.18) with  $\varepsilon = 0$ . Since  $Z$  is continuous and  $H_\varepsilon$  and  $H_0$  are closed, it is clear that  $\tau^0$ ,  $\tau_n^\varepsilon$  and  $\alpha_n^\varepsilon$  are  $\mathcal{F}_t$ -stopping times. For conciseness, we will often denote  $\tau_1^\varepsilon$  and  $\alpha_1^\varepsilon$  simply by  $\tau^\varepsilon$  and  $\alpha^\varepsilon$ , respectively. Note that this is consistent with the notation of  $\tau^\varepsilon$  given in (4.18).

**Lemma 4.2.** *For every  $\varepsilon > 0$ ,*

$$\mathbb{E}_0 \left[ e^{-L(\tau^1)} \right] \leq \frac{\mathbb{E}_0 \left[ \mathbb{P}_{Z(\tau^\varepsilon)} (\tau^0 \geq \tau^1) \right]}{\mathbb{E}_0 \left[ \mathbb{P}_{Z(\tau^\varepsilon)} (\tau^0 \geq \tau^1) \right] + \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ (1 - e^{-L(\tau^0)}) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right]}.$$

*Proof.* From the elementary inequality

$$L(\tau^1) \geq \sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1)),$$

it immediately follows that

$$(4.22) \quad \mathbb{E}_0 \left[ e^{-L(\tau^1)} \right] \leq \mathbb{E}_0 \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \right].$$

On the set  $\{\alpha^\varepsilon \geq \tau^1\}$ , we have  $\alpha_n^\varepsilon \wedge \tau^1 = \tau_n^\varepsilon \wedge \tau^1 = \tau^1$  for every  $n \geq 2$ . Therefore, the right-hand side of (4.22) can be decomposed as

$$\begin{aligned} \mathbb{E}_0 \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \right] &= \mathbb{E}_0 \left[ e^{-L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1)} \mathbb{I}_{\{\alpha^\varepsilon \geq \tau^1\}} \right] \\ &\quad + \mathbb{E}_0 \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \right]. \end{aligned}$$

Conditioning on  $\mathcal{F}_{\alpha^\varepsilon}$  and using the fact that  $\mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}}$ ,  $L(\alpha^\varepsilon \wedge \tau^1)$  and  $L(\tau^\varepsilon \wedge \tau^1)$  are  $\mathcal{F}_{\alpha^\varepsilon}$ -measurable, the last term above can be rewritten as

$$\begin{aligned}
& \mathbb{E}_0 \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \right] \\
&= \mathbb{E}_0 \left[ \mathbb{E}_0 \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \middle| \mathcal{F}_{\alpha^\varepsilon} \right] \right] \\
&= \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \mathbb{E}_0 \left[ e^{-\sum_{n=2}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \middle| \mathcal{F}_{\alpha^\varepsilon} \right] \right] \\
&= \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \mathbb{E}_{Z(\alpha^\varepsilon)} \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \right] \right] \\
&= \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \right] \mathbb{E}_0 \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \right],
\end{aligned}$$

where the second-last equality uses the strong Markov property of  $Z$  and the last equality follows because  $Z(\alpha^\varepsilon) \equiv 0$ . Combining the last two assertions and rearranging terms, we obtain

$$\mathbb{E}_0 \left[ e^{-\sum_{n=1}^{\infty} (L(\alpha_n^\varepsilon \wedge \tau^1) - L(\tau_n^\varepsilon \wedge \tau^1))} \right] = \frac{\mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon \geq \tau^1\}} \right]}{1 - \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \right]}.$$

This, together with (4.22), yields the inequality

$$(4.23) \quad \mathbb{E}_0 \left[ e^{-L(\tau^1)} \right] \leq \frac{\mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon \geq \tau^1\}} \right]}{1 - \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \right]}.$$

We now show that the upper bound stated in the lemma follows from (4.23). Indeed, using the non-negativity of  $L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1)$  and the strong Markov property of  $Z$ , we obtain

$$\begin{aligned}
(4.24) \quad \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon \geq \tau^1\}} \right] &\leq \mathbb{E}_0 \left[ \mathbb{I}_{\{\alpha^\varepsilon \geq \tau^1\}} \right] \\
&= \mathbb{E}_0 \left[ \mathbb{E}_0 \left[ \mathbb{I}_{\{\alpha^\varepsilon \geq \tau^1\}} \middle| \mathcal{F}_{\tau^\varepsilon} \right] \right] \\
&= \mathbb{E}_0 \left[ \mathbb{P}_{Z(\tau^\varepsilon)} (\tau^0 \geq \tau^1) \right],
\end{aligned}$$

where recall  $\tau^0 = \inf\{t \geq 0 : Z(t) \in H_0\}$ . Similarly, once again conditioning on  $\mathcal{F}_{\tau^\varepsilon}$  and using the strong Markov property of  $Z$ , we obtain

$$\begin{aligned}
& \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \right] \\
&= \mathbb{E}_0 \left[ \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \middle| \mathcal{F}_{\tau^\varepsilon} \right] \right] \\
&= \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\tau^0 \wedge \tau^1)} \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
(4.25) \quad & 1 - \mathbb{E}_0 \left[ e^{-(L(\alpha^\varepsilon \wedge \tau^1) - L(\tau^\varepsilon \wedge \tau^1))} \mathbb{I}_{\{\alpha^\varepsilon < \tau^1\}} \right] \\
&= \mathbb{E}_0 \left[ 1 - \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\tau^0 \wedge \tau^1)} \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right] \\
&= \mathbb{E}_0 \left[ \mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0 \geq \tau^1) \right] + \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\tau^0)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right].
\end{aligned}$$

The lemma follows from (4.23), (4.24) and (4.25).  $\square$

Next, we establish an elementary lemma that holds when the drift is zero. Recall the vector  $\vec{\nu}$  of property 2 of Definition 2.5.

**Lemma 4.3.** *When  $b \equiv 0$ , the process  $\langle Z, \vec{\nu} \rangle$  is an  $\mathcal{F}_t$ -martingale on  $[0, \tau^0]$  and for every  $\varepsilon > 0$ ,  $\mathbb{P}_0$  a.s.,*

$$(4.26) \quad \mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0 \geq \tau^1) = \varepsilon.$$

*Proof.* First, note that  $H_0 = \{0\} = \mathcal{V}$  by property 2 of Definition 2.5 and so  $T_{\mathcal{V}}$  defined in (2.10) coincides with  $\tau^0$ . From Lemma 2.8 and the continuous paths of  $Y$ , it follows that for  $t \in [0, \tau^0]$ ,  $\langle Y(t), \vec{\nu} \rangle = 0$  and so  $\mathbb{P}$  a.s.,

$$(4.27) \quad \langle Z(t), \vec{\nu} \rangle = \langle Z(0), \vec{\nu} \rangle + \tilde{M}, \quad t \in [0, \tau^0],$$

where  $\tilde{M} \doteq \langle \int_0^\cdot \sigma(Z(s)) \cdot dB(s), \vec{\nu} \rangle$  is an  $\mathcal{F}_t$  martingale on  $[0, \tau^0]$  since  $\sigma$  is uniformly bounded. This establishes the first assertion of the lemma. Now, the quadratic variation  $\langle \tilde{M} \rangle$  of  $\tilde{M}$  is given by

$$\langle \tilde{M} \rangle(t) = \int_0^t \vec{\nu}^T a(Z(s)) \vec{\nu} ds \quad \text{for } t \in [0, \infty),$$

where  $a \doteq \sigma^T \sigma$ . By property 4 of Definition 2.5,  $a(\cdot)$  is uniformly elliptic. Therefore,  $\langle \tilde{M} \rangle$  is strictly increasing and  $\langle \tilde{M} \rangle_\infty \doteq \lim_{t \rightarrow \infty} \langle \tilde{M} \rangle(t) = \infty$   $\mathbb{P}$  a.s. Let

$$T(t) \doteq \inf\{s \geq 0 : \langle \tilde{M} \rangle(s) > t\},$$

$\mathcal{G}_t \doteq \mathcal{F}_{T(t)}$  and  $\tilde{B}(t) \doteq \tilde{M}(T(t))$  for  $t \in [0, \infty)$ . Then  $\{\tilde{B}_t, \mathcal{G}_t\}_{t \geq 0}$  is a standard one-dimensional Brownian motion (see, e.g., Theorem 4.6 on page 174 of [20]). Let  $\tilde{\tau}^\varepsilon \doteq \inf\{t \geq 0 : \tilde{B}(t) = \varepsilon\}$ . Then by (4.27) we have  $\mathbb{P}_0$  a.s.,

$$\mathbb{P}_{Z(\tau^\varepsilon)}(\tau^0 \geq \tau^1) = \mathbb{P}(\tilde{\tau}^0 \geq \tilde{\tau}^1 \mid \tilde{B}(0) = \varepsilon) = \varepsilon,$$

where the latter follows from standard properties of Brownian motion. This proves (4.26).  $\square$

**Remark 4.4.** From Lemmas 4.2 and 4.3, we conclude that for every  $\varepsilon > 0$ ,

$$\mathbb{E} \left[ e^{-L(\tau^1)} \right] \leq \frac{\varepsilon}{\varepsilon + \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\tau^\varepsilon)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right]}.$$

Thus, to show (4.19), it suffices to show that

$$(4.28) \quad \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\tau^\varepsilon)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right] = \infty.$$

This is established in Section 4.1.3 using scaling arguments. In Section 4.1.2, we establish some preliminary results required for the proof. The reader may prefer to skip forward to the proof of Proposition 4.1 in Section 4.1.3 and refer back to the results in Sections 4.1.2 when required.

4.1.2. *A Weak Convergence Result.* Recall that we have assumed that the drift  $b \equiv 0$ . Now, let  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  and  $\{x_k\}_{k \in \mathbb{N}}$  be sequences such that  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$  and  $x_k \in H_{\varepsilon_k}$  for  $k \in \mathbb{N}$ . For each  $k \in \mathbb{N}$ , let  $Z_{(k)}$  be the pathwise unique solution to the associated SDER with initial condition  $x_k$ , and let  $X_{(k)}, Y_{(k)}$  and  $L_{(k)}$  be the associated processes as defined in Definition 2.2 and (2.6). For  $k \in \mathbb{N}$ , consider the scaled process

$$B^k(t) \doteq \frac{B(\varepsilon_k^2 t)}{\varepsilon_k} \quad t \in [0, \infty),$$

which is a standard Brownian motion due to Brownian scaling. Similarly, define

$$(4.29) \quad A^k(t) \doteq \frac{A_{(k)}(\varepsilon_k^2 t)}{\varepsilon_k} \quad A = X, Y, Z, L,$$

and let  $\mathcal{F}_t^k \doteq \mathcal{F}_{\varepsilon_k^2 t}$  for  $t \in [0, \infty)$ . Clearly, the processes  $Z^k, B^k, Y^k$  and  $L^k$  are  $\{\mathcal{F}_t^k\}$ -adapted and  $L^k(t) = \text{Var}_{[0,t]} Y^k$  for every  $t \geq 0$ . For  $(r, R) \in (0, \infty)^2$  such that  $r < R$ , let

$$(4.30) \quad \theta_{r,R}^k \doteq \inf \left\{ t \geq 0 : \langle Z^k(t), \vec{\nu} \rangle \notin (r, R) \right\} \quad k \in \mathbb{N}.$$

This section contains two main results. Roughly speaking, the first result (Lemma 4.6) shows that for the question under consideration, we can in effect replace the state-dependent diffusion coefficient  $\sigma(\cdot)$  by  $\sigma(0)$ . This property is then used in Corollary 4.7 to provide bounds on the total variation sequence  $L^k(\theta_{r,R}^k)$ , as  $\varepsilon_k \rightarrow 0$ , which are used to obtain two important estimates in Lemmas 4.8 and 4.9 of the next section. First, in Remark 4.5 we establish a simple equivalence between  $(X^k, Z^k, Y^k)$  and another triplet of processes that will be easier to work with.

**Remark 4.5.** For notational conciseness, we define the scaled diffusion coefficient

$$\sigma^k(x) \doteq \sigma(\varepsilon^k x) \quad x \in \mathbb{R}^J, k \in \mathbb{N}.$$

Then, by the definition of  $Z_{(k)}$  and the scaling (4.29), it follows that

$$X^k(t) = \frac{x_k}{\varepsilon_k} + \frac{1}{\varepsilon_k} \int_0^{\varepsilon_k^2 t} \sigma(Z_{(k)}(s)) dB(s) = \frac{x_k}{\varepsilon_k} + \int_0^t \sigma^k(Z_{(k)}(s)) dB^k(s),$$

where the last equality holds by the time-change theorem for stochastic integrals (see Proposition 1.4 in Chapter V of [25]). This implies  $Z^k$  is a strong solution to the SDER associated with  $(G, d(\cdot))$ ,  $b \equiv 0$ ,  $\sigma^k$  and the Brownian motion  $\{B^k(t), \mathcal{F}_t^k\}$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ , with initial condition  $x_k/\varepsilon_k$ . If  $\sigma$  satisfies properties 3 and 4 of Definition 2.5 then so does  $\sigma^k$ , and thus  $(G, d(\cdot))$ ,  $b \equiv 0$  and  $\sigma^k$  also describe a Class  $\mathcal{A}$  SDER. Therefore,

by Theorem 2.7 there exists a pathwise unique solution  $\tilde{Z}^k$  to the associated SDER for the Brownian motion  $\{B_t, \mathcal{F}_t\}$  with initial condition  $x_k/\varepsilon_k$ . Let  $\tilde{X}^k, \tilde{Y}^k$  be defined in the usual manner:

$$(4.31) \quad \tilde{X}^k(t) = \frac{x_k}{\varepsilon_k} + \int_0^t \sigma^k(\tilde{Z}^k(s)) dB(s) \quad t \in [0, \infty),$$

and  $\tilde{Y}^k = \tilde{Z}^k - \tilde{X}^k$ . From the fact that solutions to Class  $\mathcal{A}$  SDERs are unique in law by Theorem 2.7, it then follows that

$$(4.32) \quad (X^k, Z^k, Y^k) \stackrel{(d)}{=} (\tilde{X}^k, \tilde{Z}^k, \tilde{Y}^k),$$

where recall that  $\stackrel{(d)}{=}$  indicates equality in distribution.

**Lemma 4.6.** *Suppose  $b \equiv 0$  and  $x_k/\varepsilon_k \rightarrow x$  as  $k \rightarrow \infty$ . Then the following properties hold:*

(1) *As  $k \rightarrow \infty$ ,*

$$(4.33) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{Z}^k(t) - \bar{Z}(t)|^2 \right] \rightarrow 0$$

*and  $(X^k, Z^k, Y^k) \Rightarrow (\bar{X}, \bar{Z}, \bar{Y})$ , where  $(\bar{Z}, \bar{Y})$  satisfy the ESP pathwise for*

$$(4.34) \quad \bar{X} \doteq x + \sigma(0)B.$$

(2) *For all but countably many pairs  $(r, R) \in (0, \infty)^2$  such that  $r < R$ ,*

$$\max_{i=1, \dots, J} \sup_{s \in [0, \theta_{r,R}^k]} Y_i^k(s) \Rightarrow \max_{i=1, \dots, J} \sup_{s \in [0, \bar{\theta}_{r,R}]} \bar{Y}_i(s) \quad \text{as } k \rightarrow \infty,$$

*where*

$$(4.35) \quad \bar{\theta}_{r,R} \doteq \inf \{t \geq 0 : \langle \bar{Z}(t), \bar{\mathbf{v}} \rangle \notin (r, R)\}.$$

*Proof.* Note that since  $x_k/\varepsilon_k \in H_1$  for every  $k \in \mathbb{N}$  and  $H_1$  is closed, we must have  $x \in H_1$ . We first prove property 1. Let  $\tilde{X}^k, \tilde{Z}^k$  and  $\tilde{Y}^k$  be as in Remark 4.5. Then, by (4.32), it clearly suffices to show that  $(\tilde{X}^k, \tilde{Z}^k, \tilde{Y}^k) \Rightarrow (\bar{X}, \bar{Z}, \bar{Y})$ . From (4.31) and (4.34), it follows that for  $t \in [0, \infty)$ ,

$$\begin{aligned} |\tilde{X}^k(t) - \bar{X}(t)|^2 &\leq \left| \frac{x_k}{\varepsilon_k} - x + \int_0^t \left( \sigma^k(\tilde{Z}^k(s)) - \sigma(0) \right) dB(s) \right|^2 \\ &\leq \left( \left| \frac{x_k}{\varepsilon_k} - x \right| + \left| \int_0^t \left( \sigma^k(\bar{Z}(s)) - \sigma(0) \right) dB(s) \right| \right. \\ &\quad \left. + \left| \int_0^t \left( \sigma^k(\tilde{Z}^k(s)) - \sigma^k(\bar{Z}(s)) \right) dB(s) \right| \right)^2. \end{aligned}$$

Using the fact that  $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$  for all  $a, b, c \in \mathbb{R}$  and taking the supremum over  $t \in [0, T]$  and then expectations of both sides, we obtain

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{X}^k(t) - \bar{X}(t)|^2 \right] \\ & \leq 3 \left| \frac{x_k}{\varepsilon_k} - x \right|^2 + 3\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \left( \sigma^k(\bar{Z}(s)) - \sigma(0) \right) dB(s) \right|^2 \right] \\ & \quad + 3\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \left( \sigma^k(\tilde{Z}^k(s)) - \sigma^k(\bar{Z}(s)) \right) dB(s) \right|^2 \right]. \end{aligned}$$

Since  $\sigma$  is uniformly bounded, the stochastic integrals on the right-hand side are martingales. By applying the Burkholder-Davis-Gundy (BDG) inequality, the Lipschitz condition on  $\sigma$ , the definition of  $\sigma^k$  and Fubini's theorem, we obtain

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \left( \sigma^k(\tilde{Z}^k(s)) - \sigma^k(\bar{Z}(s)) \right) dB(s) \right|^2 \right] \\ & \leq C_2 \mathbb{E} \left[ \int_0^T \left| \sigma^k(\tilde{Z}^k(s)) - \sigma^k(\bar{Z}(s)) \right|^2 ds \right] \\ & \leq C_2 \tilde{K}^2 \varepsilon_k^2 \mathbb{E} \left[ \int_0^T \left| \tilde{Z}^k(s) - \bar{Z}(s) \right|^2 ds \right] \\ & \leq C_2 \tilde{K}^2 \varepsilon_k^2 \int_0^T \mathbb{E} \left[ \sup_{u \in [0, s]} \left| \tilde{Z}^k(u) - \bar{Z}(u) \right|^2 \right] ds, \end{aligned}$$

where  $C_2 < \infty$  is the universal constant in the BDG inequality. Using similar arguments, we can also obtain

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \left( \sigma^k(\bar{Z}(s)) - \sigma(0) \right) dB(s) \right|^2 \right] & \leq C_2 \tilde{K}^2 \varepsilon_k^2 \int_0^T \mathbb{E} \left[ \sup_{u \in [0, s]} |\bar{Z}(u)|^2 \right] ds \\ & \leq C_2 \tilde{K}^2 \varepsilon_k^2 T \mathbb{E} \left[ \sup_{t \in [0, T]} |\bar{Z}(t)|^2 \right]. \end{aligned}$$

Combining the last three displays, and setting  $\tilde{C}_T \doteq 3C_2 \tilde{K}_T^2 (1 \vee T) < \infty$ , we have

(4.36)

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{X}^k(t) - \bar{X}(t)|^2 \right] & \leq \tilde{C}_T \varepsilon_k^2 \int_0^T \mathbb{E} \left[ \sup_{u \in [0, s]} \left| \tilde{Z}^k(u) - \bar{Z}(u) \right|^2 \right] ds \\ & \quad + R^k(T), \end{aligned}$$

where

$$(4.37) \quad R^k(T) \doteq 3 \left| \frac{x_k}{\varepsilon_k} - x \right|^2 + \tilde{C}_T \varepsilon_k^2 \mathbb{E} \left[ \sup_{t \in [0, T]} |\bar{Z}(t)|^2 \right] \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

since  $x_k/\varepsilon_k \rightarrow x$ ,  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$  and, by the assumed Lipschitz continuity of  $\bar{\Gamma}$ ,

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} |\bar{Z}(t)|^2 \right] &\leq K_T^2 \mathbb{E} \left[ \sup_{t \in [0, T]} |x + \sigma(0)B(t)|^2 \right] \\ &\leq 2K_T^2|x|^2 + 2K_T^2|\sigma(0)|^2 \mathbb{E} \left[ \sup_{t \in [0, T]} |B(t)|^2 \right] < \infty. \end{aligned}$$

Now (4.36), along with the Lipschitz continuity of the map  $\bar{\Gamma}$ , shows that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{Z}^k(t) - \bar{Z}(t)|^2 \right] \leq K_T^2 R^k(T) + K_T^2 \tilde{C}_T \varepsilon_k^2 \int_0^T \mathbb{E} \left[ \sup_{u \in [0, s]} |\tilde{Z}^k(u) - \bar{Z}(u)|^2 \right] ds.$$

So, an application of Gronwall's lemma shows that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{Z}^k(t) - \bar{Z}(t)|^2 \right] \leq K_T^2 R^k(T) e^{K_T^2 \tilde{C}_T \varepsilon_k^2} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where the convergence to zero follows from (4.37), and the limit  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . This proves (4.33). In turn, substituting the last inequality back into (4.36) and, again using (4.37) and the fact that  $\varepsilon_k \rightarrow 0$ , we also obtain

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\tilde{X}^k(t) - \bar{X}(t)|^2 \right] \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which implies  $\tilde{X}^k \Rightarrow \bar{X}$ . Since the mapping from  $\tilde{X}^k \mapsto (\tilde{X}^k, \tilde{Z}^k, \tilde{Y}^k)$  is continuous, by the continuous mapping theorem it follows that  $(\tilde{X}^k, \tilde{Z}^k, \tilde{Y}^k) \Rightarrow (\bar{X}, \bar{Z}, \bar{Y})$  and property 1 is established.

We now turn to property 2. By property 1, we have  $(Z^k, Y^k) \Rightarrow (\bar{Z}, \bar{Y})$  as  $k \rightarrow \infty$ . This immediately implies that for all but countably main pairs  $(r, R) \in (0, \infty)^2$  such that  $r < R$ , as  $k \rightarrow \infty$ ,

$$(Z^k(\cdot \wedge \theta_{r,R}^k), Y^k(\cdot \wedge \theta_{r,R}^k), \theta_{r,R}^k) \Rightarrow (\bar{Z}(\cdot \wedge \bar{\theta}_{r,R}), \bar{Y}(\cdot \wedge \bar{\theta}_{r,R}), \bar{\theta}_{r,R}).$$

(For an argument that justifies this implication, see, e.g., the proof of Theorem 4.1 on page 354 of [13]). Using the continuity of the map  $(f, g, t) \mapsto \max_{i=1, \dots, J} \sup_{s \in [0, t]} g_i(s)$  from  $\mathcal{C}[0, \infty) \times \mathcal{C}[0, \infty) \times \mathbb{R}_+$  to  $\mathbb{R}_+$ , an application of the continuous mapping theorem yields property 2.  $\square$

**Corollary 4.7.** *Suppose  $b \equiv 0$  and  $x_k/\varepsilon_k \rightarrow x$  as  $k \rightarrow \infty$ . Then for each pair  $(r, R) \in (0, \infty)$  such that  $r < R$ , the following properties hold.*

- (1)  $\mathbb{P} \left( \sup_{k \in \mathbb{N}} L^k(\theta_{r,R}^k) < \infty \right) = 1$ .
- (2)  $\varepsilon_k L^k(\theta_{r,R}^k) \Rightarrow 0$ .
- (3)  $\mathbb{P}(\bar{L}(\bar{\theta}_{r,R}) < \infty) = 1$  and if  $r < \langle x, \bar{\mathbf{v}} \rangle < R$ ,  $\mathbb{P}(\bar{L}(\bar{\theta}_{r,R}) > 0) > 0$ .

*Proof.* If  $\langle x, \bar{\mathbf{v}} \rangle < r$  or  $\langle x, \bar{\mathbf{v}} \rangle > R$ , then  $\bar{\theta}_{r,R} = 0$  and  $\theta_{r,R}^k = 0$  for all  $k$  sufficiently large. In this case, properties (1)–(3) hold trivially. Hence, for the rest of the proof, we assume that  $r \leq \langle x, \bar{\mathbf{v}} \rangle \leq R$ .

We start with property 1. Let  $\tilde{X}^k$ ,  $\tilde{Z}^k$  and  $\tilde{Y}^k$  be defined as in Remark 4.5, and let  $\tilde{L}^k$  be defined as in (2.6), but with  $Y$  replaced by  $\tilde{Y}^k$ . By (4.32), it follows that  $(L^k, \theta_{r,R}^k)$  and  $(\tilde{L}^k, \tilde{\theta}_{r,R}^k)$  have the same distribution for each  $k \in \mathbb{N}$ , where  $\tilde{\theta}_{r,R}^k$  is defined in the obvious way:

$$\tilde{\theta}_{r,R}^k \doteq \inf \left\{ t \geq 0 : \langle \tilde{Z}^k(t), \vec{\nu} \rangle \notin (r, R) \right\}.$$

Using Lemma 2.8 and the uniform ellipticity condition, it can be shown that  $\mathbb{P}(\tilde{\theta}_{r,R}^k) = 1$ . Therefore, to prove property 1, it suffices to show that

$$\mathbb{P} \left( \sup_{k \in \mathbb{N}} \tilde{L}^k(\tilde{\theta}_{r,R}^k \wedge T) < \infty \right) = 1 \quad \text{for each } T > 0.$$

Fix  $T \in (0, \infty)$ . Since  $r > 0$ , there exists  $\delta > 0$  such that  $\langle y, \vec{\nu} \rangle < r$  for all  $y$  with  $|y| \leq \delta$ . Let  $\tilde{\kappa}_\delta^k \doteq \inf \{ t \geq 0 : |\tilde{Z}^k(t)| \leq \delta \}$ . Then  $\theta_{r,R}^k \leq \tilde{\kappa}_\delta^k$  for all  $k \in \mathbb{N}$ . Let

$$\tilde{C}^k = \sup_{t \in [0, T]} |\tilde{Z}^k(t) \vee |\tilde{X}^k(t)|.$$

By property 1 of Lemma 4.6, we have  $\tilde{Z}^k \Rightarrow \bar{Z}$  and  $\tilde{X}^k \Rightarrow \bar{X}$  as  $k \rightarrow \infty$ . Also, due to the Lipschitz continuity of the ESM  $\bar{\Gamma}$  and (4.34),  $\mathbb{P}$ -a.s., we have

$$\sup_{t \in [0, T]} |\bar{Z}(t)| \leq K_T \sup_{t \in [0, T]} |\bar{X}(t)| \leq K_T \left( |x| + |\sigma(0)| \sup_{s \in [0, T]} |B(s)| \right) < \infty,$$

and hence  $\mathbb{P}(\sup_{k \in \mathbb{N}} \tilde{C}^k < \infty) = 1$ . Moreover,  $\mathcal{V} = \{0\}$  and for each  $\omega \in \Omega$ ,  $(\bar{Z}(\cdot, \omega), \bar{Y}(\cdot, \omega))$  solves the ESP for  $\bar{X}(\cdot, \omega)$ . Therefore, from Lemma 2.8 of [22], it is easy to see that there exist  $\rho > 0$ , independent of  $k$ , a finite set  $\mathbb{I} = \{1, \dots, I\}$  and a collection of open sets  $\{\mathcal{O}_i, i \in \mathbb{I}\}$  of  $\mathbb{R}^J$  that satisfy properties 1 and 2 of Lemma 2.8 of [22]. Moreover, from the proof of Lemma 2.9 of [22], for each  $\omega \in \Omega$ , there exist integers  $\bar{N}(\omega) < \infty$  and times  $\{\bar{T}_m(\omega), m \in \mathbb{N}\}$ , defined for  $\bar{Z}(\omega)$  in the same way as  $M$  and  $\{T_m, m \in \mathbb{N}\}$  are defined in terms of  $\phi$  in Lemma 2.9 of [22], except that we replace  $\rho$  and  $\delta$  by  $\rho/2$  and  $\delta/2$ , respectively. Since  $(X^k, Z^k, Y^k) \Rightarrow (\bar{X}, \bar{Z}, \bar{Y})$  as  $k \rightarrow \infty$  and  $(\bar{X}, \bar{Z}, \bar{Y})$  has continuous paths, by invoking the Skorokhod representation theorem, we may assume without loss of generality that there exists  $\tilde{\Omega}$  with  $\mathbb{P}(\tilde{\Omega}) = 1$  such that for every  $\omega \in \tilde{\Omega}$ ,  $(X^k(\omega), Z^k(\omega), Y^k(\omega)) \rightarrow (\bar{X}(\omega), \bar{Z}(\omega), \bar{Y}(\omega))$  uniformly on  $[0, T]$  as  $k \rightarrow \infty$ . Let  $\bar{k} < \infty$  be such that for all  $k > \bar{k}$ ,  $\sup_{t \in [0, T]} |Z^k(t, \omega) - \bar{Z}(t, \omega)| < (\rho \wedge \delta)/4$ . As in the proof of Lemma 2.9 of [22], for each  $m < \bar{N}(\omega)$ , let  $k_m$  be the index of  $\mathcal{O}_k$  associated with  $\bar{T}_m$ . Then,  $Z^k(\cdot, \omega)$  will stay in  $N_\rho(\mathcal{O}_{k_m})$  during the interval  $[\bar{T}_m(\omega), \bar{T}_{m+1}(\omega))$ . Exactly as in the proof of Lemma 2.9 of [22] (note that the argument there only requires that  $\phi(t) \in N_\rho(\mathcal{O}_{k_{m-1}})$  for  $t \in [T_{m-1}, T_m)$ ), we can then argue that  $\tilde{L}^k(T \wedge \tau_\delta^k(\omega), \omega) \leq (4\tilde{C}^k(\omega)\bar{N}(\omega))/\rho$  for  $\omega \in \tilde{\Omega}$ . Together with the fact that  $\mathbb{P}(\sup_{k \in \mathbb{N}} \tilde{C}^k < \infty) = 1$  and  $\bar{N}(\omega) < \infty$  for each  $\omega \in \tilde{\Omega}$ , this shows that

$\mathbb{P}(\tilde{L}^k(\tilde{\tau}_\delta^k \wedge T) < \infty) = 1$ . Since  $\tilde{L}^k(\tilde{\theta}_{r,R}^k \wedge T) \leq \tilde{L}^k(\tilde{\tau}_\delta^k \wedge T)$ , we then have  $\mathbb{P}(\sup_{k \in \mathbb{N}} \tilde{L}^k(\tilde{\theta}_{r,R}^k \wedge T) < \infty) = 1$ . This completes the proof of property 1.

Property 2 follows directly from property 1 and the fact that  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . As for property 3, the first assertion follows from Theorem 2.7 and the fact that  $\bar{\theta}_{r,R} < T_V$ . For the second assertion, notice that, with positive probability, the Brownian motion  $x + \sigma(0)B$  will exit  $G$  before it hits one of the two levels  $H_r$  and  $H_R$ . Hence,  $\bar{L}(\bar{\theta}_{r,R})$  should be positive with positive probability in order to constrain  $x + \sigma(0)B$  to stay in  $G$ , and the second assertion follows.  $\square$

4.1.3. *Proof of Proposition 4.1.* Since the relation (4.20) follows from (4.19). in order to prove Proposition 4.1, it suffices to establish (4.19). In turn, by Remark 4.5, (4.19) is implied by the estimate (4.28), which we reproduce below:

$$(4.38) \quad \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\tau^0)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right] = \infty.$$

We will establish this estimate using the strong Markov property and scaling arguments. First, we need to introduce some additional notation. Let  $\Lambda_\varepsilon$  denote the following collection of hyperplanes:

$$(4.39) \quad \Lambda_\varepsilon \doteq \bigcup_{n \in \mathbb{Z}} H_{2^n \varepsilon}.$$

For  $x \in \Lambda_\varepsilon$ , let  $N_\varepsilon(x)$  denote the pair of hyperplanes in  $\Lambda_\varepsilon$  that are adjacent to the hyperplane on which  $x$  lies. More precisely, let

$$(4.40) \quad N_\varepsilon(x) \doteq H_{2^{n-1} \varepsilon} \cup H_{2^{n+1} \varepsilon} \quad \text{for } x \in H_{2^n \varepsilon}, n \in \mathbb{Z}.$$

Moreover, for  $\varepsilon > 0$ , let  $\{\beta_n^\varepsilon\}_{n \in \mathbb{N}}$  be the sequence of random times defined recursively by  $\beta_0^\varepsilon \doteq 0$  and for  $n \in \mathbb{N}$ ,

$$(4.41) \quad \beta_n^\varepsilon \doteq \inf\{t \geq \beta_{n-1}^\varepsilon : Z(t) \in N_\varepsilon(Z(\beta_{n-1}^\varepsilon))\}.$$

It is easy to see that  $\{\beta_n^\varepsilon\}_{n \in \mathbb{N}}$  defines a sequence of stopping times (for completeness, a proof is provided in Lemma B.1).

Fix  $\varepsilon > 0$ . Observe that  $L$  is non-decreasing and for  $x \in H_\varepsilon$ ,  $\mathbb{P}_x$ -a.s.,  $\beta_n^\varepsilon \leq \tau_0$  for every  $n \in \mathbb{N}$ . Since  $Z(\tau^\varepsilon) \neq 0$  when  $\varepsilon > 0$ , this implies that for every  $n \in \mathbb{N}$ ,

$$(4.42) \quad \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\tau^0)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \geq \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\beta_n^\varepsilon)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right].$$

Using the expansion

$$1 - e^{-L(\beta_n^\varepsilon)} = 1 - e^{-L(\beta_{n-1}^\varepsilon)} + e^{-L(\beta_{n-1}^\varepsilon)} \left( 1 - e^{-(L(\beta_n^\varepsilon) - L(\beta_{n-1}^\varepsilon))} \right),$$

conditioning on  $\mathcal{F}_{\beta_{n-1}^\varepsilon}$ , and invoking the strong Markov property of  $Z$ , the right-hand side of (4.42) can be further expanded as

$$\begin{aligned} & \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left(1 - e^{-L(\beta_n^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \\ &= \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left(1 - e^{-L(\beta_{n-1}^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \\ &\quad + \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{n-1}^\varepsilon)} \left(1 - e^{-(L(\beta_n^\varepsilon) - L(\beta_{n-1}^\varepsilon))}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \middle| \mathcal{F}_{\beta_{n-1}^\varepsilon} \right] \right] \\ &= \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left(1 - e^{-L(\beta_{n-1}^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \\ &\quad + \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{n-1}^\varepsilon)} \mathbb{E}_{Z(\beta_{n-1}^\varepsilon)} \left[ \left(1 - e^{-L(\beta_1^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right]. \end{aligned}$$

Observing that the first term on the right-hand side is identical to the term on the left-hand side, except for a shift down in the index  $n$ , we can iterate this procedure and use the relation  $L(\beta_0^\varepsilon) = L(0) = 0$  to conclude that for any  $n \in \mathbb{N}$ ,

$$(4.43) \quad \begin{aligned} & \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left(1 - e^{-L(\beta_n^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \\ &= \sum_{m=1}^n \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{m-1}^\varepsilon)} \mathbb{E}_{Z(\beta_{m-1}^\varepsilon)} \left[ \left(1 - e^{-L(\beta_1^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right]. \end{aligned}$$

We now show that each term in the sum on the right-hand side of (4.43) is  $O(\varepsilon)$  (as  $\varepsilon \downarrow 0$ ) with a constant that is independent of  $m$ . This proof relies on the estimates obtained in the next two lemmas. In both lemmas,  $Z_{(k)}, Y_{(k)}, L_{(k)}$  will denote the processes defined at the beginning of Section 4.1.2, and for  $\varepsilon > 0$ ,  $\beta_{(k),n}^\varepsilon$  is defined as  $\beta_n^\varepsilon$  is in (4.41), but with  $Z_{(k)}$  in place of  $Z$ :  $\beta_{(k),0}^\varepsilon \doteq 0$  and for  $n \in \mathbb{N}$ ,

$$(4.44) \quad \beta_{(k),n}^\varepsilon \doteq \inf \left\{ t \geq \beta_{(k),n-1}^\varepsilon : Z_{(k)}(t) \in N_1 \left( Z_{(k)} \left( \beta_{(k),n-1}^\varepsilon \right) \right) \right\}.$$

Likewise, the processes  $Z^k, Y^k, L^k$  are defined in terms of  $Z_{(k)}, Y_{(k)}, L_{(k)}$  via (4.29), and  $\beta_n^{k,\varepsilon}$  is defined as in (4.41), but with  $Z$  replaced by  $Z^k$ . With these definitions, due to the scaling, we have the equivalence

$$(4.45) \quad \varepsilon_k^2 \beta_n^{k,1} = \beta_{(k),n}^{\varepsilon_k}, \quad k, n \in \mathbb{N}.$$

**Lemma 4.8.** *Suppose  $b \equiv 0$ . Then there exists  $C > 0$  such that*

$$(4.46) \quad \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \inf_{x \in H_\varepsilon} \mathbb{E}_x \left[ \left(1 - e^{-L(\beta_1^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \geq C.$$

*Proof.* Let  $\varepsilon_k, k \in \mathbb{N}$ , with  $\varepsilon_k \downarrow 0$  as  $k \rightarrow \infty$ , and  $x_k \in H_{\varepsilon_k}, k \in \mathbb{N}$ , be sequences such that

$$(4.47) \quad \begin{aligned} & \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \inf_{x \in H_\varepsilon} \mathbb{E}_x \left[ \left(1 - e^{-L(\beta_1^\varepsilon)}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \\ & \geq \frac{1}{2} \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \mathbb{E}_{x_k} \left[ \left(1 - e^{-L(\beta_1^{\varepsilon_k})}\right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right]. \end{aligned}$$

Since  $H_1$  is compact and  $x_k/\varepsilon_k \in H_1$  for every  $k \in \mathbb{N}$ , we can assume without loss of generality (by choosing a further subsequence if necessary) that there

exists  $x \in H_1$  such that  $x_k/\varepsilon_k \rightarrow x$  as  $k \rightarrow \infty$ . Moreover, since the law of  $(Z_{(k)}, Y_{(k)}, L_k)$  under  $\mathbb{P}$  is the same as the law of  $(Z, Y, L)$  under  $\mathbb{P}_{x_k}$ , we have

$$(4.48) \quad \begin{aligned} \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \mathbb{E}_{x_k} \left[ \left( 1 - e^{-L(\beta_1^{\varepsilon_k})} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \\ = \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \mathbb{E} \left[ \left( 1 - e^{-L^{(k)}(\beta_{(k),1}^{\varepsilon_k})} \right) \mathbb{I}_{\{\tau_{(k)}^0 < \tau_{(k)}^1\}} \right], \end{aligned}$$

with  $\beta_{(k),1}^{\varepsilon}$  defined as in (4.44), and  $\tau_{(k)}^{\varepsilon}$  and  $\tau^{k,\varepsilon}$  defined as follows:

$$(4.49) \quad \tau_{(k)}^{\varepsilon} \doteq \inf \{ t \geq 0 : Z_{(k)}(t) \in H^{\varepsilon} \} \quad \tau^{k,\varepsilon} \doteq \inf \left\{ t \geq 0 : Z^k(t) \in H_{\frac{\varepsilon}{\varepsilon_k}} \right\}.$$

Assume without loss of generality that  $k$  is large enough so that  $\varepsilon_k < 1$ . Then applying the mean value theorem for the function  $f_k(x) = 1 - e^{\varepsilon_k x}$ , we infer that for  $x \geq 0$ , there exists  $\varepsilon_k^* \in (0, \varepsilon_k)$  such that

$$\frac{1 - e^{-\varepsilon_k x}}{\varepsilon_k} = x e^{-\varepsilon_k^* x} \geq x e^{-x}.$$

Thus, applying first the scaling (4.29) and then the above inequality, along with the relation  $\beta_{(k),1}^{\varepsilon_k} = \varepsilon_k^2 \beta_1^{k,1}$ , we have for all  $k$  sufficiently large,

$$\begin{aligned} \frac{1}{\varepsilon_k} \mathbb{E} \left[ \left( 1 - e^{-L^{(k)}(\beta_{(k),1}^{\varepsilon_k})} \right) \mathbb{I}_{\{\tau_{(k)}^0 < \tau_{(k)}^1\}} \right] &= \mathbb{E} \left[ \left( \frac{1 - e^{-\varepsilon_k L^k(\beta_1^{k,1})}}{\varepsilon_k} \right) \mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}} \right] \\ &\geq \mathbb{E} \left[ L^k(\beta_1^{k,1}) e^{-L^k(\beta_1^{k,1})} \mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}} \right]. \end{aligned}$$

Comparing this with (4.47) and (4.48), it is clear that to prove the lemma it suffices to show that there exists  $\tilde{C} > 0$  such that

$$(4.50) \quad \liminf_{k \rightarrow \infty} \mathbb{E} \left[ L^k(\beta_1^{k,1}) e^{-L^k(\beta_1^{k,1})} \mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}} \right] \geq \tilde{C}.$$

Choose  $r \in (1/2, 1)$  and  $R \in (1, 2)$  such that the convergence in property 2 of Lemma 4.6 holds. Then property 3 of Corollary 4.7 implies that there exists  $\delta > 0$  such that

$$\mathbb{P} \left( \max_{i=1,\dots,J} \sup_{t \in [0, \theta_{r,R}]} \bar{Y}_i(t) > \delta \right) > 2\delta.$$

Property 2 of Lemma 4.6 and the Portmanteau theorem then imply that

$$\begin{aligned} \mathbb{P} \left( \max_{i=1,\dots,J} \sup_{t \in [0, \theta_{r,R}]} \bar{Y}_i(t) > \delta \right) &\leq \liminf_{k \rightarrow \infty} \mathbb{P} \left( \max_{i=1,\dots,J} \sup_{t \in [0, \theta_{r,R}^k]} Y_i^k(t) > \delta \right) \\ &\leq \liminf_{k \rightarrow \infty} \mathbb{P} \left( L^k(\theta_{r,R}^k) > \delta \right). \end{aligned}$$

The last two statements, along with the fact that  $\beta_1^{k,1} \geq \theta_{r,R}^k$  for all  $k$ , imply that there exists  $K < \infty$  such that

$$(4.51) \quad \mathbb{P} \left( L^k \left( \beta_1^{k,1} \right) > \delta \right) \geq \delta \quad \text{for all } k \geq K.$$

Next, choose  $r' \in (0, 1/2)$  and  $R' \in (2, \infty)$ . Then  $\beta_1^{k,1} \leq \theta_{r',R'}^k$ . By property 1 of Corollary 4.7, there exists  $c < \infty$  such that

$$\mathbb{P} \left( \sup_{k \in \mathbb{N}} L^k(\theta_{r',R'}^k) < c \right) \geq 1 - \frac{\delta}{4}.$$

It follows that

$$(4.52) \quad \sup_{k \in \mathbb{N}} \mathbb{P} \left( e^{-L^k(\beta_1^{k,1})} > e^{-c} \right) \geq \sup_{k \in \mathbb{N}} \mathbb{P} \left( L^k(\theta_{r',R'}^k) < c \right) \geq 1 - \frac{\delta}{4}.$$

On the other hand, since  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ , by (4.26) of Lemma 4.3 we have

$$\mathbb{P}(\tau^{k,0} < \tau^{k,1}) = \mathbb{P}(\tau_{(k)}^0 < \tau_{(k)}^1) = 1 - \varepsilon_k \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

By choosing  $K$  larger if necessary, we can assume that for all  $k \geq K$ ,

$$(4.53) \quad \mathbb{P}(\tau^{k,0} < \tau^{k,1}) \geq 1 - \frac{\delta}{4}.$$

Now define the set

$$S_k \doteq \left\{ \tau^{k,0} < \tau^{k,1}, e^{-L^k(\beta_1^{k,1})} \geq e^{-c}, L^k(\beta_1^{k,1}) > \delta \right\}.$$

Then (4.51), (4.52) and (4.53), together show that for  $k \geq K$ ,  $\mathbb{P}(S_k) \geq \delta/2$ . Therefore, for all  $k \geq K$ ,

$$\begin{aligned} \mathbb{E} \left[ L^k(\beta_1^{k,1}) e^{-L^k(\beta_1^{k,1})} \mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}} \right] \\ \geq \mathbb{E} \left[ L^k(\beta_1^{k,1}) e^{-L^k(\beta_1^{k,1})} \mathbb{I}_{\{\tau^{k,0} < \tau^{k,1}\}} \mathbb{I}_{S_k} \right] \geq \delta e^{-c} \frac{\delta}{2}, \end{aligned}$$

and so (4.50) holds with  $\tilde{C} = \delta^2 e^{-c}/2$ . This completes the proof of the lemma.  $\square$

**Lemma 4.9.** *Suppose  $b \equiv 0$ . For every  $n \in \mathbb{N}$ ,*

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in H_\varepsilon} \mathbb{E}_x \left[ 1 - e^{-L(\beta_n^\varepsilon)} \right] = 0.$$

*Proof.* Fix  $n \in \mathbb{N}$ . We prove the lemma using an argument by contradiction. Suppose that there exists  $\delta_0 > 0$  and a sequence  $\varepsilon_k, k \in \mathbb{N}$ , such that  $\varepsilon_k \downarrow 0$  as  $k \rightarrow \infty$  and for every  $k \in \mathbb{N}$ ,

$$\sup_{x \in H_{\varepsilon_k}} \mathbb{E}_x \left[ 1 - e^{-L(\beta_n^{\varepsilon_k})} \right] \geq \delta_0.$$

For each  $k \in \mathbb{N}$ , let  $x_k \in H_{\varepsilon_k}$  be such that

$$(4.54) \quad \mathbb{E}_{x_k} \left[ 1 - e^{-L(\beta_n^{\varepsilon_k})} \right] \geq \frac{\delta_0}{2}.$$

Since, the law of  $(Z_{(k)}, Y_{(k)}, L_{(k)})$  under  $\mathbb{P}$  is the same as the law of  $(Z, Y, L)$  under  $\mathbb{P}_{x_k}$ , (4.54) is equivalent to

$$(4.55) \quad \mathbb{E} \left[ 1 - e^{-L_{(k)}(\beta_{(k),n}^{\varepsilon_k})} \right] \geq \frac{\delta_0}{2},$$

with  $\beta_{(k),n}^{\varepsilon}$  defined by (4.44). Since  $H_1$  is compact and  $x_k/\varepsilon_k \in H_1$  for every  $k \in \mathbb{N}$ , we can assume without loss of generality (by choosing an appropriate subsequence, if necessary) that there exists  $x \in H_1$  such that  $x_k/\varepsilon_k \rightarrow x$  as  $k \rightarrow \infty$ .

The definition (4.29) of the scaling and the relation  $\varepsilon_k^2 \beta_n^{k,1} = \beta_{(k),n}^{\varepsilon_k}$  show that

$$(4.56) \quad \mathbb{E} \left[ 1 - e^{-L_{(k)}(\beta_{(k),n}^{\varepsilon_k})} \right] = \mathbb{E} \left[ 1 - e^{-\varepsilon_k L^k(\beta_n^{k,1})} \right].$$

Moreover, since  $x_k \in H_{\varepsilon_k}$  implies  $Z^k(0) = x_k/\varepsilon_k \in H_1$ , it follows that, on the interval  $[0, \beta_n^{k,1}]$ ,  $\langle Z^k(t), \vec{\nu} \rangle \in [2^{-n}, 2^n]$ . Therefore, there exist  $0 < r < 2^{-n}$  and  $R > 2^n$  such that  $\beta_n^{k,1} \leq \theta_{r,R}^k$ , where  $\theta_{r,R}^k$  is defined in (4.30). As a result, we conclude that

$$\mathbb{E} \left[ 1 - e^{-\varepsilon_k L^k(\beta_n^{k,1})} \right] \leq \mathbb{E} \left[ 1 - e^{-\varepsilon_k L^k(\theta_{r,R}^k)} \right] \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

where the last limit holds due to the weak convergence  $\varepsilon_k L^k(\theta_{r,R}^k) \Rightarrow 0$  established in Corollary 4.7, and the fact that  $x \mapsto 1 - e^{-x}$  is a bounded continuous function. When combined with (4.56), this contradicts (4.54) and thus proves the lemma.  $\square$

We can now wrap up the proof of Proposition 4.1.

*Proof of Proposition 4.1.* First observe that by Lemma 4.8, there exists  $C > 0$  and  $\varepsilon^0 > 0$  such that for all  $\varepsilon < \varepsilon^0$ , the relation

$$\inf_{x \in H_\varepsilon} \mathbb{E}_x \left[ \left( 1 - e^{-L(\beta_1^\varepsilon)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \geq \frac{C}{2} \varepsilon$$

is satisfied. In turn, this relation, together with the fact that  $Z(\tau^\varepsilon) \in H_\varepsilon$  and, for any  $x \in H_\varepsilon$ ,  $\mathbb{P}_x$ -a.s.,

$$(4.57) \quad \langle Z(\beta_{n-1}^\varepsilon), \vec{\nu} \rangle \leq 2^{n-1} \varepsilon,$$

implies that for all  $\varepsilon < 2^{-(n-1)} \varepsilon_0$  and  $m = 1, \dots, n$ ,

$$(4.58) \quad \begin{aligned} & \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{m-1}^\varepsilon)} \mathbb{E}_{Z(\beta_{m-1}^\varepsilon)} \left[ \left( 1 - e^{-L(\beta_1^\varepsilon)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right] \\ & \geq \frac{C}{2} \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{m-1}^\varepsilon)} \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right]. \end{aligned}$$

When combined with (4.42) and (4.43), this shows that

$$(4.59) \quad \begin{aligned} & \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\tau^0)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right] \\ & \geq \frac{C}{2} \sum_{m=1}^n \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{m-1}^\varepsilon)} \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right]. \end{aligned}$$

Each summand on the right-hand side can be rewritten in the more convenient form

$$\begin{aligned} \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{m-1}^\varepsilon)} \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right] &= \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right] \\ &\quad - \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\beta_{m-1}^\varepsilon)} \right) \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right]. \end{aligned}$$

Since  $b \equiv 0$ , Lemma 4.3 and the uniform bound (4.57) shows that  $\langle Z, \vec{\nu} \rangle$  is a martingale on  $[0, \beta_n^\varepsilon]$ . In addition, because  $\beta_{m-1}^\varepsilon \leq \beta_n^\varepsilon$  and  $\langle Z(\tau^\varepsilon), \vec{\nu} \rangle = \varepsilon$ , it follows that

$$\mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right] \right] = \mathbb{E}_0[\varepsilon] = \varepsilon.$$

Furthermore, by (4.57), Lemma 4.9 and the bounded convergence theorem, for any  $n \in \mathbb{N}$  and  $m = 1, \dots, n$ ,

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\beta_{m-1}^\varepsilon)} \right) \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right] \right] \\ \leq 2^{n-1} \lim_{\varepsilon \downarrow 0} \mathbb{E}_0 \left[ \sup_{x \in H_\varepsilon} \mathbb{E}_x \left[ \left( 1 - e^{-L(\beta_{m-1}^\varepsilon)} \right) \right] \right] = 0. \end{aligned}$$

Combining the last three assertions, we see that for every  $n \in \mathbb{N}$  and  $m = 1, \dots, n$ ,

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ e^{-L(\beta_{m-1}^\varepsilon)} \langle Z(\beta_{m-1}^\varepsilon), \vec{\nu} \rangle \right] \right] = 1.$$

Together with (4.59), this shows that for every  $n \in \mathbb{N}$ ,

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_0 \left[ \mathbb{E}_{Z(\tau^\varepsilon)} \left[ \left( 1 - e^{-L(\tau^0)} \right) \mathbb{I}_{\{\tau^0 < \tau^1\}} \right] \right] \geq \frac{nC}{2}.$$

Taking the limit as  $n \rightarrow \infty$ , we obtain (4.38), thus completing the proof of the proposition.  $\square$

**4.2. The General Drift Case.** In this section we establish Theorem 4.1. Specifically, we generalize the case of zero drift, established in Proposition 4.1, to arbitrary bounded, Lipschitz drifts (as mentioned in Section 2.3, this can be extended to drifts satisfying the usual Lipschitz and growth conditions) by using a Girsanov transformation. As usual, let  $Z$  be the unique solution to the Class  $\mathcal{A}$  SDER, which exists by Theorem 2.7, and let  $\tau^1$  be the first hitting time to  $H_1$ , as defined in (4.18). We begin with a simple lemma that shows that  $\tau^1$  is finite with positive  $\mathbb{P}_0$  probability.

**Lemma 4.10.** *We have*

$$(4.60) \quad \mathbb{P}_0(\tau^1 < \infty) > 0.$$

*Moreover, if  $\inf_{x: \langle x, \vec{\nu} \rangle \leq 1} \langle b(x), \vec{\nu} \rangle \geq 0$ , then*

$$(4.61) \quad \mathbb{P}_0(\tau^1 < \infty) = 1.$$

*Proof.* Recall the definition of  $X$  and  $M$  given in (2.4) and (2.12) and let  $\widehat{H} \doteq \langle H, \vec{\nu} \rangle$  for  $H = Z, M, X$ . By Lemma 2.8 and Theorem 2.7, we know that  $\widehat{Z} = \Gamma_1(\widehat{X})$ , where  $\Gamma_1$  is the 1-dimensional Skorokhod map. Let  $T(t) \doteq$

$\inf\{s \geq 0 : \langle \widehat{M} \rangle_s > t\}$ . Then, due to the uniform ellipticity of  $a$ ,  $T$  is strictly increasing and  $\widehat{M}(T(\cdot))$  is a 1-dimensional Brownian motion. In turn, this implies  $\widehat{Z}$  is a one-dimensional reflected Brownian motion with drift

$$\int_0^t \langle b(Z(T(s))), \vec{\nu} \rangle dT(s) = \int_0^t \langle b(Z(T(s))), \vec{\nu} \rangle \frac{1}{\vec{\nu}^T a(Z(s)) \vec{\nu}} ds.$$

Since  $\langle b(x), \vec{\nu} \rangle / \vec{\nu}^T a(x) \vec{\nu}$  is continuous on  $G$ , there exists  $\kappa \in (-\infty, \infty)$  such that

$$\frac{\langle b(x), \vec{\nu} \rangle}{\vec{\nu}^T a(x) \vec{\nu}} > \kappa \quad \text{for all } x \in G, \langle x, \vec{\nu} \rangle \leq 1.$$

Consider the process  $\tilde{X}$  defined by  $\tilde{X}(t) \doteq \kappa t + M(T(t))$  for  $t \in [0, \infty)$  and let  $\tilde{Z} \doteq \Gamma_1(\tilde{X})$  be a one-dimensional reflected Brownian motion with constant drift  $\kappa$ . Then  $\widehat{X}(T(t)) - \widehat{X}(T(s)) \geq \tilde{X}(t) - \tilde{X}(s)$  for every  $0 \leq s \leq t$ , and so the comparison principle for  $\Gamma_1$  (see, for example, equation (4.1) in Lemma 4.1 of [21]) shows that  $\widehat{Z}(T(t)) \geq \tilde{Z}(t)$  for every  $t \in [0, \hat{\tau}^1]$ , where

$$\hat{\tau}^1 \doteq \inf\{t > 0 : \widehat{Z}(T(t)) = 1\}.$$

Since  $T(\hat{\tau}^1) = \tau^1$ , it follows that

$$\mathbb{P}_0(\tilde{Z}(t \wedge \hat{\tau}^1) \leq \widehat{Z}(T(t) \wedge \tau^1) \text{ for all } t \geq 0) = 1.$$

Since  $T$  is strictly increasing, we have  $\tau^1 = \infty$  if and only if  $\hat{\tau}^1 = \infty$ . Then on the set  $\tau^1 = \infty$ , we must have

$$\tilde{Z}(t) \leq \widehat{Z}(T(t)) < 1 \quad \text{for all } t \in [0, \infty).$$

Since  $\tilde{Z}$  will hit 1 with positive  $\mathbb{P}_0$  probability, and in fact will hit 1  $\mathbb{P}_0$  a.s. if  $\kappa \geq 0$  (see, for example, page 197 of [20]), this implies (4.60) and (4.61) and the proof of the lemma is complete.  $\square$

**Proof of Theorem 4.1.** The uniform ellipticity of  $a(\cdot)$  ensures that  $a^{-1}(\cdot)$  exists. Let  $\mu \doteq -b^T a^{-1} \sigma$ , note that  $\mu^T \mu = bab^T$ , and define

$$(4.62) \quad D(t) \doteq \exp \left\{ \int_0^t \mu(Z(s)) dB(s) - \frac{1}{2} \int_0^t b(Z(s)) a(Z(s)) b^T(Z(s)) ds \right\}$$

for  $t \in [0, \infty)$ . Property 3 of Definition 2.5 guarantees that  $\mu^T(Z(\cdot))\mu(Z(\cdot))$  has linear growth and so  $\{D(t), \mathcal{F}_t\}$  is a martingale by Corollary 5.16 of [20].

Fix  $T < \infty$ . Define a new probability measure  $\mathbb{Q}_0$  on  $(\Omega, \mathcal{F}, \{\mathcal{F}_T\})$  by setting

$$\mathbb{Q}_0(A) = \mathbb{E}[D(T)\mathbb{I}_A] \quad \text{for } A \in \mathcal{F}_T.$$

Define

$$\widehat{B}(t) \doteq B(t) + \int_0^t \sigma^T(Z(s)) a^{-1}(Z(s)) b(s) ds, \quad t \in [0, T].$$

By Girsanov's theorem (see Theorem 5.1 of [20]), under  $\mathbb{Q}_0$ ,  $\{\widehat{B}_t, \mathcal{F}_t\}_{t \in [0, T]}$  is a Brownian motion and

$$Z(t) = \int_0^t \sigma(Z(s)) d\widehat{B}(s) + Y(t), \quad t \in [0, T],$$

where  $(Z, Y)$  satisfy the ESP pathwise for  $Z - Y$ . Since, under  $\mathbb{Q}_0$ ,  $Z$  is the solution to a Class  $\mathcal{A}$  SDER with no drift, by Proposition 4.1, we know that

$$\mathbb{Q}_0(L(\tau^1) < \infty, \tau^1 \leq T) = 0.$$

Since  $\mathbb{P}_0 \ll \mathbb{Q}_0$  (with  $d\mathbb{P}_0/d\mathbb{Q}_0 = D^{-1}(T)$  on  $\mathcal{F}_T$ ), this implies

$$\mathbb{P}_0(L(\tau^1) < \infty, \tau^1 \leq T) = 0.$$

Since  $T < \infty$  is arbitrary, sending  $T \rightarrow \infty$  (along a countable sequence), we conclude that

$$\mathbb{P}_0(L(\tau^1) < \infty, \tau^1 < \infty) = 0.$$

However,  $\mathbb{P}_0(\tau^1 < \infty) > 0$  by Lemma 4.10. Hence,  $\mathbb{P}_0(L(\tau^1) = \infty, \tau^1 < \infty) > 0$ , which in turn implies that there exists  $T < \infty$  such that  $\mathbb{P}_0(L(T) = \infty) > 0$ . Note that if  $\inf_{x \in G: \langle x, \vec{v} \rangle \leq 1} \langle b(x), \vec{v} \rangle \geq 0$ , then  $\mathbb{P}_0(\tau^1 < \infty) = 1$  and so we in fact have  $\mathbb{P}_0(L(\tau^1) = \infty) = 1$ .  $\square$

**4.3. The Semimartingale Property for  $Z$ .** Recall from Theorem 2.7 that the process  $Z$  has the decomposition  $Z = M + A$ , where

$$(4.63) \quad M = \int_0^\cdot \sigma(Z(s)) dB(s) \quad A = \int_0^\cdot b(Z(s)) ds + Y,$$

and  $Y$  is the constraining term associated with the ESP.  $M$  is clearly a (local) martingale. However, as mentioned earlier, Theorem 4.1 does not immediately imply that  $Z$  is not a semimartingale because we do not know a priori that the above decomposition must be the Doob decomposition of  $Z$  if it were a semimartingale. The following result shows that this is indeed the case.

**Proposition 4.2.** *If  $Z$  were a semimartingale then its Doob decomposition must be  $Z = M + A$ .*

*Proof.* Suppose that  $Z$  is a semimartingale, and let its (unique) Doob decomposition take the form

$$Z = \tilde{M} + \tilde{A},$$

where  $\tilde{M}$  is an  $\{\mathcal{F}_t\}$ -adapted continuous local martingale and  $\tilde{A}$  is an  $\{\mathcal{F}_t\}$ -adapted continuous, process with  $\mathbb{P}$  a.s. finite variation on bounded intervals.

Fix  $R < \infty$  and let  $\theta_R \doteq \inf\{t \geq 0 : |M(t)| \geq R\}$ . For each  $\varepsilon > 0$ , define two sequences of stopping times  $\{\tau_n^\varepsilon\}_{n \in \mathbb{N}}$  and  $\{\xi_n^\varepsilon\}_{n \in \mathbb{N}}$  as follows:  $\xi_0^\varepsilon \doteq 0$  and for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \tau_n^\varepsilon &\doteq \inf\{t \geq \xi_{n-1}^\varepsilon : Z(t) \in H_\varepsilon\} \wedge \theta_R, \\ \xi_n^\varepsilon &\doteq \inf\{t \geq \tau_n^\varepsilon : Z(t) \in H_{\varepsilon/2}\} \wedge \theta_R. \end{aligned}$$

(For notational conciseness, we have suppressed the dependence of these stopping times on  $R$ .) From the equality  $Z = M + A = \tilde{M} + \tilde{A}$ , we have

$$(4.64) \quad \begin{aligned} Z(t \wedge \xi_n^\varepsilon) - Z(t \wedge \tau_n^\varepsilon) &= M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon) + A(t \wedge \xi_n^\varepsilon) - A(t \wedge \tau_n^\varepsilon) \\ &= \tilde{M}(t \wedge \xi_n^\varepsilon) - \tilde{M}(t \wedge \tau_n^\varepsilon) + \tilde{A}(t \wedge \xi_n^\varepsilon) - \tilde{A}(t \wedge \tau_n^\varepsilon). \end{aligned}$$

By uniqueness of the Doob decomposition, clearly  $Z(\cdot \wedge \xi_n^\varepsilon) - Z(\cdot \wedge \tau_n^\varepsilon)$  is an  $\{\mathcal{F}_t\}$ -adapted semimartingale, with Doob decomposition (4.64). On the other hand, since  $M$  is an  $\{\mathcal{F}_t\}$ -adapted continuous (local) martingale, and  $M$  is uniformly bounded on  $[0, \theta_R]$ , the stopped processes  $M(\cdot \wedge \xi_n^\varepsilon)$  and  $M(\cdot \wedge \tau_n^\varepsilon)$  are  $\{\mathcal{F}_t\}$ -adapted continuous martingales. Hence,  $M(\cdot \wedge \xi_n^\varepsilon) - M(\cdot \wedge \tau_n^\varepsilon)$  is also an  $\{\mathcal{F}_t\}$ -adapted continuous martingale. Moreover, Theorem 2.7 implies that  $Y(\cdot \wedge \xi_n^\varepsilon) - Y(\cdot \wedge \tau_n^\varepsilon)$ , and therefore  $A(\cdot \wedge \xi_n^\varepsilon) - A(\cdot \wedge \tau_n^\varepsilon)$ , has  $\mathbb{P}$  a.s. finite variation on each bounded time interval. By uniqueness of the Doob decomposition, we conclude that for every  $\varepsilon > 0$  and  $t \in [0, \infty)$ ,

$$M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon) = \tilde{M}(t \wedge \xi_n^\varepsilon) - \tilde{M}(t \wedge \tau_n^\varepsilon).$$

Summing over  $n \in \mathbb{N}$  on both sides of the last equation, we obtain

$$(4.65) \quad \sum_{n=1}^{\infty} (M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon)) = \sum_{n=1}^{\infty} (\tilde{M}(t \wedge \xi_n^\varepsilon) - \tilde{M}(t \wedge \tau_n^\varepsilon)).$$

On the other hand,  $\mathbb{P}$  a.s., because  $M(0) = 0$  and  $\xi_n^\varepsilon \rightarrow \theta_R$  as  $n \rightarrow \infty$ , we have the elementary relation

$$M(t \wedge \theta_R) = \sum_{n=1}^{\infty} (M(t \wedge \xi_n^\varepsilon) - M(t \wedge \xi_{n-1}^\varepsilon)), \quad t \in [0, \infty).$$

Therefore, we can write

$$\begin{aligned} M(t \wedge \theta_R) - \sum_{n=1}^{\infty} (M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon)) &= \sum_{n=1}^{\infty} (M(t \wedge \tau_n^\varepsilon) - M(t \wedge \xi_{n-1}^\varepsilon)) \\ &= \int_0^t \sum_{n=1}^{\infty} \mathbb{I}_{(\xi_{n-1}^\varepsilon, \tau_n^\varepsilon]}(s) dM(s) \\ &= \int_0^t \sum_{n=1}^{\infty} \mathbb{I}_{(\xi_{n-1}^\varepsilon, \tau_n^\varepsilon]}(s) \mathbb{I}_{[0, \varepsilon]}(\langle \vec{\nu}, Z(s) \rangle) dM(s). \end{aligned}$$

The last equality holds because  $\langle \vec{v}, Z(s) \rangle \leq \varepsilon$  for  $s \in (\xi_{n-1}^\varepsilon, \tau_n^\varepsilon]$ . So, by Doob's maximal martingale inequality, it follows that

$$\begin{aligned} & \mathbb{E} \left[ \sup_{s \in [0, t]} \left| \sum_{n=1}^{\infty} (M(s \wedge \xi_n^\varepsilon) - M(s \wedge \tau_n^\varepsilon)) - M(s \wedge \theta_R) \right|^2 \right] \\ & \leq 4\mathbb{E} \left[ \left| \sum_{n=1}^{\infty} (M(t \wedge \xi_n^\varepsilon) - M(t \wedge \tau_n^\varepsilon)) - M(t \wedge \theta_R) \right|^2 \right] \\ & = 4\mathbb{E} \left[ \left| \int_0^t \sum_{n=1}^{\infty} \mathbb{I}_{(\xi_{n-1}^\varepsilon, \tau_n^\varepsilon]}(s) \mathbb{I}_{[0, \varepsilon]}(\langle \vec{v}, Z(s) \rangle) dM(s) \right|^2 \right] \\ & \leq 4\mathbb{E} \left[ \int_0^t \mathbb{I}_{[0, \varepsilon]}(\langle \vec{v}, Z(s) \rangle) |a(Z(s))| ds \right]. \end{aligned}$$

Since  $a$  is bounded on the set  $\{x : \langle \vec{v}, x \rangle \leq \varepsilon\}$ , by the bounded convergence theorem we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{\langle \vec{v}, Z(s) \rangle \leq \varepsilon\}} |a(Z(s))| ds \right] = |a(0)| \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{\langle \vec{v}, Z(s) \rangle = 0\}} ds \right] = 0,$$

where the last equality is a consequence of the fact that  $\langle \vec{v}, Z \rangle$  is a uniformly elliptic one-dimensional reflected diffusion (see Lemma 2.8) and consequently spends zero Lebesgue time at the origin (see, for example, page 90 of [17]).

An exactly analogous argument, with  $\tilde{\theta}_R \doteq \inf\{t \geq 0 : |\tilde{M}|(t) \geq R\}$  and  $\tilde{\xi}_n^\varepsilon, \tilde{\tau}_n^\varepsilon$  defined like  $\xi_n^\varepsilon, \tau_n^\varepsilon$ , but with  $\theta_R$  replaced by  $\tilde{\theta}_R$ , shows that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{s \in [0, t]} \left| \sum_{n=1}^{\infty} (\tilde{M}(s \wedge \tilde{\xi}_n^\varepsilon) - \tilde{M}(s \wedge \tilde{\tau}_n^\varepsilon)) - \tilde{M}(s \wedge \tilde{\theta}_R) \right|^2 \right] \\ & \leq \lim_{\varepsilon \rightarrow 0} 4J^2 \sum_{i=1}^J \mathbb{E} \left[ \int_0^t \mathbb{I}_{[0, \varepsilon]}(\langle \vec{v}, Z(s) \rangle) d\langle \tilde{M}_i \rangle(s) \right] \\ & = 4J^2 \sum_{i=1}^J \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{0\}}(\langle \vec{v}, Z(s) \rangle) d\langle \tilde{M}_i \rangle(s) \right] \\ & = 4J^2 \sum_{i=1}^J \mathbb{E} \left[ \int_0^t \mathbb{I}_{\{0\}}(Z_i(s)) d\langle \tilde{M}_i \rangle(s) \right]. \end{aligned}$$

The last equality uses the property that  $Z_i(s) = 0$  for every  $i = 1, \dots, J$  if and only if  $\langle \vec{v}, Z(s) \rangle = 0$  (see property 2 of Definition 2.5). By the assumption that  $\tilde{Z}_i$  is a semimartingale with decomposition  $\tilde{M}_i + \tilde{A}_i$ , the occupation times formula for continuous semimartingales (see, e.g., Corollary 1.6 in Chapter VI of [25]) and the fact that the set  $\{x : x_i = 0\}$  has zero Lebesgue measure, we have, a.s.,

$$\int_0^t \mathbb{I}_{\{0\}}(Z_i(s)) d\langle \tilde{M}_i \rangle(s) = \int_0^t \mathbb{I}_{\{0\}}(Z_i(s)) d\langle Z_i \rangle(s) = 0, \quad i = 1, \dots, J.$$

Combining the last four displays with (4.65), we conclude that  $M(t \wedge \theta_R) = \tilde{M}(t \wedge \tilde{\theta}_R)$ ,  $\mathbb{P}_0$ -a.s., for every  $t \geq 0$ . This in turn implies that  $\theta_R = \tilde{\theta}_R$   $\mathbb{P}_0$  a.s. and, sending  $R \rightarrow \infty$  and invoking the continuity of both  $M$  and  $\tilde{M}$ , we have  $M = \tilde{M}$   $\mathbb{P}_0$ -a.s. In turn, this implies  $A = \tilde{A}$ , thus completing the proof of the theorem.  $\square$

The proof of Theorem 3.1 is now a simple consequence of Theorem 4.1 and Proposition 4.2.

**Proof of Theorem 3.1.** If  $Z$  were a semimartingale, then by Proposition 4.2  $Z = M + A$  is the Doob decomposition for  $Z$  and so, in particular, we must have  $\mathbb{P}_0(L(T) < \infty) = 1$  for every  $T \in [0, \infty)$ , where  $L(T) = \mathcal{V}ar_{[0,T]} Y$ . However, this contradicts the assertion of Theorem 4.1 that there exists  $T < \infty$  such that  $\mathbb{P}_0(L(T) = \infty) > 0$ . Thus we conclude that  $Z$  is not a semimartingale.  $\square$

**Remark 4.11.** We expect that similar, but somewhat more involved, arguments could be used to show that the semimartingale property fails to hold for a more general class of reflected diffusions in the non-negative orthant, in particular those that arise as approximations of generalized processor sharing networks (rather than just a single station, as considered in [23, 24]). Such diffusions would to satisfy properties 1, 2 and 4 of Definition 2.5 but have more complicated  $\mathcal{V}$ -sets (see [12] for a description of the ESP associated with such a network). This is a subject of future work.

## 5. DIRICHLET PROCESS CHARACTERIZATION

This section is devoted to the proof of Theorem 3.5. Specifically, here we only assume that  $(G, d(\cdot)), b(\cdot)$  and  $\sigma(\cdot)$  satisfy Assumptions 1 and 3, and let  $(Z_t, B_t), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$  be a Markov, weak solution to the associated SDER that satisfies Assumption 2 for some constants  $p > 1, q \geq 2$  and  $K_T < \infty, T \in (0, \infty)$ . As usual, let  $Y = Z - X$ , with  $X$  as defined in (2.4), and recall that  $Z$  admits the decomposition

$$Z(t) = Z(0) + \int_0^t b(Z(s)) ds + \int_0^t \sigma(Z(s)) dB(s) + Y(t), \quad t \in [0, \infty),$$

and  $\int_0^\cdot b(Z(s)) ds$  is a process of bounded variation, and therefore of bounded  $p$ -variation for any  $p > 1$  by Remark 3.4. As a result, in order to establish Theorem 3.5, it suffices to show that under  $\mathbb{P}$ ,  $Y$  has zero  $p$ -variation.

In Section 5.1, we first show that it suffices to establish a localized version (5.68) of the zero  $p$ -variation condition on  $Y$ . This is used to prove Theorem 3.5 in Section 5.2.

**5.1. Localization.** Fix  $T > 0$ , let  $\{\pi^n, n \geq 1\}$  be a sequence of partitions of  $[0, T]$  such that  $\Delta(\pi^n) \rightarrow 0$  as  $n \rightarrow \infty$ . As mentioned above, to prove

Theorem 3.5 we need to establish the following result:

$$(5.66) \quad \sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p \xrightarrow{(\mathbb{P})} 0 \quad \text{as } \Delta(\pi^n) \rightarrow \infty.$$

For each  $m \in (0, \infty)$ , let

$$(5.67) \quad \zeta^m \doteq \inf\{t > 0 : |Z(t)| \geq m\}.$$

It is easy to see that  $\mathbb{P}$ -a.s.,  $\zeta^m \rightarrow \infty$  as  $m \rightarrow \infty$ . We now show that the localized version, (5.68) below, is equivalent to (5.66).

**Lemma 5.1.** *The result (5.66) holds if and only if for each  $m \in (0, \infty)$ ,*

$$(5.68) \quad \sum_{t_i \in \pi^n} |Y(t_i \wedge \zeta^m) - Y(t_{i-1} \wedge \zeta^m)|^p \xrightarrow{(\mathbb{P})} 0 \quad \text{as } \Delta(\pi^n) \rightarrow 0.$$

*Proof.* First assume (5.68) holds for every  $m \in (0, \infty)$ . For any  $m \in (0, \infty)$  and  $\delta > 0$ ,

$$\begin{aligned} & \mathbb{P} \left( \sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p \geq \delta \right) \\ & \leq \mathbb{P} \left( \sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p \geq \delta, \zeta^m > T \right) + \mathbb{P}(\zeta^m \leq T) \\ & = \mathbb{P} \left( \sum_{t_i \in \pi^n} |Y(t_i \wedge \zeta^m) - Y(t_{i-1} \wedge \zeta^m)|^p \geq \delta, \zeta^m > T \right) + \mathbb{P}(\zeta^m \leq T) \\ & \leq \mathbb{P} \left( \sum_{t_i \in \pi^n} |Y(t_i \wedge \zeta^m) - Y(t_{i-1} \wedge \zeta^m)|^p \geq \delta \right) + \mathbb{P}(\zeta^m \leq T). \end{aligned}$$

Taking limits as  $\Delta(\pi^n) \rightarrow 0$ , the first term on the right-hand side vanishes due to (5.68). Next, sending  $m \rightarrow \infty$ , and using the fact that  $\zeta^m \rightarrow \infty$   $\mathbb{P}$  a.s., the second term also vanishes, and so we obtain (5.66). This proves the “if” part of the result.

For the converse, suppose (5.66) holds. Let  $\theta_n^m \doteq \sup\{t_i \in \pi^n : t_i \leq \zeta^m\}$ . Then

$$\sum_{t_i \in \pi^n} |Y(t_i \wedge \zeta^m) - Y(t_{i-1} \wedge \zeta^m)|^p \leq \sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p + |Y(\zeta^m \wedge T) - Y(\theta_n^m)|^p.$$

Taking limits as  $\Delta(\pi^n) \rightarrow 0$ , the last term vanishes  $\mathbb{P}$  a.s. since  $|\zeta^m \wedge T - \theta_n^m| \leq \Delta(\pi^n)$  and  $Y$  is continuous. Therefore, (5.68) follows from (5.66).  $\square$

**5.2. Proof of Theorem 3.5.** For each  $\varepsilon > 0$ , recursively define two sequences of stopping times  $\{\tau_n^\varepsilon\}_{n \in \mathbb{N}}$  and  $\{\xi_n^\varepsilon\}_{n \in \mathbb{N}}$  as follows:  $\xi_0^\varepsilon \doteq 0$  and for  $n \in \mathbb{N}$ ,

$$(5.69) \quad \begin{aligned} \tau_n^\varepsilon & \doteq \inf \{t \geq \xi_{n-1}^\varepsilon : d(Z(t), \mathcal{V}) = \varepsilon\}, \\ \xi_n^\varepsilon & \doteq \inf \{t \geq \tau_n^\varepsilon : d(Z(t), \mathcal{V}) = \varepsilon/2\}. \end{aligned}$$

For each  $\varepsilon > 0$ , we have the decomposition

$$\begin{aligned} \sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p &= \sum_{t_i \in \pi^n} \sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{(\tau_k^\varepsilon, \xi_k^\varepsilon)}(t_{i-1}) \\ &\quad + \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon]}(t_{i-1}). \end{aligned}$$

Therefore, for any given  $\delta > 0$ , we have

$$\begin{aligned} (5.70) \quad &\mathbb{P} \left( \sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p > \delta \right) \\ &\leq \mathbb{P} \left( \sum_{t_i \in \pi^n} \sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{(\tau_k^\varepsilon, \xi_k^\varepsilon)}(t_{i-1}) > \frac{\delta}{2} \right) \\ &\quad + \mathbb{P} \left( \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon]}(t_{i-1}) > \frac{\delta}{2} \right). \end{aligned}$$

Under additional uniform boundedness assumptions on  $b$  and  $\sigma$ , the proof of (5.66) is essentially a consequence of the following two lemmas, which provide estimates on the two terms on the right-hand side of (5.70).

**Lemma 5.2.** *Suppose  $b$  and  $\sigma$  are uniformly bounded. Then, as  $\Delta(\pi^n) \rightarrow 0$ , for each  $\varepsilon > 0$ , we have*

$$(5.71) \quad \mathbb{P} \left( \sum_{t_i \in \pi^n} \sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{(\tau_k^\varepsilon, \xi_k^\varepsilon)}(t_{i-1}) > \frac{\delta}{2} \right) \rightarrow 0.$$

*Proof.* Fix  $\varepsilon > 0$ ,  $n \in \mathbb{N}$  and let

$$\Omega_n^\varepsilon \doteq \left\{ Z(t) \notin \mathcal{V} \quad \forall t \in \bigcup_{k \in \mathbb{N}: \xi_k^\varepsilon \leq T} [\xi_k^\varepsilon, \xi_k^\varepsilon + \Delta(\pi^n)] \right\}.$$

Also, define

$$N^\varepsilon \doteq \inf \{k \geq 0 : \text{either } \tau_k^\varepsilon > T \text{ or } \xi_k^\varepsilon > T\}.$$

Observe that  $N^\varepsilon < \infty$   $\mathbb{P}$  a.s., since  $Z$  has  $\mathbb{P}$  a.s. continuous sample paths and therefore crosses the levels  $\{z \in G : d(z, \mathcal{V}) = \varepsilon\}$  and  $\{z \in G : d(z, \mathcal{V}) = \varepsilon/2\}$  at most a finite number of times in the interval  $[0, T]$ . The continuity of  $Z$  also implies that for each  $\varepsilon > 0$ ,

$$(5.72) \quad \mathbb{P}(\Omega_n^\varepsilon) \rightarrow 1 \quad \text{as} \quad \Delta(\pi^n) \rightarrow 0.$$

On the set  $\Omega_n^\varepsilon$ , we have

$$\begin{aligned}
(5.73) \quad & \sum_{t_i \in \pi^n} \sum_{k=1}^{\infty} |Y(t_i) - Y(t_{i-1})|^{p-1} \mathbb{I}_{[\tau_k^\varepsilon, \xi_k^\varepsilon)}(t_{i-1}) \\
& \leq \max_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{t_i \in \pi^n} \sum_{k=1}^{\infty} L(t_{i-1}, t_i] \mathbb{I}_{[\tau_k^\varepsilon, \xi_k^\varepsilon)}(t_{i-1}) \\
& = \max_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{t_i \in \pi^n} \sum_{k=1}^{\infty} L(t_{i-1}, t_i] \mathbb{I}_{[\tau_k^\varepsilon, \xi_k^\varepsilon)}(t_{i-1}) \\
& \leq \max_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{k=1}^{\infty} L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T].
\end{aligned}$$

By definition,  $\mathbb{P}$  a.s.  $(Z, Y)$  satisfy the ESP for  $X$ . Therefore by Lemma A.1, for each  $k \in \mathbb{N}$ ,  $(Z(\tau_k^\varepsilon \wedge T + \cdot), Y(\tau_k^\varepsilon \wedge T + \cdot) - Y(\tau_k^\varepsilon \wedge T))$   $\mathbb{P}$  a.s. solve the ESP for  $Z(\tau_k^\varepsilon \wedge T) + X(\tau_k^\varepsilon \wedge T + \cdot) - X(\tau_k^\varepsilon \wedge T)$ . On  $\Omega_n^\varepsilon$ ,  $Z$  is away from  $\mathcal{V}$  on  $[\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T]$  for each  $k \geq 1$ , and hence by Theorem 2.9 of [22] it follows that  $L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T] < \infty$ . Together with the fact that  $N^\varepsilon < \infty$   $\mathbb{P}$  a.s., this implies that

$$\sum_{k=1}^{\infty} L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T] < \infty \quad \mathbb{P} \text{ almost surely on } \Omega_n^\varepsilon.$$

On the other hand, since  $Y$  is continuous on  $[0, T]$  and  $p > 1$ , we have

$$\max_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \rightarrow 0 \quad \text{as } \Delta(\pi^n) \rightarrow 0.$$

Combining the above two displays with (5.72), we conclude that for every  $\delta > 0$ , as  $\Delta(\pi^n) \rightarrow 0$ ,

$$\mathbb{P} \left( \max_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^{p-1} \sum_{k=1}^{\infty} L(\tau_k^\varepsilon \wedge T, (\xi_k^\varepsilon + \Delta(\pi^n)) \wedge T] > \frac{\delta}{2} \right) \rightarrow 0.$$

Together with (5.73), this shows that (5.71) holds and completes the proof of the lemma.  $\square$

**Lemma 5.3.** *Suppose  $b$  and  $\sigma$  are uniformly bounded. Then there exists a finite constant  $C < \infty$  such that for each  $\varepsilon > 0$ ,*

$$\begin{aligned}
(5.74) \quad & \lim_{\Delta(\pi^n) \rightarrow 0} \mathbb{P} \left( \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} |Y(t_i) - Y(t_{i-1})|^{p-1} \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) > \frac{\delta}{2} \right) \\
& \leq \begin{cases} \frac{C}{\delta} \mathbb{E} \left[ \int_0^T \sum_{k=0}^{\infty} \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t) dt \right] & \text{if } q = 2, \\ 0 & \text{if } q > 2. \end{cases}
\end{aligned}$$

*Proof.* Fix  $\varepsilon > 0$ . Then by Markov's inequality (whose application is justified by (5.77) and (5.78) below) and the monotone convergence theorem,

$$(5.75) \quad \begin{aligned} & \mathbb{P} \left( \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) > \frac{\delta}{2} \right) \\ & \leq \frac{2}{\delta} \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} \mathbb{E} \left[ |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) \right]. \end{aligned}$$

Let  $a = \sigma^T \sigma$  and let  $\bar{C} > 1$  be an upper bound on  $|b|$ ,  $|\sigma|$  and  $|a|$ . By Assumption 2, the definition (2.4) of  $X$  and the elementary inequality  $|x + y|^q \leq 2^q(|x|^q + |y|^q)$ , there exists  $K_T < \infty$  such that for each  $t_i \in \pi^n$ ,

$$(5.76) \quad \begin{aligned} & \mathbb{E} \left[ |Y(t_i) - Y(t_{i-1})|^p \mid \mathcal{F}_{t_{i-1}} \right] \\ & \leq K_T \mathbb{E} \left[ \sup_{u \in [t_{i-1}, t_i]} |X(u) - X(t_{i-1})|^q \mid \mathcal{F}_{t_{i-1}} \right] \\ & \leq 2^q K_T \mathbb{E} \left[ \sup_{u \in [t_{i-1}, t_i]} \left| \int_{t_{i-1}}^u b(Z(v)) dv \right|^q \right. \\ & \quad \left. + \sup_{u \in [t_{i-1}, t_i]} \left| \int_{t_{i-1}}^u \sigma(Z(v)) dB_v \right|^q \mid \mathcal{F}_{t_{i-1}} \right] \\ & \leq 2^q K_T \mathbb{E} \left[ \bar{C}^q (t_i - t_{i-1})^q + \left( \frac{q}{q-1} \right)^q \left| \int_{t_{i-1}}^{t_i} \sigma(Z(v)) dB_v \right|^q \mid \mathcal{F}_{t_{i-1}} \right] \\ & \leq 2^q K_T \bar{C}^q \Delta(\pi^n)^{q-1} (t_{i-1} - t_i) \\ & \quad + 2^q K_T \left( \frac{q}{q-1} \right)^q \tilde{K} \mathbb{E} \left[ \left( \int_{t_{i-1}}^{t_i} |a(Z(v))| dv \right)^{q/2} \mid \mathcal{F}_{t_{i-1}} \right] \\ & \leq 2^q K_T \bar{C}^q \Delta(\pi^n)^{q-1} (t_{i-1} - t_i) + 2^q K_T \left( \frac{q}{q-1} \right)^q \tilde{K} \bar{C}^{q/2} (t_i - t_{i-1})^{q/2}. \end{aligned}$$

Here, the third inequality holds due to the uniform bound on  $b(\cdot)$ , the Markov property of  $Z$  and Doob's maximal martingale inequality, while the fourth inequality follows, with  $\tilde{K} < \infty$  a universal constant, by an application of the martingale moment inequality, which is justified since the uniform boundedness on  $a$  ensures that the stochastic integral is a martingale.

Define  $\tilde{C} \doteq 2^q K_T [\bar{C}^q \vee (q^q \bar{C}^{q/2} \tilde{K} / (q-1)^q)]$ . We now consider two cases. If  $q > 2$ , it follows from (5.76) that, for all sufficiently large  $n$  such that  $\Delta(\pi)^n < 1$ ,

$$\mathbb{E} \left[ |Y(t_i) - Y(t_{i-1})|^p \mid \mathcal{F}_{t_{i-1}} \right] \leq \tilde{C} \Delta(\pi^n)^{q/2-1} (t_{i-1} - t_i).$$

Multiplying both sides of this inequality by  $\mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1})$ , which is  $\mathcal{F}_{t_{i-1}}$ -measurable since  $\tau_k^\varepsilon$  and  $\xi_k^\varepsilon$  are stopping times, then taking expectations and

subsequently summing over  $k = 0, 1, \dots$ , and  $t_i \in \pi^n$ , it follows that

$$(5.77) \quad \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} \mathbb{E} \left[ |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) \right] \leq \tilde{C} \Delta(\pi^n)^{q/2-1} T.$$

Since  $\Delta(\pi^n)^{q/2-1} \rightarrow 0$  as  $n \rightarrow \infty$ , combining this with (5.75), we then obtain

$$\lim_{\Delta(\pi^n) \rightarrow 0} \mathbb{P} \left( \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) > \frac{\delta}{2} \right) = 0.$$

On the other hand, if  $q = 2$ , again multiplying both sides of (5.76) by  $\mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1})$ , then taking expectations, subsequently summing over  $k = 0, 1, \dots$ , and  $t_i \in \pi^n$ , and then using the monotone convergence theorem, we obtain with  $\tilde{C}$  as above,

$$(5.78) \quad \begin{aligned} & \sum_{t_i \in \pi^n} \sum_{k=0}^{\infty} \mathbb{E} \left[ |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) \right] \\ & \leq \tilde{C} \left( T \Delta(\pi^n)^{q-1} + \mathbb{E} \left[ \sum_{t_i \in \pi^n} (t_i - t_{i-1}) \sum_{k=0}^{\infty} \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) \right] \right) \\ & \leq \tilde{C} T (\Delta(\pi^n)^{q-1} + 1). \end{aligned}$$

Sending  $\Delta(\pi^n) \rightarrow 0$  on both sides of the first inequality in (5.78), and invoking the bounded convergence theorem, the right-continuity of  $\mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(\cdot)$  and the definition of the Riemann integral, we obtain

$$\lim_{\Delta(\pi^n) \rightarrow 0} \sum_{k=0}^{\infty} \mathbb{E} \left[ |Y(t_i) - Y(t_{i-1})|^p \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t_{i-1}) \right] \leq \tilde{C} \mathbb{E} \left[ \int_0^T \sum_{k=0}^{\infty} \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t) dt \right].$$

Together with (5.75), this shows that (5.74) holds with  $C = 2\tilde{C}$ .  $\square$

*Proof of Theorem 3.5.* Due to Lemma 5.1, using a localization argument and the local boundedness of  $b$  and  $\sigma$  stated in Assumption 3, we can assume without loss of generality that  $a, b$  and  $\sigma$  are bounded. Then, combining (5.70) with Lemmas 5.2 and 5.3, we have

$$\begin{aligned} & \lim_{\Delta(\pi^n) \rightarrow 0} \mathbb{P} \left( \sum_{t_i \in \pi^n} |Y(t_i) - Y(t_{i-1})|^p > \delta \right) \\ & \leq \begin{cases} \frac{C}{\delta} \mathbb{E} \left[ \int_0^T \sum_{k=0}^{\infty} \mathbb{I}_{[\xi_k^\varepsilon, \tau_{k+1}^\varepsilon)}(t) dt \right] & \text{if } q = 2, \\ 0 & \text{if } q > 2. \end{cases} \end{aligned}$$

for every  $\varepsilon > 0$ , and so the result follows if  $q > 2$ . If  $q = 2$ , sending  $\varepsilon \downarrow 0$  and using the bounded convergence theorem and the definition of the stopping times  $\xi_k^\varepsilon$  and  $\tau_k^\varepsilon$ , we see that the term on the right-hand side converges to

$$\frac{C}{\delta} \mathbb{E} \left[ \int_0^T \mathbb{I}_{\mathcal{V}}(Z(t)) dt \right] = 0,$$

where the last equality follows from the fact that  $\mathcal{V} \subset G$  and (2.5). This proves (5.66), and Theorem 3.5 then follows from the discussion at the beginning of Section 5.  $\square$

**Acknowledgments.** The authors are grateful to F. Coquet for first posing the question as to whether the generalized processor sharing reflected diffusion is a Dirichlet process. The second author would also like to thank S.R.S. Varadhan and A.S. Sznitman for useful discussions.

#### APPENDIX A. ELEMENTARY PROPERTIES OF THE ESP

**Lemma A.1.** *If  $(\phi, \eta)$  is a solution to the ESP  $(G, d(\cdot))$  for  $\psi \in \mathcal{C}_G [0, \infty)$ , then for each  $0 \leq s < \infty$ ,  $(\phi^s, \eta^s)$  is a solution to the ESP for  $\phi(s) + \psi^s$ , where  $\phi^s(\cdot) \doteq \phi(s + \cdot)$ ,*

$$\psi^s(\cdot) \doteq \psi(s + \cdot) - \psi(s) \quad \text{and} \quad \eta^s(\cdot) \doteq \eta(s + \cdot) - \eta(s).$$

*Moreover, if the ESM is well-defined and Lipschitz continuous on  $\mathcal{C}_G [0, \infty)$  then for every  $T < \infty$ , there exists  $\tilde{K}_T < \infty$  such that for every  $0 \leq s < t \leq T + s$ ,*

$$|\eta(t) - \eta(s)| \leq \tilde{K}_T \sup_{u \in [0, t-s]} |\psi(s+u) - \psi(s)|.$$

*Proof.* Fix  $s \in [0, \infty)$  and a path  $\psi \in \mathcal{D}_G [0, \infty)$ . The first statement follows from Lemma 2.3 of [22]. It implies that  $\eta^s = \bar{\Gamma}(\psi^1) - \psi^1$ , where  $\psi^1 \doteq \phi(s) + \psi^s$ . On the other hand, consider the path  $\psi^2$  which is equal to the constant  $\phi(s)$  on  $[0, \infty)$ , i.e.,  $\psi^2(u) \doteq \phi(s)$  for all  $u \in [0, \infty)$ . Then clearly  $(\phi(s), 0)$  is the unique solution to the ESP for  $\psi^2$ , i.e.,  $0 = \bar{\Gamma}(\psi^2)(u) - \psi^2(u)$  for all  $u \in [0, \infty)$ . Using the Lipschitz continuity of the ESM, for  $\delta \in [0, T-s]$  we obtain

$$\begin{aligned} |\eta^s(\delta) - 0| &\leq \sup_{u \in [0, \delta]} |\bar{\Gamma}(\psi^1)(u) - \psi^1(u) - \bar{\Gamma}(\psi^2)(u) + \psi^2(u)| \\ &\leq \sup_{u \in [0, \delta]} |\bar{\Gamma}(\psi^1)(u) - \bar{\Gamma}(\psi^2)(u)| + \sup_{u \in [0, \delta]} |\psi^1(u) - \psi^2(u)| \\ &\leq K_T \sup_{u \in [0, \delta]} |\psi^s(u)| + \sup_{u \in [0, \delta]} |\psi^s(u)|, \end{aligned}$$

where  $K_T < \infty$  is the Lipschitz constant of  $\bar{\Gamma}$  on  $[0, T]$ . The lemma follows by letting  $\tilde{K}_T \doteq K_T + 1$  and  $\delta = t - s$ .  $\square$

#### APPENDIX B. AUXILIARY RESULTS

For completeness, we provide the proof of the fact that the sequences of times defined in Section 4.1.3 are stopping times.

**Lemma B.1.**  *$\{\beta_n^\varepsilon\}_{n \in \mathbb{N}}$ ,  $\{\beta_{(k),n}^\varepsilon\}_{n \in \mathbb{N}}$ ,  $k \in \mathbb{N}$ , are sequences of  $\{\mathcal{F}_t\}$ -stopping times. Also,  $\{\beta_n^{k,\varepsilon}\}_{n \in \mathbb{N}}$ ,  $k \in \mathbb{N}$ , are sequences of  $\{\mathcal{F}_t^k\}$ -stopping times.*

*Proof.* Clearly,  $\beta_0^\varepsilon \doteq 0$  is an  $\{\mathcal{F}_t\}$ -stopping time. Now, suppose  $\beta_{n-1}^\varepsilon$  is an  $\{\mathcal{F}_t\}$ -stopping time and note that for each  $\varepsilon > 0$ ,  $n \in \mathbb{N}$  and  $t \in [0, \infty)$ ,

$$\{\beta_n^\varepsilon \leq t\} = \bigcup_{k \in \mathbb{Z}} [\{\beta_{n-1}^\varepsilon \leq t\} \cap \{Z(\beta_{n-1}^\varepsilon) \in H_{2^{k\varepsilon}}\} \cap A_{k,n}^\varepsilon(t)],$$

where

$$A_{k,n}^\varepsilon(t) \doteq \left\{ \sup_{s \in [\beta_{n-1}^\varepsilon, t]} \langle Z(s), \vec{\nu} \rangle \geq 2^{k+1}\varepsilon \right\} \cup \left\{ \inf_{s \in [\beta_{n-1}^\varepsilon, t]} \langle Z(s), \vec{\nu} \rangle \leq 2^{k-1}\varepsilon \right\}.$$

Then  $\{\beta_{n-1}^\varepsilon \leq t\} \in \mathcal{F}_t$  because  $\beta_{n-1}^\varepsilon$  is an  $\{\mathcal{F}_t\}$ -stopping time. Since  $Z$  is continuous we also know that  $\{\beta_{n-1}^\varepsilon \leq t\} \cap \{Z(\beta_{n-1}^\varepsilon) \in H_{2^{k\varepsilon}}\}$  lies in  $\mathcal{F}_t$ . In addition, the continuity of  $\langle Z, \vec{\nu} \rangle$  and the fact that  $[2^{k+1}\varepsilon, \infty)$  and  $(-\infty, 2^{k-1}\varepsilon]$  are closed show that  $\{\beta_{n-1}^\varepsilon \leq t\} \cap A_{k,n}^\varepsilon(t) \in \mathcal{F}_t$ . When combined, this implies that  $\{\beta_n^\varepsilon \leq t\} \in \mathcal{F}_t$  or, equivalently, that  $\beta_n^\varepsilon$  is an  $\{\mathcal{F}_t\}$ -stopping time, and the first assertion follows by induction. The proof for the other sequences is exactly analogous.  $\square$

## REFERENCES

- [1] J. Bertoin. Les processus de Dirichlet et tant qu'espace de banach. *Stochastics* (1986) **18** 2:155–168.
- [2] J. Bertoin. Sur une integrale pour les processus a  $\alpha$ -variation bornée. *Ann. Probab.* (1989) **17** 4:1521–1535.
- [3] K. Burdzy, W. N. Kang and K. Ramanan. The Skorokhod problem in a time-dependent interval, to appear in *Stoch. Proc. Appl.* (2008).
- [4] K. Burdzy and E. Toby, A Skorokhod-type lemma and a decomposition of reflected Brownian motion, *Ann. Probab.*, **23** (1995) 586–604.
- [5] Z.-Q. Chen, On reflecting diffusion processes and Skorokhod decompositions. *Probab. Theor. Rel. Fields*, 1993, **94** 281–315.
- [6] Z. Q. Chen, P. J. Fitzsimmons, K. Kuwae and T. S. Zhang. Stochastic calculus for Dirichlet processes. *Ann. Probab.* **36** (2008) 931–970.
- [7] F. Coquet, A. Jakubowski, J. Mémin and L. Slomiński. Natural decomposition of processes and weak Dirichlet processes. in *Séminaire de Probabilités XXXIX*, Lecture Notes in Math, Springer Berlin-Heidelberg, **1874** (2006) 81–116.
- [8] C. Dellacherie and P.A. Meyer. *Probabilités et Potentiel*. Hermann, Paris (1980).
- [9] P. Dupuis and K. Ramanan. A Skorokhod problem formulation and large deviation analysis of the generalized processor sharing model. *Queueing Systems*, **28** (1998) 109–124.
- [10] P. Dupuis and K. Ramanan. Convex duality and the Skorokhod problem – Part I. *Probab. Theor. Rel. Fields*, **115** (1999) 2:153–195.
- [11] P. Dupuis and K. Ramanan. Convex duality and the Skorokhod problem – Part II. *Probab. Theor. Rel. Fields*, **115** (1999) 2:197–236.
- [12] P. Dupuis and K. Ramanan. A multiclass feedback queueing network with a regular Skorokhod problem. *Queueing Systems*, **36** (2000) 327–349.
- [13] S. Ethier and T. Kurtz. *Markov Processes: Characterization and Convergence*. Wiley, 1986.
- [14] H. Föllmer. Calcul d'Ito sans Probabilités, in *Séminaire de Probabilités XV*, Lecture Notes in Maths **850**, Springer-Verlag (1981) 143–150.
- [15] H. Föllmer. Dirichlet Processes, in “*Stochastic Integrals*”, *Proc. LMS Durham Symp.*, Lecture Notes in Maths **851**, Springer-Verlag (1981), 476–478.

- [16] H. Föllmer, P. Protter and A. N. Shiryaev. Quadratic covariation and an extension of Itô's formula. *Bernoulli*, **1** (1995) (1-2):149–169.
- [17] M. Freidlin. *Functional Integration and Partial Differential Equations*. Annals of Mathematics Studies Vol. 109, Princeton University Press, Princeton, 1985.
- [18] M. Fukushima, Y. Oshima and T. Takeda. *Dirichlet Forms and Symmetric Markov Processes*. DeGruyter Studies in Mathematics, **19**, Walter de Gruyter & Co., Berlin, 1994.
- [19] W. N. Kang and K. Ramanan. Stationary distributions of reflected diffusions in polyhedral domains. *Preprint*
- [20] I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*. Springer-Verlag, New York, 1988.
- [21] L. Kruk, J. Lehoczky, K. Ramanan, and S. Shreve, An explicit formula for the Skorokhod map on  $[0, a]$ , *Ann. Probab.*, **35** (2007) 5:1740–1768.
- [22] K. Ramanan. Reflected diffusions defined via the extended Skorokhod map, *Elec. Jour. Probab.*, **11** (2006) 934–992.
- [23] K. Ramanan and M. Reiman. Fluid and heavy traffic diffusion limits for a generalized processor sharing model, *Ann. Appl. Probab.*, **13** (2003) 100–139.
- [24] K. Ramanan and M. Reiman. The heavy traffic limit of an unbalanced generalized processor sharing model, *Ann. Appl. Probab.*, **18** (2008) 1:22–58.
- [25] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Third Edition, Springer, Berlin, 1999.
- [26] A. Rozkosz. On a decomposition of symmetric diffusions with reflecting boundary conditions. *Stoch. Proc. Appl.* (2003) **103** 101–122.
- [27] A. Rozkosz and L. Slominski. Diffusion processes corresponding to uniformly elliptic divergence form operators with reflecting boundary conditions. *Studia Math.* (2000) **129** 141–174.
- [28] F. Russo and P. Vallois. Elements of stochastic calculus via regularization, in *Séminaire de Probabilités XL*, Lecture Notes in Math, **1899**, Springer (2007), 147–186.
- [29] A.V. Skorokhod. Stochastic equations for diffusions in a bounded region, *Theor. of Prob. and Appl.*, **6** (1961) 264–274.
- [30] D. W. Stroock and S. R. S. Varadhan. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.*, (1971) 24:147–225.
- [31] R. J. Williams. Reflected Brownian motion in a wedge: semimartingale property, (1985) *Probab. Theor. Rel. Fields*, 69:161–176.
- [32] R. J. Williams. Semimartingale reflecting Brownian motions in an orthant, in *Stochastic Networks*, IMA Volumes in Mathematics and its Applications, F.P. Kelly and R.J. Williams, editors, (1995) 125–137.

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA

*E-mail address:* `weikang@andrew.cmu.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213, USA

*E-mail address:* `kramanan@math.cmu.edu`