

# MINKOWSKI CONTENT AND NATURAL PARAMETRIZATION FOR THE SCHRAMM-LOEWNER EVOLUTION

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We prove the existence and nontriviality of the  $d$ -dimensional Minkowski content for the Schramm Loewner evolution ( $SLE_\kappa$ ) with  $\kappa < 8$  and  $d = 1 + \frac{\kappa}{8}$ . We show that this is a multiple of the natural parametrization.

**1. Introduction.** A number of measures on paths or clusters on two-dimensional lattices arising from critical statistical mechanical models are believed to exhibit some kind of conformal invariance in the scaling limit. Schramm introduced a one-parameter family of such processes, now called the (*chordal*) *Schramm-Loewner evolution with parameter  $\kappa$*  ( $SLE_\kappa$ ) and showed that these give the only possible limits for conformally invariant processes in simply connected domains satisfying a certain “domain Markov property”. He defined the process as a probability measure on curves from 0 to  $\infty$  in  $\mathbb{H}$  and then used conformal invariance to define the process in other simply connected domains.

The definition of the process in  $\mathbb{H}$  uses the half-plane Loewner equation. Suppose  $\gamma : (0, t] \rightarrow \overline{\mathbb{H}}$  is a curve with  $\gamma(0) = 0$ , and let  $\gamma_t = \gamma(0, t]$ . Let  $H_t$  denote the unbounded component of  $\mathbb{H} \setminus \gamma_t$ . We assume that  $\gamma$  is non-crossing in the sense that for all  $s < t$ ,  $\gamma[s, \infty) \subset \overline{H_s}$ , and  $\gamma[s, t] \cap H_s$  is nonempty. Let  $g_t : H_t \rightarrow \mathbb{H}$  be the unique conformal transformation with  $g_t(z) - z = o(1)$  as  $z \rightarrow \infty$ . Then for every  $a > 0$ , there exists a reparametrization of the curve such that the following holds.

- For  $z \in \mathbb{H}$ , the map  $t \mapsto g_t(z)$  is a smooth flow and satisfies the Loewner differential equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where  $U_t$  is a continuous function on  $\mathbb{R}$ . This equation is valid up to a time  $T_z \in (0, \infty]$ .

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Under the reparametrization, the transformation  $g_t$  satisfies

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \rightarrow \infty.$$

We say that the curve is *parametrized by (half-plane) capacity*. Schramm defined chordal  $SLE_\kappa$  to be the solution to the Loewner equation with  $a = 2$  and  $U_t$  a Brownian motion with variance parameter  $\kappa$ . An equivalent definition (up to a linear time change) is to choose  $U_t$  to be a standard Brownian motion and  $a = 2/\kappa$ . It has been shown that a number of discrete random models have  $SLE$  as the scaling limit provided that the discrete models are parameterized using (half-plane) capacity. Examples are loop-erased random walk for  $\kappa = 2$  [9], Ising interfaces for  $\kappa = 3$  [18], harmonic explorer for  $\kappa = 4$  [16], percolation interfaces on the triangular lattice for  $\kappa = 6$  [17], and uniform spanning trees for  $\kappa = 8$  [9].

If  $D$  is a simply connected domain with distinct boundary points  $w_1, w_2$  where  $\partial D$  is nice in neighborhoods of  $w_1, w_2$ , then chordal  $SLE_\kappa$  from  $w_1$  to  $w_2$  in  $D$  is defined by taking the conformal image of  $SLE_\kappa$  in the upper half plane under a transformation  $F : \mathbb{H} \rightarrow D$  with  $F(0) = w_1, F(\infty) = w_2$ . The map  $F$  is not unique, but scale invariance of  $SLE$  in  $\mathbb{H}$  shows that the distribution on paths is independent of the choice. This can be considered as a measure on the curves  $F \circ \gamma$  with the induced parametrization or as a measure on curves modulo reparametrization.

While the capacity parametrization is useful for analyzing the curve, it is not the scaling limit of the “natural” parametrization of the discrete models. For example, for loop-erased walks, it is natural to parameterize by the length of the random walk. One can ask whether the curves parameterized by a normalized version of this “natural length” converge to  $SLE$  with a different parametrization. The Hausdorff dimension of the  $SLE$  paths [2] is  $d = 1 + \min\{\frac{\kappa}{8}, 1\}$ . It was conjectured in [8] that the “natural length” of an  $SLE$  path might be given by the  $d$ -dimensional Minkowski content defined as follows. Let

$$\text{Cont}_d(\gamma_t; r) = e^{r(2-d)} \text{Area} \{z : \text{dist}(z, \gamma_t) \leq e^{-r}\}.$$

Then the  $d$ -dimensional content is

$$\text{Cont}_d(\gamma_t) = \lim_{r \rightarrow \infty} \text{Cont}_d(\gamma_t; r),$$

provided that the limit exists. If  $\kappa \geq 8$ , then  $d = 2$ , and the 2-dimensional Minkowski content is the same as the area and the limit clearly exists. If  $\kappa < 8$ , it is not at all obvious that the limit exists and is positive for  $t > 0$ . The main goal of this paper is to prove this.

Before stating the theorem, we will set some notational conventions for this paper. Let  $0 < \kappa < 8$  and let

$$a = \frac{2}{\kappa} \in (1/4, \infty), \quad d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a} \in (1, 2).$$

Recall that  $\gamma_t = \gamma(0, t]$  and we write  $\gamma = \gamma_\infty = \gamma(0, \infty)$  for the entire path of the curve. The Green's function for  $\kappa < 8$  is defined by

$$G(z) = \lim_{r \rightarrow \infty} e^{r(2-d)} \mathbb{P}\{\text{dist}(z, \gamma) \leq e^{-r}\}.$$

This limit exists (see Section 2.3) and there exists  $c = c_\kappa$  such that

$$G(z) = c [\Im z]^{d-2} [\sin \arg z]^{4a-1}.$$

Our definition of the Green's function differs by a multiplicative constant from that in other papers. If  $F : D \rightarrow \mathbb{H}$  is a conformal transformation with  $F(w_1) = 0, F(w_2) = \infty$ , we define

$$G_D(z; w_1, w_2) = |F'(z)|^{2-d} G(F(z)).$$

There is also a two-point Green's function (see Section 2.4)

$$G(z, w) = \lim_{r \rightarrow \infty} e^{2r(2-d)} \mathbb{P}\{\text{dist}(z, \gamma) \leq e^{-r}, \text{dist}(w, \gamma) \leq e^{-r}\}.$$

If  $D \subset \mathbb{H}$ , let

$$G(D) = \int_D G(z) dA(z), \quad G^2(D) = \int_D \int_D G(z, w) dA(z) dA(w),$$

where  $dA$  denotes integration with respect to area. We call  $\gamma(t)$  a *double point* for the  $SLE_\kappa$  path if there exists  $s < t$  such that  $\gamma(t) \in \partial H_s$ . If  $0 < \kappa \leq 4$ , the  $SLE$  path has no double points while they exist for  $4 < \kappa < 8$ .

**THEOREM 1.1.** *If  $0 < \kappa < 8$  there exists  $\beta > 0$ , such that if  $\gamma(t)$  is an  $SLE_\kappa$  curve from 0 to  $\infty$  in  $\mathbb{H}$  parametrized by capacity, then with probability one, the following holds.*

- For every  $t > 0$ , the Minkowski content

$$\Theta_t = \text{Cont}_d(\gamma_t) = \lim_{r \rightarrow \infty} \text{Cont}_d(\gamma_t; r),$$

*exists.*

- The function  $t \mapsto \Theta_t$  is strictly increasing and if  $s < t$ ,

$$\Theta_t - \Theta_s = \text{Cont}_d(\gamma[s, t]) = \text{Cont}_d(\gamma(s, t] \cap H_s).$$

- On every bounded interval  $[0, t_0]$ ,  $\Theta_t$  is Hölder continuous of order  $\beta$ .

Moreover, if  $D \subset \mathbb{H}$  is a bounded domain with piecewise smooth boundary, then

$$\begin{aligned}\mathbb{E}[\text{Cont}_d(\gamma \cap D)] &= G(D), \\ \mathbb{E}[\text{Cont}_d(\gamma \cap D)^2] &= G^2(D),\end{aligned}$$

and if  $t > 0$ ,

$$\mathbb{E}[\text{Cont}_d(\gamma \cap D) \mid \gamma_t] = \text{Cont}_d(\gamma_t \cap D) + \int_D G_{H_t}(z; \gamma(t), \infty) dA(z).$$

The proof will show that we can choose any  $\beta < \alpha_0 d/2$  where

$$\beta < \frac{d}{2} \min \left\{ 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8 + \kappa}}, \frac{1}{2} \right\} > 0.$$

The theorem allows us to define  $SLE_\kappa$  with the *natural parametrization* by letting

$$\tilde{\gamma}(t) = \gamma(\sigma_t), \quad \sigma_t = \inf\{s : \Theta_s = t\}.$$

Under this parametrization with probability one for all  $t$ ,

$$\text{Cont}_d(\tilde{\gamma}_t) = t.$$

If  $F : \mathbb{H} \rightarrow D$  with  $F(0) = w_1, F(\infty) = w_2$  is a conformal transformation, then as in [7] the natural parametrization in  $D$  can be defined by saying that the time to traverse  $F(\tilde{\gamma}[s, t])$  is

$$(1) \quad \int_s^t |F'(\tilde{\gamma}(r))|^d dr.$$

If  $\tilde{\gamma}[s, t] \subset \mathbb{H}$ , we can see that this is the same as  $\text{Cont}_d[F \circ \tilde{\gamma}[s, t]]$ . We expect this to be true for all nice  $D$ . The only question is the intersection of the curve with the boundary for  $4 < \kappa < 8$  with  $D$  having a nonsmooth boundary, perhaps of large dimension.

As an example, let  $D$  be the unit disk  $\mathbb{D}$  and let  $w_1 = 1, w_2 = -1$ . In this case the map  $F : \mathbb{H} \rightarrow \mathbb{D}$  extends analytically to  $\mathbb{R}$  and there is no problem establishing that (1) equals  $\text{Cont}_d[F \circ \tilde{\gamma}[s, t]]$ . Let  $\gamma(t)$  be the  $SLE_\kappa$  path in  $\mathbb{H}$  with the capacity parametrization, and let  $\eta(t) = F(\tilde{\gamma}(t))$  which is an  $SLE_\kappa$  curve from 1 to  $-1$  in  $\mathbb{D}$ . Let  $\Theta_t = \text{Cont}_d[\eta_t]$ . In this case  $\Theta_\infty$  is an integrable random variable with

$$\mathbb{E}[\Theta_\infty] = \int_{\mathbb{D}} G_{\mathbb{D}}(z; 1, -1) dA(z) < \infty.$$

Moreover,

$$\mathbb{E}[\Theta_\infty \mid \eta_t] = \Theta_t + \Psi_t,$$

where

$$\Psi_t = \int_{D_t} G_{D_t}(z; \eta(t), -1) dA(z).$$

Since  $M_t := \mathbb{E}[\Theta_\infty \mid \eta_t]$  is a martingale, we can see that  $\Theta_t$  is the unique increasing process such that  $\Psi_t + \Theta_t$  is a martingale. This is a Doob-Meyer decomposition.

In [8], the natural parametrization was *defined* to be the unique process  $\Theta_t$  which makes  $\Psi_t + \Theta_t$  a martingale. While this is a simple definition, it requires moment bounds in order to make sure that the process exists (uniqueness is easy). Indeed, it is not hard to see that  $M_t(z) := G_{D_t}(z; \eta(t), -1)$  is a local martingale, and hence  $\Psi_t$  is an integral of positive local martingales. If  $\Psi_t$  were also a local martingale, then no nontrivial  $\Theta_t$  could exist.

- In [8], it was shown that for  $\kappa < 5.0\dots$ , the process  $\Theta_t$  exists in  $\mathbb{H}$  (the definition has to be modified slightly in  $\mathbb{H}$  because  $\Psi_0$  as we have defined it above is infinite— this is not very difficult). The necessary second moment bounds were obtained using the reverse Loewner flow. It was shown that for this range of  $\kappa$ , there exists  $\alpha_0 = \alpha_0(\kappa) > 0$  such that the function  $t \mapsto \Theta_t$  is Hölder continuous of order  $\alpha$  for  $\alpha < \alpha_0$ .
- In [11], the natural parametrization was shown to exist for all  $\kappa < 8$ . There the necessary two-point estimates were obtained from estimates on the two-point Green's function [2, 10]. However, the estimates were not strong enough to determine Hölder continuity of the function  $\Theta_t$ .
- In [7] a new proof was given for all  $\kappa < 8$  combining ideas in [8, 11] with known results about the Hölder continuity of the Schramm-Loewner evolution (with respect to the capacity parametrization). This established continuity and Hölder continuity of the natural parametrization for all  $\kappa$ .

Let us discuss some conclusions that we can derive. If  $\Theta_t = \text{Cont}_d(\gamma_t)$ , then clearly  $\Theta_t$  is increasing and measurable with respect to  $\gamma_t$ . The conditional distribution of  $\text{Cont}_d[\gamma(t, \infty)]$  given  $\gamma_t$  is the same as the distribution of the Minkowski content for *SLE* from  $\gamma(t)$  to  $-1$  in  $D_t$ . In particular, using the fact that  $\Theta_t - \Theta_s = \text{Cont}_d(\gamma(s, t] \cap D_s)$ , we have

$$\mathbb{E}[\Theta_\infty - \Theta_t \mid \gamma_t] = \int_{D_t} \hat{G}_t(z) dA(z),$$

where  $\hat{G}_t(z) = G_{D_t}(z; \gamma(t), -1)$ . Therefore,

$$\Theta_t + \int_{D_t} \hat{G}_t(z) dA(z)$$

is a martingale. Uniqueness of the Doob-Meyer decomposition shows that our  $\Theta_t$  must be the same as the natural parametrization as discussed in [8, 11, 7]. Using the Minkowski content as the definition, we immediately get independence of domain as well as reversibility of the natural parametrization, that is, the time to traverse  $\gamma[s, t]$  is the same as the time to traverse the path in the reverse direction. By independence of domain we mean that if  $\gamma$  is an  $SLE_\kappa$  curve in  $\mathbb{H}$  and  $D \subset \mathbb{H}$  with  $\gamma(0, \infty) \subset D$ , then the natural parametrization for  $\gamma$  considered as an  $SLE$  curve in  $D$  is the same as that for the  $SLE$  curve in  $\mathbb{H}$ . While this is clearly a property that we would expect from a “natural” parametrization, it is not at all obvious using the definition in [8].

Another possible candidate for the “natural length” of an  $SLE$  curve might be the  $d$ -dimensional Hausdorff measure. However, it has been proved [13] that this is zero with probability one. It is unknown whether one can find a Hausdorff measure with a different gauge function which would give a nontrivial quantity.

1.1. *Outline of the paper.* Section 2 sets notation for the paper and reviews previous work. We define the Minkowski content in Section 2.2 and derive some simple properties. The Green’s function for chordal  $SLE_\kappa$  is reviewed in Section 2.3. This is a normalized limit of the probability of getting near a point  $z$ . We also discuss estimates in [10] concerning the probability that an  $SLE_\kappa$  path gets close to two points. In the following subsection we discuss some of the ideas used to prove two-point estimates; in particular, some precise formulations are made of the rough statement “after an SLE curve gets close to  $z$  it is unlikely to get close again”. This section uses ideas from [6, 10].

The proof of the main result is in the remainder of the paper. Before going into specifics, let us outline the basic idea of the proof. For ease let us fix a square, say  $\Gamma = [0, 1] + i[1, 2]$  and consider  $\gamma \cap \Gamma$ . For each  $z \in \Gamma$  and  $r > 0$ , let  $\tau_r(z) = \inf\{t : |\gamma(t) - z| \leq e^{-r}\}$  and let  $J_r(z)$  be  $e^{r(2-d)}$  times the indicator function of the event  $\{\tau_r(z) < \infty\}$ . Let  $T_r(z)$  be the first time that the conformal radius of  $z$  in  $\mathbb{H} \setminus \gamma(0, t]$  equals  $e^{-r+2}$ . The Koebe 1/4-theorem implies that  $T_r(z) < \tau_r(z)$ . By comparison with “two-sided radial” ( $SLE$  conditioned to go through  $z$ ), one can show that there exists  $c_1$  such that  $\mathbb{P}\{T_r(z) < \infty\} \sim c_1 G(z) e^{r(d-2)}$ . If  $r$  is large, and we view the path

$\gamma[0, T_r(z)]$  near  $z$ , then locally it appears like a path with the distribution of two-sided radial *SLE*. Using this, one can see that

$$\mathbb{P}\{J_r(z) > 0 \mid T_r(z) < \infty\} = \rho + o(1), \quad r \rightarrow \infty,$$

where  $\rho$  is independent of  $z$ , and using this in turn, we get a one-point estimate

$$(2) \quad \mathbb{E}[J_r(z)] = c_1 \rho G(z) + o(1),$$

If we fix  $\delta > 0$ , we can see similarly that there exists  $\rho'$  such that

$$\mathbb{P}\{J_{r+\delta}(z) > 0 \mid T_r(z) < \infty\} = \rho' + o(1), \quad r \rightarrow \infty,$$

and by using (2), we see that  $\rho' = e^{\delta(d-2)} \rho$ . In other words,  $\mathbb{E}[J_{r+\delta}(z) - J_r(z)] = o(1)$ . The conditional distribution of  $J_{r+\delta}(z) - J_r(z)$  given  $\gamma[0, T_r(z)]$  is determined (up to a small error) by the way the curve  $\gamma$  looks near  $\gamma(T_r(z))$ , and this latter distribution is understood through two-sided radial *SLE* $_{\kappa}$ . If  $z, w$  are not very close together and the *SLE* curve gets close to both  $z$  and  $w$ , we might hope (and, indeed, this is what we show) that the local behavior of  $\gamma$  near  $\gamma(T_r(z))$  and near  $\gamma(T_r(w))$  are almost independent. The upshot of this is that if we consider the random variable

$$Y_r = \int_{\Gamma} [J_{r+\delta}(z) - J_r(z)] dA(z),$$

then  $\mathbb{E}[Y_r^2]$  is small. We show that  $\mathbb{E}[Y_r^2] \leq c e^{-\beta r}$ , from which we conclude that

$$\lim_{r \rightarrow \infty} \int_{\Gamma} J_r(z) dA(z)$$

exists as a limit in  $L^2$  and with probability one.

This outline is carried out in Section 3.1 assuming a moment bound, Theorem 3.2 which is proved later. This establishes that with probability one  $\text{Cont}_d[\gamma \cap \Gamma]$  exists for every dyadic square  $\Gamma$ . Section 3.2 uses this to prove the statements in Theorem 1.1, again leaving one fact for the last section.

The main estimates are proved in the final section. Section 4.2 analyzes the one-point estimate, that is, the estimate for getting close to a single point  $z$ . See Theorem 4.2. A key to the two-point estimate is to understand the one-point estimate very well. For ease we consider *SLE* in the disk between boundary points and choose the origin to be the target point. Two-sided radial, which is an example of what are sometimes called *SLE* $(\kappa, \rho)$

processes, describes chordal *SLE* “conditioned to go through  $z$ ”. It can be analyzed by a one-dimensional SDE. We use this to study *SLE* conditioned to get near  $z$ . To do the two-point estimate, we start in Section 2.4 by reviewing the basic idea that after one gets close to a point, one tends not to return to it. This statement requires care to make precise. See Lemmas 2.5 and 2.6. In the final section we complete the proof giving a rigorous version of the rough outline above.

## 2. Preliminaries.

2.1. *Notations and distortion.* We fix  $\kappa < 8$  and allow all constants to depend implicitly on  $\kappa$ . Recall that  $a = 2/\kappa$  and  $d = 1 + \frac{\kappa}{8}$ . If  $\gamma$  is an *SLE* $_{\kappa}$  curve from  $w_1$  to  $w_2$  in a simply connected domain  $D$ , we write  $\gamma_t = \gamma(0, t] = \{\gamma(s) : 0 < s \leq t\}$ .

If  $n, j, k$  are integers, we write  $\Gamma_n(j, k)$  for the dyadic square

$$\Gamma_n(j, k) = [j2^{-n}, (j+1)2^{-n}) \times i[k2^{-n}, (k+1)2^{-n}).$$

Let

$$\mathcal{Q}_n = \{\Gamma_n(j, k) : j \in \mathbb{Z}, k \geq 0\}, \quad \mathcal{Q}_n^+ = \{\Gamma_n(j, k) \in \mathcal{Q}_n : k > 0\},$$

$$\mathcal{Q} = \bigcup_{n \in \mathbb{Z}} \mathcal{Q}_n, \quad \mathcal{Q}^+ = \bigcup_{n \in \mathbb{Z}} \mathcal{Q}_n^+.$$

We will need the following simple distortion estimate.

LEMMA 2.1. *There exists  $\delta > 0$  such that if  $f : \mathbb{D} \rightarrow f(\mathbb{D})$  is a conformal transformation with  $f(0) = 0$ ,  $|f'(0)| = \lambda$  and  $|z| \leq \delta$*

$$|z| \exp\{-4|z|\} \leq |f^{-1}(\lambda z)| \leq |z| \exp\{4|z|\}.$$

PROOF. By scaling we may assume that  $f'(0) = 1$ . The growth theorem (see, e.g., [4, Theorem 3.23]), states that for all  $|z| < 1$ ,

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Since  $(1 \pm |z|)^{-2} = 1 \mp 2|z| + O(|z|^2)$  and  $\exp\{\pm 4|z|\} = 1 \pm 4|z| + O(|z|^2)$ , we get the lemma.  $\square$



2.2. *Minkowski content.* The  $d$ -dimensional Minkowski content is one way to “measure” the size of a  $d$ -dimensional fractal. We use the quotes because the content is not technically a measure. Its definition is in some ways more natural than  $d$ -dimensional Hausdorff measure; however, it has the disadvantage that it is defined in terms of a limit that does not always exist. We will restrict our consideration to  $1 < d < 2$  and  $V \subset \mathbb{C}$ .

Let

$$\begin{aligned} \text{Cont}_d(V; r) &= e^{r(2-d)} \text{Area}\{z : \text{dist}(z, V) \leq e^{-r}\} \\ &= e^{r(2-d)} \int_{\mathbb{C}} 1\{\text{dist}(z, V) \leq e^{-r}\} dA(z). \end{aligned}$$

Here, and throughout this paper,  $dA$  denotes integration with respect to two-dimensional Lebesgue measure. The *upper and lower  $d$ -dimensional Minkowski contents* are defined by

$$\begin{aligned} \text{Cont}_d^+(V; r) &= \sup_{s \geq r} \text{Cont}_d(V; s), & \text{Cont}_d^+(V) &= \lim_{r \rightarrow \infty} \text{Cont}_d^+(V; r). \\ \text{Cont}_d^-(V; r) &= \inf_{s \geq r} \text{Cont}_d(V; s), & \text{Cont}_d^-(V) &= \lim_{r \rightarrow \infty} \text{Cont}_d^-(V; r). \end{aligned}$$

The  *$d$ -dimensional Minkowski content* is defined if  $\text{Cont}_d^+(V) = \text{Cont}_d^-(V)$  in which case

$$\text{Cont}_d(V) = \lim_{r \rightarrow \infty} \text{Cont}_d(V; r).$$

The following simple lemma lists the basic properties of Minkowski content that we will use.

LEMMA 2.2.

- If  $\text{Cont}_d(V)$ ,  $\text{Cont}_d(V')$  exist and  $\text{dist}(V, V') > 0$ , then  $\text{Cont}_d(V \cup V')$  exists and

$$\text{Cont}_d(V \cup V') = \text{Cont}_d(V) + \text{Cont}_d(V').$$

- If  $\text{Cont}_d(V)$  exists, then

$$\text{Cont}_d(V) \leq \text{Cont}_d^+(V \cup V') \leq \text{Cont}_d(V) + \text{Cont}_d^+(V').$$

- If  $d > 1$  and  $D$  is a bounded domain whose boundary is a piecewise analytic curve, then  $\text{Cont}_d(D) = 0$ . If  $V \subset \overline{D}$ , then

$$(3) \quad \text{Cont}_d(V) = \lim_{r \rightarrow \infty} e^{r(2-d)} \int_D 1\{\text{dist}(z, V) \leq e^{-r}\} dA(z),$$

provided that either side exists.

- Suppose  $V_1, V_2, \dots$  are bounded sets for which  $\text{Cont}_d(V_n)$  is well defined. Let  $V$  be a bounded set such that

$$\lim_{n \rightarrow \infty} [\text{Cont}_d^+(V \setminus V_n) + \text{Cont}_d^+(V_n \setminus V)] = 0.$$

Then  $\text{Cont}_d(V)$  exists and

$$(4) \quad \text{Cont}_d(V) = \lim_{n \rightarrow \infty} \text{Cont}_d(V_n).$$

PROOF. We leave this to the reader. The last conclusion uses

$$\begin{aligned} \text{Cont}_d(V_n; r) - \text{Cont}_d(V_n \setminus V; r) &\leq \text{Cont}_d(V; r) \\ &\leq \text{Cont}_d(V_n; r) + \text{Cont}_d(V \setminus V_n; r). \end{aligned}$$

□

2.3. *Green's function.* The Green's function for chordal  $SLE_\kappa$  is the normalized probability that the path gets near a point  $z$ . By nature it is defined up to a multiplicative constant and we choose the constant in a way that will be convenient for us. The precise definition uses the following theorem. If  $D$  is a simply connected domain and  $z \in D$  we let  $\text{crad}_D(z)$  denote the conformal radius of  $z$  in  $D$ , that is, if  $f : \mathbb{D} \rightarrow D$  is a conformal transformation with  $f(0) = z$ , then  $\text{crad}_D(z) = |f'(0)|$ .

THEOREM 2.3. *For every  $\kappa < 8$ , there exists  $c' = c'(\kappa), \hat{c} = \hat{c}(\kappa), \alpha < \infty$  such that if  $w = e^{2i\theta} \in \partial\mathbb{D}$  and  $\gamma$  is a chordal  $SLE_\kappa$  path from 1 to  $w$  in  $\mathbb{D}$ , then*

$$(5) \quad \begin{aligned} \mathbb{P}\{\text{crad}_A(0) \leq e^{-r}\} &= c' [\sin \theta]^{4a-1} e^{r(d-2)} [1 + O(e^{-\alpha r})]. \\ \mathbb{P}\{\text{dist}(0, \partial A) \leq e^{-r}\} &= \hat{c} [\sin \theta]^{4a-1} e^{r(d-2)} [1 + O(e^{-\alpha r})]. \end{aligned}$$

Here  $A$  denotes the connected component of  $\mathbb{D} \setminus \gamma$  containing the origin.

PROOF. For the first expression see, for example, [10]. The proof gives an explicit form for  $c'$  but we will not need it. The second was proved in [7], but we reprove it here in Theorem 4.2. This proof does not give an explicit expression for the constant  $\hat{c}$ . □

To be precise, let  $\mathbb{P}_\theta$  denote the probability distribution on paths  $\gamma = \gamma[0, \infty)$  given by chordal  $SLE_\kappa$  from 1 to  $e^{2i\theta}$  in  $\mathbb{D}$ . Then there exist  $c', \hat{c}, \alpha, c$ , depending only on  $\kappa$ , such that for all  $\theta$  and all  $r \geq 1/2$ ,

$$\left| e^{r(2-d)} [\sin \theta]^{1-4a} \mathbb{P}_\theta\{\text{crad}_A(0) \leq e^{-r}\} - c' \right| \leq c e^{-\alpha r},$$

$$\left| e^{r(2-d)} [\sin \theta]^{1-4a} \mathbb{P}_\theta \{ \text{dist}(0, \gamma) \leq e^{-r} \} - \hat{c} \right| \leq c e^{-\alpha r}.$$

From the previously proven (5) and the Koebe (1/4)-theorem, we can easily deduce the following estimate which we will use before deriving Theorem 4.2.

- If  $\gamma$  is an  $SLE_\kappa$  path from 0 to  $\infty$  in  $\mathbb{H}$ ,  $\Im(z) \geq 1$  and  $r \geq 0$ ,

$$(6) \quad \mathbb{P}\{\text{dist}(z, \gamma) \leq e^{-r}\} \asymp [\Im(z)]^{d-2} [\sin \arg z]^{4a-1} e^{r(d-2)}.$$

We also use the following estimate, see [1].

- If  $x > 0$ ,

$$(7) \quad \mathbb{P}\{\text{dist}(x, \gamma) \leq e^{-r}\} \leq c [e^{-r}/x]^{4a-1}.$$

If  $w_1, w_2$  are distinct boundary points of a simply connected domain  $D$ , let  $S_D(z; w_1, w_2)$  denote the sine of the argument of  $z$  with respect to  $w_1, w_2$ , that is, if  $F : D \rightarrow \mathbb{H}$  is a conformal transformation with  $F(w_1) = 0, F(w_2) = \infty$ , then  $S_D(z; w_1, w_2) = \sin[\arg F(z)]$ . Note that  $S_D(z; w_1, w_2)$  is a conformal invariant and  $\text{crad}_D(z)$  is conformally covariant,  $\text{crad}_{f(D)}(f(z)) = |f'(z)| \text{crad}_D(z)$ . The *chordal Green's function* is defined by

$$(8) \quad G_D(z; w_1, w_2) = \hat{c} \text{crad}_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1}.$$

Here we choose the constant  $\hat{c}$  from Theorem 3.1; our definition differs from the definition elsewhere (for example in [10]) by a multiplicative constant. Previously it was defined so that  $G_{\mathbb{H}}(z; 0, \infty) = \Im(z)^{d-2} [\sin \arg z]^{4a-1} = [\text{crad}_{\mathbb{H}}(z)/2]^{d-2} S_{\mathbb{H}}(z; 0, \infty)^{4a-1}$ . The Green's function satisfies the conformal covariance rule

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)).$$

We choose the definition (8) so that we do not need to keep writing the constant  $\hat{c}$ . Theorem 2.3 extends immediately to other simply connected domains by conformal invariance of  $SLE$ .

**THEOREM 2.4.** *If  $\kappa < 8$ ,  $\gamma$  is a chordal  $SLE_\kappa$  path from  $w_1$  to  $w_2$  in a simply connected domain  $D$ , then for  $z \in D$  with  $\text{dist}(z, \partial D) \geq 2e^{-r}$ ,*

$$\mathbb{P}\{\text{dist}(z, \gamma) \leq e^{-r}\} = G_D(z; w_1, w_2) e^{r(d-2)} [1 + O(e^{-\alpha r})],$$

for some  $\alpha > 0$  which depends only on  $\kappa$ .

Most of our computations will be in the upper half plane or in the disk. For notational ease we will write

$$G(z) = G_{\mathbb{H}}(z; 0, \infty), \quad G(z; \theta) = G_{\mathbb{D}}(z; 1, e^{2\theta i}).$$

If  $V \subset \mathbb{H}$ , we define

$$G(V) = \int_V G(z) dA(z).$$

Note that if  $\Gamma \in \mathcal{Q}_n$  and  $z$  is the center point of  $\Gamma$ , then

$$G(\Gamma) \asymp 2^{-2n} G(z).$$

(If  $\Gamma = \Gamma_n(j, 0)$ , this requires a simple estimate of an integral.)

*2.4. Two-point estimates.* A basic principle in proving two-point estimates for *SLE* is the idea that if a path gets very close to a point  $z$  and then gets away from  $z$ , then it is unlikely to get even closer to  $z$ . While this is the heuristic, as just stated the principle is not always valid. Since this idea is important in several of our proofs, we will spend some time to formulate and prove a precise version. We are expanding on ideas in [6, 7]. Let  $\gamma$  be an *SLE* $_{\kappa}$  curve from 0 to  $\infty$  in  $\mathbb{H}$ . As before, if  $z \in \mathbb{H}$ , let

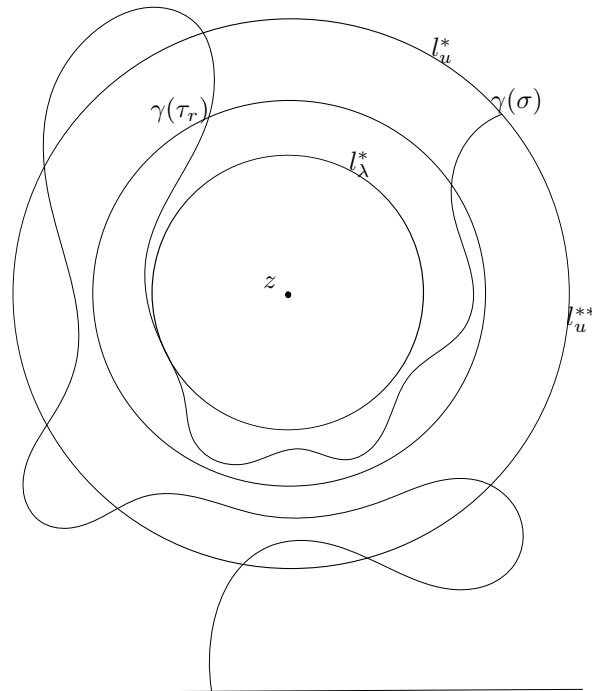
$$\tau_r(z) = \inf\{t : |\gamma(t) - z| = e^{-r}\}.$$

If  $\tau_r = \tau_r(z) < \infty$ , let  $H = H_{\tau_r}$  denote the unbounded component of  $\mathbb{H} \setminus \gamma_{\tau_r}$ , let  $\mathcal{B}_u = \mathcal{B}_u(r, z)$  denote the disk of radius  $e^{-ur}$  containing  $z$ , and let  $\mathcal{B} = \mathcal{B}_1$  denote the disk of radius  $e^{-r}$  about  $z$ . Let  $V_u = V_u(r, z)$  denote the connected component of  $\mathcal{B}_u \cap H$  containing  $z$ . If  $u \leq 1$ , the intersection of  $\partial V_u$  with  $H$  is a disjoint union of open arcs in  $\partial \mathcal{B}_u$  each of whose endpoints is in  $\partial H$ . There is a unique such arc  $l$ , that we denote by  $l_u = l_u(r, z)$ , such that  $z$  is contained in the bounded component of  $H \setminus l$ . Simple connectedness of  $H$  is used to see that this arc is unique. However, one may note the following facts.

- The bounded component of  $H \setminus l_u$  does not need to be contained in  $\mathcal{B}_u$ . Indeed, we have no universal bound on the diameter of the bounded component.
- There may be other subarcs  $l$  of  $\partial \mathcal{B}_u \cap H$  such that  $z$  is in the bounded component of  $H \setminus l$ . However, these arcs are not on  $\partial V_u$ .

For  $0 < u \leq 1$ , let

$$\sigma = \sigma_u(r, z) = \inf\{t \geq \tau_r : \gamma(t) \in \overline{l_u}\}.$$

FIG 1. *Quantities in Section 2.4*

Then a correct, although still imprecise, version of our heuristic principle is: if  $\tau_r < \infty$ , then after time  $\sigma$  the path is unlikely to get closer to  $z$ . We will now be more precise. Note that for fixed  $z, u, r$ , with probability one  $\gamma(\sigma) \in l_u$ . In this case (which we now assume),  $l_u \setminus \{\gamma(\sigma)\}$  consists of two crosscuts of  $H_\sigma$  that we denote by  $l_u^*$  and  $l_u^{**}$ . If  $z \in H_\sigma$ , which is always true if  $\kappa \leq 4$ , we let  $l_u^*$  be the crosscut such that  $z$  lies in the bounded component of  $H_\sigma \setminus l_u^*$ . If  $\tau_r < \infty$ , define  $\lambda = \lambda(r, z, u) \geq 1$  by

$$\text{dist}[z, \gamma_\sigma] = e^{-\lambda r}.$$

Let  $l_\lambda^*$  denote the connected subarc of  $\partial\mathcal{B}_\lambda \cap H_\sigma$  that separates  $z$  from infinity. (If the intersection of  $\gamma_\sigma$  with  $\mathcal{B}_\lambda$  is a single point, which we expect to be the case with probability one, then  $l_\lambda^*$  is a circle with a single point deleted.)

Since we will use it in several proofs, we recall that if  $D$  is a domain and  $\eta^1, \eta^2$  are disjoint subarcs of  $\partial D$ , then the (Brownian) *excursion measure* between  $\eta^1, \eta^2$  is given by

$$\mathcal{E}_D(\eta^1, \eta^2) = \int_{\eta^1} \partial_n \phi(z) |dz|,$$

where  $\partial_n$  denotes the inward normal derivative and  $\phi = \phi_{D, \eta^2}$  is the harmonic function on  $D$  with boundary value  $1_{\eta^2}$ . The above expression assumes that  $\eta^1$  is smooth, but one can check that  $\mathcal{E}_D(\eta^1, \eta^2)$  is a conformal invariant and hence can be defined for all domains. Also,  $\mathcal{E}_D(\eta^1, \eta^2) = \mathcal{E}_D(\eta^2, \eta^1)$ . See [4, Chapter 5] for more details. When estimating excursion measures, we will use the following estimate that follows from the strong Markov property. Suppose  $\eta$  is a crosscut of  $D$  that separates  $\eta^1$  from  $\eta^2$ . Then

$$(9) \quad \mathcal{E}_D(\eta^1, \eta^2) \leq \mathcal{E}_{D \setminus \eta}(\eta^1, \eta) \sup_{z \in \eta} \phi(z).$$

If  $D$  is simply connected, so that  $(D, \eta^1, \eta^2)$  is a conformal rectangle, and  $\mathcal{E}_D(\eta^1, \eta^2) \leq 1$ , then

$$(10) \quad \mathcal{E}_D(\eta^1, \eta^2) \asymp \max_{z \in D} \phi_1(z) \phi_2(z),$$

where  $\phi_j = \phi_{D, \eta^j}$ . One can check this by verifying it for a rectangle  $[0, L] + i[0, \pi]$  by direct computation and using conformal invariance.

LEMMA 2.5. *There exists  $c$  such that for all  $0 < u \leq 1$ ,*

$$\mathbb{P}\{\text{dist}[z, \gamma_\infty] < \text{dist}[z, \gamma_\sigma] \mid \gamma_\sigma\} \leq c e^{\alpha(u-\lambda)r},$$

where  $\alpha = (4a - 1)/2 > 0$ . In particular,

$$(11) \quad \mathbb{P}\{\text{dist}[z, \gamma_\infty] < \text{dist}[z, \gamma_\sigma] \mid \gamma_\sigma\} \leq c e^{\alpha(u-1)r}.$$

PROOF. Let  $g : H_\sigma \rightarrow \mathbb{H}$  be a conformal transformation with  $g(\gamma(\sigma)) = 0, g(\infty) = \infty$ . The image  $\eta = g \circ l_u^*$  is a crosscut of  $\mathbb{H}$  with one endpoint on the origin and one on the real line which without loss of generality we will assume is on the positive real line. The curve  $\eta' = g \circ l_\lambda^*$  is a crosscut of  $\mathbb{H}$  contained in the bounded component of  $\mathbb{H} \setminus \eta$  with positive endpoints  $x_1 \leq x_2$ . Let us consider the conformal rectangle given by the component of  $\mathbb{H} \setminus (\eta \cup \eta')$  that contains both  $\eta$  and  $\eta'$  on its boundary and with  $\eta, \eta'$  as two of the boundary arcs of the rectangle. The excursion measure between  $\eta$  and  $\eta'$  in this rectangle is the same as the excursion measure between  $l_u^*$  and  $l_\lambda^*$  for the corresponding rectangle in  $H_\sigma \setminus (l_u^* \cup l_\lambda^*)$ . The Beurling estimate (see, e.g., [4, Theorem 3.76]) implies that the latter is bounded above by  $c e^{-(\lambda-u)r/2}$ . Since  $\eta$  separates  $\eta'$  from the negative real line, we see that the excursion measure between  $\eta'$  and  $(-\infty, 0]$  in the unbounded component of  $\mathbb{H} \setminus \eta'$  is bounded above by  $c e^{-(\lambda-u)r/2}$ . By standard estimates of the Poisson kernel in  $\mathbb{H}$ , this shows that  $\text{diam}(\eta') \leq c e^{-(\lambda-u)r/2} x_1$ , and hence by (7), the probability that an *SLE* path hits it is  $O(e^{-(\lambda-u)(4a-1)r/2})$ .  $\square$

The next lemma strengthens (11) for  $\kappa \leq 4$ . We do not know if it is true for  $4 < \kappa < 8$ . Let  $\mathcal{B} = \mathcal{B}_1$  denote the disk of radius  $e^{-r}$  about  $z$ .

LEMMA 2.6. *If  $\kappa \leq 4$ , there exists  $c$  such that if  $0 < u \leq 1$ ,*

$$(12) \quad \mathbb{P}\{\gamma[\sigma, \infty) \cap \mathcal{B} \neq \emptyset \mid \gamma_\sigma\} \leq c e^{\alpha(u-1)r},$$

where  $\alpha = (4a - 1)/2 > 0$ .

PROOF. Let  $V$  denote the unbounded component of  $H_\sigma \setminus \overline{\mathcal{B}}$  and note that  $l_u^*, l_u^{**} \subset V$ . Let  $L = \partial V \cap H_\sigma \cap \partial \mathcal{B}$  which is a disjoint (finite or countable) union of open subarcs of  $\partial \mathcal{B}$ , which we denote by  $L_1, L_2, \dots$ . Let  $h_j$  denote the angular length of  $L_j$ , that is to say, if  $L_j = \{z + e^{-r+i\theta}, \theta_1 < \theta < \theta_2\}$ , then  $h_j = \theta_2 - \theta_1$ . Then

$$\sum_{j=1}^{\infty} h_j \leq 2\pi.$$

The probability on the left-hand side of (12) is the probability that  $\gamma[\sigma, \infty) \cap L \neq \emptyset$ . Hence it suffices to show that

$$\sum_{j=1}^{\infty} \mathbb{P}\{\gamma[\sigma, \infty) \cap L_j \neq \emptyset \mid \gamma_\sigma\} \leq c e^{\alpha(u-1)r}.$$

For each arc  $L_j$ , either  $l_u^*$  or  $l_u^{**}$  disconnects  $L_j$  from infinity in  $H_\sigma$ , that is,  $L_j$  is in the bounded component of  $H_\sigma \setminus l_u^*$  or the bounded component of  $H_\sigma \setminus l_u^{**}$ . We will assume the former, but the argument is the same in the latter case. We write  $\mathcal{R}_j$  for this bounded component which we view as a rectangle including boundary arcs  $l_u^*$  and  $L_j$ .

Let  $g, \eta$  be as in the previous proof, and let  $\eta_j = g \circ L_j$ . For fixed  $j$ , consider the conformal rectangle given by the component of  $\mathbb{H} \setminus (\eta \cup \eta_j)$  that contains  $\eta, \eta_j$  on its boundary and with  $\eta, \eta_j$  as two boundary arcs of the rectangle. Then the excursion measure is the same as the that between  $L_j$  and  $l_u^*$  in  $\mathcal{R}_j$  and that it exits at  $l_u^*$  is less than  $1 - \rho$ .

We claim that this latter measure is  $O(h_j e^{r(u-1)/2})$ . To see this, let  $l''$  be a crosscut of  $\mathcal{R}_j$  that disconnects  $L_j$  and  $l_u^*$  in  $\mathcal{R}_j$  and that is also a subarc of the circle of radius  $e^{ru-1}$ . Consider a Brownian motion starting at  $z \in l''$ . By the Beurling estimate, the probability to get to the circle of radius  $e^{1-r}$  about  $z$  without leaving  $\mathcal{R}$  is  $O(e^{(u-1)r/2})$ . The Poisson kernel in  $\mathcal{C} \setminus \overline{\mathcal{B}}$  shows that the conditional probability that it leaves  $\mathcal{R}_j$  at  $L_j$  given this is  $O(h_j)$ . Hence the probability that a Brownian motion starting at  $l''$  exits at  $L_j$  is  $O(h_j e^{(u-1)r/2})$ . But there is clearly a uniform upper bound on the excursion measure between  $l_u^*$  and  $l''$  in  $\mathcal{R}_j$ . Hence the estimate follows from (9).

As in the previous proof, we conclude by (7) that

$$(13) \quad \mathbb{P}\{\gamma[\sigma, \infty) \cap L_j \neq \emptyset \mid \gamma_\sigma\} \leq c e^{\alpha(u-1)r} h_j^{4a-1}.$$

The argument up to this point has not used the fact that  $\kappa \leq 4$ . However, if  $\kappa \leq 4$ , we know that  $4a - 1 \geq 1$  and hence

$$\sum_{j=1}^{\infty} h_j^{4a-1} \leq \left[ \sum_{j=1}^{\infty} h_j \right]^{4a-1} \leq (2\pi)^{4a-1}.$$

Therefore, by summing (13) over  $j$  we get (12).  $\square$

While we do not know if the last lemma holds for  $\kappa > 4$ , the next lemma will suffice for our needs.

LEMMA 2.7. *If  $4 < \kappa < 8$ , there exist  $c < \infty, \beta > 0$  such that if  $\tau_r < \infty$  and  $0 < u \leq 1$ , then*

$$\mathbb{P}\{\mathcal{B} \cap H_\sigma \neq \emptyset \mid \gamma_{\tau_r}\} \leq c e^{\beta(u-1)r}.$$

PROOF. Let  $\zeta = \gamma(\tau_r)$ . Let  $g$  be a conformal transformation of  $H_{\tau_r}$  onto  $\mathbb{H}$  with  $g(\zeta) = 0, g(\infty) = \infty$ . Let  $\eta = g \circ l_u, \eta' = g \circ [\partial\mathcal{B} \setminus \{\zeta\}]$ . Then  $\eta$  is a crosscut of  $\mathbb{H}$  with one endpoint positive and one endpoint negative, and  $\eta'$  is a simple loop rooted at the origin lying in the bounded component of  $\mathbb{H} \setminus \eta$ . By choosing a multiple of  $g$  if necessary, we may assume that  $\max\{|w| : w \in \eta'\} = 1$ .

We claim that there exists  $c'$  such that  $\text{dist}(0, \eta) \geq c' e^{(1-u)r/2}$ . To see this, let  $\mathcal{R}$  denote the component of  $H_\sigma \setminus (\partial\mathcal{B} \cup l_u)$  whose boundary contains both  $\partial\mathcal{B}$  and  $l_u$ . Then using the Beurling estimate as in the previous lemma we see that  $\mathcal{E}_{\mathcal{R}}(\partial\mathcal{B}, l_u) \leq c e^{-(u-1)/2}$ . Therefore,

$$\mathcal{E}_{g(\mathcal{R})}(\eta, \eta') = O(e^{-(u-1)r/2}).$$

But  $\eta'$  is a connected set containing the origin of radius 1. If  $v = \text{dist}(0, \eta)$ , then by setting  $z = i\sqrt{v}$  in (10) we get the bound

$$\mathcal{E}_{g(\mathcal{R})}(\eta, \eta') \geq c v^2.$$

By conformal invariance,  $\mathbb{P}\{\mathcal{B} \cap H_\sigma \neq \emptyset \mid \gamma_{\tau_r}\}$  is bounded above by the probability that an  $SLE_\kappa$  path starting at the origin has not separated the unit circle from infinity before it reaches the circle of radius  $c' e^{(1-u)r/2}$ . Using scaling and the fact that  $SLE_\kappa$  has double points, it is not hard to show that this is  $O(e^{\beta(u-1)r})$  for some  $\beta$ .  $\square$



COROLLARY 2.8. *If  $\kappa < 8$ , there exists  $c < \infty$  and  $\beta > 0$  such that if  $|z| > e^{-r/2}$  and  $0 < u \leq 1$ , then if  $\tau_r < \infty$ ,*

$$\mathbb{P}\{\gamma[\sigma, \infty) \cap \bar{\mathcal{B}} \neq \emptyset \mid \gamma_{\tau_r}\} \leq c e^{\beta(u-1)r},$$

$$\mathbb{P}\{\tau_r < \infty, \gamma[\sigma, \infty) \cap \bar{\mathcal{B}} \neq \emptyset\} \leq c G(z) e^{(d-2)r} e^{\beta(u-1)r}.$$

The other estimates we need will deal with upper bounds for the probabilities that the *SLE* curves gets close to two different points  $z, w$ . For the remainder of this section, we assume that  $\gamma$  is an *SLE* curve from 0 to  $\infty$  in  $\mathbb{H}$ . If  $z \in \bar{\mathbb{H}}$ , we let

$$\tau_r(z) = \inf \{t : |\gamma(t) - z| \leq e^{-r}\}.$$

LEMMA 2.9. *There exists  $c < \infty$  such that if  $|z|, |w| \geq e^{-u}$  and  $|z - w| \geq e^{-u}$ , then for  $0 < s < r$ ,*

$$(14) \quad \mathbb{P}\{\tau_{s+u}(w) < \infty, \tau_{r+u}(z) < \infty\} \leq c e^{(s+r)(d-2)},$$

$$(15) \quad \mathbb{P}\{\tau_{s+u}(w) < \tau_{r+u}(z) < \tau_{r+u}(w) < \infty\} \leq c e^{2r(d-2)} e^{-\alpha s},$$

where  $\alpha = (4a - 1)/2$ .

PROOF. By scaling it suffices to prove the lemma for  $u = 0$ . The first estimate is Theorem 2 in [10]. The second estimate follows from the ideas in [10, Lemma 4.10], but we will redo the proof using some ideas from this section.

Without loss of generality we assume that  $s, r$  are integers. Let  $E = E(s, r, z, w)$  denote the event  $\{\tau_s(w) < \tau_r(z) < \tau_r(w) < \infty\}$ , and for  $j, k$  nonnegative integers, define

$$E_j^0 = E \cap \{\tau_s(w) \leq \tau_j(w) < \tau_1(z) < \tau_{j+1}(w)\},$$

$$E_j^k = E \cap \{\tau_k(z) < \tau_s(w) \leq \tau_j(w) < \tau_{k+1}(z) < \tau_{j+1}(w)\}, \quad k \geq 1,$$

$$E_j = \bigcup_{k=0}^{r-1} E_j^k.$$

Note that

$$E = \bigcup_{j=s}^{r-1} E_j.$$

We will show that  $\mathbb{P}(E_j) \leq c e^{2r(d-2)} e^{-\alpha j}$  from which (15) follows by summing over  $j \geq s$ .

Let  $H = H_{\tau_j(w)}$ ,  $g = g_{\tau_j(w)}$ . Let  $D$  denote the intersection of  $\mathbb{H}$  and the open disk of radius  $1/2$  centered at  $w$ . Let  $V$  be the connected component of  $D \cap H$  that contains  $w$ . As before, there is a unique subarc  $l$  of  $\partial D$  that is a crosscut of  $V$  such that  $w$  and  $z$  lie in different components of  $V \setminus l$ . Let  $A_z = A_z^j$  (resp.,  $A_w = A_w^j$ ) denote the event that  $\tau_j(w) < \infty$  and that  $z$  (resp.,  $w$ ) is in the bounded component of  $V \setminus l$ .

On the event  $A_z$ , consider the open disk of radius  $\text{dist}(z, \partial H)$  about  $z$ . For any point in this disk, the probability that a Brownian motion hits the “other side” of  $\partial H$  is bounded above by the probability that a Brownian motion reaches the disk of radius  $e^{-j}$  about  $w$  without leaving  $H$ . By the Beurling estimates we see that this probability is  $O(e^{-(j+k)/2})$  (There is a probability  $O(e^{-k/2})$  to get to distance  $1/2$  of  $z$  and then given this there is a probability of  $O(e^{-j/2})$  to reach the disk around  $w$ .) We can use the boundary estimate (7) to see that the conditional probability that  $\text{dist}(z, \gamma_\infty) < \text{dist}(z, \partial H)$  given  $A_z \cap \{\text{dist}(z, \partial H) \leq e^{-k}\}$  is  $O(e^{-(j+k)\alpha})$ . We now estimate  $\mathbb{P}(E_j^k)$  from above by the product of three terms:

$$\mathbb{P}\{\tau_k(z), \tau_j(w) < \infty\},$$

$$\mathbb{P}\{\tau_{k+1}(z) < \infty \mid \tau_k(z) < \tau_s(w) \leq \tau_j(w) < \tau_{k+1}(z), A_z\},$$

$$\mathbb{P}\{\tau_r(z), \tau_r(w) < \infty \mid \tau_k(z) < \tau_s(w) \leq \tau_j(w) < \tau_{k+1}(z) < \tau_{j+1}(w), A_z\}.$$

Using the estimate (14), we see that the first term is  $O(e^{(d-2)(k+j)})$  and the third term is  $O(e^{(d-2)(r-k+r-j)})$ . Hence, we see that  $\mathbb{P}(E_j^k \cap A_z) \leq c e^{2r(d-2)} e^{-\alpha(j+k)}$ . By summing over  $k$ , this gives  $\mathbb{P}(E_j \cap A_z) \leq c e^{2r(d-2)} e^{-\alpha j}$ .

If  $m \leq r$  is an integer, let  $A_w^m$  be the union over all  $k < r$  of the events

$$E \cap A_w \cap \{\tau_k(z) < \tau_j(w) < \tau_{k+1}(z) \leq \tau_m(z) < \tau_{j+1}(w) < \tau_{m+1}(z)\}.$$

If  $k = 0$  we replace  $\tau_0(z)$  in the above expression with 0. If  $m = r$  it may be the case that  $\tau_{m+1}(z) = \infty$ . Since  $w$  is in the bounded component, we can argue as in the previous paragraph to see that

$$\mathbb{P}\{\tau_{j+1}(w) < \infty \mid \tau_j(w) < \tau_{k+1}(z) \leq \tau_m(z) < \tau_{j+1}(w), A_w\} \leq c e^{-\alpha(j+m)}.$$

By splitting into three terms as before, we see that

$$\begin{aligned} \mathbb{P}[E \cap A_w \cap \{\tau_k(z) < \tau_j(w) < \tau_{k+1}(z) \leq \tau_m(z) < \tau_{j+1}(w) < \tau_{m+1}(z)\}] \\ \leq c e^{-\alpha(j+m)} e^{2r(d-2)}, \end{aligned}$$

and by summing over  $m$ , we see that  $\mathbb{P}[E_j \cap A_w] \leq c e^{-\alpha j} e^{2r(d-2)}$ .  $\square$

Given this estimate one also shows that if  $\Im(z), \Im(w) \geq 1$  with  $|z-w| \leq 1$ , then

$$(16) \quad \mathbb{P}\{\tau_r(z) < \infty, \tau_r(w) < \infty\} \leq c e^{2r(d-2)} |z-w|^{d-2}.$$

Indeed, if  $\rho = \inf\{t : |\gamma(t) - z| \leq 2|z-w|\}$ , Then,

$$\mathbb{P}\{\rho < \infty\} \leq c|z-w|^{2-d},$$

and by conformal invariance,

$$\mathbb{P}\{\tau_r(z) < \infty, \tau_r(w) < \infty \mid \rho < \infty\} \leq [e^{-r}/|z-w|]^{2(2-d)}.$$

In [10], it was shown that the limit,

$$\lim_{\epsilon, \delta \downarrow 0} \epsilon^{d-2} \delta^{d-2} \mathbb{P}\{\text{crad}_{\mathbb{H} \setminus \gamma}(z_1) \leq \epsilon, \text{crad}_{\mathbb{H} \setminus \gamma}(z_2) \leq \delta\},$$

exists and defines a two-point Green's function. In Section 4.2 we show how to adapt this argument to show existence of

$$(17) \quad G(z_1, z_2) = \lim_{\epsilon, \delta \downarrow 0} \epsilon^{d-2} \delta^{d-2} \mathbb{P}\{\text{dist}(z_1, \gamma) \leq \epsilon, \text{dist}(z_2, \gamma) \leq \delta\}.$$

In fact, we can write  $G(z_1, z_2) = \hat{G}(z_1, z_2) + \hat{G}(z_2, z_1)$  where

$$\hat{G}(z, w) = G(z) \mathbb{E}^*[G_{H_T}(w; z, \infty)],$$

and  $\mathbb{E}^*$  denotes expectation with respect to two-sided radial  $SLE_\kappa$  from 0 to  $z$  stopped at

$$T = \inf\{t : \gamma(t) = z\}.$$

See Section 4.2 for a review of two-sided radial  $SLE$ .

### 3. Existence of Minkowski content.

3.1. *Main theorem.* If  $\Gamma \in \mathcal{Q}_n$  as defined in Section 2.1, and  $\gamma$  is an  $SLE_\kappa$  curve from 0 to  $\infty$  in  $\mathbb{H}$ , let

$$Z(\Gamma) = \text{Cont}_d^+(\gamma \cap \Gamma; n \log 2),$$

$$J_r(z) = e^{(2-d)r} \mathbf{1}\{\tau_r(z) < \infty\}, \quad J_r(V) = \int_V J_r(z) dA(z).$$

Note that if  $s > 0$ , then  $J_{r+s}(z) \leq e^{s(2-d)} J_r(z)$ .

**THEOREM 3.1.** *Suppose  $\kappa < 8$  and  $\gamma$  is an  $SLE_\kappa$  curve from 0 to  $\infty$  in  $\mathbb{H}$ . Then the following holds for all  $\Gamma \in \mathcal{Q}^+$ .*

- *The limit*

$$\mu(\Gamma) := \lim_{r \rightarrow \infty} J_r(\Gamma)$$

*exists with probability one and in  $L^2$ .*

- *With probability one,*

$$(18) \quad \text{Cont}_d(\gamma \cap \Gamma) = \mu(\Gamma).$$

- *Let  $\partial_n \Gamma = \{z \in \mathbb{H} : \text{dist}(z, \partial \Gamma) \leq 2^{-n}\}$ . Then with probability one,*

$$(19) \quad \lim_{n \rightarrow \infty} \text{Cont}_d(\gamma \cap \partial_n \Gamma; n \log 2) = 0.$$

- *The following moment relations holds.*

$$(20) \quad \mathbb{E}[\mu(\Gamma)] = \int_{\Gamma} G(z) dA(z),$$

$$(21) \quad \mathbb{E}[\mu(\Gamma)^2] = \int_{\Gamma} \int_{\Gamma} G(z, w) dA(z) dA(w),$$

$$(22) \quad \mathbb{E} [Z(\Gamma)^2] < \infty.$$

Since  $\mathcal{Q}^+$  is countable, all the with probability one statements can be restated as “with probability one, for all  $\Gamma \in \mathcal{Q}^+, \dots$ ”. The bulk of the work in proving the theorem is to prove Theorem 3.2 below. Let  $0 < \delta < 1/10$ . Since we want to take a limit of  $J_r$  as  $r \rightarrow \infty$ , we will look at

$$\begin{aligned} Q_r^\delta(z) &= J_r(z) - J_{r+\delta}(z) \\ &= e^{r(2-d)} \left[ 1_{\{\tau_r(z) < \infty\}} - e^{\delta(2-d)} 1_{\{\tau_{r+\delta}(z) < \infty\}} \right]. \end{aligned}$$

The random variable  $Q_r^\delta(z)$  is normalized so that  $|Q_r^\delta(z)|$  is of order 1 but  $\mathbb{E}[Q_r^\delta(z)]$  is nearly zero. The main estimate shows that if  $r$  is large and  $z, w$  are not too close, then  $Q_r(z)$  and  $Q_r(w)$  are almost independent.

**THEOREM 3.2.** *Suppose  $\kappa < 8$  and  $\gamma$  is  $SLE_\kappa$  from 0 to  $\infty$  in  $\mathbb{H}$ . There exists  $c < \infty, \beta > 0$  such that if  $0 < \delta < 1/10$ ,  $\Im(z), \Im(w) \geq 1$  and  $r \geq 0$ , then*

$$(23) \quad \mathbb{E} \left[ Q_r^\delta(z) Q_r^\delta(w) \right] \leq c e^{-\beta r} |z - w|^{\beta-2}.$$

We will not try to find the optimal  $c, \beta$  in our proof.

PROOF OF THEOREM 3.1 GIVEN THEOREM 3.2. By scaling we may assume that  $\Gamma = [j, j+1) \times i[k, k+1) \in \mathcal{Q}_0^+$  with  $k \geq 1$ . Suppose that  $0 < \delta < 1/10$ . Let  $J_r = J_r(\Gamma)$  and

$$Q_r = Q_r^\delta = Q_r^\delta(\Gamma) = J_r - J_{r+\delta} = \int_{\Gamma} Q_r^\delta(z) dA(z),$$

By integrating (23), we see that if  $r \geq 0$ , then  $\mathbb{E}[Q_r^2] \leq c e^{-\beta r}$ . Let

$$X_n = X_n^\delta = J_0 + \sum_{j=1}^n |Q_{j\delta}|.$$

Then  $X_n$  converges in  $L^2$  to a random variable  $X_\infty$ . For each positive integer  $n$ ,  $|J_{n\delta}| \leq X_\infty$  and hence

$$(24) \quad \sup_{r \geq 0} J_r \leq e^{\delta(2-d)} \sup_n J_{n\delta} \leq e^{1/10} X_\infty,$$

Also, if  $n \leq m$ ,

$$|J_{n\delta} - J_{m\delta}| \leq X_\infty - X_n.$$

Therefore  $\{J_{n\delta}\}$  is a Cauchy sequence in  $L^2$  and has an  $L^2$ -limit which we call  $J_\infty$ . If  $r\delta \leq s < (r+1)\delta$ , we similarly have

$$(25) \quad \mathbb{E}[(J_s - J_{r\delta})^2] = \mathbb{E}[(Q_{r\delta}^{s-r\delta})^2] \leq c e^{-\beta r\delta},$$

so we see that

$$\lim_{s \rightarrow \infty} \mathbb{E}[(J_s - J_\infty)^2] \leq \lim_{s \rightarrow \infty} \mathbb{E}[(J_s - J_{r\delta})^2] + \lim_{s \rightarrow \infty} \mathbb{E}[(J_{r\delta} - J_\infty)^2] = 0.$$

Hence  $J_s \rightarrow J_\infty$  in  $L^2$ ; in particular,  $J_\infty$  does not depend on  $\delta$ .

Chebyshev's inequality shows that

$$\sum_{n=1}^{\infty} \mathbb{P}\{|Q_{n\delta}| \geq e^{-\beta n\delta/4}\} \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}[Q_{n\delta}^2]}{e^{-\beta n\delta/2}} < \infty.$$

Hence for each  $\delta$ , by the Borel-Cantelli lemma, with probability one for all  $n$  sufficiently large,

$$|J_{n\delta} - J_{(n+1)\delta}| \leq 2 e^{-\beta n\delta/4}.$$

This shows that with probability one, the sequence  $\{J_{n\delta}\}$  is a Cauchy sequence and hence with probability one, for all  $\delta = 2^{-m}$ ,

$$\lim_{n \rightarrow \infty} J_{n\delta} = J_\infty.$$

If  $n\delta \leq r \leq (n+1)\delta$ , then

$$(26) \quad e^{\delta(d-2)} J_{(n+1)\delta} \leq J_r \leq e^{\delta(2-d)} J_{n\delta},$$

from which we conclude that with probability one, for all  $\delta = 2^{-m}$ ,

$$e^{\delta(d-2)} J_\infty \leq \liminf_{r \rightarrow \infty} J_r \leq \limsup_{r \rightarrow \infty} J_r \leq e^{\delta(2-d)} J_\infty.$$

Since this holds for all  $\delta$ ,  $J_r \rightarrow J_\infty$ .

Note that for  $r > 0$ ,

$$\mathbb{E}[J_r] = \int_{\Gamma} e^{(2-d)r} \mathbb{P}\{\tau_r(z) < \infty\} dA(z),$$

$$\mathbb{E}[J_r^2] = \int_{\Gamma} \int_{\Gamma} e^{2(2-d)r} \mathbb{P}\{\tau_r(z), \tau_r(w) < \infty\} dA(z) dA(w).$$

Since  $J_r \rightarrow J_\infty$  in  $L^2$ , we know that  $\mathbb{E}[J_\infty] = \lim_{r \rightarrow \infty} \mathbb{E}[J_r]$ ,  $\mathbb{E}[J_\infty^2] = \lim_{r \rightarrow \infty} \mathbb{E}[J_r^2]$ . Hence (20) and (21) follow from Theorem 2.3 and (17). Indeed, the definition of the Green's function (including the choice of multiplicative constant) was made in order for these equalities to hold.

Note that if  $n \log 2 \leq r \leq (n+1) \log 2$ ,

$$|\text{Cont}_d[\gamma \cap \Gamma; r] - J_r(\Gamma)| \leq c J_{n \log 2}(\partial_n \Gamma).$$

Using (6), we see that  $\mathbb{E}[\text{Cont}_d(\gamma \cap \partial_n \Gamma; n \log 2)] \leq c \text{Area}(\partial_n \Gamma) \leq c 2^{-n}$ . Hence, using the Markov inequality and the Borel-Cantelli lemma, we see that with probability one for all  $n$  sufficiently large  $\text{Cont}_d(\gamma \cap \partial_n \Gamma; n \log 2) \leq 2^{-n/2}$ . This gives (19). Also,

$$\begin{aligned} \text{Cont}_d[\gamma \cap \Gamma; n \log 2] &\leq J_{n \log 2} \\ &\leq \text{Cont}_d[\gamma \cap \Gamma; n \log 2] + \text{Cont}_d(\gamma \cap \partial_n \Gamma; n \log 2). \end{aligned}$$

This gives (18).

Given  $\Gamma \in \mathcal{Q}_0$ , let  $\Gamma_1, \dots, \Gamma_{12}$  denote the twelve squares in  $\mathcal{Q}_1$  whose interior does not intersect  $\Gamma$  but whose boundary does. Note that these squares are in  $\mathcal{Q}_1^+$ . Any point within distance  $1/2$  of  $\gamma \cap \Gamma$  is contained in  $\Gamma \cup \Gamma_1 \cup \dots \cup \Gamma_{12}$ , and hence for  $r \geq 1$ ,

$$\text{Cont}_d(\gamma \cap \Gamma; r) \leq J_r(\Gamma) + J_r(\Gamma_1) + \dots + J_r(\Gamma_{12}).$$

This implies that

$$\text{Cont}_d^+(\gamma \cap \Gamma; 0) \leq c + \sup_{r \geq 1} [J_r(\Gamma) + J_r(\Gamma_1) + \dots + J_r(\Gamma_{12})].$$

Since  $\Gamma_j \in \mathcal{Q}_1^+$ , the argument as in (24), we see that for each  $j$ ,

$$\sup_{r \geq 1} J_r(\Gamma_j)$$

is square integrable. Hence,  $\text{Cont}_d^+(\gamma \cap \Gamma; 0)$  is square integrable which gives (22). □

**3.2. Natural length.** By Theorem 3.1, with probability one we can define a function on  $\mathcal{Q}$  by

$$\mu(\Gamma) = \text{Cont}_d(\gamma \cap \Gamma) = \text{Cont}_d(\gamma \cap \text{int}(\Gamma)) = \text{Cont}_d(\gamma \cap \bar{\Gamma}).$$

**PROPOSITION 3.3.** *On this event,  $\mu$  extends to be a Borel measure.*

**PROOF.** For each  $\Gamma \in \mathcal{Q}^+$  and positive integer  $r$ , we can define a Borel measure  $\mu_r$  by stating that the Radon-Nikodym derivative with respect to Lebesgue measure is  $J_r(z)$ . Since  $\mu_r(\Gamma) \rightarrow \mu(\Gamma)$  and  $\Gamma$  is compact, for each subsequence  $\{r_j\}$ , there is a subsubsequence  $\{r_{jk}\}$  that converges to a measure  $\mu'$  with total mass  $\mu(\Gamma)$ . By using a diagonalization argument, we can find a single subsubsequence such that the convergence holds for all  $\Gamma \in \mathcal{Q}^+$ . By (19), we see that  $\mu'(\partial\Gamma) = 0$ . Any open set  $U$  can be written as a countable union of squares  $\Gamma \in \mathcal{Q}^+$  such that the interiors of the squares are disjoint. Hence, we can determine  $\mu'(U)$  for any open set, and hence we can see that  $m' = \mu$  is unique. □

We call  $\mu$  the (*natural*) *occupation measure* for the *SLE* curve  $\gamma$ . If  $D$  is an open set, then we can find  $D_n \in \mathcal{S}_{\mathbb{H}}$  increasing to  $D$  and hence

$$\mathbb{E}[\mu(D)] = \int_D G(z) dA(z), \quad \mathbb{E}[\mu(D)^2] = \int_{D \times D} G(z, w) dA(z) dA(w).$$

It is not immediately obvious, but we will now show that, with probability one, for all  $0 \leq s < t < \infty$ ,

$$(27) \quad \mu(\gamma[s, t]) = \mu(\gamma_t \setminus \gamma_s) = \text{Cont}_d(\gamma[s, t]).$$

The bulk of the work is in the following lemma. Recall that  $H_s$  is the unbounded component of  $\mathbb{H} \setminus \gamma_s$ , and let

$$\partial_n H_s = \{z \in \bar{H}_s : \text{dist}(z, \partial H_s) \leq 2^{-n}\}.$$

**LEMMA 3.4.** *There exists  $\alpha > 0$  such that the following holds with probability one.*

- For each  $t_0$ , there exists  $n_0 < \infty$  such that if  $0 \leq s \leq t_0$  and  $n \geq n_0$ , then

$$(28) \quad \text{Cont}_d^+(\gamma[s, s + 2^{-n}]) \leq 2^{-n\alpha}.$$

- Suppose that  $0 \leq s < t$ , and  $u > 0$ . Then

$$(29) \quad \lim_{n \rightarrow \infty} \text{Cont}_d^+[\gamma[s + u, t] \cap \partial_n H_s] = 0.$$

The limit (29) is immediate for  $\kappa \leq 4$  since  $\gamma[s + u, t] \cap \partial_n H_s$  is empty if  $n$  is large. Before proving the lemma, we will show how to deduce (27) from the lemma. We approximate  $\gamma[s, t]$  by intersections of  $\gamma$  with finite unions of dyadic squares. If  $s < t$  and  $n$  is a positive integer, let  $V_n(s, t)$  denote the union of all  $\Gamma \in \mathcal{Q}_n$  satisfying  $\Gamma \subset H_s \setminus \partial_n H_s$  and  $\gamma[s, t] \cap \Gamma \neq \emptyset$ . Let  $O_n(s, t) = \gamma \cap V_n(s, t)$ . Note that  $O_n(s, t) \subset \gamma \setminus \gamma_s$ , but it is possible for  $\gamma(t, \infty) \cap O_n(s, t)$  to be nonempty. Note that if  $u > 0$ , then

$$\begin{aligned} O_n(s, t) \setminus \gamma[s, t] &\subset \gamma[t, t + u] \cup (\gamma[t + u, \infty) \cap \partial_{n-1} H_t), \\ \gamma[s, t] \setminus O_n(s, t) &\subset \gamma[s, s + u] \cup [\gamma(s + u, \infty) \cap \partial_{n-2} H_s]. \end{aligned}$$

Here we use the simple geometric facts that  $O_n(s, t) \cap \bar{H}_t \subset \partial_{n-1} H_t$ , and that if  $\Gamma \in \mathcal{Q}_n$ , then either  $\Gamma \subset H_s \setminus \partial_n H_s$  or  $\Gamma \subset \partial_{n-2} H_s$ . The lemma implies that

$$\lim_{n \rightarrow \infty} \text{Cont}_d^+[\gamma[s, t] \setminus O_n(s, t)] + \lim_{n \rightarrow \infty} \text{Cont}_d^+[O_n(s, t) \setminus \gamma[s, t]] = 0.$$

Then (27) follows from (4). The remainder of this subsection will be devoted to proving the lemma. There is some technical work involved here and are the basic reasons why the lemma holds.

- For (28) we use the Hölder continuity of an *SLE* path to say that that the diameter of  $\gamma[s, s + 2^{-n}]$  is not very big. We also use moment estimates to show that the Minkowski content is not very big on any set of small diameter.
- For (29) we use the fact that  $\gamma[s + u, t] \cap \partial_n H_s$  consists of points of the curve that are either near the real line or are nearly double points of the curve. We estimate moments for the content of such paths.

We start by

LEMMA 3.5. *Let  $Z(\Gamma)$  be defined as before Theorem 3.1. There exists  $c < \infty$  such that if  $\Gamma \in \mathcal{Q}_n^+$ , then*

$$\mathbb{E}[Z(\Gamma)] \leq cG(\Gamma), \quad \mathbb{E}[Z(\Gamma)^2] \leq c2^{-dn} G(\Gamma).$$



PROOF. If  $\tilde{\Gamma} \in \mathcal{Q}_n^+$ , then  $\Gamma = 2^n \tilde{\Gamma} \in \mathcal{Q}_0^+$  with  $G(\Gamma) = 2^{dn} G(\tilde{\Gamma})$ . Also, the distribution of  $Z(\Gamma)$  is the same as that of  $2^{dn} Z(\tilde{\Gamma})$ . Hence, we may assume that  $\Gamma \in \mathcal{Q}_0^+$ .

If  $\text{dist}(0, \Gamma) \leq 10$ , then  $G(\Gamma) \asymp 1$  and we can use (22). Otherwise, let  $\tau$  be the first time that  $\text{dist}(\Gamma, \gamma(t)) = 8$ . By (6),  $\mathbb{P}\{\tau < \infty\} \asymp G(\Gamma)$ , and by distortion estimates we can see that

$$\mathbb{E}[Z(\Gamma) \mid \tau < \infty] \leq c, \quad \mathbb{E}[Z(\Gamma)^2 \mid \tau < \infty] \leq c.$$

□

COROLLARY 3.6. *With probability one, if  $R < \infty$ , then for  $n$  sufficiently large,  $\Gamma \in \mathcal{Q}_n$  with  $\text{dist}(0, \Gamma) \leq R$ ,*

$$Z(\Gamma) \leq n 2^{-dn/2}.$$

PROOF. By Chebyshev's inequality, if  $\Gamma \in \mathcal{Q}_n$ ,

$$\mathbb{P}\{Z(\Gamma) \geq n 2^{-nd/2}\} \leq n^{-2} 2^{nd} \mathbb{E}[Z(\Gamma)^2] \leq c n^{-2} G(\Gamma).$$

Hence if  $V$  is any bounded set,

$$\sum_{n=0}^{\infty} \sum_{\Gamma \in \mathcal{Q}_n, \Gamma \subset V} \mathbb{P}\{Z(\Gamma) \geq n 2^{-nd/2}\} \leq c \int_V G(z) dA(z) < \infty.$$

The result follows from the Borel-Cantelli lemma. □

Note that

$$\partial_n \mathbb{H} = \{z \in \mathbb{H} : \Im(z) \leq 2^{-n}\}.$$

LEMMA 3.7. *With probability one, if  $R < \infty$  and  $u > 0$ , then for all  $n$  sufficiently large*

$$\text{Cont}_d^+(\gamma \cap \partial_n \mathbb{H} \cap \{|z| \leq R\}) \leq u 2^{-n}.$$

*In particular, for each  $t_0$ , for all  $n$  sufficiently large,*

$$\text{Cont}_d^+(\gamma[0, t_0] \cap \partial_n \mathbb{H}) \leq u 2^{-n}.$$

PROOF. The argument is the same for all  $R$ ; for ease, we let  $R = 1$  and write  $V_n = \partial_n \mathbb{H} \cap \{|z| \leq 1\}$ . We will first show that

$$(30) \quad \sum_{n=1}^{\infty} \mathbb{P}\{\text{Cont}_d(\gamma \cap V_n; n \log 2) \geq 2^{-n}\} < \infty.$$

Since  $V_n \subset \bigcup_{|j| \leq 2^n} \Gamma_n(j, 0)$ ,

$$\begin{aligned} \text{Cont}_d(\gamma \cap V_n; n \log 2) &\leq \sum_{|j| \leq 2^n} \text{Cont}_d(\gamma \cap \Gamma_n(j, 0); n \log 2) \\ &\leq 6 \cdot 2^{-2n} \cdot 2^{(2-d)n} \sum_{|j| \leq 2^n} 1_{\{\gamma \cap \Gamma_n(j, 0) \neq \emptyset\}}. \end{aligned}$$

The estimate (7) implies that  $\mathbb{P}\{\gamma \cap \Gamma_n(j, 0) \neq \emptyset\} \leq c j^{1-4a}$ . If we choose  $\beta$  with  $1 < \beta < d - (2 - 4a)_+$ , we can see that

$$\mathbb{E}[\text{Cont}_d(\gamma \cap V_n; n \log 2)] \leq c 2^{-n\beta},$$

$$\mathbb{P}\{\text{Cont}_d(\gamma \cap V_n; n \log 2) \geq 2^{-n}\} \leq c 2^{-n(\beta-1)}.$$

This gives (30), and by the Borel-Cantelli lemma with probability one for all  $n$  sufficiently large,

$$\text{Cont}_d(\gamma \cap V_n; n \log 2) \leq 2^{-n}.$$

It follows that for  $n$  sufficiently large, if  $m \geq n$ ,

$$\text{Cont}_d(\gamma \cap V_n; m \log 2) \leq 2^{-m} + \text{Cont}_d(\gamma \cap (V_n \setminus V_m); m \log 2),$$

and hence

$$\text{Cont}_d^+(\gamma \cap V_n) \leq 2^{2-d} \sup_{m \geq n} \text{Cont}_d(\gamma \cap (V_n \setminus V_m); m \log 2),$$

where the supremum on the right is restricted to integers  $m$ . Let  $\mathcal{A}_n$  denote the set of all squares of the form  $\Gamma_l(j, 1)$ ,  $l, j \in \mathbb{Z}$ , that intersect  $V_n$ . These squares are disjoint and

$$\text{Cont}_d(\gamma \cap (V_n \setminus V_m); m \log 2) \leq \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma).$$

Hence,

$$\text{Cont}_d^+(\gamma \cap V_n) \leq 2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma).$$

By Lemma 3.5,

$$\sum_{\Gamma \in \mathcal{A}_n} \mathbb{E}[Z(\Gamma)] \leq c \sum_{\Gamma \in \mathcal{A}_n} G(\Gamma) \leq c G(V_n).$$

As above, we find  $\beta > 1$  such that  $G(V_n) \leq c 2^{-n\beta}$ , and hence

$$\begin{aligned} \mathbb{P} \left\{ 2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma) \geq u 2^{-n} \right\} &\leq u^{-1} 2^n \mathbb{E} \left[ 2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma) \right] \\ &\leq c u^{-1} 2^{n(1-\beta)}. \end{aligned}$$

Hence, by the Borel-Cantelli lemma, with probability one, for all  $n$  sufficiently large and all  $m \geq n$ ,

$$2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma) \leq u 2^{-n}.$$

□

The next proposition establishes the Hölder continuity of the function  $t \mapsto \text{Cont}_d(\gamma(0, t])$  and completes the proof of (28).

PROPOSITION 3.8. *There exists  $\alpha > 0$  such that with probability one for every  $t < \infty$  for all  $n$  sufficiently large and all  $s \leq t$ ,*

$$\text{Cont}_d^+(\gamma[s, s + 2^{-n}]) \leq 2^{-n\alpha}.$$

PROOF. It is known [3, 12] that for  $\kappa \neq 8$ , the  $SLE_\kappa$  curve is Hölder continuous with respect to the capacity parametrization. That is to say, there exists  $\beta = \beta_\kappa > 0$  such that with probability one, if  $t < \infty$ , then for  $n$  sufficiently large, and all  $0 \leq s \leq t$ ,

$$\text{diam}(\gamma[s, s + 2^{-n}]) \leq 2^{-n\beta}.$$

Let  $m$  be the largest integer less than  $\beta n$ . Then  $\gamma[s, s + 2^{-n}]$  is contained in the union of four rectangles  $\Gamma_1, \dots, \Gamma_4 \in \mathcal{Q}_m$ . For  $n$  sufficiently large, if  $\Gamma_j \in \mathcal{Q}_m^+$ , then Corollary 3.6 implies that  $\text{Cont}_d^+(\Gamma_j \cap \gamma) \leq m 2^{-dm/2}$ . If  $\Gamma_j \in \mathcal{Q}_m \setminus \mathcal{Q}_m^+$ , then Lemma 3.7 implies that  $\text{Cont}_d^+(\Gamma_j) \leq c 2^{-m}$ . The result follows for  $\alpha < \beta d/2$ .

□

In the remainder of this section, we prove (29) which is

$$\lim_{n \rightarrow \infty} \text{Cont}_d^+(\gamma[s + u, t] \cap \partial_n H_s) = 0.$$

Let  $U = U_{j,k} = \{x + iy : -2^k \leq x < 2^k, y \geq 2^{-j}\}$ . Using Lemma 3.7 and compactness of  $\gamma[t, u]$ , we see that it suffices to prove that with probability one for every  $s < u$  and all positive integers  $j, k$ ,

$$(31) \quad \lim_{n \rightarrow \infty} \text{Cont}_d^+(\gamma[u, \infty) \cap \partial_n H_s \cap U_{j,k}) = 0.$$

It suffices to consider rational  $s, u$ , and hence we need to show that for fixed  $s, u, j, k$ , (31) holds with probability one. By scaling it suffices to prove this for  $j = 0$  which we now assume. So we have

$$U = U_{0,k} = \{x + iy : -2^k \leq x < 2^k, y \geq 1\}.$$

We fix integer  $k > 0$  and allow constants to depend on  $k$ . We only consider  $n \geq k + 4$ . Let  $U = U_{0,k}$ , and let  $\mathcal{Q}_n(U)$  denote the set of  $\Gamma \in \mathcal{Q}_n$  with  $\Gamma \subset U$ . Note that  $G(\Gamma) \leq c 2^{-2n}$  if  $\Gamma \in \mathcal{Q}_n(U)$ .

We will now define a quantity  $\hat{Z}(\Gamma)$  for  $\Gamma \in \mathcal{Q}$  that is an upper bound for the Minkowski content of the intersection of the path with  $\Gamma$  “after it has gotten close to the square and then gotten away from the square”. To be precise, suppose that  $\Gamma \in \mathcal{Q}_n$  with center point  $z$ , and define the following quantities.

- $\xi_1 = \xi_1(\Gamma)$  is the first time  $t$  such that  $|z - \gamma(t)| = 2^{-n+3}$ . If  $\xi_1 < \infty$ , let  $l = l(\Gamma)$  denote a subarc of the circle of radius  $2^{-n/2}$  about  $z$  such that  $z$  is in the bounded component of  $H_{\xi_1} \setminus l$ . See Section 2.4 where a particular such arc  $l$  was selected. To be specific, we will make that choice here.
- $\xi_2 = \xi_2(\Gamma)$  is the first time  $t > \xi_1$  such that  $\gamma(t) \in \bar{l}$ .
- $\xi_3 = \xi_3(\Gamma)$  is the first time  $t > \xi_1$  such that  $|z - \gamma(t)| = 2^{-n+1}$ .
- $\xi_4 = \xi_4(\Gamma)$  is the first time  $t > \xi_2$  such that  $|z - \gamma(t)| = 2^{-n+1}$ .

We think of time  $\xi_4$  as the time of the “second return” to the (neighborhood of the) square.

LEMMA 3.9. *There exists  $n_0$  such that if  $n \geq n_0$ ,  $\Gamma \in \mathcal{Q}_n$ ,  $\Gamma \cap U \neq \emptyset$ , and  $\xi_1 \leq s$ , then  $\xi_2 < u$ .*

PROOF. The curve  $\gamma$  is parametrized so that  $\text{hcap}(\gamma[s_1, s_2]) = a(s_2 - s_1)$  where  $\text{hcap}$  is the half-plane capacity which can be defined by

$$\text{hcap}(V) = \lim_{y \rightarrow \infty} y \mathbb{E}^{iy} [\mathfrak{S}(B_\tau)],$$

where  $B_t$  is a standard Brownian motion and  $\tau = \tau_V = \inf\{t : B_t \in \mathbb{R} \cup V\}$ . In particular, if  $V_1 \subset V_2$ ,

$$\text{hcap}(V_2) - \text{hcap}(V_1) \leq \lim_{y \rightarrow \infty} y \mathbb{E}^{iy} [\mathfrak{S}(B_{\tau_2}); \tau_2 < \tau_1], \quad \tau_j = \tau_{V_j}.$$

Since the half-plane capacity is monotone,

$$\text{hcap}(\gamma[0, \xi_2]) \leq \text{hcap}(\gamma[0, \xi_1] \cup l).$$

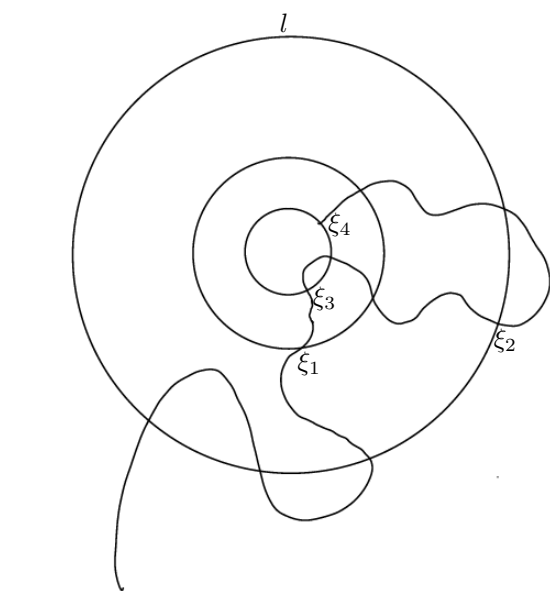


FIG 2. *the quantities in Lemma 3.9 in the case  $\xi_3 < \xi_2$*

Using the Beurling estimate and the fact that  $\Gamma \cap U \neq \emptyset$ , we can see that if  $V_1 = \gamma(0, \xi_1]$ ,  $V_2 = \gamma(0, \xi_1] \cap l$ , then

$$\lim_{y \rightarrow \infty} y \mathbb{E}^{iy} [\mathfrak{S}(B_{\tau_2}); \tau_2 < \tau_1] \leq \lim_{y \rightarrow \infty} y \mathbb{P}^{iy} \{\tau_2 < \tau_1\} \leq c \operatorname{diam}[l]^{1/2} \leq c 2^{-n/4}.$$

If  $n_0$  is chosen sufficiently large, then the right-hand side is less than  $u - s$  and hence  $\xi_2 - \xi_1 < u - s$ . □

We let  $E_1(\Gamma)$  be the event  $\{\xi_1 < \xi_3 < \xi_2 < \xi_4 < \infty\}$ ,  $E_2(\Gamma)$  the event  $\{\xi_1 < \xi_2 < \xi_3 = \xi_4 < \infty\}$ , and  $E(\Gamma) = E_1(\Gamma) \cup E_2(\Gamma) = \{\xi_4 < \infty\}$ . We define  $\hat{Z}(\Gamma)$  as follows.

$$\hat{Z}(\Gamma) = 0 \quad \text{on the complement of } E(\Gamma),$$

$$\hat{Z}(\Gamma) = 2^{(n+1)(2-d)} \quad \text{on the event } E_1(\Gamma),$$

$$\hat{Z}(\Gamma) = \operatorname{Cont}_d^+(\Gamma \cap \gamma; n \log 2) \quad \text{on the event } E_2(\Gamma).$$

Take  $n_0$  as in the previous lemma and recall that  $U = U_{0,k} = \{x + iy : -2^k \leq x < 2^k, y \geq 1\}$ . The definition is such that the following holds.

- If  $m > n_0$  and  $\Gamma \in \mathcal{Q}_m$  with  $\Gamma \cap U \neq \emptyset$ , then on the event  $E(\Gamma)$ ,

$$\operatorname{Cont}_d[\gamma[u, \infty) \cap \Gamma; m \log 2] \leq \hat{Z}(\Gamma).$$

- If  $m \geq n > n_0$  and  $\Gamma \in \mathcal{Q}_n$  with  $\Gamma \cap U \neq \emptyset$ , then on the event  $E_2(\Gamma)$ ,

$$\operatorname{Cont}_d[\gamma[u, \infty) \cap \Gamma; m \log 2] \leq \hat{Z}(\Gamma).$$

We will use the following fact which states that once one gets close to  $z$  and then leaves, one is unlikely to return. It is a quantitative expression of the fact that the double points of  $SLE_\kappa$  curve have strictly smaller fractal dimension than the curve itself. It is an immediate corollary of Corollary 2.8.

LEMMA 3.10. *There exist  $c, \beta$  such that if  $\Gamma \in \mathcal{Q}_n$  with  $\Gamma \subset \{\mathfrak{S}(z) \geq 1\}$ , then  $\mathbb{P}[E(\Gamma)] \leq c 2^{n(d-2)} 2^{-n\beta}$ .*

Arguing as in the proof of Lemma 3.5, we have on the event  $E_2(\Gamma)$ ,

$$\mathbb{E}[\hat{Z}(\Gamma) \mid \gamma_{\xi_4}] \leq c 2^{n(2-d)}.$$

Then we see that

$$\mathbb{E}[\hat{Z}(\Gamma) \mid \xi_1(\Gamma) < \infty] \leq c 2^{n(d-2)},$$

$$\mathbb{E}[\hat{Z}(\Gamma)] \leq \mathbb{P}\{\xi_1(\Gamma) < \infty\} \mathbb{E}[\hat{Z}(\Gamma) \mid \xi_1(\Gamma) < \infty] \leq cG(\Gamma) 2^{-n\beta}.$$

Let

$$\hat{Z}_n = \hat{Z}_n(U) = \sum_{m=n}^{\infty} \sum_{\Gamma \in \mathcal{Q}_m, \Gamma \subset U} \hat{Z}(\Gamma).$$

Then  $\mathbb{E}[\hat{Z}_n] \leq c2^{-\beta n}$ , and hence using the Borel-Cantell lemma, with probability one for all  $n$  sufficiently large,  $\hat{Z}_n \leq 2^{-\beta n/2}$ . To establish (31), it therefore suffices to show that there exists  $c$  such that for  $n > n_0$ ,

$$\text{Cont}_d^+[(\gamma \setminus [\gamma_u \cup \partial_n H_s]) \cap U] \leq c \hat{Z}_n.$$

To show this it suffices to show for all integers  $m \geq n$

$$\text{Cont}_d[(\gamma \setminus [\gamma_u \cup \partial_n H_s]) \cap U; m \log 2] \leq c \hat{Z}_n.$$

For each  $m \geq n > n_0$ , will cover  $\gamma \setminus [\gamma_u \cup \partial_n H_s]$  by squares  $\Gamma \in \mathcal{Q}_j$  with  $n \leq j \leq m$ . We will choose all squares  $\Gamma \in \mathcal{Q}_m$  that intersect  $H_s$  and are within distance  $2^{-m+2}$  of  $\partial H_s$ . This includes squares that intersect  $\partial H_s$ . However, for  $n \leq j < m$ , we only choose squares whose distance from  $\partial H_s$  is comparable to  $2^{-j}$ . In particular, these squares do not intersect  $\partial H_s$ .

To be precise, let  $s < u$  and assume that  $n > n_0$ . For fixed  $n < m$ , let  $\mathcal{A} = \mathcal{A}_{s,m,n}$  denote the set of  $\Gamma \in \mathcal{Q}_j(U), j = n, \dots, m$  that satisfy  $2^{-j+1} \leq \text{dist}(\Gamma, \partial H_s) \leq 2^{-j+3}$ . Let  $\mathcal{C} = \mathcal{C}_{s,m}$  denote the set of  $\Gamma \in \mathcal{Q}_m(U)$  that satisfy  $\text{dist}(\Gamma, \partial H_s) \leq 2^{-m+2}$ . We claim that for each  $m$ , the squares in  $\mathcal{A} \cup \mathcal{C}$  cover  $U \cap \partial_n H_s$ . To see this, suppose that  $z \in U \cap \partial_n H_s$ . Then  $\text{dist}(z, \partial H_s) \leq 2^{-n}$ . If  $\text{dist}(z, \partial H_s) \leq 2^{-m+2}$ , then the unique  $\Gamma \in \mathcal{Q}_m(U)$  containing  $z$  is in  $\mathcal{C}$ . If  $\text{dist}(z, \partial H_s) > 2^{-m+2}$  find  $j$  such that  $2^{-j+2} < \text{dist}(z, \partial H_s) \leq 2^{-j+3}$ . Let  $\Gamma$  be the unique square in  $\mathcal{Q}_j(U)$  that contains  $z$  and note that  $2^{-j+1} \leq \text{dist}(\Gamma, \partial_n H_s) \leq 2^{-j+3}$ .

If  $\Gamma \in \mathcal{A} \cup \mathcal{C}$ , then  $\xi_1 \leq s$ . Since  $n > n_0$ , by Lemma 3.9  $\xi_2 < u$ . Hence  $\gamma[u, \infty) \cap \Gamma \subset \gamma[\xi_4, \infty)$ . If  $\Gamma \in \mathcal{A}$  with  $\gamma[u, \infty) \cap \Gamma \neq \emptyset$ , then  $\xi_2 < \xi_3$  which means that the event  $E_2(\Gamma)$  has occurred. If  $\Gamma \in \mathcal{C}$  and  $\gamma[u, \infty) \cap \Gamma \neq \emptyset$ , we know that  $E(\Gamma)$  has occurred. Either way, we see that

$$\text{Cont}_d[\gamma[u, \infty) \cap \Gamma; m \log 2] \leq \hat{Z}_n(\Gamma).$$

**4. Proof of Theorem 3.2 .** Throughout this section  $0 < \delta \leq 1/10$ , but constants are independent of  $\delta$ .

4.1. *Some reductions*. Suppose  $\Im(z), \Im(w) \geq 1$ , and let  $J_r(z), Q_r(z) = Q_r^\delta(z)$  as in Section 3.1. Since  $Q_r(z)Q_r(w) = 0$  if  $\tau_r(z) = \infty$  or  $\tau_r(w) = \infty$ , in order to prove (23) it suffices by symmetry to prove that

$$(32) \quad \mathbb{E}[Q_r(z)Q_r(w); \tau_r(z) < \tau_r(w) < \infty] \leq c e^{-\beta r} |z - w|^{\beta-2}.$$

By (16), we know that if  $|z - w| \leq e^{-ur}$ ,

$$\begin{aligned} \mathbb{E}[Q_r(z)Q_r(w)] &\leq c e^{2r(2-d)} \mathbb{P}\{\tau_r(z) < \infty, \tau_r(w) < \infty\} \\ &\leq c |z - w|^{d-2} \\ &\leq c |z - w|^{(d-1)-2} e^{-ur}. \end{aligned}$$

Hence it suffices to find  $u > 0$  and  $c, \beta$  such that (32) holds for  $|z - w| \geq e^{-ur}$ . Let  $\alpha$  be as in (15), and suppose that  $s > 0$ . Choose  $u > 0$  with  $u[2(2-d) + \alpha] \leq \alpha s/2$ .

Then if  $|z - w| \geq e^{-ur}$ ,

$$\begin{aligned} &\mathbb{E}[Q_r(z)Q_r(w); \tau_{sr}(w) \leq \tau_r(z) \leq \tau_r(w) < \infty] \\ &\leq c e^{2r(2-d)} \mathbb{P}\{\tau_{sr}(w) \leq \tau_r(z) \leq \tau_r(w) < \infty\} \\ &\leq c e^{2r(2-d)} \mathbb{P}\{\tau_{sr-ur+ur}(w) \leq \tau_{r-ur+ur}(z) \leq \tau_{r-ur+ur}(w) < \infty\} \\ &\leq c e^{2r(2-d)} e^{2(r-ur)(d-2)} e^{-\alpha(sr-ur)} \\ &\leq c e^{ru[2(2-d)+\alpha]} e^{-\alpha sr} \leq c e^{-\alpha sr/2}. \end{aligned}$$

From this we see that in order to prove (32) it suffices to prove the following. There exist  $u > 0, s > 0, \beta > 0, c < \infty$  such that if  $\Im(z), \Im(w) \geq 1$  and  $|z - w| \geq e^{-ur}$ , then

$$(33) \quad \mathbb{E}[Q_r(z)Q_r(w); \tau_r(z) < \tau_{sr}(w) < \tau_r(w) < \infty] \leq c e^{-\beta r} |z - w|^{\beta-2}.$$

This is what we will establish in this section.

4.2. *One-point estimate*. As is often the case, an important step in getting a two-point estimate is to get a sharp one-point estimate with good control on the error terms. Much of the necessary analysis has been done for *SLE*, and we review some of the methods here.

We will consider chordal *SLE* $_\kappa$  from 1 to  $w = e^{2i\theta}$  in the unit disk  $\mathbb{D}$ , and we will study how close the path gets to the origin. We parametrize the *SLE* $_\kappa$  path  $\gamma$  using the radial parametrization. To be specific, we let  $D_t$  denote the component of  $\mathbb{D} \setminus \gamma_t$  containing the origin and  $g_t : D_t \rightarrow \mathbb{D}$  the unique conformal transformation with  $g_t(0) = 0, g_t(\gamma(t)) = 1$ . The radial parametrization is defined so that  $|g_t'(0)| = e^t$ . The total lifetime of the curve



in this parametrization,  $T$ , is finite with probability one. (If  $\kappa > 4$ ,  $T$  is not the time that the curve reaches  $w$ , but rather the time at which the curve disconnects the origin from  $w$ . Although the  $SLE$  curves continues after this time, the domain  $D_t$  does not change so we do not need to consider the path after time  $T$ .) We write  $w_t = e^{2i\theta_t} = g_t(w)$ . The path  $\gamma_t$ , and hence the transformations  $g_t$ , are determined by  $\theta_s, 0 \leq s \leq t$ . If  $t > T$ , then  $g_t = g_T$ . We let  $\mathbb{P}_\theta, \mathbb{E}_\theta$  denote probabilities and expectations given by chordal  $SLE_\kappa$  from 1 to  $e^{2i\theta}$ . The angle  $\theta_t$  satisfies a simple one-dimensional SDE. Its form is a little nicer if we consider a linear time change. If  $\hat{\theta}_t = \theta_{2at}$ , then  $\hat{\theta}_t$  satisfies the ‘‘radial Bessel equation’’

$$d\hat{\theta}_t = (1 - 2a) \cot \hat{\theta}_t dt + dB_t,$$

where  $B_t$  is a standard Brownian motion. This equation is valid until the time  $\hat{T} = T/2a$  at which  $\hat{\theta}_{\hat{T}} = \theta_T \in \{0, \pi\}$ .

The Koebe (1/4)-theorem and Schwarz lemma implies that for  $0 \leq t \leq T$ ,

$$(34) \quad e^{-t - \log 4} \leq \text{dist}[0, \gamma_t] \leq e^{-t}.$$

Let  $S_t = S_{D_t}(0; w, \gamma(t)) = S_{\mathbb{D}}(0; w_t, 1) = \sin \theta_t$ . Itô’s formula shows that

$$M_t = 1\{T > t\} e^{t(2-d)} S_t^{4a-1}$$

is a local martingale; more precisely,  $\hat{M}_t = M_{2at}$  satisfies

$$d\hat{M}_t = (4a - 1) [\cot \hat{\theta}_t] \hat{M}_t dB_t, \quad t < \hat{T}.$$

In fact,  $M_t$  is a continuous martingale with  $\mathbb{P}\{M_T = 0\} = 1$ .

Let  $\mathbb{D}_r$  denote the open disk of radius  $e^{-r}$  about the origin with closure  $\overline{\mathbb{D}_r}$ , and

$$\tau_r = \inf\{t : \text{dist}[0, \gamma_t] = e^{-r}\} = \inf\{t : \gamma(t) \in \overline{\mathbb{D}_r}\}.$$

Note that (34) implies that

$$r - 2 < r - \log 4 \leq \tau_r \leq r.$$

The measure obtained by tilting by the martingale  $M_t$  is called *two-sided radial  $SLE_\kappa$*  (from 0 to  $e^{2i\theta}$  in  $\mathbb{D}$  going through the origin stopped when it reaches the origin). We will write  $\mathbb{P}^*, \mathbb{E}^*$  for probabilities and expectations with respect to this measure. These measures depend on the initial angle  $\theta$  and we will write  $\mathbb{P}_\theta^*, \mathbb{E}_\theta^*$  if we wish to make this explicit. The quantity  $\mathbb{E}_\theta^*$  is defined by saying that if  $X$  is a random variable that depends only on  $\gamma_t$ , then

$$\mathbb{E}_\theta^*(X) = M_0^{-1} \mathbb{E}_\theta[X M_t] = [\sin \theta]^{1-4a} e^{t(2-d)} \mathbb{E}_\theta[X S_t^{4a-1} 1\{T > t\}],$$

or equivalently,

$$(35) \quad \mathbb{E}_\theta [X \mathbf{1}\{T > t\}] = e^{(d-2)t} [\sin \theta]^{4a-1} \mathbb{E}_\theta^* [X S_t^{1-4a}].$$

The Girsanov theorem shows that under the measure  $\mathbb{E}_\theta^*$ ,

$$(36) \quad d\hat{\theta}_t = 2a \cot \hat{\theta}_t dt + dW_t,$$

where, as before,  $\hat{\theta}_t = \theta_{2at}$  and  $W_t$  is a standard Brownian motion with respect to the tilted measure. This equation has an invariant probability density

$$\phi(\theta) = C_{4a}^{-1} [\sin \theta]^{4a}, \quad C_{4a} = \int_0^\pi \sin^{4a} \theta d\theta.$$

Moreover, the rate of convergence to equilibrium is exponential (see, e.g., [11, Section 2.1.1]). To be more explicit, there exists  $\alpha > 0$  such that if  $\phi_t(\theta; \theta_0)$  is the density at time  $t$  given initial condition  $\theta_0$ , then

$$(37) \quad \phi_t(\theta; \theta_0) = \phi(\theta) [1 + O(e^{-\alpha t})].$$

Implicit in this formulation is the fact that for every  $t_0 > 0$  there exists  $C = C(t_0) < \infty$  such that if  $t \geq t_0$ ,  $C^{-1} \phi(\theta) \leq \phi_t(\theta; \theta_0) \leq C \phi(\theta)$ .

If we apply this to (35) with  $X \equiv 1$ , we get

$$\mathbb{P}_\theta \{T > t\} = c_* e^{(d-2)t} [\sin \theta]^{4a-1} [1 + O(e^{-\alpha t})],$$

where

$$c_* = \int_0^\pi [\sin \theta]^{1-4a} \phi(\theta) d\theta = 2 C_{4a}^{-1}.$$

In particular, we see that for  $r \geq 1/10$ ,

$$\mathbb{P}_\theta \{T > \tau_r\} \asymp [\sin \theta]^{4a-1} e^{(d-2)r},$$

and if  $r \geq 3, 0 \leq s \leq 1/10$ , and  $T > r - 2$ , conformal invariance implies that

$$\begin{aligned} \mathbb{P}_\theta \{T > \tau_{r+s} \neq \emptyset \mid \gamma_{r-2}\} &= \mathbb{P}_{\theta_{r-2}} \{\gamma \cap g_{r-2}(\mathbb{D}_{r+s}) \neq \emptyset\} \\ &\asymp [\sin \theta_{r-2}]^{4a-1}. \end{aligned}$$

We can also phrase this in terms of the quasi-stationary distribution for  $\theta_t$ . Let  $\psi(\theta) = \frac{1}{2} \sin \theta$ . Under the measure  $\mathbb{P}$ , The random variable  $\theta_t \mathbf{1}\{T > t\}$  has an atom at 0 and has a density  $\psi_t(\theta)$  for  $0 < \theta < \pi$  satisfying

$$\mathbb{P}\{T > t\} = \int_0^\pi \psi_t(\theta) d\theta.$$

The results of the previous paragraph show that  $\psi$  is a quasi-stationary density in the sense that if  $\phi_0 \equiv \psi$ , then

$$\psi_t(\theta) = e^{t(d-2)} \psi(\theta).$$

Moreover, if  $\psi_t(\theta; \theta_0)$  denotes the density assuming initial condition  $\theta_0$ ,

$$(38) \quad \psi_t(\theta; \theta_0) = \mathbb{P}_{\theta_0}\{T > t\} \psi(\theta) [1 + O(e^{-t\alpha})] = c_* e^{t(d-2)} \psi(\theta) [1 + O(e^{-t\alpha})].$$

We write  $\mathbb{P}_\psi$  for probabilities assuming the initial density  $\psi$ . We can see

$$\mathbb{P}_\psi\{T > r\} = e^{r(d-2)}$$

PROPOSITION 4.1. *There exists  $0 < c_1 < \infty$  such that*

$$\mathbb{P}_\psi\{\tau_r < \infty\} = c_1 e^{r(d-2)} [1 + O(e^{-r})].$$

PROOF. If  $r > 0, u > 2$ , then since  $\tau_r > r - 2$ ,

$$\mathbb{P}_\psi\{\tau_{r+u} < \infty\} = \mathbb{P}_\psi\{T > r\} \mathbb{P}_\psi\{\tau_{r+u} < \infty \mid T > r\}.$$

The conformal Markov property implies that if  $T > r$ , then

$$\mathbb{P}_\psi\{\tau_{r+u} < \infty \mid \gamma_r\} = \mathbb{P}_{\theta_r}\{\gamma[0, \infty) \cap g_r(\mathbb{D}_{r+u}) \neq \emptyset\},$$

where  $\theta_r$  started according to  $\psi$ . By Lemma 2.1, there exists  $u_0$  such that if  $u > u_0$ , then on the event  $T > r$ , if  $|z| = e^{-r-u}$ ,

$$e^{-u} \exp\{-4e^{-u}\} \leq |g_r(z)| \leq e^{-u} \exp\{4e^{-u}\}.$$

Combining the last two expressions, we see that if  $r > 0$  and  $u > u_0$ , then if  $T > r$ ,

$$(39) \quad \mathbb{P}_{\theta_r}\{\tau_{u+4e^{-u}} < \infty\} \leq \mathbb{P}_{\theta_r}\{\gamma[0, \infty) \cap g_r(\mathbb{D}_{r+u}) \neq \emptyset\} \leq \mathbb{P}_{\theta_r}\{\tau_{u-4e^{-u}} < \infty\}.$$

Since  $\psi$  is the quasi-stationary density, the conditional density on  $\theta_r$  given  $T > r$  is  $\psi$ . Therefore,

$$e^{r(d-2)} \mathbb{P}_\psi\{\tau_{u+4e^{-u}} < \infty\} \leq \mathbb{P}_\psi\{\tau_{r+u} < \infty\} \leq e^{r(d-2)} \mathbb{P}_\psi\{\tau_{u-4e^{-u}} < \infty\}$$

If we replace  $r$  with  $s = r - 5e^{-u}$  and  $u$  with  $v = u + 5e^{-u}$ , we get for  $u$  sufficiently large so that  $e^{-v} \geq (4/5)e^{-u}$ ,

$$\begin{aligned} \mathbb{P}_\psi\{\tau_{r+u} < \infty\} &= \mathbb{P}_\psi\{\tau_{s+v} < \infty\} \\ &\leq e^{s(d-2)} \mathbb{P}_\psi\{\tau_{v-4e^{-v}} < \infty\} \\ &\leq e^{s(d-2)} \mathbb{P}_\psi\{\tau_u < \infty\} \\ &= e^{r(d-2)} \mathbb{P}_\psi\{\tau_u < \infty\} [1 + O(e^{-u})]. \end{aligned}$$

We get a bound in the other direction by choosing  $s = r + 5e^{-u}$  and  $v = r = e^{-5u}$ . Hence,

$$\mathbb{P}_\psi\{\tau_{r+u} < \infty\} = e^{r(d-2)}\mathbb{P}_\psi\{\tau_u < \infty\} [1 + O(e^{-u})],$$

where the error term is bounded uniformly independent of  $r$ . If we define  $L_r = \log[e^{r(2-d)}\mathbb{P}_\psi\{\tau_r < \infty\}]$ , then the above expression can be written as

$$\sup_{r \geq u} |L_r - L_u| = O(e^{-u}),$$

which implies that the limit  $L_\infty = \lim_{u \rightarrow \infty} L_u \in (-\infty, \infty)$  exists, and  $|L_u - L_\infty| = O(e^{-u})$ . The proposition follows with  $c_1 = e^{L_\infty}$ .  $\square$

**THEOREM 4.2.** *There exists  $0 < \hat{c} < \infty$  and  $\beta > 0$ , such that*

$$(40) \quad \mathbb{P}_\theta\{\tau_r < \infty\} = \hat{c} [\sin \theta]^{4a-1} e^{r(d-2)} [1 + O(e^{-r\beta})].$$

**PROOF.** As in (39),

$$\mathbb{P}_{\theta_r}\{\tau_{r+4e^{-r}} < \infty\} \leq \mathbb{P}_{\theta_r}\{\gamma[0, \infty) \cap g_r(\mathbb{D}_{2r}) \neq \emptyset\} \leq \mathbb{P}_{\theta_r}\{\tau_{r-4e^{-r}} < \infty\}.$$

By Proposition 4.1, if  $\psi$  is the invariant distribution,

$$\mathbb{P}_\psi\{\tau_{r \pm 4e^{-r}} < \infty\} = c_1 e^{r(d-2)} [1 + O(e^{-r})].$$

Combining this with (38), we see that

$$\begin{aligned} \mathbb{P}_\theta\{\tau_{2r} < \infty\} &= \mathbb{P}_\theta\{T > r\} \mathbb{P}_\theta\{\tau_{2r} < \infty \mid T > r\} \\ &= c_1 c_* e^{2r(d-2)} [1 + O(e^{-ar})]. \end{aligned}$$

$\square$

With this theorem we could *define* the chordal Green's function on  $\mathbb{D}$  by

$$G_{\mathbb{D}}(0; 1, e^{2i\theta}) = \hat{c} [\sin \theta]^{4a-1},$$

and define it for other simply connected domains by

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{\mathbb{D}}(0; 1, e^{2i\theta}),$$

where  $f : D \rightarrow \mathbb{D}$  is a conformal transformation with  $f(z) = 0$ ,  $f(w_1) = 1$ ,  $f(w_2) = e^{2i\theta}$ . In fact,

$$G_D(z; w_1, w_2) = \hat{c} \text{crad}_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1}.$$

PROPOSITION 4.3. *Let  $0 < \kappa < 8$ . There exists  $c < \infty, \alpha > 0$  such that the following is true. Suppose that  $D$  is a simply connected domain and  $\gamma$  is a chordal  $SLE_\kappa$  path from  $w_1$  to  $w_2$  in  $D$ . Suppose that  $z \in D$ ,  $R = \text{dist}(z, \partial D)$  and  $G = G_D(z; w_1, w_2)$ . Then if  $e^{-r} \leq R/2$ ,*

$$\left| G^{-1} e^{r(2-d)} \mathbb{P} \{ \text{dist}(\gamma, z) \leq e^{-r} \} - 1 \right| \leq c [e^{-r}/R]^\alpha.$$

*In particular, there exists  $c < \infty$  such that if  $0 < r < s$  then,*

$$(41) \quad \left| \mathbb{P} \{ \text{dist}(\gamma, z) \leq e^{-r} \} - e^{(s-r)(2-d)} \mathbb{P} \{ \text{dist}(\gamma, z) \leq e^{-s} \} \right| \leq c [e^{-r}/R]^{2-d+\alpha}.$$

PROOF. Without loss of generality, we assume  $z = 0$ , and by scaling we may assume that  $R = 1$ . Let  $F : D \rightarrow \mathbb{D}$  be the conformal transformation with  $F(0) = 0, F(w_1) = 1, F(w_2) = e^{2i\theta}$  where  $\sin \theta = S_D(z; w_1, w_2)$ . The Schwarz lemma and Koebe (1/4)-theorem imply that  $1/4 \leq |F'(0)| \leq 1$ . Note that  $G = \hat{c} |F'(0)|^{2-d} [\sin \theta]^{4a-1}$ . Proposition 2.1 implies that there exists universal  $r_0$  such that if  $r > r_0$ ,

$$|F'(0)| |z| \exp\{-4|z|\} \leq |F(z)| \leq |F'(0)| |z| \exp\{4|z|\}.$$

Therefore, by conformal invariance, if  $q = -\log |F'(0)|$

$$\mathbb{P}_\theta \{ \tau_{r+4e^{-r+q}} < \infty \} \leq \mathbb{P} \{ \text{dist}(\gamma, z) \leq e^{-r} \} \leq \mathbb{P}_\theta \{ \tau_{r-4e^{-r+q}} < \infty \}.$$

But (40) tells us that

$$\begin{aligned} \mathbb{P}_\theta \{ \tau_{r \pm 4e^{-r+q}} < \infty \} &= \hat{c} [\sin \theta]^{4a-1} e^{(r+q)(d-2)} [1 + O(e^{-\alpha r})] \\ &= G e^{(r+q)(d-2)} [1 + O(e^{-\alpha r})]. \end{aligned}$$

□

With these results we can follow the proof in [10, Section 3], which proves the corresponding result with distance replaced by conformal radius, to conclude (17). We need to replace Lemma [10, 2.16], with the corresponding result for the distance. The necessary lemma, written in the notation of this paper, is the following.

LEMMA 4.4. *There exist  $\alpha > 0, c < \infty$  such that if  $0 < s < u < 1$  and  $r \geq 3$ ,*

$$\mathbb{P}_\theta \{ \tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr} \} \leq c [\sin \theta]^{4a-1} e^{r(d-2)} e^{-\alpha tr},$$

*where  $t = \min\{1 - u, u - s\}$ .*

PROOF. A corresponding result was proved for two-sided radial  $SLE_\kappa$  in [6]. In particular, there exist  $c, \alpha$  such that

$$\mathbb{P}_\theta^* \{ \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr} \} \leq c e^{\alpha(s-u)r}.$$

In particular, since  $\tau_r > r - 2$ ,

$$\mathbb{P}_\theta^* \{ \gamma(\tau_{ur}, r - 2) \not\subset \mathbb{D}_{sr} \} \leq c e^{\alpha(s-u)r}.$$

Using the definition of the measure  $\mathbb{P}_\theta^*$  we see that this implies that

$$\begin{aligned} \mathbb{E}_\theta [ [\sin \theta_{r-2}]^{1-4a}; T > r - 2, \gamma(\tau_{ur}, r - 2) \not\subset \mathbb{D}_{sr} ] \\ \leq c [\sin \theta]^{4a-1} e^{r(d-2)} e^{\alpha(s-u)r}. \end{aligned}$$

However, if  $T > r - 2$ ,  $\mathbb{P}\{\tau_r < \infty \mid \gamma_{r-2}\} \asymp [\sin \theta_{r-2}]^{4a-1}$ . Hence,

$$(42) \quad \mathbb{P}_\theta \{ \tau_r < \infty, \gamma(\tau_{ur}, r - 2) \not\subset \mathbb{D}_{sr} \} \leq c [\sin \theta]^{4a-1} e^{r(d-2)} e^{\alpha(s-u)r}.$$

On the event  $E := \{T > r - 2, \gamma(\tau_{ur}, r - 2) \subset \mathbb{D}_{sr}\}$ , topological considerations (see [6, Lemma 2.3]) imply that there is a unique subarc  $l$  of  $\partial\mathbb{D}_{ur} \cap D_{r-2}$  such that removal of  $l$  disconnects 0 from  $\partial\mathbb{D}_{sr}$  in  $D_{r-2}$ . The point  $\gamma(r - 2)$  may be in  $l$  or in either of the connected components of  $D_{r-2} \setminus l$ . In any of these cases, if  $\sigma = \inf\{t \geq r - 2 : \gamma(t) \in \bar{l}\}$ , then the event  $E \cap \{\tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr}\}$  is contained in the event  $E \cap \{\sigma < \tau_r < \infty\}$ . As in (11), on the event  $E \cap \{\sigma < \infty, \sigma < \tau_r\}$ ,

$$\mathbb{P}\{\tau_r < \infty \mid \gamma_\sigma\} \leq c e^{\alpha(u-1)r}.$$

Hence,

$$\begin{aligned} & \mathbb{P}_\theta \{ \tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr}, \gamma(\tau_{ur}, r - 2) \subset \mathbb{D}_{sr} \} \\ & \leq \mathbb{P}_\theta(E) \mathbb{P}_\theta \{ \tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr} \mid E \} \\ & \leq \mathbb{P}_\theta\{T > r - 2\} \mathbb{P}_\theta \{ \tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr} \mid E \} \\ (43) \quad & \leq c [\sin \theta]^{4a-1} e^{r(d-2)} e^{\alpha r(u-1)}. \end{aligned}$$

The lemma follows from (42) and (43). □

The proof of (17) follows that in [10, Section 3]. We will not give all the details, but we sketch the argument using the notation of this paper. We need to prove the existence of the limit

$$G(z, w) = \lim_{r, s \rightarrow \infty} e^{r(2-d)} e^{s(2-d)} \mathbb{P}\{\tau_r(z), \tau_s(w) < \infty\},$$

PROPOSITION 4.5. *For every  $z, w \in \mathbb{H}$ ,*

$$\lim_{r,s \rightarrow \infty} e^{r(2-d)} e^{s(2-d)} \mathbb{P}\{\tau_r(z) < \tau_s(w) < \infty\} = G(z) \mathbb{E}^* [G_{H_T}(w; z, \infty)],$$

where  $\mathbb{E}^*$  denotes expectation with respect to two-sided radial SLE to  $z$ .

PROOF (SKETCH). Let  $\tau_r = \tau_r(z)$ . Arguing as in (15), we see that

$$\lim_{r,s \rightarrow \infty} e^{r(2-d)} e^{s(2-d)} \mathbb{P}\{\tau_{s/2}(w) < \tau_r < \tau_s(w) < \infty\} = 0.$$

Also, by (15) and Proposition 4.3.

$$\lim_{r,s \rightarrow \infty} e^{r(2-d)} \mathbb{P}\{\tau_r < \tau_{s/2}(w)\} = \lim_{r \rightarrow \infty} e^{r(2-d)} \mathbb{P}\{\tau_r < \infty\} = G(z).$$

By Proposition 4.3, there exists  $\alpha$  such that if  $\tau < \tau_{s/2}(w)$ ,

$$\mathbb{P}\{\tau_s(w) < \infty \mid \gamma_\tau\} = e^{s(d-2)} G_{H_{\tau_r}}(w; \gamma(\tau_r), \infty) [1 + O(e^{-\alpha s})].$$

Therefore,

$$\begin{aligned} \lim_{r,s \rightarrow \infty} e^{r(2-d)} e^{s(2-d)} G(z)^{-1} \mathbb{P}\{\tau_r < \tau_s(w) < \infty\} = \\ \lim_{r,s \rightarrow \infty} \mathbb{E} [G_{H_{\tau_r}}(w; \gamma(\tau_r), \infty) 1\{\tau_r < \tau_{s/2}(w)\} \mid \tau_r < \infty]. \end{aligned}$$

Hence we need to show that the right-hand side equals  $\mathbb{E}^*[G_{H_T}(w; z, \infty)]$ .

We assume that the curve has the radial parametrization heading to  $z$ . We use Lemma 4.4 to see that as  $r, s \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{E} [G_{H_{\tau_r}}(w; \gamma(\tau_r), \infty) 1\{\tau_r < \tau_{s/2}(w)\} \mid \tau_r < \infty] \\ \sim \mathbb{E} [G_{H_{r/2}}(w; \gamma(r/2), \infty) 1\{\tau_r < \tau_{s/2}(w)\} \mid \tau_r < \infty] \\ \sim \mathbb{E} [G_{H_{r/2}}(w; \gamma(r/2), \infty) \mathbb{P}\{\tau_r < \tau_{s/2}(w) \mid \gamma_{r/2}\} \mid \tau_r < \infty]. \end{aligned}$$

We now use Proposition 4.3 to see that the weighting by  $\mathbb{P}\{\tau_r < \tau_{s/2}(w) \mid \gamma_{r/2}\}$  is the same up to small error as weighting by  $G_{H_{s/2}}(z; \gamma(s/2), \infty)$  which is the weighting which defines two-sided SLE going to  $z$ . The arguments for justifying this are the same whether one uses conformal radius or  $\tau_r$  as the stopping time, so the proof in [10, Section 3] works here.  $\square$

**Remark** The same method shows that we can define  $n$ -point Green's function and we expect that

$$\mathbb{E}[\Theta(D)^n] = \int_{D^n} G(z_1, \dots, z_n) dA(z_1) \cdots dA(z_n).$$

At the moment we cannot prove it because we have no upper bound for  $G(z_1, \dots, z_n)$ .

4.3. *Proof of (33).* We will now prove (33) for an appropriate  $0 < u \leq 1/4$  that we will define below. We assume that  $\Im(z), \Im(w) \geq 1$  and  $|z - w| > e^{-r/4}$ . It suffices to prove the result for  $r > 4$  and hence  $|z - w| > e^{-(r-2)/2}$ .

Let  $0 < q < 1/8$  be a parameter that we will choose later. Let  $\tau_r = \tau_r(z)$ ,  $H = H_{\tau_r}(z)$ ,  $l_{3/4} = l_{3/4}(r, z)$ ,  $\lambda = \lambda(r, z, 3/4)$ ,  $\mathcal{B}_u = \mathcal{B}_u(r, z)$ ,  $V_u = V_u(r, z)$  be as in Section 2.4. Recall that we are assuming that  $|z - w| > e^{-r/2}$  and hence  $w \notin \mathcal{B}_{1/2}$ . Let  $\mathcal{I}_r(z, w)$  be the indicator function of the event that  $\tau_r < \tau_{qr}(w)$  and  $w$  is in the unbounded component of  $H \setminus l_{3/4}$  and let  $\mathcal{J}_r(z, w)$  be the indicator function of the event that  $w$  is in the bounded component of  $H \setminus l_{3/4}$ .

We consider two cases. First suppose that  $\mathcal{J}_r(z, w) = 1$ . Then  $w, z$  are both in the bounded component of  $H \setminus l_{3/4}$ . Since  $w \notin \mathcal{B}_{1/2}$ , there is a unique subarc  $l'$  of  $\partial V_{3/4} \cap H$  such that  $z, w$  are in different components of  $H \setminus l'$ . Since  $w$  is in the bounded component of  $H \setminus l_{3/4}$ ,  $l' \neq l_{3/4}$ . In particular,  $z$  is in the unbounded component of  $H \setminus l'$ . The Beurling estimate implies that the probability that a Brownian motion starting at  $w$  reaches  $l'$  without leaving  $H$  is bounded above by  $c e^{-r/8}$ . Hence  $S_{\tau_r}(w) \leq c e^{-r/8}$ , and therefore,

$$\begin{aligned} \mathbb{P}\{\tau_r(w) < \infty \mid \gamma_{\tau_r}\} &\leq c G_H(w; \gamma_{\tau_r}, \infty) e^{r(d-2)} \\ &\leq c S_{\tau_r}(w)^{4a-1} \text{dist}(w, \gamma_{\tau_r})^{d-2} e^{r(d-2)} \\ &\leq c e^{-pr} \text{dist}(w, \gamma_{\tau_r})^{d-2} e^{r(d-2)}. \end{aligned}$$

Where  $p = (4a - 1)/8 > 0$ . We know from (15) that

$$\mathbb{P}\{\text{dist}(w, \gamma_{\tau_r}) \leq e^s, \tau_r < \infty\} \leq c |w - z|^{d-2} e^{r(d-2)} e^{s(d-2)}.$$

By summing over positive integers  $s \leq r$ , we get

$$\mathbb{P}\{\tau_r < \tau_r(w) < \infty, \mathcal{J}_r(z, w) = 1\} \leq cr |w - z|^{d-2} e^{2r(d-2)} e^{-pr}.$$

In particular, if  $|z - w| \geq e^{-ur}$ , where  $u = p/[3(2 - d)]$ ,

$$\mathbb{P}\{\tau_r(z) < \tau_r(w) < \infty, \mathcal{J}_r(z, w) = 1\} \leq c e^{2r(d-2)} e^{-pr/2},$$

which implies that if  $|z - w| \geq e^{-ur}$ ,

$$\mathbb{E}[Q_r(z) Q_r(w) \mathcal{J}_r(z, w)] \leq c e^{-pr/2}.$$

For the remainder we will assume that  $\mathcal{I}_r(z, w) = 1$ . Let  $\sigma = \sigma_{3/4}(r - 2, z)$  as in Section 2.4. Let  $\tilde{Q}_r(z)$  be the analogue of  $Q_r(z)$  for the curve stopped at time  $\sigma$ ,

$$\tilde{Q}_r(z) = e^{r(2-d)} \left[ 1 \{\tau_r(z) < \sigma\} - e^{\delta(2-d)} 1 \{\tau_{r+\delta}(z) < \sigma\} \right].$$



To establish our estimate, we will show that

$$(44) \quad \left| \mathbb{E} \left[ \tilde{Q}_r(z) Q_r(w) \mathcal{I}_r(z, w) \right] \right| \leq c e^{-\beta r},$$

and

$$(45) \quad \mathbb{E} \left[ |Q_r(z) - \tilde{Q}_r(z)| |Q_r(w)| \mathcal{I}_r(z, w) \right] \leq c e^{-\beta r},$$

which together imply that

$$|\mathbb{E} [Q_r(z) Q_r(w) \mathcal{I}_r(z, w)]| \leq c e^{-\beta r}.$$

To prove (44), note that since  $\tilde{Q}_r(z) \mathcal{I}_r(z, w)$  is  $\gamma_\sigma$  measurable,

$$\mathbb{E} \left[ \tilde{Q}_r(z) Q_r(w) \mathcal{I}_r(z, w) \right] = \mathbb{E} \left[ \tilde{Q}_r(z) \mathcal{I}_r(z, w) \mathbb{E}(Q_r(w) \mid \gamma_\sigma) \right],$$

and hence,

$$\left| \mathbb{E} \left[ \tilde{Q}_r(z) Q_r(w) \mathcal{I}_r(z, w) \right] \right| \leq \mathbb{E} \left[ |\tilde{Q}_r(z)| \mathcal{I}_r(z, w) |\mathbb{E}(Q_r(w) \mid \gamma_\sigma)| \right].$$

We appeal to (41) to see that

$$\begin{aligned} |\mathbb{E}(Q_r(w) \mid \gamma_\sigma)| &\leq c e^{(2-d)r} [e^{-r} / \text{dist}(w, \partial H_\sigma)]^{2-d+\alpha} \\ &\leq c \exp\{[(2-d) + (q-1)(2-d+\alpha)]r\}. \end{aligned}$$

In particular, if  $q$  is chosen sufficiently small so that  $q(2-d) \leq \alpha(1-q)/2$ ,

$$|\mathbb{E}(Q_r(w) \mid \gamma_\sigma)| \leq c e^{-\alpha r/2},$$

and hence

$$\begin{aligned} \left| \mathbb{E} \left[ \tilde{Q}_r(z) Q_r(w) \mathcal{I}_r(z, w) \right] \right| &\leq c e^{-r\alpha/2} \mathbb{E} \left[ |\tilde{Q}_r(z)| \mathcal{I}_r(z, w) \right] \\ &\leq c e^{-r\alpha/2} e^{r(2-d)} \mathbb{P}[\mathcal{I}_r(z, w)] \leq c e^{-r\alpha/2}. \end{aligned}$$

For (45), we observe that if  $Q_r(z) \neq \tilde{Q}_r(z)$ , then  $\text{dist}(z, \gamma) < \text{dist}(z, \gamma_\sigma)$ . In other words,

$$\begin{aligned} \mathbb{E} \left[ |Q_r(z) - \tilde{Q}_r(z)| |Q_r(w)| \mathcal{I}_r(z, w) \right] &\leq \\ &c e^{2r(2-d)} \mathbb{P}\{\mathcal{I}_r(z, w) = 1, \text{dist}(z, \gamma) < \text{dist}(z, \gamma_\sigma), \tau_r(w) < \infty\}. \end{aligned}$$

We know that  $\mathbb{P}\{\mathcal{I}_r(z, w) = 1\} \leq c e^{r(d-2)}$ . Hence it suffices to show that we can find  $q, \beta > 0$  and  $c < \infty$  such that that on the event  $\{\mathcal{I}_r(z, w) = 1\}$ ,

$$\mathbb{P}\{\rho < \infty, \tau_r(w) < \infty \mid \gamma_\sigma\} \leq c e^{r(2-d)} e^{-r\beta},$$

where  $\rho = \inf\{t \geq \sigma : |\gamma(t) - z| = \text{dist}(z, \gamma_\sigma)\}$ . For every integer  $k$  with  $qr + 1 < k < r - 1$  we consider the event

$$E_k = \{\tau_k(w) < \rho < \tau_{k+1}(w) < \tau_r(w) < \infty\}.$$

We claim that there exists  $c, \alpha$  such that on the event  $\{\mathcal{I}_r(z, w) = 1\}$

$$(46) \quad \mathbb{P}(E_k \mid \gamma_\sigma) \leq c e^{(2-d)(q-1)r} e^{-\alpha r}.$$

Indeed, recall that on the event on the event  $\{\mathcal{I}_r(z, w) = 1\}$ ,  $\text{dist}(w, \partial H_\sigma) \geq e^{-qr}$ . if we use  $\tilde{\mathbb{P}}$  to denote conditional probabilities given  $\gamma_\sigma$ , then

$$\tilde{\mathbb{P}}\{\tau_k(w) < \infty\} \leq c e^{(d-2)(k-qr)},$$

$$\tilde{\mathbb{P}}\{\rho < \infty \mid \tau_k(w) < \infty\} \leq c e^{-\alpha r},$$

$$\tilde{\mathbb{P}}\{\tau_r(w) < \infty \mid \tau_k(w) < \rho < \tau_{k+1}(w) < \infty\} \leq c e^{(d-2)(r-k)}.$$

By summing (46) over  $k$ , and then choosing  $q$  sufficiently small, we see that

$$\tilde{\mathbb{P}}\{\rho < \infty, \tau_r(w) < \infty\} \leq c r e^{(2-d)(q-1)r} e^{-\alpha r} \leq c e^{r(d-2)} e^{-\alpha r/2}.$$

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