A Probabilistic Solution to the Stroock-Williams Equation

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We consider the initial boundary value problem

$$\begin{aligned} &u_t = \mu u_x + \tfrac{1}{2} u_{xx} \quad (t > 0,\, x \geq 0) \\ &u(0,x) = f(x) \quad (x \geq 0) \\ &u_t(t,0) = \nu\, u_x(t,0) \quad (t > 0) \end{aligned}$$

of Stroock and Williams [12] where $\mu, \nu \in I\!R$ and the boundary condition is not of Feller's type when $\nu < 0$. We show that when f belongs to C_b^1 with $f(\infty) = 0$ then the following probabilistic representation of the solution is valid

$$u(t,x) = \mathsf{E}_x\big[f(X_t)\big] - \mathsf{E}_x\Big[f'(X_t) \int_0^{\ell_t^0(X)} e^{-2(\nu-\mu)s}\, ds\Big]$$

where X is a reflecting Brownian motion with drift μ and $\ell^0(X)$ is the local time of X at 0. The solution can be interpreted in terms of X and its creation in 0 at rate $\kappa \ell^0(X)$ where $\kappa = 2|\nu - \mu| \vee 0$. Invoking the law of $(X_t, \ell_t^0(X))$ this also yields a closed integral formula for u expressed in terms of μ, ν and f.

1. Introduction

In this paper we consider the initial boundary value problem

$$u_t = \mu u_x + \tfrac{1}{2} u_{xx} \quad (t > 0,\, x \geq 0) \tag{1.1}$$

$$u(0,x) = f(x) \quad (x \geq 0) \tag{1.2}$$

$$u_t(t,0) = \nu\, u_x(t,0) \quad (t > 0) \tag{1.3}$$

of Stroock and Williams [12] (see also [13], [14], [8], [9]) where $\mu, \nu \in I\!R$ and the boundary condition is not of Feller's type when $\nu < 0$ (cf. [2], [3], [4]). If $\nu > 0$ then it is known that the solution to (1.1)-(1.3) with $f \in C_b([0,\infty))$ can be represented as

$$u(t,x) = \mathsf{E}_x\big[f(\tilde{X}_t)\big] \tag{1.4}$$

where $\tilde{X}$ starts at x under P_x, behaves like Brownian motion with drift μ when in $(0,\infty)$, and exhibits a sticky boundary behaviour at 0. The process $\tilde{X}$ can be constructed by a familiar time change of the reflecting Brownian motion X with drift μ (the inverse of the

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running time plus the local time of X at 0 divided by ν) forcing it to spend more time at 0 (cf. [6, p. 186]). If $\nu = 0$ then (1.4) remains valid with $\tilde{X}$ being absorbed at 0 (corresponding to the limiting case of infinite stickiness). If $\nu < 0$ then Feller's semigroup approach (cf. [5], [15], [2], [3], [4]) is no longer applicable since the speed measure of $\tilde{X}$ cannot be negative. Stroock and Williams [12] show that the minimum principle breaks down in this case (non-negative f can produce negative u) so that the solution to (1.1)-(1.3) cannot be represented by (1.4) where $\tilde{X}$ is a strong Markov process which behaves like Brownian motion with drift μ when in $(0, \infty)$ (for connections with Feller's Brownian motions see [7, Section 5.7]).

Motivated by this peculiarity Stroock and Williams [12] show that the solution to (1.1)-(1.3) is still generated by a semigroup of operators when $\nu < 0$ and they characterise non-negative solutions by means of the Riccati equation. This leads to subspaces of functions f for which (1.4) remains valid with the same time-changed Brownian motion X with drift μ that now jumps into $(0, \infty)$ or possibly to a coffin state just before hitting 0. This representation of the solution is applicable when $f(0) = \int_0^\infty f(y)g(y)\,dy$ where g is the minimal non-negative solution to the Riccati equation. For more details and further fascinating developments along these lines see [12], [13], [14], [8], [9].

Inspired by these insights in this paper we develop an entirely different approach to solving (1.1)-(1.3) probabilistically that applies to smooth initial data f vanishing at ∞ with no further requirement on its shape. Firstly, exploiting higher degrees of smoothness of the solution u in the interior of the domain (which is a well-known fact from the theory of parabolic PDEs) we reduce the *sticky* boundary behaviour at 0 to (i) a *reflecting* boundary behaviour when $\nu = \mu$ and (ii) an *elastic* boundary behaviour when $\nu \neq \mu$. Secondly, writing down the probabilistic representations of the solutions to the resulting initial boundary value problems expressed in terms of the reflecting Brownian motion with drift μ and its local time at 0, choosing joint realisations of these processes where the initial point is given explicitly so that the needed algebraic manipulations are possible (making use of the extended Lévy's distributional theorem), we find that the following probabilistic representation of the solution is valid

$$u(t,x) = \mathsf{E}_x\big[F(X_t, \ell_t^0(X))\big] \tag{1.5}$$

where X is a reflecting Brownian motion with drift μ starting at x under P_x, and $\ell^0(X)$ is the local time of X at 0. The function F is explicitly given by

$$F(x,\ell) = f(x) - f'(x)\int_0^\ell e^{-2(\nu-\mu)s}\,ds \tag{1.6}$$

for $x \geq 0$ and $\ell \geq 0$. The derivation applies simultaneously to all μ and ν with no restriction on the sign of ν, and the process X (with its local time) plays the role of a fundamental solution in this context (a building block for all other solutions).

Since $(X, \ell^0(X))$ is a Markov process we see that the solution u is generated by the semigroup of transition operators $(\mathsf{P}_t)_{t\geq 0}$ acting on f by means of (1.5) and (1.6) (in the reverse order). Moreover, it is clear from (1.5) and (1.6) that the solution can be interpreted in terms of X and its creation in 0 at rate $\kappa\ell^0(X)$ where $\kappa = 2|\nu-\mu| \vee 0$. Note that this also holds when $\nu < 0$ in which case the Feller's semigroup approach based on the probabilistic representation (1.4) is not applicable. Finally, invoking the law of $(X_t, \ell_t^0(X))$ we derive a closed integral formula for u expressed in terms of μ, ν and f. Integrating further by parts yields a closed formula for u where smoothness of f is no longer needed.

2. Result and proof

Consider the initial boundary value problem (1.1)-(1.3) and recall that $C_b^1([0,\infty))$ denotes the family of C^1 functions f on $[0,\infty)$ such that f and f' are bounded on $[0,\infty)$. Recall also that the standard normal density and tail distribution functions are given by $\varphi(x) = (1/\sqrt{2\pi})\, e^{-x^2/2}$ and $\Psi(x) = 1-\Phi(x) = \int_x^\infty \varphi(y)\, dy$ for $x \in I\!R$ respectively. The main result of the paper may be stated as follows.

Theorem 1. (i) *If* $f \in C_b^1([0,\infty))$ *with* $f(\infty) = 0$ *then there exists a unique solution* u *to* (1.1)-(1.3) *satisfying* $u \in C^\infty((0,\infty)\times[0,\infty))$ *with* $u, u_x \in C_b([0,T]\times[0,\infty))$ *for* $T > 0$ *and* $u(t,\infty) = 0$ *for* $t > 0$.

(ii) *The solution* u *admits the following probabilistic representation*

$$u(t,x) = \mathsf{E}_x\big[f(X_t)\big] - \mathsf{E}_x\Big[f'(X_t)\int_0^{\ell_t^0(X)} e^{-2(\nu-\mu)s}\, ds\Big] \tag{2.1}$$

where X *is a reflecting Brownian motion with drift* μ *starting at* x *under* P_x, *and* $\ell^0(X)$ *is the local time of* X *at* 0.

(iii) *The solution* u *admits the following integral representation*

$$u(t,x) = \int_0^\infty f(y)\, G(t;x,y)\, dy - \int_0^\infty f'(y)\, H(t;x,y)\, dy \tag{2.2}$$

where the kernels G *and* H *are given by*

$$G(t;x,y) = \frac{1}{\sqrt{t}}\Big[e^{2\mu y}\,\varphi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big) + \varphi\Big(\frac{x-y+\mu t}{\sqrt{t}}\Big) - 2\mu\, e^{2\mu y}\,\Psi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big)\Big] \tag{2.3}$$

$$\begin{aligned} H(t;x,y) &= \frac{e^{2\mu y}}{\nu-\mu}\Big[(2\nu-\mu)\, e^{2(\nu-\mu)(x+y+\nu t)}\,\Psi\Big(\frac{x+y+(2\nu-\mu)t}{\sqrt{t}}\Big) \\ &\qquad - \mu\Psi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big)\Big] \quad \text{if } \nu \neq \mu \\ &= 2e^{2\mu y}\Big[\big(1+\mu(x+y+\mu t)\big)\,\Psi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big) - \mu\sqrt{t}\,\varphi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big)\Big] \quad \text{if } \nu = \mu \end{aligned} \tag{2.4}$$

for $t > 0$ *and* $x, y \geq 0$.

Proof. Let $f \in C_b^1([0,\infty))$ with $f(\infty) = 0$ be given and fixed. We first show that any solution u to (1.1)-(1.3) satisfying $u \in C^\infty((0,\infty)\times[0,\infty))$ with $u, u_x \in C_b([0,T]\times[0,\infty))$ for $T > 0$ and $u(t,\infty) = 0$ for $t > 0$ admits the probabilistic representation (2.1).

1. Setting $v = u_x$ and differentiating both sides in (1.1) with respect to x we see that v solves the same equation

$$v_t = \mu v_x + \tfrac{1}{2} v_{xx} \quad (t > 0,\ x \geq 0)\,. \tag{2.5}$$

Moreover, differentiating both sides in (1.2) with respect to x we find that

$$v(0,x) = f'(x) \quad (x \geq 0)\,. \tag{2.6}$$

Finally, combining (1.3) with (1.1) we see that (1.3) reads as follows

$$v_x(t,0) = \lambda v(t,0) \quad (t>0) \tag{2.7}$$

where we set $\lambda = 2(\nu-\mu)$. In this way we have obtained the initial boundary value problem (2.5)-(2.7) for v. Note that the boundary condition (2.7) corresponds to (i) a reflecting boundary behaviour when $\lambda = 0$ and (ii) an elastic boundary behaviour when $\lambda \neq 0$. Setting

$$B_t^{-\mu} = B_t - \mu t \quad \& \quad S_t^{-\mu} = \sup_{0\le s\le t} B_s^{-\mu} \tag{2.8}$$

for $t \ge 0$ where B is a standard Brownian motion, and denoting by $R^{\mu,x}$ a reflecting Brownian motion with drift μ starting at x in $[0,\infty)$, it is known that the classic Lévy's distributional theorem (see [11, p. 240]) extends as follows

$$(x\vee S^{-\mu} - B^{-\mu}, x\vee S^{-\mu} - x) \stackrel{\text{law}}{=} (R^{\mu,x}, \ell^0(R^{\mu,x})) \tag{2.9}$$

where $\ell^0(R^{\mu,x})$ is the local time of $R^{\mu,x}$ at 0 (for a formal verification based on Skorokhod's lemma see the proof of Theorem 3.1 in [10]). Identifying

$$X_t^x := x\vee S_t^{-\mu} - B_t^{-\mu} \quad \& \quad \ell_t^0(X^x) = x \vee S_t^{-\mu} - x \tag{2.10}$$

in accordance with (2.9) above, we claim (cf. [6, pp. 183–184]) that the solution v to the problem (2.5)-(2.7) admits the probabilistic representation

$$v(t,x) = \mathsf{E}\big[e^{-\lambda \ell_t^0(X^x)} f'(X_t^x)\big] \tag{2.11}$$

for $t \ge 0$ and $x \ge 0$ (for multi-dimensional extensions see [1, Section 2]).

2. To verify (2.11) we can make use of standard arguments by letting time run backwards and applying Itô's formula to v composed with $(t-s, X_s^x)$ and multiplied by $e^{-\lambda \ell_s^0(X^x)}$ for $s \in [0,t)$ where $t>0$ and $x \ge 0$ are given and fixed. This yields

$$\begin{aligned}
e^{-\lambda \ell_s^0(X^x)} v(t-s, X_s^x) &= v(t,x) + \int_0^s (-\lambda) e^{-\lambda \ell_r^0(X^x)} v(t-r, X_r^x)\, d\ell_r^0(X^x) \\
&\quad + \int_0^s e^{-\lambda \ell_r^0(X^x)} (-v_t)(t-r, X_r^x)\, dr \\
&\quad + \int_0^s e^{-\lambda \ell_r^0(X^x)} v_x(t-r, X_r^x)\, d(x\vee S_r^{-\mu} - B_r^{-\mu}) \\
&\quad + \frac{1}{2}\int_0^s e^{-\lambda \ell_r^0(X^x)} v_{xx}(t-r, X_r^x)\, d\big\langle X^x, X^x\big\rangle_r \\
&= v(t,x) + \int_0^s e^{-\lambda \ell_r^0(X^x)} (-\lambda v + v_x)(t-r, X_r^x)\, d(x\vee S_r^{-\mu}) \\
&\quad + \int_0^s e^{-\lambda \ell_r^0(X^x)} (-v_t + \mu v_x + \tfrac{1}{2} v_{xx})(t-r, X_r^x)\, dr \\
&\quad - \int_0^s e^{-\lambda \ell_r^0(X^x)} v_x(t-r, X_r^x)\, dB_r
\end{aligned} \tag{2.12}$$

$$= v(t,x) - \int_0^s e^{-\lambda \ell_r^0(X^x)} v_x(t-r, X_r^x)\, dB_r$$

since $d(x \vee S_r^{-\mu})$ is zero off the set of all r at which $X_r^x \neq 0$, while $(-\lambda v + v_x)(t-r, X_r^x) = 0$ for $X_r^x = 0$ by (2.7) above, so that the integral with respect to $d(x \vee S_r^{-\mu})$ is equal to zero. Note also that $d\langle X^x, X^x \rangle_r = dr$ since $r \mapsto x \vee S_r^{-\mu}$ is increasing and thus of bounded variation while in the final equality we also use (2.5). From (2.12) we see that

$$v(t,x) = e^{-\lambda \ell_s^0(X^x)} v(t-s, X_s^x) + M_s \tag{2.13}$$

where $M_s = \int_0^s e^{-\lambda \ell_r^0(X^x)} v_x(t-r, X_r^x) dB_r$ is a continuous local martingale for $s \in [0,t)$. Choose a localisation sequence of stopping times $(\sigma_n)_{n \geq 1}$ for M (meaning that M stopped at σ_n is a martingale for each $n \geq 1$ and $\sigma_n \uparrow \infty$ as $n \to \infty$), take any sequence $s_n \uparrow t$ as $n \to \infty$, and set $\tau_n := \sigma_n \wedge s_n$ for $n \geq 1$. Then the optional sampling theorem yields

$$\begin{aligned} v(t,x) &= \mathsf{E}\big[e^{-\lambda \ell_{\tau_n}^0(X^x)} v(t-\tau_n, X_{\tau_n}^x)\big] + \mathsf{E} M_{\tau_n} \\ &= \mathsf{E}\big[e^{-\lambda \ell_{\tau_n}^0(X^x)} v(t-\tau_n, X_{\tau_n}^x)\big] \to \mathsf{E}\big[e^{-\lambda \ell_t^0(X^x)} v(0, X_t^x)\big] = \mathsf{E}\big[e^{-\lambda \ell_t^0(X^x)} f'(X_t^x)\big] \end{aligned} \tag{2.14}$$

as $n \to \infty$ by the dominated convergence theorem and (2.6) above where we use that $v \in C_b([0,T] \times [0,\infty))$ for $T \geq t$ and $\mathsf{E} e^{|\lambda| \ell_t^0(X^x)} < \infty$ for $t > 0$ in view of (2.10) above. This establishes (2.11) as claimed.

3. Recalling that $v = u_x$ and $u(t,\infty) = 0$ we find using (2.10) and (2.11) that

$$\begin{aligned} u(t,x) &= -\int_x^\infty u_x(t,y)\, dy + u(t,\infty) \\ &= -\int_x^\infty v(t,y)\, dy = -\int_x^\infty \mathsf{E}\big[e^{-\lambda(y \vee S_t^{-\mu} - y)} f'(y \vee S_t^{-\mu} - B_t^{-\mu})\big]\, dy \\ &= -\int_x^\infty \mathsf{E}\big[f'(y - B_t^{-\mu})\, I(S_t^{-\mu} \leq y) + e^{-\lambda(S_t^{-\mu} - y)} f'(S_t^{-\mu} - B_t^{-\mu})\, I(S_t^{-\mu} > y)\big]\, dy \\ &= -\mathsf{E}\Big[\int_{x \vee S_t^{-\mu}}^\infty f'(y - B_t^{-\mu})\, dy\Big] - \mathsf{E}\Big[\int_x^{x \vee S_t^{-\mu}} e^{-\lambda(S_t^{-\mu} - y)} f'(S_t^{-\mu} - B_t^{-\mu})\, dy\Big] \\ &= -\mathsf{E}\Big[\int_{x \vee S_t^{-\mu} - B_t^{-\mu}}^\infty f'(z)\, dz\Big] - \mathsf{E}\Big[f'(x \vee S_t^{-\mu} - B_t^{-\mu}) \int_x^{x \vee S_t^{-\mu}} e^{-\lambda(x \vee S_t^{-\mu} - y)}\, dy\Big] \\ &= \mathsf{E}\Big[f(x \vee S_t^{-\mu} - B_t^{-\mu})\Big] - \mathsf{E}\Big[f'(x \vee S_t^{-\mu} - B_t^{-\mu}) \int_0^{x \vee S_t^{-\mu} - x} e^{-\lambda s}\, ds\Big] \end{aligned} \tag{2.15}$$

for $t \geq 0$ and $x \geq 0$, where in the second last equality we use that $S_t^{-\mu} = x \vee S_t^{-\mu}$ since otherwise the integral from x to $x \vee S_t^{-\mu}$ equals zero, and in the last equality we use that $f(\infty) = 0$. Making use of (2.9) in (2.15) establishes the probabilistic representation (2.1) as claimed in the beginning of the proof.

4. Focusing on (2.1) and recalling (2.10) we see that an explicit calculation of the right-hand side in (2.1) is possible since the probability density function g of $(B_t^{-\mu}, S_t^{-\mu})$ is known and can be readily derived from the known probability density function of (B_t, S_t) when μ is zero

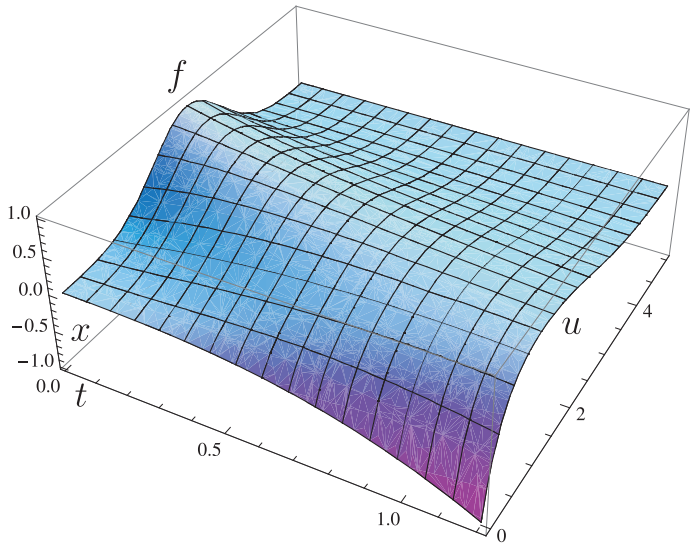

Figure 1. The solution u to the initial boundary value problem (1.1)-(1.3) when $\mu = 1$, $\nu = -1/2$ and $f(x) = e^{-(x-5/2)^2}$ for $x \geq 0$. Note that u takes negative values even though f is positive so that the classic semigroup representation (1.4) of u is not possible in this case. The probabilistic representation (2.1) is valid and this also yields the integral representation (2.2). The solution can be interpreted in terms of a reflecting Brownian motion X with drift μ and its creation in 0 at rate $\kappa \ell^0(X)$ where $\kappa = 2|\nu - \mu| \vee 0 = 3$.

(see e.g. [7, p. 27] or [11, p. 110]) using a standard change-of-measure argument. This yields the following closed form expression

$$g(t;b,s) = \sqrt{\frac{2}{\pi}}\,\frac{1}{t^{3/2}}\,(2s-b)\,\exp\Big[-\frac{(2s-b)^2}{2t} - \mu\Big(b+\frac{\mu t}{2}\Big)\Big] \tag{2.16}$$

for $t > 0$ and $b \leq s$ with $s \geq 0$. It follows that the functions on the right-hand side of (2.1) can be given the following integral representations

$$\begin{aligned} u^1(t,x) &:= \mathsf{E}_x\big[f(X_t)\big] = \mathsf{E}\big[f(x \vee S_t^{-\mu} - B_t^{-\mu})\big] \\ &= \int_0^\infty \int_{-\infty}^s f(x \vee s - b)\, g(t;b,s)\, db\, ds \end{aligned} \tag{2.17}$$

$$\begin{aligned} u^2(t,x) &:= \mathsf{E}_x\Big[f'(X_t) \textstyle\int_0^{\ell_t^0(X)} e^{-\lambda r}\, dr\Big] \\ &= \mathsf{E}_x\Big[f'(x \vee S_t^{-\mu} - B_t^{-\mu}) \textstyle\int_0^{x \vee S_t^{-\mu} - x} e^{-\lambda r}\, dr\Big] \\ &= \int_0^\infty \int_{-\infty}^s \big(f'(x \vee s - b) \textstyle\int_0^{x \vee s - x} e^{-\lambda r}\, dr\big)\, g(t;b,s)\, db\, ds \end{aligned} \tag{2.18}$$

for $t > 0$ and $x \geq 0$ where $\lambda = 2(\nu - \mu)$. A lengthy elementary calculation then shows that

$$u^1(t,x) = \int_0^\infty f(y)\, G(t;x,y)\, dy \tag{2.19}$$

$$u^2(t,x) = \int_0^\infty f'(y)\, H(t;x,y)\, dy \tag{2.20}$$

for $t > 0$ and $x \geq 0$ where G and H are given in (2.3) and (2.4) above. Noting that

$$u(t,x) = u^1(t,x) - u^2(t,x) \tag{2.21}$$

we see that this establishes the integral representation (2.2) as claimed.

5. A direct analysis of the integral representations (2.19) and (2.20) with G and H from (2.3) and (2.4) then shows that u from (2.21) belongs to both $C^\infty((0,\infty)\times[0,\infty))$ and $C_b([0,T]\times[0,\infty))$ for $T > 0$ and $u(t,\infty) = 0$ for $t > 0$. A similar analysis also shows that both u_x^1 and u_x^2 belong to $C_b(([0,T]\times[0,\infty))\setminus\{(0,0)\})$ for $T > 0$. Moreover, it can be directly verified that (i) $u_x^1(t,x) \to f'(x)$ as $t \downarrow 0$ for all $x > 0$ but $u_x^1(t,0) = 0$ for all $t > 0$ so that u_x^1 is not continuous at $(0,0)$ unless $f'(0) = 0$; and (ii) $u_x^2(t,x) \to 0$ as $t \downarrow 0$ for all $x > 0$ but $u_x^2(t,0) \to -f'(0)$ as $t \downarrow 0$ so that u_x^2 is not continuous at $(0,0)$ either unless $f'(0) = 0$. Despite the possibility that both u_x^1 and u_x^2 are discontinuous at $(0,0)$ it turns out that when acting in cohort to form $u_x = u_x^1 - u_x^2$ the resulting function u_x is continuous at $(0,0)$ so that u_x belongs to $C_b([0,T]\times[0,\infty))$ for $T > 0$. It follows therefore from the construction and these arguments that the function u defined by (2.2) with G and H from (2.3) and (2.4) solves the initial boundary problem (1.1)-(1.3) and satisfies $u \in C^\infty((0,\infty)\times[0,\infty))$ with $u, u_x \in C_b([0,T]\times[0,\infty))$ for $T > 0$ and $u(t,\infty) = 0$ for $t > 0$. Placing then any such u at the beginning of the proof and repeating the same arguments as above we can conclude that u admits the probabilistic representation (2.1). These arguments therefore establish both the existence and uniqueness of the solution u to the initial boundary problem (1.1)-(1.3) satisfying the specified conditions and the proof is complete. □

Remark 2 (*Non-smooth initial data*). The integral representation (2.2) requires that f is differentiable. Integrating by parts we find that

$$\int_0^\infty f'(y)\, H(t;x,y)\, dy = -f(0)\, H(t;x,0) - \int_0^\infty f(y)\, H_y(t;x,y)\, dy\,. \tag{2.22}$$

Inserting this back into (2.2) we find that u admits the following integral representation

$$u(t,x) = \int_0^\infty f(y)\, (G+H_y)(t;x,y)\, dy + f(0)\, H(t;x,0) \tag{2.23}$$

where the first function is given by

$$\begin{aligned}(G+H_y)(t;x,y) &= \frac{1}{\sqrt{t}}\Big[\varphi\Big(\frac{x-y+\mu t}{\sqrt{t}}\Big) - e^{2\mu y}\,\varphi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big)\Big] \\ &\quad - \frac{2\nu\, e^{2\mu y}}{\nu-\mu}\Big[\mu\Psi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big) + (\mu-2\nu)\, e^{2(\nu-\mu)(x+y+\nu t)} \\ &\qquad \times \Psi\Big(\frac{x+y+(2\nu-\mu)t}{\sqrt{t}}\Big)\Big] \quad \text{if } \nu \neq \mu \\ &= \frac{1}{\sqrt{t}}\,\varphi\Big(\frac{x-y+\mu t}{\sqrt{t}}\Big) - \frac{e^{2\mu y}}{\sqrt{t}}\Big[(1+4\mu^2 t)\,\varphi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big)\end{aligned} \tag{2.24}$$

$$- 4\mu(1+\mu(x+y)+\mu^2 t)\sqrt{t}\,\Psi\Big(\frac{x+y+\mu t}{\sqrt{t}}\Big)\Big] \quad \text{if } \nu = \mu$$

and the second function is given by

$$H(t;x,0) = \frac{1}{\nu-\mu}\Big[(2\nu-\mu)\,e^{2(\nu-\mu)(x+\nu t)}\,\Psi\Big(\frac{x+(2\nu-\mu)t}{\sqrt{t}}\Big) - \mu\Psi\Big(\frac{x+\mu t}{\sqrt{t}}\Big)\Big] \quad \text{if } \nu \neq \mu \tag{2.25}$$

$$= 2\Big[\big(1+\mu(x+\mu t)\big)\,\Psi\Big(\frac{x+\mu t}{\sqrt{t}}\Big) - \mu\sqrt{t}\,\varphi\Big(\frac{x+\mu t}{\sqrt{t}}\Big)\Big] \quad \text{if } \nu = \mu$$

for $t > 0$ and $x, y \geq 0$. Note that smoothness of f is no longer needed in the integral representation (2.23) and this formula for u can be used when $f \in C_b([0,\infty))$ for instance.

References

[1] Burdzy, K. Chen, Z. Q. *and* Sylvester, J. (2004). The heat equation and reflected Brownian motion in time-dependent domains. *Ann. Probab.* 32 (775–804).

[2] Feller, W. (1952). The parabolic differential equations and the associated semi-groups of transformations. *Ann. of Math. (2)* 55 (468–519).

[3] Feller, W. (1954). Diffusion processes in one dimension. *Trans. Amer. Math. Soc.* 77 (1–31).

[4] Feller, W. (1957). Generalized second order differential operators and their lateral conditions. *Illinois J. Math.* 1 (459–504).

[5] Hille, E. (1948). *Functional Analysis and Semi-Groups.* Amer. Math. Soc.

[6] Itô, K. *and* McKean, H. P. Jr. (1963). Brownian motions on a half line. *Illinois J. Math.* 7 (181–231).

[7] Itô, K. *and* McKean, H. P. Jr. (1965). *Diffusion Processes and their Sample Paths.* Springer.

[8] Pang, H. *and* Stroock, D. W. (2007). A peculiar two point boundary value problem. *Ann. Probab.* 35 (1623–1641).

[9] Pang, H. *and* Stroock, D. W. (2008). A highly unstable initial value boundary value problem. *J. Funct. Anal.* 255 (2579–2605).

[10] Peskir, G. (2006). On reflecting Brownian motion with drift. *Proc. Symp. Stoch. Syst.* (Osaka 2005), *ISCIE Kyoto* (1–5).

[11] Revuz, D. *and* Yor, M. (1999). *Continuous Martingales and Brownian Motion.* Springer.

[12] Stroock, D. W. *and* Williams, D. (2005). A simple PDE and Wiener-Hopf Riccati equations. *Comm. Pure Appl. Math.* 58 (1116–1148).

[13] Stroock, D. W. *and* Williams, D. (2006). Further study of a simple PDE. *Illinois J. Math.* 50 (961–989).

[14] Williams, D. *and* Andrews, S. (2006). Indefinite inner products: a simple illustrative example. *Math. Proc. Cambridge Philos. Soc.* 141 (127–159).

[15] Yosida, K. (1949). An operator-theoretical treatment of temporally homogeneous Markoff process. *J. Math. Soc. Japan* 1 (244–253).

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