A simplified proof of the relation between scaling exponents in first-passage percolation

Antonio Auffinger* Michael Damron†

September 2011

Abstract

In a recent breakthrough work, Chatterjee [5] proved a long standing conjecture that relates the transversal exponent ξ and the fluctuation exponent χ in first-passage percolation on $\mathbb{Z}^d$. The purpose of this paper is to introduce a new and intuitive idea that replaces the main argument of [5] and gives an alternative proof of this relation. Specifically, we show that under the assumption that exponents defined in [5] exist, one has the relation $\chi \leq 2\xi - 1$. One advantage of our argument is that it does not require the 'nearly Gamma' assumption of [5].

1 Introduction

We consider first-passage percolation (FPP) on $\mathbb{Z}^d$ with non-negative i.i.d. weights τ_e on edges with common distribution μ. For a review and a description of known results on the model we refer the reader to [3, 8, 12].

The random variable τ_e is called the *passage time* of the edge e, a nearest-neighbor edge in $\mathbb{Z}^d$. A *path* γ is a sequence of edges $e_1, e_2, \ldots$ in $\mathbb{Z}^d$ such that for each $n \geq 1$, e_n and e_{n+1} share exactly one endpoint. For any finite path γ we define the *passage time* of γ to be $\tau(\gamma) = \sum_{e\in\gamma} \tau_e$ and given two points $x, y \in \mathbb{R}^d$ we set

$$\tau(x,y) = \inf_{\gamma} \tau(\gamma) \ .$$

The infimum is over all finite paths γ that contain both x' and y', and x' is the unique vertex in $\mathbb{Z}^d$ such that $x \in x' + [0,1)^d$ (similarly for y'). A minimizing path for $\tau(x,y)$ is called a *geodesic* from x to y. We assume throughout the paper that μ has no mass larger than or equal to $p_c(d)$, the critical probability of bond percolation, at the infimum of its support.

*Mathematics Department, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637. Email:auffing@math.uchicago.edu.

†Mathematics Department, Princeton University, Fine Hall, Washington Rd., Princeton, NJ 08544. Email: mdamron@math.princeton.edu; Research funded by an NSF Postdoctoral Fellowship.

Consider a geodesic from the origin to a point v with passage time $\tau(0, v)$. One of the central questions [8, 12] in this model (and in related ones) is to prove the following statement. There exists an intrinsic relation between the magnitude of deviation of $\tau(0, v)$ from its mean and the magnitude of deviation of the geodesic $\tau(0, v)$ from a straight line joining 0 and v. This relation is *universal*; that is, it is independent of the dimension d and of the law of the weights (as long they satisfy certain moment assumptions).

The fluctuations of the passage time $\tau(0, v)$ about $\mathbb{E}\tau(0, v)$ should be of order $|v|^{\chi}$, where χ is called the *fluctuation exponent*. Analogously, a *transversal exponent* ξ should measure the maximal Euclidean distance of a geodesic from 0 to v from the straight line that joins 0 to v. The intrinsic relation described above should be given as

$$\chi = 2\xi - 1. \tag{1.1}$$

Despite numerous citations (both in mathematics and physics papers [10, 13, 14, 15, 16]) and the mystery surrounding (1.1), the existence and the 'correct' definition of these exponents is still not established and these issues form part of the above conjecture.

For a certain definition of the exponents, the inequality $\chi \geq 2\xi - 1$ was proved and understood in the 1995 work of Newman-Piza [15]. The other inequality, however, has remained elusive for more than twenty years. A recent breakthrough work of Chatterjee [5] proposed a stronger definition of the exponents that allows a complete proof of the equality (1.1). Of its many contributions, it was the first paper to introduce any idea about how to prove the inequality $\chi \leq 2\xi - 1$. The proof relies on a construction similar to that in [6]. One first breaks a geodesic into smaller segments and then uses an approximation scheme to compare the passage time to a sum of nearly i.i.d. random variables. The proof is then a trade-off between minimizing the error while maximizing the variance of the passage time. Assuming that the distribution is 'nearly Gamma' (see [4] for a definition), the optimization can be achieved by carefully choosing different parameters in the approximation.

The main goal of this paper is to show a simple new idea that enables us to replace the main argument of [5] to prove the inequality $\chi \leq 2\xi - 1$ in the case $\chi > 0$. Our proof does not use a 'nearly Gamma' assumption on the passage times, and so applies to all distributions for which Chatterjee's exponents exist. Furthermore, this idea allow us to extend our theorem to related models like Directed Polymers in Random Environments, Last Passage Percolation and Euclidean First Passage Percolation. These questions will be addressed in a forthcoming paper. A secondary goal of this paper is to explain how the simplicity of our proof could allow to extract weaker assumptions on the model to guarantee that different versions of (1.1) hold.

We close this section by discussing earlier works related to (1.1) and sketching the idea of Newman-Piza [15]. In 1993, Kesten [11] showed that $\chi \leq \frac{1}{2}$. In 1996, Licea, Newman and Piza [14] proved that $\xi \geq \frac{1}{2}$ in all dimensions for one definition of ξ and $\xi \geq \frac{3}{5}$ in two dimensions for another definition. In Section 4.2, we use their cylinder construction [14] as a fundamental tool to obtain the proof of (1.1) . The most well-known conjecture after (1.1) is, however, that in two dimensions one should have the exact values $\xi = \frac{2}{3}$ and $\chi = \frac{1}{3}$.

The proof of Newman-Piza [15] was based on an argument of Aizemann-Wehr [1] (in a different context) with important contributions from Alexander [2] and Kesten [11] . Their main tool was an assumption of curvature of the limit shape $\mathcal{B}_\mu$ (defined in (3.1)) and the following argument (see Figure 1): Let v be a unit vector. Assume that a geodesic from 0 to nv leaves a box of height n^ξ centered on the straight line that joins the origin to nv through a point w. Furthermore, assume that the limit shape has shape curvature 2 in the direction of v; that is, there exists a positive constant c such that

$$g(v) := \lim_{n\to\infty} \frac{\mathbb{E}(\tau(0,nv))}{n} \quad \text{satisfies} \quad c|z|^2 \leq |g(v+z) - g(v)| \tag{1.2}$$

for all vectors z orthogonal to v of small length. (A precise definition of the shape curvature will be given in Section 3.) The passage time being additive in a geodesic implies that

$$\tau(0,nv) = \tau(0,w) + \tau(w,nv). \tag{1.3}$$

Alexander's subadditive approximation theorem [2] (see (4.1)) guarantees that $\tau(0,w) + \tau(w,nv) - \mathbb{E}\tau(0,nv)$ is within $O(n^\chi)$ of $g(w) + g(nv-w) - g(nv)$, which is equal to $g(\lambda v - (\lambda v - w)) + g((n-\lambda)v - (w-\lambda v)) - g(nv)$ (see Figure 1). By the curvature assumption (1.2) and by linearity of g in the direction of v this term is now of order at least $\frac{|w-\lambda v|^2}{n} = n^{2\xi-1}$ regardless of the choice of w. This would contradict the fact that $\tau(0,nv) - \mathbb{E}\tau(0,nv)$ has order n^χ if $\chi < 2\xi - 1$, proving the lower bound.

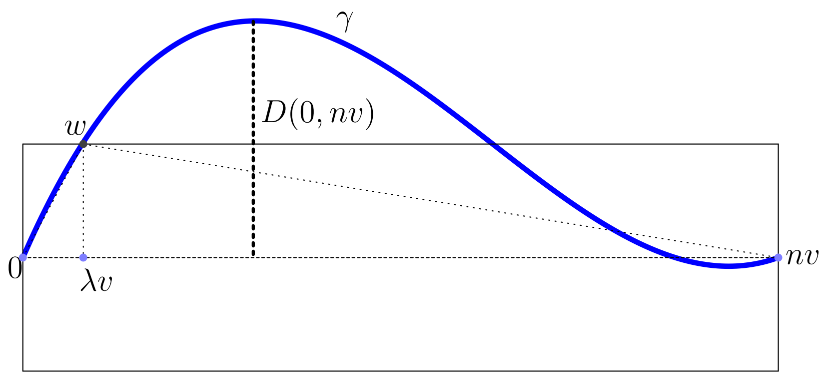

Figure 1: A geodesic γ from 0 to nv leaving a box of height n^ξ at a point w.

1.1 Outline of the paper

In the next section, we state our main result and give a sketch of the proof. Next, in Section 3, we discuss how this new proof could give rise to important extensions. In Section 4 we prove Theorem 2.1.

2 Results

Let $D(0,v)$ be the maximum Euclidean distance between the set of all geodesics from 0 to v and the line segment joining 0 to v. We say that the FPP model has *global exponents in the sense of Chatterjee* if there exist real numbers χ_a, χ_b and ξ_a, ξ_b such that:

1. For each choice of $\chi' > \chi_a$ and $\xi' > \xi_a$, there exists $\alpha > 0$ so that

$$\sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} \exp\left(\alpha \frac{|\tau(0,v) - \mathbb{E}\tau(0,v)|}{|v|^{\chi'}}\right) < \infty \text{ and } \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} \exp\left(\alpha \frac{D(0,v)}{|v|^{\xi'}}\right) < \infty. \tag{2.1}$$

2. For each choice of $\chi'' < \chi_b$ and all $\xi'' < \xi_b$,

$$\inf_{v \in \mathbb{Z}^d \setminus \{0\}} \frac{\mathrm{Var}(\tau(0,v))}{|v|^{2\chi''}} > 0 \text{ and } \inf_{v \in \mathbb{Z}^d \setminus \{0\}} \frac{\mathbb{E}(D(0,v))}{|v|^{\xi''}} > 0. \tag{2.2}$$

Remark 1. It is not difficult to prove (see [5]) that if such exponents exist then $0 \leq \xi_b \leq \xi_a \leq 1$ and $0 \leq \chi_b \leq \chi_a \leq \frac{1}{2}$.

Our main result is the following.

Theorem 2.1. *Assume that the FPP model has global exponents in the sense of Chatterjee and $\chi := \chi_a = \chi_b > 0$. Then*

$$\chi \leq 2\xi_a - 1. \tag{2.3}$$

Remark 2. Our proof does not require to assume that the distribution μ of the $\tau_e's$ is *nearly Gamma* as in [5]. The case $\chi = 0$ was treated with a separate and elegant argument in [5]. It does not require this assumption on μ and although it is stated for continuous distributions only, the arguments hold under our condition on the support of μ.

In [5] it was shown using the ideas of Newman-Piza [15] and Howard [8] that for this definition of exponents, the lower bound holds:

$$\chi_a \geq 2\xi_b - 1. \tag{2.4}$$

This fact, combined with Theorem 2.1 and with the assumption $\xi_a = \xi_b$, implies

Theorem 2.2. *Assume that the FPP model has global exponents in the sense of Chatterjee with $\chi := \chi_a = \chi_b$ and $\xi := \xi_a = \xi_b$. Then* (1.1) *holds.*

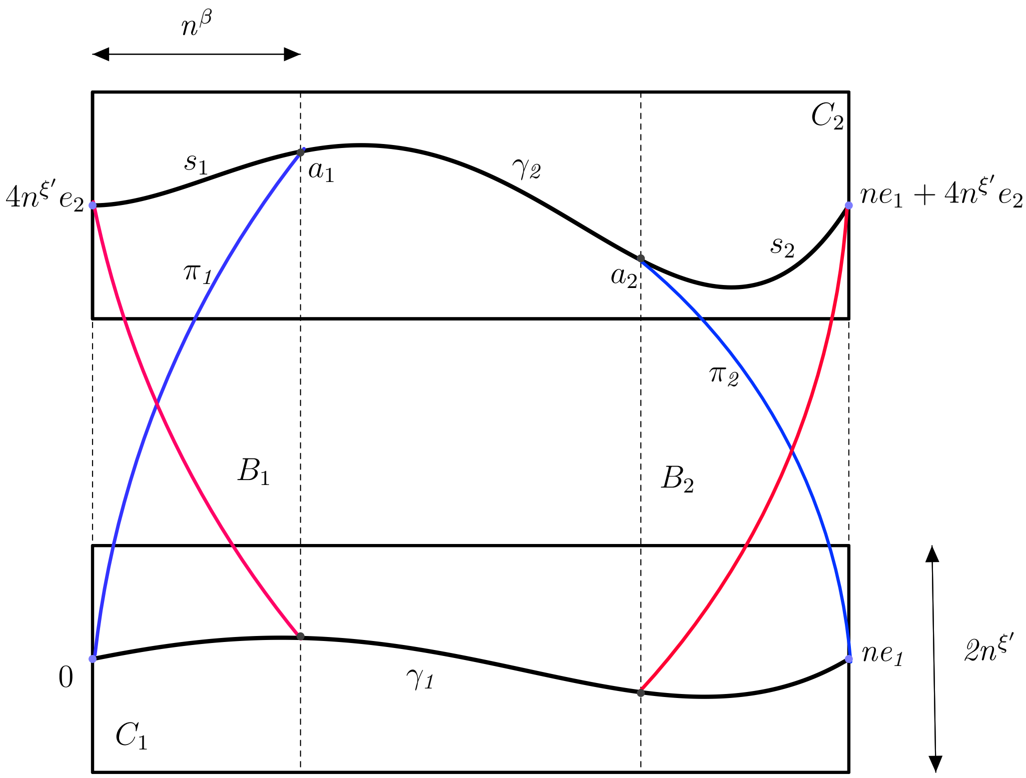

Figure 2: B_1 and B_2 are boxes of length n^β and radius $3n^{\xi'}$ and $\pi_i, i = 1, 2$ are geodesics joining the points 0 to a_1 and a_2 to ne_1 respectively. Since γ_1 is a geodesic, the path from 0 to ne_1 using π_1, π_2 and the middle part of γ_2 has larger passage time than γ_1. Note β is chosen larger than ξ' so the boxes B_1 and B_2 are not drawn to scale.

2.1 Sketch of the proof

In this subsection we sketch the proof of Theorem 2.1. It will follow from the picture below.

Look at the cylinders C_1 and C_2 in Figure 2.1. They are both of length n and radius $n^{\xi'}$ for some $\xi' > \xi_a$. The top cylinder is identical to the bottom but shifted up (in direction e_2, the second coordinate vector) by $4n^{\xi'}$. The dark paths γ_1 and γ_2 joining 0 to ne_1 and their shifted points are geodesics.

Since we chose $\xi' > \xi_a$ it is possible to show that the passage times $\tau(\gamma_1)$ and $\tau(\gamma)$ are almost independent. Using (2.2), this implies that for any $\chi'' < \chi$ and n large

$$n^{2\chi''} \leq \mathrm{Var}(\tau(\gamma_2) - \tau(\gamma_1)). \tag{2.5}$$

Assuming $\xi_a < 1$ and $\xi' < \beta < 1$, build two cylinders B_1 and B_2 of length n^β and radius $3n^{\xi'}$as in the picture. Let a_1 and a_2 be the last and first points of intersection of the geodesic γ_2 with these cylinders. Consider the geodesics π_1 joining 0 to a_1 and π_2 joining a_2 to ne_1 (in blue). Note that the concatenation of π_1, the piece of γ_2 from a_1 to a_2, and π_2 is a path from 0 to ne_1. Therefore, if s_1 and s_2 are the other two parts of γ_2 (as in the picture),

$$\tau(\gamma_1) \leq \tau(\pi_1) + \tau(a_1, a_2) + \tau(\pi_2) = \tau(\pi_1) + \Big(\tau(\gamma_2) - \tau(s_1) - \tau(s_2)\Big) + \tau(\pi_2).$$

which implies

$$\tau(\gamma_1) - \tau(\gamma_2) \leq \tau(\pi_1) - \tau(s_1) + \tau(\pi_2) - \tau(s_2). \tag{2.6}$$

The difference $\tau(\pi_1) - \tau(s_1)$ is bounded above by

$$X := \max_{u,v,u',v' \in \partial B_1} \tau(u, v) - \tau(u', v'),$$

where u and u' are points on the left boundary of the box B_1 while v and v' are points on the right boundary of the box. Using the box B_2 one can similarly bound the difference of $\tau(\pi_2) - \tau(s_2)$ by a random variable with same distribution as X. Using the red paths instead of the blue ones and reversing the roles of γ_1 and γ_2 in (2.6), we get an inequality for the absolute value of the left side of (2.6). Combining these bounds,

$$\mathrm{Var}(\tau(\gamma_2) - \tau(\gamma_1)) \leq 4\mathbb{E}X^2.$$

For $\mathbb{E}X^2$ it suffices to bound (independently of u and v) the second moment of

$$\begin{aligned} |\tau(u, v) - \tau(0, n^\beta e_1)| \leq\ & |\tau(u, v) - \mathbb{E}\tau(u, v)| + |\tau(0, n^\beta e_1) - \mathbb{E}\tau(0, n^\beta e_1)| \\ & + |g(v - u) - \mathbb{E}\tau(u, v)| + |g(ne_1) - \mathbb{E}\tau(0, n^\beta e_1)| \\ & + |g(v - u) - g(n^\beta e_1)|. \end{aligned} \tag{2.7}$$

The first two lines above are bounded above by $n^{\beta\chi'}$ for any $\chi' > \chi$ (by assumption and Alexander's subadditive approximation) while the third is of order $n^{2\xi' - \beta}$ by the curvature of the limit shape $\mathcal{B}_\mu$ (see (3.1)). This implies by (2.5) and the above computation

$$n^{2\chi''} \leq \mathrm{Var}(\tau(\gamma_2) - \tau(\gamma_1)) \leq C(n^{2\beta\chi'} + n^{2(2\xi' - \beta)}). \tag{2.8}$$

Now choosing χ'' and χ' close enough to χ and recalling that $\beta < 1$ we get $n^{2\chi''} \leq Cn^{2(2\xi'-\beta)}$ for large n. This implies $\chi'' \leq 2\xi' - \beta$. Taking $\beta \uparrow 1$, $\chi'' \uparrow \chi$ and $\xi' \downarrow \xi$ ends the proof.

3 Extensions

In this section we discuss how to improve Theorem 2.1. There are two main directions. The first one is to establish a relation for directionally defined exponents. This would weaken our assumptions, allowing to prove the existence of both exponents more easily. The second is to add shape curvature into the relation (1.1).

One can define the exponents ξ_a, χ_a, χ_b directionally as follows. For a unit vector u, define the cylinder $\mathcal{C}(u, a, b)$ of length a and radius b in the direction u as the set of points in $\mathbb{R}^d$ at most ℓ_∞ distance b away from the line segment connecting 0 to au. We denote $\partial^f \mathcal{C}(u, a, b)$ as the set of all points $x \in \mathcal{C}(u, a, b)$ with $|\langle u, x\rangle| \geq a$.

The exponent ξ_a^u is now defined as in (2.1) with v taken as a non-zero multiple of u instead of an arbitrary vector in $\mathbb{Z}^d \setminus \{0\}$. χ_a^u is defined similarly to (2.1) but as a function of ξ_a^u; it is the smallest real number such that for any $\chi' > \chi_a^u$ there exists α so that

$$\inf_{\xi' > \xi_a^u} \sup_{n \in \mathbb{N}} \sup_{v \in \partial^f C(u, n, n^{\xi'})} \mathbb{E} \exp\left(\alpha \frac{|\tau(0, v) - \mathbb{E}\tau(0, v)|}{|v|^{\chi'}}\right) < \infty.$$

χ_b^u is defined as the largest real number such that for any $\chi'' < \chi_b^u$,

$$\inf_{n \in \mathbb{N}} \frac{\operatorname{Var} \tau(0, nu)}{n^{2\chi''}} > 0.$$

One can go through the proof of Theorem 2.1 and see that the scaling relation (1.1) holds with these new exponents as long one is able to prove that Alexander's subadditive exponent can be made directional. Namely, the question becomes the following:

Question 3.1. *Is it true that for any $\chi' > \chi_a^u$ there exists $\xi' > \xi_a^u$ and a constant $C = C(\chi', \xi') > 0$ such that for all $x \in \partial^f \mathcal{C}(u, n, n^{\xi'})$ and all n,*

$$|\mathbb{E}\tau(0, x) - g(x)| \leq C|x|^{\chi'} .$$

We currently do not know the answer to this question.

Another way to generalize the relation (1.1) is to add curvature. Let

$$\mathcal{B}_\mu := \{x \in \mathbb{R}^d, g(x) \leq 1\} \tag{3.1}$$

be the limit shape of the model (see [12]). Let u be a unit vector of $\mathbb{R}^d$ and let H_0 be a hyperplane such that $u + H_0$ is tangent to $g(u)\mathcal{B}_\mu$ at u. We introduce a third exponent, called the *curvature exponent* as follows.

Definition 3.1. The curvature exponent κ^u in the direction u is a real number such that there exist positive constants c, C and ε such that for any $z \in H_0$ with $|z| < \varepsilon$, one has

$$c|z|^{\kappa^u} \leq g(u + z) - g(u) \leq C|z|^{\kappa^u}. \tag{3.2}$$

The directional approach mentioned above together with the definition of the curvature exponent allows us to generalize the relation (1.1) to one that includes all three of these exponents. Assume that Question 3.1 is answered affirmatively and that $\chi^u := \chi^u_a = \chi^u_b$ ($\geq$ 0). Then it would follow directly from the proof of Theorem 2.1 that (2.3) generalizes to

$$\chi^u \leq \kappa^u \xi^u_a - (\kappa^u - 1). \tag{3.3}$$

Moreover, if $\xi^u := \xi^u_a = \xi^u_b$ then one would have

$$\chi^u = \kappa^u \xi^u - (\kappa^u - 1). \tag{3.4}$$

Remark 3. Note that when $\kappa^u = 2$ and the exponents are global, (3.4) is the same as (1.1). This is believed to be true in the case where the weights τ have a continuous distribution with finite exponential moments. It would be of interest to find examples, maybe of other growth models, where (3.3) holds for $\kappa^u \neq 2$.

Remark 4. It is unclear if Chatterjee's exponents exist and, if so, what are the implications. For example, existence immediately implies that $\kappa^u \leq 2$ in all directions where κ^u is defined. In particular the limit shape can not contain flat pieces as in [7]. However if the statement in Question 3.1 holds and if one uses directional exponents (provided they exist) then it is possible to show that the upper bound in (3.2) holds for all κ (this is true, for example, if there is a flat edge in direction u) if and only if $\xi^u_a = 1$.

4 Proof of Theorem 2.1

4.1 Preliminary lemmas

Recall the definition of the function g from (1.2). We first state a bound on the "non-random fluctuations" from [2]. The proof of the lemma, as stated, can be found in [5].

Lemma 4.1. *For any $\chi' > \chi_a$, there exists $C_1 = C_1(\chi') > 0$ such that for all $x \in \mathbb{R}^d$,*

$$|\mathbb{E}\tau(0,x) - g(x)| \leq C_1 |x|^{\chi'} \ .$$

For a unit vector x_0, let H_0 be as in Definition 3.1 (taking $u = x_0$). For $m, n \geq 1$ and $i = 1, 2$, set

$$S_i(x_0; m, n) = \{x \in (i-1)nx_0 + H_0 : |x - (i-1)nx_0| \leq m\}$$

and

$$X(x_0; m, n) = \max_{\substack{v_1, v_2 \in S_1(x_0; m, n) \\ w_1, w_2 \in S_2(x_0; m, n)}} |\tau(v_1, w_1) - \tau(v_2, w_2)| \ .$$

The following proposition is a slight modification of arguments in [5]. For a random variable G, write $\|G\|_2$ for the L^2 norm $(\mathbb{E}G^2)^{1/2}$.

Proposition 4.2. *Let $|x_0| = 1$ and assume (3.2) holds for some C_{x_0}, κ and ε_{x_0}. For each $\chi' > \chi_a$ there exists $C_2 = C_2(d, \chi')$ such that if m, n have $m \leq (\varepsilon_{x_0}/2\sqrt{d-1})n$, then*

$$\|X(x_0; m, n)\|_2 \leq C_2 n^{1-\kappa} m^\kappa + C_2 n^{\chi'} \ .$$

Proof. By the triangle inequality, it suffices to bound the variable Y:

$$Y(x_0; m, n) = \max_{\substack{v \in S_1(x_0;m,n) \\ w \in S_2(x_0;m,n)}} |\tau(v, w) - \tau(0, nx_0)| \ .$$

For $v \in S_1(x_0; m, n)$ and $w \in S_2(x_0; m, n)$, the idea is to use the following decomposition:

$$\begin{aligned} |\tau(0, nx_0) - \tau(v, w)| &\leq |\tau(0, nx_0) - \mathbb{E}\tau(0, nx_0)| + |\tau(v, w) - \mathbb{E}\tau(v, w)| && (4.1)\\ &+ |\mathbb{E}\tau(0, nx_0) - g(nx_0)| + |\mathbb{E}\tau(v, w) - g(w - v)| && (4.2)\\ &+ |g(nx_0) - g(w - v)| \ . && (4.3) \end{aligned}$$

We first estimate (4.3):

$$|g(nx_0) - g(w - v)| = n|g(x_0) - g(x_0 + (w - v)/n - x_0)| \ .$$

By assumption, $|(w - v)/n - x_0| = (1/n)|w - v - nx_0| \leq 2(m/n)\sqrt{d-1} \leq \varepsilon_{x_0}$. Therefore we can apply (3.2) and find C_3 such that

$$|g(nx_0) - g(w - v)| \leq C_{x_0} n |(w - v)/n - x_0|^\kappa \leq C_3 n^{1-\kappa} m^\kappa \ . \tag{4.4}$$

For (4.2), we note that $|w - v| \leq 2n$ for all w, v. So by Lemma 4.1,

$$|\mathbb{E}\tau(0, nx_0) - g(nx_0)| + |\mathbb{E}\tau(v, w) - g(w - v)| \leq 3C_1 n^{\chi'} \ . \tag{4.5}$$

We turn to contributions to $Y(x_0; m, n)$ from terms in (4.1). Pick $\hat{\chi} = (1/2)(\chi_a + \chi')$ and

$$X := \max_{\substack{v \in S_1(x_0;m,n) \\ w \in S_2(x_0;m,n)}} \frac{|\tau(v, w) - \mathbb{E}\tau(v, w)|}{|w - v|^{\hat{\chi}}} \ .$$

By the fact that $\hat{\chi} > \chi_a$, for some C_4 and C_5,

$$\mathbb{E}e^{\alpha X} \leq \sum_{\substack{v \in S_1(x_0;m,n) \\ w \in S_2(x_0;m,n)}} \mathbb{E}\left(\exp\left[\alpha \frac{|\tau(v, w) - \mathbb{E}\tau(v, w)|}{|w - v|^{\hat{\chi}}}\right]\right) \leq C_4 |S_1(x_0; m, n)|^2 \leq C_5 m^{2(d-1)} \ .$$

Since $\alpha > 0$ and X is positive, we may use Jensen's inequality to get

$$e^{\alpha\|X\|_2} = 1 + \alpha\|X\|_2 + \sum_{n=2}^\infty \frac{(\alpha\|X\|_2)^n}{n!} \leq 1 + \alpha\|X\|_2 + \mathbb{E}\sum_{n=2}^\infty \frac{(\alpha X)^n}{n!} \leq \alpha\|X\|_2 + \mathbb{E}e^{\alpha X} \ . \tag{4.6}$$

Because $e^{\alpha t} \geq 2\alpha t$ for all $t \in \mathbb{R}$, it cannot be that $\alpha\|X\|_2$ is the maximum of the two terms on the right side of (4.6). Thus an upper bound is $2\mathbb{E}e^{\alpha X}$ and, taking logarithms of both sides, we find $\|X\|_2 \leq \frac{1}{\alpha}\log 2\mathbb{E}e^{\alpha X}$. So $\|X\|_2 \leq C_6 \log m$ for some C_6. Let

$$X' := \max_{\substack{v \in S_1(x_0;m,n) \\ w \in S_2(x_0;m,n)}} |\tau(v,w) - \mathbb{E}\tau(v,w)| \ .$$

Since $|w-v| \leq 2n$ for all $v \in S_1(x_0;m,n)$ and $w \in S_2(x_0;m,n)$, $X' \leq C_7 n^{\hat{\chi}} X$. Therefore $\|X'\|_2 \leq C_8 n^{\chi'}$. We finish by putting this together with (4.4) and (4.5):

$$\|Y(x_0;m,n)\|_2 \leq C_3 n^{1-\kappa} m^{\kappa} + C_9 n^{\chi'} \ .$$

$\square$

To end the section, we give one general lemma about random variables. Denote by $I(A)$ the indicator function of the event A.

Lemma 4.3. *Let X and Y be random variables with $\|X\|_4, \|Y\|_4 < \infty$ and let B be an event such that*

$$(X-Y)I(B) = 0 \text{ almost surely.}$$

Then

$$|\operatorname{Var}\ X - \operatorname{Var}\ Y| \leq (\|X\|_4 + \|Y\|_4)^2 \mathbb{P}(B^c)^{1/4} \ . \tag{4.7}$$

Proof. Let $\widetilde{X} = X - \mathbb{E}X$ and $\widetilde{Y} = Y - \mathbb{E}Y$. The left side of (4.7) equals

$$\begin{aligned}\left|\|\widetilde{X}\|_2^2 - \|\widetilde{Y}\|_2^2\right| = \left|\|\widetilde{X}\|_2 - \|\widetilde{Y}\|_2\right| \ \left|\|\widetilde{X}\|_2 + \|\widetilde{Y}\|_2\right| &\leq \|X-Y\|_2(\|X\|_2 + \|Y\|_2) \\ &\leq \|(X-Y)I(B^c)\|_2(\|X\|_4 + \|Y\|_4) \\ &\leq \|X-Y\|_4(\|X\|_4 + \|Y\|_4)\mathbb{P}(B^c)^{1/4} \ ,\end{aligned}$$

which implies the lemma. $\square$

4.2 Cylinder construction

Pick x_0 of unit norm and H_0 a hyperplane as in Definition 3.1. Fix an orthonormal basis $x_1, \ldots, x_{d-1}$ of H_0. Let $T_1(x_0;n) = \tau(0, nx_0)$, $T_2(x_0;n,\xi') = \tau(4n^{\xi'}x_1, nx_0 + 4n^{\xi'}x_1)$, and

$$\delta T(x_0;n,\xi') = T_1(x_0;n) - T_2(x_0;n,\xi') \ .$$

The idea will be to give a lower bound for the variance of δT (Section 4.2.1) and then an upper bound (Section 4.2.2). Comparing them, we obtain the desired inequalities (2.3) and (3.3). This idea was introduced by Licea-Newman-Piza in [14] and also used in [9] and [16].

4.2.1 Lower bound on Var δT

We will now assume that

$$\xi_a < 1 \text{ and } \chi_b > 0 \tag{4.8}$$

so that we can choose ξ' and χ'' such that

$$\xi_a < \xi' < 1 \text{ and } 0 < \chi'' < \chi_b \ . \tag{4.9}$$

Proposition 4.4. *Assume* (4.8)*. For each ξ' and χ'' chosen as in* (4.9)*, there exists $C = C(\xi', \chi'')$ such that for all n,*

$$\text{Var } \delta T(x_0; n, \xi') \geq C n^{2\chi''} \ .$$

Proof. Define $\mathcal{C}_1$ as the set of points in $\mathbb{R}^d$ at most ℓ_∞ distance $n^{\xi'}$ away from the line segment connecting 0 to nx_0. Define $\mathcal{C}_2$ as the set of points at most ℓ_∞ distance $n^{\xi'}$ away from the line segment connecting $4n^{\xi'}x_1$ to $nx_0 + 4n^{\xi'}x_1$. Let $T_1(x_0; n)'$ and $T_2(x_0; n, \xi')'$ be as follows:

1. $T_1(x_0; n)'$ is the passage time from 0 to nx_0 using only edges with endpoints in $\mathcal{C}_1$.
2. $T_2(x_0; n, \xi')'$ is the passage time from $4n^{\xi'}x_1$ to $nx_0 + 4n^{\xi'}x_1$ using only edges with endpoints in $\mathcal{C}_2$.

Let B be the event $\{T_1(x_0; n) = T_1(x_0; n)' \text{ and } T_2(x_0; n, \xi') = T_2(x_0; n, \xi')'\}$. Note that if $T_1(x_0; n) \neq T_1(x_0; n)'$ then $D(0, nx_0) \geq n^{\xi'}$. A similar statement holds for $T_2(x_0; n, \xi')$ and $T_2(x_0; n, \xi')'$. Therefore $\mathbb{P}(B^c) \leq 2\mathbb{P}(D(0, nx_0) \geq n^{\xi'})$. Picking $\xi'' = (1/2)(\xi' + \xi_a)$, so that $\xi_a < \xi'' < \xi' < 1$, we find from the definition of ξ_a (from (2.1)) that there exists $C_1 > 0$ such that for all n, $\mathbb{P}(D(0, nx_0) \geq n^{\xi'}) \leq e^{-C_1 n^{\xi' - \xi''}}$. Therefore

$$\mathbb{P}(B^c) \leq 2e^{-C_1 n^{\xi' - \xi''}} \ . \tag{4.10}$$

By Lemma 4.3 with $X = \delta T(x_0; n, \xi')$ and $Y = \delta T(x_0; n, \xi')' := T_1(x_0; n)' - T_2(x_0; n, \xi')'$:

$$\begin{aligned} \text{Var } \delta T(x_0; n, \xi') &\geq \text{Var } \delta T(x_0; n, \xi')' - (\|\delta T(x_0; n, \xi')\|_4 + \|\delta T(x_0; n, \xi')'\|_4)^2 \mathbb{P}(B^c)^{1/4} \\ &\geq \text{Var } \delta T(x_0; n, \xi')' - C_2 n^2 e^{-\frac{C_1}{4} n^{\xi' - \xi''}} \end{aligned}$$

for some C_2. Here we have used inequality (4.10) and that each δT is a difference of two passage times, each of which has L^4 norm bounded above by Cn (compare for example to a deterministic path). Therefore there exists C_3 such that for all n,

$$\text{Var } \delta T(x_0; n, \xi') \geq \text{Var } \delta T(x_0; n, \xi')' - C_3 \ . \tag{4.11}$$

But $\delta T(x_0; n, \xi')'$ is the difference of i.i.d. random variables distributed as $T_1(x_0; n)'$, so

$$\text{Var } \delta T(x_0; n, \xi')' = 2 \, \text{Var } T_1(x_0; n)' \ . \tag{4.12}$$

By exactly the same argument as that given above, we can find C_4 such that for all n,

$$\text{Var } T_1(x_0; n)' \geq \text{Var } T_1(x_0; n) - C_4 = \text{Var } \tau(0, nx_0) - C_4 \ .$$

Using the definition of χ'', we can find another C_5 such that for all n, Var $\tau(0, nx_0) \geq C_5 n^{2\chi''}$. Combining this with (4.12) and (4.11), we complete the proof. $\square$

4.2.2 Upper bound on Var δT

In this section we continue to assume (4.8) and we work with the same choice of ξ' that satisfies (4.9). We will prove the following.

Proposition 4.5. *Assume* (4.8) *and that* (3.2) *holds for some* C, ε_{x_0} *and* κ. *For each* β *satisfying* $\xi' < \beta < 1$ *and each* $\chi' > \chi_a$, *there exists* $C = C(\beta, \chi')$ *such that for all* n,

$$\text{Var } \delta T(x_0; n, \xi') \leq Cn^{2\beta(1-\kappa)+2\xi'\kappa} + Cn^{2\beta\chi'} .$$

Proof. Define the hyperplanes

$$H_1 = n^\beta x_0 + H_0 \text{ and } H_2 = (n - n^\beta)x_0 + H_0 .$$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be as in the proof of the lower bound. For two points a and b in $\mathbb{R}^d$, let $S(a, b)$ be the set of finite paths P from a to b (or their closest lattice points) such that for both $i = 1$ and 2, $P \cap H_i \cap [\mathcal{C}_1 \cup \mathcal{C}_2] \neq \varnothing$. Define $T_1(x_0; n)''$, $T_2(x_0; n, \xi')''$ as follows:

1. $T_1(x_0; n)''$ is the minimum passage time of all paths in $S(0, nx_0)$.
2. $T_2(x_0; n, \xi')''$ is the minimum passage time of all paths in $S(4n^{\xi'}x_1, nx_0 + 4n^{\xi'}x_1)$.

Again we set B equal to the event $\{T_1(x_0; n) = T_1(x_0; n)'' \text{ and } T_2(x_0; n, \xi') = T_2(x_0; n, \xi')''\}$. Because B^c implies that $D(0, nx_0) \geq n^{\xi'}$ (or the corresponding statement for $T_2(x_0; n, \xi')$), we may choose C_1 such that for all n, $\mathbb{P}(B^c) \leq 2e^{-C_1 n^{\xi'-\xi''}}$, where $\xi'' = (1/2)(\xi' + \xi_a)$. Therefore we can argue exactly as in the previous section to find C_2 such that for all n,

$$\text{Var } \delta T(x_0; n, \xi') \leq \text{Var } \delta T(x_0; n, \xi')'' + C_2 , \tag{4.13}$$

where $\delta T(x_0; n, \xi')'' = T_1(x_0; n)'' - T_2(x_0; n, \xi')''$.

For almost every passage time realization, we may define a path $\gamma_1 \in S(0, nx_0)$ (in a measurable and deterministic way when there are not unique geodesics) from 0 to nx_0 such that $\tau(\gamma_1) = T_1(x_0; n)''$ and $\gamma_2 \in S(4n^{\xi'}x_1, nx_0 + 4n^{\xi'}x_1)$ such that $\tau(\gamma_2) = T_2(x_0; n, \xi')''$. Let a_1 be the last lattice point on γ_2 before it intersects $H_1 \cap (\mathcal{C}_1 \cup \mathcal{C}_2)$ and a_2 the last lattice point of γ_2 before it intersects $H_2 \cap (\mathcal{C}_1 \cup \mathcal{C}_2)$. Similarly let a_1' be the last lattice point of γ_1 before it intersects $H_1 \cap (\mathcal{C}_1 \cup \mathcal{C}_2)$ and a_2' the last lattice point of γ_1 before it intersects $H_2 \cap (\mathcal{C}_1 \cup \mathcal{C}_2)$. Write s_1 for the piece of γ_2 (seen as an oriented path) from $4n^{\xi'}x_1$ to a_1, t_2 for the piece of γ_2 from a_1 to a_2, and s_2 for the piece of γ_2 from a_2 to $nx_0 + 4n^{\xi'}x_1$. Similarly, write s_1' for the piece of γ_1 from 0 to a_1' and s_2' for the piece of γ_1 from a_2' to nx_0. By definition of $T_1(x_0; n)''$, we have the following almost surely:

$$\begin{aligned} T_1(x_0; n)'' &\leq \tau(0, a_1) + \tau(t_2) + \tau(a_2, ne_1) \\ &= \tau(0, a_1) - \tau(s_1) + \tau(a_2, ne_1) - \tau(s_2) + T_2(x_0; n, \xi')'' . \end{aligned} \tag{4.14}$$

Set $H_3 = nx_0 + H_0$ and let $\mathcal{C}$ be the set of all points in $\mathbb{R}^d$ that are ℓ_∞ distance at most $5n^{\xi'}$ from the line segment connecting 0 to nx_0. Last, let $V_i = H_i \cap \mathcal{C}$ for $i = 0, \ldots, 3$ and

$$X_i(n, \xi', \beta) = \max_{\substack{v_1, v_2 \in V_{2i} \\ w_1, w_2 \in V_{2i+1}}} |\tau(v_1, w_1) - \tau(v_2, w_2)| \ , \ i = 0, 1 .$$

Using this notation and (4.14), we can give an upper bound for $T_1(x_0;n)''$ of

$$T_1(x_0;n)'' \le T_2(x_0;n,\xi')'' + X_0(n,\xi',\beta) + X_1(n,\xi',\beta) \ .$$

To bound $T_2(x_0;n,\xi')''$, we can similarly write

$$T_2(x_0;n,\xi')'' \le \tau(4n^{\xi'}x_1, a_1') - \tau(s_1') + \tau(a_2', nx_0 + 4n^{\xi'}x_1) - \tau(s_2') + T_1(x_0;n)' \ .$$

Therefore $T_2(x_0;n,\xi')'' \le X_0(n,\xi',\beta) + X_1(n,\xi',\beta) + T_1(x_0;n)''$. Putting these together,

$$|\delta T(x_0;n,\xi')''| \le X_0(n,\xi',\beta) + X_1(n,\xi',\beta) \text{ almost surely,}$$

and consequently

$$\mathrm{Var}\ \ \delta T(x_0;n,\xi')'' \le \|\delta T(x_0;n,\xi')''\|_2^2 \le 2(\|X_0(n,\xi',\beta)\|_2^2 + \|X_1(n,\xi',\beta)\|_2^2) \ .$$

The variables X_0 and X_1 are identically distributed, so $\mathrm{Var}\ \ \delta T(x_0;n,\xi')'' \le 4\|X_0(n,\xi',\beta)\|_2^2$. Finally, we combine with (4.13) to get

$$\mathrm{Var}\ \ \delta T(n,\xi') \le 4\|X_0(n,\xi',\beta)\|_2^2 + C_2 \ . \tag{4.15}$$

The last step is to invoke Proposition 4.2. The variable $X_0(n,\xi',\beta)$ is the same as $X(x_0;5n^{\xi'},n^\beta)$ there. Because β was chosen to be larger than ξ', the condition $5n^{\xi'} \le \frac{\varepsilon_{x_0}}{2\sqrt{d-1}}n^\beta$ holds for large n. Thus there exists C_3 such that for all large n,

$$\|X_0(n,\xi',\beta)\|_2^2 \le C_3 n^{2\beta(1-\kappa)} n^{2\xi'\kappa} + C_3 n^{2\beta\chi'} \ ,$$

where κ is from the statement of this proposition. With (4.15), this completes the proof. □

4.3 Proof of Theorem 2.1

We now prove Theorem 2.1. Assume that $\chi_a = \chi_b = \chi > 0$. Further, we may assume $\xi_a < 1$ because if $\xi_a = 1$, the relation holds by the bound $\chi \le 1/2$ (see Remark 1).

Choose $|x_0| = 1$ such that (3.2) holds for some ε_{x_0} and $C_{x_0} > 0$ for $\kappa = 2$. (The existence of such a point is proved in [5, Proposition 5.1].) From the previous two sections, for each choice of χ', χ'', ξ' and β satisfying

$$0 < \chi'' < \chi < \chi' \text{ and } \xi_a < \xi' < \beta < 1 \ , \tag{4.16}$$

there exist constants $C_i = C_i(\chi',\chi'',\xi',\beta)$ $(i = 1,2)$ such that for all n,

$$C_1 n^{2\chi''} \le \mathrm{Var}\ \ \delta T(n,\xi') \le C_2 n^{-2\beta+4\xi'} + C_2 n^{2\beta\chi'} \ .$$

For any β with $\xi' < \beta < 1$, we may choose $\chi'' = \chi''(\beta)$ and $\chi' = \chi'(\beta)$ that satisfy (4.16) and are so close to χ that $2\beta\chi' < 2\chi''$. For such a choice of χ'' and χ' we then have

$$(1/2)C_1 n^{2\chi''} \le C_2 n^{-2\beta+4\xi'} \text{ for all large } n \ ,$$

and therefore $\chi'' \le -\beta + 2\xi'$. Taking $\beta \uparrow 1$ and noting that $\chi''(\beta) \uparrow \chi$, we find $\chi \le -1 + 2\xi'$. This is true for all $\xi' > \xi_a$, so $\chi \le -1 + 2\xi_a$.

5 Acknowledgements

We are very grateful to S. Chatterjee for his advice and comments on the presentation in an earlier version of the paper. Also we thank G. Ben Arous and C. Newman for suggestions on the introduction and A. Sapozhnikov for comments and careful readings of the paper.

References

[1] Aizenman, M. and Wehr, J. (1987). Fluctuations of extensive functions of quenched random couplings. *J. Stat. Phys.* **60**, 287–306.

[2] Alexander, K. (1997). Approximation of subadditive functions and convergence rates in limiting-shape results. *Ann. Probab.* **1**, 30–55.

[3] Blair-Stahn, N. D. (2010). First passage percolation and competition models. *arXiv:1005.0649*.

[4] Benaim, M. and Rossignol, R. (2008). Exponential concentration for first passage percolation through modified Poincaré inequalities. *: Ann. Inst. H. Poincaré Probab. Statist.* **44**, 544–573.

[5] Chatterjee, S. (2011). The universal relation between scaling exponents in first-passage percolation. *arXiv: 1105.4566*.

[6] Chatterjee, S. and Dey. P. (2011). Central limit theorem for first-passage percolation time across thin cylinders. *arXiv: 0911.5702*.

[7] Durrett, R. and Liggett, T. (1981). The shape of the limit set in Richardson's growth model. *Ann. Probab.* **9**, 186-193.

[8] Howard, C. D. Models of first-passage percolation. *Probability on discrete structures,* 125–173, Encyclopaedia Math. Sci., 110, *Springer, Berlin,* 2004.

[9] Johansson, K. (2000) Transversal fluctuations for increasing subsequences on the plane. *Probab. Th. Rel. Fields* **116**, 445–456.

[10] Kardar, M. and Zhang, Y.-C. (1987). Scaling of directed polymers in random media. *Phys. Rev. Lett.* **56**, 2087-2090.

[11] Kesten, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.* **3**, 296-338.

[12] Kesten, H. First-passage percolation. *From classical to modern probability,* 93–143, Progr. Probab., 54, *Birkhäuser, Basel,* 2003.

[13] Krug, J. (1987). Scaling relation for a growing surface. *Phys. Rev. A* **36**, 5465–5466.

[14] Licea, C. and Newman, C. and Piza, M. (1996). Superdiffusivity in first-passage percolation. *Probab. Theory Related Fields* **106**, 559–591.

[15] Newman, C. and Piza, M. (1995). Divergence of shape fluctuations in two dimensions. *Ann. Probab.* **23**, 977–1005.

[16] Wuthrich, M. V. (1998). Scaling identity for crossing Brownian motion in a Possonian potential. *Probab. Theory Related Fields,* **112**, 299–319.