Scaling window for mean-field percolation of averages

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April 11, 2012

Abstract

For a complete graph of size n, assign each edge an i.i.d. exponential variable with mean n. For $\lambda > 0$, consider the length of the longest path whose average weight is at most λ. It was shown by Aldous (1998) that the length is of order $\log n$ for $\lambda < 1/\mathrm{e}$ and of order n for $\lambda > 1/\mathrm{e}$. Aldous (2003) posed the question on detailed behavior at and near criticality $1/\mathrm{e}$. In particular, Aldous asked whether there exist scaling exponents μ, ν such that for λ within $1/\mathrm{e}$ of order $n^{-\mu}$, the length for the longest path of average weight at most λ has order n^{ν}.

We answer this question by showing that the critical behavior is far richer: For λ around $1/\mathrm{e}$ within a window of $\alpha(\log n)^{-2}$ with a small absolute constant $\alpha > 0$, the longest path is of order $(\log n)^3$. Furthermore, for $\lambda \geqslant 1/\mathrm{e} + \beta(\log n)^{-2}$ with β a large absolute constant, the longest path is at least of length a polynomial in n. An interesting consequence of our result is the existence of a second transition point in $1/\mathrm{e} + [\alpha(\log n)^{-2}, \beta(\log n)^{-2}]$. In addition, we demonstrate a smooth transition from subcritical to critical regime. Our results were not known before even in a heuristic sense.

1 Introduction

In this work, we study the stochastic mean-field distance model. For a complete graph $G = (V, E)$ of size n, associate each edge $e \in E$ a non-negative weight X_e which is an independent exponential variable with mean n. For $\lambda > 0$, let $L(n, \lambda)$ be the length of the longest path whose average weight is at most λ. It was shown by Aldous [2] that with high probability (with probability tending to 1 as $n \to \infty$) $L(n, \lambda) = O(\log n)$ for $\lambda < 1/\mathrm{e}$ and $L(n, \lambda) = \Theta(n)$ for $\lambda > 1/\mathrm{e}$. Aldous [3] posed the question on the detailed behavior of $L(n, \lambda)$ at and near criticality $1/\mathrm{e}$. In particular, Aldous asked whether there exist scaling exponents μ, ν such that

$$n^{-\mu} L(n, \mathrm{e}^{-1} + x n^{-\nu}) \to m(x) \tag{1}$$

in probability for some deterministic function $m(x)$ satisfying

$$\lim_{x \to \infty} m(x) = \infty, \lim_{x \to -\infty} m(x) = 0\,.$$

We show in this work that the critical behavior for the stochastic mean-field model is different from and far richer than that questioned as in (1). Our first result determines the order of $L(n, \lambda)$ at criticality as well as establishes the right order for the critical window, as incorporated below.

Theorem 1.1. *There exist absolute constants $\alpha, C, c > 0$ such that for all $\mathrm{e}^{-1} - (\log n)^{-2} \leqslant \lambda \leqslant \mathrm{e}^{-1} + \alpha(\log n)^{-2}$*

$$\mathbb{P}\left(c(\log n)^3 \leqslant L(n, \lambda) \leqslant C(\log n)^3\right) \to 1\,,$$

Remark. In a recent private communication, Aldous made a guess that $L(n, 1/\mathrm{e}) = n^{o(1)}$, which is confirmed by the preceding theorem.

Our second result shows a lower bound of polynomial in n on $L(n, \lambda)$ if $(\lambda - 1/\mathrm{e})/(\log n)^2$ exceeds a large absolute constant.

Theorem 1.2. *There exist absolute constants $\beta, C > 0$ such that for all $\lambda \geqslant \mathrm{e}^{-1} + \beta(\log n)^{-2}$*

$$\mathbb{P}\left(n^{1/4} \leqslant L(n, \lambda) \leqslant Cn(\lambda - \mathrm{e}^{-1})\right) \to 1\,,$$

Remark. It seems more careful analysis can improve the lower bound to $n^{1/2+o(1)}$. We chose not to do so because we believe it is still far away from being tight and thus the improvement is only technical.

Interestingly, Theorems 1.1 and 1.2 imply that there is yet another phase transition occurring somewhere in $1/\mathrm{e} + [\alpha(\log n)^{-2}, \beta(\log n)^{-2}]$. In addition, we demonstrate a smooth transition from subcritical to critical regime.

Theorem 1.3. *There exist absolute constants $C, c > 0$ such that for all $\lambda \leqslant \mathrm{e}^{-1} - (\log n)^{-2}$,*

$$\mathbb{P}\left(c(\mathrm{e}^{-1} - \lambda)^{-1} \log n \leqslant L(n, \lambda) \leqslant C(\mathrm{e}^{-1} - \lambda)^{-1} \log n\right) \to 1\,.$$

Related work. While our work focuses on the second order behavior (or finite-size scaling in the language of statistical physics), the first-order behavior was studied by Aldous [4]. It is believed that $L(n, \lambda)/n \to \delta(\lambda)$ in probability as $n \to \infty$ for some function $\delta(\lambda)$. Indeed, we see that $\delta(\lambda) = 0$ for $\lambda < 1/\mathrm{e}$. In [4], a non-rigorous derivation of $\delta(\lambda)$ using a reformulation of the cavity method gives that $\delta(\lambda) \asymp (\lambda - 1/\mathrm{e})^3$.

In addition, the quantity $L(n, \lambda)$ studied in this paper is a natural variant of several other objects that were studied before. If we consider the path of small maximal weight other than average weight, this is an extensively studied question of the longest path in Erdős-Rényi random graphs. For $G \sim G(n, c/n)$ (a random graph obtained by preserving each edge in complete graph with probability c/n independently), Ajtai, Komlós, and Szemerédi [1] proved that there is a path of length $\alpha(c)n$ where $\alpha(c) > 0$ for $c > 1$ and $\alpha(c) \to 1$ as $c \to \infty$; a similar and slightly weaker result was shown independently by Fernandez de la Vega [7]. Later, the attention was shifted to the asymptotics of $1 - \alpha(c)$. Improving a previous work of Bollobás [5], Frieze [8] obtained a sharp estimate on the asymptotics of $1 - \alpha(c)$ as $c \to \infty$. In addition, it is not hard to see that for $c < 1$ the longest path is of order $\log n$, and for $c = 1$ it can be deduced from a result of Nachmias and Peres [13] that the longest path is of order $n^{1/3}$.

Here we fix a maximum λ for the average weight and try to maximize the length for the longest path that satisfies this constraint. If we reverse the optimization (i.e., we insist on a path that visits every vertex and minimize the average weight), it becomes the classic traveling salesman problem in the mean-field setting. For this question, Wästlund [15] established the sharp asymptotics for more general distributions on the edge weight, confirming the Krauth-Mézard-Parisi conjecture [11, 12, 9].

Finally, the maximal size $T(n, \lambda)$ of the subtree whose average weight is at most λ was studied in [2]. It was shown that $T(n, \lambda)$ transitions from $o(n)$ to $\Theta(n)$ at some critical point λ_0 whose value can be specified in terms of a fixed point of a mapping on probability distributions.

Remark. While our work was in review, Mathieu and Wilson posted an article [10] studying the minimal mean-weight cycles in the same setting, where they demonstrated a transition at $1/\mathrm{e}$.

Main ideas of the proofs. We view the problem from a slightly different perspective. We first fix a length ℓ and compute the minimal average weight of all paths of length ℓ; then we vary ℓ to match this minimal average weight with λ. A simple and useful fact is that the minimal average weight is increasing with ℓ at least in a coarse sense (c.f. Claim 2.4).

With the aforementioned perspective in mind, our proof ideas can be traced back to Bramson's celebrated work [6], which gives a very precise evaluation on the minimal displacement of the branching Brownian motion. The main obstacle is that, we do not have a real tree structure in the mean-field setting (one could argue that it is locally tree-like, but certainly not globally), which is a crucial component in Bramson's argument. The lacking of a tree structure poses the challenges on how to control the correlation between different paths (say, of the same length) and how to select the truncation function for the second moment calculation, where the two issues are intrinsically related to each other. In what follows, we discuss the solution to these challenges focusing on the case of $\lambda = 1/\mathrm{e}$.

The solution arises from the following observations. Note that there are two opposite forces on the (maximal) deviation of the partial sums from the expectation for a typical path with small average weight. First, the deviation cannot be too large since otherwise there would exist a path whose average weight is too small, but that is unlikely due to a first moment calculation (c.f., Lemma 2.1). Second, the deviation cannot be too small for long paths since conditioning on the average weight of a path, the partial sums behave like a Brownian bridge and thus typically exhibits a deviation of order $\sqrt{\text{path length}}$ (c.f., Lemma 2.3). These two forces together, leaves no other possibility but that the path of small average weight is short.

Crucially, the aforementioned deviation of the path serves well as the truncation function (for the proof of the lower bound). Observe that a bad event which produces large probability for the average weights of both two paths to be small, is that the weights on the common edges for these two paths are unusually small. However, once restricted to paths of small deviation, the total weight of the common edges cannot differ much from the expectation (given the average weight of the path) and it is indeed bounded by the maximal deviation multiplied with the number of segments induced by these common edges (c.f., Definition 2.8). Another important ingredient is that the number of pairs (of the paths) decreases rapidly with the number of segments of the common edges (c.f., Lemma 2.10). Altogether, this allows us to control the correlation globally, and thus provides a way to prove the lower bound.

Discussions and further questions. Our work suggests a number of open questions. Naturally, one could ask what is the location and behavior for the second phase transition. The main obstacle for identifying the transition location seems to be that we have to select different truncation functions for the upper and lower bounds in the proof. More importantly, the probability costs for these two truncations are hugely different. The argument of Bramson also adopts different truncations (the so-called upper and lower envelops), but the probability costs for these two turn out to be of the same order in that case.

It would also be interesting to determine the right order of $L(n, \lambda)$ in the regime of Theorem 1.2. The lower bound we obtained there seems to be far away from being tight. The main limitation of our arguments is that, we rely heavily on the fact that the number of pairs (of paths) decreases rapidly with the number of segments for the common edges assuming a fixed number of common edges. This stops being true once the path under consideration gets too long.

An alternative direction is on the refined estimate at criticality. In particular, does

$$L(n, \mathrm{e}^{-1})/(\log n)^3 \to \xi \tag{2}$$

in probability for some $\xi > 0$? If so, what is the limit and what is the variance of $L(n, 1/\mathrm{e})$?

A word on notation. Throughout the paper, we denote by $C, c > 0$ absolute constants whose value could vary from line to line. Other absolute constants like $\alpha, \beta, C^\star, c^\star$ are fixed once for all. As we have different regimes to consider, we usually fix the value of the parameter λ and possibly other parameters in each section/subsection, and all of these settings for values will appear at the very beginning at each section/subsection.

2 Critical behavior within scaling window

In this section, we study the critical behavior within scaling window and prove Theorem 1.1.

2.1 Deviation of typical light path

For a path $\gamma = v_0, e_1, v_1, \ldots, e_\ell, v_\ell$ where v_{i-1} and v_i are endpoints of e_i for all $i \in [\ell]$ (of course a sequence of edges would already uniquely specify a path, but we purposely choose to emphasize both vertices and edges for a path in this work), let

$$X(\gamma) = \textstyle\sum_{i=1}^{\ell} X_{e_i} \tag{3}$$

be the (total) weight of γ. Clearly, $X(\gamma)$ follows Gamma distribution, which is of central importance throughout the work. Let $Z \sim \Gamma(\theta, k)$ be a Gamma variable with parameter (θ, k), that is to say, Z has the same law as a sum of k i.i.d. exponential variables with mean θ. We will repeatedly use the density function $f_{\theta,k}(z)$ of Z, where

$$f_{\theta,k}(z) = z^{k-1} \frac{\mathrm{e}^{-z/\theta}}{\theta^k (k-1)!} \text{ for all } z \geqslant 0, \theta > 0, k \in \mathbb{N}\,. \tag{4}$$

We first show that the average weight of a path cannot be significantly smaller than $1/\mathrm{e}$. For convenience of notation, denote by Γ_ℓ the collection of all paths of length ℓ, for any $\ell \in [n]$.

Lemma 2.1. *Let E_n be the event that there is a path of length ℓ with weight at most $\ell/\mathrm{e} - \log n$ by*

$$E_n = \cup_{\ell=1}^{n} \cup_{\gamma \in \Gamma_\ell} \left\{ X(\gamma) \leqslant \mathrm{e}^{-1}\ell - \log n \right\}. \tag{5}$$

Then $\mathbb{P}(E_n) \to 0$, as $n \to \infty$.

Proof. For any $\ell \in [n]$, we have $|\Gamma_\ell| \leqslant n^{\ell+1}$. In addition, by (4), the probability for each $\gamma \in \Gamma_\ell$ has total weight less than $\mathrm{e}^{-1}\ell - \log n$ is bounded by

$$\mathbb{P}\left(X(\gamma) \leqslant \mathrm{e}^{-1}\ell - \log n\right) \leqslant 10 \cdot (\mathrm{e}^{-1}\ell - \log n)^{\ell-1} \frac{\mathrm{e}^{-\ell/n}}{n^\ell (\ell-1)!} = O(n^{-(\ell+\mathrm{e})})\,,$$

where the last equality follows from Stirling's formula. An application of a union bound over Γ_ℓ and then over $\ell \in [n]$ yields the lemma. □

Define $M(\gamma)$ to be the deviation of γ away from the linear interpolation between the starting and ending edges, by

$$M(\gamma) = \sup_{1 \leqslant k \leqslant \ell} \left| \textstyle\sum_{i=1}^{k} X_{e_i} - \frac{k}{\ell} X(\gamma) \right|. \tag{6}$$

By Donsker's theorem, it is not hard to say that the deviation process $\{\sum_{i=1}^{k} X_{e_i} - \frac{k}{\ell} X(\gamma)\}$ conditioned on the value of $X(\gamma)$ converges to a Brownian bridge after suitable normalization. The deviation of a Brownian bridge, i.e., the maximum of the absolute values, is known to have Kolmogorov distribution, where the law can be written down explicitly as a sum of series. In particular, its left tail area has been obtained in [14] as follows:

$$\mathbb{P}\left(\max_{0\leqslant t\leqslant 1}|B_t| \leqslant \delta\right) = \frac{\sqrt{2\pi} + o_\delta(1)}{\delta} \mathrm{e}^{-\frac{\pi^2}{8\delta^2}}\,, \tag{7}$$

where $(B_t)_{0\leqslant t\leqslant 1}$ is a standard Brownian bridge, and $0 \leqslant o_\delta(1) \downarrow 0$ as $\delta \to 0$. The analog to deviation of Brownian bridge gives convincing evidence for the type of decay for the lower tail of $M(\gamma)$. However, as we are trying to analyze the tiny probability for a rare event, the desired estimate could not follow directly by convergence in law. We give a proof in what follows, without aiming at optimizing the exponents for the decay. We start with the next simple claim.

Claim 2.2. *For i.i.d. exponential variables Z_i with mean $\theta > 0$ and $m \leqslant n/2$, let $Z = \sum_{i=1}^{n} Z_i$ and $Z' = \sum_{i=1}^{m} Z_i$. Let $g(\cdot)$ be the density function of Z' conditioned on $Z = \theta n$. Then for all $1 \leqslant |z - \theta m| \leqslant 10\sqrt{\theta m}$,*

$$\tfrac{1}{10^6\sqrt{\theta m}} \leqslant g(z) \leqslant \tfrac{2}{\sqrt{\theta m}}\,.$$

Proof. Let $f_{k,\theta}(\cdot)$ be density function of Gamma distribution as in (4). By Bayesian formula, we obtain that

$$g(z) = \frac{f_{m,\theta}(z) f_{n-m,\theta}(\theta n - z)}{f_{n,\theta}(\theta n)} = \frac{z^{m-1}\mathrm{e}^{-z/\theta}}{\theta^m (m-1)!} \frac{(\theta n - z)^{n-m-1}\mathrm{e}^{-(\theta n - z)/\theta}}{\theta^{n-m}(n-m-1)!} \frac{\theta^n (n-1)!}{(\theta n)^{n-1}\mathrm{e}^{-n}}\,.$$

Now the claim follows from a direct computation with an application of Stirling's formula. □

Lemma 2.3. *Let Z_i be i.i.d. exponential variables with mean $\theta > 0$ for $1 \leqslant i \leqslant n$. For $1/4 \leqslant \rho \leqslant 4$, consider the variable*

$$M = M_n = \sup_{1\leqslant k\leqslant n} |\textstyle\sum_{i=1}^{k} Z_i - \rho k|. \tag{8}$$

Then, there exist absolute constants $c^\star, C^\star > 0$ such that for all $r \geqslant 1$ and $n \geqslant r^2$,

$$\mathrm{e}^{-C^\star n/r^2} \leqslant \mathbb{P}(M \leqslant r \mid \textstyle\sum_{i=1}^{n} Z_i = \rho n) \leqslant \mathrm{e}^{-c^\star n/r^2}\,.$$

Proof. First observe a useful property for exponential variables: for i.i.d. exponential variables Y_i with mean θ_1 and i.i.d. exponential variables Z_i with mean θ_2 for any $\theta_1, \theta_2 > 0$, we have that for all $k \in \mathbb{N}$ and $z > 0$

$$(Y_1, Y_2, \ldots, Y_k \mid \textstyle\sum_{i=1}^{k} Y_i = z) \overset{law}{=} (Z_1, Z_2, \ldots, Z_k \mid \textstyle\sum_{i=1}^{k} Z_i = z)\,, \tag{9}$$

since both vectors are uniformly distributed over $\{(z_1, \ldots, z_k) \in \mathbb{R}_+^k : \sum_i z_i = z\}$ conditioning on the sums being z (Note that this property is known in Statistics as "sufficiency" of the sample mean for the parameter in the family of Exponential distributions index by the mean). Therefore, we can in what follows assume that $\theta = \rho$.

We now give a proof for the upper bound. For convenience, we assume that r is a positive integer. The intuition (also for the lower bound) is that we can divide n into blocks of size r^2, and in every such a block the fluctuation of the path is of order r and thus the probability for

the path in this block to stay within $[-r, r]$ is bounded away from 0 and 1. Since the number of blocks is n/r^2, this gives the right type of decay for the lower tail of fluctuation. Precisely, for $j = 1, \ldots, \lfloor n/r^2 \rfloor - 1$, define the event Q_j by

$$Q_j = \cap_{(j-1)r^2 \leqslant k \leqslant jr^2} \{ |\textstyle\sum_{i=1}^k Z_i - \rho k| \leqslant r \} .$$

It is clear that

$$\mathbb{P}(Q_{j+1} \mid Q_1, \ldots, Q_j) \leqslant \mathbb{P}\left(|\textstyle\sum_{(j-1)r^2 \leqslant i \leqslant jr^2} Z_i - \rho r| \leqslant 2r \mid Q_1, \ldots Q_j \right) \leqslant 1 - 10^{-7} ,$$

where the last step follows from Claim 2.2. This yields that

$$\mathbb{P}\left(M \leqslant r \mid \textstyle\sum_{i=1}^n Z_i = \rho n \right) \leqslant \mathbb{P}\left(Q_j : \forall 1 \leqslant j \leqslant \lfloor n/r^2 \rfloor - 1 \right) \leqslant (1 - 10^{-7})^{\lfloor n/r^2 \rfloor - 1} .$$

Now we turn to the proof of lower bound. For $j = 1, 2, \ldots, \lfloor n/r^2 \rfloor$, define the event

$$R_j = \{ |\textstyle\sum_{i=1}^{jr^2} Z_i - \rho j r^2| \leqslant r/2 \}, \text{ and } S_j = \cap_{(j-1)r^2 \leqslant k \leqslant jr^2} \{ |\textstyle\sum_{i=(j-1)r^2}^k Z_i - \rho(k - (j-1)r^2)| \leqslant r/2 \}.$$

It is clear from triangle inequality that

$$\cap_{1 \leqslant j \leqslant n/r^2 - 1} R_j \cap \cap_{1 \leqslant j \leqslant n/r^2} S_j \subseteq \{ M \leqslant r \} . \tag{10}$$

Furthermore, by Claim 2.2 again, we get that (write $\mathcal{R} = \cap_{1 \leqslant j \leqslant n/r^2 - 1} R_j$)

$$\mathbb{P}(\mathcal{R}) \geqslant \textstyle\prod_{1 \leqslant j \leqslant n/r^2 - 1} \mathbb{P}(R_j \mid R_1, \ldots, R_{j-1}) \geqslant 10^{-8n/r^2} . \tag{11}$$

In addition, conditioned on $\sum_{i=(j-1)r^2}^{jr^2} Z_i = s$, we see that $\frac{1}{\sqrt{r}} (\sum_{i=(j-1)r^2}^{(j-1)r^2 + tr^2} Z_i)$ for $1 \leqslant t \leqslant 1$ converges to a standard Brownian bridge, and thus (see (7)) for $r \geqslant r_0$ where r_0 is a large absolute constant, we have that $\mathbb{P}(S_j \mid \mathcal{R}, \sum_{i=(j-1)r^2}^{jr^2} Z_i = s) \geqslant 10^{-2}$. Trivially, for $r \leqslant r_0$ we have $\mathbb{P}(S_j \mid \mathcal{R}, \sum_{i=(j-1)r^2}^{jr^2} Z_i = s) \geqslant 10^{-r_0}$. Since given the sum in each block, the variables in different blocks are independent, we can then deduce that $\mathbb{P}(\cap_{j=1}^{n/r^2} S_j \mid \mathcal{R}) \geqslant 10^{-r_0 n/r^2}$. Combined with (10) and (11), this gives the desired lower bound with $C^\star = 3r_0$. □

2.2 Upper bound

In this subsection, we prove the upper bound for Theorem 1.1. Recall that $c^\star > 0$ is the absolute constant defined in Lemma 2.3. Set

$$\alpha = \tfrac{c^\star}{27\mathrm{e}} . \tag{12}$$

Fix in this subsection

$$\lambda = \lambda_\alpha = \mathrm{e}^{-1} + \alpha (\log n)^{-2} .$$

By monotonicity, it suffices to give an upper bound on $L(n, \lambda)$. We start with a simple claim, reducing the consideration to paths of length between $[\ell, 2\ell)$ for the purpose of showing $L(n, \lambda) \leqslant \ell$.

Claim 2.4. *The following holds deterministically. Suppose that there exists a path of length $L \geqslant \ell$ such that the average weight is ζ for some $\zeta > 0$. Then there exists a path of length between $[\ell, 2\ell)$ such that the average weight is at most ζ.*

Proof. Let γ be a path of length $L \geqslant \ell$ with average weight ζ. Suppose that γ consists a sequence of consecutive edges $e_1, \dots, e_L$. Write $k = \lfloor L/\ell \rfloor$, and we divide γ into a collection of k edge-disjoint paths where γ_i consists of edges $e_{i\ell+1}, \dots, e_{(i+1)\ell}$ for $0 \leqslant i < k-1$ and γ_{k-1} consists of edges $e_{(k-1)\ell+1}, \dots, e_L$. Obviously $|\gamma_i| = \ell$ for all $0 \leqslant i < k-1$ and $|\gamma_{k-1}| \in [\ell, 2\ell)$. Since $\gamma = \cup_{i=0}^{k-1} \gamma_i$, we see that at least one of paths γ_i must have average weight at most ζ, as required. □

We now show that there cannot exist a long path with small average weight but even moderately large deviation.

Lemma 2.5. *For $\ell = (\log n)^3/\alpha$, we have*

$$\mathbb{P}\left(\exists \ell \leqslant \ell' < 2\ell, \gamma \in \Gamma_{\ell'} : X(\gamma) \leqslant \lambda \ell', M(\gamma) \geqslant 3 \log n\right) \to 0\,.$$

Proof. Suppose there exists $\ell \leqslant \ell' < 2\ell$ and $\gamma \in \Gamma_{\ell'}$ such that $X(\gamma) \leqslant \lambda \ell'$ and $M(\gamma) \geqslant 3 \log n$. Denote by $\gamma = v_0, e_1, v_1, \dots, e_{\ell'}, v_{\ell'}$, and let ℓ^* be such that

$$M(\gamma) = |\textstyle\sum_{i=1}^{\ell^*} X_{e_i} - \frac{\ell^*}{\ell'} X(\gamma)|\,.$$

Consider two sub-paths $\gamma_1 = v_0, e_1, \dots, e_{\ell^*}, v_{\ell^*}$ and $\gamma_2 = v_{\ell^*}, e_{\ell^*+1}, \dots, e_{\ell'}, v_{\ell'}$. By our assumption on γ and definition of ℓ^*, we have

$$\text{either} \quad X(\gamma_1) \leqslant \mathrm{e}^{-1}\ell^* - \log n \quad \text{or} \quad X(\gamma_2) \leqslant \mathrm{e}^{-1}(\ell' - \ell^*) - \log n\,.$$

This implies that

$$\{\exists \gamma \in \Gamma_{\ell'} : X(\gamma) \leqslant \lambda \ell', M(\gamma) \geqslant 3 \log n\} \subseteq E_n\,,$$

where E_n is the event defined in (5). The desired estimate now follows from Lemma 2.1. □

We next turn to control paths with small deviation.

Lemma 2.6. *For $\ell = (\log n)^3/\alpha$, we have*

$$\mathbb{P}\left(\exists \ell \leqslant \ell' < 2\ell, \gamma \in \Gamma_{\ell} : X(\gamma) \leqslant \lambda \ell, M(\gamma) \leqslant 3 \log n\right) \to 0\,.$$

Proof. Fix an ℓ' with $\ell \leqslant \ell' < 2\ell$ and fix $\gamma \in \Gamma_{\ell'}$. By (4) and Lemma 2.3, we obtain that

$$\mathbb{P}\left(X(\gamma) \leqslant \lambda \ell', M(\gamma) \leqslant 3 \log n\right) = O(1) n^{-\ell'} \ell'^{-1/2} \mathrm{e}^{2\mathrm{e} \log n} \mathrm{e}^{-c^\star \log n/(9\alpha)} = O(1) n^{-\ell'} n^{-\mathrm{e}}\,,$$

where the last equality follows from the definition of α in (12). Noting that $|\Gamma'_\ell| \leqslant n^{\ell'+1}$, we deduce the desired result by first applying a union bound over $\gamma \in \Gamma_{\ell'}$ and then over $\ell \leqslant \ell' < 2\ell$. □

The upper bound for Theorem 1.1 is an immediate consequence of Lemmas 2.5 and 2.6, together with Claim 2.4.

2.3 Lower bound

In this subsection, we prove the lower bound for Theorem 1.1. Fix $\lambda = \mathrm{e}^{-1} - (\log n)^{-2}$ in this subsection, and it suffices to establish the lower bound on $L(n, \lambda)$. Let $c, \delta > 0$ be two small absolute constants to be selected. For $1 \leqslant \ell \leqslant n$ and $\gamma \in \Gamma_\ell$, define

$$F_\gamma = \{\lambda\ell - 1 \leqslant X(\gamma) \leqslant \lambda\ell, M(\gamma) \leqslant \delta \log n\}\,. \tag{13}$$

By (4) and Lemma 2.3, we obtain that for all $\gamma \in \Gamma_\ell$ with $\ell = c(\log n)^3$

$$\mathbb{P}(F_\gamma) \geqslant \tfrac{1}{100} n^{-\ell} \ell^{-1/2} n^{-ec - cC^\star/\delta^2}\,, \tag{14}$$

where $C^\star$ is the absolute constant from Lemma 2.3. Defining

$$N = \textstyle\sum_{\gamma \in \Gamma_\ell} \mathbf{1}_{F_\gamma}\,,$$

we see that the first moment of N would be large if we select c, δ properly. The key issue here is to bound the second moment of N.

Lemma 2.7. *Consider $\ell = c(\log n)^3$. For any $\gamma \in \Gamma_\ell$, we have that*

$$\textstyle\sum_{\gamma' \in \Gamma_\ell} \mathbb{P}(F_\gamma \cap F_{\gamma'}) \leqslant \mathbb{P}(F_\gamma) \cdot (\mathbb{E}N + O(1)\ell^3 n^\delta)\,.$$

In order to prove the preceding lemma, one needs to study the correlation structure between γ and all other paths in Γ_ℓ. In order to have a global control of the correlation between γ and all other paths, a natural strategy is to first select different scales of correlations and then estimate the cardinality of the paths that fall into each scale. This strategy was implemented in the case of branching Brownian motion where the scale for correlation is chosen to be the number of common edges between a path and the path γ. In the mean-field setting, we do not really have a tree structure (as two paths can bifurcate and merge and then bifurcate...). In addition, merely the number of common edges does not seem to fully characterize the correlation between two paths. Therefore, we need to choose an auxiliary quantity which together with the number of common edges, offers an effective measurement of the correlation. We elaborate in what follows.

For a path γ, denote by $E(\gamma)$ the collection of edges in γ. For $S \subseteq E(\gamma)$, we call a segment of γ an S-component if it is a maximal segment of γ where all the edges belong to S.

Definition 2.8. *For two paths γ and γ', we define a functional $\theta(\gamma, \gamma')$ to be the number of S-components of γ where $S = E(\gamma) \cap E(\gamma')$.*

The functional $\theta(\gamma, \gamma')$ turns out to be a good additional measurement for the correlation between γ and γ'. Given a collection of edges S, denote by $V(S)$ the collection of vertices which are endpoints for edges in S. The next simple observation is of crucial importance for our proof.

Lemma 2.9. *For $1 \leqslant \ell \leqslant n$ and $\gamma, \gamma' \in \Gamma_\ell$, Write $S = E(\gamma) \cap E(\gamma')$. We have*

$$|V(S)| = |S| + \theta(\gamma, \gamma')\,.$$

Proof. By definition, there exist no edge in S that crosses different S-components of γ, and there exists no vertex in $V(S)$ that is belonging to different S-components. Therefore, we can analyze each S-component separately. In addition, it is obvious that for each such S-component, the number of vertices is larger than the number of edges by 1. Summing over all the S-components, we complete the proof of the lemma. □

Given a path $\gamma' \in \Gamma_\ell$, we now partition Γ_ℓ based on its correlation with γ, i.e., based on the tuple $(\theta(\gamma,\gamma'), |E(\gamma)\cap E(\gamma')|)$. Precisely, for all integers $i \leqslant j$, define

$$A_{i,j} \equiv A_{i,j}(\gamma) := \{\gamma' \in \Gamma_\ell : \theta(\gamma,\gamma') = i, E(\gamma)\cap E(\gamma') = j\}\,. \tag{15}$$

It is now natural to control the cardinality of $A_{i,j}$. One could prove more precise estimate on $A_{i,j}$, but for our purpose, the following is sufficient.

Lemma 2.10. *For any $1 \leqslant \ell \leqslant n$ and any $\gamma \in \Gamma_\ell$, we have that for any non-negative integers $i \leqslant j$*

$$|A_{i,j}(\gamma)| \leqslant \binom{\ell+1}{2i}\binom{n-i-j}{\ell+1-i-j} 2^i(\ell+1-j)! \leqslant \ell^{3i} n^{\ell+1-i-j}\,.$$

Proof. In order to bound $|A_{i,j}|$, we consider the following procedure to generate a path in $A_{i,j}$:

1. Select i vertex disjoint segments from γ such that the total number of edges is j (and thus the total number of vertices is $i+j$ by Lemma 2.9).
2. Select $\ell+1-i-j$ vertices from remaining $n-i-j$ vertices in the graph.
3. Choose a direction for each of the segment (two options for every segment). Then take each of the i segments as an individual element and permute these i elements and $\ell+1-i-j$ vertices selected from step 2, such that no more edges in γ will be introduced when the permutation is viewed as a path.

Clearly, the cardinality of $A_{i,j}$ is bounded by the product of the number of choices N_k (for $k = 1,2,3$) in each step. For Step 1, we see that the choice for the edges is complete determined by the $2i$ endpoints selected from the path γ and vice versa, and thus $N_1 \leqslant \binom{\ell+1}{2i}$. For Step 2, we have $N_2 = \binom{n-i-j}{\ell+1-i-j}$. For Step 3, it is obvious that $N_3 \leqslant 2^i(\ell+1-j)!$. Taking a product for N_1, N_2 and N_3, we complete the proof of the lemma. □

We are now ready to give

Proof of Lemma 2.7. For $\ell = c(\log n)^3$ and $\gamma \in \Gamma_\ell$, we consider Γ_ℓ as a union over $A_{i,j}$. For $i = 0$, the only legitimate choice of j is also 0, and in this case $E(\gamma')\cap E(\gamma) = \emptyset$ for all $\gamma' \in A_{0,0}$. Therefore, the events F_γ and $F(\gamma')$ are independent. Thus,

$$\textstyle\sum_{\gamma'\in A_{0,0}} \mathbb{P}(F_\gamma \cap F_{\gamma'}) = \sum_{\gamma'\in A_{0,0}} \mathbb{P}(F_\gamma)\cdot\mathbb{P}(F_{\gamma'}) \leqslant \mathbb{P}(F_\gamma)\mathbb{E}N\,. \tag{16}$$

Since $|E(\gamma)\cap E(\gamma')| \geqslant \theta(\gamma,\gamma')$ always, we next consider $1 \leqslant i \leqslant j \leqslant \ell$. Take $\gamma' \in A_{i,j}$, and we wish to bound the probability for the event $F_{\gamma'}$ conditioned on F_γ. Write $S = E(\gamma)\cap E(\gamma')$ and $S' = E(\gamma')\setminus S$. Conditioned on F_γ (see the definition of F_γ in (13)), we have that

$$\textstyle\sum_{e\in S} X_e \geqslant |S|\lambda - 1 - 2i\delta\log n\,.$$

Therefore, we obtain that

$$\mathbb{P}(F_{\gamma'} \mid F_\gamma) \leqslant \mathbb{P}\left(\textstyle\sum_{e\in S'} X_e \leqslant \lambda|S'| + 1 + 2i\delta\log n \mid F_\gamma\right) = \mathbb{P}\left(\textstyle\sum_{e\in S'} X_e \leqslant \lambda|S'| + 1 + 2i\delta\log n\right)\,, \tag{17}$$

where in the last inequality we used independence of these exponential variables X_e. Recalling that $|S'| = \ell - j$, we get from (4) that

$$\mathbb{P}(F_{\gamma'} \mid F_\gamma) \leqslant O(1) n^{-(\ell-j)} n^{2i\delta}\,.$$

By Lemma 2.10, we see that $A_{i,j} \leqslant \ell^{3i} n^{\ell+1-i-j}$. A simple union bound then yields that

$$\sum_{\gamma' \in A_{i,j}} \mathbb{P}(F_{\gamma'} \mid F_\gamma) \leqslant O(n)\ell^{3i} n^{-(1-2\delta)i}\,.$$

Summing over $1 \leqslant i \leqslant j \leqslant \ell$, we then obtain that

$$\sum_{1 \leqslant i \leqslant j \leqslant \ell} \sum_{\gamma' \in A_{i,j}} \mathbb{P}(F_{\gamma'} \mid F_\gamma) \leqslant O(1)\ell^5 n^{2\delta}\,.$$

Combined with (16), it completes the proof of the lemma. □

We next conclude this subsection with the proof for the lower bound on $L(n, \lambda)$.

Proof of Theorem 1.1: lower bound. Set $\delta = \frac{1}{10}\min(1, 1/C^\star)$ and $c = \delta^3$. Since $|\Gamma_\ell| = (1 + o(1))n^{\ell+1}$, the estimate (14) then gives that for sufficiently enough n

$$\mathbb{E}N \geqslant n^{4/5}\,.$$

Meanwhile, by Lemma 2.7, we have that

$$\mathbb{E}N^2 \leqslant \mathbb{E}N(\mathbb{E}N + n^{1/5}) = (1 + o(1))(\mathbb{E}N)^2\,.$$

At this point, a simple application of Chebyshev's inequality gives that $\mathbb{P}(N > 0) \to 1$ as $n \to \infty$, completing the proof for the lower bound. □

3 The existence of second transition point

Throughout this section, we let $\lambda = \mathrm{e}^{-1} + \varepsilon$ and assume that $\varepsilon \geqslant \beta(\log n)^{-2}$ for an absolute large constant $\beta \geqslant 10$ to be specified later. The goal of this subsection is to demonstrate the existence of another phase transition at a point in $[\mathrm{e}^{-1} + \alpha(\log n)^{-2}, \mathrm{e}^{-1} + \beta(\log n)^{-2}]$. To this end, we prove Theorem 1.2 in this section, whose main content is a polynomial lower bound on $L(n, \lambda)$ when $\lambda - \mathrm{e}^{-1} \geqslant \beta(\log n)^{-2}$.

The upper bound for Theorem 1.2 follows from a straightforward first moment computation, as quickly incorporated in what follows.

Proof of Theorem 1.2: upper bound. Consider $\ell \geqslant 6\varepsilon n$. For any $\gamma \in \Gamma_\ell$, we have from (4) that

$$\mathbb{P}(X(\gamma) \leqslant \lambda\ell) = O(1)((\mathrm{e}^{-1} + \varepsilon)\ell)^{\ell-1} \tfrac{1}{n^\ell(\ell-1)!} = O(1/\sqrt{\ell})\mathrm{e}^{\varepsilon \mathrm{e}\ell}\,.$$

In addition, we get that $|\Gamma_\ell| = \prod_{i=0}^{\ell}(n - i) \leqslant n^{\ell+1}\mathrm{e}^{-\frac{\ell^2}{2n}}$. A simple union bound over Γ_ℓ and $6\varepsilon n \leqslant \ell \leqslant 12\varepsilon n$ then gives that

$$\mathbb{P}\left(\exists 6\varepsilon n \leqslant \ell \leqslant 12\varepsilon n, \gamma \in \Gamma_\ell : X(\gamma) \leqslant \lambda\ell\right) \to 0\,.$$

Combined with Claim 2.4, this gives the upper bound. □

3.1 Deviation of typical light path: revisited

In view of the lower tail of the deviation as in Lemma 2.3, it is obvious that when the length of the path gets large, the tail gets extremely small and thus needs to be tracked down carefully. In particular, we need an estimate for the lower tail of the deviation given the values of some of the variables along the path. We handle this delicate issue in this subsection.

Lemma 3.1. *Let Z_i be i.i.d. exponential variables for $i \in \mathbb{N}$. Let $1/4 \leqslant \rho \leqslant 1$ and M_n be defined as in* (8)*. Write for all $s \in \mathbb{N}$*

$$p_s = \mathbb{P}\left(M_s \leqslant r \mid \textstyle\sum_{i=1}^{s} Z_i = \rho s\right) . \tag{18}$$

Then for $j, k \in \mathbb{N}$, we have

$$p_{j+k} \geqslant \frac{1}{10^8 r\sqrt{j \wedge k}} p_j p_k \, .$$

Proof. Assume that $j \leqslant k$. The proof follows from a natural idea: conditioning on the partial sum of the first j variables. Given that this partial sum is close to the expectation within a window of size 1 (which occurs with probability $1/\sqrt{j \wedge k}$ as $j \wedge k$ is the variance for this partial sum), the two segments are independent and the probability for each of them to have deviation smaller than r is very close to p_j and p_k. Noting that the probability for the whole sequence to have deviation smaller than r is larger than the product of the three aforementioned probabilities, we can then complete the argument. In what follows, we carry out the technical details.

Denote by $\Omega_\delta = \{(z_i) : \sum_{i=1}^{j+k} z_i = \rho(j+k), \sum_{i=1}^{j} z_i = \rho j + \delta\rho\}$, and by $\Omega = \cup_{0 \leqslant \delta \leqslant 1/2} \Omega_\delta$. By Claim 2.2, we see that

$$\mathbb{P}((Z_i) \in \Omega) \geqslant \tfrac{1}{2 \cdot 10^6 \sqrt{j}} \, .$$

Just for the technical reason (which will be clear later), we partition Ω_0 (Ω_0 is defined as the aforementioned Ω_δ with $\delta = 0$) into a union of sets $\Omega_{0,\tau}$ such that $(z_i) \in \Omega_{0,\tau}$ if and only if $\tau = \min\{i \geqslant j+1 : z_i \geqslant \delta\rho\}$. Let $\Xi \in \mathbb{R}^{j+k}$ be such that for all $(z_i) \in \Xi$

$$|\textstyle\sum_{i=1}^{s} z_i - \rho s| \leqslant r \text{ for all } 1 \leqslant s \leqslant j+k \, . \tag{19}$$

Since sequences in Ξ has small deviation, we see that $\Xi \subseteq \cup_{\tau=1}^{2r} \Omega_{0,\tau}$. Choose τ^* such that

$$\mathbb{P}\left((Z_i) \in \Xi \cap \Omega_{0,\tau^*}\right) = \max_{1 \leqslant \tau \leqslant 2r} \mathbb{P}\left((Z_i) \in \Xi \cap \Omega_{0,\tau}\right) \geqslant \tfrac{1}{2r} \mathbb{P}((Z_i) \in \Xi \cap \Omega_0) \, .$$

Next, we show that for all $0 \leqslant \delta \leqslant 1/2$, we have

$$\mathbb{P}((Z_i) \in \Xi \mid (Z_i) \in \Omega_\delta) \geqslant \tfrac{1}{10} \mathbb{P}\left((Z_i) \in \Xi \cap \Omega_{0,\tau^*} \mid (Z_i) \in \Omega_0\right) . \tag{20}$$

For $(z_i) \in \Omega_{0,\tau^*} \cap \Xi$, we map (z_i) to (z_i') by let $(z_i') \in \Omega_\delta$ be such that $z_i' = z_i$ for $i \neq j, \tau^*$ and $z_j' = z_j + \delta\rho$ and $z_{\tau^*}' = z_{\tau^*} - \delta\rho$ (the assumption that $z_{\tau^*} \geqslant \delta\rho$ guarantees that $z_{j+1} \geqslant 0$). Since $0 \leqslant \delta \leqslant 1/2 \leqslant r$, it is clear that the sequence (z_i') also satisfies (19). Also, we see that the determinant of the Jacobian matrix of this mapping is 1. It remains to compare the densities for (Z_i) at (z_i) and (z_i') given $(Z_i) \in \Omega_0$ and $(Z_i) \in \Omega_\delta$, respectively. It is obvious and straightforward to check that the ratio of these two densities are within a constant factor, say, 10. This yields that

$$p_{j+k} \geqslant \tfrac{1}{10} \mathbb{P}((Z_i) \in \Omega) \mathbb{P}\left((Z_i) \in \Xi \cap \Omega_{0,\tau^*} \mid \Omega_0\right) \geqslant \tfrac{1}{10^7 \cdot 2r\sqrt{j}} p_j p_k \, ,$$

where the last inequality we used conditional independence given Ω_0. Altogether, this completes the proof of the lemma. □

Lemma 3.2. *Let Z_i be i.i.d. exponential variables for $i \in \mathbb{N}$. Consider $1 \leqslant r \leqslant \sqrt{n}$ and $1 \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant \ldots \leqslant a_m \leqslant b_m \leqslant n$ such that $q = \sum_{i=1}^m (b_i - a_i + 1) \leqslant n - 10r$. Let $1/4 \leqslant \rho \leqslant 1$ and M_n be defined as in (8). Then for all z_j such that*

$$\textstyle\sum_{j=a_i}^{b_i} z_j - \rho(b_i - a_i + 1) \leqslant 2r\,,$$

we have (write $A = \cup_{i=1}^m [a_i, b_i] \cap \mathbb{N}$ and use the notation of p_s as in (18))

$$\mathbb{P}\left(M_n \leqslant r \mid \textstyle\sum_{i=1}^n Z_i = \rho n, Z_j = z_j \forall j \in A\right) \leqslant O(r\sqrt{q \wedge n - q}) p_n 10^{100mr} \mathrm{e}^{C^\star q/r^2}\,,$$

where $C^\star$ is the absolute constant from Lemma 2.3.

Proof. We first sketch the outline of the proof. Since we are conditioned on the sum of Z_i, by (9) the mean of Z_i is irrelevant. Just for convenience, some times we assume that Z_i has mean ρ. We will consider a new sequence of i.i.d. exponential variables (Z_i'), whose average will also be conditioned to be ρ. In addition, the size n' of the new sequence is larger than the number of free variables (the variables that are not conditioned to be a given value) in the old sequence (Z_i) by $10mr$. For each segment $[a_i, b_i]$, we force a segment of size $10r$ in (Z_i') such that its partial sum (biased by ρ for each variable) grows almost linearly with end points being 0 and $\sum_{j \in [a_i, b_i]} (z_j - \rho)$ (say the probability cost is p). Given this linear interpolation, the free variables in (Z_i) and (Z_i') will have almost the same distribution and we couple them together. Furthermore, if the free variables are such that the deviation in (Z_i) is less than r (say this occurs with probability p'), so should it be in (Z_i') by our construction. But we know that the probability for (Z_i') to have deviation smaller than r is $p_{n'}$. Therefore, we can deduce the bound $p_{n'} \geqslant p \cdot p'$. The technical details are carried out in what follows.

Let $(Z_i')_{1 \leqslant i \leqslant n'}$ be i.i.d. exponential variables where $n' = n - q + 10mr$. We first define the mapping $\phi(\cdot)$ between the coordinates of the original sequence (Z_i) and our new sequence (Z_i'), by

$$\phi(t) = |i \leqslant t : i \notin A| + 10r|i : b_i \leqslant t|\,.$$

Note that $\phi(t)$ remains constant over $[a_i, b_i)$. The intuition behind is that we replace each segment $[a_i, b_i]$ in the original sequence by a segment of size $10r$. Write $s_i = \sum_{j=a_i}^{b_i} z_j - \rho(b_i - a_i + 1)$, for $1 \leqslant i \leqslant m$. By definition, we have $|s_i| \leqslant 2r$. For $0 \leqslant \delta_i \leqslant 1/20$, we define $\Omega_{(\delta_i)} \subseteq \mathbb{R}^{n'}$ such that $(x_j)_{1 \leqslant j \leqslant n'} \in \Omega_{(\delta_i)}$ if for all $1 \leqslant i \leqslant m$

$$\begin{gathered}\forall 1 \leqslant k < 10r : -1/20 \leqslant \textstyle\sum_{j=1}^k x_{\phi(b_i) - 10r + j} - k(\rho + s_i/10r) \leqslant 0, \\ \textstyle\sum_{j=1}^{10r} x_{\phi(b_i) - 10r + j} = 10r\rho + s_i - \delta_i\,.\end{gathered}$$

The idea, as we outlined before, is that we use the added segment to linearly interpolate the (biased) endpoints of the old segment while we keep the total sum (biased by mean) to be very close. Denote by $\Omega = \cup_{(\delta_i) \in [0, 1/20]^m} \Omega_{(\delta_i)}$. A repeated application of Claim 2.2 gives that

$$\mathbb{P}((Z_j')_{1 \leqslant j \leqslant n'} \in \Omega) \geqslant 10^{-80mr}\,.$$

Next, consider $\Xi \subseteq \mathbb{R}^n$ such that $(x_j) \in \Xi$ if,

$$\forall j \in A : x_j = z_j, \text{ and } \forall k \in [n] : |\textstyle\sum_{j=1}^k x_k - \rho k| \leqslant r, \text{ and } \sum_{j=1}^n x_j = \rho n\,.$$

Just as a translation of our definition of Ξ, we have

$$\mathbb{P}\left(M_n \leqslant r \mid \sum_{j=1}^n Z_j = \rho n, Z_j = z_j \forall j \in A\right) = \mathbb{P}\left((Z_j)_{1\leqslant j\leqslant n} \in \Xi \mid \sum_{i=j}^n Z_j = \rho n, Z_i = z_i \forall i \in A\right). \tag{21}$$

Now for each (δ_i) and $(\omega_j)_{1\leqslant j\leqslant n'} \in \Omega_{(\delta_i)}$, we construct a mapping $\psi_{(\delta_i),(\omega_j)} : \Xi \mapsto \Omega_{(\delta_i)}$ such that it maps $(x_j)_{1\leqslant j\leqslant n} \in \Xi$ to $(y_j)_{1\leqslant j\leqslant n'}$

$$\forall j \in A : y_j = \omega_j\,, \forall j \notin A \cup \{b_1, \ldots, b_m\} : y_{\phi_j} = x_j, \text{ and } y_{\phi_{b_i}+1} = x_{b_i+1} + \delta_i\,.$$

Crucially, the density of $(Z_j)_{1\leqslant j\leqslant n}$ at $(x_j)_{1\leqslant j\leqslant n}$ given that $\{Z_j = z_j \forall j \in A\}$ and $\sum_{j\in[n]} Z_j = \rho n$, is comparable with the density of $(Z'_j)_{1\leqslant j\leqslant n'}$ at any $(y_j)_{1\leqslant j\leqslant n'}$ given that $(Z'_j)_{1\leqslant j\leqslant n'} \in \Omega_{(\delta_i)}$ and $y_j = \omega_j \forall j \in A$ and $\sum_{j=1}^{n'} Z'_j = \rho n'$. Indeed, the ratio between these two densities can be directed computed and founded to be within a factor of 10^m. In order to see this, note that given these conditions, the two random vectors have the same number of free variables and the sum of these free variables differ by amount of order m.

Define $\Xi' \subseteq \mathbb{R}^{n'}$ such that if $(y_j) \in \Xi'$,

$$(y_j) \in \Omega, \forall k \in [n'] : |\textstyle\sum_{j=1}^k y_j - k\rho| \leqslant r, \text{ and } \sum_{j=1}^{n'} y_j = \rho n'\,.$$

By definition, we can verify that for every $(\omega_j) \in \Omega_{(\delta_i)}$

$$\psi_{(\delta_i),(\omega_j)}(\Xi) \in \Xi' \text{ for all } (\delta_i) \in [0, 1/20]^m\,.$$

We can then finally conclude that

$$\begin{aligned}
&\mathbb{P}\left((Z'_j)_{1\leqslant j\leqslant n'} \in \Xi' \mid \textstyle\sum_{j=1}^{n'} Z'_j = \rho n'\right) \\
&\geqslant \mathbb{P}\left((Z'_j)_{1\leqslant j\leqslant n'} \in \Omega) \min_{(\delta_i)} \mathbb{P}((Z'_j)_{1\leqslant j\leqslant n'} \in \Xi' \mid \textstyle\sum_{j=1}^{n'} Z'_j = \rho n', (Z'_j)_{1\leqslant j\leqslant n'} \in \Omega_{(\delta_i)}\right) \\
&\geqslant 10^{-8rm} 10^{-m} \mathbb{P}\left((Z_j)_{1\leqslant j\leqslant n} \in \Xi \mid Z_j = z_j \forall j \notin A, \textstyle\sum_{j=1}^n Z_n = \rho n\right).
\end{aligned}$$

Combined with (21), it follows that

$$\mathbb{P}\left(M_n \leqslant r \mid \textstyle\sum_{i=1}^n Z_i = \rho n, Z_i = z_i \forall i \in A\right) \leqslant 10^{(8r+1)m} p_{n'}\,.$$

Now the desired estimates follows from Lemma 3.1 and Lemma 2.3. □

3.2 Lower bound

Throughout this subsection, fix $\lambda = \mathrm{e}^{-1} + \beta(\log n)^{-2}$ for a large absolute constant $\beta > 0$ to be specified. Write $\ell = n^{1/4}$. Let $\zeta > 0$ be a small constant to be specified later. For $\gamma \in \Gamma_\ell$, define

$$G_\gamma = \{\lambda\ell - 1 \leqslant X(\gamma) \leqslant \lambda\ell, M(\gamma) \leqslant \zeta \log n \cdot X(\gamma)/\lambda\ell\}\,. \tag{22}$$

Remark. Note that given that the event $G(\gamma)$ occurs, we always have $M(\gamma) \leqslant \zeta \log n + 1$. The extra seemingly funny factor of $X(\gamma)/\lambda\ell$ is not crucial in the definition of G_γ. It is merely for the purpose to have the following: (which will save us some tedious effort)

$$\mathbb{P}(M(\gamma) \leqslant \zeta \log n \cdot X(\gamma)/\lambda\ell \mid X(\gamma) = z) \equiv \text{constant for all } \lambda\ell - 1 \leqslant z \leqslant \lambda\ell\,. \tag{23}$$

Property (23) follows from the fact that for all $z > 0$:

$$\{\tfrac{1}{z}(X_e)_{e\in\gamma} \mid X(\gamma) = z\} \overset{law}{=} \{(X_e)_{e\in\gamma} \mid X(\gamma) = 1\}\,,$$

which one can verify by definition of exponential variables and (4).

By (4) and Lemma 2.3, we obtain that

$$\mathbb{P}(G_\gamma) \geqslant \tfrac{1}{100} n^{-\ell}\ell^{-1/2} \exp\left((\mathrm{e}\beta - C^\star/\zeta^2)\ell(\log n)^{-2}\right), \tag{24}$$

where $C^\star$ is the absolute constant from Lemma 2.3. Defining $N = \sum_{\gamma\in\Gamma_\ell} \mathbf{1}_{G_\gamma}$, we see that the first moment of N would be large if we select c, δ properly. In particular, we have

$$\mathbb{E}N = \mathbb{P}(G_\gamma)|\Gamma_\ell| = (1+o(1))\mathbb{P}(G_\gamma)n^{\ell+1} \geqslant \tfrac{(1+o(1))n}{100}\ell^{-1/2}\exp\left((\mathrm{e}\beta - C^\star/\zeta^2)\ell(\log n)^{-2}\right). \tag{25}$$

As in Section 2.3, the key issue is to bound the second moment of N. We use the basic ideas in Section 2.3, with more delicate analysis. One of the main difficulties is that now the probability cost for the truncation on the deviation is so large such that it has to be tracked down throughout, requiring delicate estimates on the deviations of light paths (as incorporated in Section 3.1) as well as a careful treatment when patching estimates together.

Lemma 3.3. *For any $\gamma \in \Gamma_\ell$ and $\gamma' \in A_{i,j}$ with $1 \leqslant i \leqslant j$, we have that*

$$\mathbb{P}(G_{\gamma'} \mid G_\gamma) \leqslant \mathbb{P}(G_\gamma) O(\sqrt{\ell/(\ell-j)}) n^j n^{300\zeta i} \mathrm{e}^{(C^\star/\zeta^2 - \mathrm{e}\beta)j(\log n)^{-2}}\,.$$

Proof. Write $S = E(\gamma)\cap E(\gamma')$ and $S' = E(\gamma')\setminus S$. Conditioned on G_γ (see the definition of G_γ in (22)), we have that

$$\textstyle\sum_{e\in S} X_e \geqslant \lambda|S| - 1 - 2i\zeta\log n = \lambda j - 1 - 2i\zeta\log n\,.$$

Therefore, we obtain that

$$\mathbb{P}(G_{\gamma'} \mid G_\gamma) = \mathbb{P}\left(\lambda\ell - 1 \leqslant X(\gamma') \leqslant \lambda\ell \mid G_\gamma\right)\cdot\mathbb{P}\left(M(\gamma') \leqslant \zeta\log n\cdot X(\gamma')/\lambda\ell \mid G_\gamma, \lambda\ell - 1 \leqslant X(\gamma') \leqslant \lambda\ell\right)\,.$$

It is clear that

$$\begin{aligned}\mathbb{P}\left(\lambda\ell - 1 \leqslant X(\gamma') \leqslant \lambda\ell \mid G_\gamma\right) &\leqslant \mathbb{P}\left(\textstyle\sum_{e\in S'} X_e \leqslant \lambda|S'| + 1 + 2i\zeta\log n \mid G_\gamma\right)\\ &= \mathbb{P}\left(\textstyle\sum_{e\in S'} X_e \leqslant \lambda|S'| + 1 + 2i\zeta\log n\right)\,.\end{aligned}$$

Recalling (4) and that $|S'| = \ell - j$, we get that

$$\mathbb{P}\left(\lambda\ell - 1 \leqslant X(\gamma') \leqslant \lambda\ell \mid G_\gamma\right) \leqslant O((\ell-j)^{-1/2}) n^{-(\ell-j)} \mathrm{e}^{\mathrm{e}\beta(\ell-j)(\log n)^{-2}} n^{2\zeta i}\,.$$

By Lemma 3.2 and property (23), we obtain that

$$\begin{aligned}&\mathbb{P}\left(M(\gamma') \leqslant \zeta\log n\cdot X(\gamma')/\lambda\ell \mid G_\gamma, \ell\lambda - 1 \leqslant X(\gamma')\right)\\ &\quad\leqslant \mathbb{P}\left(M(\gamma) \leqslant \zeta\log n\cdot X(\gamma)/\lambda\ell \mid \lambda\ell - 1 \leqslant X(\gamma) \leqslant \lambda\ell\right)\sqrt{j\wedge(n-j)}\, n^{300\zeta i}\mathrm{e}^{C^\star j(\zeta\log n)^{-2}}\,.\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{P}(G_\gamma) &= \mathbb{P}\left(\lambda\ell - 1 \leqslant X(\gamma) \leqslant \lambda\ell\right)\mathbb{P}\left(M(\gamma) \leqslant \zeta\log n\cdot X(\gamma)/\lambda\ell \mid \lambda\ell - 1 \leqslant X(\gamma) \leqslant \lambda\ell\right)\,,\\ \mathbb{P}\left(G_{\gamma'} \mid G_\gamma\right) &= \mathbb{P}\left(\lambda\ell - 1 \leqslant X(\gamma') \leqslant \lambda\ell \mid G_\gamma\right)\mathbb{P}\left(M(\gamma') \leqslant \zeta\log n\cdot X(\gamma')/\lambda\ell \mid G_\gamma, \lambda\ell - 1 \leqslant X(\gamma') \leqslant \lambda\ell\right)\,.\end{aligned}$$

Combining the last four displays together, we complete the proof of the lemma. □

We next conclude this subsection with the proof for the lower bound on $L(n,\lambda)$.

Proof of Theorem 1.1: lower bound. Set

$$\zeta = 1/10^4, \text{ and } \beta = 10^8 \max(C^\star, 1)\,. \tag{26}$$

By (25), we see that

$$\mathbb{E}N \geqslant n^{3/4}\,.$$

Next, we turn to bound the second moment of N. For $\gamma \in \Gamma_\ell$, consider Γ_ℓ as a union of $A_{i,j}$ over $0 \leqslant i \leqslant j \leqslant \ell$. For all $\gamma' \in A_{0,0}$, the events G_γ and $G(\gamma')$ are independent. Thus,

$$\textstyle\sum_{\gamma' \in A_{0,0}} \mathbb{P}(G_\gamma \cap G_{\gamma'}) = \sum_{\gamma' \in A_{0,0}} \mathbb{P}(G_\gamma) \cdot \mathbb{P}(G_{\gamma'}) \leqslant \mathbb{P}(G_\gamma)\mathbb{E}N\,. \tag{27}$$

We next consider $1 \leqslant i \leqslant j \leqslant \ell$. For $\gamma' \in A_{i,j}$, Lemma 3.3 and (26) gives that

$$\mathbb{P}(G_{\gamma'} \mid G_\gamma) \leqslant \mathbb{P}(G_\gamma) O(\sqrt{\ell/\ell - j}) n^j n^{i/10} \mathrm{e}^{-10j(\log n)^{-2}}\,.$$

Combined with Lemma 2.10, it follows that

$$\textstyle\sum_{\gamma' \in A_{i,j}} \mathbb{P}(G_{\gamma'} \mid G_\gamma) \leqslant \mathbb{P}(G_\gamma) n^{\ell+1} n^{-i/8} \mathrm{e}^{-8j(\log n)^{-2}} = \mathbb{E}N \cdot n^{-i/8} \mathrm{e}^{-8j(\log n)^{-2}}\,.$$

Summing over $1 \leqslant i \leqslant j \leqslant \ell$ and recalling (27), we obtain that

$$\mathbb{E}(N \mid G_\gamma) \leqslant (1 + O(n^{-1/10}))\mathbb{E}N\,,$$

and therefore

$$\mathbb{E}N^2 = \textstyle\sum_{\gamma \in \Gamma_\ell} \mathbb{P}(G_\gamma)\mathbb{E}(N \mid G_\gamma) = (1 + O(n^{-1/10}))(\mathbb{E}N)^2\,.$$

At this point, a simple application of Chebyshev's inequality gives that $\mathbb{P}(N > 0) \to 1$ as $n \to \infty$, completing the proof for the lower bound. □

4 Smooth interpolation through near sub-critical regime

In this section, we demonstrate a smooth interpolation from sub-criticality to criticality by proving Theorem 1.3. The proof uses similar ideas as within the critical window and is also simpler. We write a separate proof in order to reduce distractions for the presentation of the core ideas in the critical regime. The proof in this section will be presented in a concise way. Throughout, we let $\lambda = \mathrm{e}^{-1} - \varepsilon \leqslant \mathrm{e}^{-1} - (\log n)^{-2}$.

To prove the upper bound, we see that for $\ell = \varepsilon^{-1} \log n$ (by Claim 2.4),

$$\mathbb{P}(L(n,\lambda) \geqslant \ell) \leqslant \mathbb{P}\left(\exists \ell \leqslant \ell' < 2\ell, \gamma \in \Gamma_{\ell'} : X(\gamma) \leqslant \lambda \ell'\right) \leqslant \textstyle\sum_{\ell \leqslant \ell' < 2\ell} n^{\ell'+1} n^{-\ell'} \mathrm{e}^{-\mathrm{e}\varepsilon\ell'} = o(1)\,.$$

For the lower bound, consider $\ell = \frac{\log n}{10^4 (C^\star \vee 1)\varepsilon}$, and for $\gamma \in \Gamma_\ell$ define

$$H_\gamma = \{\lambda\ell - 1 \leqslant X(\gamma) \leqslant \lambda\ell, M(\gamma) \leqslant \log n/10\}\,.$$

Then, by (4) and Lemma 2.3, we have that

$$\mathbb{E}N \geqslant n^{\ell+1} \tfrac{1}{100\sqrt{\ell}} n^{-\ell} \mathrm{e}^{-\mathrm{e}\varepsilon\ell - 100C^\star \ell/(\log n)^2} \geqslant \sqrt{n}\,.$$

It remains to control the second moment of N. Analogous to derivation of (17), we obtain that for $\gamma' \in A_{i,j}(\gamma)$

$$\mathbb{P}(H_{\gamma'} \mid H_\gamma) = O(1) n^{-(\ell-j)} n^{i/5} .$$

Thus, by Lemma 2.10, we obtain that for all $1 \leqslant i \leqslant j$

$$\textstyle\sum_{\gamma' \in A_{i,j}} \mathbb{P}(H_{\gamma'} \mid H_\gamma) = O(\ell^3 n) n^{-4i/5} .$$

Summing over $0 \leqslant i \leqslant j \leqslant \ell$, we obtain that

$$\mathbb{E}N^2 = (\mathbb{E}N)^2 + \mathbb{E}N \cdot \ell^5 n n^{-4/5} = (1 + o(1))(\mathbb{E}N)^2 .$$

The lower bound follows immediately.

Acknowledgements

We are grateful to David Aldous for helpful communications, and we thank an anonymous referee for helpful comments correcting numerous typos in an earlier manuscript.

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