On the uniform convergence of random series in Skorohod space and representations of càdlàg infinitely divisible processes

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May 1, 2012

Abstract

Let X_n be independent random elements in the Skorohod space $D([0,1];E)$ of càdlàg functions taking values in a separable Banach space E. Let $S_n = \sum_{j=1}^n X_j$. We show that if S_n converges in finite dimensional distributions to a càdlàg process, then $S_n + y_n$ converges a.s. pathwise uniformly over $[0,1]$, for some $y_n \in D([0,1];E)$. This result extends the Itô-Nisio Theorem to the space $D([0,1];E)$, which is surprisingly lacking in the literature even for $E = \mathbb{R}$. The main difficulties of dealing with $D([0,1];E)$ in this context are its non-separability under the uniform norm and the discontinuity of addition under Skorohod's J_1-topology.

We use this result to prove the uniform convergence of various series representations of càdlàg infinitely divisible processes. As a consequence, we obtain explicit representations of the jump process, and of related path functionals, in a general non-Markovian setting. Finally, we illustrate our results on an example of stable processes. To this aim we obtain new criteria for such processes to have càdlàg modifications, which may also be of independent interest.

Keywords: Itô-Nisio Theorem; Skorohod Space; Infinitely Divisible Processes; Series Representations
AMS Subject Classification: 60G50, 60G52, 60G17

1 Introduction

The Itô-Nisio Theorem [7] plays a fundamental role in the study of series of independent random vectors in separable Banach spaces, see, e.g., Araujo and Giné [1], Linde [15], Kwapień and Woyczyński [13], and Ledoux and Talagrand [14]. In particular, it implies that various series expansions of a Brownian motion, and of other sample continuous Gaussian processes, converge uniformly pathwise, which was the original motivation for the theorem, see Ikeda and Taniguchi [6].

In order to obtain the corresponding results for series expansions of sample discontinuous processes, it is natural to consider an extension of the Itô-Nisio Theorem to the Skorohod space $D[0,1]$ of càdlàg functions. A deep pioneering work in this direction was done by Kallenberg [11]. Among other results, he showed that if a series of independent random elements in $D[0,1]$ converges in distribution in the Skorohod topology, then it "usually" converges a.s. uniformly on $[0,1]$ (see Section 2 for more details). Notice that $D[0,1]$ under the uniform norm $\|\cdot\|$ is not separable and such basic random elements in $D[0,1]$ as a Poisson process, are not strongly measurable functions. Therefore, we may formulate our problem concerning $(D[0,1], \|\cdot\|)$ in a more general framework of non-separable Banach spaces as follows.

Consider a Banach space $(F, \|\cdot\|)$ of functions from a set T into $\mathbb{R}$ such that all evaluation functionals $\delta_t \colon x \mapsto x(t)$ are continuous. Assume, moreover, that the map $x \mapsto \|x\|$ is measurable with respect to the cylindrical σ-algebra $\mathcal{C}(F) = \sigma(\delta_t : t \in T)$ of F. Let $\{X_j\}$ be a sequence of independent and *symmetric* stochastic processes indexed by T with paths in F and set $S_n = \sum_{j=1}^n X_j$. That is, S_n are $\mathcal{C}(F)$-measurable random vectors in F. We will say that the Itô-Nisio Theorem holds for F if the following two conditions are equivalent:

(i) S_n converges in finite dimensional distributions to a process with paths in F,

(ii) S_n converges a.s. in $(F, \|\cdot\|)$,

for all sequences $\{X_j\}$ as above.

If F is separable, the Itô-Nisio Theorem gives the equivalence of (i) and (ii), and in this case $\mathcal{C}(F) = \mathcal{B}(F)$. For non-separable Banach spaces we have only examples but not a general characterization of spaces for which the Itô-Nisio Theorem holds, despite the fact that many interesting path spaces occurring in probability theory are non-separable. For instance, the Itô-Nisio Theorem holds for BV_1, the space of right-continuous functions of bounded variation, which can be deduced from the proof of Jain and Monrad [8, Theorem 1.2] by a conditioning argument. However, this theorem fails to hold for $F = \ell^\infty(\mathbb{N})$ and it is neither valid for BV_p, the space of right-continuous functions of bounded p-variation with $p > 1$, or for $C^{0,\alpha}([0,1])$, the space of Hölder continuous functions of order $\alpha \in (0,1]$, see Remark 2.4. The case of $F = D[0,1]$ under the uniform norm has been open. Notice that Kallenberg's result [11] can not be applied because the convergence in (i) is much weaker than the convergence in the Skorohod topology, see also Remark 2.5.

In this paper we show that the Itô-Nisio Theorem holds for the space $D([0,1]; E)$ of càdlàg functions from $[0,1]$ into a separable Banach space E under the uniform norm

(Theorem 2.1). From this theorem we derive a simple proof of the above mentioned result of Kallenberg (Corollary 2.2 below). Furthermore, using Theorem 2.1 we establish the uniform convergence of shot noise-type expansions of càdlàg Banach space-valued infinitely divisible processes (Theorem 3.1). In the last part of this paper, we give applications to stable processes as an example, see Section 4. To this aim, we establish a new sufficient criterion for the existence of càdlàg modifications of general symmetric stable processes (Theorem 4.3) and derive explicit expressions and distributions for several functionals of the corresponding jump processes.

Definitions and Notation: In the following $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(E, |\cdot|_E)$ is a separable Banach space, and $D([0,1];E)$ is the space of càdlàg functions from $[0,1]$ into E. (Càdlàg means right-continuous with left-hand limits.) The space $D([0,1];E)$ is equipped with the cylindrical σ-algebra, i.e., the smallest σ-algebra under which all evaluations $x \mapsto x(t)$ are measurable for $t \in [0,1]$. A random element in $D([0,1];E)$ is a random function taking values in $D([0,1];E)$ measurable for the cylindrical σ-algebra. $\|x\| = \sup_{t\in[0,1]}|x(t)|_E$ denotes the uniform norm of $x \in D([0,1];E)$ and $\Delta x(t) = x(t) - x(t-)$ is the size of jump of x at t; the mappings $x \mapsto \|x\|$ and $x \mapsto \Delta x(t)$ are measurable. For more information on $D([0,1];E)$ we refer to Billingsley [2] and Kallenberg [12]. Integrals of E-valued functions are defined in the Bochner sense. By $\xrightarrow{d}, \xrightarrow{w}, \stackrel{d}{=}$ and $\mathcal{L}(X)$ we denote, respectively, convergence in distribution, convergence in law, equality in distribution, and the law of the random element X.

2 Itô-Nisio Theorem for $D([0,1];E)$

Let $\{X_j\}$ be a sequence of independent random elements in $D([0,1];E)$ and let $S_n = \sum_{j=1}^n X_j$. We study the convergence of S_n in $D([0,1];E)$ with respect to the uniform topology.

Kallenberg [11] proved that in $D[0,1]$ endowed with the Skorohod J_1-topology ($E = \mathbb{R}$), convergence a.s. and in distribution of S_n are equivalent. Moreover, if S_n converges in distribution relative to the Skorohod topology, then it converges uniformly a.s. under mild conditions, such as e.g. when the limit process does not have a jump of nonrandom size and location. In concrete situations, however, a verification of the assumption that S_n converges in distribution in the Skorohod topology can perhaps be as difficult as a direct proof of the uniform convergence. We prove the uniform convergence of S_n under much weaker conditions.

Theorem 2.1. *Suppose there exist a random element Y in $D([0,1];E)$ and a dense subset T of $[0,1]$ such that $1 \in T$ and for any $t_1, \ldots, t_k \in T$*

$$(S_n(t_1), \ldots, S_n(t_k)) \xrightarrow{d} (Y(t_1), \ldots, Y(t_k)) \quad \text{as } n \to \infty. \tag{2.1}$$

Then there exists a random element S in $D([0,1];E)$ with the same distribution as Y such that

(i) $S_n \to S$ a.s. uniformly on $[0,1]$, provided X_n are symmetric.

(ii) If X_n are not symmetric, then

$$S_n + y_n \to S \quad \text{a.s. uniformly on } [0,1]. \tag{2.2}$$

for some $y_n \in D([0,1];E)$ such that $\lim_{n\to\infty} y_n(t) = 0$ for every $t \in T$.

(iii) Moreover, if the family $\{|S(t)|_E : t \in T\}$ is uniformly integrable and the functions $t \mapsto \mathbb{E}\left(X_n(t)\right)$ belong to $D([0,1];E)$, then one can take in (2.2) y_n given by

$$y_n(t) = \mathbb{E}\left(S(t) - S_n(t)\right). \tag{2.3}$$

The next corollary gives an alternative and simpler proof of the above mentioned result of Kallenberg [11]. Our proof relies on Theorem 2.1. Recall that the Skorohod J_1-topology on $D([0,1];E)$ is determined by a metric

$$d(x,y) = \inf_{\lambda\in\Lambda} \max\Big\{ \sup_{t\in[0,1]} |x(t) - y\circ\lambda(t)|_E,\ \sup_{t\in[0,1]} |\lambda(t) - t| \Big\},$$

where Λ is the class of strictly increasing, continuous mappings of $[0,1]$ onto itself (see, e.g., [2, page 124]).

Corollary 2.2. *If $S_n \xrightarrow{d} Y$ in the Skorohod J_1-topology and Y does not have a jump of nonrandom size and location, then S_n converges a.s. uniformly on $[0,1]$.*

Proof of Corollary 2.2. Since $S_n \xrightarrow{d} Y$, condition (2.1) holds for

$$T = \{t \in (0,1) :\ \mathbb{P}(\Delta Y(t) = 0) = 1\} \cup \{0,1\},$$

see [2, Sec. 13]. By Theorem 2.1(ii), there exist $\{y_n\} \subseteq D([0,1];E)$ and $S \stackrel{d}{=} Y$ such that $\|S_n + y_n - S\| \to 0$ a.s. Moreover, $\lim_{n\to\infty} y_n(t) = 0$ for every $t \in T$. We want to show that $\|y_n\| \to 0$.

Assume to the contrary that $\limsup_{n\to\infty}\|y_n\| > \epsilon > 0$. Then there exist a subsequence $N' \subseteq \mathbb{N}$, and a monotone sequence $\{t_n\}_{n\in N'} \subset [0,1]$ with $t_n \to t$ such that $|y_n(t_n)|_E \geq \epsilon$ for all $n \in N'$. Assume that $t_n \uparrow t$ (the case $t_n \downarrow t$ follows similarly). From the uniform convergence we have that $S_n(t_n) + y_n(t_n) \to S(t-)$ a.s. ($n \to \infty$, $n \in N'$), and since $S_n + y_n \xrightarrow{d} S$ also in $D([0,1];E)$ endowed with the Skorohod topology, the sequence

$$W_n := \big(S_n, S_n + y_n, S_n(t_n) + y_n(t_n)\big), \qquad n \in N',$$

is tight in $D([0,1];E)^2 \times E$ in the product topology. Passing to a further subsequence, if needed, we may assume that $\{W_n\}_{n\in N'}$ converges in distribution. By the Skorohod Representation Theorem, see, e.g. [2, Theorem 6.7], there exist random elements $\{Z_n\}_{n\in N'}$ and Z in $D([0,1];E)^2 \times E$ such that $Z_n \stackrel{d}{=} W_n$ and $Z_n \to Z$ a.s. From the measurability of addition and the evaluation maps it follows that Z_n are on the form

$$Z_n = \big(U_n, U_n + y_n, U_n(t_n) + y_n(t_n)\big)$$

for some random elements $U_n \stackrel{d}{=} S_n$ in $D([0,1];E)$. We claim that Z is on the form

$$Z = \big(U, U, U(t-)\big) \tag{2.4}$$

for some random element $U \stackrel{d}{=} S$ in $D([0,1];E)$. To show this write $Z = (Z^1, Z^2, Z^3)$ and note that $Z^1 \stackrel{d}{=} Z^2 \stackrel{d}{=} S$. Since the evaluation map $x \mapsto x(s)$ is continuous at any x such that $\Delta x(s) = 0$, see Billingsley [2, Theorem 12.5], for each $s \in T$ with probability one

$$Z^1(s) = \lim_{n\to\infty,\ n\in N'} U_n(s) = \lim_{n\to\infty,\ n\in N'} \big[U_n(s) + y_n(s)\big] = Z^2(s),$$

which shows that $Z^1 = Z^2$ a.s. Since $(S_n + y_n, S_n(t_n) + y_n(t_n)) \stackrel{d}{\to} (S, S(t-))$ we have that $(S, S(t-)) \stackrel{d}{=} (Z^2, Z^3)$. The latter yields $(S(t-), S(t-)) \stackrel{d}{=} (Z^2(t-), Z^3)$, so that $Z^3 = Z^2(t-)$ a.s. This shows (2.4) with $U := Z^1 \stackrel{d}{=} S$, and with probability one we have that

$$U_n \to U \quad \text{and} \quad U_n(t_n) + y_n(t_n) \to U(t-) \qquad n \to \infty,\ n \in N'.$$

We may choose a sequence $\{\lambda_n(\cdot,\omega)\}_{n\in N'}$ in Λ such that as $n \to \infty$,

$$\sup_{s\in[0,1]} |U_n(s) - U(\lambda_n(s))|_E + \sup_{s\in[0,1]} |\lambda_n(s) - s| \to 0 \qquad \text{a.s.}$$

Therefore,

$$\begin{aligned} &|U(\lambda_n(t_n)) - U(t-) + y_n(t_n)| \\ &\quad \le |U(\lambda_n(t_n)) - U_n(t_n)| + |U_n(t_n) + y_n(t_n) - U(t-)| \to 0 \quad \text{a.s.} \end{aligned} \tag{2.5}$$

Since $\lambda_n(t_n) \to t$ a.s. as $n \to \infty$, $n \in N'$, the sequence $\{U(\lambda_n(t_n))\}_{n\in N'}$ is relatively compact in E with at most two cluster points, $U(t)$ or $U(t-)$. By (2.5), the cluster points for $\{y_n(t_n)\}_{n\in N'}$ are $-\Delta U(t)$ or 0 and since $|y_n(t_n)|_E \ge \epsilon$ we have that $y_n(t_n) \to -\Delta U(t)$ a.s., $n \in N'$. This shows that $\Delta U(t) = c$ for some non-random $c \in E \setminus \{0\}$ and since $U \stackrel{d}{=} S \stackrel{d}{=} Y$, we have a contradiction. □

To prove Theorem 2.1 we need the following lemma:

Lemma 2.3. *Let $\{x_i\} \subseteq D([0,1];E)$ be a deterministic sequence and let $\{\epsilon_i\}$ be i.i.d. symmetric Bernoulli variables. Assume that there is a dense set $T \subseteq [0,1]$ and a random element S in $D([0,1];E)$ such that for each $t \in T$*

$$S(t) = \sum_{i=1}^{\infty} \epsilon_i x_i(t) \quad a.s. \tag{2.6}$$

Then

$$\lim_{i\to\infty} \|x_i\| = 0. \tag{2.7}$$

Proof. Suppose to the contrary, there is an $\epsilon > 0$ such that

$$\limsup_{i\to\infty} \|x_i\| > \epsilon. \tag{2.8}$$

Choose $i_1 \in \mathbb{N}$ and $t_1 \in T$ such that $|x_{i_1}(t_1)|_E > \epsilon$ and then inductively choose $i_n \in \mathbb{N}$ and $t_n \in T$, $n \geq 2$, such that

$$|x_{i_n}(t_n)|_E > \epsilon \quad \text{and} \quad |x_{i_n}(t_k)|_E < \epsilon/2 \quad \text{for all } k < n.$$

This is always possible in view of (2.6) and (2.8) because $\lim_{i\to\infty} x_i(t) = 0$ for each $t \in T$. It follows that all t_n's are distinct. The sequence $\{t_n\}_{n\in\mathbb{N}}$ contains a monotone convergent subsequence $\{t_n\}_{n\in N'}$, $\lim\limits_{n\to\infty,\ n\in N'} t_n = t$. Then for every $n > k$, $k, n \in N'$,

$$\begin{aligned}\mathbb{P}\big(|S(t_n) - S(t_k)|_E > \epsilon/2\big) &= \mathbb{P}\Big(\Big|\sum_{i=1}^{\infty} \epsilon_i[x_i(t_n) - x_i(t_k)]\Big|_E > \epsilon/2\Big) \\ &\geq \frac{1}{2}\mathbb{P}\big(|\epsilon_{i_n}[x_{i_n}(t_n) - x_{i_n}(t_k)]|_E > \epsilon/2\big) = \frac{1}{2}\,, \end{aligned} \tag{2.9}$$

which follows from the fact that if $(X, Y) \stackrel{d}{=} (X, -Y)$, then for all $\tau > 0$, $\mathbb{P}(\|X\| > \tau) = \mathbb{P}(\|(X+Y)+(X-Y)\| > 2\tau) \leq 2\mathbb{P}(\|X+Y\| > \tau)$. The bound (2.9) contradicts the fact that S is càdlàg and thus proves (2.7). □

Proof of Theorem 2.1. First we construct a random element S in $D([0,1]; E)$ such that $S \stackrel{d}{=} Y$ and

$$S(t) = \lim_{n\to\infty} S_n(t) \quad \text{a.s. for every } t \in T. \tag{2.10}$$

By the Itô-Nisio Theorem [7], $S^*(t) = \lim\limits_{n\to\infty} S_n(t)$ exists a.s. for $t \in T$. Put $S^*(t) = \lim\limits_{r\downarrow t,\ r\in T} S^*(r)$ when $t \in [0,1] \setminus T$, where the limit is in probability (the limit exists since $(S^*(r), S^*(s)) \stackrel{d}{=} (Y(r), Y(s))$ for all $r, s \in T$ and Y is right-continuous). Therefore, the process $\{S^*(t)\}_{t\in[0,1]}$ has the same finite dimensional distributions as $\{Y(t)\}_{t\in[0,1]}$ whose paths are in $D([0,1]; E)$. Since the cylindrical σ-algebra of $D([0,1]; E)$ coincides with the Borel σ-algebra under the Skorohod topology, by Kallenberg [12, Lemma 3.24], there is a process $S = \{S(t)\}_{t\in[0,1]}$, on the same probability space as S^*, with all paths in $D([0,1]; E)$ and such that $\mathbb{P}(S(t) = S^*(t)) = 1$ for every $t \in [0,1]$.

(i): Let $n_1 < n_2 < \cdots$ be an arbitrary subsequence in $\mathbb{N}$ and $\{\epsilon_i\}$ be i.i.d. symmetric Bernoulli variables defined on $(\Omega', \mathcal{F}', \mathbb{P}')$. By the symmetry, W_k in $D([0,1]; E)$ given by

$$W_k(t) = \sum_{i=1}^{k} \epsilon_i(S_{n_i}(t) - S_{n_{i-1}}(t)), \quad t \in [0,1],$$

($S_{n_0} \equiv 0$) has the same distribution as S_{n_k}. By the argument stated at the beginning of the proof, there is a process $W = \{W(t)\}_{t\in[0,1]}$ with paths in $D([0,1]; E)$, defined on

$(\Omega' \times \Omega, \mathcal{F}' \otimes \mathcal{F}, \mathbb{P}' \otimes \mathbb{P})$, such that $W \stackrel{d}{=} Y$ and

$$W(t) = \sum_{i=1}^{\infty} \epsilon_i (S_{n_i}(t) - S_{n_{i-1}}(t)) \quad \text{a.s. for every } t \in T.$$

Choose a countable set $T_0 \subset T$, dense in $[0,1]$, and $\Omega_0 \subseteq \Omega$, $\mathbb{P}(\Omega_0) = 1$, such that for each $\omega \in \Omega_0$, $\mathbb{P}'\{\omega' : W(\cdot, \omega', \omega) \in D([0,1]; E)\} = 1$ and

$$W(t, \cdot, \omega) = \sum_{i=1}^{\infty} \epsilon_i (S_{n_i}(t, \omega) - S_{n_{i-1}}(t, \omega)) \quad \mathbb{P}'\text{-a.s. for every } t \in T_0.$$

By Lemma 2.3, $\lim_{i\to\infty} \|S_{n_i}(\omega) - S_{n_{i-1}}(\omega)\| = 0$, which implies that $\|S_n - S\| \to 0$ in probability. By the Lévy-Octaviani inequality [13, Proposition 1.1.1(i)], which holds for measurable seminorms on linear measurable spaces, $\|S_n - S\| \to 0$ almost surely.

(ii): Define on the product probability space $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \otimes \mathbb{P})$ the following: $\tilde{X}_n(t; \omega, \omega') = X_n(t, \omega) - X_n(t, \omega')$, $\tilde{S}(t; \omega, \omega') = S(t, \omega) - S(t, \omega')$, and $\tilde{S}_n = \sum_{k=1}^{n} \tilde{X}_k$, where the random element S in $D([0,1]; E)$ is determined by (2.10). By (i), $\tilde{S}_n \to \tilde{S}$ a.s. in $\|\cdot\|$. From Fubini's theorem we infer that there is an ω' such that the functions $x_n(\cdot) = X_n(\cdot, \omega')$ and $y(\cdot) = S(\cdot, \omega')$ belong to $D([0,1]; E)$ and $\sum_{k=1}^{n}(X_k - x_k) \to S - y$ a.s. in $\|\cdot\|$. Thus (2.2) holds with $y_n = y - \sum_{k=1}^{n} x_k$, which combined with (2.10) yields $\lim_{n\to\infty} y_n(t) = 0$ for every $t \in T$.

(iii): Let us assume for a moment that $\mathbb{E}S(t) = \mathbb{E}S_n(t) = 0$ for all $t \in T$ and $n \in \mathbb{N}$. We want to show that $y_n = 0$ satisfies (2.2). Since $S(t) \in L^1(E)$ we have that $S_n(t) \to S(t)$ in $L^1(E)$ (cf. [13], Theorem 2.3.2) and hence $S_n(t) = \mathbb{E}[S(t) \,|\, \mathcal{F}_n]$ where $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. This shows that $\{S_n(t) \colon t \in T, n \in \mathbb{N}\}$ is uniformly integrable, cf. [5], (6.10.1). First we will prove that the sequence $\{y_n\}$ is uniformly bounded, that is

$$\sup_{n\in\mathbb{N}} \|y_n\| < \infty. \tag{2.11}$$

Assume to the contrary that there exists an increasing subsequence $n_i \in \mathbb{N}$ and $t_i \in T$ such that

$$|y_{n_i}(t_i)|_E > i^3, \quad i \in \mathbb{N}. \tag{2.12}$$

Define

$$\mathbf{V}_n = (S_n(t_1), \dots, i^{-2} S_n(t_i), \dots).$$

$\mathbf{V}_n$ are random vectors in $c_0(E)$ since

$$\mathbb{E} \limsup_{k\to\infty} |k^{-2} S_n(t_k)|_E \le \lim_{k\to\infty} \sum_{i=k}^{\infty} i^{-2} \mathbb{E}|S_n(t_i)|_E \le \lim_{k\to\infty} M \sum_{i=k}^{\infty} i^{-2} = 0,$$

where $M = \sup_{t\in T} \mathbb{E}|S(t)|_E$. By the same argument

$$\mathbf{V} = (S(t_1), \dots, i^{-2} S(t_i), \dots).$$

is a random vector in $c_0(E)$, and since $S_n(t_i) \to S(t_i)$ in $L^1(E)$, $\mathbb{E}\|\mathbf{V}_n - \mathbf{V}\|_{c_0(E)} \to 0$. Thus $\mathbf{V}_n \to \mathbf{V}$ a.s. in $c_0(E)$ by Itô and Nisio [7, Theorem 3.1].

Since each y_n is a bounded function,

$$\mathbf{a_n} = (y_n(t_1), \dots, i^{-2} y_n(t_i), \dots) \in c_0(E).$$

Also $\mathbf{V}_n + \mathbf{a}_n \to \mathbf{V}$ a.s. in $c_0(E)$ because

$$\|\mathbf{V}_n + \mathbf{a}_n - \mathbf{V}\|_{c_0(E)} \le \|S_n + y_n - S\| \to 0.$$

Hence $\mathbf{a}_n = (\mathbf{V}_n + \mathbf{a}_n) - \mathbf{V}_n \to 0$ in $c_0(E)$. Since $\lim_{i\to\infty} \|\mathbf{a}_{n_i}\|_{c_0(E)} = \infty$ by (2.12), we have a contradiction. Thus (2.11) holds.

Now we will show that

$$\lim_{n\to\infty} \|y_n\| = 0. \tag{2.13}$$

Assume to the contrary that there exists an $\epsilon > 0$, an increasing subsequence $n_i \in \mathbb{N}$, and $t_i \in T$ such that

$$|y_{n_i}(t_i)|_E > \epsilon, \qquad i \in \mathbb{N}. \tag{2.14}$$

Since (2.11) holds, $\{S_n(t) + y_n(t) : t \in T, n \in \mathbb{N}\}$ is uniformly integrable. Passing to a subsequence, if necessary, we may assume that $\{t_i\}$ is strictly monotone and converges to some $t \in [0,1]$. It follows from (2.2) that $S_{n_i}(t_i) + y_{n_i}(t_i) \to Z$ a.s. in E, where $Z = S(t)$ or $Z = S(t-)$. By the uniform integrability the convergence also holds in $L^1(E)$, thus $y_{n_i}(t_i) \to \mathbb{E}Z = 0$, which contradicts (2.14).

We proved (2.13), so that (2.2) holds with $y_n = 0$ when $\mathbb{E}S(t) = \mathbb{E}S_n(t) = 0$ for all $t \in T$ and $n \in \mathbb{N}$. In the general case, notice that $\mathbb{E}S(\cdot) \in D([0,1];E)$, so that $S - \mathbb{E}S \in D([0,1];E)$. From the already proved mean-zero case

$$\sum_{k=1}^{n} (X_k - \mathbb{E}X_k) \to S - \mathbb{E}S \qquad \text{a.s. uniformly on [0,1]},$$

which gives (2.2)–(2.3). □

Next we will show that the Itô-Nisio Theorem does not hold in many interesting non-separable Banach spaces. From this perspective, the spaces BV_1 and $(D([0,1];E), \|\cdot\|)$ are exceptional. We will use the following notation.

For $p \ge 1$, BV_p is the space of right-continuous functions $f\colon [0,1] \to \mathbb{R}$ of bounded p-variation with $f(0) = 0$ equipped with the norm

$$\|f\|_{BV_p} = \sup\Big\{ \Big(\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|^p \Big)^{1/p} : n \in \mathbb{N},\, 0 = t_0 \le \dots \le t_n = 1 \Big\}.$$

For $\alpha \in (0,1]$, $C^{0,\alpha}([0,1])$ is the space of α-Hölder continuous functions $f\colon [0,1] \to \mathbb{R}$ with $f(0) = 0$ equipped with the norm

$$\|f\|_{C^{0,\alpha}} = \sup_{s,t\in[0,1]:\, s\neq t} \frac{|f(t) - f(s)|}{|t-s|^\alpha}.$$

Moreover, $\ell^\infty(\mathbb{N})$ is the space of real sequences $\mathbf{a} = \{a_k\}_{k\in\mathbb{N}}$ with the norm $\|\mathbf{a}\|_{\ell^\infty} := \sup_{k\in\mathbb{N}}|a_k| < \infty$.

Remark 2.4. In the following we will show that the Itô-Nisio Theorem is not valid for the following non-separable Banach spaces: $\ell^\infty(\mathbb{N})$, BV_p for $p > 1$ and $C^{0,\alpha}([0,1])$ for $\alpha \in (0,1]$.

For all $p > 1$ set $r = 4^{[p/(p-1)+1]}$ where $[\,\cdot\,]$ denotes the integer part. For $j \in \mathbb{N}$ let

$$f_j(t) = r^{-j/p}\log^{-1/2}(j+1)\sin(r^j\pi t), \qquad t \in [0,1],$$

$\{Z_j\}$ be i.i.d. standard Gaussian random variables, and $X = \{X(t)\}_{t\in[0,1]}$ be given by

$$X(t) = \sum_{j=1}^{\infty} f_j(t)Z_j \qquad \text{a.s.} \tag{2.15}$$

According to Jain and Monrad [9, Proposition 4.5], X has paths in BV_p but the series (2.15) does not converge in BV_p. This shows that the Itô-Nisio Theorem is not valid for BV_p for $p > 1$. A closer inspection of [9, Proposition 4.5] reveals that X, given by (2.15), has paths in $C^{0,1/p}([0,1])$ and since $\|\cdot\|_{BV_p} \le \|\cdot\|_{C^{0,1/p}}$, the Itô-Nisio Theorem is neither valid for $C^{0,\alpha}([0,1])$ for $\alpha \in (0,1)$.

For fixed $p > 1$ choose a sequence $\{x_n^*\}_{n\in\mathbb{N}}$ of continuous linear mappings from BV_p into $\mathbb{R}$ each of the form

$$x \mapsto \sum_{i=1}^{k} \alpha_i\big(x(t_i) - x(t_{i-1})\big)$$

where $k \in \mathbb{N}$, $(\alpha_i)_{i=1}^k \subseteq \mathbb{R}$, $\sum_{i=1}^k|\alpha_i|^q \le 1$ with $q := p/(p-1)$ and $0 = t_0 < \cdots < t_k = 1$, such that

$$\|f\|_{BV_p} = \sup_{n\in\mathbb{N}}|x_n^*(f)| \quad \text{for all } f \in BV_p.$$

Set $Y(n) = x_n^*(X)$ and $b_j(n) = x_n^*(f_j)$ for all $n, j \in \mathbb{N}$. Then $Y = \{Y(n)\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ a.s., $b_j = \{b_j(n)\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ and since each x_n^* only depends on finitely many coordinate variables we have that

$$Y(n) = \sum_{j=1}^{\infty} Z_j b_j(n) \quad \text{a.s. for all } n \in \mathbb{N}.$$

By the identity

$$\Big\|\sum_{j=r}^{m} Z_j b_j\Big\|_{\ell^\infty} = \Big\|\sum_{j=r}^{m} Z_j f_j\Big\|_{BV_p} \qquad \text{for } 1 \le r < m$$

we see that the sequence $\{\sum_{j=1}^n Z_j b_j\}$ is not Cauchy in $\ell^\infty(\mathbb{N})$ a.s. and therefore not convergent in $\ell^\infty(\mathbb{N})$. This shows that the Itô-Nisio Theorem is not valid for $\ell^\infty(\mathbb{N})$.

Next we will consider $C^{0,1}([0,1])$. A function $f\colon [0,1] \to \mathbb{R}$ with $f(0) = 0$ is in $C^{0,1}([0,1])$ if and only if it is absolutely continuous with a derivative f' in $L^\infty([0,1]) = L^\infty([0,1], ds)$, and in this case we have

$$\|f\|_{C^{0,1}} = \|f'\|_{L^\infty}. \tag{2.16}$$

Let $Y = \{Y(n)\}_{n\in\mathbb{N}}$ and b_j, for $j \in \mathbb{N}$, be defined as above and choose a Borel measurable partition $\{A_j\}_{j\in\mathbb{N}}$ of $[0,1]$ generating $\mathscr{B}([0,1])$. For all $j, n \in \mathbb{N}$ and $t \in A_n$ let $h_j(t) = b_j(n)$ and $U(t) = Y(n)$. Then $h_j \in L^\infty([0,1])$ for all $j \in \mathbb{N}$ and $U \in L^\infty([0,1])$ a.s. For all $n \in \mathbb{N}$ let y_n^* denote the continuous linear functional on $L^1([0,1])$ given by $f \mapsto \int_{A_n} f(s)\, ds$. Since $\{y_n^*\}$ separates points on $L^1([0,1])$ and

$$y_n^*(U) = Y(n) \int_{A_n} 1\, ds = \sum_{j=1}^\infty y_n^*(h_j) Z_j \quad \text{a.s.}$$

it follows by the Itô-Nisio Theorem that the series $\sum_{j=1}^\infty h_j Z_j$ converges a.s. in the separable Banach space $L^1([0,1])$ to U, and hence for all $t \in [0,1]$

$$V(t) := \int_0^t U(s)\, ds = \sum_{j=1}^\infty Z_j \int_0^t h_j(s)\, ds \quad \text{a.s.}$$

Process $V = \{V(t)\}_{t\in[0,1]} \in C^{0,1}([0,1])$ a.s. and for all $1 \le r \le v$ we have by (2.16)

$$\Big\| \sum_{j=r}^v \Big(\int_0^{\cdot} h_j(s)\, ds \Big) Z_j \Big\|_{C^{0,1}} = \Big\| \sum_{j=r}^v h_j Z_j \Big\|_{L^\infty} = \Big\| \sum_{j=r}^v b_j Z_j \Big\|_{\ell^\infty}.$$

This shows that the Itô-Nisio Theorem is not valid for $C^{0,1}([0,1])$.

Remark 2.5. Here we will indicate why the usual arguments in the proof of the Itô-Nisio Theorem do not work for $D[0,1]$ equipped with Skorohod's J_1-topology. Such arguments rely on the fact that all probability measures μ on a separable Banach space F are convex tight, that is, for all $\epsilon > 0$ there exists a convex compact set $K \subseteq F$ such that $\mu(K^c) < \epsilon$ (see e.g. [13], Theorem 2.1.1). This is not the case in $D[0,1]$. We will show that if X is a continuous in probability process with paths in $D[0,1]$ having convex tight distribution, then X must have continuous sample paths a.s. Indeed, let K be a convex compact subset of $D[0,1]$ relative to Skorohod's J_1-topology. According to Daffer and Taylor [3, Theorem 6], for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and $t_1, \dots, t_n \in [0,1]$ such that for all $x \in K$ and $t \in [0,1] \setminus \{t_1, \dots, t_n\}$ we have $|\Delta x(t)| \le \epsilon$. In particular,

$$\mathbb{P}(X \in K) \le \mathbb{P}\big(\sup_{t\in[0,1]\setminus\{t_1,\dots,t_n\}} |\Delta X(t)| \le \epsilon \big) = \mathbb{P}\big(\sup_{t\in[0,1]} |\Delta X(t)| \le \epsilon \big) \tag{2.17}$$

where the last equality uses that X is continuous in probability. Letting $\epsilon \to 0$ on the right-hand side of (2.17) and taking K such that the left-hand side is close to 1 we prove that $\mathbb{P}(\sup_{t\in[0,1]} |\Delta X(t)| = 0) = 1$. Therefore, the only convex tight random elements in $D[0,1]$, which are continuous in probability, are sample continuous. In particular, a Lévy process with a nontrivial jump part is not convex tight.

3 Series representations of infinitely divisible processes

In this section we study infinitely divisible processes with values in a separable Banach space E. Recall that an infinitely divisible probability measure μ on E, without Gaussian component, admits a Lévy-Khintchine representation of the form

$$\hat{\mu}(x^*) = \exp\Big\{ i\langle x^*, b\rangle + \int_E \big(e^{i\langle x^*, x\rangle} - 1 - i\langle x^*, [\![x]\!]\rangle\big)\, \nu(dx)\Big\}, \quad x^* \in E^*, \tag{3.1}$$

where $b \in E$, ν is a σ-finite measure on E with $\nu(\{0\}) = 0$ and $[\![x]\!] = x/(1 \vee \|x\|)$ is a continuous truncation function. Vector b will be called the shift and ν the Lévy measure of μ. Here E^* denotes the dual of E and $\langle x^*, x\rangle := x^*(x)$, $x^* \in E^*$ and $x \in E$. We refer the reader to [1] for more information on infinitely divisible distributions on Banach spaces.

Let T be an arbitrary set. An E-valued stochastic process $X = \{X(t)\}_{t\in T}$ is called infinitely divisible if for any $t_1, \dots, t_n \in T$ the random vector $(X(t_1), \dots, X(t_n))$ has infinitely divisible distribution in E^n. We can write its characteristic function in the form

$$\begin{aligned}\mathbb{E}\exp\Big\{ i\sum_{j=1}^n \langle x_j^*, X(t_j)\rangle\Big\} &= \exp\Big\{ i\sum_{j=1}^n \langle x_j^*, b(t_j)\rangle \\ &\quad + \int_{E^n} \big(e^{i\sum_{j=1}^n \langle x_j^*, x_j\rangle} - 1 - i\sum_{j=1}^n \langle x_j^*, [\![x_j]\!]\rangle\big)\, \nu_{t_1,\dots,t_n}(dx_1\cdots dx_n)\Big\},\end{aligned} \tag{3.2}$$

where $\{x_j^*\} \subseteq E^*$, $\{b(t_j)\} \subseteq E$ and $\nu_{t_1,\dots,t_n}$ are Lévy measures on E^n. Below we will work with $T = [0,1]$; extensions to $T = [0,a]$ or $T = [0,\infty)$ are obvious.

In this section $\{V_j\}$ will stand for an i.i.d. sequence of random elements in a measurable space $\mathcal{V}$ with the common distribution η. $\{\Gamma_j\}$ will denote a sequence of partial sums of standard exponential random variables independent of the sequence $\{V_j\}$. Put $V = V_1$.

Theorem 3.1. *Let $X = \{X(t)\}_{t\in[0,1]}$ be an infinitely divisible process without Gaussian part specified by* (3.2) *and with trajectories in $D([0,1];E)$. Let $H\colon [0,1]\times\mathbb{R}_+\times\mathcal{V} \to E$ be a measurable function such that for every $t_1,\dots,t_n \in [0,1]$ and $B \in \mathcal{B}(E^n)$*

$$\int_0^\infty \mathbb{P}\big((H(t_1,r,V),\dots,H(t_n,r,V)) \in B\setminus\{0\}\big)\, dr = \nu_{t_1,\dots,t_n}(B), \tag{3.3}$$

$H(\cdot,r,v) \in D([0,1];E)$ for every $(r,v) \in \mathbb{R}_+\times\mathcal{V}$, and $r \mapsto \|H(\cdot,r,v)\|$ is nonincreasing for every $v \in \mathcal{V}$. Define for $u > 0$

$$Y^u(t) = b(t) + \sum_{j\colon \Gamma_j \le u} H(t,\Gamma_j,V_j) - A^u(t),$$

where

$$A^u(t) = \int_0^u \mathbb{E}[\![H(t,r,V)]\!]\, dr.$$

Then, with probability 1 as $u \to \infty$,

$$Y^u(t) \to Y(t) \tag{3.4}$$

uniformly in $t \in [0,1]$, where the process $Y = \{Y(t)\}_{t\in[0,1]}$ has the same finite dimensional distributions as X and paths in $D([0,1];E)$.

Moreover, if the probability space on which the process X is defined is rich enough, so that there exists a standard uniform random variable independent of X, then the sequences $\{\Gamma_j, V_j\}$ can be defined on the same probability space as X, such that with probability 1, X and Y have identical sample paths.

The proof of Theorem 3.1 will be preceded by corollaries, remarks, and a crucial lemma.

Corollary 3.2. *Under assumptions and notation of Theorem 3.1, with probability 1*

$$Y(t) = b(t) + \sum_{j=1}^{\infty} \left[H(t,\Gamma_j,V_j) - C_j(t)\right] \quad \textit{for all } t \in [0,1] \tag{3.5}$$

where the series converges a.s. uniformly on $[0,1]$ and $C_j(t) = A^{\Gamma_j}(t) - A^{\Gamma_{j-1}}(t)$.

Moreover, if b and A^u, for sufficiently large u, are continuous functions of $t \in [0,1]$, then with probability 1

$$\Delta Y(t) = \sum_{j=1}^{\infty} \Delta H(t,\Gamma_j,V_j) \quad \textit{for all } t \in [0,1] \tag{3.6}$$

where the series converges a.s. uniformly on $[0,1]$. ($\Delta f(t) = f(t) - f(t-)$ denotes the jump of a function $f \in D([0,1];E)$.)

Proof of Corollary 3.2. Since the convergence in (3.4) holds for a continuous index u, we may take $u = \Gamma_n$, which gives

$$Y(t) = \lim_{n\to\infty} Y^{\Gamma_n}(t) = \lim_{n\to\infty} \left(b(t) + \sum_{j=1}^{n} H(t,\Gamma_j,V_j) - A^{\Gamma_n}(t) \right) \quad \text{a.s. in } \|\cdot\|,$$

proving (3.5). This argument and our assumptions imply (3.6) as well. □

Corollary 3.3. *Suppose that the process X in Theorem 3.1 is symmetric and H satisfies stated conditions except that* (3.3) *holds for some measures $\nu^0_{t_1,\dots,t_n}$ in place of $\nu_{t_1,\dots,t_n}$ such that*

$$\nu_{t_1,\dots,t_n}(B) = \frac{1}{2}\nu^0_{t_1,\dots,t_n}(B) + \frac{1}{2}\nu^0_{t_1,\dots,t_n}(-B)$$

for every $B \in \mathcal{B}(E^n)$. Let $\{\epsilon_j\}$ be i.i.d. symmetric Bernoulli variables independent of $\{\Gamma_j, V_j\}$. Then, with probability 1, the series

$$Y(t) = \sum_{j=1}^{\infty} \epsilon_j H(t,\Gamma_j,V_j) \tag{3.7}$$

converges uniformly in $t \in [0,1]$. The process $Y = \{Y(t)\}_{t\in[0,1]}$ has the same finite dimensional distributions as process X and paths in $D([0,1];E)$.

Proof of Corollary 3.3. Apply Theorem 3.1 for $\tilde{H} : [0,1] \times \mathbb{R}_+ \times \tilde{\mathcal{V}} \mapsto E$ defined by

$$\tilde{H}(t,r,\tilde{v}) = sH(t,r,v),$$

where $\tilde{v} = (s,v) \in \tilde{\mathcal{V}} := \{-1,1\} \times \tilde{V}$, and $\tilde{V}_j = (\epsilon_j, V_j)$ in the place of H and V_j.

An alternative way to establish the uniform convergence in (3.7) is to use Theorem 2.1(i) conditionally on the sequence $\{\Gamma_j, V_j\}$. □

Remark 3.4. There are several ways to find H and V for a given process such that (3.3) is satisfied, see Rosiński [19] and [20]. They lead to different series representations of infinitely divisible processes. One of such representations will be given in the next section.

Lemma 3.5. *In the setting of Theorem 3.1, the assumption that X has paths in $D([0,1];E)$ implies that $b \in D([0,1];E)$,*

$$\int_0^\infty \mathbb{P}(\|H(\cdot,r,V)\| > 1)\, dr < \infty \tag{3.8}$$

and

$$\lim_{j\to\infty} \|H(\cdot,\Gamma_j,V_j)\| = 0 \qquad a.s. \tag{3.9}$$

Proof. By the uniqueness, $b = b(\mu)$ in (3.1) and by [17, Lemma 2.1.1], $\mu_n \xrightarrow{w} \mu$ implies $b(\mu_n) \to b(\mu)$ in E. Since X has paths in $D([0,1];E)$, the function $t \mapsto \mathcal{L}(X(t))$ is càdlàg, so that $b = b(\mathcal{L}(X(t))) \in D([0,1];E)$.

To prove (3.8) consider $\tilde{X}(t) = X(t) - X'(t)$, where X' is an independent copy of X. Let $\{\epsilon_j\}$ be i.i.d. symmetric Bernoulli variables independent of $\{(\Gamma_j, V_j)\}$. Using [19, Theorem 2.4] and (3.3), we can easily verify that the series

$$\sum_{j=1}^\infty \epsilon_j H(t, 2^{-1}\Gamma_j, V_j)$$

converges a.s. for each $t \in [0,1]$ to a process $\tilde{Y} = \{\tilde{Y}(t)\}_{t\in[0,1]}$ which has the same finite dimensional distributions as $\tilde{X}$. Thus we can and do assume that $\tilde{Y}$ has trajectories in $D([0,1];E)$ a.s. Applying Lemma 2.3 conditionally, for a fixed realization of $\{(\Gamma_j, V_j)\}$, we obtain that

$$\lim_{j\to\infty} \|H(\cdot, 2^{-1}\Gamma_j, V_j)\| = 0 \qquad \text{a.s.} \tag{3.10}$$

Observe that for each $\theta \in (2^{-1}, 1)$, $\Gamma_j < 2\theta j$ eventually a.s. Thus, by (3.10) and the monotonicity of H,

$$\lim_{j\to\infty} \|H(\cdot, \theta j, V_j)\| = 0 \qquad \text{a.s.}$$

By the Borel-Cantelli lemma

$$\sum_{j=1}^{\infty} \mathbb{P}(\|H(\cdot, \theta j, V_j)\| > 1) < \infty. \tag{3.11}$$

Hence

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{P}(\|H(\cdot,\Gamma_j, V_j)\| > 1) &\le \sum_{j=1}^{\infty} \mathbb{P}(\|H(\cdot, \Gamma_j, V_j)\| > 1, \Gamma_j > \theta j) + \sum_{j=1}^{\infty} \mathbb{P}(\Gamma_j \le \theta j) \\ &\le \sum_{j=1}^{\infty} \mathbb{P}(\|H(\cdot, \theta j, V_j)\| > 1) + (1-\theta)^{-1} + \sum_{j \ge (1-\theta)^{-1}} \frac{(\theta j)^j}{(j-1)!} e^{-\theta j} < \infty,\end{aligned}$$

where the last inequality follows from (3.11) and the following bound for $j \ge (1-\theta)^{-1}$

$$\mathbb{P}(\Gamma_j \le \theta j) = \int_0^{\theta j} \frac{x^{j-1}}{(j-1)!} e^{-x}\, dx \le \frac{(\theta j)^j}{(j-1)!} e^{-\theta j},$$

which holds because the function under the integral is increasing on the interval of integration. Now we observe that

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{P}(\|H(\cdot, \Gamma_j, V_j)\| > 1) &= \sum_{j=1}^{\infty} \int_0^{\infty} \mathbb{P}(\|H(\cdot, r, V_j)\| > 1) \frac{r^{j-1}}{(j-1)!} e^{-r}\, dr \\ &= \int_0^{\infty} \mathbb{P}(\|H(\cdot, r, V_j)\| > 1) \sum_{j=1}^{\infty} \frac{r^{j-1}}{(j-1)!} e^{-r}\, dr \\ &= \int_0^{\infty} \mathbb{P}(\|H(\cdot, r, V)\| > 1)\, dr,\end{aligned}$$

which proves (3.8). We also notice that (3.10) and the monotonicity of H imply (3.9). □

Proof of Theorem 3.1. Define a bounded function H_0 by

$$H_0(t, r, v) = H(t, r, v)\mathbf{1}(\|H(\cdot, r, v)\| \le 1)$$

and let

$$A_0^u(t) = \int_0^u \mathbb{E}\{H_0(t, r, V)\}\, dr.$$

Consider for $u \ge 0$

$$Y_0^u(t) = \sum_{j:\, \Gamma_j \le u} H_0(t, \Gamma_j, V_j) - A_0^u(t). \tag{3.12}$$

Let $\rho_{t_1,\dots,t_n}$ be defined by the left-hand side of (3.3) with H replaced by H_0, $0 \le t_1 < \cdots < t_n \le 1$. $\rho_{t_1,\dots,t_n}$ is a Lévy measure on E^n because $\rho_{t_1,\dots,t_n} \le \nu_{t_1,\dots,t_n}$, see [1, Chapter 3.4, Exercise 4]. Referring to the proof of Theorem 2.4 in [19], we infer that for each $t \in [0, 1]$,

$$Y_0(t) = \lim_{u \to \infty} Y_0^u(t)$$

exists a.s. Moreover, the finite dimensional distributions of $\{Y_0(t)\}_{t\in[0,1]}$ are given by (3.2) with $b \equiv 0$ and $\nu_{t_1,\dots,t_n}$ replaced by $\rho_{t_1,\dots,t_n}$.

Let

$$b_0(t) = b(t) - \int_0^\infty \mathbb{E}[\![H(t,r,V)]\!]\mathbf{1}(\|H(\cdot,r,V)\| > 1)\,dr.$$

Using Lemma 3.5 we infer that the integral is well defined and $b_0 \in D([0,1];E)$. In view of (3.9), the process

$$Z(t) = b_0(t) + \sum_{j=1}^\infty \{H(t,\Gamma_j,V_j) - H_0(t,\Gamma_j,V_j)\}$$

is also well defined, as the series has finitely many terms a.s., and Z has paths in $D([0,1];E)$. Processes Y_0 and Z are independent because they depend on a Poisson point process $N = \sum_{j=1}^\infty \delta_{(\Gamma_j,V_j)}$ restricted to disjoint sets $\{(r,v) : \|H(\cdot,r,v)\| \le 1\}$ and its complement, respectively. Finite dimensional distributions of $Z - b_0$ are compound Poisson as $(\nu_{t_1,\dots,t_n} - \rho_{t_1,\dots,t_n})(E^n) < \infty$ due to (3.8). We infer that

$$Y_0 + Z \stackrel{d}{=} X,$$

where the equality holds in the sense of finite dimensional distributions. Thus Y_0 has a modification with paths in $D([0,1];E)$ a.s.

The family $\{\mathcal{L}(Y_0(t))\}_{t\in[0,1]}$ is relatively compact because $\mathcal{L}(Y_0(t))$ is a convolution factor of $\mathcal{L}(X(t))$ and $\{\mathcal{L}(X(t))\}_{t\in[0,1]}$ is relatively compact, use Theorem 4.5, Chapter 1 together with Corollary 4.6, Chapter 3 from [1]. The latter claim follows from the fact that the function $t \mapsto \mathcal{L}(X(t))$ is càdlàg. Since $\rho_t(x : |x|_E > 1) = 0$ for all $t \in [0,1]$, $\{|Y_0(t)|_E : t \in [0,1]\}$ is also uniformly integrable (see [10, Theorem 2]).

It follows from (3.12) that the $D([0,1];E)$-valued process $\{Y_0^u\}_{u\ge 0}$ has independent increments and $\mathbb{E}Y_0^u(t) = 0$ for all t and u. By Theorem 2.1(iii)

$$\|Y_0^u - Y_0\| \to 0 \quad \text{a.s.} \tag{3.13}$$

as $u = u_n \uparrow \infty$. Since for each $t \in [0,1]$, the process $\{Y_0^u(t)\}_{u\ge 0}$ is càdlàg (3.13) holds also for the continuous parameter $u \in \mathbb{R}_+$, $u \to \infty$, cf. [19, Lemma 2.3].

Therefore, with probability 1 as $u \to \infty$,

$$\begin{aligned}
&\|Y^u - Y_0 - Z\| \le \|Y^u - Y_0^u - Z\| + \|Y_0^u - Y_0\| \\
&\quad \le \Big\| \sum_{j:\,\Gamma_j > u} \{H(\cdot,\Gamma_j,V_j) - H_0(\cdot,\Gamma_j,V_j)\} \Big\| \\
&\qquad + \Big\| \int_u^\infty \mathbb{E}[\![H(\cdot,r,V)]\!]\mathbf{1}(\|H(\cdot,r,V)\| > 1)\,dr \Big\| + \|Y_0^u - Y_0\| \\
&\quad \le \sum_{j:\,\Gamma_j > u} \|H(\cdot,\Gamma_j,V_j)\|\mathbf{1}(\|H(\cdot,\Gamma_j,V_j)\| > 1) + \int_u^\infty \mathbb{P}(\|H(\cdot,r,V)\| > 1)\,dr + \|Y_0^u - Y_0\| \\
&\quad = I_1(u) + I_2(u) + I_3(u) \to 0.
\end{aligned}$$

Indeed, $I_1(u) = 0$ for sufficiently large u by (3.9), $I_2(u) \to 0$ by (3.8), and $I_3(u) \to 0$ by (3.13). The proof is complete. □

4 Symmetric stable processes with càdlàg paths

In this section we illustrate applications of results of Section 3 to stable processes. Let $X = \{X(t)\}_{t\in[0,1]}$ be right-continuous in probability symmetric α-stable process, $\alpha \in (0,2)$. Any such process has a stochastic integral representation of the form

$$X(t) = \int_S f(t,s)\, M(ds) \qquad \text{a.s. for each } t \in [0,1], \tag{4.1}$$

where M is an independently scattered symmetric α-stable random measure defined on some measurable space $(S, \mathcal{S})$ with a finite control measure m, that is, for all $A \in \mathcal{S}$

$$\mathbb{E}\exp\{i\theta M(A)\} = \exp\{-|\theta|^\alpha m(A)\}, \tag{4.2}$$

and $f(t,\cdot) \in L^\alpha(S,m)$ for all $t \in [0,1]$, see Rajput and Rosiński [16, Theorem 5.2] for the almost sure representation in (4.1). Therefore, all symmetric α-stable process are Volterra processes. Conversely, a process given by (4.1)–(4.2) is symmetric α-stable.

A trivial case of (4.1) is when X is a standard symmetric Lévy process. In that case, M is a random measure generated by the increments of X, $S = [0,1]$, m is the Lebesgue measure, and $f(t,s) = \mathbf{1}_{(0,t]}(s)$.

A process X given by (4.1) has many series representations of the form (3.5), because there are many ways to construct a function H satisfying (3.3), see [20]. A particularly nice representation, called the LePage representation, is the following. Let $\{V_j\}$ be an i.i.d. sequence of random elements in S with the common distribution $m/m(S)$. Let $\{\Gamma_j\}$ be a sequence of partial sums of standard exponential random variables independent of the sequence $\{V_j\}$. Let $\{\epsilon_j\}$ be an i.i.d. sequence of symmetric Bernoulli random variables. Assume that the random sequences $\{V_j\}$, $\{\Gamma_j\}$, and $\{\epsilon_j\}$ are independent. Then for each $t \in [0,1]$

$$X(t) = c_\alpha m(S)^{1/\alpha} \sum_{j=1}^\infty \epsilon_j \Gamma_j^{-1/\alpha} f(t, V_j) \qquad \text{a.s.} \tag{4.3}$$

(the almost sure representation is obtained by combining [20] and [18, Proposition 2]). Here $c_\alpha = [-\alpha\cos(\pi\alpha/2)\Gamma(-\alpha)]^{-1/\alpha}$ for $\alpha \neq 1$ and $c_1 = 2/\pi$.

Corollary 4.1. *Let $X = \{X(t)\}_{t\in[0,1]}$ be a symmetric α-stable process of the form (4.1), where $\alpha \in (0,2)$. Assume that X is càdlàg and continuous in probability and also that $f(\cdot,s) \in D[0,1]$ for all s. Then with probability 1,*

$$X(t) = c_\alpha m(S)^{1/\alpha} \sum_{j=1}^\infty \epsilon_j \Gamma_j^{-1/\alpha} f(t, V_j) \quad \textit{for all } t \in [0,1]$$

where the series converges a.s. uniformly on $[0,1]$. Therefore, with probability 1

$$\Delta X(t) = c_\alpha m(S)^{1/\alpha} \sum_{j=1}^\infty \epsilon_j \Gamma_j^{-1/\alpha} \Delta f(t, V_j), \quad t \in [0,1], \tag{4.4}$$

where the series has no more than one nonzero term for each t. That is,

$$\mathbb{P}\left(\Delta f(t,V_j)\Delta f(t,V_k) = 0 \textit{ for all } j \neq k \textit{ and } t \in [0,1]\right) = 1. \tag{4.5}$$

Proof of Corollary 4.1. In view of Corollary 3.2 we only need to show (4.5). $f(\cdot, V_j)$ are i.i.d. càdlàg processes. Since X is continuous in probability, from (4.3) by a symmetrization inequality we get $\mathbb{P}(\Delta f(t, V_j) = 0) = 1$ for each $t \in [0,1]$. Thus for each $j \neq k$ and $\mu = \mathcal{L}\big(f(\cdot, V_k)\big)$ we have

$$\mathbb{P}\left(\sup_{1\le t\le 1} |\Delta f(t,V_j)\Delta f(t,V_k)| = 0\right) = \int_{D[0,1]} \mathbb{P}\left(\sup_{1\le t\le 1} |\Delta f(t,V_j)\Delta x(t)| = 0\right) \mu(dx) = 1,$$

because $\Delta x(t) \neq 0$ for at most countably many t. This implies (4.5). □

Next we consider some functionals of the jump process ΔX. Let $V_p(g)$ be defined as

$$V_p(g) = \sum_{t\in[0,1]} |\Delta g(t)|^p,$$

where $g \in D[0,1]$ and $p > 0$. Recall that a random variable Z is Fréchet distributed with shape parameter $\alpha > 0$ and scale parameter $\sigma > 0$ if for all $x > 0$, $\mathbb{P}(Z \le x) = e^{-(x/\sigma)^{-\alpha}}$. The results below are well known for a Lévy stable process. Below we give their versions for general càdlàg symmetric stable processes.

Corollary 4.2. *Under the assumptions of Corollary 4.1 we have the following.*

(i) $V_p(X) < \infty$ *a.s. if and only if either* $f(\cdot, s)$ *is continuous for* m*-a.a.* s*, in which case* $V_p(X) = 0$ *a.s. or* $p > \alpha$ *and* $\int V_p(f(\cdot,s))^{\alpha/p}\, m(ds) \in (0,\infty)$*. In the latter case,* $V_p(X)$ *is a positive* (α/p)*-stable random variable with shift parameter* 0 *and scale parameter*

$$c_\alpha^p c_{\alpha/p}^{-1}\Big(\int V_p(f(\cdot,s))^{\alpha/p}\, m(ds)\Big)^{p/\alpha}.$$

(ii) The largest in the absolute value jump of X*,* $\sup_{t\in[0,1]}|\Delta X(t)|$*, is Fréchet distributed with shape parameter* α *and scale parameter*

$$c_\alpha\Big(\int \sup_{t\in[0,1]} |\Delta f(t,s)|^\alpha\, m(ds)\Big)^{1/\alpha}.$$

(iii) The largest jump of X*,* $\sup_{t\in[0,1]} \Delta X(t)$*, is Fréchet distributed with shape parameter* α *and scale parameter*

$$2^{-1}c_\alpha\Big[\Big(\int \big|\sup_{t\in[0,1]} \Delta f(t,s)\big|^\alpha\, m(ds)\Big)^{1/\alpha} + \Big(\int \big|\inf_{t\in[0,1]} \Delta f(t,s)\big|^\alpha\, m(ds)\Big)^{1/\alpha}\Big].$$

Proof of Corollary 4.2. (i): By (4.4)–(4.5) we have a.s.

$$\begin{aligned}\sum_{t\in[0,1]} |\Delta X(t)|^p &= c_\alpha^p m(S)^{p/\alpha} \sum_{t\in[0,1]} \sum_{j=1}^\infty \Gamma_j^{-p/\alpha} |f(t,V_j)|^p \\ &= c_\alpha^p m(S)^{p/\alpha} \sum_{j=1}^\infty \Gamma_j^{-1/(\alpha/p)} V_p(f(\cdot,V_j)),\end{aligned}$$

which show (i), see e.g. [22].

(ii): By (4.4)–(4.5) we have a.s.

$$\sup_{t\in[0,1]} |\Delta X(t)| = c_\alpha m(S)^{1/\alpha} \sup_{t\in[0,1]} \sup_{j\in\mathbb{N}} \Gamma_j^{-1/\alpha} |\Delta f(t,V_j)|$$
$$= c_\alpha m(S)^{1/\alpha} \sup_{j\in\mathbb{N}} \Gamma_j^{-1/\alpha} W_j,$$

where $W_j = \sup_{t\in[0,1]} |\Delta f(t,V_j)|$ are i.i.d. random variables. Let $\xi_j = \Gamma_j^{-1/\alpha} W_j$. Then $\sum_{j=1}^\infty \delta_{\xi_j}$ is a Poisson point process on $\mathbb{R}_+$ with the intensity measure $\mu(dx) = \alpha \mathbb{E} W_1^\alpha \, x^{-\alpha-1}\, dx$, $x > 0$. Let $\eta_j = (\mathbb{E}W_1^\alpha)^{1/\alpha} \Gamma_j^{-1/\alpha}$. Since the Poisson point processes $\sum_{j=1}^\infty \delta_{\xi_j}$ and $\sum_{j=1}^\infty \delta_{\eta_j}$ have the same intensity measures, the distributions of their measurable functionals are equal. That is $\sup_j \xi_j \stackrel{d}{=} \sup_j \eta_j$, so that

$$\sup_{t\in[0,1]} |\Delta X(t)| \stackrel{d}{=} c_\alpha m(S)^{1/\alpha} \sup_{j\in\mathbb{N}} (\mathbb{E}W_1^\alpha)^{1/\alpha} \Gamma_j^{-1/\alpha} = c_\alpha m(S)^{1/\alpha} (\mathbb{E}W_1^\alpha)^{1/\alpha} \Gamma_1^{-1/\alpha}.$$

This shows (ii).

(iii): By (4.4)–(4.5) we have a.s.

$$\sup_{t\in[0,1]} \Delta X(t) = c_\alpha m(S)^{1/\alpha} \sup_{t\in[0,1]} \sup_{j\in\mathbb{N}} \epsilon_j \Gamma_j^{-1/\alpha} \Delta f(t,V_j)$$
$$= c_\alpha m(S)^{1/\alpha} \sup_{j\in\mathbb{N}} \Gamma_j^{-1/\alpha} W_j,$$

where

$$W_j = \begin{cases} \sup_{t\in[0,1]} \Delta f(t,V_j) & \text{if } \epsilon_j = 1, \\ -\inf_{t\in[0,1]} \Delta f(t,V_j) & \text{if } \epsilon_j = -1. \end{cases}$$

Observe that $W_j \geq 0$ is an i.i.d. sequence. Proceeding as in (ii) we get

$$\sup_{t\in[0,1]} \Delta X(t) \stackrel{d}{=} c_\alpha m(S)^{1/\alpha} (\mathbb{E}W_1^\alpha)^{1/\alpha} \Gamma_1^{-1/\alpha},$$

which completes the proof. □

It can be instructive to examine how Corollaries 4.1–4.2 apply to the above mentioned standard symmetric stable Lévy process.

The crucial assumption in the above corollaries is that a stable process has càdlàg paths. To this end we establish a sufficient criterion which extends a recent result of Davydov and Dombry [4] obtained by different methods (see Remark 4.5).

Theorem 4.3. *Let $X = \{X(t)\}_{t\in[0,1]}$ be given by (4.1) and let $\alpha \in (1,2)$. Assume that there exist $\beta_1, \beta_2 > 1/2$, $p_1 > \alpha$, $p_2 > \alpha/2$ and increasing continuous functions $F_1, F_2 \colon [0,1] \to \mathbb{R}$ such that for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,*

$$\int |f(t_2,s) - f(t_1,s)|^{p_1}\, m(ds) \leq |F_1(t_2) - F_1(t_1)|^{\beta_1}, \tag{4.6}$$

$$\int |(f(t,s) - f(t_1,s))(f(t_2,s) - f(t,s))|^{p_2}\, m(ds) \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}. \tag{4.7}$$

Then X has a càdlàg modification.

Proof. Decompose M as $M = N + N'$, where N and N' are independent, independently scattered random measures given by

$$\mathbb{E}\exp\{i\theta N(A)\} = \exp\Big\{k_\alpha m(A)\int_0^1 \big(\cos(\theta x) - 1\big)x^{-1-\alpha}\,dx\Big\}, \tag{4.8}$$

and

$$\mathbb{E}\exp\{i\theta N'(A)\} = \exp\Big\{k_\alpha m(A)\int_1^\infty \big(\cos(\theta x) - 1\big)x^{-1-\alpha}\,dx\Big\},$$

where $A \in \mathcal{S}$ and $k_\alpha = \alpha c_\alpha^\alpha$. Treating $f = \{f(t,\cdot)\}_{t\in[0,1]}$ as a stochastic process defined on $(S, m/m(S))$, observe that by [2, Theorem 13.6], (4.6)–(4.7) imply that f has a modification with paths in $D[0,1]$. Therefore, without affecting (4.1), we may choose f such that $t \mapsto f(t,s)$ is càdlàg for all s. Since N' has finite support a.s. ($N'(S)$ has a compound Poisson distribution), it suffices to show that a process $Y = \{Y(t)\}_{t\in[0,1]}$ given by

$$Y(t) = \int_S f(t,s)\,N(ds),$$

has a càdlàg modification. To this end, invoking again [2, Theorem 13.6], it is enough to show that Y is right-continuous in probability and there exist a continuous increasing function $F\colon [0,1] \to \mathbb{R}$, $\beta > \frac{1}{2}$ and $p > 0$ such that for all $0 \le t_1 \le t \le t_2 \le 1$ and $\lambda \in (0,1)$

$$\mathbb{P}\big(|Y(t) - Y(t_1)| \wedge |Y(t_2) - Y(t)| > \lambda\big) \le \lambda^{-p}[F(t_2) - F(t_1)]^{2\beta}. \tag{4.9}$$

(Notice that [2, Theorem 13.6] assumes that (4.9) holds for all $\lambda > 0$, but the proof reveals that $\lambda \in (0,1)$ suffices.)

Set

$$Z_1 = Y(t) - Y(t_1) = \int_S h_1\,dN \quad \text{and} \quad Z_2 = Y(t_2) - Y(t) = \int_S h_2\,dN,$$

where $h_1(s) = f(t,s) - f(t_1,s)$ and $h_2(s) = f(t_2,s) - f(t,s)$. Below C will stand for a constant that is independent of λ, t_1, t, t_2 but may be different from line to line. Applying (4.10) of Lemma 4.4 and assumptions (4.6)–(4.7) we get

$$\begin{aligned}\mathbb{P}\big(|Y(t) - Y(t_1)| \wedge |Y(t_2) - Y(t)| > \lambda\big) &= \mathbb{P}(|Z_1| \wedge |Z_2| > \lambda) \le \mathbb{P}(|Z_1 Z_2| > \lambda^2)\\ &\le C\Big(\lambda^{-2p_1}\int |h_1|^{p_1}\,dm \int |h_2|^{p_1}\,dm + \lambda^{-2p_2}\int |h_1 h_2|^{p_2}\,dm\Big)\\ &\le C\Big(\lambda^{-2p_1}|F_1(t_2) - F_1(t_1)|^{2\beta_1} + \lambda^{-2p_2}|F_2(t_2) - F_2(t_1)|^{2\beta_2}\Big).\end{aligned}$$

Thus (4.9) holds for $\lambda \in (0,1)$ with $p = 2(p_1 \vee p_2)$, $\beta = \beta_1 \wedge \beta_2$ and $F = C(F_1 + F_2)$. The last bound in Lemma 4.4 together with (4.6) imply continuity of Y in L^{p_1}. The proof will be complete after proving the following lemma. □

Lemma 4.4. *Let N be given by (4.8) and let $Z_k = \int_S h_k \, dN$, where h_k is a deterministic function integrable with respect to N, $k = 1, 2$. For all $p_1 > \alpha$ and $p_2 > \alpha/2$ there exists a constant $C > 0$, depending only on p_1, p_2 and α, such that for all $\lambda > 0$*

$$\mathbb{P}(|Z_1 Z_2| > \lambda) \le C\Big(\lambda^{-p_1} \int |h_1|^{p_1} \, dm \int |h_2|^{p_1} \, dm + \lambda^{-p_2} \int |h_1 h_2|^{p_2} \, dm\Big). \tag{4.10}$$

Moreover, $E|Z_1|^{p_1} \le C \int |h_1|^{p_1} \, dm$.

Proof. To show (4.10) we may and do assume that h_1 and h_2 are simple functions of the form $h_1 = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$ and $h_2 = \sum_{j=1}^n b_j \mathbf{1}_{A_j}$, where $(A_j)_{j=1}^n$ are disjoint measurable sets and $(a_j)_{j=1}^n, (b_j)_{j=1}^n \subseteq \mathbb{R}$. We have

$$Z_1 Z_2 = \sum_{j,k=1:\, k \ne j}^n a_j b_k N(A_j) N(A_k) + \sum_{k=1}^n a_k b_k N(A_k)^2 = T + D,$$

and hence

$$\mathbb{P}(|Z_1 Z_2| > \lambda) \le \mathbb{P}(|T| > \lambda/2) + \mathbb{P}(|D| > \lambda/2), \quad \lambda > 0. \tag{4.11}$$

For all $(u_j)_{j=1}^n \subseteq \mathbb{R}$ set $h = \sum_{j=1}^n u_j \mathbf{1}_{A_j}$ and $\mathbf{X} = (u_1 N(A_1), \ldots, u_n N(A_n))$. The Euclidean norm on $\mathbb{R}^n$ is denoted $|\mathbf{x}|_n = (\sum_{j=1}^n x_j^2)^{1/2}$. We claim that for all $p > \alpha$ there exists a constant C_1, only depending on p, α and $m(S)$, such that

$$\mathbb{E}|\mathbf{X}|_n^p \le C_1 \int |h|^p \, dm, \tag{4.12}$$

$$\mathbb{E}\Big| \int_S h(s) \, N(ds) \Big|^p \le C_1 \int |h|^p \, dm. \tag{4.13}$$

We will show (4.12)–(4.13) at the end of this proof. Now we notice that for $p_2 > \alpha/2$ and $u_j = |a_j b_j|^{1/2}$, $j = 1, \ldots, n$ bound (4.12) yields

$$\mathbb{E}|D|^{p_2} \le \mathbb{E}|\mathbf{X}|_n^{2p_2} \le C_1 \int \big(|h_1 h_2|^{1/2}\big)^{2p_2} \, dm = C_1 \int |h_1 h_2|^{p_2} \, dm. \tag{4.14}$$

Now let $p_1 > \alpha$. By a decoupling inequality, see [13, Theorem 6.3.1], there exists a constant C_2, only depending on p, such that

$$\mathbb{E}|T|^{p_1} \le C_2 \int_\Omega \mathbb{E}\Big(\Big| \int_S \phi(s, \omega') \, N(ds) \Big|^{p_1} \Big) \, \mathbb{P}(d\omega')$$

where $\phi(s, \omega') = \sum_{j=1}^n \tilde{a}_j(\omega') \mathbf{1}_{A_j}(s)$ and $\tilde{a}_j(\omega') = a_j \sum_{k=1:\, k \ne j}^n b_k N(A_k)(\omega')$. By (4.13) we have

$$\mathbb{E}\Big| \int_S \phi(s, \omega') \, N(ds) \Big|^{p_1} \le C_1 \sum_{j=1}^n |a_j|^{p_1} m(A_j) \Big| \sum_{k=1:\, k \ne j}^n b_k N(A_k)(\omega') \Big|^{p_1},$$

and hence by another application of (4.13),

$$\mathbb{E}|T|^{p_1} \le C_1^2 C_2 \int |h_1|^{p_1} \, dm \int |h_2|^{p_1} \, dm. \tag{4.15}$$

Combining (4.11), (4.14) and (4.15) with Markov's inequality we get (4.10).

To show (4.12) we use Rosiński and Turner [21]. Notice that the Lévy measure of $\mathbf{X}$ is given by

$$\nu(B) = \frac{1}{2}k_\alpha \int_{-1}^{1} \Big(\int_{\mathbb{R}^n} \mathbf{1}_B(r\theta)\, \kappa(d\theta) \Big) |r|^{-1-\alpha}\, dr \qquad B \in \mathcal{B}(\mathbb{R}^n),$$

where $\kappa = \sum_{j=1}^n m(A_j)\delta_{u_j \mathbf{e}_j}$ and $(\mathbf{e}_j)_{j=1}^n$ is the standard basis in $\mathbb{R}^n$. For all $l > 0$ set

$$\xi_p(l) = \int_{\mathbb{R}^n} |xl^{-1}|_n^p \mathbf{1}_{\{|xl^{-1}|_n > 1\}}\, \nu(dx) + \int_{\mathbb{R}^n} |xl^{-1}|_n^2 \mathbf{1}_{\{|xl^{-1}|_n \le 1\}}\, \nu(dx) = V_1(l) + V_2(l).$$

According to [21, Theorem 4], $c_p l_p \le (\mathbb{E}|\mathbf{X}|_n^p)^{1/p} \le C_p l_p$ for some constants c_p, C_p depending only on p, where $l = l_p$ is the unique solution of the equation $\xi_p(l) = 1$. From the above decomposition we have either $V_1(l_p) \ge 1/2$ or $V_2(l_p) \ge 1/2$. In the first case

$$\frac{1}{2} \le V_1(l_p) \le \int_{\mathbb{R}^n} |xl_p^{-1}|_n^p\, \nu(dx) = C_3 l_p^{-p} \int |h|^p\, dm,$$

where $C_3 = k_\alpha/(p-\alpha)$. Thus

$$\mathbb{E}|\mathbf{X}|_n^p \le 2C_p^p C_3 \int |h|^p\, dm,$$

proving (4.12). If $V_2(l_p) \ge 1/2$, then we consider two cases. First assume that $p \in (\alpha, 2]$. We have

$$\frac{1}{2} \le V_2(l_p) \le \int_{\mathbb{R}^n} |xl_p^{-1}|_n^p\, \nu(dx) = C_3 l_p^{-p} \int |h|^p\, dm,$$

which yields (4.12) as above. Now we assume that $p > 2$. We get

$$\frac{1}{2} \le V_2(l_p) \le \int_{\mathbb{R}^n} |xl_p^{-1}|_n^2\, \nu(dx) = C_4 l_p^{-2} \int |h|^2\, dm,$$

where $C_4 = k_\alpha/(2-\alpha)$. Applying Jensen's inequality to the last term we get

$$\frac{1}{2} \le C_4 m(S)^{1-2/p}\, l_p^{-2} \Big(\int |h|^p\, dm \Big)^{2/p},$$

which yields the desired bound for l_p, establishing (4.12) for all $p > \alpha$. The proof of (4.13) is similar and it is therefore omitted. This completes the proof of the lemma. □

Remark 4.5. In a recent paper Davydov and Dombry [4] obtained sufficient conditions for the uniform convergence in $D[0,1]$ of the LePage series (4.3), which in turn yield criteria for a symmetric stable process to have càdlàg paths. Their result is a special case

of our Theorem 4.3 combined with Corollary 4.1, when one takes $p_1 = p_2 = 2$ and assumes additionally that $\mathbb{E}\|f(\cdot, V)\|^\alpha < \infty$. The methods are also different from ours.

In our approach, we established the existence of a càdlàg version first, using special distributional properties of the process. Then the uniform convergence of the LePage series, and also of other shot noise series expansions, follows automatically by Corollary 3.3. This strategy applies to other infinitely divisible processes as well. Here we provided only an example of possible applications of the results of Section 3.

Acknowledgments

The authors are grateful to Jørgen Hoffmann–Jørgensen for discussions and interest in this work and to the anonymous referee for careful reading of the manuscript and helpful suggestions.

References

[1] Araujo, A. and E. Giné (1980). *The Central Limit Theorem for Real and Banach Valued Random Variables.* John Wiley & Sons, New York-Chichester-Brisbane. Wiley Series in Probability and Mathematical Statistics.

[2] Billingsley, P. (1999). *Convergence of Probability Measures* (Second ed.). Wiley Series in Probability and Statistics: Probability and Statistics. New York: John Wiley & Sons Inc. A Wiley-Interscience Publication.

[3] Daffer, P. Z. and R. L. Taylor (1979). Laws of large numbers for $D[0, 1]$. *Ann. Probab. 7*(1), 85–95.

[4] Davydov, Y. and C. Dombry (2011). On the convergence of Le Page series in Skohorod space. *arXiv:1107.2193v1*.

[5] Hoffmann-Jørgensen, J. (1994). *Probability with a View toward Statistics. Vol. I.* Chapman & Hall Probability Series. New York: Chapman & Hall.

[6] Ikeda, N. and S. Taniguchi (2010). The Itô-Nisio theorem, quadratic Wiener functionals, and 1-solitons. *Stochastic Process. Appl. 120*(5), 605–621.

[7] Itô, K. and M. Nisio (1968). On the convergence of sums of independent Banach space valued random variables. *Osaka J. Math. 5*, 35–48.

[8] Jain, N. C. and D. Monrad (1982). Gaussian quasimartingales. *Z. Wahrsch. Verw. Gebiete 59*(2), 139–159.

[9] Jain, N. C. and D. Monrad (1983). Gaussian measures in B_p. *Ann. Probab. 11*(1), 46–57.

[10] Jurek, Z. J. and J. Rosiński (1988). Continuity of certain random integral mappings and the uniform integrability of infinitely divisible measures. *Teor. Veroyatnost. i Primenen. 33*(3), 560–572.

[11] Kallenberg, O. (1974). Series of random processes without discontinuities of the second kind. *Ann. Probab. 2*, 729–737.

[12] Kallenberg, O. (2002). *Foundations of Modern Probability* (Second ed.). Probability and its Applications (New York). New York: Springer-Verlag.

[13] Kwapień, S. and W. A. Woyczyński (1992). *Random Series and Stochastic Integrals: Single and Multiple*. Probability and its Applications. Boston, MA: Birkhäuser Boston Inc.

[14] Ledoux, M. and M. Talagrand (1991). *Probability in Banach Spaces*, Volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Berlin: Springer-Verlag. Isoperimetry and processes.

[15] Linde, W. (1986). *Probability in Banach Spaces—Stable and Infinitely Divisible Distributions* (Second ed.). A Wiley-Interscience Publication. Chichester: John Wiley & Sons Ltd.

[16] Rajput, B. S. and J. Rosiński (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields 82*(3), 451–487.

[17] Rosiński, J. (1987). Bilinear random integrals. *Dissertationes Math. (Rozprawy Mat.) 259*, 71.

[18] Rosiński, J. (1989). On path properties of certain infinitely divisible processes. *Stochastic Process. Appl. 33*(1), 73–87.

[19] Rosiński, J. (1990). On series representations of infinitely divisible random vectors. *Ann. Probab. 18*(1), 405–430.

[20] Rosiński, J. (2001). Series representations of Lévy processes from the perspective of point processes. In *Lévy Processes*, pp. 401–415. Boston, MA: Birkhäuser Boston.

[21] Rosiński, J. and M. Turner (2011). Explicit L^p-norm estimates of infinitely divisible random vectors in Hilbert spaces. *Preprint*.

[22] Samorodnitsky, G. and M. S. Taqqu (1994). *Stable Non-Gaussian Random Processes*. Stochastic Modeling. New York: Chapman & Hall. Stochastic models with infinite variance.