Variance of partial sums of stationary sequences

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Abstract. Let $X_1, X_2, \dots$ be a centred sequence of weakly stationary random variables with spectral measure F and partial sums $S_n = X_1 + \cdots + X_n$, , and let $G(x) = \int_{-x}^{x} F(\mathrm{d}x)$. We show that $\mathrm{var}(S_n)$ is regularly varying of index γ at infinity, if and only if $G(x)$ is regularly varying of index $2-\gamma$ at the origin $(0 < \gamma < 2)$.
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1 Introduction

Let $X_1, X_2, \dots$ be a sequence of centered weakly stationary random variables with finite second moments and spectral measure F, such that $r_k := \mathrm{cov}(X_0, X_k) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}tk} \mathrm{d}F(t)$, where to simplify calculations we assume that F is a symmetric measure about the origin and $G(x) = \int_{-x}^{x} F(\mathrm{d}x)$. Denote by S_n the sequence of partial sums $S_n = X_1 + \cdots + X_n$.
The main result of the paper is the following

Theorem 1.1. *For $\gamma \in (0, 2)$, define $C(\gamma) = \Gamma(1+\gamma)\sin(\gamma\pi/2)/[\pi(2-\gamma)]$. Let $L(x)$ be a positive function, slowly varying at infinity. Then,*
(i) $G(x) \sim C(\gamma) K_0 x^{2-\gamma} L(1/x)$ as $x \to 0$ if and only if (ii) $\mathrm{var}(S_n) \sim K_0 n^{\gamma} L(n)$ as $n \to \infty$.
In particular, $\mathrm{var}(S_n)/n \to K_0$ if and only if $G(x)/x \to K_0/\pi$.

The rate of growth of the variance of the partial sums S_n has received considerable attention in the literature due its key role in the limit theory of stationary random sequences (see Bradley [4, Chapter 8] and Samorodnitsky[16, Chapter 5] for comprehensive reviews).

Asymptotically linear behaviour $\mathrm{var}(S_n) \sim K_0 n$. To prove asymptotic normality a common restriction on the dependence structure is to assume that the growth $\mathrm{var}(S_n)$ is asymptotically linear (Merlevède, Peligrad and Utev [13]).
For this particular situation, there exist several results which guarantee the convergence of $\mathrm{var}(S_n)/n$ under sufficient conditions given in terms of mixing coefficients, linear dependence coefficients, or in terms of the covariances where it is well-known that if $\lim_n \sum_{k=-n}^{n} \mathbb{E} X_0 X_k$ exists, then $\lim_n \mathrm{var}(S_n)/n$ also exists in $[0, \infty)$ and the two limits are equal ([4, Chapter 5]).
In terms of the spectral measure F, Ibragimov's [10] result states that when F is absolutely continuous, a sufficient condition is the continuity of the spectral density f at the origin, in which case $\lim_n \mathrm{var}(S_n)/n = 2\pi f(0)$–this follows from the following representation

$$\frac{\mathrm{var}(S_n)}{n} = \frac{1}{n} \int_{-\pi}^{\pi} \frac{\sin^2(\frac{nt}{2})}{\sin^2(\frac{t}{2})} f(t) \mathrm{d}t,$$

and Fejer's theorem on the Cesàro summability of the Fourier series of the spectral density f.

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The continuity of the spectral density f at the origin is by no means necessary, and in fact Hardy and Littlewood [9, Theorem C] in 1924 proved that a necessary and sufficient condition is the convergence of

$$\frac{1}{t}\int_{-t}^{t} f(s)\mathrm{d}s \to c_0 \in (0,\infty).$$

which has appeared before in a probabilistic context ([5]).

General case, $\mathrm{var}(S_n) \sim K_0 n^{\gamma} L(n)$. Necessary conditions usually require restrictive assumptions on the covariances such as regular variation in the Zygmund sense. Several sufficient conditions are stated either in terms of the covariances or of the spectral density, e.g. $f(x) \sim |x|^{-\alpha} L(1/|x|)$, for $\alpha \in (0,1)$, implies that $\mathrm{var}(S_n) \sim n^{1+\alpha} L(n)$, ([16]).

Even when $\gamma = 1$, the asymptotically non-linear behaviour of the variance has appeared often in the limit theorems for dependent variables, such as under general mixing conditions (see e.g. [10, 13]) or specific models such as random walk in random scenery (see [11, 3]).

The case $\gamma > 1$ frequently occurs in *long-range dependent* time-series when the covariances are not summable or the spectral density has an appropriate singularity at the origin which often results in non-Gaussian limiting behavior ([15, 18, 8, 17, 16]). The case $\gamma < 1$ occurs when the spectral density vanishes at the origin in which case both non-Gaussian ([15]), and Gaussian limits ([18]), have appeared in the literature.

Technique. It is not clear whether the approach of [9] (and [19]), can handle the slowly varying function and the case $\gamma \neq 1$. We suggest an alternative technique which is based on weak convergence and Fourier analysis of tempered distributions which also allows us to work directly with spectral measures, without assuming absolute continuity.

Subsequences. Subsequences $\mathrm{var}(S_{2^n})/2^n$ have often been applied through the use of dyadic induction and stationarity, in the context of mixing conditions, martingale approximations, central limit theorems, and invariance principles, ([13]). The question whether convergence along a subsequence is enough to guarantee convergence of the full sequence has been around for some time now, and it was presented to us as a conjecture by M. Peligrad. Although the answer is positive under extra conditions such as ρ-mixing, the necessary and sufficient condition stated in Theorem 1.1 allows us to construct a counterexample proving that convergence along dyadic subsequences does not imply convergence over the full sequence.

Proposition 1.2. *There exists a stationary process such that* $\mathrm{var}(S_{2^r})/2^r$ *converges, but the full sequence* $\mathrm{var}(S_n)/n$ *does not.*

The proofs of Theorem 1.1 and Proposition 1.2 are given in the next section along with several auxiliary results which are of independent interest.

2 Proofs

We start by proving three auxiliary Lemmas. By C we denote a generic positive constant.

Auxiliary results. Our starting point is the following inequality.

Lemma 2.1. *For any* $A > 0$

$$\frac{4}{\pi^2} n^2 G(1/n) \leq \mathrm{var}(S_n) \leq G(\pi) + \frac{\pi^2}{4} n^2 G(A/n) + \pi^2 \int_{A/n}^{\pi} \frac{G(y)}{y^3}\mathrm{d}y.$$

Proof. Define the positive Fejer kernel $I_n(y) = \sin^2(ny/2)/\sin^2(y/2)$. To prove the lower bound, notice that $I_n(y) \geq 4n^2/\pi^2$ for $0 < y < 1/n$, and hence

$$\mathrm{var}(S_n) = \int_0^\pi I_n(y)G(\mathrm{d}y) \geq \int_0^{1/n} \frac{4}{\pi^2}n^2 G(\mathrm{d}y) \geq \frac{4}{\pi^2}n^2 G(1/n).$$

To prove the upper bound, let $A \leq n$ and apply the bounds $I_n(y) \leq n^2\pi^2/4$ for $y \leq A/n$ and $I_n(y) \leq \pi^2/y^2$ for $y \geq A/n$ and integration by parts, to derive

$$\begin{aligned}\mathrm{var}(S_n) &= \int_0^{A/n} I_n(y)G(\mathrm{d}y) + \int_{A/n}^\pi I_n(y)G(\mathrm{d}y) \leq \int_0^{A/n} [n^2\pi^2/4]G(\mathrm{d}y) + \int_{A/n}^\pi [\pi^2/y^2]G(\mathrm{d}y) \\ &\leq \frac{\pi^2}{4}n^2 G(A/n) + G(\pi) + \pi^2 \int_{A/n}^\pi \frac{G(y)}{y^3}\mathrm{d}y. \qquad \square\end{aligned}$$

The next result establishes that upper bounds of $\mathrm{var}(S_n)/g(n)$, where $g(n) = n^\gamma L(n)$ for $\gamma \in (0, 2)$, are equivalent to upper bounds for the spectral measure G.

Lemma 2.2. *Suppose $\{n_k\}_{k\geq 0}$ is a positive non-decreasing integer sequence such that $n_k \to \infty$, and $\sup n_{k+1}/n_k = \kappa < \infty$. Then the following are equivalent:*
1. $\exists C > 0$ and such that $\mathrm{var}(S_{n_k}) \leq Cg(n_k)$;
2. $\exists C > 0$ such that $G(x) \leq Cx^{2-\gamma}L(1/x)$;
3. $\exists C > 0$ such that $\mathrm{var}(S_n) \leq Cg(n)$.

Proof. (1)⇒(2). From Lemma 2.1 and our assumptions, we have that for some positive constant $C > 0$, $G(1/n_k) \leq \pi^2 C n_k^{\gamma-2} L(n_k)/4$. Thus, by monotonicity of G and properties of slowly varying functions ([2]), for $1/n_{k+1} < x \leq 1/n_k$,

$$G(x) \leq G(1/n_k) \leq \frac{\pi^2 C}{4}\kappa^{2-\gamma}x^{2-\gamma}L(1/x) \sup_{1/\kappa \leq \lambda \leq 1} \frac{L(\lambda/x)}{L(1/x)} \leq C' x^{2-\gamma}L(1/x).$$

(2) ⇒ (3). We apply Lemma 2.1 with $A = 1$ to get

$$\mathrm{var}(S_n) \leq G(\pi) + \frac{\pi^2}{4}n^2 G(1/n) + \pi^2 \int_{1/n}^\pi \frac{G(y)}{y^3}\mathrm{d}y \leq C\left(1 + g(n) + \int_{1/n}^\pi y^{-\gamma-1}L(1/y)\mathrm{d}y\right).$$

Using the change of variables $x = 1/y$,

$$\int_{1/n}^\pi y^{-\gamma-1}L(1/y)\mathrm{d}y = \int_{1/\pi}^n x^{\gamma-1}L(x)\mathrm{d}x \sim \frac{n^\gamma L(n)}{\gamma},$$

as $n \to \infty$ by the Tauberian theorem ([2, Proposition 1.5.8]), since $\gamma - 1 > -1$. Therefore there is a constant C such that $\mathrm{var}(S_n) \leq Cg(n)$ which completes the proof since (3) ⇒ (1) is obvious. $\square$

The situation is similar when one considers lower bounds for $\mathrm{var}(S_n)$ and $G(x)$.

Lemma 2.3. *Suppose that there exist positive constants C_1 and C_2 such that $G(x) \leq C_1 x^{2-\gamma}L(1/x)$ and $\mathrm{var}(S_n) \geq C_2 g(n)$. Then there exists a positive constant C_3 such that $G(x) > C_3 x^{2-\gamma}L(1/x)$.*

Proof. We proceed by absurd and assume that there is a sequence $0 < y_k \to 0$ such that $G(y_k)/y_k^{2-\gamma}L(1/y_k) \to 0$, as $k \to \infty$. Then we can construct a further sequence $1 \le A_k \to \infty$ slowly enough so that
$A_k y_k \to 0\ ,\ G(A_k y_k)/y_k^{2-\gamma}L(1/y_k) \to 0\ ,\ L(1/A_k y_k)/L(1/y_k) \to 1\ .$
Then, for $n_k = [1/y_k] + 1 \to \infty$ we have by Lemma 2.1, the Tauberian theorem, and the monotonicity of G, for generic positive constants $C, C' > 0$, as $k \to \infty$,

$$\begin{aligned}\frac{\mathrm{var}(S_{n_k})}{g(n_k)} &\le C\left(\frac{1}{g(n_k)} + \frac{n_k^2 G(A_k/n_k)}{n_k^\gamma L(n_k)} + \frac{1}{n_k^\gamma L(n_k)}\int_{A_k/n_k}^{\pi} {y_k}^{-3} G(y)\mathrm{d}y\right)\\ &\le C'\left(\frac{1}{g(n_k)} + \frac{G(A_k y_k)}{{y_k}^{2-\gamma}L(1/y_k)} + \frac{L(1/y_k A_k)}{A_k^\gamma L(1/y_k)}\right) \to 0\end{aligned}$$

which contradicts assumptions of the lemma. □

Remark 2.4. From Lemma 2.1 it is obvious that the converse of Lemma 2.3 is also true.

Remark 2.5. For the boundary case $\gamma = 0$, Theorem 1.1 does not hold in general. For example, for $G(x) = x^2$ the direct calculations show that $\mathrm{var}(S_n) = 4\ln(n) + O(1)$ which is not bounded. Actually, Robinson [14] proved that

$$\sup_n \mathrm{var}(S_n) < \infty \text{ iff } \int_0^\pi x^{-2} dG(x) < \infty$$

When $\mathrm{cov}(X_0, X_n) \to 0$, the Leonov dichotomy holds; either $\mathrm{var}(S_n) \to \infty$ or $\sup_n \mathrm{var}(S_n) < \infty$ (see Bradley [4, Chapter 8]). However, the dichotomy is not true in general even for ergodic sequences as it follows e.g. from the Aaronson and Weiss construction [1] on Chacon's ergodic transformations.
A non-ergodic counterexample for Gaussian measures easily follows by taking $G(\{2\pi 2^{-k}\}) = 4^{-k}$ $k \ge 2$, and following the calculations similar to Samorodnitsky (see [16, Chapter 5]). More exactly, then

$$\sup_k \mathrm{var}(S_{2^k}) < \infty \quad \text{and} \quad \sup_n \mathrm{var}(S_n) = \infty\ .$$

Remark 2.6. For the boundary case $\gamma = 2$, the following dichotomy easily follows from Lemma 2.1

Corollary 2.7. *Either* $\liminf_{n\to\infty} \mathrm{var}(S_n)/n^2 > 0$ *or* $\lim_{n\to\infty} \mathrm{var}(S_n)/n^2 = 0$.

The fact also follows from the von Neumann L_2 ergodic theorem which states that $\mathbb{E}(S_n^2)/n^2$ vanishes if and only if the spectral measure has no atom at the origin (see [6]).
Also from Lemma 2.1, the following corollary easily follows.

Corollary 2.8. *Let* $\gamma \in (0, 2]$, *and* $\{n_k\}_{k\in\mathbb{Z}}$ *a non-negative increasing integer sequence such that* $\sup_k n_{k+1}/n_k < \infty$. *Then the following are equivalent:*
(1) $\mathrm{var}(S_{n_k})/n_k^\gamma \to 0$; *(2)* $x^{2-\gamma}G(x) \to 0$ *as* $0 < x \to 0$ *and* *(3)* $\mathrm{var}(S_n)/n^\gamma \to 0$.

Unlike the case $\gamma = 2$, for $\gamma \ne 2$, the equivalence does not hold in general without the assumption $\sup_k n_{k+1}/n_k < \infty$ as it follows from Theorem 1.1 by using a slowly varying function L such that $\liminf_{n\to\infty} L(n) = 0$ and $\limsup_{n\to\infty} L(n) = \infty$.

Proof of the main results. We are now ready to prove the main results. Let L be a positive function slowly varying at infinity, $2-\gamma \in (0,2)$ and $g(n) = n^{\gamma} L(n)$. The sufficiency part in Theorem 1.1 will be stated as independent Lemma.

Lemma 2.9. *Let $G(x) \sim x^{2-\gamma} L(1/x)$ as $x \to 0$. Then, $\lim_{n\to\infty} \mathrm{var}(S_n)/g(n) = 1/C(\gamma)$.*

Proof of Lemma 2.9. Start with the representation

$$\mathrm{var}(S_n) = \int_0^M \frac{\sin^2(y)}{n^2 \sin^2(y/n)} n^2 G(2\mathrm{d}y/n) + \int_M^{n\pi/2} \frac{\sin^2(y)}{n^2 \sin^2(y/n)} n^2 G(2\mathrm{d}y/n) =: I_{n,M} + J_{n,M},$$

for fixed $M \leq n$.
The inequalities $n^2 \sin^2(y/n) \geq 4y^2/\pi^2$, $G(x) \leq Cx^{2-\gamma} L(1/x)$, and integration by parts give

$$J_{n,M} \leq \frac{\pi^2}{4} \int_M^{n\pi/2} y^{-2} n^2 G(2\mathrm{d}y/n) \leq \frac{\pi^2}{4} \left[4\frac{G(\pi)}{\pi^2} + C2^{3-\gamma} \int_M^{\infty} n^{\gamma} \frac{L(n/2y)}{y^{1+\gamma}} \mathrm{d}y \right],$$

Bounding the integral term by using the change $x = n/2y$ and the Tauberian theorem, we then derive

$$J_{n,M} \leq \frac{\pi^2}{4} \left[4\frac{G(\pi)}{\pi^2} + C2^{3-\gamma} \frac{2^{\gamma}}{\gamma} \Big(\frac{n}{2M}\Big)^{\gamma} L\Big(\frac{n}{2M}\Big) \right] = O(1) + O(g(n/2M))$$

By regular variation $g(n/2M)/g(n) \to (2M)^{-\gamma}$ and therefore

$$\frac{\mathrm{var}(S_n)}{g(n)} = \frac{I_{n,M}}{g(n)} + O(1/g(n)) + O(M^{-\gamma}). \tag{1}$$

Notice that for $y \leq M \leq n$, $\sin^2(y)/n^2 \sin^2(y/n) = \sin^2(y)/y^2 + O(M^2/n^2)$, and thus

$$\frac{I_{n,M}}{g(n)} = \int_0^M \frac{\sin^2(y)}{y^2} \frac{n^{2-\gamma} G(2\mathrm{d}y/n)}{L(n)} + O(M^2/n^{\gamma}).$$

By regular variation of G, it follows that for $y \leq M$

$$\mu_n([0,y)) := \frac{n^{2-\gamma} G(2y/n)}{L(n)(2M)^{2-\gamma}} \to \Big(\frac{y}{M}\Big)^{2-\gamma},$$

which defines a probability measure on $[0,M]$ and hence, by weak convergence, since $\sin^2(y)/y^2$ is continuous and bounded, there exists a sequence $\mathcal{E}_M(n) \to 0$, as $n \to \infty$ for all M, such that

$$\begin{aligned} \frac{I_{n,M}}{g(n)} &= 2^{2-\gamma}(2-\gamma) \int_0^M \frac{\sin^2(y)}{y^{1+\gamma}} \mathrm{d}y + \mathcal{E}_M(n) + O(M^{-\gamma}) \\ &= 2^{2-\gamma}(2-\gamma) \int_0^{\infty} \frac{\sin^2(y)}{y^{1+\gamma}} \mathrm{d}y + \mathcal{E}_M(n) + O(M^{-\gamma}) = (1/C(\gamma)) + \mathcal{E}_M(n) + O(M^{-\gamma}) \end{aligned}$$

(see [7] for the integral). This together with (1) implies the lemma. □

Proof of Theorem 1.1. Implication (2) $\Rightarrow$ (1) immediately follows from Lemma 2.9.
For (1) $\Rightarrow$ (2), let $t_j \to \infty$ be a positive increasing integer sequence. Similar to Lemma 2.9, we derive

$$\frac{\mathrm{var}(S_{t_j})}{g(t_j)} = \int_0^M \frac{\sin^2(y)}{y^2} \frac{t_j^{2-\gamma} G(2\mathrm{d}y/t_j)}{L(t_j)} + O(M^2/t_j^\gamma) + O(M^{-\gamma}).$$

For $y \le M$ we have

$$\frac{t_j^{2-\gamma} G(2M/t_j)}{L(t_j)} \le CM^{2-\gamma} \frac{L(t_j/2M)}{L(t_j)} \le CM^{2-\gamma}.$$

Helly's principle and a diagonal argument imply that there exists a monotone increasing function h, defined on $[0, \infty)$, and a subsequence j' such that

$$F_{t_{j'}}(y) := \frac{t_{j'}^{2-\gamma} G(2y/t_{j'})}{L(t_{j'})} \to h(y), \tag{2}$$

as $j' \to \infty$ for all continuity points y of h. Since $h(y) \le CM^{2-\gamma}$ for $y \le M$, and $\sin^2(y)/y^2$ is continuous and bounded on $[0, M]$, by weak convergence we have that

$$\int_0^M \frac{\sin^2(y)}{y^2} F_{t_{j'}}(\mathrm{d}y) \to \int_0^M \frac{\sin^2(y)}{y^2} h(\mathrm{d}y).$$

Therefore, writing an identity for arbitrary $M > 0$ and then letting $M \to \infty$

$$K_0 = \lim_{j' \to \infty} \frac{\mathrm{var}(S_{t_{j'}})}{g(t_{j'})} = \int_0^M \frac{\sin^2(y)}{y^2} h(\mathrm{d}y) + O(M^{-\gamma}) \ = \int_0^\infty \frac{\sin^2(y)}{y^2} h(\mathrm{d}y).$$

Let $[x]$ denote the integer part of x, and notice that from (2) and regular variation of G, we also have

$$F([rt_{j'}]) = \frac{[rt_{j'}]^{2-\gamma} G(2y/[rt_{j'}])}{L([rt_{j'}])} \to r^{2-\gamma} h(y/r),$$

as $j' \to \infty$ for arbitrary $r > 0$ and all continuity points y/r of h. Since $\mathrm{var}(S_n)/g(n)$ converges on the full sequence, it follows that

$$K_0 = \lim_{j' \to \infty} \frac{\mathrm{var}(S_{[rt_{j'}]})}{g([rt_{j'}])} = \int_0^\infty \frac{\sin^2(y)}{y^2} r^{2-\gamma} h(\mathrm{d}y/r),$$

for any $r > 0$, implying that

$$\int_0^\infty \frac{\sin^2(rx)}{x^2} h(\mathrm{d}x) = r^\gamma K_0. \tag{3}$$

For $y > 0$, let $\psi(y) := \lim_{N \to \infty} \int_y^N x^{-2} h(\mathrm{d}x)$, which is well defined since by integration by parts we have

$$\int_y^\infty x^{-2} h(\mathrm{d}x) = \lim_{N \to \infty} \left[2 \int_y^N \frac{h(x) - h(y)}{x^3} \mathrm{d}x \right] = 2 \int_y^\infty \frac{h(x) - h(y)}{x^3} \mathrm{d}x < \infty.$$

The idea is to identify ψ from its Fourier transform using the convolution type equation (3); along these lines we continue by calculating the sine-transform of ψ by interchanging the integrals, which is allowed since the positive function ψ is bounded away from 0 and $|\psi(y)| \le Cy^{-\gamma}$,

$$\int_0^a \sin(ry)\psi(y)\mathrm{d}y = \int_0^a \Big(\int_{x=y}^\infty \sin(ry)\frac{h(\mathrm{d}x)}{x^2}\Big)\mathrm{d}y = \frac{2}{r}\int_0^a \frac{\sin^2(rx/2)}{x^2}h(\mathrm{d}x) + \frac{2}{r}\int_a^\infty \frac{\sin^2(ax/2)}{x^2}h(\mathrm{d}x)$$
$$\to \Big(\frac{r}{2}\Big)^{\gamma-1}K_0 = \lim_{a\to\infty}\int_0^a \sin(ry)\psi(y)\mathrm{d}y$$

as $a \to \infty$ for all $r > 0$.
For $y < 0$, we define $\psi(y) = -\psi(-y)$ so that for any $r \in \mathbb{R}$ we have

$$\lim_{a\to\infty}\int_{-a}^a \sin(ry)\psi(y)\mathrm{d}y = 2^{2-\gamma}\,\mathrm{sgn}(r)|r|^{\gamma-1}K_0. \tag{4}$$

To identify the function ψ and therefore h, we apply the Fourier analysis and treat ψ as a distribution acting on the Schwartz space of test functions $\mathcal{S} = \mathcal{S}(\mathbb{R})$ such that $\sup_x |x^\alpha \phi_n^{(\beta)}(x)| < \infty$ for all non-negative integers α, β. More exactly, we define a linear functional on $\mathcal{S}$ by

$$\Psi[\phi] = \int_0^\infty \psi(y)(\phi(y) - \phi(-y))\mathrm{d}y\ ,$$

which is continuous since

$$|\Psi[\phi]| \le 4\sup_y|\phi'(y)|\Big(\int_0^1 \psi(y)ydy\Big) + 4\sup_y|y\phi(y)|\Big(\int_1^\infty(\psi(y)/y)dy\Big)$$
$$\le C_\psi(\sup_y|\phi'(y)| + \sup_y|y\phi(y)|)$$

The next step is to calculate the Fourier transform of the tempered distribution Ψ, through the formula $\hat{\Psi}[\phi] = \Psi[\hat{\phi}]$. Then given $\phi \in \mathcal{S}$ we have

$$\Psi[\hat{\phi}] = \int_{y=0}^\infty \psi(y)\Big(\int_{t=-\infty}^\infty(\mathrm{e}^{\mathrm{i}ty} - \mathrm{e}^{-\mathrm{i}ty})\phi(t)\mathrm{d}t\Big)\mathrm{d}y = \mathrm{i}\int_{y=-\infty}^\infty \psi(y)\Big(\int_{t=-\infty}^\infty \sin(yt)\phi(t)\mathrm{d}t\Big)\mathrm{d}y$$

Observe that $|\sin(yt)| \le |yt|$, $|\psi(y)| \le C|y|^{-\gamma}$, and $1-\gamma > -1$ and so for fixed a

$$\int_{-a}^a\int_{-\infty}^\infty |\psi(y)||\sin(yt)|\phi(t)|\mathrm{d}t\mathrm{d}y \le \int_{-a}^a\int_{-\infty}^\infty |y|^{1-\gamma}|t\phi(t)|\mathrm{d}t\mathrm{d}y < \infty\ .$$

Therefore by Fubini's theorem

$$\Psi[\hat{\phi}] = \mathrm{i}\lim_{a\to\infty}\int_{-a}^a \psi(y)\int_{t=-\infty}^\infty \sin(yt)\phi(t)\mathrm{d}t\mathrm{d}y = \mathrm{i}\lim_{a\to\infty}\int_{-\infty}^\infty \phi(t)\int_{y=-a}^a \psi(y)\sin(yt)\mathrm{d}y\mathrm{d}t$$

We next bound the integrand in order to use dominated convergence. Let $\tau = [ta/\pi]$, and write

$$I := \int_{y=0}^a \psi(y)\sin(yt)\mathrm{d}y = \frac{1}{t}\int_{x=0}^{ta}\sin(x)\psi(x/t)\mathrm{d}x$$
$$= \frac{1}{t}\sum_{j=0}^{\tau-1}\int_{j\pi}^{(j+1)\pi}\sin(x)\psi(x/t)\mathrm{d}x + \frac{1}{t}\int_{\tau\pi}^{ta}\sin(x)\psi(x/t)\mathrm{d}x.$$

Since $\tau\pi$ is the largest multiple of π less than ta, for $x \in [\tau\pi, ta]$ $\sin(x)$ does not change sign and therefore

$$\left|\frac{1}{t}\int_{\tau\pi}^{ta} \sin(x)\psi(x/t)\mathrm{d}x\right| \leq |t|^{\gamma-1}\int_{\tau\pi}^{ta} \frac{|\sin(x)|}{|x|^{\gamma}}\mathrm{d}x \leq C|t|^{\gamma-1}.$$

The other term can be written as an alternating sum

$$Q := \frac{1}{t}\sum_{j=0}^{\tau-1}(-1)^j \int_{j\pi}^{(j+1)\pi} |\sin(x)|\psi(x/t)\mathrm{d}x.$$

From the fact that ψ is decreasing we can then show that

$$\begin{aligned} c_j := \int_{j\pi}^{(j+1)\pi} |\sin(x)|\psi(x/t)\mathrm{d}x &\geq \int_{j\pi}^{(j+1)\pi} |\sin(x)|\psi\big((j+1)\pi/t\big)\mathrm{d}x \\ &= \int_{(j+1)\pi}^{(j+2)\pi} |\sin(x)|\psi\big((j+1)\pi/t\big)\mathrm{d}x \geq \int_{(j+1)\pi}^{(j+2)\pi} |\sin(x)|\psi\big(x/t\big)\mathrm{d}x = c_{j+1}, \end{aligned}$$

and thus the sum Q is conditionally convergent and in absolute value less than its first term,

$$|Q| \leq \frac{1}{t}\left|\int_0^{\pi} \sin(x)\psi(x/t)\mathrm{d}x\right| = \frac{1}{t}\int_0^{\pi} \sin(x)\psi(x/t)\mathrm{d}x \leq \frac{C}{t}\int_0^{\pi} \sin(x)\frac{t^{\gamma}}{x^{\gamma}}\mathrm{d}x = Ct^{\gamma-1}.$$

Overall the above calculations imply that for all $a > 0$

$$\left|\phi(t)\int_{y=-a}^{a} \sin(yt)\psi(y)\mathrm{d}y\right| \leq C|\phi(t)|t^{\gamma-1},$$

where C does not depend on a. Furthermore since $\gamma - 1 > -1$, the function $|\phi(t)|t^{\gamma-1}$ has at most an integrable singularity at the origin and is integrable. Therefore by dominated convergence and (4)

$$\hat{\Psi}[\phi] = \mathrm{i}\int_{-\infty}^{\infty} \phi(t)\left(\lim_{a\to\infty}\int_{y=-a}^{a} \psi(y)\sin(yt)\mathrm{d}y\right)\mathrm{d}t = \int_{-\infty}^{\infty} \phi(t)\left(\mathrm{i}2^{2-\gamma}K_0 \operatorname{sgn}(t)|t|^{\gamma-1}\right)\mathrm{d}t.$$

Then inverting the Fourier transform of the distribution Ψ (see [7] for example) we identify the function $\psi(y)$, and by standard calculations $h(x)$, for $x, y > 0$, $\gamma \in (0, 2)$

$$\psi(y) = K_0 D(\gamma)y^{-\gamma},\ h(x) = (\gamma/(2-\gamma))K_0 D(\gamma)x^{2-\gamma}\,, \text{where}\ \ D(\gamma) = \Gamma(\gamma)2^{2-\gamma}\sin(\gamma\pi/2)/\pi\ .$$

We have shown that for every integer sequence $t_j \to 0$, there exists a subsequence $t_{j'} \to \infty$ such that for $x, r > 0$ and $\gamma \in (0, 2)$

$$\frac{[rt_{j'}]^2 G(x/[rt_{j'}])}{g([rt_{j'}])} \to r^{2-\gamma}h(x/2r) = (\gamma/(2-\gamma))K_0 D(\gamma)(x/2)^{2-\gamma}\ .$$

From this by standard limiting arguments we now deduce that

$$\lim_{x\to 0}\frac{G(x)}{x^{2-\gamma}L(1/x)} = (\gamma/(2-\gamma))K_0 D(\gamma)(1/2)^{2-\gamma} = C(\gamma)K_0\ ,$$

which proves the Theorem. □

Proof of Proposition 1.2. The proof is through a counterexample. Let $G(x) = 2^{-k}$, for $x \in (2^{-(k+1)}, 2^{-k}]$, for $k \geq 1$. Then obviously $\lim_{x\to 0} G(x)/x$ does not exist, as different subsequences give different limits. Therefore by Theorem 1.1 the limit of the full sequence $\mathrm{var}(S_n)/n$ cannot exist.
On the other hand, by direct calculation on the subsequence 2^r,

$$\frac{\mathrm{var}(S_{2^r})}{2^r} = \sum_{k=1}^{\infty} \sin^2(2^{r-k-1})2^{k+1-r} \to \sum_{k=0}^{\infty} \frac{\sin^2(2^k)}{2^k} + \sum_{k=1}^{\infty} 2^k \sin^2(2^{-k}) \in (0, \infty).$$

completing the proof of the proposition. □

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