ANNEALED BROWNIAN MOTION IN A HEAVY TAILED POISSONIAN POTENTIAL

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Consider a d-dimensional Brownian motion in a random potential defined by attaching a non-negative and polynomially decaying potential around Poisson points. We introduce a repulsive interaction between the Brownian path and the Poisson points by weighting the measure by the Feynman-Kac functional. We show that under the weighted measure, the Brownian motion tends to localize around the origin. We also determine the scaling limit of the path and also the limit shape of the random potential.

1. Introduction.

1.1. *The model.* Consider a random potential defined by attaching the shape function $\hat{v}(x) = |x|^{-\alpha} \wedge 1$ $(d < \alpha < d+2)$ around a Poisson point process $(\omega = \sum_i \delta_{\omega_i}, \mathbb{P})$ with unit intensity as follows:

$$V_\omega(x) = \sum_i \hat{v}(x - \omega_i).$$

Suppose we are also given the standard Brownian motion $(\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$ on $\mathbb{R}^d$. In this paper, we are interested in the long time behavior of a Brownian motion under the *annealed path measure* defined by

$$Q_t(A) = \frac{1}{Z_t} \mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} : A \right],$$

where Z_t denotes the normalizing constant

$$Z_t = \mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} \right].$$

The weight $\exp\{-\int_0^t V_\omega(X_s)\mathrm{d}s\}$ introduces a repulsive interaction between the Brownian path and Poisson points. Since the averages are taken over

both the path and the configuration, it is natural to expect that ω tends to rarefy in a region around the origin and the path favors to stay there. However, it is often challenging to prove that such a localization is typical under Q_t.

1.2. *Earlier studies.* We mention some early studies which are related to ours. When $\alpha > d + 2$, which is referred to as the light tailed case, Donsker and Varadhan [4] determined the asymptotics of the normalizing constant

$$(1.1) \quad \begin{aligned} &\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} \right] \\ &= \exp\left\{ -\inf_{U:\,\mathrm{open}} \{ \lambda_1(U) + |U| \} \, t^{\frac{d}{d+2}} (1 + o(1)) \right\} \end{aligned}$$

as t goes to ∞, where $|U|$ and $\lambda_1(U)$ stand for the volume of U and the smallest Dirichlet eigenvalue of $-\Delta/2$ in U, respectively. It follows from Faber-Krahn's inequality that the unique minimizer of the above variational problem is the ball with a certain radius R_0, up to translation. This result suggests that the dominant contribution to the right hand side of (1.1) comes from the following strategy: there exists $x \in \mathbb{R}^d$ such that $\omega(B(x, R_0 t^{1/(d+2)})) = 0$ and the Brownian motion $\{X_s\}_{0 \le s \le t}$ stays in the ball. Indeed, one can easily see that this specific event gives the correct lower bound. Motivated by this observation, Sznitman [11] $(d = 2)$ and Povel [10] $(d \ge 3)$ proved that the above confinement is typical under the annealed path measure when $\hat{v}$ has compact support. They also proved that the scaled process $\{t^{-1/(d+2)} X_{t^{2/(d+2)}s}\}_{s \ge 0}$ converges to a Brownian motion conditioned to stay in a ball with radius R_0 and random center. See also Bolthausen [1] for a similar result in two-dimensional discrete space setting.

On the other hand, in the heavy tailed case $d < \alpha < d + 2$, Pastur [9] (first term) and the author [**?**] (second term) determined the asymptotics of the normalizing constant

$$(1.2) \quad \mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} \right] = \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - (a_2 + o(1)) t^{\frac{\alpha+d-2}{2\alpha}} \right\}$$

as $t \to \infty$, where

$$a_1 = \omega_d \Gamma\left(\frac{\alpha - d}{\alpha} \right),$$

$$(1.3) \quad a_2 = \inf_{\substack{\phi \in W^{1,2}(\mathbb{R}^d), \\ \|\phi\|_{L^2} = 1}} \left\{ \int \frac{1}{2} |\nabla \phi(x)|^2 + C(d, \alpha) |x|^2 \phi(x)^2 \mathrm{d}x \right\}$$

with ω_d being the volume of the unit ball, $C(d, \alpha) > 0$ a constant, and $W^{1,2}(\mathbb{R}^d)$ the usual Sobolev space. The constant $C(d, \alpha)$ admits an explicit expression $\frac{\alpha \sigma_d}{2d} \Gamma\left(\frac{2\alpha - d + 2}{\alpha}\right)$ with σ_d the surface area of the unit sphere. As in the light tailed case, the correct lower bound of (1.2) can be given by considering a specific strategy:

(St1) $V_\omega(0) = a_1 \frac{d}{\alpha} t^{-\frac{\alpha - d}{\alpha}} + o(t^{-\frac{\alpha - d + 2}{2\alpha}})$,

(St2) $V_\omega(x) - V_\omega(0) = C(d, \alpha) t^{-\frac{\alpha - d + 2}{\alpha}} |x|^2 + o(t^{-\frac{\alpha - d + 2}{2\alpha}})$ for $|x| < Mt^{\frac{\alpha - d + 2}{4\alpha}}$,

(St3) $\sup_{0 \leq s \leq t} |X_s| < Mt^{\frac{\alpha - d + 2}{4\alpha}}$, and

(St4) $L_t = \frac{1}{t} \int_0^t \delta_{X_s} ds$ is, after the diffusive scaling with spatial factor $t^{\frac{\alpha - d + 2}{4\alpha}}$, weakly close to $\phi_1(x)^2 dx$, where

$$\phi_1(x) = \left(\frac{\sqrt{2C(d, \alpha)}}{\pi}\right)^{\frac{d}{4}} \exp\left\{-\sqrt{\frac{C(d, \alpha)}{2}} |x|^2\right\}$$

is the unique minimizer of (1.3).

Let us informally explain how this strategy gives the correct lower bound. The first event (St1) has probability

$$\mathbb{P}\left(V_\omega(0) = a_1 \frac{d}{\alpha} t^{-\frac{\alpha - d}{\alpha}} + o\left(t^{-\frac{\alpha - d + 2}{2\alpha}}\right)\right) \geq \exp\left\{-a_1 \frac{\alpha - d}{\alpha} t^{\frac{d}{\alpha}} + o\left(t^{\frac{\alpha + d - 2}{2\alpha}}\right)\right\}.$$

The second event (St2) introduces no extra cost since conditioned on (St1), its probability is not too small. In fact, it is close to 1 when $\alpha > 2$ and even when $\alpha \leq 2$, it is only polynomially small. As for the third and fourth events, Donsker and Varadhan's large deviation principle shows that

$$P_0((\text{St3}) \text{ and } (\text{St4}) \text{ hold}) \geq \exp\left\{-t^{\frac{\alpha + d - 2}{2\alpha}} \int \frac{1}{2} |\nabla \phi_1(x)|^2 dx + o\left(t^{\frac{\alpha + d - 2}{2\alpha}}\right)\right\}$$

after letting $M \to \infty$. Summarizing the above, we obtain the correct lower bound

$$\mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^t V_\omega(X_s) ds\right\} : (\text{St1})-(\text{St4}) \text{ hold}\right]$$
$$\geq \exp\left\{-a_1 t^{\frac{d}{\alpha}} - t^{\frac{\alpha + d - 2}{2\alpha}} \left(\int \frac{1}{2} |\nabla \phi_1(x)|^2 + C(d, \alpha)|x|^2 \phi_1(x)^2 dx + o(1)\right)\right\}.$$

From this observation, it is natural to ask whether the above strategy is typical under Q_t or not.

REMARK 1.1. Our model can be viewed as an example of diffusion in a spatially correlated random potential. There are several results, mainly concerning the asymptotics of Z_t, for such models. See e.g. [7] (discrete space) and [5, 6] (continuous space).

1.3. *Results.*   The results of the present paper show that the events (St1)–(St4) consisting the optimal strategy in the last subsection are, with appropriate modifications, typical under the annealed path measure Q_t. Moreover, the scaling limit of the path is also identified. The first result is about the path localization (St3).

THEOREM 1.1.   *For any $\delta > 0$,*

$$\lim_{t \to \infty} Q_t \left(\sup_{0 \le s \le t} |X_s| \le t^{\frac{\alpha - d + 2}{4\alpha}} (\log t)^{\frac{1}{2} + \delta} \right) = 1.$$

REMARK 1.2.   The logarithmic correction corresponds to the $M \to \infty$ operation in (1.2). The power $1/2$ is natural in view of Theorem 1.5 below.

The second result says that the random potential viewed from its local minimum looks like the quadratic function

$$p_t(x) = C(d, \alpha) t^{-\frac{\alpha - d + 2}{\alpha}} |x|^2,$$

i.e. (St2) is typical. It also turns out that the barycenter m_{L_t} of the occupation time measure $L_t = \frac{1}{t} \int_0^t \delta_{X_s} \mathrm{d}s$ is close to a *local minimizer* of V_ω.

THEOREM 1.2.   *Let $m_t(\omega)$ be the point where V_ω attains the minimum in $B(0, t^{\frac{\alpha - d + 2}{4\alpha}} \log t)$. Then there exists $\epsilon_0 > 0$ such that*

(1.4)
$$\lim_{t \to \infty} Q_t \Big(\sup \Big\{ |V_\omega(x) - V_\omega(m_t(\omega)) - p_t(x - m_t(\omega))| :$$
$$|x - m_t(\omega)| \le t^{\frac{\alpha - d + 2}{4\alpha}} \log t \Big\} \le t^{-\frac{\alpha - d + 2}{2\alpha} - \epsilon_0} \Big) = 1.$$

Moreover,

(1.5)
$$\lim_{t \to \infty} Q_t \left(|m_t(\omega) - m_{L_t}| < \epsilon t^{\frac{\alpha - d + 2}{4\alpha}} \right) = 1$$

for any $\epsilon > 0$.

REMARK 1.3.   A statement similar to (1.4) is proved by Grüninger and König [8] for the parabolic Anderson model on $\mathbb{Z}^d$ with so-called "almost bounded potentials". They call this phenomenon "potential confinement".

From Theorems 1.1 and 1.2, we can deduce scaling limits of the path, occupation time measure, and local minimum of the potential. We introduce the scale function $r(t) = t^{\frac{\alpha - d + 2}{4\alpha}}$ and define the scaled process by $\tilde{X}_s =$

$r(t)^{-1}X_{r(t)^2s}$. (The dependence of $\tilde{X}$ on t is omitted but this does not seem to cause any confusion.) Let us begin with the result for the occupation time measure $\tilde{L}_t = \frac{1}{tr(t)^{-2}} \int_0^{tr(t)^{-2}} \delta_{\tilde{X}_s} ds$, which implies that (St4) is typical. Let $\tilde{m}_t(\omega) = r(t)^{-1}m_t(\omega)$ and ν_m denote the measure with density

$$\phi_1(x-m)^2 = \left(\frac{\sqrt{2C(d,\alpha)}}{\pi} \right)^{\frac{d}{2}} \exp \left\{ -\sqrt{2C(d,\alpha)}|x-m|^2 \right\}.$$

THEOREM 1.3. *For any $f \in C_c(\mathbb{R}^d)$ and $\epsilon > 0$,*

(1.6) $$\lim_{t\to\infty} Q_t \left(\left| \int f d\tilde{L}_t - \int f d\nu_{\tilde{m}_t(\omega)} \right| < \epsilon \right) = 1.$$

Next we state results on the local minimum of V_ω. It turns out that there is a gap between the expected value of $V_\omega(m_t(\omega))$ and the right hand side of (St1). Moreover, the fluctuation of $V_\omega(m_t(\omega))$ is even larger than the gap when $\alpha < 2$.

THEOREM 1.4. *The following hold:*

(i) $Q_t[V_\omega(m_t(\omega)] = a_1 \frac{d}{\alpha} t^{-\frac{\alpha-d}{\alpha}} - \left(\sqrt{\frac{C(d,\alpha)}{8}} + o(1) \right) t^{-\frac{\alpha-d+2}{2\alpha}}$,

(ii) $t^{\frac{2\alpha-d}{2\alpha}} \{V_\omega(m_t(\omega)) - Q_t[V_\omega(m_t(\omega)]\}$ *converges in law to the Gaussian random variable with variance* $\alpha\sigma_d\Gamma(\frac{3\alpha-d+1}{\alpha})$.

We finally state the result on the scaling limit of the path. We write R_0^m for the law of the Ornstein-Uhlenbeck process starting at the origin with generator $-\Delta/2 + \sqrt{2C(d,\alpha)}\langle x-m, \nabla \rangle$. We call m the *center*.

THEOREM 1.5. *The process $\{\tilde{X}_s\}_{s\geq 0}$ under Q_t converges as $t \to \infty$ in law to the Ornstein-Uhlenbeck processes with random centers*

$$\int dm \left(\frac{\sqrt{C(d,\alpha)}}{\sqrt{2\pi}} \right)^{\frac{d}{2}} \exp \left\{ -\sqrt{\frac{C(d,\alpha)}{2}}|m|^2 \right\} R_0^m.$$

REMARK 1.4. It becomes clear in the proof that the random center m corresponds to $\tilde{m}_t(\omega) = r(t)^{-1}m_t(\omega)$. Hence roughly speaking, this theorem means that $\{\tilde{X}_s\}_{s\geq 0}$ behaves like the Ornstein-Uhlenbeck process centered at $\tilde{m}_t(\omega)$.

1.4. *Organization of the paper.* We briefly comment on the outline of the proof and the organization of the paper. The main difficulty lies in the proof of Theorem 1.1 and 1.2. They are closely related in the sense that the path localization implies the potential confinement and vice versa. But of course we need another input to leave the circular argument. The key idea is that we actually need only a weaker result than Theorem 1.1 to deduce the potential confinement. We prove such a weak localization in Section 2 by using crude estimates. Then in Section 3, we show that it indeed implies a potential confinement (Proposition 3.1), which is a slight modification of Theorem 1.2. Given the potential confinement, Theorem 1.1 follows by repeating a part of argument in Section 2 and it is done in Section 4. In the end of Section 4, we show (1.5) of Theorem 1.2 by combining Theorem 1.1 and Proposition 3.1 and it completes the proof of Theorem 1.2. Sections 5, 6, and 7 are devoted to prove Theorems 1.3, 1.4, and 1.5, respectively.

1.5. *Notation.* For functions $f, g : [0, \infty) \to \mathbb{R}$, we say $f(t) = \mathcal{O}(\exp\{g(t)\})$ if there exists $c > 0$ such that $f(t) = O(\exp\{cg(t)\})$ as $t \to \infty$.

2. Weak localization. In this section, we prove the following weaker version of Theorem 1.1.

PROPOSITION 2.1. *There exists $M_1 > 0$ such that*

$$\lim_{t \to \infty} Q_t \left(\sup_{0 \le s \le t} |X_s| \le M_1 t^{\frac{\alpha - d + 6}{8\alpha}} \right) = 1.$$

We use the notation

$$h_t = a_1 \frac{d}{\alpha} t^{-\frac{\alpha - d}{\alpha}},$$

which is expected to be the typical height of the bottom of V_ω under Q_t as explained in the introduction. We first show that $\min_{0 \le s \le t} V_\omega(X_s)$ can not be too far from h_t.

LEMMA 2.1. *For any $M \in \mathbb{R}$ and sufficiently large t,*

$$\mathbb{P}\left(\min_{x \in (-t,t)^d} V_\omega(x) - h_t < Mt^{-\frac{3\alpha - 3d + 2}{4\alpha}} \right)$$
$$\le \exp\left\{ -a_1 \frac{\alpha - d}{\alpha} t^{\frac{d}{\alpha}} + Mt^{\frac{\alpha + 3d - 2}{4\alpha}} - |M| t^{\frac{\alpha + d - 2}{2\alpha}} \right\}$$

PROOF. The event in the left-hand side concerns infinitely many points but we can reduce it to finite points since V_ω is smooth where it is small. Indeed, if $V_\omega(x) < 1$, then obviously

$$V_\omega(x) = \sum_i |x - \omega_i|^{-\alpha} \wedge 1 = \sum_i |x - \omega_i|^{-\alpha}$$

and in particular $\min_i |x - \omega_i| > 1$. Since there exists $c_0 > 0$ such that $|\nabla |x|^{-\alpha}| < c_0 |x|^{-\alpha}$ for $|x| > 1$, it follows that

$$|\nabla V_\omega(x)| \le \sum_i |\nabla |x - \omega_i|^{-\alpha}| \le \sum_i c_3 |x - \omega_i|^{-\alpha} = c_0 V_\omega(x) < c_0$$

when $V_\omega(x) < 1$. Therefore one can find $N > 0$ such that

$$\mathbb{P}\left(\min_{x \in (-t,t)^d} V_\omega(x) - h_t < Mt^{-\frac{3\alpha - 3d + 2}{4\alpha}} \right)$$

$$\le \mathbb{P}\left(\min_{x \in (-t,t)^d \cap t^{-N}\mathbb{Z}^d} V_\omega(x) - h_t < (M + t^{-1})t^{-\frac{3\alpha - 3d + 2}{4\alpha}} \right)$$

$$\le (2t + 1)^{dN} \mathbb{P}\left(V_\omega(0) - h_t < (M + t^{-1})t^{-\frac{3\alpha - 3d + 2}{4\alpha}} \right).$$

To bound the probability in the last line, we use the asymptotics of the moment generating function

$$(2.1) \qquad \mathbb{E}[\exp\{-sV_\omega(0)\}] = \exp\left\{ -a_1 s^{\frac{d}{\alpha}} + O(e^{-s}) \right\} \quad \text{as } s \to \infty,$$

which is proved in Lemma 1 of [?]. Taking $s = t(1 + t^{-\frac{d+2-\alpha}{4\alpha}})$ in (2.1) and using Chebyshev's inequality, one can see by a straightforward calculation that

$$\mathbb{P}\left(V_\omega(0) - h_t < (M + t^{-1})t^{-\frac{3\alpha - 3d + 2}{4\alpha}} \right)$$

$$\le \exp\left\{ -a_1 \frac{\alpha - d}{\alpha} t^{\frac{d}{\alpha}} + Mt^{\frac{\alpha + 3d - 2}{4\alpha}} - (|M| + \epsilon)t^{\frac{\alpha + d - 2}{2\alpha}} \right\}$$

for some small $\epsilon > 0$. $\qquad \square$

LEMMA 2.2.

$$Q_t\left(\min_{0 \le s \le t} V_\omega(X_s) - h_t \notin t^{-\frac{3\alpha - 3d + 2}{4\alpha}} [-2a_2, 2a_2] \right) = \mathcal{O}\left(\exp\left\{ -t^{\frac{\alpha + d - 2}{2\alpha}} \right\} \right).$$

PROOF. We shall only give the proof of

$$Q_t \left(\min_{0 \leq s \leq t} V_\omega(X_s) - h_t > 2a_2 t^{-\frac{3\alpha-3d+2}{4\alpha}} \right) = \mathcal{O} \left(\exp \left\{ -t^{\frac{\alpha+d-2}{2\alpha}} \right\} \right).$$

The other half can be shown by a similar argument. (See also Lemma 2.5 where a stronger statement is proved.) Note first that

$$P_0 \left(X_{[0,t]} \not\subset (-t,t)^d \right) = \mathcal{O} \left(\exp\{-t\} \right)$$

by a simple application of the reflection principle and that

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(X_s) \mathrm{d}s \right\} : \min_{0 \leq s \leq t} V_\omega(X_s) > 2a_1 t^{-\frac{\alpha-d}{\alpha}} \right]$$
$$\leq \exp \left\{ -2a_1 t^{\frac{d}{\alpha}} \right\}.$$

From these bound and (1.2), it follows that

$$Q_t \left(\{ X_{[0,t]} \not\subset (-t,t)^d \} \cup \left\{ \min_{0 \leq s \leq t} V_\omega(X_s) > 2a_1 t^{-\frac{\alpha-d}{\alpha}} \right\} \right)$$
$$= \mathcal{O} \left(\exp \left\{ -t^{\frac{\alpha+d-2}{2\alpha}} \right\} \right).$$

Next, for each $k \in \mathbb{N}$, let us introduce two events

$$E_k = \left\{ \min_{0 \leq s \leq t} V_\omega(X_s) - h_t - 2a_2 t^{-\frac{3\alpha-3d+2}{4\alpha}} \in t^{-1}[k, k+1) \right\},$$
$$F_k = \left\{ \min_{x \in (-t,t)^d} V_\omega(x) - h_t < 2a_2 t^{-\frac{3\alpha-3d+2}{4\alpha}} + t^{-1}(k+1) \right\}.$$

Then, it follows by the fact $E_k \cap \{ X_{[0,t]} \subset (-t,t)^d \} \subset F_k$ and Lemma 2.1 that

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(X_s) \mathrm{d}s \right\} : E_k \cap \{ X_{[0,t]} \subset (-t,t)^d \} \right]$$
$$\leq \exp \left\{ -a_1 \frac{d}{\alpha} t^{\frac{d}{\alpha}} - 2a_2 t^{\frac{\alpha+3d-2}{4\alpha}} - k \right\} \mathbb{P}(F_k)$$
$$\leq \exp \left\{ -a_1 t^{\frac{d}{\alpha}} - 2a_2 t^{\frac{\alpha+d-2}{2\alpha}} + 1 \right\}.$$

Dividing both sides by Z_t and summing over $k \in [0, 2a_1 t^{d/\alpha} + 1] \cap \mathbb{Z}$, we find

$$Q_t \left(\{ X_{[0,t]} \subset (-t,t)^d \} \cap \left\{ \min_{0 \leq s \leq t} V_\omega(X_s) - h_t \in \left[2a_2 t^{-\frac{3\alpha-3d+2}{4\alpha}}, 2a_1 t^{-\frac{\alpha-d}{\alpha}} \right] \right\} \right)$$
$$= \mathcal{O} \left(\exp \left\{ -t^{\frac{\alpha+d-2}{2\alpha}} \right\} \right)$$

and the proof is complete.   □

Let us define a *low level set* of V_ω by

$$\mathcal{L}_{t,\omega} = \left\{ x \in \mathbb{R}^d : V_\omega(x) \leq h_t + 2a_2 t^{-\frac{3\alpha - 3d + 2}{4\alpha}} \right\}.$$

Then Lemma 2.2 above shows that $Q_t(X_{[0,t]} \cap \mathcal{L}_{t,\omega} \neq \emptyset) \to 1$. In what follows, we are going to show that if $x \in \mathcal{L}_{t,\omega}$, then V_ω is bounded below by a certain quadratic function in an annulus around x. This has two consequences which lead us to the weak localization bound:

(i) each connected component of $\mathcal{L}_{t,\omega}$ is not too large,
(ii) once the Brownian motion hits a connected component of $\mathcal{L}_{t,\omega}$, it is difficult to escape from a neighborhood of it.

Let us start by evaluating the Laplace transform of $V_\omega(x) + V_\omega(y)$. We do it as a special case of a more general asymptotic formula for the Laplace functional which we shall make repeated use of in the sequel. We write m_μ for the barycenter $\int x\mu(\mathrm{d}x)$ for a probability measure μ on $\mathbb{R}^d$.

PROPOSITION 2.2. *Let $\epsilon \in (0, 1/\alpha)$. For any probability measure μ supported inside $B(0, t^{1/\alpha - \epsilon})$,*

$$\mathbb{E}[\exp\{-t\langle \mu, V_\omega \rangle\}]$$
$$= \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - (C(d, \alpha) + o(1)) t^{\frac{d-2}{\alpha}} \int |z - m_\mu|^2 \mu(\mathrm{d}z) \right\}$$

as $t \to \infty$. In particular,

$$(2.2) \quad \mathbb{E}\left[\exp\left\{ -\frac{t}{2}(V_\omega(x) + V_\omega(y)) \right\} \right] \leq \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - c_1 t^{\frac{d-2}{\alpha}} |x - y|^2 \right\}$$

for $c_1 = C(d, \alpha)/5$ and $x, y \in \mathbb{R}^d$ with $|x - y| < t^{1/\alpha - \epsilon}$ when t is sufficiently large.

PROOF. We may assume $m_\mu = 0$ by translation invariance of Poisson point process. By a well-known formula for Laplace functional of Poisson point process,

$$-\log \mathbb{E}[\exp\{-t\langle \mu, V_\omega \rangle\}]$$
$$= \int \left(1 - e^{-t\int \hat{v}(z-y)\mu(\mathrm{d}z)} \right) \mathrm{d}y$$
$$= \int \left(1 - e^{-t\hat{v}(-y)} \right) \mathrm{d}y + \int \left(e^{-t\hat{v}(-y)} - e^{-t\int \hat{v}(z-y)\mu(\mathrm{d}z)} \right) \mathrm{d}y.$$

This first term is easily shown to be $a_1 t^{d/\alpha} + O(e^{-t})$, see Lemma 1 of [?]. We pick a positive constant

$$0 < \delta < \frac{\epsilon}{\alpha + 2}$$

and divide the second term into the integrals over $\{|y| < t^{1/\alpha - \delta}\}$ and $\{|y| \geq t^{1/\alpha - \delta}\}$. For the first region, we know

$$\min\left\{ t\hat{v}(-y), t\int \hat{v}(z-y)\mu(\mathrm{d}z) \right\} \geq t^{\alpha\epsilon}$$

by the assumption on μ and hence the contribution form this part is negligible. For the second region, we may replace $\hat{v}$ by $v(\cdot) = |\cdot|^{-\alpha}$ and by a change of variable, it follows that

$$\int_{|y| \geq t^{1/\alpha - \delta}} \left(e^{-t\hat{v}(-y)} - e^{-t\int \hat{v}(z-y)\mu(\mathrm{d}x)} \right) \mathrm{d}y$$
$$= t^{\frac{d}{\alpha}} \int_{|\eta| \geq t^{-\delta}} e^{-|\eta|^{-\alpha}} \left(1 - e^{-\int |\eta - t^{-\frac{1}{\alpha}}z|^{-\alpha} - |\eta|^{-\alpha}\mu(\mathrm{d}z)} \right) \mathrm{d}\eta$$

By Taylor's theorem, we can find bounded functions R_2 and R_3 such that

$$|\eta - t^{-\frac{1}{\alpha}}z|^{-\alpha} - |\eta|^{-\alpha}$$
$$= t^{-\frac{1}{\alpha}}\langle z, \nabla v(\eta)\rangle + R_2(z,\eta)t^{-\frac{2}{\alpha}}|z|^2|\eta|^{-\alpha-2}$$
$$= t^{-\frac{1}{\alpha}}\langle z, \nabla v(\eta)\rangle + \frac{1}{2}t^{-\frac{2}{\alpha}}\langle z, \mathrm{Hess}_v(\eta)z\rangle + R_3(z,\eta)t^{-\frac{3}{\alpha}}|z|^3|\eta|^{-\alpha-3}$$

for $|\eta| \geq t^{-\delta}$ and $|z| \leq t^{1/\alpha - \epsilon}$. Using this second line and recalling that $m_\mu = 0$, we get

$$\left| \int |\eta - t^{-\frac{1}{\alpha}}z|^{-\alpha} - |\eta|^{-\alpha}\mu(\mathrm{d}z) \right| \leq \|R_2\|_\infty O(t^{-2\epsilon + \delta(\alpha+2)})$$

when $|\eta| \geq t^{-\delta}$. This right-hand side is $o(1)$ thanks to our choice of δ and thus we can use the inequality $|1 - e^{-a} - a| < a^2$ which holds for small a to obtain

$$\int_{|\eta| \geq t^{-\delta}} e^{-|\eta|^{-\alpha}} \left(1 - e^{-\int |\eta - t^{-\frac{1}{\alpha}}z|^{-\alpha} - |\eta|^{-\alpha}\mu(\mathrm{d}z)} \right)$$

(2.3)
$$= \frac{1}{2} \int_{|\eta| \geq t^{-\delta}} e^{-|\eta|^{-\alpha}} t^{-\frac{2}{\alpha}} \int \langle z, \mathrm{Hess}_v(\eta)z\rangle \mu(\mathrm{d}z)\mathrm{d}\eta$$
$$+ \int_{|\eta| \geq t^{-\delta}} e^{-|\eta|^{-\alpha}} t^{-\frac{3}{\alpha}} \int R_3(z,\eta)|z|^3\mu(\mathrm{d}z)|\eta|^{-\alpha-3}\mathrm{d}\eta$$
$$+ O\left(\int_{|\eta| \geq t^{-\delta}} e^{-|\eta|^{-\alpha}} \left(t^{-\frac{2}{\alpha}} \int R_2(z,\eta)|z|^2\mu(\mathrm{d}z) \right)^2 |\eta|^{-2\alpha-4}\mathrm{d}\eta \right).$$

A computation shows that the first term in the right-hand side of (2.3) equals

$$\frac{1}{2}t^{-\frac{2}{\alpha}}\int\left\langle z, \int_{|\eta|\geq t^{-\delta}}\mathrm{Hess}_v(\eta)e^{-|\eta|^{-\alpha}}\mathrm{d}\eta z\right\rangle\mu(\mathrm{d}z)$$

$$= (C(d,\alpha)+o(1))t^{-\frac{2}{\alpha}}\int|z|^2\mu(\mathrm{d}z).$$

The other terms in (2.3) turn out to be smaller order than this: indeed by the assumption on μ,

(i) the second term is bounded by $t^{-2/\alpha}\int|z|^2\mu(\mathrm{d}z)$ multiplied by

$$O\left(t^{-\frac{1}{\alpha}}\right)\sup\{|z| : z\in\mathrm{supp}\mu\} = o(1),$$

(ii) $t^{-2/\alpha}\int|z|^2\mu(\mathrm{d}z)$ itself is of $o(1)$ and the third term has smaller order than it as its square.

$\square$

From the above lemma, we can deduce controls on $V_\omega(x) + V_\omega(y)$ for all x, y within an intermediate distance.

LEMMA 2.3. *For any $\epsilon\in(0,1/\alpha)$, there exists $R > 0$ such that*

(2.4)
$$\lim_{t\to\infty}Q_t\Big(V_\omega(x)+V_\omega(y) > 2h_t + c_1 t^{-\frac{\alpha-d+2}{\alpha}}|x-y|^2$$

$$\textit{for all } x,y\in(-2t,2t)^d \textit{ with } Rt^{\frac{\alpha-d+6}{8\alpha}} < |x-y| < t^{1/\alpha-\epsilon}\Big) = 1$$

PROOF. We prove that for sufficiently large $R > 0$,
(2.5)
$$Q_t\left(V_\omega(x)+V_\omega(y)\leq 2h_t + c_1 t^{-\frac{\alpha-d+2}{\alpha}}|x-y|^2\right) = \mathcal{O}\left(\exp\left\{-t^{\frac{\alpha+d-2}{2\alpha}}\right\}\right)$$

uniformly in $x, y\in\mathbb{R}^d$ with $Rt^{\frac{\alpha-d+6}{8\alpha}} < |x-y| < t^{1/\alpha-\epsilon}$. One can deduce (2.4) from this by dividing $(-2t,2t)^d$ into small cubes and using the union bound, just as in the proof of Lemma 2.1.

To prove (2.5), we may restrict our consideration to the event

$$E = \left\{\min_{0\leq s\leq t}V_\omega(X_s) - h_t \geq -2a_2 t^{-\frac{3\alpha-3d+2}{4\alpha}}\right\}$$

by Lemma 2.2. Then denoting the event in (2.5) by $F_{x,y}$, we have

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ - \int_0^t V_\omega(X_s) \mathrm{d}s \right\} : E \cap F_{x,y} \right]$$
$$\leq \exp \left\{ -a_1 \frac{d}{\alpha} t^{\frac{d}{\alpha}} + 2a_2 t^{\frac{\alpha+3d-2}{4\alpha}} \right\} \mathbb{P}(F_{x,y}).$$

Now a simple large deviation bound with the help of (2.2) shows

$$\mathbb{P}(F_{x,y}) \leq \exp \left\{ -a_1 \frac{\alpha - d}{\alpha} t^{\frac{d}{\alpha}} - \frac{c_1}{2} R^2 t^{\frac{\alpha+3d-2}{4\alpha}} \right\}$$

uniformly in $x, y \in \mathbb{R}^d$ with $Rt^{\frac{\alpha-d+6}{8\alpha}} < |x - y| < t^{1/\alpha-\epsilon}$. Therefore by taking sufficiently large R, we obtain

$$Q_t(E \cap F_{x,y}) = \mathcal{O} \left(\exp \left\{ -t^{\frac{\alpha+3d-2}{4\alpha}} \right\} \right).$$

Since $\frac{\alpha+3d-2}{4\alpha} > \frac{\alpha+d-2}{2\alpha}$ for $\alpha < d + 2$, this completes the proof. $\qquad \square$

Let $A_R(x)$ denote the annulus $B\left(x, 2Rt^{\frac{\alpha-d+6}{8\alpha}}\right) \setminus B\left(x, Rt^{\frac{\alpha-d+6}{8\alpha}}\right)$ around $x \in \mathbb{R}^d$. The above lemma implies that

(2.6)
$$\lim_{t \to \infty} Q_t \Big(\text{For any } x \in \mathcal{L}_{t,\omega} \cap (-t, t)^d,$$
$$V_\omega > h_t + c_1 R^2 t^{-\frac{3\alpha-3d+2}{4\alpha}} \text{ on } A_R(x) \Big) = 1.$$

As a consequence, if $c_1 R^2 > 1$ then every connected component of $\mathcal{L}_{t,\omega} \cap (-t, t)^d$ is contained in a ball with radius $Rt^{\frac{\alpha-d+6}{8\alpha}}$ with high probability. Therefore Proposition 2.1 reduces to the following.

PROPOSITION 2.3.   *There exists $R > 0$ such that*

$$\lim_{t \to \infty} Q_t \left(\sup_{0 \leq s \leq t} \mathrm{dist}(X_s, \mathcal{L}_{t,\omega}) \leq 2Rt^{\frac{\alpha-d+6}{8\alpha}} \right) = 1.$$

PROOF.   We use two lemmas whose proofs are given later. The first one is a lower bound for Z_t in terms of a random eigenvalue, which is not explicit but much more precise than (1.2).

LEMMA 2.4.   *There exists $c_2 > 0$ such that*

$$Z_t \geq c_2 t^{-2d} \mathbb{E} \left[\exp \left\{ -t\lambda_1^\omega((-t, t)^d) \right\} \right].$$

The second one is an upper bound for the eigenvalue.

LEMMA 2.5. *For any $\epsilon > 0$,*

$$\lim_{t \to \infty} Q_t \left(\lambda_1^\omega((-t,t)^d) \leq h_t + \epsilon t^{-\frac{3\alpha-3d+2}{4\alpha}} \right) = 1.$$

Let G be the intersection of $\{X_{[0,t]} \subset (-t,t)^d\}$, $\{X_{[0,t]} \cap \mathcal{L}_{t,\omega} \neq \emptyset\}$, and the events in (2.6) and in Lemma 2.5. As we have proved $\lim_{t \to \infty} Q_t(G) = 1$ above, we restrict ourselves to G. Though we drop $\cap G$ from the notation for simplicity, it is assumed throughout the proof. Now, let us introduce stopping times

$$H_0 = \inf \left\{ s \in [0,t] : \operatorname{dist}(X_s, \mathcal{L}_{t,\omega}) > 2Rt^{\frac{\alpha-d+6}{8\alpha}} \right\},$$
$$H_1 = \inf \left\{ s \in [0,t] : X_s \in \mathcal{L}_{t,\omega} \right\}.$$

We first consider the case $H_0 > H_1$. In this case, we further introduce random times defined by

$$H_2 = \inf \left\{ s \in [H_1,t] : X_s \notin B(X_{H_1}, 2Rt^{\frac{\alpha-d+6}{8\alpha}}) \right\},$$
$$H_3 = \sup \left\{ s \in [H_1,H_2] : X_s \in B(X_{H_1}, Rt^{\frac{\alpha-d+6}{8\alpha}}) \right\},$$

which satisfy $H_1 < H_3 < H_2 < H_0 < t$. Let us define $E_k = \{H_3 \in [k,k+1)\}$, $S = \{H_2 - H_3 > t^{\frac{\alpha+d-2}{2\alpha}}, H_2 < t\}$ (slow crossing), and $F = \{H_2 - H_3 \leq t^{\frac{\alpha+d-2}{2\alpha}}, H_2 < t\}$ (fast crossing). Note that for $s \in [H_3, H_2]$, we have $X_s \in A_R(X_{H_1})$ and hence

$$V_\omega(X_s) \geq h_t + c_1 R^2 t^{-\frac{3\alpha-3d+2}{4\alpha}}$$
$$\geq \lambda_1^\omega((-t,t)^d) + t^{-\frac{3\alpha-3d+2}{4\alpha}}.$$

for sufficiently large $R > 0$. We use the strong Markov property at H_2 to see

(2.7)
$$\mathbb{E} \otimes E_0 \left[\exp \left\{ -\int_0^t V_\omega(X_s) \mathrm{d}s \right\} : S \cap E_k \right]$$
$$\leq \mathbb{E} \left[E_0 \left[\exp \left\{ -\int_0^{k+1} V_\omega(X_s) \mathrm{d}s \right\} \right. \right.$$
$$\exp \left\{ -(H_2 - k - 1) \left(\lambda_1^\omega((-t,t)^d) + t^{-\frac{3\alpha-3d+2}{4\alpha}} \right) \right\} 1_{S \cap E_k}$$
$$\left. \left. E_{X_{H_2}} \left[\exp \left\{ -\int_0^{t-H_2} V_\omega(X_s) \mathrm{d}s \right\} : X_{[0,t-H_2]} \subset (-t,t)^d \right] \right] \right].$$

Now from a uniform bound

$$
(2.8) \quad
\begin{aligned}
\sup_{x \in \mathbb{R}^d} E_x & \left[\exp \left\{ - \int_0^s V_\omega(X_s) \mathrm{d}s \right\} : X_{[0,s]} \subset (-t,t)^d \right] \\
& \leq c(d) (1 + (s\lambda_1^\omega((-t,t)^d))^{d/2}) \exp \left\{ -s\lambda_1^\omega((-t,t)^d) \right\}
\end{aligned}
$$

on Feynman-Kac semigroup (see Theorem 3.1.2 in [12]) and the fact $H_2 - k - 1 > t^{\frac{\alpha+d-2}{2\alpha}} - 1$ on $S \cap E_k$, it follows that

$$
(2.9) \quad
\frac{\mathbb{E} \otimes E_0 \left[\exp \left\{ - \int_0^t V_\omega(X_s) \mathrm{d}s \right\} : S \cap E_k \right]}{\mathbb{E} \left[\exp \left\{ -t\lambda_1^\omega((-t,t)^d) \right\} \right]} = \mathcal{O} \left(\exp \left\{ -t^{\frac{d+2-\alpha}{4\alpha}} \right\} \right).
$$

As for the fast crossing, we proceed as in (2.7) and use the Markov property at time k to get

$$
\begin{aligned}
\mathbb{E} \otimes E_0 & \left[\exp \left\{ - \int_0^t V_\omega(X_s) \mathrm{d}s \right\} : F \cap E_k \right] \\
& \leq \mathbb{E} \left[E_0 \left[\exp \left\{ - \int_0^k V_\omega(X_s) \mathrm{d}s \right\} \exp \left\{ -(H_2 - k - 1)\lambda_1^\omega((-t,t)^d) \right\} 1_{E_k} \right. \right. \\
& \qquad P_{X_k} \left(\sup \left\{ |X_s - X_0| : 0 \leq s \leq t^{\frac{\alpha+d-2}{2\alpha}} \right\} > Rt^{\frac{\alpha-d+6}{8\alpha}} \right) \\
& \qquad \left. \left. E_{X_{H_2}} \left[\exp \left\{ - \int_0^{t-H_2} V_\omega(X_s) \mathrm{d}s \right\} : X_{[0,t-H_2]} \subset (-t,t)^d \right] \right] \right].
\end{aligned}
$$

Since we have

$$
(2.10) \quad
\begin{aligned}
P_{X_k} & \left(\sup \left\{ |X_s - X_0| : 0 \leq s \leq t^{\frac{\alpha+d-2}{2\alpha}} \right\} > Rt^{\frac{\alpha-d+6}{8\alpha}} \right) \\
& = P_0 \left(\sup \left\{ |X_s| : 0 \leq s \leq t^{\frac{\alpha+d-2}{2\alpha}} \right\} > Rt^{\frac{\alpha-d+6}{8\alpha}} \right) \\
& = \mathcal{O} \left(\exp \left\{ -t^{\frac{d+2-\alpha}{4\alpha}} \right\} \right),
\end{aligned}
$$

it follows that

$$
\frac{\mathbb{E} \otimes E_0 \left[\exp \left\{ - \int_0^t V_\omega(X_s) \mathrm{d}s \right\} : F \cap E_k \right]}{\mathbb{E} \left[\exp \left\{ -t\lambda_1^\omega((-t,t)^d) \right\} \right]} = \mathcal{O} \left(\exp \left\{ -t^{\frac{d+2-\alpha}{4\alpha}} \right\} \right)
$$

just as before. Summing (2.9) and (2.10) over $k \in [0,t] \cap \mathbb{Z}$ and using Lemma 2.4, we obtain

$$
(2.11) \quad
\lim_{t \to \infty} Q_t \left(\sup_{s \in [H_1, t]} |X_s| > 2Rt^{\frac{\alpha-d+6}{8\alpha}} \right) = 0.
$$

In the other case $H_0 < H_1$, we introduce random times defined by

$$\tilde{H}_2 = \sup\left\{ s \in [0,t] : \sup_{0 \le s \le t} \mathrm{dist}(X_s, \mathcal{L}_{t,\omega}) > 2Rt^{\frac{\alpha-d+6}{8\alpha}} \right\},$$

$$\tilde{H}_3 = \inf\left\{ s \in [\tilde{H}_2, t] : \mathrm{dist}(X_s, \mathcal{L}_{t,\omega}) \le Rt^{\frac{\alpha-d+6}{8\alpha}} \right\},$$

which satisfy $H_0 < \tilde{H}_2 < \tilde{H}_3 < H_1 < t$. By using $\tilde{H}_2$ and $\tilde{H}_3$ instead of H_2 and H_3 respectively, one can show that

$$\lim_{t \to \infty} Q_t\left(\sup_{s \in [0,H_1]} |X_s| > 2Rt^{\frac{\alpha-d+6}{8\alpha}} \right) = 0.$$

by the same argument as above. □

PROOF OF LEMMA 2.4. We start by introducing the notation

$$T_t^\omega f(x) = E_x\left[f(X_t) \exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} : X_{[0,t]} \subset (-t,t)^d \right].$$

Since we know from (2) that

$$Z_t = \mathbb{E}[T_t^\omega 1(0)] + o(Z_t),$$

we consider the first term on the right-hand side.

Using translation invariance with respect to $\mathbb{P}$, we find

$$\begin{aligned}
Z_t &= \mathbb{E} \otimes E_0\left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} \right] \\
&= \frac{1}{(2t)^d}\mathbb{E}\left[\int_{(-t,t)^d} E_x\left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} \right] \mathrm{d}x \right] \\
&\ge \frac{1}{(2t)^d}\mathbb{E}\left[\int_{(-t,t)^d} T_t^\omega 1(x)\mathrm{d}x : \lambda_1^\omega((-t,t)^d) \le 1 \right] \\
&= \frac{1}{(2t)^d}\mathbb{E}\left[\langle T_t^\omega 1, 1 \rangle : \lambda_1^\omega((-t,t)^d) \le 1 \right] \\
&\ge \frac{1}{(2t)^d}\mathbb{E}\left[\langle \phi_1^\omega, 1 \rangle^2 \exp\left\{ -t\lambda_1^\omega((-t,t)^d) \right\} : \lambda_1^\omega((-t,t)^d) \le 1 \right],
\end{aligned}$$

(2.12)

where ϕ_1^ω is the L^2-normalized non-negative eigenfunction associated with $\lambda_1^\omega((-t,t)^d)$. Since $\|T_1^\omega\|_{1\to\infty} \le (2\pi)^{-\frac{d}{2}}$, we have

$$0 \le \exp\left\{ -\lambda_1^\omega((-t,t)^d) \right\} \phi_1^\omega(z) = T_1^\omega \phi_1^\omega(z) \le (2\pi)^{-\frac{d}{2}} \langle \phi_1^\omega, 1 \rangle$$

for any $z \in (-t, t)^d$. Therefore on $\{\lambda_1^\omega((-t, t)^d) \le 1\}$,

$$\langle \phi_1^\omega, 1 \rangle^2 \ge (2\pi)^d e^{-2} \sup_{z \in (-t,t)^d} \phi_1^\omega(z)^2 \ge (2\pi)^d e^{-2}(2t)^{-d},$$

where we have used $\operatorname{supp} \phi_1^\omega \subset (-t, t)^d$ together with $\|\phi_1^\omega\|_2 = 1$ in the second inequality. Coming back to (2.12), we obtain

$$Z_t \ge \frac{(2\pi)^d e^{-2}}{(2t)^{2d}} \mathbb{E}\Big[\exp\big\{-t\lambda_1^\omega((-t, t)^d)\big\} : \lambda_1^\omega((-t, t)^d) \le 1\Big].$$

for $t \ge 1$. We can drop $\{\lambda_1^\omega((-t, t)^d) \le 1\}$ from the right-hand side since

$$\mathbb{E}\Big[\exp\big\{-t\lambda_1^\omega((-t, t)^d)\big\} : \lambda_1^\omega((-t, t)^d) > 1\Big] \le e^{-t} = o(Z_t).$$

$\square$

PROOF OF LEMMA 2.5. The argument is similar to that for Lemma 2.2. We may restrict ourselves to $\{X_{[0,t]} \subset (-t, t)^d\}$ thanks to (2). Then by (2.8), it follows that

$$\mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^t V_\omega(X_s)\mathrm{d}s\Big\} : X_{[0,t]} \subset (-t, t)^d, \lambda_1^\omega((-t, t)^d) > 3a_1 t^{-\frac{\alpha-d}{\alpha}}\Big]$$
$$\le \exp\Big\{-2a_1 t^{\frac{d}{\alpha}}\Big\} c(d) \sup_{\lambda > 0}(1 + \lambda^{d/2})e^{-\frac{\lambda}{3}}$$
$$= o(Z_t).$$

Next we show that

$$\mathbb{E} \otimes E_0\Big[\exp\Big\{-\int_0^t V_\omega(X_s)\mathrm{d}s\Big\} : X_{[0,t]} \subset (-t, t)^d,$$

(2.13)
$$\lambda_1^\omega((-t, t)^d) - h_t - \epsilon t^{-\frac{3\alpha-3d+2}{4\alpha}} \in t^{-1}[k-1, k)\Big]$$

$$= Z_t \mathcal{O}\Big(\exp\Big\{-t^{\frac{\alpha+d-2}{2\alpha}}\Big\}\Big)$$

uniformly in $k \in \mathbb{N} \cap [0, 3a_1 t^{d/\alpha} + 1]$. Using (2.8) again, we can bound the above left-hand side by

$$\mathbb{E}\Big[c(d)(1 + (t\lambda_1^\omega((-t, t)^d))^{d/2}) \exp\big\{-t\lambda_1^\omega((-t, t)^d)\big\} :$$

(2.14)
$$\lambda_1^\omega((-t, t)^d) - h_t - \epsilon t^{-\frac{3\alpha-3d+2}{4\alpha}} \in t^{-1}[k-1, k)\Big]$$
$$\le t^d \exp\Big\{-th_t - \epsilon t^{\frac{\alpha+3d-2}{4\alpha}} - k + 1\Big\}$$
$$\mathbb{P}\Big(\lambda_1^\omega((-t, t)^d) \le h_t + \epsilon t^{-\frac{3\alpha-3d+2}{4\alpha}} + t^{-1}k\Big)$$

for sufficiently large t. To estimate the last probability, we use an inequality

$$\mathbb{P}\left(\lambda_1^\omega((-t,t)^d) \leq \lambda\right) \leq (2t)^d N(\lambda),$$

where $N(\lambda)$ is the integrated density of states of $-1/2\Delta + V_\omega$ (see, e.g. [2], Chapter VI for the definition of integrated density of states and also the above inequality). The asymptotics of $N(\lambda)$ has been determined up to the second term in Theorem 3 of [?]:

$$N(\lambda) = \exp\left\{-l_1\lambda^{-\frac{d}{\alpha-d}} - (l_2 + o(1))\lambda^{-\frac{\alpha+d-2}{2(\alpha-d)}}\right\}$$

as $\lambda \downarrow 0$, where

$$l_1 = \frac{\alpha-d}{\alpha}\left(\frac{d}{\alpha}\right)^{\frac{d}{\alpha-d}} a_1^{\frac{\alpha}{\alpha-d}}, \quad l_2 = a_2\left(\frac{da_1}{\alpha}\right)^{\frac{\alpha+d-2}{2(\alpha-d)}}.$$

Substituting $h_t + \epsilon t^{-\frac{3\alpha-3d+2}{4\alpha}} + t^{-1}k$ into λ, one can find a constant $c_3 > 0$ satisfying

$$\mathbb{P}\left(\lambda_1^\omega((-t,t)^d) \leq h_t + \epsilon t^{-\frac{3\alpha-3d+2}{4\alpha}} + t^{-1}k\right)$$
$$\leq \exp\left\{-a_1\frac{\alpha-d}{\alpha}t^{\frac{d}{\alpha}} + \epsilon t^{\frac{\alpha+3d-2}{4\alpha}} + k - (a_2 + c_3\epsilon^2)t^{\frac{\alpha+d-2}{2\alpha}}\right\}$$

for all $k \in \mathbb{N} \cap [0, 3a_1t^{d/\alpha} + 1]$ by a straightforward calculation. (It suffices to take c_3 so small that

$$(1+x)^{-\frac{d}{\alpha-d}} \geq 1 - \frac{d}{\alpha-d}x + c_3x^2$$

for all $x \in [0, 4\alpha/d]$.) From this and (2.14), the desired bound (2.13) follows. Summing over k, we obtain

$$\mathbb{E} \otimes E_0\left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\} : X_{[0,t]} \subset (-t,t)^d,\right.$$
$$\left.\lambda_1^\omega((-t,t)^d) \in \left(h_t + \epsilon t^{-\frac{3\alpha-3d+2}{4\alpha}}, 3a_1t^{-\frac{\alpha-d}{d}}\right]\right]$$
$$= Z_t\mathcal{O}\left(\exp\left\{-t^{\frac{\alpha+d-2}{2\alpha}}\right\}\right)$$

and we are done. $\qquad\square$

As a corollary to the weak localization, we obtain an upper bound on the variance of L_t. In contrast to the Proposition 2.1, the bound is of correct order and this will be crucial in the next section.

COROLLARY 2.1.   *For any* $M_2 \geq \frac{2a_2}{C(d,\alpha)}$,

$$\lim_{t\to\infty} Q_t \left(\int |x - m_{L_t}|^2 L_t(\mathrm{d}x) < M_2 t^{\frac{\alpha-d+2}{2\alpha}} \right) = 1.$$

PROOF. In view of Proposition 2.1, we only need to show

$$\lim_{t\to\infty} Q_t \bigg(\int |x - m_{L_t}|^2 L_t(\mathrm{d}x) \geq \frac{2a_2}{C(d,\alpha)} t^{\frac{\alpha-d+2}{2\alpha}},$$

$$\operatorname{supp} L_t \subset B(0, M_1 t^{\frac{\alpha-d+6}{8\alpha}}) \bigg) = 0.$$

But it follows from Proposition 2.2 that if L_t lies in the above event,

$$\mathbb{E}\left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} \right]$$

$$= \mathbb{E}\left[\exp\left\{ -t\langle L_t, V_\omega \rangle \right\} \right]$$

$$= \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - (C(d,\alpha) + o(1)) t^{\frac{d-2}{\alpha}} \int |x - m_{L_t}|^2 L_t(\mathrm{d}x) \right\}$$

$$\leq \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - (2a_2 + o(1)) t^{\frac{\alpha+d-2}{2\alpha}} \right\}$$

$$= o(Z_t).$$

$\square$

3. Potential confinement. We prove the following version of Theorem 1.2 in this section by using the weak localization result. It will be used in next section to complete the proof of Theorem 1.1. The proof of Theorem 1.2 will be completed at the end of next section with the help of Theorem 1.1.

PROPOSITION 3.1.   *There exists an* $\epsilon > 0$ *such that*

$$\lim_{t\to\infty} Q_t \Big(\sup\Big\{ |V_\omega(x) - V_\omega(m_{L_t}) - p_t(x - m_{L_t})| :$$

$$|x - m_{L_t}| \leq 2t^{\frac{\alpha-d+2}{4\alpha}} \log t \Big\} < t^{-\frac{\alpha-d+2}{2\alpha} - \epsilon} \Big) = 1.$$

We define the weighted measure by

$$\frac{\mathrm{d}\mathbb{P}_t^\mu}{\mathrm{d}\mathbb{P}} = \frac{e^{-t\langle\mu, V_\omega\rangle}}{\mathbb{E}[e^{-t\langle\mu, V_\omega\rangle}]}$$

for each probability measure μ on $\mathbb{R}^d$. Under $\mathbb{P}_t^\mu$, ω is a Poisson point process with intensity $e^{-t \int \hat{v}(x-y)\mu(\mathrm{d}x)}\mathrm{d}y$. We take M_2 as in Corollary 2.1 and define a class of probability measures on $\mathbb{R}^d$ by

$$\mathcal{P}_{\mathrm{wl}} = \left\{ \mu : \operatorname{supp}\mu \subset B(0, M_1 t^{\frac{\alpha-d+6}{8\alpha}}), \ \int |x - m_\mu|^2 \mu(\mathrm{d}x) < M_2 t^{-\frac{\alpha-d+2}{2\alpha}} \right\},$$

where "wl" stands for "weakly localized".

PROPOSITION 3.2.   *There exists an $\epsilon > 0$ such that uniformly in $\mu \in \mathcal{P}_{\mathrm{wl}}$,*

$$\lim_{t\to\infty} \mathbb{P}_t^\mu \Big(\sup\Big\{ |V_\omega(x) - V_\omega(m_\mu) - p_t(x - m_\mu)| :$$
$$|x - m_\mu| \le 2t^{\frac{\alpha-d+2}{4\alpha}} \log t \Big\} < t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon} \Big) = 1$$

Let us first see that this immediately implies Proposition 3.1. Indeed, denoting the event in Proposition 3.2 by $\mathrm{PC}_t(\mu)$, we obtain

$$\begin{aligned} Q_t(\mathrm{PC}_t(L_t)) &\ge Q_t\left(\mathrm{PC}_t(L_t),\, L_t \in \mathcal{P}_{\mathrm{wl}}\right) \\ &= \frac{1}{Z_t} E_0\Big[\mathbb{E}[e^{-t\langle L_t, V_\omega \rangle}] \mathbb{P}_t^{L_t}(\mathrm{PC}_t(L_t)) : L_t \in \mathcal{P}_{\mathrm{wl}} \Big] \\ &\sim \frac{1}{Z_t} E_0\Big[\mathbb{E}[e^{-t\langle L_t, V_\omega \rangle}] : L_t \in \mathcal{P}_{\mathrm{wl}} \Big] \\ &= Q_t\left(L_t \in \mathcal{P}_{\mathrm{wl}}\right) \to 1 \end{aligned}$$

as $t \to \infty$ by Proposition 2.1 and Corollary 2.1.

To prove Proposition 3.2, we first compute the expectation of $V_\omega(x) - V_\omega(m_\mu)$ and then bound the variance.

LEMMA 3.1.   *There exists an $\epsilon > 0$ such that uniformly in $x \in B(0, 2t^{\frac{\alpha-d+2}{4\alpha}} \log t)$ and $\mu \in \mathcal{P}_{\mathrm{wl}}$,*

$$\mathbb{E}_t^\mu[V_\omega(x) - V_\omega(m_\mu)] = p_t(x - m_\mu) + o\left(t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon}\right)$$

as $t \to \infty$.

PROOF. By a well known formula for Poisson point process,

$$\begin{aligned} &\mathbb{E}_t^\mu[V_\omega(x) - V_\omega(m_\mu)] \\ &\quad = \int (\hat{v}(x - y) - \hat{v}(m_\mu - y)) e^{-t \int \hat{v}(z-y)\mu(\mathrm{d}z)} \mathrm{d}y. \end{aligned}$$

We assume $m_\mu = 0$ by translation. Pick $\delta > 0$ so small that

$$(3.1) \qquad \frac{\alpha - d + 6}{8\alpha} < \frac{1}{\alpha} - \delta \quad \text{and} \quad \delta(\alpha + 2) < \frac{d + 2 - \alpha}{2\alpha}.$$

Then uniformly in $\mu \in \mathcal{P}_{\mathrm{wl}}$,

$$\left| \int_{|y| \le t^{\frac{1}{\alpha} - \delta}} (\hat{v}(x - y) - \hat{v}(-y)) e^{-t \int \hat{v}(z-y)\mu(\mathrm{d}z)} \mathrm{d}y \right|$$
$$\le \exp\left\{ -t \inf\left\{ |z - y|^{-\alpha} : |z| \le M_1 t^{\frac{\alpha - d + 6}{8\alpha}}, |y| \le t^{\frac{1}{\alpha} - \delta} \right\} + O(\log t) \right\}$$
$$= \mathcal{O}\left(\exp\left\{ -t^{\alpha\delta} \right\} \right)$$

and hence this part is negligible. On the other hand, if $|y| > t^{1/\alpha - \delta}$, we can replace $\hat{v}$ in the integrand by $v(\cdot) = |\cdot|^{-\alpha}$ and hence

$$\int_{|y| > t^{\frac{1}{\alpha} - \delta}} (\hat{v}(x - y) - \hat{v}(-y)) e^{-t \int \hat{v}(z-y)\mu(\mathrm{d}z)} \mathrm{d}y$$
$$= t^{-\frac{\alpha - d}{\alpha}} \int_{|\eta| > t^{-\delta}} (|\eta - t^{-\frac{1}{\alpha}} x|^{-\alpha} - |\eta|^{-\alpha}) e^{-\int |\eta - t^{-\frac{1}{\alpha}} z|^{-\alpha} \mu(\mathrm{d}z)} \mathrm{d}\eta.$$

Let us write $D_t(\eta, x) = |\eta - t^{-\frac{1}{\alpha}} x|^{-\alpha} - |\eta|^{-\alpha}$ for short and further decompose the integral in the last line as

$$(3.2)$$
$$t^{-\frac{\alpha - d}{\alpha}} \int_{|\eta| > t^{-\delta}} D_t(\eta, x) e^{-|\eta|^{-\alpha}} + D_t(\eta, x) \left(e^{-\int D_t(\eta, z)\mu(\mathrm{d}z)} - 1 \right) e^{-|\eta|^{-\alpha}} \mathrm{d}\eta.$$

Let us first see that the first term of (3.2) gives us the desired quantity. We use the following expansions of $D_t(\eta, \cdot)$, which immediately follow from Taylor's theorem.

LEMMA 3.2. *For* $|z| \le M_1 t^{\frac{\alpha - d + 6}{8\alpha}}$ *and* $|\eta| > t^{-\delta}$, *there exist bounded functions* R_1, R_2, *and* R_3 *such that*

$$D_t(\eta, z)$$
$$(3.3) \quad = R_1(z, \eta) t^{-\frac{1}{\alpha}} |z| |\eta|^{-\alpha - 1}$$
$$(3.4) \quad = t^{-\frac{1}{\alpha}} \langle z, \nabla v(\eta) \rangle + R_2(z, \eta) t^{-\frac{2}{\alpha}} |z|^2 |\eta|^{-\alpha - 2}$$
$$(3.5) \quad = t^{-\frac{1}{\alpha}} \langle z, \nabla v(\eta) \rangle + \frac{1}{2} t^{-\frac{2}{\alpha}} \langle z, \mathrm{Hess}_v(\eta) z \rangle + R_3(z, \eta) t^{-\frac{3}{\alpha}} |z|^3 |\eta|^{-\alpha - 3}.$$

Using (3.5), one can deduce

$$t^{-\frac{\alpha-d}{\alpha}} \int_{|\eta|>t^{-\delta}} D_t(\eta, x) \exp\left\{-|\eta|^{-\alpha}\right\} \mathrm{d}\eta$$
$$= t^{-\frac{\alpha-d+2}{\alpha}} C(d,\alpha)|x|^2 + O\left(t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon}\right)$$

for some $\epsilon > 0$ by a straightforward calculation. (Note that $\int \nabla v(\eta) e^{-|\eta|^{-\alpha}} \mathrm{d}\eta = 0$ by symmetry.)

Now let us turn to the estimate of the second term of (3.2). As a consequence of (3.4), we have

(3.6)
$$\left|\int D_t(\eta, z)\mu(\mathrm{d}z)\right| = \|R_2\|_\infty |\eta|^{-\alpha-2} t^{-\frac{2}{\alpha}} \int |z|^2 \mu(\mathrm{d}x)$$
$$= O\left(t^{\delta(\alpha+2)-\frac{d+2-\alpha}{2\alpha}}\right)$$

uniformly in $|\eta| > t^{-\delta}$ and $\mu \in \mathcal{P}_{\mathrm{wl}}$, where we have used $m_\mu = 0$. Thanks to our choice of δ, the last line goes to 0 as $t \to \infty$. Therefore we may use an inequality $|e^{-a} - 1| \leq 2|a|$ valid for small a in the second term of (3.2) to obtain

$$t^{-\frac{\alpha-d}{\alpha}} \left|\int_{|\eta|>t^{-\delta}} D_t(\eta, x) \left(e^{-\int D_t(\eta,z)\mu(\mathrm{d}z)} - 1\right) e^{-|\eta|^{-\alpha}} \mathrm{d}\eta\right|$$
$$\leq 2t^{-\frac{\alpha-d}{\alpha}} \int_{|\eta|>t^{-\delta}} |D_t(\eta, x)| \left|\int D_t(\eta, z)\mu(\mathrm{d}z)\right| e^{-|\eta|^{-\alpha}} \mathrm{d}\eta.$$

Applying (3.3) and (3.4) to $D_t(\eta, x)$ and $D_t(\eta, z)$ respectively and then using the variance bound for μ, one can easily conclude that the above right-hand side is of order $O(t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon})$ for some $\epsilon > 0$. $\qquad\square$

PROOF OF PROPOSITION 3.2. Throughout the proof we assume $\mu \in \mathcal{P}_{\mathrm{wl}}$ and all the estimates below are supposed to be uniform in μ. We further assume $m_\mu = 0$ for simplicity. That this causes no loss of generality will become clear in the argument below. In view of Lemma 3.1, it suffices to estimate the maximum of

$$|V_\omega(x) - V_\omega(0) - \mathbb{E}_t^\mu[V_\omega(x) - V_\omega(0)]|$$
$$= \int (\hat{v}(x-y) - \hat{v}(-y)) \left(\omega(\mathrm{d}y) - e^{-t\int \hat{v}(z-y)\mu(\mathrm{d}z)}\mathrm{d}y\right)$$

among $|x| \leq r(t) \log t$ (recall $r(t) = t^{\frac{\alpha-d+2}{4\alpha}}$). Fix $\delta > 0$ as in (3.1). We first show that the region $\{|y| \leq t^{1/\alpha-\delta}\}$ makes only negligible contribution. Let us introduce the notation

$$\bar{\omega}_t^\mu(\mathrm{d}y) = \omega(\mathrm{d}y) - e^{-\int \hat{v}(z-y)\mu(\mathrm{d}z)}\mathrm{d}y.$$

Observe that

$$\left| \int_{|y| \le t^{\frac{1}{\alpha} - \delta}} (\hat{v}(x - y) - \hat{v}(-y)) \bar{\omega}_t^\mu(\mathrm{d}y) \right|$$

$$\le \int_{|y| \le t^{\frac{1}{\alpha} - \delta}} |\hat{v}(x - y) - \hat{v}(-y)| \left(\omega(\mathrm{d}y) + e^{-t \int \hat{v}(z-y)\mu(\mathrm{d}z)} \mathrm{d}y \right)$$

$$\le \int_{|y| \le t^{\frac{1}{\alpha} - \delta}} \bar{\omega}_t^\mu(\mathrm{d}y) + 2 \int_{|y| \le t^{\frac{1}{\alpha} - \delta}} e^{-t \int \hat{v}(z-y)\mu(\mathrm{d}z)} \mathrm{d}y$$

since $0 < \hat{v} \le 1$. The first term has zero $\mathbb{E}_t^\mu$-mean and its variance, which equals the half of second term, is seen to be of $\mathcal{O}(\exp\{-t^{\alpha\delta}\})$ by the same argument as in the proof of Lemma 3.1. Therefore, Chebyshev's inequality yields

$$\lim_{t \to \infty} \mathbb{P}_t^\mu \left(\sup_{x \in \mathbb{R}^d} \left| \int_{|y| \le t^{\frac{1}{\alpha} - \delta}} (\hat{v}(x - y) - \hat{v}(-y)) \bar{\omega}_t^\mu(\mathrm{d}y) \right| < t^{-\frac{\alpha - d + 2}{2\alpha} - \epsilon} \right) = 1$$

for any $\epsilon > 0$.

On the remaining part $\{|y| > t^{1/\alpha - \delta}\}$, we may replace $\hat{v}$ by $v(\cdot) = |\cdot|^{-\alpha}$. Then it follows by Lemma 3.2 that

$$\left| \int_{|y| > t^{\frac{1}{\alpha} - \delta}} (\hat{v}(x - y) - \hat{v}(-y)) \bar{\omega}_t^\mu(\mathrm{d}y) \right|$$

$$\le \left| \left\langle x, \int_{|y| > t^{\frac{1}{\alpha} - \delta}} \nabla v(y) \bar{\omega}_t^\mu(\mathrm{d}y) \right\rangle \right|$$

(3.7)

$$+ \left| \left\langle x, \int_{|y| > t^{\frac{1}{\alpha} - \delta}} \mathrm{Hess}_v(y) \bar{\omega}_t^\mu(\mathrm{d}y) x \right\rangle \right|$$

$$+ |x|^3 \left| \int_{|y| > t^{\frac{1}{\alpha} - \delta}} R_3(x, t^{-\frac{1}{\alpha}} y) |y|^{-\alpha - 3} \bar{\omega}_t^\mu(\mathrm{d}y) \right|$$

$$=: I_1(x) + I_2(x) + I_3(x).$$

Let us start with $I_3(x)$. As we did for $\{|y| \le t^{1/\alpha - \delta}\}$, we can bound the integral as

$$\sup_{|x| \le r(t) \log t} \left| \int_{y > t^{\frac{1}{\alpha} - \delta}} R_3(x, t^{-\frac{1}{\alpha}} y) |y|^{-\alpha - 3} \bar{\omega}_t^\mu(\mathrm{d}y) \right|$$

(3.8)

$$\le \int_{|y| > t^{\frac{1}{\alpha} - \delta}} \|R_3\|_\infty |y|^{-\alpha - 3} \bar{\omega}_t^\mu(\mathrm{d}y)$$

$$+ 2 \int_{|y| > t^{\frac{1}{\alpha} - \delta}} \|R_3\|_\infty |y|^{-\alpha - 3} e^{-\int |z - y|^{-\alpha} \mu(\mathrm{d}z)} \mathrm{d}y.$$

The second term on the right-hand side can be seen to be of $O(t^{-\frac{\alpha-d+3}{\alpha}})$, by using the change of variable $y = t^{1/\alpha}\eta$ and then (3.6) to replace $\int|\eta - t^{-1/\alpha}z|^{-\alpha}\mu(\mathrm{d}z)$ by $|\eta|^{-\alpha}$. Since $|x|^3 t^{-\frac{\alpha-d+3}{\alpha}} = O\left(t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon}\right)$ uniformly in $|x| \le r(t)\log t$ for some $\epsilon > 0$, the second term is negligible. By the same way, we can show that

$$
(3.9) \quad
\begin{aligned}
&\mathbb{V}\mathrm{ar}_t^\mu\left(\int_{|y|>t^{\frac{1}{\alpha}-\delta}}\|R_3\|_\infty|y|^{-\alpha-3}\bar\omega_t^\mu(\mathrm{d}y)\right) \\
&= \int_{|y|>t^{\frac{1}{\alpha}-\delta}}\|R_3\|_\infty^2|y|^{-2\alpha-6}e^{-\int|z-y|^{-\alpha}\mu(\mathrm{d}z)}\mathrm{d}y
\end{aligned}
$$

is of order $O(t^{-\frac{2\alpha-d+6}{\alpha}})$. Therefore by using Chebyshev's inequality, we obtain

$$
\begin{aligned}
&\mathbb{P}_t^\mu\left(\sup_{|x|\le r(t)\log t}|x|^3\left|\int_{|y|>t^{\frac{1}{\alpha}-\delta}}\|R_3\|_\infty|y|^{-\alpha-3}\bar\omega_t^\mu(\mathrm{d}y)\right| \ge t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon}\right) \\
&\le \mathbb{P}_t^\mu\left(\left|\int_{|y|>t^{\frac{1}{\alpha}-\delta}}\|R_3\|_\infty|y|^{-\alpha-3}\bar\omega_t^\mu(\mathrm{d}y)\right| \ge t^{-\frac{5(\alpha-d+2)}{4\alpha}-\frac{\epsilon}{2}}\right) \\
&= O\left(t^{-\frac{3d+2-\alpha}{2\alpha}+\epsilon}\right),
\end{aligned}
$$

which goes to 0 as $t \to \infty$ for sufficiently small ϵ.

As for I_1 and I_2, we use variance bounds

$$
\mathbb{V}\mathrm{ar}_t^\mu\left(\int_{|y|>t^{\frac{1}{\alpha}-\delta}}\nabla v(y)\bar\omega_t^\mu(\mathrm{d}y)\right) = O\left(t^{-\frac{2\alpha-d+2}{\alpha}}\right),
$$

$$
\mathbb{V}\mathrm{ar}_t^\mu\left(\int_{|y|>t^{\frac{1}{\alpha}-\delta}}\mathrm{Hess}_v(y)\bar\omega_t^\mu(\mathrm{d}y)\right) = O\left(t^{-\frac{2\alpha-d+4}{\alpha}}\right),
$$

which follow by routine arguments. Given these bounds, one can easily conclude that

$$
\lim_{t\to\infty}\mathbb{P}_t^\mu\left(\sup_{|x|\le r(t)\log t}(I_1(x) + I_2(x)) < t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon}\right) = 1
$$

for some $\epsilon > 0$. $\qquad\square$

REMARK 3.1. Inspecting the above proof, one can easily improve the bound as

$$
\lim_{t\to\infty}Q_t\Big(\sup\Big\{|V_\omega(x) - V_\omega(m_{L_t}) - \mathbb{E}_t^\mu[V_\omega(x) - V_\omega(m_{L_t})]| :
$$

$$
|x - m_{L_t}| \le 2t^{\frac{\alpha-d+2}{4\alpha}}\log t\Big\} < t^{-\beta}\Big) = 1
$$

for any $\beta < \min\{\frac{3\alpha-d+2}{4\alpha}, \frac{\alpha-d+6}{4\alpha}\}$. This will be used in Section 6.

4. Strong localization. In this section, we accomplish the proof of Theorem 1.1. Loosely speaking, we do this by simply repeating the argument for Proposition 2.1 but in the correct scale. Recall that we used an annulus A_R which was much larger than the correct scale $r(t) = t^{\frac{\alpha-d+2}{4\alpha}}$ in the proof of Proposition 2.1. It was because we had control of V_ω only at points in the intermediate distance from $\mathcal{L}_{t,\omega}$ (see Lemma 2.3). Now that Proposition 3.1 is available, we have the control of V_ω near m_{L_t} and can work in the correct scale.

PROOF OF THEOREM 1.1. Set $G = \{L_t \in \mathcal{P}_{\mathrm{wl}}\} \cap \mathtt{PC}(L_t)$. As we have already shown $\lim_{t\to\infty} Q_t(G) = 1$, we restrict ourselves to G. Though we drop $\cap G$ from the notation for simplicity, it is assumed throughout the proof.

We first fix $q \in \mathbb{Z}^d$ and assume $m_{L_t} \in q + [0,1)^d$. Let us set $R = \sqrt{2M_2}$ and define

$$H_1(q) = \inf\left\{ s \geq 0 : |X_s - q| \leq Rt^{\frac{\alpha-d+2}{4\alpha}} \right\}.$$

Clearly $\{H_1(q) < t\} \supset \{L_t \in \mathcal{P}_{\mathrm{wl}}\} \cap \{m_{L_t} \in q + [0,1)^d\}$. Note also that on $\mathtt{PC}(L_t)$, we have

$$(4.1) \qquad \lambda_1^\omega((-t,t)^d) \leq V_\omega(m_{L_t}) + (a_2 + o(1))t^{-\frac{\alpha-d+2}{2\alpha}}$$

by an application of Rayleigh-Ritz's variational formula together with a simple cut-off argument.

Now, fix $\delta \in (0, 1/2)$ and define

$$H_2(q) = \inf\left\{ s \geq 0 : |X_s - q| > t^{\frac{\alpha-d+2}{4\alpha}} (\log t)^{\frac{1}{2}+\delta} \right\}.$$

We first consider the case $H_1(q) < H_2(q) \leq t$. In this case, set

$$H_3(q) = \sup\left\{ s \leq H_2(q) : |X_s - q| \leq \frac{1}{2} t^{\frac{\alpha-d+2}{4\alpha}} (\log t)^{\frac{1}{2}+\delta} \right\}.$$

Then for $s \in [H_3(q), H_2(q)]$ and $c < C(d,\alpha)/4$,

$$V_\omega(X_s) \geq V_\omega(m_{L_t}) + ct^{-\frac{\alpha-d+2}{2\alpha}} (\log t)^{1+2\delta}$$
$$\geq \lambda_1^\omega((-t,t)^d) + \frac{c}{2} t^{-\frac{\alpha-d+2}{2\alpha}} (\log t)^{1+2\delta}$$

by Proposition 3.1 and (4.1). Let us define $E_{k,l}(q) = \{H_3(q) \in [k, k+1), H_2(q) \in [l, l+1)\}$ for integers $0 \leq k \leq l \leq t-1$. Suppose first that $l - k > t^{\frac{\alpha-d+2}{2\alpha}}$ (slow crossing). Then using the Markov property at time l

and then (2.8) exactly the same way as in the proof of Proposition 2.1, we obtain

$$
\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} : E_{k,l}(q) \right]
$$

$$
\leq \mathbb{E}\Big[c(d)^2 (1 + (t\lambda_1^\omega((-t,t)^d))^{\frac{d}{2}})^2 \exp\left\{ -(k+1)\lambda_1^\omega((-t,t)^d) \right\}
$$

(4.2)
$$
\exp\left\{ -(l-k-1)\left(\lambda_1^\omega((-t,t)^d) + \frac{c}{2}t^{-\frac{\alpha-d+2}{2\alpha}}(\log t)^{1+2\delta} \right) \right\}
$$

$$
\exp\left\{ -(t-l)\lambda_1^\omega((-t,t)^d) \right\} \Big]
$$

$$
= \mathcal{O}\left(\exp\left\{ -(\log t)^{1+2\delta} \right\} \right) \mathbb{E}\left[\exp\left\{ -t\lambda_1^\omega((-t,t)^d) \right\} \right].
$$

For the case $l - k \leq t^{\frac{\alpha-d+2}{2\alpha}}$ (fast crossing), we again proceed as in the proof of (2.7) to see

$$
\mathbb{E}\left[\exp\left\{ -\int_0^t V_\omega(X_s)\mathrm{d}s \right\} : E_{k,l}(q) \right]
$$

$$
\leq \mathbb{E}\Big[c(d)^2 (1 + (t\lambda_1^\omega((-t,t)^d))^{\frac{d}{2}})^2 \exp\left\{ -t\lambda_1^\omega((-t,t)^d) \right\}
$$

(4.3)
$$
P_0\left(\sup\left\{ |X_s| : 0 \leq s \leq t^{\frac{\alpha-d+2}{2\alpha}} \right\} > t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\delta} \right) \Big]
$$

$$
\leq \mathcal{O}\left(\exp\left\{ -(\log t)^{1+2\delta} \right\} \right) \mathbb{E}\left[\exp\left\{ -t\lambda_1^\omega((-t,t)^d) \right\} \right].
$$

Summing (4.2) and (4.3) over $0 \leq k \leq l \leq t-1$ and using Lemma 2.4, we find that

$$
Q_t\left(H_1(q) < H_2(q) \leq t, m_{L_t} \in q + [0,1)^d \right) = \mathcal{O}\left(\exp\left\{ -(\log t)^{1+2\delta} \right\} \right).
$$

The other case $H_2(q) < H_1(q) \leq t$ can also be treated in the same way as above by using

$$
\tilde{H}_3(q) = \sup\left\{ s \in [0, H_1(q)) : |X_s - q| > t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\delta} \right\},
$$

$$
\tilde{H}_2(q) = \inf\left\{ s \geq \tilde{H}_3(q) : |X_s - q| \leq \frac{1}{2}t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\delta} \right\}
$$

instead of $H_3(q)$ and $H_2(q)$ respectively. Consequently, we obtain

$$
Q_t\left(H_2(q) \leq t, m_{L_t} \in q + [0,1)^d \right) = \mathcal{O}\left(\exp\left\{ -(\log t)^{1+2\delta} \right\} \right).
$$

Since the possible values of q is only polynomially many due to the weak localization result, we can sum over q to see

$$\lim_{t\to\infty} Q_t\left(\sup_{s\in[0,t]}|X_s - m_{L_t}| \le 2t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\delta}\right) = 1.$$

Finally observe that on the event in the left-hand side, we have $|m_{L_t}| \le 2t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\delta}$ since $X_0 = 0$. Therefore we arrive at the desired conclusion

$$\lim_{t\to\infty} Q_t\left(\sup_{s\in[0,t]}|X_s| \le 4t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\delta}\right) = 1.$$

$\square$

By the strong localization result, it in particular follows that m_{L_t} is close to $m_t(\omega)$, which completes the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Fix $\epsilon > 0$ so small that Proposition 3.1 holds. Then, for sufficiently large t,

$$t^{\frac{\alpha-d+2}{4\alpha}-\frac{\epsilon}{4}} < |x - m_{L_t}| < 2t^{\frac{\alpha-d+2}{4\alpha}}\log t$$

implies $V_\omega(x) > V_\omega(m_{L_t})$. Therefore, m_{L_t} is within $t^{\frac{\alpha-d+2}{4\alpha}-\frac{\epsilon}{4}}$ from the minimizer of V_ω in $B(m_{L_t}, 2t^{\frac{\alpha-d+2}{4\alpha}}\log t)$. On the other hand, Theorem 1.1 implies $m_{L_t} \in B(0, t^{\frac{\alpha-d+2}{4\alpha}}\log t)$ and thus the above minimizer is nothing but $m_t(\omega)$. It is now easy to deduce Theorem 1.2 from Proposition 3.1. $\square$

5. Scaling limit of the occupation time measure. We prove Theorem 1.3 in this section. Given Theorem 1.1 and 1.2, it is more or less straightforward. Indeed, what we do is, replacing V_ω by a quadratic function, using the Girsanov formula, and applying the large deviation principle for Ornstein-Uhlenbeck process.

PROOF OF THEOREM 1.3. Let us introduce the event

$$G_1 = \left\{\sup_{0\le u\le t}|X_u| < r(t)(\log t)^{\frac{3}{4}}\right\}.$$

By Theorem 1.1 and Proposition 3.1, we know that

$$Q_t\left(\tilde{L}_t \in A\right)$$

$$= Q_t\left(\tilde{L}_t \in A, G_1 \cap \mathtt{PC}(L_t)\right) + o(1)$$

$$= \frac{1}{Z_t}\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\} : \tilde{L}_t \in A, G_1 \cap \mathtt{PC}(L_t)\right] + o(1)$$

and thus we may concentrate on $G_1 \cap \mathtt{PC}(L_t)$. On this event, we have

$$
\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\} 1_{\mathtt{PC}(L_t)\cap G_1}
$$
$$
\leq \exp\left\{-tV_\omega(m_{L_t}) + o\left(t^{\frac{\alpha+d-2}{2\alpha}}\right)\right\}
$$
$$
\exp\left\{-C(d,\alpha)\int_0^{tr(t)^{-2}} |\tilde{X}_s - m_{\tilde{L}_t}|^2 \mathrm{d}s\right\} 1_{G_1}.
$$

Let $q \in t^{-1}\mathbb{Z}^d$ and $f \in C_c(\mathbb{R}^d)$. We integrate the above over

$$
F(q,f) = \left\{\|m_{\tilde{L}_t} - q\|_\infty \leq (2t)^{-1}, \left|\int f \mathrm{d}\tilde{L}_t - \int f \mathrm{d}\nu_q\right| > \epsilon\right\}.
$$

Note that if $|q| > 2\log t$, then $F(q,f) \cap G_1 = \emptyset$. Hence we assume $|q| \leq 2\log t$ in what follows. Since

$$
\int_0^{tr(t)^{-2}} |\tilde{X}_s - m_{\tilde{L}_t}|^2 \mathrm{d}s = \int_0^{tr(t)^{-2}} |\tilde{X}_s - q|^2 \mathrm{d}s + o\left(t^{\frac{\alpha+d-2}{2\alpha}}\right)
$$

on $G_1 \cap F(q,f)$, we have

$$
\mathbb{E} \otimes E_0\left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\} : \mathtt{PC}(L_t) \cap G_1 \cap F(q,f)\right]
$$
$$
\leq \exp\left\{-a_1 t^{\frac{d}{\alpha}} + o\left(t^{\frac{\alpha+d-2}{2\alpha}}\right)\right\}
$$
$$
E_0\left[\exp\left\{-C(d,\alpha)\int_0^{tr(t)^{-2}} |\tilde{X}_s - q|^2 \mathrm{d}s\right\} : G_1 \cap F(q,f)\right].
$$

By the scaling invariance of Brownian motion and the Girsanov formula, it follows that the right-hand side equals

$$
\exp\left\{-a_1 t^{\frac{d}{\alpha}} - (a_2 + o(1))t^{\frac{\alpha+d-2}{2\alpha}}\right\} R_0^q\left[\frac{\psi(-q)}{\psi(X_{tr(t)^{-2}} - q)} : \tilde{G}_1 \cap \tilde{F}(q,f)\right],
$$

where

$$
\psi(x) = \exp\left\{-\sqrt{\frac{C(d,\alpha)}{2}}|x|^2\right\},
$$
$$
\tilde{G}_1 = \left\{\sup_{s\in[0,tr(t)^{-2}]} |X_s| \leq \log t\right\},
$$
$$
\tilde{F}(q,f) = \left\{\|m_{L_{tr(t)^{-2}}} - q\|_\infty \leq (2t)^{-1}, \left|\int f \mathrm{d}L_{tr(t)^{-2}} - \int f \mathrm{d}\nu_q\right| > \epsilon\right\}.
$$

Recalling $|q| \le 2 \log t$, we have

$$\frac{\psi(-q)}{\psi(X_{tr(t)^{-2}} - q)} = \mathcal{O}\left(\exp\left\{(\log t)^2\right\}\right)$$

on $\tilde{G}_1$. It is well known that L_s under R_0^q satisfies the full large deviation principle with rate s in the space of probability measures equipped with weak topology and the rate function has unique zero at ν_q, see e.g. [3]. Therefore, we arrive at

$$\frac{1}{Z_t} \mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\} : \mathtt{PC}(L_t) \cap G_1 \cap F(q, f)\right]$$
$$\le \exp\left\{o\left(t^{\frac{\alpha+d-2}{2\alpha}}\right)\right\} R_0^q\left(\left|\int f\mathrm{d}L_{tr(t)^{-2}} - \int f\mathrm{d}\nu_q\right| > \epsilon\right)$$
$$\le \exp\left\{\delta(\epsilon)t^{\frac{\alpha+d-2}{2\alpha}}\right\}$$

for some $\delta(\epsilon) > 0$. Summing over $q \in t^{-1}\mathbb{Z}^d \cap B(0, 2\log t)$ and replacing m_{L_t} by $m_t(\omega)$ using Theorem 1.2, we obtain (1.6). $\square$

6. Fluctuation of the local minimum of the potential. We prove Theorem 1.4 in this section. The argument is similar to that in Section 3 but we need a better control on $\int |x - m_{\tilde{L}_t}|^2 \tilde{L}_t(\mathrm{d}x)$. We deduce it from Theorem 1.3.

PROOF OF THEOREM 1.4. We first prove

(6.1) $$\lim_{t\to\infty} Q_t\left(\left|\int |x - m_{\tilde{L}_t}|^2 \tilde{L}_t(\mathrm{d}x) - (8C(d, \alpha))^{-\frac{1}{2}}\right| > \epsilon\right) = 0$$

for any $\epsilon > 0$. To this end, recall that Corollary 2.1 allows us to assume

$$\int |x - m_{\tilde{L}_t}|^2 \tilde{L}_t(\mathrm{d}x) \le M_2.$$

Under this assumption,

$$\left|\int |x - m_{\tilde{L}_t}|^2 \tilde{L}_t(\mathrm{d}x) - \int |x - m_{\tilde{L}_t}|^2 \nu_{m_{\tilde{L}_t}}(\mathrm{d}x)\right| > \epsilon$$

implies

$$\left|\int \left(|x - m_{\tilde{L}_t}|^2 \wedge M\right) \tilde{L}_t(\mathrm{d}x) - \int \left(|x - m_{\tilde{L}_t}|^2 \wedge M\right) \nu_{m_{\tilde{L}_t}}(\mathrm{d}x)\right| > \frac{\epsilon}{2}$$

for sufficiently large M. Since $\int |x|^2 \nu_0(\mathrm{d}x) = (8C(d,\alpha))^{-\frac{1}{2}}$, (6.1) follows from Theorem 1.3. Now let us define the class of probability measures on $\mathbb{R}^d$ by

$$\mathcal{P}_{\mathrm{loc}}^\epsilon := \left\{ \mu : \mathrm{supp}\mu \subset B(0, t^{\frac{\alpha-d+2}{4\alpha}} \log t), \right.$$

$$\left. \left| \int |x - m_\mu|^2 \mu(\mathrm{d}x) - (8C(d,\alpha))^{-\frac{1}{2}} \right| < \epsilon t^{-\frac{\alpha-d+2}{2\alpha}} \right\}.$$

By Theorem 1.1 and (6.1), we know that $\lim_{t\to\infty} Q_t(L_t \notin \mathcal{P}_{\mathrm{loc}}^\epsilon) = 0$ for any $\epsilon > 0$.

Now let us prove Theorem 1.4-(i). It suffices to show the assertion with $m_t(\omega)$ replaced by m_{L_t} in view of (1.5) and Lemma 3.1. For $\mu \in \mathcal{P}_{\mathrm{loc}}^\epsilon$, the same argument as for Lemma 3.1 yields

(6.2)
$$\mathbb{E}_t^\mu[V_\omega(m_\mu)]$$
$$= \int \hat{v}(m_\mu - y) e^{-t \int \hat{v}(z-y)\mu(\mathrm{d}z)} \mathrm{d}y$$
$$= a_1 \frac{d}{\alpha} t^{-\frac{\alpha-d}{\alpha}} - \left(\frac{\alpha-d+2}{\alpha} C(d,\alpha) + o(1) \right) t^{-\frac{\alpha-d+2}{\alpha}} \int |x - m_\mu|^2 \mu(\mathrm{d}x),$$

where $o(1)$ is uniform in μ. Then it follows

$$Q_t[V_\omega(m_{L_t})]$$
$$\sim \frac{1}{Z_t} E_0 \left[\mathbb{E} \left[e^{-t\langle L_t, V_\omega \rangle} \right] \mathbb{E}_t^{L_t}[V_\omega(m_{L_t})] : L_t \in \mathcal{P}_{\mathrm{loc}}^\epsilon \right]$$
$$= a_1 \frac{d}{\alpha} t^{-\frac{\alpha-d}{\alpha}} - \frac{\alpha-d+2}{\alpha} \sqrt{\frac{C(d,\alpha)}{8}} t^{-\frac{\alpha-d+2}{2\alpha}} + \epsilon O \left(t^{-\frac{\alpha-d+2}{2\alpha}} \right)$$

and letting $\epsilon \downarrow 0$, we get Theorem 1.4-(i).

Let us turn to Theorem 1.4-(ii). It suffices to consider the assertion with $m_t(\omega)$ replaced by m_{L_t} again. Indeed, as is mentioned in Remark 3.1, we have

$$\lim_{t\to\infty} Q_t \left(|V_\omega(m_t(\omega)) - V_\omega(m_{L_t}) - \mathbb{E}_t^\mu[V_\omega(m_t(\omega)) - V_\omega(m_{L_t})]| < t^{-\beta} \right) = 1$$

for any

$$\frac{2\alpha - d}{2\alpha} < \beta < \min \left\{ \frac{3\alpha - d + 2}{4\alpha}, \frac{\alpha - d + 6}{4\alpha} \right\}.$$

Now it follows by a well-known formula for the characteristic functional for Poisson point process that for $\mu \in \mathcal{P}_{\text{loc}}^{\epsilon}$,

$$\log \mathbb{E}_t^{\mu} \left[\exp \left\{ i\theta t^{\frac{2\alpha-d}{2\alpha}} \left(V_{\omega}(m_{\mu}) - \mathbb{E}_t^{\mu}[V_{\omega}(m_{\mu})] \right) \right\} \right]$$
$$= \int \left(e^{i\theta t^{\frac{2\alpha-d}{2\alpha}} \hat{v}(m_{\mu}-y)} - 1 - i\theta t^{\frac{2\alpha-d}{2\alpha}} \hat{v}(m_{\mu}-y) \right) e^{-t \int \hat{v}(z-y)\mu(\mathrm{d}z)} \mathrm{d}y.$$

Using Taylor's theorem, replacing $\int \hat{v}(z-y)\mu(\mathrm{d}z)$ by $v(m_{\mu}-y)$, and changing variable, one can show that

$$\log \mathbb{E}_t^{\mu} \left[\exp \left\{ i\theta t^{\frac{2\alpha-d}{2\alpha}} \left(V_{\omega}(m_{\mu}) - \mathbb{E}_t^{\mu}[V_{\omega}(m_{\mu})] \right) \right\} \right]$$
$$\sim -\frac{\theta^2}{2} \int t^{\frac{2\alpha-d}{\alpha}} |y|^{-2\alpha} e^{-t|y|^{-\alpha}} \mathrm{d}y$$
$$= -\frac{\theta^2}{2} \alpha \sigma_d \Gamma \left(\frac{3\alpha - d + 1}{\alpha} \right).$$

We leave the details to the reader. From this, Theorem 1.4-(ii) follows by the same way as above. $\qquad\square$

7. Scaling limit of the process. In this section, we prove Theorem 1.5. Given the strong localization and the potential confinement, we can basically follow the argument in [11]. We write B_t for $B(m_t(\omega), r(t)\log t)$.

PROOF OF THEOREM 1.5. We define the good events by

$$G_1(s) = \left\{ \sup_{0 \leq u \leq s} |X_u| < r(t)(\log t)^{\frac{3}{4}} \right\},$$
$$G_2 = \left\{ \sup_{x \in B_t} |V_{\omega}(x) - V_{\omega}(m_t(\omega)) - p_t(x - m_t(\omega))| \leq t^{-\frac{\alpha-d+2}{2\alpha}-\epsilon_0}, \right.$$
$$\left. |m_t(\omega)| \leq r(t)(\log t)^{\frac{3}{4}} \right\}.$$

Note that $X_{[0,t]} \subset B_t$ on $G_1(t) \cap G_2$. Due to Theorems 1.1 and 1.2, it suffices to show that for any $T > 0$ and f a bounded continuous function on $C([0,T], \mathbb{R}^d)$,

$$\lim_{t \to \infty} Q_t[f(\tilde{X}) : G_1(t) \cap G_2]$$
$$= \int \mathrm{d}m \left(\frac{\sqrt{C(d,\alpha)}}{\sqrt{2\pi}} \right)^{\frac{d}{2}} \exp \left\{ -\sqrt{\frac{C(d,\alpha)}{2}} |m|^2 \right\} R_0^m[f(X)],$$

with an obvious abuse of notation. Let $\lambda_i^\omega(B_t)$ denote the i-th smallest eigenvalue of $-\Delta/2 + V_\omega(x) - V_\omega(m_t(\omega))$ in B_t with the Dirichlet boundary condition and ϕ_ω the L^2-normalized non-negative principal eigenfunction. Let us further denote by ψ_ω the L^2-normalized non-negative principal eigenfunctions of $-\Delta/2 + p_t(x - m_t(\omega))$ on $\mathbb{R}^d$.

LEMMA 7.1. (i) $\lambda_i^\omega(B_t) = V_\omega(m_t(\omega)) + (a_2 + O(t^{-\epsilon_0}))t^{-\frac{\alpha-d+2}{2\alpha}}$ uniformly in $i \in \{1, 2\}$ and $\omega \in G_2$.
(ii) There exists $c_3 > 0$ such that for any $\omega \in G_2$,

$$\lambda_2^\omega(B_t) - \lambda_\omega^1(B_t) \ge c_3 t^{-\frac{\alpha-d+2}{2\alpha}}.$$

(iii) $\sup_{\omega \in G_2} \|\phi_\omega - \psi_\omega\|_2 \to 0$ as $t \to \infty$.

PROOF. It may be assumed that $m_t(\omega) = 0$ by a spatial shift. Let $\lambda_i^{p_t}(U)$ denote the i-th smallest eigenvalue of $-\Delta/2 + p_t(x)$ in U with the Dirichlet boundary condition. Recall that when $U = \mathbb{R}^d$, the corresponding eigenfunctions are products of the Hermite polynomials multiplied by the Gaussian density with variance $\sqrt{r(t)/2C(d,\alpha)}$. In particular, they are of $\mathcal{O}(\exp\{-(\log t)^2\})$ near ∂B_t. Thus, by using the Rayleigh-Ritz variational formula and a standard cut-off argument, one can show that

$$(7.1) \qquad \lambda_i^{p_t}(B_t) = \lambda_i^{p_t}(\mathbb{R}^d) + \mathcal{O}\left(\exp\left\{-(\log t)^2\right\}\right).$$

Since

$$(7.2) \qquad |(\lambda_i^\omega(B_t) - V_\omega(m_t(\omega))) - \lambda_i^{p_t}(B_t)| \le t^{-\frac{\alpha-d+2}{2\alpha} - \epsilon_0}$$

on G_2, the first assertion follows. Moreover, we know

$$(7.3) \qquad \begin{aligned} \lambda_2^{p_t}(\mathbb{R}^d) - \lambda_1^{p_t}(\mathbb{R}^d) &= r(t)^{-2}\lambda_2^{p_1}(\mathbb{R}^d) - r(t)^{-2}\lambda_1^{p_1}(\mathbb{R}^d) \\ &\ge c_3 t^{-\frac{\alpha-d+2}{2\alpha}} \end{aligned}$$

for some $c_3 > 0$ by a scaling and the fact that $-\Delta/2 + C(d,\alpha)|x|^2$ has positive spectral gap. From (7.1)–(7.3), the second assertion follows.

Next, we introduce the Dirichlet form $(\mathcal{E}_\omega, \mathcal{D}(\mathcal{E}_\omega))$ associated with $-\Delta/2 + p_t(x)$ on $L^2(\mathbb{R}^d)$. Then $\phi_\omega \in \mathcal{D}(\mathcal{E}_\omega)$ since ϕ_ω has compact support and $\int |\nabla \phi_\omega|^2 dx < \infty$. Let us decompose ϕ_ω as $\gamma_t \psi_\omega + \sqrt{1 - \gamma_t^2}\psi_2$ by using $\psi_2 \in \mathcal{D}(\mathcal{E}_\omega)$ with unit L^2-norm and orthogonal to ψ_ω with respect to $\mathcal{E}_\omega(\cdot, \cdot) + (\cdot, \cdot)_{L^2}$. Note that we have

$$\mathcal{E}_\omega(\psi_\omega, \psi_2) = ((-\Delta/2 + p_t)\psi_\omega, \psi_2)_{L^2} = \lambda_1^{p_t}(\mathbb{R}^d)(\psi_\omega, \psi_2)_{L^2}$$

since ψ_ω is an eigenfunction. Hence in fact $\mathcal{E}_\omega(\psi_\omega, \psi_2) = (\psi_\omega, \psi_2)_{L^2} = 0$ and it follows that

$$
\begin{aligned}
(7.4) \qquad \mathcal{E}_\omega(\phi_\omega, \phi_\omega) &= \gamma_t^2 \mathcal{E}_\omega(\psi_\omega, \psi_\omega) + (1 - \gamma_t^2) \mathcal{E}_\omega(\psi_2, \psi_2) \\
&= \gamma_t^2 \lambda_1^{p_t}(\mathbb{R}^d) + (1 - \gamma_t^2) \mathcal{E}_\omega(\psi_2, \psi_2).
\end{aligned}
$$

Now for $\omega \in G_2$, one can easily find

$$
\begin{aligned}
\mathcal{E}_\omega(\phi_\omega, \phi_\omega) \\
&= \frac{1}{2} \int |\nabla \phi_\omega(x)|^2 \mathrm{d}x + \int (V_\omega(x) - V_\omega(0)) \phi_\omega(x)^2 \mathrm{d}x + o\left(t^{-\frac{\alpha-d+2}{2\alpha}}\right) \\
&= \lambda_1^\omega(B_t) - V_\omega(0) + o\left(t^{-\frac{\alpha-d+2}{2\alpha}}\right) \\
&= \lambda_1^{p_t}(\mathbb{R}^d) + o\left(t^{-\frac{\alpha-d+2}{2\alpha}}\right),
\end{aligned}
$$

where the last line is due to part (i). On the other hand, by the variational formula and (7.3),

$$
\mathcal{E}_\omega(\psi_2, \psi_2) \geq \lambda_2^{p_t}(\mathbb{R}^d) \geq \lambda_1^{p_t}(\mathbb{R}^d) + c_3 t^{-\frac{\alpha-d+2}{2\alpha}}.
$$

Substituting these relations into (7.4), we obtain $\gamma_t^2 \to 1$ as $t \to \infty$. Finally, since both ϕ_ω and ψ_ω are non-negative, γ_t must converge to 1 and the last assertion is proved. $\qquad \square$

Let us define $\tau = t^{\frac{\alpha-d+2}{2\alpha} + \frac{\epsilon_0}{2}}$ and

$$
\begin{aligned}
F_1(x, \omega) &= E_x\left[\exp\left\{-\int_0^{t-\tau} V_\omega(X_s)\mathrm{d}s\right\} : G_1(t - \tau)\right], \\
F_2(x, \omega) &= \langle \phi_\omega, 1\rangle \phi_\omega(x) \exp\left\{-(t - \tau)\lambda_1^\omega(B_t)\right\}
\end{aligned}
$$

for the ease of notation. Note that $\tau > Tr(t)^2$ for sufficiently large t. Applying the Markov property at time τ, we find that the numerator of $Q_t[f(\tilde{X}) : G_1(t) \cap G_2]$ is

$$
\begin{aligned}
\mathbb{E} \otimes E_0 &\left[f(\tilde{X}) \exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\} : G_1(t) \cap G_2\right] \\
(7.5) \qquad = \mathbb{E}&\left[E_0\left[f(\tilde{X}) \exp\left\{-\int_0^\tau V_\omega(X_s)\mathrm{d}s\right\}\right.\right. \\
&\left.\left. \qquad F_1(X_\tau, \omega) : G_1(\tau)\right] : G_2\right].
\end{aligned}
$$

To replace F_1 by F_2 in this expression, we estimate the difference as

$$
(7.6) \quad
\begin{aligned}
&\left| E_0 \left[f(\tilde{X}) \exp \left\{ - \int_0^\tau V_\omega(X_s) \mathrm{d}s \right\} \right. \right. \\
&\qquad \left. \left. (F_1(X_\tau, \omega) - F_2(X_\tau, \omega)) : G_1(\tau) \right] \right| \\
&\leq \|f\|_\infty \int p(1,0,x) S_{\tau-1}^\omega |F_1(\cdot, \omega) - F_2(\cdot, \omega)|(x) \mathrm{d}x \\
&\leq \|f\|_\infty \left(\int p(1,0,x)^2 \mathrm{d}x \right)^{\frac{1}{2}} \|S_{\tau-1}^\omega |F_1(\cdot, \omega) - F_2(\cdot, \omega)| \|_2,
\end{aligned}
$$

where $p(t,x,y)$ denotes the transition kernel of the Brownian motion and $\{S_s^\omega\}_{s \geq 0}$ the semigroup generated by $-\Delta/2 + V_\omega$ in B_t with the Dirichlet boundary condition. Now, it is well known that $\|S_s^\omega\|_{L^2 \to L^2} = \exp\{-s\lambda_1^\omega(B_t)\}$ and also by considering eigenfunction expansion, one can deduce from Lemma 7.1-(ii) that

$$
\|F_1(\cdot, \omega) - F_2(\cdot, \omega)\|_2 \leq |B_t|^{\frac{1}{2}} \exp\left\{ -(t-\tau)\left(\lambda_1^\omega(B_t) + c_3 t^{-\frac{\alpha-d+2}{2\alpha}} \right) \right\}
$$

on G_2. Therefore for sufficiently large t,

$$
(7.7) \qquad \text{RHS of } (7.6) \leq \exp\left\{ -t\lambda_1^\omega(B_t) - \frac{c_3}{2} t^{\frac{\alpha+d-2}{2\alpha}} \right\}.
$$

Coming back to (7.5) and replacing F_1 by F_2, we arrive at

$$
(7.8) \quad
\begin{aligned}
\text{LHS of } (7.5) &= \mathbb{E}\left[E_0 \left[f(\tilde{X}) \phi_\omega(X_\tau) \exp\left\{ -\int_0^\tau V_\omega(X_s) \mathrm{d}s \right\} : G_1(\tau) \right] \right. \\
&\qquad \left. \langle \phi_\omega, 1 \rangle e^{-(t-\tau)\lambda_1^\omega(B_t)} : G_2 \right] \\
&\quad + \mathcal{O}\left(\exp\left\{ -t^{\frac{\alpha+d-2}{2\alpha}} \right\} \right) \mathbb{E}\left[\exp\left\{ -t\lambda_1^\omega(B_t) \right\} \right].
\end{aligned}
$$

Recall that the last term is of $o(Z_t)$ due to Lemma 2.4.

LEMMA 7.2. *Uniformly in $\omega \in G_2$,*

$$
(7.9) \quad
\begin{aligned}
&E_0\left[f(\tilde{X}) \phi_\omega(X_\tau) \exp\left\{ -\int_0^\tau V_\omega(X_s)\mathrm{d}s \right\} : G_1(\tau) \right] \\
&= e^{-\tau\lambda_1^\omega(B_t) + o(1)} \left(\psi_\omega(0) R_0^{\tilde{m}_t(\omega)}[f(X)] + \mathcal{O}\left(\exp\left\{ -(\log t)^2 \right\} \right) \right).
\end{aligned}
$$

PROOF. Let us use the orthogonal decomposition $\phi_\omega = \gamma_t \psi_\omega + \sqrt{1 - \gamma_t^2}\psi_2$ in the proof of Lemma 7.1. Denoting the left-hand side of (7.9) by $T(\phi_\omega)$, we have

$$(7.10) \qquad T(\phi_\omega) = \gamma_t T(\psi_\omega) + \sqrt{1 - \gamma_t^2}T(\psi_2).$$

We begin with the first term in the right-hand side. We know $\gamma_t \to 1$ by Lemma 7.1-(iii) and on G_2,

$$
\begin{aligned}
T(\psi_\omega) &= E_0\left[f(\tilde{X})\psi_\omega(X_\tau)\exp\left\{-\int_0^\tau V_\omega(X_s)\mathrm{d}s\right\} : G_1(\tau)\right] \\
&\leq \exp\left\{-\tau V_\omega(m_t(\omega)) + t^{-\frac{\epsilon_0}{2}}\right\} \\
&\qquad E_0\left[f(\tilde{X})\psi_\omega(X_\tau)\exp\left\{-\int_0^\tau p_t(X_s - m_t(\omega))\mathrm{d}s\right\} : G_1(\tau)\right]
\end{aligned}
$$

thanks to our choice of τ. On the other hand, it follows by the Girsanov transform and Brownian scaling that

$$
\begin{aligned}
R_0^{\tilde{m}_t(\omega)}&\left[f(X) : X_{[0,t^{\epsilon_0}]} \subset r(t)^{-1}B_t\right] \\
&= e^{a_2}E_0\left[f(X)\frac{\psi_\omega(r(t)X_{r(t)^{-2}\tau})}{\psi_\omega(0)}\right.\\
&\qquad\qquad \left. \exp\left\{-\int_0^\tau p_t(r(t)X_{r(t)^{-2}s} - m_t(\omega))\mathrm{d}s\right\} : G_1(\tau)\right].
\end{aligned}
$$

Combining the above estimates and using Lemma 7.1-(i), we obtain

$$\gamma_t T(\psi_\omega) = e^{-\tau\lambda_1^\omega(B_t)+o(1)}\psi_\omega(0)R_0^{\tilde{m}_t(\omega)}\left[f(X) : X_{[0,t^{\epsilon_0}/2]} \subset r(t)^{-1}B_t\right].$$

Finally, by the very same argument as for the proof of the strong localization, it follows that

$$R_0^{\tilde{m}_t(\omega)}\left(X_{[0,t^{\epsilon_0}/2]} \not\subset r(t)^{-1}B_t\right) = \mathcal{O}\left(\exp\left\{-(\log t)^2\right\}\right)$$

on G_2. (We have used the second condition in G_2 to control $\psi_\omega(0)$.)

Next, we estimate the second term on the right-hand side of (7.10). This is rather easy since by the same argument as for (7.7), it follows

$$
\begin{aligned}
|T(\psi_2)| &\leq \|f\|_\infty \exp\left\{-(\tau - 1)\left(\lambda_1^\omega(B_t) + c_3 t^{-\frac{\alpha-d+2}{2\alpha}}\right)\right\} \\
&\leq \|f\|_\infty \exp\left\{-\tau\lambda_1^\omega(B_t) - \frac{c_3}{2}t^{\epsilon_0}\right\}.
\end{aligned}
$$

$\square$

Substituting (7.9) into (7.8) and dropping $o(Z_t)$ term, we obtain

$$Q_t[f(\tilde{X}) : G_1(t) \cap G_2]$$
$$\sim \frac{1}{Z_t} \mathbb{E}\Big[e^{-t\lambda_1^\omega(B_t)}\langle\phi_\omega, 1\rangle\Big(\psi_\omega(0)R_0^{\tilde{m}_t(\omega)}[f(X)]$$
$$+ \mathcal{O}\left(\exp\left\{-(\log t)^2\right\}\right)\Big) : G_2\Big].$$

The term $\mathcal{O}(\exp\left\{-(\log t)^2\right\})$ is negligible in view of Lemma 2.4 and the fact that $|\langle\phi_\omega, 1\rangle| \leq \|\phi_\omega\|_{L^2(B_t)}\|1\|_{L^2(B_t)}$ grows at most polynomially fast. Therefore we arrive at the expression

$$Q_t[f(\tilde{X}) : G_1(t) \cap G_2]$$
(7.11)
$$\sim \int \frac{1}{Z_t} \mathbb{E}\left[e^{-t\lambda_1^\omega(B_t)}\langle\phi_\omega, 1\rangle : G_2, \tilde{m}_t(\omega) \in \mathrm{d}m\right]$$
$$\left(\frac{\sqrt{C(d,\alpha)}}{\sqrt{2}\pi r(t)}\right)^{\frac{d}{4}} \exp\left\{-\sqrt{\frac{C(d,\alpha)}{2}}|m|^2\right\} R_0^m[f(X)].$$

Since $\mathbb{E}[e^{-t\lambda_1^\omega(B_t)}\langle\phi_\omega, 1\rangle : G_2, \tilde{m}_t(\omega) \in \mathrm{d}m]$ defines a translation invariant measure on $B(0, (\log t)^{3/4})$, it is a constant multiple of the Lebesgue measure. We can determine the constant asymptotically by setting $f = 1$ and it follows that the right-hand side of (7.11) converges to

$$\int \mathrm{d}m \left(\frac{\sqrt{C(d,\alpha)}}{\sqrt{2}\pi}\right)^{\frac{d}{2}} \exp\left\{-\sqrt{\frac{C(d,\alpha)}{2}}|m|^2\right\} R_0^m[f(X)].$$

$\square$

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